

Chapter 2

Loop Measures

2.1 A Measure on Based Loops

We denote by \mathbb{P}^x the family of probability laws on piecewise constant paths defined by P_t .

$$\mathbb{P}^x(\gamma(t_1) = x_1, \dots, \gamma(t_h) = x_h) = P_{t_1}(x, x_1)P_{t_2-t_1}(x_1, x_2) \dots P_{t_h-t_{h-1}}(x_{h-1}, x_h)$$

The corresponding process is a Markov chain in continuous time. It can also be constructed as the process ξ_{N_t} , where ξ_n is the discrete time Markov chain starting at x , with transition matrix P , and N_t an independent Poisson process.

In the transient case, the lifetime is a.s. finite and denoting by $p(\gamma)$ the number of jumps and T_i the jump times, we have:

$$\begin{aligned} \mathbb{P}^x(p(\gamma) = k, \gamma_{T_1} = x_1, \dots, \gamma_{T_k} = x_k, T_1 \in dt_1, \dots, T_k \in dt_k) \\ = \frac{C_{x, x_1} \dots C_{x_{k-1}, x_k} K_{x_k}}{\lambda_x \lambda_{x_1} \dots \lambda_{x_k}} 1_{\{0 < t_1 < \dots < t_k\}} e^{-t_k} dt_1 \dots dt_k \end{aligned}$$

For any integer $p \geq 2$, let us define a *based loop* with p points in X as a couple $l = (\xi, \tau) = ((\xi_m, 1 \leq m \leq p), (\tau_m, 1 \leq m \leq p+1))$ in $X^p \times \mathbb{R}_+^{p+1}$, and set $\xi_{p+1} = \xi_1$ (equivalently, we can parametrize the associated discrete based loop by $\mathbb{Z}/p\mathbb{Z}$). The integer p represents the number of points in the discrete based loop $\xi = (\xi_1, \dots, \xi_{p(\xi)})$ and will be denoted $p(\xi)$, and the τ_m are holding times. Note however that two time parameters are attached to the base point since the based loops do not in general end or start with a jump.

Based loops with one point ($p = 1$) are simply given by a pair (ξ, τ) in $X \times \mathbb{R}_+$.

Based loops have a natural time parametrization $l(t)$ and a time period $T(\xi) = \sum_{i=1}^{p(\xi)+1} \tau_i$. If we denote $\sum_{i=1}^m \tau_i$ by T_m : $l(t) = \xi_m$ on $[T_{m-1}, T_m)$ (with by convention $T_0 = 0$ and $\xi_1 = \xi_{p+1}$).

Let $\mathbb{P}_t^{x,y}$ denote the (non normalized) “bridge measure” on piecewise constant paths from x to y of duration t constructed as follows:

If $t_1 < t_2 < \dots < t_h < t$,

$$\mathbb{P}_t^{x,y}(l(t_1) = x_1, \dots, l(t_h) = x_h) = [P_{t_1}]_{x_1}^{x_h} [P_{t_2-t_1}]_{x_2}^{x_1} \dots [P_{t-t_h}]_y^{x_h} \frac{1}{\lambda_y}$$

Its mass is $p_t^{x,y} = \frac{[P_t]_y^x}{\lambda_y}$. For any measurable set A of piecewise constant paths indexed by $[0, t]$, we can also write

$$\mathbb{P}_t^{x,y}(A) = \mathbb{P}_x(A \cap \{x_t = y\}) \frac{1}{\lambda_y}.$$

Exercise 8. Prove that $\mathbb{P}_t^{y,x}$ is the image of $\mathbb{P}_t^{x,y}$ by the operation of time reversal on paths indexed by $[0, t]$.

A σ -finite measure μ is defined on based loops by

$$\mu = \sum_{x \in X} \int_0^\infty \frac{1}{t} \mathbb{P}_t^{x,x} \lambda_x dt$$

Remark 4. The introduction of the factor $\frac{1}{t}$ will be justified in the following. See in particular formula (2.3). It can be interpreted as the normalization of the uniform measure on the loop, according to which the base point is chosen.

From the expression of the bridge measure, we see that by definition of μ , if $t_1 < t_2 < \dots < t_h < t$,

$$\mu(l(t_1) = x_1, \dots, l(t_h) = x_h, T \in dt) = [P_{t_1+t-t_h}]_{x_1}^{x_h} [P_{t_2-t_1}]_{x_2}^{x_1} \dots [P_{t_h-t_{h-1}}]_{x_h}^{x_{h-1}} \frac{1}{t} dt. \quad (2.1)$$

Note also that for $k > 1$, using the second expression of $\mathbb{P}_t^{x,y}$ and the fact that conditionally on $N_t = k$, the jump times are distributed like an increasingly reordered k -uniform sample of $[0, t]$

$$\begin{aligned} \lambda_x \mathbb{P}_t^{x,x}(p = k, \xi_1 = x_1, \xi_2 = x_2, \dots, \xi_k = x_k, T_1 \in dt_1, \dots, T_k \in dt_k) \\ = 1_{\{x=x_1\}} e^{-t} \frac{t^k}{k!} P_{x_2}^{x_1} P_{x_3}^{x_2} \dots P_{x_1}^{x_k} 1_{\{0 < t_1 < \dots < t_k < t\}} \frac{k!}{t^k} dt_1 \dots dt_k \\ = 1_{\{x=x_1\}} P_{x_2}^{x_1} P_{x_3}^{x_2} \dots P_x^{x_k} 1_{\{0 < t_1 < \dots < t_k < t\}} e^{-t} dt_1 \dots dt_k \end{aligned}$$

Therefore,

$$\mu(p = k, \xi_1 = x_1, \dots, \xi_k = x_k, T_1 \in dt_1, \dots, T_k \in dt_k, T \in dt) \quad (2.2)$$

$$= P_{x_2}^{x_1} \dots P_{x_1}^{x_k} \frac{1_{\{0 < t_1 < \dots < t_k < t\}}}{t} e^{-t} dt_1 \dots dt_k dt \quad (2.3)$$

for $k > 1$.

