Chapter 2
Simultaneous Diagonalisation (Modal Damping)

In this chapter we describe undamped and modally damped systems. They are wholly explained by the knowledge of the mass and the stiffness matrix. This is the broadly known case and we shall outline it here, not only because it is an important special case, but because it is often used as a starting position in the analysis of general damped systems.

2.1 Undamped Systems

The system (1.1) is called *undamped*,, if the damping vanishes: $C = 0$.

The solution of an undamped system is best described by *the generalised eigenvalue decomposition* of the matrix pair K, M:

$$\Phi^T K \Phi = \operatorname{diag}(\mu_1, \dots, \mu_n), \quad \Phi^T M \Phi = I. \tag{2.1}$$

We say that the matrix Φ *reduces the pair* K, M *of symmetric matrices to diagonal form by congruence*. This reduction is always possible, if the matrix M is positive definite. Instead of speaking of the matrix pair one often speaks of *the matrix pencil*, (that is, matrix function) $K - \lambda M$. If $M = I$ then (2.1) reduces to *the (standard) eigenvalue decomposition* valid for any symmetric matrix K, in this case the matrix Φ is orthogonal.

An equivalent way of writing (2.1) is

$$K\Phi = M\Phi \operatorname{diag}(\mu_1, \dots, \mu_n), \quad \Phi^T M \Phi = I. \tag{2.2}$$

or also

$$K\phi_j = \mu_j M \phi_j, \quad \phi_j^T M \phi_k = \delta_{kj}.$$

Thus, the columns ϕ_j of Φ form an M-orthonormal basis of eigenvectors of the generalised eigenvalue problem

$$K\phi = \mu M \phi, \tag{2.3}$$

K. Veselić, *Damped Oscillations of Linear Systems*,
Lecture Notes in Mathematics 2023, DOI 10.1007/978-3-642-21335-9_2,
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whereas μ_k are the zeros of the characteristic polynomial

$$\det(K - \mu M)$$

of the pair K, M. Hence

$$\mu = \frac{\phi^T K \phi}{\phi^T M \phi}, \text{ in particular, } \mu_k = \frac{\phi_k^T K \phi_k}{\phi_k^T M \phi_k}$$

shows that all μ_k are positive, if *both* K and M are positive definite as in our case. So we may rewrite (2.1) as

$$\Phi^T K \Phi = \Omega^2, \quad \Phi^T M \Phi = I \tag{2.4}$$

with

$$\Omega = \operatorname{diag}(\omega_1, \dots, \omega_n), \quad \omega_k = \sqrt{\mu_k} \tag{2.5}$$

The quantities ω_k will be called *the eigenfrequencies* of the system (1.1) with $C = 0$. The generalised eigenvalue decomposition can be obtained by any common matrix computation package (e.g. by calling eig(K,M) in MATLAB).

The solution of the homogeneous equation

$$M\ddot{x} + Kx = 0 \tag{2.6}$$

is given by the formula

$$x(t) = \Phi \begin{bmatrix} a_1 \cos \omega_1 t + b_1 \sin \omega_1 t \\ \vdots \\ a_n \cos \omega_n t + b_n \sin \omega_n t \end{bmatrix}, \quad a = \Phi^{-1} x_0, \quad \omega_k b_k = (\Phi^{-1} \dot{x}_0)_k, \tag{2.7}$$

which is readily verified. The values ω_k are of interest even if the damping C does not vanish and in this context they are called *the undamped frequencies* of the system (1.1).

In physical language the formula (2.7) is oft described by the phrase 'any oscillation is a superposition of harmonic oscillations or eigenmodes' which are

$$\phi_k(a_k \cos \omega_k t + b_k \sin \omega_k t), \quad k = 1, \dots, n.$$

Exercise 2.1 *Show that the eigenmodes are those solutions $x(t)$ of the equation (2.6) in which 'all particles oscillate in the same phase' that is,*

$$x(t) = x_0 T(t),$$

where x_0 is a fixed non-zero vector and $T(t)$ is a scalar-valued function of t (the above formula is also well known under the name 'Fourier ansatz').

The eigenvalues μ_k, taken in the non-decreasing ordering, are given by the known *minimax formula*

$$\mu_k = \max_{S_{n-k+1}} \min_{\substack{x \in S_{n-k+1} \\ x \neq 0}} \frac{x^T K x}{x^T M x} = \min_{S_k} \max_{\substack{x \in S_k \\ x \neq 0}} \frac{x^T K x}{x^T M x}, \tag{2.8}$$

where S_j denotes any subspace of dimension j. We will here skip proving these – fairly known – formulae, valid for any pair K, M of symmetric matrices with M positive definite. We will, however, provide a proof later within a more general situation (see Chap. 10 below).

