Chapter 1
Laplace Operators on Networks and Trees

Abstract This work is an autonomous study of functions on infinite networks reflecting potential theory on locally compact spaces, influenced by the function theory associated with random walks and electrical networks. Starting with an overview of the contents of the five chapters presented here, this chapter introduces harmonic and superharmonic functions and their basic properties in networks. A discrete version of the Green's formula is given and the Minimum Principle for superharmonic functions is proved. Infinite trees as a special case are seen to provide examples and motivations for the development of an abstract discrete function theory on infinite networks.

1.1 Introduction

A *graph* consists of a finite number of points (called vertices) and a finite number of lines (called edges) joining some of them. The graph theory studies the inter-relation between the vertices and the edges (for example, [66]). Now for some problems, the edges have to be oriented in which case the graph is called a *digraph*. It would be easier to represent a digraph by its *incidence matrix* of order $n \times m$, where m is the number of edges and n is the number of vertices, with entries $-1, 0$ or 1. The interest in graph theory comes from the fact that many real-life situations can be represented as graphs.

Take for example, the postman problem: The postman collects the post from the post office and walks through all the streets in his beat, distributing the letters and finally returns to the post office. His problem is how well to choose a route so that, if possible, he does not go through any street more than once, yet covers all the streets. To solve this, we can think of each street corner as a vertex and each street as an edge, thus getting the model of a graph; the problem now reduces to finding a path that contains all the edges once and once only. Like this, there are other problems connected with chemical bonding, bus routes, work assignments etc. In some situations like bus routes, the distance between two vertices (that is, between

V. Anandam, *Harmonic Functions and Potentials on Finite or Infinite Networks*,
Lecture Notes of the Unione Matematica Italiana 12, DOI 10.1007/978-3-642-21399-1_1,

two bus stops) may be important. That is, each edge has a real number associated with it and then we have *weighted graphs*. It is interesting to study these *geometrical structures* of a graph for their own merit. But it would be more fruitful to represent a *physical problem* as a graph theory problem and try to solve it.

Though graph theory generally deals with a finite number of objects and their inter-connectedness where the geometrical aspects of graphs play a decisive role, yet there are also problems that involve functions on finite graphs. For example, consider a *finite electrical network*. This can be represented as a graph [32] provided with a voltage-current regime subject to *Ohm's law* and *Kirchhoff's voltage and current laws*. Here we are interested not only in the geometrical properties of a finite graph but also on functions defined on nodes and branches satisfying certain conditions. In this context, the incidence matrix of the graph takes care of the geometrical properties of the graph and for the function-theoretic aspects one introduces the *Laplace operator* Δ dependent on the incidence matrix and its transpose which can be considered as operators on functions defined on its edges and vertices.

There is another development which requires the study of *infinite graphs*. Consider finite difference approximations of equations in physics; some of them lead to partial difference equations [14]. The approximations to find a solution involve horizontal and vertical displacements and so can be treated as functions on an infinite grid in the context of electrical networks. Take for example, *wave equations*; the domain of existence of the solution may be unbounded, suggesting a problem in a graph with infinite vertices [73]. Another example of an infinite graph arises in the study of *Markov chains* [68]. A Markov chain consists of a countable number of states provided with a *transition probability* and the *Markov property* which says that given the present, the past and the future are independent.

The study of functions on infinite networks has thus far been carried out on the background of Markov chains and random walks or on the requirements of extending results from finite electrical networks to infinite networks. There are many common features in these two developments. Actually, a close connection has been established between the concepts like transition probabilities, transience, recurrence, hitting time etc. used in the probabilistic study of functions on infinite networks and the concepts like effective resistance, equilibrium principle, capacity etc. used in a current-voltage regime in electrical networks. The effective resistance has a close relation to the escape probability for a reversible Markov chain [59, 64] which is characterized by the transition probability from one state to another. The similarity between the conductance and the transition probability is obvious. Consequently, it is not uncommon to see a problem arising in the context of random walks being solved by electrical methods and conversely. The electrical methods make use of functional analysis techniques, starting with the Dirichlet norm (the discrete analogue of the energy integral in the classical case) and its associated inner-product. Thus, the abstract potential theory on infinite networks, as developed by Yamasaki [70], Soardi [63] and others, is a study of Dirichlet finite functions (modeled after Dirichlet functions in the classical potential theory) dealing with discrete analogues of the solution of Poisson equation, Green's function, extremal length, Royden decomposition and Royden compactification.

The current work presents an autonomous study of functions on infinite networks influenced by potential theory on locally compact spaces which does not assign any direct role for the notion of derivatives of functions. Initially, finite networks provided with the Laplacian operator are taken up for the study, the development depending on algebraic methods starting with the incidence matrix. We can call this *algebraic potential theory* because of the association of the Laplacian (represented as a matrix) with electric potentials [24]. Later, infinite networks, with the Laplace operator defined as in the finite case, are taken up for consideration. In this situation, the development resembles the study of harmonic and subharmonic functions in the complex plane or more generally in $\Re^n, n \geq 2$ [10, 27, 53], and in the Brelot axiomatic potential theory [17, 28, 40]. Here the Dirichlet norm does not play a dominant role; nor are the probabilistic interpretations considered. However, in both the finite and the infinite cases, the important basic properties and significant results like the equilibrium principle, the condenser principle, the capacity etc. that are related to an electrical network come as solutions to the following *Dirichlet-Poisson problem* on the (finite or infinite) graph X, namely: Let F be a subset of the vertex set X. Suppose f and g are real-valued functions on X. Then, find u defined on the vertex set X satisfying the conditions $\Delta u = f$ at each vertex in F and $u = g$ on $X \setminus F$.

The present work is rather like a discrete version of function theory on Riemann surfaces [3, 39, 58, 65] and Riemannian manifolds [60], devoid of any attempt to connect it to any of the many important works on electrical networks and random walks. We develop a function theory on networks similar to the classical and the axiomatic potential theory on Euclidean spaces and on locally compact spaces. The basic definitions of potentials, Green's kernel, balayage etc. are introduced here as in the case of the Brelot axiomatic theory rather than as in the theory of probability ([31] for example). Yamasaki [69, 70] also has proved many potential theoretic results in an infinite network without involving the methods used in the study of random walks. However his study is based on Dirichlet norms and functional analysis methods, resembling potential theory on Dirichlet spaces studied by Deny [41, 42], Beurling and Deny [22, 23], Fukushima [46], Bouleau and Hirsch [26] and others. These methods are not convenient if we have to study potential theory on infinite networks in which the only non-negative potential is 0. Thus a deeper study of infinite networks without positive potentials as in the case of parabolic Riemann surfaces becomes cumbersome. Under these circumstances, the approach we have adopted here is easy to deal with situations in infinite networks that resemble those of Riemann surfaces that are hyperbolic or parabolic.

Chapter 1 is devoted to some preliminary remarks concerning networks and trees, the interior and the boundary of subsets in a network, inner and outer normal derivatives, Green's formulas, the definition and some properties of superharmonic functions and the minimum principle.