Moreover, for one point-loops, $\mu\{p(\xi) = 1, \xi_1 = x_1, \tau_1 \in dt\} = \frac{e^{-t}}{t} dt$.

It is clear on these formulas that for any positive constant c , the energy forms e and ce define the same loop measure.

2.2 First Properties

Note that the loop measure is invariant under time reversal.

If D is a subset of X , the restriction of μ to loops contained in D , denoted μ^D is clearly the loop measure induced by the Markov chain killed at the exit of D . This can be called the *restriction property*.

Let us recall that this killed Markov chain is defined by the restriction of λ to D and the restriction P^D of P to D^2 (or equivalently by the restriction e_D of the Dirichlet form e to functions vanishing outside D).

As $\int \frac{t^{k-1}}{k!} e^{-t} dt = \frac{1}{k}$, it follows from (2.2) that for $k > 1$, on based loops,

$$\mu(p(\xi) = k, \xi_1 = x_1, \dots, \xi_k = x_k) = \frac{1}{k} P_{x_2}^{x_1} \dots P_{x_1}^{x_k}. \quad (2.4)$$

In particular, we obtain that, for $k \geq 2$

$$\mu(p = k) = \frac{1}{k} Tr(P^k)$$

and therefore, as $Tr(P) = 0$, in the transient case:

$$\mu(p > 1) = \sum_2^\infty \frac{1}{k} Tr(P^k) = -\log(\det(I - P)) = \log(\det(G) \prod_x \lambda_x) \quad (2.5)$$

since (denoting M_λ the diagonal matrix with entries λ_x), we have

$$\det(I - P) = \frac{\det(M_\lambda - C)}{\det(M_\lambda)}$$

Note that $\det(G)$ is defined as the determinant of the matrix $G^{x,y}$. It is the determinant of the matrix representing the scalar product defined on $\mathbb{R}^{|X|}$ (more precisely, on the space of measures on X) by G in any basis, orthonormal with respect to the natural euclidean scalar product on $\mathbb{R}^{|X|}$.

Moreover

$$\int p(l) 1_{\{p>1\}} \mu(dl) = \sum_2^{\infty} Tr(P^k) = Tr((I - P)^{-1}P) = Tr(GC)$$

2.3 Loops and Pointed Loops

It is clear on formula (2.1) that μ is invariant under the time shift that acts naturally on based loops.

A loop is defined as an equivalence class of based loops for this shift. Therefore, μ induces a *measure on loops also denoted by μ* .

A loop is defined by the discrete loop ξ° formed by the ξ_i in circular order, (i.e. up to translation) and the associated holding times. We clearly have:

$$\mu(\xi^\circ = (x_1, x_2, \dots, x_k)^\circ) = P_{x_2}^{x_1} \dots P_{x_1}^{x_k}$$

provided the loop is primitive i.e. does not have a non trivial period, as it is in this case formed by p equivalent based loops. Otherwise, the right hand side should be divided by the mutiplicity. However, loops are not easy to parametrize, that is why we will work mostly with based loops or with *pointed loops*. These are defined as based loops ending with a jump, or equivalently as loops with a starting point. They can be parametrized by a based discrete loop and by the holding times at each point. Calculations are easier if we work with based or pointed loops, even though we will deal only with functions independent of the base point.

The parameters of the pointed loop naturally associated with a based loop are ξ_1, \dots, ξ_p and

$$\tau_1 + \tau_{p+1} = \tau_1^*, \tau_i = \tau_i^*, \quad 2 \leq i \leq p$$

An elementary change of variables, shows the expression of μ on pointed loops can be written:

$$\mu(p = k, \xi_i = x_i, \tau_i^* \in dt_i) = P_{x_2}^{x_1} \dots P_{x_1}^{x_k} \frac{t_1}{\sum t_i} e^{-\sum t_i} dt_1 \dots dt_k. \quad (2.6)$$

Trivial ($p = 1$) pointed loops and trivial based loops coincide.

Note that loop functionals can be written

$$\Phi(l^\circ) = \sum 1_{\{p=k\}} \Phi_k((\xi_i, \tau_i^*), i = 1, \dots, k)$$

with Φ_k invariant under circular permutation of the variables (ξ_i, τ_i^*) .

Then, for non negative Φ_k

$$\int \Phi_k(l^\circ) \mu(dl) = \sum \int \Phi_k((x_i, t_i) i = 1, \dots, k) P_{x_2}^{x_1} \dots P_{x_1}^{x_k} e^{-\sum t_i} \frac{t_1}{\sum t_i} dt_1 \dots dt_k$$

and by invariance under circular permutation, the term t_1 can be replaced by any t_i . Therefore, adding up and dividing by k , we get that

$$\int \Phi_k(l^\circ) \mu(dl) = \sum \int \frac{1}{k} \Phi_k((x_i, t_i) i = 1, \dots, k) P_{x_2}^{x_1} \dots P_{x_1}^{x_k} e^{-\sum t_i} dt_1 \dots dt_k.$$

The expression on the right side, applied to any pointed loop functional defines a different measure on pointed loops, we will denote by μ^* . It induces the same measure as μ on loops.