The eigenfrequencies have an important monotonicity property. We introduce the relation called *relative stiffness* in the set of all pairs of positive definite symmetric matrices K, M as follows. We say that the pair $\hat{K}, \hat{M}$ is *relatively stiffer* than K, M, if the matrices $\hat{K} - K$ and $M - \hat{M}$ are positive semidefinite (that is, if stiffness is growing and the mass is falling).

Theorem 2.2 *Increasing relative stiffness increases the eigenfrequencies. More precisely, if $\hat{K} - K$ and $M - \hat{M}$ are positive semidefinite then the corresponding non-decreasingly ordered eigenfrequencies satisfy*

$$\omega_k \leq \hat{\omega}_k.$$

Proof. Just note that

$$\frac{x^T K x}{x^T M x} \leq \frac{x^T \hat{K} x}{x^T \hat{M} x}$$

for all non-vanishing x. Then take first minimum and then maximum and the statement follows from (2.8). Q.E.D.

If in Example 1.1 the matrix $\hat{K}$ is generated by the spring stiffnesses $\hat{k}_j$ then by (1.10) for $\delta K = \hat{K} - K$ we have

$$x^T \delta K x = \delta k_1 x_1^2 + \sum_{j=2}^{n} \delta k_j (x_j - x_{j-1})^2 + \delta k_{n+1} x_n^2, \tag{2.9}$$

where

$$\delta k_j = \hat{k}_j - k_j.$$

So, $\hat{k}_j \geq k_j$ implies the positive semidefiniteness of δK, that is the relative stiffness is growing. The same happens with the masses: take $\delta M = \hat{M} - M$, then

$$x^T \delta M x = \sum_{j=1}^{n} \delta m_j x_j^2, \quad \delta m_j = \hat{m}_j - m_j$$

and $\hat{m}_j \leq m_j$ implies the negative semidefiniteness of δM – the relative stiffness is again growing. Thus, our definition of the relative stiffness has deep physical roots.

The next question is: how do small changes in the system parameters k_j, m_j affect the eigenvalues? We make the term 'small changes' precise as follows

$$|\delta k_j| \leq \epsilon k_j, \quad |\delta m_j| \leq \eta m_j \tag{2.10}$$

with $0 \leq \epsilon, \eta < 1$. This kind of relative error is typical both in physical measurements and in numerical computations, in fact, in floating point arithmetic $\epsilon, \eta \approx 10^{-d}$ where d is the number of significant digits in a decimal number.

The corresponding errors in the eigenvalues will be an immediate consequence of (2.10) and Theorem 2.2. Indeed, from (2.9) and (2.10) it follows

$$|x^T \delta K x| \leq \epsilon x^T K x, \quad |x^T \delta M x| \leq \eta x^T M x. \tag{2.11}$$

Then

$$(1-\epsilon) x^T K x \leq x^T \hat{K} x \leq (1+\epsilon) x^T K x$$

and

$$(1-\eta) x^T M x \leq x^T \hat{M} x \leq (1+\eta) x^T M x$$

such that the pairs

$$(1-\epsilon)K,\ (1+\eta)M; \quad \hat{K},\ \hat{M}; \quad (1+\epsilon)K,\ (1-\eta)M$$

are ordered in growing relative stiffness. Therefore by Theorem 2.2 the corresponding eigenvalues

$$\frac{1-\epsilon}{1+\eta}\mu_k, \quad \hat{\mu}_k, \quad \frac{1+\epsilon}{1-\eta}\mu_k$$

satisfy

$$\frac{1-\epsilon}{1+\eta}\mu_k \leq \hat{\mu}_k \leq \frac{1+\epsilon}{1-\eta}\mu_k \tag{2.12}$$

(and similarly for the respective $\omega_k, \hat{\omega}_k$). In particular, for $\delta\mu_k = \hat{\mu}_k - \mu_k$ the relative error estimates

$$|\delta\mu_k| \leq \frac{\epsilon+\eta}{1-\eta}\mu_k \tag{2.13}$$

are valid. Note that both (2.12) and (2.13) are quite general. They depend only on the bounds (2.11), the only requirement is that both matrices K, M be symmetric and positive definite.