Chapter 2 brings into focus certain aspects of potential theory on *finite networks*. The Laplacian is represented by a matrix and the properties of this matrix lead to the minimum principle, domination principle, equilibrium principle and solutions to some mixed boundary-value problems like Poisson-Dirichlet problem and Neumann problem in a finite network. It is easy then to consider in a similar vein the Schrödinger operators in finite networks. These results in a finite network are already proved in Bendito et al. [20] by assuming the symmetry of conductance and then constructing equilibrium measures appropriate to each principle. Ours is a unified method based on the inverses of certain sub-matrices of the Laplace matrix.

Chapter 3 deals with the classification theory of infinite networks. It starts with the first broad division of networks into *hyperbolic and parabolic networks*, depending on whether it is possible or not to define the *Green kernel* on the network. A hyperbolic network is further classified based on the existence of non-constant positive and bounded harmonic functions on the network. This leads to the *Riesz-Martin representation* of positive superharmonic functions on a hyperbolic network. Later a study of parabolic networks is taken up, starting with the construction of a kernel like $\log|x-y|$ in the plane. The notion of *flux at infinity* of a superharmonic function is discussed in detail. Balayage and Dirichlet problem on arbitrary subsets of a parabolic network receive attention. Then, an introductory study of *pseudo-potentials* (similar to logarithmic potentials in the plane) follows.

Chapter 4 is devoted to the potential theory on an infinite network X associated to the *Schrödinger operator* $\Delta u(x) - q(x)u(x)$, for an arbitrary real-valued function $q(x)$ and then more specifically when $q(x) \geq \dfrac{\Delta\xi(x)}{\xi(x)}$ for some function $\xi > 0$ on X. This condition implies that q can take negative values, but ensures that there exists a positive q-superharmonic function on X. With respect to this operator, the topics like generalised Dirchlet problem, balayage, condenser principle, equilibrium principle, etc. are investigated. This example of two related harmonic structures, one from the Laplace operator and the other from the Schrödinger operator, in the same infinite network is later generalised to study the relation between a basic harmonic structure and a subordinate harmonic structure on X.

Chapter 5 takes up the study of polyharmonic functions on an infinite tree T. A real-valued function s on X is said to be polysuperharmonic of order m or simply m-superharmonic (for an integer $m \geq 1$) if $(-\Delta)^m s \geq 0$. Actually, to characterize m-superharmonic and m-harmonic functions, we build up on the solutions u to the Poisson equation $\Delta u = f$ for an arbitrary real-valued function f on X. Since the solutions to this equation are not easily available in an arbitrary network, we have to confine our study of polyharmonic functions defined on a tree T only. For polysuperharmonic functions on T, Laurent decomposition, m-harmonic Green functions, domination principle, balayage etc. are obtained. Finally, the Riesz-Martin representation for positive m-superharmonic functions is exhibited.

1.2 Preliminaries

Let X be a countable (finite or infinite) set of points, called vertices, some of them pair-wise joined by *edges*; we say that the edge $[x, y]$ joins the vertices x and y. Let Y denote the set of edges which are assumed to be countable. Denote $x \sim y$ to mean that there is an edge $[x, y]$ joining x and y, in which case the vertices x and y are said to be *neighbours*. A vertex e is named *terimal* if it has only one neighbour. A *walk* from x to y is a collection of vertices $\{x = x_0, x_1, ..., x_n = y\}$ where $x_i \sim x_{i+1}$ if $0 \leq i \leq n-1$; for this walk, the *length* is n. If the vertices in the walk are distinct, the walk is referred to as a *path*. The shortest length $d(x, y)$ between x and y is called *the distance between x and y*. We also assume that given any two vertices x and y, there exists *an associated non-negative number*, called *conductance*, $t(x, y) \geq 0$ such that t(x, y) > 0 if and only if $x \sim y$. Then $N = \{X, Y, t\}$ is called *a network* if the following conditions are also satisfied:

1. There is no *self-loop* in N, that is no edge of the form $[x, x]$ in Y.
2. Given any vertices x and y in X, there is a path connecting x and y. (That is, X is *connected*.)
3. Every $x \in X$ has only a finite number of neighbours. (That is, X is *locally finite*.)

Instead of writing $N = \{X, Y, t\}$, we simply write X to refer to a network. If $t(x, y) = t(y, x)$ for every pair of vertices x and y, then we say that X is a network with *symmetric conductance*. A network X is called a *tree* if there is no *cycle* in X, that is there is no closed path $\{x_1, x_2, ..., x_n, x_1\}$ with $n \geq 3$. An infinite tree T is said to be *homogeneous* of degree $q + 1$, if each vertex in T has $(q + 1)$ neighbours. A tree T is said to be a *standard homogenous tree of degree* $q + 1$ if every vertex in T has exactly $(q + 1)$ neighbours and $t(x, y) = (q + 1)^{-1}$ if $x \sim y$. If a tree T is considered in the context of probability, we denote the conductance as $p(x, y)$ instead of $t(x, y)$, so that $\sum\limits_{y \sim x} p(x, y) = 1$ for any $x \in T$. We refer to $p(x, y)$ as the *transition probability* from x to y. It is important to note that in a tree T, if x and y are any two vertices, then there exists a unique path joining x and y.

For any subset E of a network X, we write $\overset{0}{E} = \{x : x$ and all its neighbours are in $E\}$ and $\partial E = E \setminus \overset{0}{E}$. $\overset{0}{E}$ is referred to as the *interior* of E and ∂E is referred to as the *boundary* of E. This definition of boundary differs from the one used by Chung and Yau [35] and Bendito et al. [18]. According to them, $y \notin E$ is a boundary point of E if and only if there exists a vertex x in E such that $x \sim y$ and the collection of these boundary points is the boundary δE of E. However, the definition of the boundary ∂E given here is preferable in the case of infinite networks, since for many boundary-value problems like the Dirichlet problem the boundary function will be defined on E only. So it is convenient to define the boundary ∂E as a subset of E rather than as a subset lying outside E. Note that for a non-empty subset E, we have $E = \overset{0}{E}$ if and only if $E = X$. An arbitrary set E in X is said to be *circled* if every vertex in ∂E has at least one neighbour in $\overset{0}{E}$. That is, E is circled if and only if $E = \overset{0}{E} \cup \delta \overset{\circ}{E}$, if we use the notation of Bendito et al. [20].

Example: Let e be a fixed vertex. For any vertex x, let $|x|$ denote the distance between e and x. Then $B_m = \{x : |x| \leq m\}$ is circled. For an example of a non-circled set, we can consider in a homogeneous tree of degree $q + 1$, $q \geq 2$, the set

E consisting of B_m and one more vertex z, $|z| = m+1$, and the edge connecting z to B_m. Write $V(E)$ to denote the union of E and all the neighbours of each vertex of E, that is $V(E) = E \cup \{y : y \sim x \text{ for some } x \in E\}$. In particular, $V(x)$ denotes the set consisting of x and all its neighbours. Remark that if E is connected, $V(E)$ also is connected. Also note that for any set E, $V(E)$ is circled. For, if $F = V(E)$, then $E \subset \overset{0}{F}$. Hence if $z \in \partial F$, then by definition, z has a neighbour in E and hence in $\overset{0}{F}$. Note that $V(E)$ is the same as $\overline{E} = E \cup \delta E$ in the notations of [20].

