# Preface

The purpose of these notes is to explore some simple relations between Markovian path and loop measures, the Poissonian ensembles of loops they determine, their occupation fields, uniform spanning trees, determinants, and Gaussian Markov fields such as the free field. These relations are first studied in complete generality in the finite discrete setting, then partly generalized to specific examples in infinite and continuous spaces.

These notes contain the results published in [27] where the main emphasis was put on the study of occupation fields defined by Poissonian ensembles of Markov loops. These were defined in [18] for planar Brownian motion in relation with SLE processes and in [19] for simple random walks. They appeared informally already in [52]. For half integral values $\frac{k}{2}$ of the intensity parameter $\alpha$, these occupation fields can be identified with the sum of squares of $k$ copies of the associated free field (i.e. the Gaussian field whose covariance is given by the Green function). This is related to Dynkin's isomorphism (cf. [6, 23, 33]).

As in [27], we first present the theory in the elementary framework of symmetric Markov chains on a finite space. After some generalities on graphs and symmetric Markov chains, we study the $\sigma$-finite loop measure associated to a field of conductances. Then we study geodesic loops with an exposition of results of independent interest, such as the calculation of Ihara's zeta function. After that, we turn our attention to the Poisson process of loops and its occupation field, proving also several other interesting results such as the relation between loop ensembles and spanning trees given by Wilson algorithm and the reflection positivity property. Spanning trees are related to the fermionic Fock space as Markovian loop ensembles are related to the bosonic Fock space, represented by the free field. We also study the decompositions of the loop ensemble induced by the excursions into the complement of any given set.

Then we show that some results can be extended to more general Markov processes defined on continuous spaces. There are no essential difficulties for the occupation field when points are not polar but other cases are more problematic. As for the square of the free field, cases for which the Green function is Hilbert Schmidt such as those corresponding to two and three dimensional Brownian motion can be dealt with through appropriate renormalization.

We show that the renormalized powers of the occupation field (i.e. the self intersection local times of the loop ensemble) converge in the case of the two dimensional Brownian motion and that they can be identified with higher even Wick powers of the free field when $\alpha$ is a half integer.

At first, we suggest the reader could omit a few sections which are not essential for the understanding of the main results. These are essentially some of the generalities on graphs, results about wreath products, infinite discrete graphs, boundaries, zeta functions, geodesics and geodesic loops. The section on reflexion positivity, and, to a lesser extent, the one on decompositions are not central. The last section on continuous spaces is not written in full detail and may seem difficult to the least experienced readers.

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