We see on this expression that conditionally on the discrete loop, the holding times of the loop are independent exponential variables.

$$\mu^*(p = k, \xi_i = x_i, \tau_i^* \in dt_i) = \frac{1}{k} \prod_{i \in \mathbb{Z}/p\mathbb{Z}} \frac{C_{x_i, x_{i+1}}}{\lambda_{x_i}} e^{-t_i} dt_i \quad (2.7)$$

Conditionally on $p(\xi) = k$, T is a gamma variable of density $\frac{t^{k-1}}{(k-1)!} e^{-t}$ on \mathbb{R}_+ and $(\frac{\tau_i^*}{T}, 1 \leq i \leq k)$ an independent ordered k -sample of the uniform distribution on $(0, T)$ (whence the factor $\frac{1}{t}$). Both are independent, conditionally on the number of points p of the discrete loop. We see that μ on based loops is obtained from μ on the loops by choosing the base point uniformly. On the other hand, it induces a choice of ξ_1 biased by the size of the τ_i^* 's, different from μ^* for which this choice is uniform (whence the factor $\frac{1}{k}$). But we will consider only loop functionals for which μ and μ^* coincide.

It will be convenient to rescale the holding time at each ξ_i by λ_{ξ_i} and set

$$\widehat{\tau}_i = \frac{\tau_i^*}{\lambda_{\xi_i}}.$$

The discrete part of the loop is the most important, though we will see that to establish a connection with Gaussian fields it is necessary to consider occupation times. The simplest variables are the number of jumps from x to y , defined for every oriented edge (x, y)

$$N_{x,y} = \#\{i : \xi_i = x, \xi_{i+1} = y\}$$

(recall the convention $\xi_{p+1} = \xi_1$) and

$$N_x = \sum_y N_{x,y}$$

Note that $N_x = \#\{i \geq 1 : \xi_i = x\}$ except for trivial one point loops for which it vanishes.

Then, the measure on pointed loops (2.6) can be rewritten as:

$$\mu^*(p = 1, \xi = x, \widehat{\tau} \in dt) = e^{-\lambda_x t} \frac{dt}{t} \text{ and} \quad (2.8)$$

$$\mu^*(p = k, \xi_i = x_i, \widehat{\tau}_i \in dt_i) = \frac{1}{k} \prod_{x,y} C_{x,y}^{N_{x,y}} \prod_x \lambda_x^{-N_x} \prod_{i \in \mathbb{Z}/p\mathbb{Z}} \lambda_{\xi_i} e^{-\lambda_{\xi_i} t_i} dt_i. \quad (2.9)$$

Another *bridge measure* $\mu^{x,y}$ can be defined on paths γ from x to y :

$$\mu^{x,y}(d\gamma) = \int_0^\infty \mathbb{P}_t^{x,y}(d\gamma) dt.$$

Note that the mass of $\mu^{x,y}$ is $G^{x,y}$. We also have, with similar notations as the one defined for loops, p denoting the number of jumps

$$\begin{aligned} \mu^{x,y}(p(\gamma) = k, \gamma_{T_1} = x_1, \dots, \gamma_{T_{k-1}} = x_{k-1}, T_1 \in dt_1, \dots, T_k \in dt_k, T \in dt) \\ = \frac{C_{x,x_1} C_{x_1,x_2} \dots C_{x_{k-1},y}}{\lambda_x \lambda_{x_1} \dots \lambda_y} 1_{\{0 < t_1 < \dots < t_k < t\}} e^{-t} dt_1 \dots dt_k dt. \end{aligned}$$

From now on, we will assume, unless otherwise specified, that we are in the *transient case*.

For any $x \neq y$ in X and $s \in [0, 1]$, setting $P_v^{(s),u} = P_v^u$ if $(u, v) \neq (x, y)$ and $P_y^{(s),x} = sP_y^x$, we can prove in the same way as (2.5) that:

$$\mu(s^{N_{x,y}} 1_{\{p>1\}}) = -\log(\det(I - P^{(s)})).$$

Differentiating in $s = 1$, and remembering that for any invertible matrix function $M(s)$, $\frac{d}{ds} \log(\det(M(s))) = \text{Tr}(M'(s)M(s)^{-1})$, it follows that:

$$\mu(N_{x,y}) = [(I - P)^{-1}]_x^y P_y^x = G^{x,y} C_{x,y}$$

and

$$\mu(N_x) = \sum_y \mu(N_{x,y}) = \lambda_x G^{x,x} - 1 \quad (2.10)$$

(as $G(M_\lambda - C) = Id$).

Exercise 9. Show that more generally

$$\mu(N_{x,y}(N_{x,y} - 1) \dots (N_{x,y} - k + 1)) = (k - 1)! (G^{x,y} C_{x,y})^k.$$

Hint: Show that if $M''(s)$ vanishes,

$$\frac{d^n}{ds^n} \log(\det(M(s))) = (-1)^{n-1} (n - 1)! \text{Tr}((M'(s)M(s)^{-1})^n).$$

Exercise 10. Show that more generally, if x_i, y_i are n distinct oriented edges:

$$\mu\left(\prod N_{x_i, y_i}\right) = \prod C_{x_i, y_i} \frac{1}{n} \sum_{\sigma \in S_n} \prod G^{y_{\sigma(i)}, x_{\sigma(i+1)}}$$

Hint: Introduce $[P^{(s_1, \dots, s_n)}]_y^x$ equal to P_y^x if $(x, y) \neq (x_i, y_i)$ for all i , and equal to $s_i P_{y_i}^{x_i}$ if $(x, y) = (x_i, y_i)$.

We finally note that if $C_{x, y} > 0$, any path segment on the graph starting at x and ending at y can be naturally extended into a loop by adding a jump from y to x . We have the following

Proposition 4. For $C_{x, y} > 0$, the natural extension of $\mu^{x, y}$ to loops coincides with $\frac{N_{y, x}(l)}{C_{x, y}} \mu(dl)$.