For $x \in X$, write $E_1 = V(x)$. Then, E_1 is connected and circled. Let $E_2 = V(E_1)$ which is also connected and circled. Inductively, define $E_n = V(E_{n-1})$ for $n \geq 3$. Then $\{E_n\}$ is an increasing sequence of finite, connected and circled sets such that $\overset{\circ}{E}_{n+1} \supset E_n$ and $X = \bigcup\limits_n E_n$, which is referred to as *a regular exhaustion* of X. For any subset A in X, A^c denotes $X \setminus A$, the complement of A in X.

Proposition 1.2.1. *Let A be circled, and $B = \overset{\circ}{A}{}^c$. Then $\partial B = \partial A$ and $\overset{0}{B} = A^c$; also, B is circled.*

Proof. Note $B = \overset{\circ}{A}{}^c = A^c \bigcup \partial A$. Let $z \in \partial A$. Then, for some $y \in \overset{0}{A}$, $y \sim z$; thus $z \in \partial A \subset B$, but a neighbour y of z is not in B, hence $z \in \partial B$. Conversely, let $b \in \partial B$. Then, $b \sim a$ for some $a \in X \setminus B = \overset{0}{A}$. Since $a \in \overset{0}{A}$ and $a \sim b$, we should have $b \in A \setminus \overset{0}{A}$, which means $b \in \partial A$. Consequently, $\partial B = \partial A$ and $\overset{0}{B} = A^c$.

To show that B is circled, take $b \in \partial B$. Since $\partial B = \partial A$, b should have a neighbour $a \in A^c = \overset{0}{B}$. Hence B is circled. □

Proposition 1.2.2. *Let E be an arbitrary set and $F = V(\overset{0}{E})$. Then $\overset{0}{E} = \overset{0}{F}$ and $\partial F \subset \partial E$; also F is the largest circled set contained in E.*

Proof. By definition, $\overset{0}{E} \subset \overset{0}{F}$. Since $F \subset E$, we also have $\overset{0}{F} \subset \overset{0}{E}$. Hence $\overset{0}{E} = \overset{0}{F}$. To see $\partial F \subset \partial E$, first note that $\partial F \cap \overset{0}{E} = \partial F \cap \overset{0}{F} = \phi$. Then, $\overset{0}{E} \cup \partial F = \overset{0}{F} \cup \partial F = F \subset E = \overset{0}{E} \cup \partial E$ implies that $\partial F \subset \partial E$. $V(\overset{0}{E})$ is circled by definition and $V(\overset{0}{E}) \subset E$.

Suppose the circled set $A \subset E$. Then $\overset{0}{A} \subset \overset{0}{E}$. Let $x \in \partial A$. Then there exists $z \in \overset{0}{A}$ such that $x \sim z$. Since $z \in \overset{0}{E}$ and $z \sim x$, we find that $x \in F$. Hence $\partial A \subset F$. Then, $A = \overset{0}{A} \cup \partial A \subset \overset{0}{E} \cup F = \overset{0}{F} \cup F = F$. □

A lower semi-continuous function $u > -\infty$ on an open set ω in the Euclidean space $\Re^n$, $n \geq 2$, is said to be *superharmonic* on ω if u is not identically ∞ and if for each $a \in \omega$, there exists a closed ball $\overline{B}(a, R) \subset \omega$ such that $u(a) \geq \int\limits_S u(a + r\xi) d\sigma(\xi)$ if $0 < r \leq R$. Here S is the unit sphere and σ is the normalised surface area measure on S, so that $\sigma(S) = 1$. Suppose u is a locally Lebesgue integrable

function on ω such that $\Delta u \leq 0$ *in the sense of distributions*. Then, there exists a superharmonic function v on ω such that $v = u$ a.e. [27, p.43]. In analogy with this result, superharmonic functions on a network are defined as follows:
We consider now functions defined on subsets of a network X. In this presentation, we assume that *all functions are real-valued.*

1. For $x \in X$, and for a function f defined on $V(x)$, define the *Laplacian*

$$\Delta f(x) = \sum_{x \sim x_i} t(x, x_i)[f(x_i) - f(x)] = -t(x)f(x) + \sum_{x \sim x_i} t(x, x_i)f(x_i),$$

   where $t(x) = \sum\limits_{x \sim x_i} t(x, x_i)$; note $t(x) > 0$ for any $x \in X$.
2. We say that f is *harmonic* (respectively *superharmonic, subharmonic*) at x if $\Delta f(x) = 0$ (respectively $\Delta f(x) \leq 0$, $\Delta f(x) \geq 0$). *A function f defined on an arbitrary set E is said to be harmonic (respectively superharmonic, subharmonic) on E if and only if $\Delta f(x) = 0$ (respectively $\Delta f(x) \leq 0$, $\Delta f(x) \geq 0$) for every $x \in \overset{0}{E}$.*

Remark 1.2.1. 1. Some authors prefer to define a real-valued function u as a harmonic function on E provided its Laplacian is 0 at every vertex of E. This presupposes that u is defined on $V(E)$. But for the topics we discuss here, such as the Dirichlet problem, the minimum principle, the condenser principle etc. u is either not defined outside E or its value outside E is not of consequence. Hence, it becomes important to distinguish between the interior and the boundary vertices of E.
2. It is possible, in the definition of a network X, to leave out the condition that each vertex has only a finite number neighbours. But then, we should assume that $\sum\limits_{y \sim x} t(x, y) < \infty$ for all x in X; also, instead of defining the Laplacian Δf for any real-valued function on f on X, we should confine ourselves to those real-valued functions u on X for which $\sum\limits_{y \sim x} t(x, y)\,|u(y)| < \infty$ for any x in X. In that case, $\sum\limits_{y \sim x} t(x, y)\,[u(y) - u(x)]$ will make sense, since

$$\sum_{y \sim x} t(x, y)\,|u(y) - u(x)| \leq \sum_{y \sim x} t(x, y)\,|u(y)| + |u(x)| \sum_{y \sim x} t(x, y)$$

   for any x in X.

Proposition 1.2.3. *If u and v are superharmonic on a subset E of X and if α, β are two non-negative constants, then $\alpha u + \beta v$ and $\inf(u, v)$ are superharmonic on E. Moreover, if u_n is a convergent sequence of superharmonic functions on E such that $u(x) = \lim u_n(x)$ is finite for every vertex in E, then u is superharmonic on E.*

Proof. If $x \in \overset{0}{E}$, then by hypothesis $\Delta u(x) \leq 0$ and $\Delta v(x) \leq 0$. Hence

$$\Delta(\alpha u + \beta v)(x) = \alpha \Delta u(x) + \beta \Delta v(x) \leq 0.$$

Hence, $\alpha u + \beta v$ is superharmonic on X.