In the case $\hat{M} = M = I$ the more commonly known error estimate holds

$$\mu_k + \min \sigma(\hat{K} - K) \leq \hat{\mu}_k \leq \mu_k + \max \sigma(\hat{K} - K) \tag{2.14}$$

and in particular

$$|\delta \mu_k| \leq \|\hat{K} - K\|. \tag{2.15}$$

The proof again goes by immediate application of Theorem 2.2 and is left to the reader.

2.2 Frequencies as Singular Values

There is another way to compute the eigenfrequencies ω_j. We first make the decomposition

$$K = L_1 L_1^T, \quad M = L_2 L_2^T, \tag{2.16}$$
$$y_1 = L_1^T x, \quad y_2 = L_2^T \dot{x},$$

(here L_1, L_2 may, but need not be Cholesky factors). Then we make the singular value decomposition

$$L_2^{-1} L_1 = U \boldsymbol{\Sigma} V^T \tag{2.17}$$

where U, V are real orthogonal matrices and $\boldsymbol{\Sigma}$ is diagonal with positive diagonal elements. Hence

$$L_2^{-1} L_1 L_1^T L_2^{-T} = U \boldsymbol{\Sigma}^2 U^T$$

or

$$K \Phi = M \Phi \boldsymbol{\Sigma}^2, \quad \Phi = L_2^{-T} U$$

Now we can identify this Φ with the one from (2.4) and $\boldsymbol{\Sigma}$ with Ω. Thus *the eigenfrequencies of the undamped system are the singular values of the matrix* $L_2^{-1} L_1$.[1] The computation of Ω by (2.17) may have advantages over the one by (2.2), in particular, if ω_j greatly differ from each other. Indeed, by setting in Example 1.1 $n = 3$, $k_4 = 0$, $m_i = 1$ the matrix L_1 is directly obtained as

$$L_1 = \begin{bmatrix} \kappa_1 & -\kappa_2 & 0 \\ 0 & \kappa_2 & -\kappa_3 \\ 0 & 0 & \kappa_3 \end{bmatrix}, \quad \kappa_i = \sqrt{k_i}. \tag{2.18}$$

If we take $k_1 = k_2 = 1$, $k_3 \gg 1$ (that is, the third spring is almost rigid) then the way through (2.2) may spoil the lower frequency. For instance, with

[1]Equivalently we may speak of ω_j as *the generalised singular values* of the pair L_1, L_2.

the value $k_3 = 9.999999 \cdot 10^{15}$ the double-precision computation with Matlab gives the frequencies

sqrt(eig(K,M))	svd(L_2\L_1)
7.962252170181258e-01	4.682131924621356e-01
1.538189001320851e+00	1.510223959022110e+00
1.414213491662415e+08	1.414213491662415e+08

The singular value decomposition gives largely correct low eigenfrequencies. This phenomenon is independent of the eigenvalue or singular value algorithm used and it has to do with the fact that standard eigensolution algorithms compute the lowest eigenvalue of (2.2) with the relative error $\approx \epsilon\kappa(KM^{-1})$, that is, the machine precision ϵ is amplified by the condition number $\kappa(KM^{-1}) \approx 10^{16}$ whereas the same error with (2.17) is $\approx \epsilon\kappa(L_2^{-1}L_1) = \epsilon\sqrt{\kappa(KM^{-1})}$ (cf. e.g. [19]). In the second case the amplification is the square root of the first one!

2.3 Modally Damped Systems

Here we study those damped systems which can be completely explained by their undamped part. In order to do this it is convenient to make a *coordinate transformation*; we set

$$x = \Phi x', \tag{2.19}$$

where Φ is any real non-singular matrix. Thus (1.1) goes over into

$$M'\ddot{x}' + C'\dot{x}' + K'x' = g(t), \tag{2.20}$$

with

$$M' = \Phi^T M\Phi, \quad C' = \Phi^T C\Phi, \quad K' = \Phi^T K\Phi, \quad g = \Phi^T f. \tag{2.21}$$

Choose now the matrix Φ as in the previous section, that is,

$$\Phi^T M\Phi = I, \quad \Phi^T K\Phi = \Omega = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2).$$

(The right hand side $f(t)$ in (2.20) can always be taken into account by the Duhamel's term as in (3.1) so we will mostly restrict ourselves to consider $f = 0$ which corresponds to a 'freely oscillating' system.)