Proof. The first assertion follows from the formulas, noticing that a loop l can be associated to $N_{y, x}(l)$ distinct bridges from x to y , obtained by “cutting” one jump from y to x . \square

Note that a) shows that the loop measure induces bridge measures $\mu^{x, y}$ when $C_{x, y} > 0$. If $C_{x, y}$ vanishes, an arbitrarily small positive perturbation creating a non vanishing conductance between x and y allows to do it. More precisely, denoting by $e^{(\varepsilon)}$ the energy form equal to e except for the additional conductance $C_{x, y}^{(\varepsilon)} = \varepsilon$, $\mu^{x, y}$ can be represented as $\frac{d}{d\varepsilon} \mu^{e^{(\varepsilon)}}|_{\varepsilon=0}$.

2.4 Occupation Field

To each loop l° we associate local times, i.e. an occupation field $\{\widehat{l}_x, x \in X\}$ defined by

$$\widehat{l}^x = \int_0^{T(l)} 1_{\{l(s)=x\}} \frac{1}{\lambda_{l(s)}} ds = \sum_{i=1}^{p(l)} 1_{\{\xi_i=x\}} \widehat{\tau}_i$$

for any representative $l = (\xi_i, \tau_i^*)$ of l° .

For a path γ , $\widehat{\gamma}$ is defined in the same way. Note that

$$\mu((1 - e^{-\alpha \widehat{l}^x}) 1_{\{p=1\}}) = \int_0^\infty e^{-t} (1 - e^{-\frac{\alpha}{\lambda_x} t}) \frac{dt}{t} = \log(1 + \frac{\alpha}{\lambda_x}). \quad (2.11)$$

The proof goes by expanding $1 - e^{-\frac{\alpha}{\lambda_x} t}$ before the integration, assuming first that α is small and then by analyticity of both members, or more elegantly, noticing that $\int_a^b (e^{-cx} - e^{-dx}) \frac{dx}{x}$ is symmetric in (a, b) and (c, d) , by Fubini's theorem.

In particular, $\mu(\widehat{l}^x 1_{\{p=1\}}) = \frac{1}{\lambda_x}$.

From formula (2.7), we get easily that the joint conditional distribution of $(\widehat{l}^x, x \in X)$ given $(N_x, x \in X)$ is a product of gamma distributions. In particular, from the expression of the moments of a gamma distribution, we get that for any function Φ of the discrete loop and $k \geq 1$,

$$\mu((\widehat{l}^x)^k 1_{\{p > 1\}} \Phi) = \lambda_x^{-k} \mu((N_x + k - 1) \dots (N_x + 1) N_x \Phi).$$

In particular, by (2.10) $\mu(\widehat{l}^x) = \frac{1}{\lambda_x} [\mu(N_x) + 1] = G^{x,x}$.

Note that functions of \widehat{l} are not the only functions naturally defined on the loops. Other such variables of interest are, for $n \geq 2$, the multiple local times, defined as follows:

$$\widehat{l}^{x_1, \dots, x_n} = \sum_{j=0}^{n-1} \int_{0 < t_1 < \dots < t_n < T} 1_{\{l(t_1)=x_{1+j}, \dots, l(t_{n-j})=x_n, \dots, l(t_n)=x_j\}} \prod \frac{1}{\lambda_{x_i}} dt_i.$$

It is easy to check that, when the points x_i are distinct,

$$\widehat{l}^{x_1, \dots, x_n} = \sum_{j=0}^{n-1} \sum_{1 \leq i_1 < \dots < i_n \leq p(l)} \prod_{l=1}^n 1_{\{\xi_{i_l} = x_{l+j}\}} \widehat{\tau}_{i_l}. \quad (2.12)$$

Note that in general $\widehat{l}^{x_1, \dots, x_k}$ cannot be expressed in terms of \widehat{l} , but

$$\widehat{l}^{x_1} \dots \widehat{l}^{x_n} = \frac{1}{n} \sum_{\sigma \in \mathcal{S}_n} \widehat{l}^{x_{\sigma(1)}, \dots, x_{\sigma(n)}}.$$

In particular, $\widehat{l}^{x, \dots, x} = \frac{1}{(n-1)!} [\widehat{l}^x]^n$. It can be viewed as a n -th self intersection local time.

One can deduce from the definitions of μ the following:

Proposition 5. $\mu(\widehat{l}^{x_1, \dots, x_n}) = G^{x_1, x_2} G^{x_2, x_3} \dots G^{x_n, x_1}$.

In particular, $\mu(\widehat{l}^{x_1} \dots \widehat{l}^{x_n}) = \frac{1}{n} \sum_{\sigma \in \mathcal{S}_n} G^{x_{\sigma(1)}, x_{\sigma(2)}} G^{x_{\sigma(2)}, x_{\sigma(3)}} \dots G^{x_{\sigma(n)}, x_{\sigma(1)}}$.

Proof. Let us denote $\frac{1}{\lambda_y} [P_t]_y^x$ by $p_t^{x,y}$ or $p_t(x, y)$. From the definition of $\widehat{l}^{x_1, \dots, x_n}$ and μ , $\mu(\widehat{l}^{x_1, \dots, x_n})$ equals:

$$\sum_x \lambda_x \sum_{j=0}^{n-1} \int \int_{\{0 < t_1 < \dots < t_n < t\}} \frac{1}{t} p_{t_1}(x, x_{1+j}) \dots p_{t-t_n}(x_{n+j}, x) \prod dt_i dt.$$

where sums of indices $k+j$ are computed mod (n) . By the semigroup property, it equals

$$\sum_{j=0}^{n-1} \int \int_{\{0 < t_1 < \dots < t_n < t\}} \frac{1}{t} p_{t_2-t_1}(x_{1+j}, x_{2+j}) \dots p_{t_1+t-t_n}(x_{n+j}, x_{1+j}) \prod dt_i dt.$$