To show that $s = \inf(u, v)$ also is superharmonic on E, that is $s(x) = \inf(u(x), v(x))$ for $x \in E$, assume without loss of generality that $s(x) = u(x)$. Then,

$$\begin{aligned}
\Delta s(x) &= \sum_{x_i \sim x} t(x, x_i)[s(x_i) - s(x)] \\
&= \sum_{x_i \sim x} t(x, x_i)[s(x_i) - u(x)] \\
&\leq \sum_{x_i \sim x} t(x, x_i)[u(x_i) - u(x)] \\
&= \Delta u(x) \leq 0.
\end{aligned}$$

Hence, $s = \inf(u, v)$ is superharmonic on E.

Finally concerning the limit of superharmonic functions, for any $x \in \overset{0}{E}$,

$$t(x) u_n(x) \geq \sum_{y \sim x} t(x, y) u_n(y).$$

Since the summation is over a finite number of terms, taking limits we conclude

$$t(x) u(x) \geq \sum_{y \sim x} t(x, y) u(y).$$

That is, u is superharmonic on E. □

Remark 1.2.2. In the classical potential theory, we should pay more attention when we consider the limit of superharmonic functions on an open set. If v_n is an increasing sequence of superharmonic functions on a domain ω, and if $v = \sup v_n$ is finite at one point in ω, then v is superharmonic on ω. But if $u_n \geq 0$ is a decreasing sequence of superharmonic functions on ω, then $u = \inf u_n$ need not be superharmonic on ω. However, there is a superharmonic function s on ω such that $s = u$ a.e. on ω. In fact, the second statement is a simplified version of the *convergence theorem* for superharmonic functions in the classical potential theory [27, pp. 74–78].

1.2.1 Examples of Superharmonic Functions on Networks

1. *The constants are the only superharmonic functions on a finite network X.* For, let s be a superharmonic function on X. Let z be a vertex in X such that $s(z) \leq s(x)$

for every x in X. Now $\Delta s(z) \leq 0$ which implies that
$0 \geq \Delta s(z) = \sum_{x_i \sim z} t(z, x_i)[s(x_i) - s(z)] \geq 0$, since $s(z) \leq s(x)$.
Consequently, $s(x_i) = s(z)$ for all $x_i \sim z$. Since X is connected, any x in X can be connected by a path $\{z, x_1, x_2, ..., x_n = x\}$. Then, the above argument shows that $s(x_1) = s(z)$, then $s(x_2) = s(x_1) = s(z)$ and so on. This permits us to conclude that finally $s(x) = s(z)$. We conclude therefore that s is a constant on X.

2. *Example of an infinite network in which every harmonic function is constant, but non-constant superharmonic functions exist.* Let $X = \{0, 1, 2, ...\}$ be an infinite network, with $t(0, 1) = 1$, $t(1, 0) = \frac{1}{2}$ and $t(n, n+1) = t(n+1, n) = \frac{1}{2}$ if $n \geq 1$.

   (a) Any harmonic function on X is a constant. For, let h be a harmonic function on X, and suppose $h(0) = a$. Then,

   $$0 = \Delta h(0) = t(0, 1)[h(1) - h(0)], \text{ so that } h(1) = h(0) = a. \text{ Again,}$$

   $$\begin{aligned} 0 = \Delta h(1) &= t(1, 0)[h(0) - h(1)] + t(1, 2)[h(2) - h(1)], \\ &\text{so that } h(2) = h(1) = a. \end{aligned}$$

   Similar calculations show that $h(n) = a$ for all $n \geq 0$.

   (b) However, non-constant superharmonic functions exist on X. For example, set $s(n) = a - n\alpha, n \geq 0, \alpha > 0$. Then, $\Delta s(0) = -\alpha < 0$ and $\Delta s(n) = 0$ if $n \geq 1$. We say that s is a superharmonic function on X with *point harmonic support* at 0, since s is harmonic at every vertex except 0 and at the vertex 0, s is superharmonic but not harmonic.

3. *Example of an infinite tree with non-constant positive superharmonic functions.* Let T be a standard homogeneous tree of degree 3, that is every vertex has exactly 3 neighbours and the transition probability $p(x, y) = \frac{1}{3}$ if $x \sim y$. Let us fix a vertex e in T. For any vertex x in T, there is a unique path from e to x and let us denote the length of this path as $|x| = d(e, x)$. Note that for any vertex x in T, with $|x| = n \geq 1$, there is one vertex y with $|y| = n - 1$ and two vertices z with $|z| = n + 1$. Let $s(x) = 2^{1-|x|}$. Then,

   $$\Delta s(x) = \frac{1}{3}\left[2^{1-(n-1)} - 2^{1-n}\right] + \frac{2}{3}\left[2^{1-(n+1)} - 2^{1-n}\right] = 0,$$

   if $|x| = n \geq 1$, and

   $$\begin{aligned} \Delta s(e) &= \sum_{y \sim e} p(e, y)\,[s(y) - s(e)] \\ &= \sum_{y \sim e} \tfrac{1}{3}\,[1 - 2] = -1. \end{aligned}$$

Hence, s is a positive superharmonic function on T with point harmonic support at e.

4. *Example of a tree with non-constant positive harmonic functions.* Let $T = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ be a tree, each vertex n having only two neighbours, with the transition probabilities $p(n, n+1) = \frac{3}{4}$ and $p(n, n-1) = \frac{1}{4}$ for any n. Then, $h(n) = 3^{-n}$ is a positive harmonic function on T. For,

$$\begin{aligned}\Delta h(n) &= \tfrac{3}{4}\,[h(n+1) - h(n)] + \tfrac{1}{4}\,[h(n-1) - h(n)] \\ &= \tfrac{3}{4}[3^{-n-1} - 3^{-n}] + \tfrac{1}{4}[3^{-n+1} - 3^{-n}] = 0.\end{aligned}$$

Let $s(n) = 1$ if $n \leq 0$ and $s(n) = 3^{-n}$ if $n > 0$, that is $s = \inf(h(n), 1)$. Then, $\Delta s(0) = -\frac{1}{2}$ and $\Delta s(n) = 0$ if $n \neq 0$. Hence, s is a bounded positive superharmonic function with point harmonic support at the vertex 0.

5. *Example of a lattice in which non-constant positive superharmonic functions exist but every positive harmonic function is constant.* Let X be the set of lattice points in $\Re^3$, of the form (a, b, c) where a, b and c take the values $\ldots,$ $-2, -1, 0, 1, 2, \ldots$. Take $t(x, y) = 1$ if $x \sim y$, otherwise $t(x, y) = 0$. Then, Courant constructs in the context of random walk and diffusion in a lattice [45, Sect. 3] the Green function $g(a, b, c) > 0$ such that $\Delta g(0, 0, 0) = -1$, $\Delta g(a, b, c) = 0$ otherwise, and $g(a, b, c) \to 0$ when $a^2 + b^2 + c^2 \to \infty$. Duffin obtains various asymptotic properties of $g(a, b, c)$ and proves that any positive harmonic function on X is a constant.