Now, if

$$D = (d_{jk}) = \Phi^T C\Phi \tag{2.22}$$

is diagonal as well then (1.1) is equivalent to

$$\ddot{\xi}_k + d_{kk}\dot{\xi}_k + \omega_k^2 \xi_k = 0, \quad x = \Phi\,\xi$$

with the known solution

$$\xi_k = a_k u^+(t, \omega_k, d_{kk}) + b_k u^-(t, \omega_k, d_{kk}), \tag{2.23}$$

$$u^+(t, \omega, d) = e^{\lambda^+(\omega,d)\,t},$$

$$u^-(t, \omega, d) = \begin{cases} e^{\lambda^-(\omega,d)\,t}, & \delta(\omega, d) \neq 0 \\ t\,e^{\lambda^+(\omega,d)\,t}, & \delta(\omega, d) = 0 \end{cases}$$

where

$$\delta(\omega, d) = d^2 - 4\omega^2,$$

$$\lambda^\pm(\omega, d) = \frac{-d \pm \sqrt{\delta(\omega, d)}}{2}.$$

The constants a_k, b_k in (2.23) are obtained from the initial data $x_0, \dot{x}_0$ similarly as in (2.7).

However, the simultaneous diagonalisability of the three matrices M, C, K is rather an exception, as shown by the following theorem.

Theorem 2.3 *Let M, C, K be as in (1.1). A non-singular Φ such that the matrices $\Phi^T M \Phi$, $\Phi^T C \Phi$, $\Phi^T K \Phi$ are diagonal exists, if and only if*

$$C K^{-1} M = M K^{-1} C. \tag{2.24}$$

Proof. The 'only if part' is trivial. Conversely, using (2.4) the identity (2.24) yields

$$C \Phi \Omega^{-2} \Phi^{-1} = \Phi^{-T} \Omega^{-2} \Phi^T C,$$

hence

$$\Phi^T C \Phi \Omega^{-2} = \Omega^{-2} \Phi^T C \Phi$$

and then

$$\Omega^2 \Phi^T C \Phi = \Phi^T C \Phi \Omega^2$$

i.e. the two real symmetric matrices Ω^2 and $D = \Phi^T C \Phi$ commute and Ω^2 is diagonal, so there exists a real orthogonal matrix U such that $U^T \Omega^2 U = \Omega^2$, and $U^T D U = \operatorname{diag}(d_{11}, \ldots, d_{nn})$. Indeed, since the diagonal elements of Ω are non-decreasingly ordered we may write

$$\Omega = \operatorname{diag}(\Omega_1, \ldots, \Omega_p),$$

where $\Omega_1, \ldots, \Omega_p$ are scalar matrices corresponding to distinct spectral points of Ω. Now, $D\Omega^2 = \Omega^2 D$ implies $D\Omega = \Omega D$ and therefore

$$D = \mathrm{diag}(D_1, \ldots, D_p),$$

with the same block partition. The matrices $D_1, \ldots, D_p$ are real symmetric, so there are orthogonal matrices $U_1, \ldots, U_p$ such that all $U_j^T D_j U_j$ are diagonal. By setting $\Phi_1 = \Phi\,\mathrm{diag}(D_1, \ldots, D_p)$ all three matrices $\Phi_1^T M \Phi_1$, $\Phi_1^T C \Phi_1$, $\Phi_1^T K \Phi_1$ are diagonal. Q.E.D.

Exercise 2.4 *Show that Theorem 2.3 remains valid, if M is allowed to be only positive semidefinite.*

Exercise 2.5 *Prove that (2.24) holds, if*

$$\alpha M + \beta C + \gamma K = 0,$$

where not all of α, β, γ vanish (proportional damping). When is this the case with C from (1.4)–(1.6)?

Exercise 2.6 *Prove that (2.24) is equivalent to $CM^{-1}K = KM^{-1}C$ and also to $KC^{-1}M = MC^{-1}K$, provided that these inverses exist.*

Exercise 2.7 *Try to find a necessary and sufficient condition that C from (1.4)–(1.6) satisfies (2.24).*

If C satisfies (2.24) then we say that the system (1.1) is *modally damped.*

http://www.springer.com/978-3-642-21334-2

Damped Oscillations of Linear Systems
A Mathematical Introduction
Veselić, K.
2011, XV, 200 p. 8 illus., Softcover
ISBN: 978-3-642-21334-2