Performing the change of variables $v_2 = t_2 - t_1, \dots, v_n = t_n - t_{n-1}, v_1 = t_1 + t - t_n$, and $v = t_1$, we obtain:

$$\begin{aligned}
& \sum_{j=0}^{n-1} \int_{\{0 < v < v_1, 0 < v_i\}} \frac{1}{v_1 + \dots + v_n} p_{v_2}(x_{1+j}, x_{2+j}) \dots p_{v_1}(x_{n+j}, x_{1+j}) \prod dv_i dv \\
&= \sum_{j=0}^{n-1} \int_{\{0 < v_i\}} \frac{v_1}{v_1 + \dots + v_n} p_{v_2}(x_{1+j}, x_{2+j}) \dots p_{v_1}(x_{n+j}, x_{1+j}) \prod dv_i \\
&= \sum_{j=1}^n \int_{\{0 < v_i\}} \frac{v_j}{v_1 + \dots + v_n} p_{v_2}(x_1, x_2) \dots p_{v_1}(x_n, x_1) \prod dv_i \\
&= \int_{\{0 < v_i\}} p_{v_2}(x_1, x_2) \dots p_{v_1}(x_n, x_1) \prod dv_i \\
&= G^{x_1, x_2} G^{x_2, x_3} \dots G^{x_n, x_1}.
\end{aligned}$$

Note that another proof can be derived from formula (2.12). \square

Exercise 11. (Shuffle product) Given two positive integers $n > k$, let $\mathcal{P}_{n,k}$ be the family of partitions of $\{1, 2, \dots, n\}$ into k consecutive non empty intervals $I_l = (i_l, i_l + 1, \dots, i_{l+1} - 1)$ with $i_1 = 1 < i_2 < \dots < i_k < i_{k+1} = n + 1$. Show that

$$\widehat{l}^{x_1, \dots, x_n} \widehat{l}^{y_1, \dots, y_m} = \sum_{j=0}^{m-1} \sum_{k=1}^{\inf(n, m)} \sum_{I \in \mathcal{P}_{n,k}} \sum_{J \in \mathcal{P}_{m,k}} \widehat{l}^{x_{I_1}, y_{j+J_1}, x_{I_2}, \dots, y_{j+J_k}}$$

where for example the term y_{j+J_1} appearing in the upper index should be read as $j + j_1, \dots, j + j_2 - 1$.

Similarly, we can define $N_{(x_1, y_1), \dots, (x_n, y_n)}$ to be

$$\sum_{j=0}^{n-1} \sum_{1 \leq i_1 < \dots < i_n \leq p(l)} \prod_{l=1}^n 1_{\{\xi_{i_l} = x_{l+j}, \xi_{i_l+1} = y_{l+j}\}}.$$

If $(x_i, y_i) = (x, y)$ for all i , it equals $\frac{N_{x,y}(N_{x,y}-1) \dots (N_{x,y}-n+1)}{(n-1)!}$.

Notice that

$$\prod N_{(x_i, y_i)} = \frac{1}{n} \sum_{\sigma \in \mathcal{S}_n} N_{(x_{\sigma(1)}, y_{\sigma(1)}), \dots, (x_{\sigma(n)}, y)}.$$

Then we have the following:

Proposition 6. $\int N_{(x_1, y_1), \dots, (x_n, y_n)}(l) \mu(dl) = \left(\prod C_{x_i, y_i} \right) G^{y_1, x_2} G^{y_2, x_3} \dots G^{y_n, x_1}.$

The proof is left as exercise.

Exercise 12. For $x_1 = x_2 = \dots = x_k$, we could define different self intersection local times

$$\widehat{l}^{x, (k)} = \sum_{1 \leq i_1 < \dots < i_k \leq p(l)} \prod_{l=1}^k 1_{\{\xi_{i_l} = x\}} \widehat{\tau}_{i_l}$$

which vanish on $N_x < k$. Note that

$$\widehat{l}^{x, (2)} = \frac{1}{2} ((\widehat{l}^x)^2 - \sum_{i=1}^{p(l)} 1_{\{\xi_i = x\}} (\widehat{\tau}_i)^2).$$

1. For any function Φ of the discrete loop, show that

$$\mu(\widehat{l}^{x, 2} \Phi) = \lambda_x^{-2} \mu \left(\frac{N_x(N_x - 1)}{2} 1_{\{N_x \geq 2\}} \Phi \right).$$

2. More generally prove in a similar way that

$$\mu(\widehat{l}^{x, (k)} \Phi) = \lambda_x^{-k} \mu \left(\frac{N_x(N_x - 1) \dots (N_x - k + 1)}{k!} 1_{\{N_x \geq k\}} \Phi \right).$$

Let us come back to the occupation field to compute its Laplace transform. From the Feynman–Kac formula, it comes easily that, denoting $M_{\frac{\chi}{\lambda}}$ the diagonal matrix with coefficients $\frac{\chi_x}{\lambda_x}$

$$\mathbb{P}_t^{x, x}(e^{-\langle \widehat{l}, \chi \rangle} - 1) = \frac{1}{\lambda_x} \left(\exp(t(P - I - M_{\frac{\chi}{\lambda}}))_x^x - \exp(t(P - I))_x^x \right).$$

Integrating in t after expanding, we get from the definition of μ (first for χ small enough):

$$\begin{aligned} \int (e^{-\langle \widehat{l}, \chi \rangle} - 1) d\mu(l) &= \sum_{k=1}^{\infty} \int_0^{\infty} [Tr((P - M_{\frac{\chi}{\lambda}})^k) - Tr((P)^k)] \frac{t^{k-1}}{k!} e^{-t} dt \\ &= \sum_{k=1}^{\infty} \frac{1}{k} [Tr((P - M_{\frac{\chi}{\lambda}})^k) - Tr((P)^k)] \\ &= -Tr(\log(I - P + M_{\frac{\chi}{\lambda}})) + Tr(\log(I - P)). \end{aligned}$$