1.3 Green's Formulas

We shall prove now a version of the Green's formula in an infinite network and in a tree, motivated by a result of Duffin's in the above-mentioned discrete situation of lattice points in the Euclidean space $\Re^3$, of the form (a, b, c) where a, b and c take on the values $..., -2, -1, 0, 1, 2, ...$ For a real-valued function u on these lattice points, Duffin defines the Laplace operator D as

$$\begin{aligned}Du(a, b, c) &= u(a+1, b, c) + u(a-1, b, c) + u(a, b+1, c) + u(a, b-1, c) \\ &\quad + u(a, b, c+1) + u(a, b, c-1) - 6u(a, b, c).\end{aligned}$$

Concerning the operator D, Duffin [45, p.233] remarks: *In physical problems the operator D is often used as an approximation to Δ. There are, however, some problems in which D appears directly, such as random walk and diffusion in a lattice. This concerns the motion of a particle which at each lattice point has an equal probability of jumping to a neighbouring lattice point. Another direct application arises if the lattice lines are regarded as metallic wires. This gives an infinite electrical network. If the electric potential of the lattice points is denoted by* *u, then at every insulated lattice point, $Du = 0$. At a source point, $Du = -w$ where w is the current entering this lattice point. These physical models are of suggestive value for the analysis of D.*

Using this operator D, the classical Green's formula

$$\iint\limits_{\omega} (f\Delta g - g\Delta f) d\sigma = \int\limits_{\partial\omega} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) ds$$

is shown to take the following form in the discrete case [45, Lemma 1]: Let u and v be lattice functions and let E be a finite set of lattice points. Then

$$\sum_E (uDv - vDu) = \sum_S [v(p)u(q) - u(p)v(q)].$$

Here $\sum\limits_E$ denotes the summation over E and $\sum\limits_S$ denotes the summation over the set S of points (p, q) where p and q are neighbours, p being in E and q being in the complement of E.

Such a formula in the discrete analysis on graphs [67] and in the framework of finite networks [18, Proposition 3.1] is known. Now, we have defined the Laplacian only for the interior vertices of E. A useful similar operator at the boundary vertices is the inner normal derivative [1]. If f is a function defined on a subset E of X, and if $\xi \in \partial E$, then the *inner normal derivative* of f at ξ is defined as

$$\frac{\partial}{\partial n^-} f(\xi) = \sum_{y \in E} t(\xi, y)[f(y) - f(\xi)].$$

If f is defined on X, and for a subset E, if $\xi \in \partial E$ then the *outer normal derivative* of f at ξ is defined as

$$\frac{\partial}{\partial n^+} f(\xi) = \sum_{y \notin E} t(\xi, y)[f(y) - f(\xi)].$$

Note that in the case of f being defined on X, if $\xi \in \partial E$, then

$$\Delta f(\xi) = \frac{\partial}{\partial n^+} f(\xi) + \frac{\partial}{\partial n^-} f(\xi).$$

Let E be a subset of X. Let us say that an edge L in X is an edge in E if and only if both the ends of L are vertices in E. Then, E can be considered as a graph, maybe not connected. Let us denote by $\Delta_E^\bullet$ the restriction of Δ to the subset E. That is, for any function f on E, $\Delta_E^\bullet f(x) = \Delta f(x)$ if $x \in \overset{0}{E}$, and $\Delta_E^\bullet f(\xi) = \frac{\partial}{\partial n^-} f(\xi)$ if $\xi \in \partial E$.

Proposition 1.3.1. *Let E be a finite subset of a network X with symmetric conductance . Let f be a function on E. Then,* $\sum_{x\in \overset{0}{E}} \Delta f(x) = -\sum_{\xi\in\partial E} \frac{\partial}{\partial n^-} f(\xi)$.

Proof. Note that $\sum_{z\in E} \Delta_E^\bullet f(z) = \sum_{x\in \overset{0}{E}} \Delta f(x) + \sum_{\xi\in\partial E} \Delta_E^\bullet f(\xi) = \sum_{x\in \overset{0}{E}} \Delta f(x) + \sum_{\xi\in\partial E} \frac{\partial}{\partial n^-} f(\xi)$. But $\sum_{z\in E} \Delta_E^\bullet f(z) = \sum_{y,z\in E} t(z,y)[f(y)-f(z)] = 0$, for in the second sum, corresponding to each term $t(z,y)[f(y)-f(z)]$ appearing in the sum, there is a term $t(y,z)[f(z)-f(y)]$ which cancels it out since $t(z,y)=t(y,z)$, with the assumption of the symmetric conductance. Hence we obtain the proposition. □

Note. If s is a harmonic function on a finite subset E, the expression $\sum_{\xi\in\partial E} \frac{\partial}{\partial n^-} s(\xi)$ is known as the *total inward flux of s on E*. Thus, the total inward flux of a harmonic function h on a finite set E should be 0. Given two functions f and g on a finite subset E of X, we write

$$(f,g)_E = \frac{1}{2}\sum_{x,y\in E} t(x,y)[f(x)-f(y)][g(x)-g(y)].$$

The following is a slight variant of the results given in Bendito et al. [18].

Theorem 1.3.2. *(Green's formula) Let f and g be two functions defined on a finite set E in a network X with symmetric conductance. Then,* $\sum_{x\in \overset{0}{E}} f(x)\Delta g(x) + (f,g)_E = -\sum_{\xi\in\partial E} f(\xi)\frac{\partial}{\partial n^-} g(\xi)$.

Proof. Extend f and g arbitrarily outside E. Then, for $x\in E$, $\Delta g(x) = \sum_{y\in X} t(x,y)[g(y)-g(x)]$ so that

$$\begin{aligned}\sum_{x\in E} f(x)\Delta g(x) &= \sum_{x\in E}\sum_{y\in X} t(x,y)f(x)[g(y)-g(x)] \\ &= \sum_{x\in E}\sum_{y\in E} t(x,y)f(x)[g(y)-g(x)] \\ &\quad + \sum_{x\in E}\sum_{y\notin E} t(x,y)f(x)[g(y)-g(x)].\end{aligned}$$

On the right side, in the first sum there are two terms $t(x,y)f(x)[g(y)-g(x)]$ and $t(y,x)f(y)[g(x)-g(y)]$ whose sum is $-t(x,y)[f(y)-f(x)][g(y)-g(x)]$; in the second sum, $t(x,y)>0$ if and only if $x\in\partial E$ since $y\notin E$, so that it can be rewritten as

$$\sum_{\xi \in \partial E} f(\xi) \frac{\partial}{\partial n^+} g(\xi) = \sum_{\xi \in \partial E} f(\xi)[\Delta g(\xi) - \frac{\partial}{\partial n^-} g(\xi)].$$

Consequently, $\sum_{x \in \overset{0}{E}} f(x)\Delta g(x) = -(f,g)_E - \sum_{\xi \in \partial E} f(\xi) \frac{\partial}{\partial n^-} g(\xi)$. □

Corollary 1.3.3. *Let f and g be two functions defined on a finite set E in a network X with symmetric conductance. Then, $\sum_{\overset{0}{E}}[f\Delta g - g\Delta f] = -\sum_{\partial E}[f\frac{\partial g}{\partial n^-} - g\frac{\partial f}{\partial n^-}]$.*

Proof. Since $(f,g)_E = (g,f)_E$, the corollary follows from the above theorem. □

Remark 1.3.1. Suppose X is an infinite network with symmetric conductance. Then $\partial X = \phi$. The question is whether the above Green's Formula will yield the *dissipation formula* $(f,f) = -\sum_{x \in X} f(x)\Delta f(x)$. Kayano and Yamasaki [51] study the class of functions f for which this formula is satisfied.

1.4 Minimum Principle

Proposition 1.4.1. *Let E be circled and $\overset{0}{E}$ be connected. If s is a superharmonic function on E that attains its minimum at a vertex in $\overset{0}{E}$, then s is constant.*

Proof. Let $\alpha = \inf_E s(x)$. By hypothesis, there is a vertex $y \in \overset{0}{E}$ such that $s(y) = \alpha$. Let $z \in \overset{0}{E}$ be an arbitrary vertex. Since $\overset{0}{E}$ is connected, there is a path $\{y, z_1, ..., z_n = z\}$ connecting y and z. Since $0 \geq \Delta s(y) = \sum t(y, y_i)[s(y_i) - s(y)]$, and $s(y)$ is the minimum value, we conclude that $s(y_i) = s(y) = \alpha$ for every $y_i \sim y$. In particular, $s(z_1) = \alpha$. Proceeding step-by-step, we arrive at the value $s(z) = \alpha$. Since $z \in \overset{0}{E}$ is an arbitrary vertex, $s(x) = \alpha$ for all $x \in \overset{0}{E}$.

Let $\xi \in \partial E$. Since E is circled, for some $x \in \overset{0}{E}$, $x \sim \xi$. Hence, by the earlier remarks, $s(x) = \alpha$ and $s(\xi) = \alpha$. Consequently, $s \equiv \alpha$ on E. □

Remark 1.4.1. In particular, if $s \geq 0$ is superharmonic on X and if $s(x) = 0$ at a vertex x in X, then $s \equiv 0$.

A variation of the above Proposition 1.4.1 is the following.

Theorem 1.4.2. *(Minimum Principle) Let E be an arbitrary proper subset of X. Let s be a superharmonic function on E, attaining its minimum on E. Then $\inf_{\partial E} s = \inf_E s$.*

Proof. Let $\alpha = \inf_{\partial E} s$ and $\beta = \inf_{E} s$. Then, $\alpha \geq \beta > -\infty$. Suppose $\alpha > \beta$. Then $s(z) = \beta$ for some $z \in \overset{0}{E}$, by hypothesis. Choose $y \notin$E. There is a path $\{z = x_0, x_1, ..., x_n = y\}$ connecting z and y. Let i be such that $x_k \in \overset{0}{E}$ for all $k \leq i$ and $x_{i+1} \notin \overset{0}{E}$. Then $i < n$.

Now, $\beta t(z) = t(z)s(z) \geq \sum_{z_i \sim z} t(z, z_i)s(z_i) \geq \beta \sum t(z, z_i) = \beta t(z)$.
It is clear then $s(z_j) = \beta$ for every $z_j \sim z$. In particular $s(x_1) = \beta$. Continuing this process, we see that $s(x_{i+1}) = \beta$. Now $x_{i+1} \notin \overset{0}{E}$, but $x_{i+1} \sim x_i \in \overset{0}{E}$. Hence $x_{i+1} \in \partial E$. Consequently, $\inf_{x \in \partial E} s(x) \leq \beta$, which is a contradiction. This proves $\alpha = \beta$. □

Corollary 1.4.3. *Let E be a finite subset of X. Let u be superharmonic on E and h be subharmonic on E such that $u \geq h$ on ∂E. Then, $u \geq h$ on E.*

Proof. Let $s = u - h$ on E. Then, s is a superharmonic function on E such that $s \geq 0$ on ∂E. Since E is a finite set, s attains its minimum value on E. Hence, by the Minimum Principle, $s \geq 0$ on E; that is, $u \geq h$ on E. □

The above corollary is very useful on many occasions in the following forms:

(a) Let E be a finite subset of X. Let u be a superharmonic function on E. Then, for any x in E, $u(x) \geq \inf_{z \in \partial E} u(z)$.
(b) Let h_1, h_2 be two harmonic functions on a finite subset E of X. If $h_1 = h_2$ on ∂E, then $h_1 = h_2$ on E.
(c) Let h be a harmonic function defined on a finite subset E of X. Then for any $x \in E$, $\inf_{z \in \partial E} h(z) \leq h(x) \leq \sup_{z \in \partial E} h(z)$.

However, for the above assertions, the assumption that E is a finite set is important. For, consider the following example of an infinite subset E where the minimum principle in the form stated above is not valid:
Let $X = \{0, 1, 2, 3, ...\}$ and $E = \{1, 2, 3, ...\}$ with $t(n, n+1) = a$ if $n \geq 0$, $t(n, n-1) = b$ if $n \geq 1$, $a + b = 1$, and $a > b$. Then, $\overset{0}{E} = \{2, 3, 4, ...\}$ and $\partial E = \{1\}$. Let $h(n) = \left(\frac{b}{a}\right)^n$ for $n \geq 0$. Then, for $n \geq 2$,

$$\begin{aligned} ah(n+1) + bh(n-1) &= a\left(\frac{b}{a}\right)^{n+1} + b\left(\frac{b}{a}\right)^{n-1} \\ &= \left(\frac{b}{a}\right)^n \left[a.\frac{b}{a} + b.\frac{a}{b}\right] \\ &= \left(\frac{b}{a}\right)^n [b + a] = \left(\frac{b}{a}\right)^n \\ &= h(n). \end{aligned}$$

Hence, h is harmonic on E and bounded also. But $h(x) \leq h(1) = \inf_{z \in \partial E} h(z)$ for all x in E.

It is possible however to obtain useful modified minimum principles as given below. (Remark that corresponding to the minimum principle for superharmonic functions, we can prove a maximum principle for subharmonic functions. This is because a function u is subharmonic if and only if $-u$ is superharmonic.)