Hence, as $Tr(\log) = \log(\det)$

$$\begin{aligned} \int (e^{-\langle \hat{l}, x \rangle} - 1) d\mu(l) &= \log[\det(-L(-L + M_{\chi/\lambda})^{-1})] \\ &= -\log \det(I + VM_{\frac{\chi}{\lambda}}) = \log \det(I + GM_{\chi}) \end{aligned}$$

which now holds for all non negative χ as both members are analytic in χ . Besides, by the “resolvent” equation (1.1):

$$\det(I + GM_{\chi})^{-1} = \det(I - G_{\chi}M_{\chi}) = \frac{\det(G_{\chi})}{\det(G)}. \quad (2.13)$$

Note that $\det(I + GM_{\chi}) = \det(I + M_{\sqrt{\chi}}GM_{\sqrt{\chi}})$ and $\det(I - G_{\chi}M_{\chi}) = \det(I - M_{\sqrt{\chi}}G_{\chi}M_{\sqrt{\chi}})$, so we can deal with symmetric matrices. Finally we have

Proposition 7. $\mu(e^{-\langle \hat{l}, x \rangle} - 1) = -\log(\det(I + M_{\sqrt{\chi}}GM_{\sqrt{\chi}})) = \log(\frac{\det(G_{\chi})}{\det(G)})$

Note that in particular $\mu(e^{-t\hat{l}^x} - 1) = -\log(1 + tG^{x,x})$. Consequently, the image measure of μ by \hat{l}^x is $1_{\{s>0\}} \frac{1}{s} \exp(-\frac{s}{G^{x,x}}) ds$.

Considering the Laguerre-type polynomials D_k with generating function

$$\sum_{k=0}^{\infty} t^k D_k(u) = e^{\frac{ut}{1+t}} - 1$$

and setting $\sigma_x = G^{x,x}$, we have:

Proposition 8. *The variables $\frac{1}{\sqrt{k}} \sigma_x^k D_k(\frac{\hat{l}^x}{\sigma_x})$ are orthonormal in $L^2(\mu)$ for $k > 0$, and more generally*

$$\mathbb{E}(\sigma_x^k D_k(\frac{\hat{l}^x}{\sigma_x}) \sigma_y^j D_j(\frac{\hat{l}^y}{\sigma_y})) = \frac{1}{k} \delta_{k,j} (G^{x,y})^{2k}.$$

Proof. By Proposition 7,

$$\begin{aligned} &\int (1 - e^{\frac{\hat{l}^x t}{1+\sigma_x t}}) (1 - e^{\frac{\hat{l}^y s}{1+\sigma_y s}}) \mu(dl) \\ &= \log(1 - \frac{\sigma_x t}{1 + \sigma_x t}) + \log(1 - \frac{\sigma_y s}{1 + \sigma_y s}) - \log \det \left(\begin{array}{cc} 1 - \frac{\sigma_x t}{1 + \sigma_x t} & -\frac{tG^{x,y}}{1 + \sigma_x t} \\ -\frac{sG^{x,y}}{1 + \sigma_y s} & 1 - \frac{\sigma_y s}{1 + \sigma_y s} \end{array} \right) \\ &= -\log(1 - st(G^{x,y})^2). \end{aligned}$$

The proposition follows by expanding both sides in powers of s and t , and identifying the coefficients. \square

Note finally that if χ has support in D , by the restriction property

$$\mu(1_{\{\widehat{l}(X \setminus D)=0\}}(e^{-\langle \widehat{l}, \chi \rangle} - 1)) = -\log(\det(I + M_{\sqrt{\chi}} G^D M_{\sqrt{\chi}})) = \log\left(\frac{\det(G_{\chi}^D)}{\det(G^D)}\right).$$

Here the determinants are taken on matrices indexed by D and G^D denotes the Green function of the process killed on leaving D .

For paths we have $\mathbb{P}_t^{x,y}(e^{-\langle \widehat{l}, \chi \rangle}) = \frac{1}{\lambda_y} \exp(t(L - M_{\frac{\chi}{\lambda}}))_{x,y}$. Hence

$$\mu^{x,y}(e^{-\langle \widehat{l}, \chi \rangle}) = \frac{1}{\lambda_y} ((I - P + M_{\chi/\lambda})^{-1})_{x,y} = [G_{\chi}]^{x,y}.$$

In particular, note that from the resolvent equation (1.1), we get that

$$G^{y,x} = [G_{\varepsilon \delta_x}]^{y,x} + \varepsilon [G_{\varepsilon \delta_x}]^{y,x} G^{x,x}.$$

Hence $\frac{[G_{\varepsilon \delta_x}]^{y,x}}{G^{y,x}} = \frac{1}{1 + \varepsilon G^{x,x}}$ and therefore, we obtain:

Proposition 9. *Under the probability $\frac{\mu^{y,x}}{G^{y,x}}$, \widehat{l}^x follows an exponential distribution of mean $G^{x,x}$.*

Also $\mathbb{E}^x(e^{-\langle \widehat{l}, \chi \rangle}) = \sum_y [G_{\chi}]^{x,y} \kappa_y$ i.e. $[G_{\chi} \kappa]^x$.

Finally, let us note that a direct calculation shows the following result, analogous to Proposition 4 in which the case $x = y$ was left aside.

Proposition 10. *On loops passing through x , $\mu^{x,x}(dl) = \widehat{l}^x \mu(dl)$.*

An alternative way to prove the proposition is to check it on multiple local times, using Exercise 11. It can be shown that the algebra formed by linear combinations of multiple local times generates the loop σ -field. Indeed, the discrete loop can be recovered by taking the multiple local time it indexes and noting it is the unique one of maximal index length among non vanishing multiple local times indexed by multiples in which consecutive points are distinct. Then it is easy to get the holding times as the product of any of their powers can be obtained from a multiple local time.