Proposition 1.4.4. *Assume that there exists a function $p > 0$ on X such that for any superharmonic function s on X, if $p + s \geq 0$ on X, then $s \geq 0$ on X. Let u be a subharmonic function on an arbitrary proper subset E of X such that $u \leq p$ on E. If $u \leq 0$ on ∂E, then $u \leq 0$ on E.*

Proof. Let $v = \sup\{u, 0\}$. Then, v is subharmonic on E and $v = 0$ on ∂E. If v is thought of as a function defined on X by taking values 0 outside E, then v is subharmonic on X and $v \leq p$ on X. Since $p - v \geq 0$ on X, by the assumption $-v \geq 0$, that is $v \leq 0$ on X and hence $u \leq 0$ on E. □

Proposition 1.4.5. *Suppose every positive superharmonic function on X is a constant. Let s be a lower bounded superharmonic function on an arbitrary proper subset E of X such that $s \geq m$ on ∂E. Then, $s \geq m$ on E.*

Proof. Suppose $s \geq \lambda$ on E. Let $v = \inf\{s, m\}$. Then, v is superharmonic on E and $v = m$ on ∂E. By giving values m outside E, we shall assume that v is defined on X. Then, v is superharmonic and also lower bounded on X. Hence by the assumption, v is a constant which should be m. This implies that $s \geq m$ on E. □

Example of a tree in which any positive superharmonic function is a constant: Let T be a homogeneous tree, each vertex having exactly 3 neighbours. Fix a vertex e and for $x \in T$, write $|x| = d(e,x) =$ the length of the unique path joining e to x. Note that for any $n \geq 1$, if $E_n = \{x : |x| \leq n\}$, then $\overset{0}{E_n} = E_{n-1}$ and $\partial E_n = \{x : |x| = n\}$. For $|x| = n \geq 1$, define $p(x,y) = \frac{1}{2}$ if $y \sim x$ and $|y| = n-1$ and $p(x,y) = \frac{1}{4}$ if $y \sim x$ and $|y| = n+1$; define $p(e,y) = \frac{1}{3}$ for each $y \sim e$. Let $h(x) = |x|$. It is easy to check that $\Delta h(e) = 1$ and $\Delta h(x) = 0$ if $x \neq e$.

Suppose $s > 0$ is a superharmonic function on T. Let $\alpha = \inf_{|x| \leq m} s(x)$, so that for some z, $|z| \leq m$, $\alpha = s(z) \leq s(x)$ for all $x \in E_m$. Let y be an arbitrary vertex in T such that $|y| > m$. Let n be a large integer, $n > |y|$. Define $h_n(x) = \alpha \frac{n-|x|}{n-m}$, which is harmonic at every $x \neq e$. Note then on ∂E_m, $s(x) \geq \alpha = h_n(x)$ and on ∂E_n, $s(x) > 0 = h_n(x)$. Then, by the Minimum Principle (Corollary 1.4.3), $s(x) \geq h_n(x)$ on $E_n \setminus \overset{0}{E_m}$. In particular, $s(y) \geq h_n(y)$. Allow $n \to \infty$ to conclude that $s(y) \geq \alpha$. As a consequence, we conclude that $s(x) \geq \alpha$ for all $x \in T$. But $s(z) = \alpha$, that is the superharmonic function attains its minimum on T. Hence s is a constant (Proposition 1.4.1).

1.5 Infinite Trees

A network need not have a symmetric conductance as in the case of the important example of random walks where the probability of transition $p(x,y)$ from a state x to a state y may not be the same as the probability of transition $p(y,x)$ from y to x. In such cases, defining $t(x,y) = \frac{1}{2}[p(x,y)+p(y,x)]$, we can obtain some useful results. In some other occasions, the network may be a tree, in which case a different method of obtaining symmetric conductance can be used.

An infinite tree T is a connected, locally finite, infinite graph without self-loops; moreover, if x and y are neighbours, there is no path $\{x = s_0, s_1, ..., s_n = y\}, n \geq 2$, with distinct vertices such that $s_i \sim s_{i+1}$ for $0 \leq i \leq n-1$. On T, it is assumed that transition probabilities $p(x,y) \geq 0$ are given such that $p(x,y) > 0$ if and only if x and y are neighbours and $\sum\limits_{y \sim x} p(x,y) = 1$ for any x in T.

T is said to be *a standard homogeneous tree* of degree $q+1$, $q \geq 2$, if each vertex has exactly $(q+1)$ neighbours and $p(x,y) = (q+1)^{-1}$ if $x \sim y$. Analogously, we can think of a standard homogeneous tree of degree 2 as follows: $T = \{..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...\}$, each vertex x_i having only two neighbours x_{i-1} and x_{i+1}, and $p(x_i, x_{i+1}) = p(x_i, x_{i-1}) = \frac{1}{2}$ for every x_i. (In the study of superharmonic functions on a standard homogeneous tree T of degree $q+1$, there will be many differences when the order $q = 1$ and $q \geq 2$.) Fix a vertex e in T and for a vertex x in T, denote $|x| = d(e,x)$, the distance of x measured from e. That is, if $\{e, x_1, x_2, ..., x_n = x\}$ is the (geodesic) path connecting x to e, then $|x| = d(e,x) = n$. Define the function s on T such that $s(e) = 1$ and $s(x) = q^{-i}$ if $|x| = i \geq 1$. Note that if $|x| = i$, then x has only one neighbour y with $|y| = i-1$ and q neighbours $z_j, 1 \leq j \leq q$, such that $|z_j| = i+1$. In a standard homogeneous tree T, it is easy to assert the existence of many non-proportional harmonic functions on T, with the help of the following proposition.

Proposition 1.5.1. *Let e be a fixed vertex in a standard homogeneous tree T of degree $q+1$, $q \geq 1$, and write $|x| = d(e,x)$ for a vertex x in T. Let $u(x)$ be a function defined when $|x| = n-1$ and $|x| = n$, for a fixed $n \geq 1$. Then, there exists a function $v(x)$ defined for $|x| \geq n-1$ such that $v(x) = u(x)$ if $|x| = n-1$ and $|x| = n$; and $\Delta v(x) = 0$ if $|x| \geq n$.*

Proof. Let $|z| = n$. Then, z has q neighbours $y_1, y_2, ..., y_q$ such that $|y_j| = n+1$ if $1 \leq j \leq q$ and one neighbour y_0 such that $|y_0| = n-1$. Take $\lambda = \frac{1}{q}[(q+1)u(z) - u(y_0)]$ and set $v(y_j) = \lambda$,$1 \leq j \leq q$, $v(y_0) = u(y_0)$ and $v(z) = u(z)$ so that $\Delta v(z) = 0$. Similarly, extend the definition of $v(x)$ for all $|x| = n+1$, so that $\Delta v(x) = 0$ if $|x| = n$ and $v(x) = u(x)$ if $|x| = n-1$ or n. An analogous procedure yields a function $v(x)$ for all x, $|x| \geq n-1$, such that $v(x) = u(x)$ if $|x| = n-1$ or n, and $\Delta v(x) = 0$ if $|x| \geq n$. $\square$

Corollary 1.5.2. *Let e be a fixed vertex in a standard homogeneous tree T of degree $q+1$, $q \geq 1$. Let $u(x)$ be a harmonic function defined on $E = \{x : |x| \leq n, n \geq 1\}$. Then, there exists a harmonic function v on T such that $v = u$ on E.*