Remark 5. Propositions 4 and 10 can be generalized: For example, if x_i are n points, $\widehat{l}^{x_1, \dots, x_n} \mu(dl)$ can be obtained as the image by circular concatenation of the product of the bridge measures $\mu^{x_i, x_{i+1}}(dl)$ and $\prod \widehat{l}^{x_i} \mu(dl)$ can be obtained as the sum of the images, by concatenation in all circular orders, of the product of the bridge measures $\mu^{y_{\sigma(i)}, x_{\sigma(i+1)}}(dl)$. If (x_i, y_i) are n oriented edges, $\prod \frac{N_{x_i, y_i}(l)}{C_{x_i, y_i}} \mu(dl)$ can be obtained as the sum of the images, by concatenation in all circular orders σ , of the product of the bridge measures $\mu^{y_{\sigma(i)}, x_{\sigma(i+1)}}(dl)$. One can also evaluate expressions of the form $\prod \widehat{l}^{z_j} \prod \frac{N_{x_i, y_i}(l)}{C_{x_i, y_i}} \mu(dl)$ as a sum of images, by concatenation in all circular orders, of a product of bridge measures.

2.5 Wreath Products

The following construction gives an interesting information about the number of distinct points visited by the loop, which is more difficult to evaluate than the occupation measure.

Associate to each point x of X an integer n_x . Let Z be the product of all the groups $\mathbb{Z}/n_x\mathbb{Z}$. On the wreath product space $X \times Z$, define a set of conductances $\tilde{C}_{(x,z),(x',z')}$ by:

$$\tilde{C}_{(x,z),(x',z')} = \frac{1}{n_x n_{x'}} C_{x,x'} \prod_{y \neq x, x'} 1_{\{z_y = z'_y\}}$$

and set $\tilde{\kappa}_{(x,z)} = \kappa_x$. This means in particular that in the associated Markov chain, the first coordinate is an autonomous Markov chain on X and that in a jump, the Z -configuration can be modified only at the point from which or to which the first coordinate jumps.

Denote by \tilde{e} the corresponding energy form. Note that $\tilde{\lambda}_{(x,z)} = \lambda_x$.

Then, denoting $\tilde{\mu}$ the loop measure and \tilde{P} the transition matrix on $X \times Z$ defined by \tilde{e} , we have the following

Proposition 11.

$$\prod_{x \in X} n_x \int 1_{\{p > 1\}} \prod_{x, N_x(l) > 0} \frac{1}{n_x} \mu(dl) = \tilde{\mu}(p > 1) = -\log(\det(I - \tilde{P})).$$

In particular, if $n_x = n$ for all x ,

$$n^{|X|} \int 1_{\{p > 1\}} n^{-\#\{x, N_x(l) > 0\}} \mu(dl) = \tilde{\mu}(p > 1) = -\log(\det(I - \tilde{P})).$$

Proof. Each time the Markov chain on $X \times Z$ defined by \tilde{e} jumps from a point above x to a point above y , z_x and z_y are resampled according to the uniform distribution on $\mathbb{Z}/n_x\mathbb{Z} \times \mathbb{Z}/n_y\mathbb{Z}$, while the other indices z_w are unchanged. It follows that

$$[\tilde{P}^k]_{(x,z)}^{(x,z)} = \sum_{x_1, \dots, x_{k-1}} P_{x_1}^x P_{x_2}^{x_1} \dots P_x^{x_{k-1}} \prod_{y \in \{x, x_1, \dots, x_{k-1}\}} \frac{1}{n_y}.$$

Note that in the set $\{x, x_1, \dots, x_{k-1}\}$, distinct points are counted only once, even if the path visit them several times. There are $\prod_{x \in X} n_x$ possible values for z . The detail of the proof is left as an exercise. \square

In the case where X is a group and P defines a random walk, \tilde{P} is associated with a random walk on $X \times Z$ equipped with its wreath product structure (Cf. [38]).

2.6 Countable Spaces

The assumption of finiteness of X can of course be relaxed but we will not do it in detail in these notes, though some infinite examples will be considered. On countable spaces, the previous results can be extended under transience conditions. In this case, the Dirichlet space \mathbb{H} is the space of all functions f with finite energy $e(f)$ which are limits in energy norm of functions with finite support, and the energy defines a Hilbertian scalar product on \mathbb{H} .

The energy of a measure is defined as $\sup_{f \in \mathbb{H}} \frac{\mu(f)^2}{e(f)}$. Finitely supported measures have finite energy. Measures of finite energy are elements of the dual \mathbb{H}^* of the Dirichlet space. The potential $G\mu$ is well defined for all finite energy measures μ , by the identity $e(f, G\mu) = \langle f, \mu \rangle$, valid for all f in the Dirichlet space. The energy of the measure μ equals $e(G\mu) = \langle G\mu, \mu \rangle$ (see [10] for more information).

It should also be noted that the submarkovianity of P (i.e. the non negativity of κ) is not essential in the construction of the loop measure μ . It has only to be positive and $I - P$ has to be invertible.

Most important examples of countable graphs are the non ramified covering of finite graphs (Recall that non ramified means that the projection is locally one to one, i.e. that the projection on X of each vertex v of the covering space has the same number of incident edges as v). Consider a non ramified covering graph (Y, F) defined by a normal subgroup H_{x_0} of Γ_{x_0} . The conductances C and the measure λ can be lifted in an obvious way to Y as $H_{x_0} \backslash \Gamma_{x_0}$ -periodic functions but the associated Green function \hat{G} or semigroup are non trivial. By applying $M_\lambda - C$, it is easy to check the following:

Proposition 12. $G^{x,y} = \sum_{\gamma \in H_{x_0} \backslash \Gamma_{x_0}} \hat{G}^{i(x), \gamma(i(y))}$ for any section i of the canonical projection from Y onto X .

Let us consider the universal covering (then H_{x_0} is trivial). It is easy to check it will be transient even in the recurrent case as soon as (X, E) is not circular.

The expression of the Green function \hat{G} on a universal covering can be given exactly when it is a regular tree, i.e. in the regular graph case. In fact a more general result can be proved as follows:

Given a graph (X, E) , set $d_x = \sum_y 1_{\{x,y\} \in E}$ (degree or valency of the vertex x), $D_{x,y} = d_x \delta_{x,y}$ and denote $A_{x,y}$ the incidence matrix $1_E(\{x, y\})$.

Consider the Green function associated with $\lambda_x = (d_x - 1)u + \frac{1}{u}$, with $0 < u < \inf(\frac{1}{d_x - 1}, x \in X)$ and for $\{x, y\} \in E$, $C_{x,y} = 1$.

Proposition 13. On the universal covering \mathfrak{T}_{x_0} , $\hat{G}^{x,y} = u^{d(x,y)} \frac{u}{1-u^2}$.

Proof. Note first that as $\frac{1}{u} > d_x - 1$, κ_x is positive for all x . Then $\hat{G} = (M_\lambda - C)^{-1}$ can be written $\hat{G} = [u^{-1}I + (D - I)u - A]^{-1}$. Moreover, since we are on a tree,

$$\sum_x A_{z,x} u^{d(x,y)} = (d_z - 1) u^{d(z,y)+1} + u^{d(z,y)-1}$$

for $z \neq y$, hence $\sum_x (\lambda_z \delta_x^z - A_{z,x}) u^{d(x,y)} = 0$ for $z \neq y$ and one checks it equals $\frac{1}{u} - u$ for $z = y$. \square

It follows from Proposition 12 that for any section i of the canonical projection from \mathfrak{T}_{x_0} onto X ,

$$\sum_{\gamma \in \Gamma_{x_0}} u^{d(i(x), \gamma(i(y)))} = \left(\frac{1}{u} - u\right) G^{x,y}.$$

2.7 Zeta Functions for Discrete Loops

We present briefly the terminology of symbolic dynamics (see for example [36]) in this simple framework: Setting $f(x_0, x_1, \dots, x_n, \dots) = \log(P_{x_0, x_1})$, P induces the Ruelle operator L_f associated with f .

The pressure is defined as the logarithm of the highest eigenvalue β of P . It is associated with a unique positive eigenfunction h (normalized in $L^2(\lambda)$), by Perron Frobenius theorem. Note that $Ph = \beta h$ implies $\lambda h P = \beta \lambda h$ by duality and that in the recurrent case, the pressure vanishes and $h = \frac{1}{\sqrt{\lambda(X)}}$.

In continuous time, the lowest eigenvalue of $-L$ i.e. $1 - \beta$ plays the role of the pressure.

The equilibrium measure associated with f , $m = h^2 \lambda$ is the law of the stationary Markov chain defined by the transition probability $\frac{1}{\beta h_x} P_y^x h_y$.

If $P1 = 1$, i.e. $\kappa = 0$, we can consider a Feynman Kac type perturbation $P^{(\varepsilon\kappa)} = PM_{\frac{\lambda}{\lambda + \varepsilon\kappa}}$, with $\varepsilon \downarrow 0$ and κ a positive measure. Perturbation theory (Cf. for example [13]) shows that $\beta^{(\varepsilon\kappa)} - 1 = \frac{1}{\lambda(X)} \sum_x \frac{\lambda_x}{1 + \varepsilon\kappa_x} - 1 + o(\varepsilon) = -\frac{\varepsilon\kappa(X)}{\lambda(X)} + o(\varepsilon)$ and that $h^{(\varepsilon\kappa)} = \frac{1}{\sqrt{\lambda(X)}} + o(\varepsilon)$.

We deduce from that the asymptotic behaviour of

$$\int (e^{-\varepsilon \langle \hat{l}, \chi \rangle} - 1) d\mu^{(\varepsilon\kappa)}(l) = \log(\det(I - P^{(\varepsilon\kappa)})) - \log(\det(I - P^{(\varepsilon(\kappa + \chi))}))$$

which is equivalent to $-\log(1 - \beta^{(\varepsilon(\kappa + \chi))}) + \log(1 - \beta^{(\varepsilon\kappa)})$ and therefore to $\log(\frac{\kappa(X)}{\kappa(X) + \chi(X)})$.

The study of relations between the loop measure μ and the zeta function $(\det(I - sP))^{-1}$ and more generally $(\det(I - M_f P))^{-1}$ with f a function on $[0, 1]$ can be done in the context of discrete loops.

$$\exp \left(\sum_{\substack{\text{based} \\ \text{discrete loops}}} \frac{1}{p(\xi)} s^{p(\xi)} \mu(\xi) \right) = (\det(I - sP))^{-1}$$

can be viewed as a type of zeta function defined for $s \in [0, 1/\beta)$

Primitive non trivial (based) discrete loops are defined as discrete based loops which cannot be obtained by the concatenation of $n \geq 2$ identical based loops. Loops are primitive iff they are classes of primitive based loops.

The zeta function has an Euler product expansion: if we denote by ξ° this discrete loop defined by the based discrete loop ξ , and set, for $\xi = (\xi_1, \dots, \xi_k)$, $\mu(\xi^\circ) = P_{\xi_2}^{\xi_1} P_{\xi_3}^{\xi_2} \dots P_{\xi_1}^{\xi_k}$, it can be seen, by taking the logarithm, that:

$$(\det(I - sP))^{-1} = \exp \left(\sum_{\substack{\text{based} \\ \text{discrete loops}}} \frac{1}{p(\xi)} s^{p(\xi)} \mu(\xi) \right) = \prod_{\substack{\text{primitive} \\ \text{discrete loops}}} \left(1 - \int s^{p(\xi^\circ)} \mu(\xi^\circ) \right)^{-1}$$

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