Proof. Extend u as in the theorem to find a function v such that $v = u$ on E and $\Delta v(x) = 0$ if $|x| \geq n$. Since by the assumption $\Delta u(x) = 0$ if $|x| \leq n-1$, we conclude that v is harmonic on T and $v = u$ on E. □

Corollary 1.5.3. *For any vertex z in a standard homogeneous tree T of degree $q+1$, $q \geq 1$, there exists a superharmonic function $q_z(x)$ on T such that $\Delta q_z(x) = -\delta_z(x)$ for all x in T.*

Proof. Measuring distances from z, define a function $u(x)$ on $E = \{x : |x| \leq 1\}$ such that $u(z) = 2$ and $u(y) = 1$ if $y \sim z$. Then, by the above theorem, there exists a function v on T such that $\Delta v(x) = 0$ if $|x| \geq 1$ and $v(x) = u(x)$ if $|x| \leq 1$. Since, $\Delta v(z) = \sum\limits_{y\sim z} (q+1)^{-1}\,[v(y) - v(z)] = \sum\limits_{y\sim z} (q+1)^{-1}\,[u(y) - u(x)] = \sum\limits_{y\sim z} (q+1)^{-1}\,[1-2] = -1$, we conclude that $\Delta v(x) = -\delta_z(x)$ for all x in T. □

Remark 1.5.1. 1. Actually, given any vertex e in a standard homogeneous tree of degree $q+1$, $q \geq 2$, we can exhibit a positive superharmonic function s on T such that $\Delta s(x) = -\delta_e(x)$ for every x in T. For, define the function s on T such that $s(e) = 1$ and $s(x) = q^{-i}$ if $|x| = i \geq 1$. Note that if $|x| = i$, then x has only one neighbour y with $|y| = i-1$ and q neighbours z_j, $1 \leq j \leq q$, such that $|z_j| = i+1$. Consequently, if $|x| = i \geq 1$, then

$$\Delta s(x) = (q+1)^{-1}[q^{-i+1} - q^{-i}] + q(q+1)^{-1}[q^{-i-1} - q^{-i}] = 0, \text{ and}$$

$$\Delta s(e) = (q+1)(q+1)^{-1}(q^{-1} - 1) = (q^{-1} - 1) < 0.$$

Hence, $G_e(x) = \frac{s(x)}{(1-q^{-1})}$ is a positive function on the standard homogeneous tree T such that $\Delta G_e(x) = -\delta_e(x)$ for all x in T.

2. *Bounded harmonic functions on a standard homogeneous tree of degree $q+1$, $q \geq 2$.* Let T be a homogeneous tree of degree $q+1$, $q \geq 2$. Fix a vertex e and one of its neighbours a. Let $E = \{x :$ the path joining x to e passes through $a\}$. Then, $a \in E$ and $e \notin E$. Denote by $d(x,y)$ the distance between two vertices x and y. Define a function u on T such that $u(x) = q + 1 - q^{-d(e,x)}$ if $x \notin E$ and $u(x) = q^{-d(a,x)}$ if $x \in E$. Then, $u(e) = q$ and $u(a) = 1$.

$$\begin{aligned}
\Delta u(e) &= q(q+1)^{-1}\left[(q+1-q^{-1}) - q\right] + (q+1)^{-1}\,[1-q] \\
&= (q+1)^{-1}\,[(q-1) + (1-q)] = 0, \text{ and} \\
\Delta u(a) &= q(q+1)^{-1}\left[q^{-1} - 1\right] + (q+1)^{-1}\,[q-1] = 0. \\
\Delta u(x) &= q(q+1)^{-1}\left[q^{-n-1} - q^{-n}\right] + (q+1)^{-1}\left[q^{-n+1} - q^{-n}\right] \\
&= 0, \text{ if } x \in E \text{ and } d(a,x) = n.
\end{aligned}$$

$$\begin{aligned}\Delta u(x) &= q(q+1)^{-1}\left[(q+1-q^{-n-1})-(q+1-q^{-n})\right]\\ &\quad +(q+1)^{-1}\left[(q+1-q^{-n+1})-(q+1-q^{-n})\right]\\ &= 0,\ \text{if } x\notin E \text{ and } d(e,x)=n.\end{aligned}$$

Hence, $u(x)$is a non-constant bounded positive harmonic function on T. Since there are $(q+1)$ neighbours for e, there are at least $(q+1)$ non-proportional bounded positive harmonic functions on T.

3. Let T be a standard homogeneous tree of degree $q+1$, $q\geq 2$. Let $u=0$ on $E=\{x:|x|\leq n, n\geq 1\}$. If $z\in\partial E=\{x:|x|=n\}$, let $y_1,y_2,...,y_q$ be the q neighbours of z such that $|y_j|=n+1$, $1\leq j\leq q$. Let $v(y_j)=\lambda_j$, where all λ_j are not 0 but $\sum_1^q\lambda_j=0$, and $v=0$ on E. Then, $\Delta v(z)=0$. Proceeding with similar constructions as in the proof of Proposition 1.5.1, we arrive at a harmonic function v on T such that v is not identically 0 but $v=0$ on E. Note that this is in contrast to the situation in the classical potential theory where if a harmonic function h on a domain ω in the Euclidean space $\Re^n, n\geq 2$, is 0 in a neighbourhood of a point in ω, then $h\equiv 0$ on ω. However, note that if $T^\bullet$ is a standard homogeneous tree of degree 2 and if $h^\bullet$ is a harmonic function on $T^\bullet$ vanishing at two consecutive vertices (in particular, if $h^\bullet=0$ in a neighbourhood of a vertex in $T^\bullet$), then $h^\bullet\equiv 0$ on $T^\bullet$.

   In fact, on a standard homogeneous tree T degree 2, if h is a function such that $\Delta h(x)=0$ for every $x\in T$, then h is determined by its values at two adjacent vertices and is of the form $h(x_i)=a+i(b-a)$ for all i, where a and b are constants. For, suppose $\Delta h\equiv 0$. Let $h(x_0)=a$ and $h(x_1)=b$. Then, to find the value $h(x_2)$ for example, use the condition

$$0=\Delta h(x_1)=\frac{1}{2}[h(x_2)-h(x_1)]+\frac{1}{2}[h(x_0)-h(x_1)]$$

   to conclude that $h(x_2)=2h(x_1)-h(x_0)=a+2(b-a)$. Similar calculations give the values of $h(x_3),h(x_4)...,h(x_{-1}),h(x_{-2})...$ to arrive at the conclusion $h(x_i)=a+i(b-a)$, for all i. Consequently, the only positive harmonic functions in this tree are the positive constants and every bounded harmonic function is a constant.

   Actually, in the case of a standard homogeneous tree of degree 2, it is not possible to find a non-constant positive function s on T such that $\Delta s(x)\leq 0$ for all x. For, let $s\geq 0, \Delta s\leq 0$ on T. Since s is a non-negative superharmonic function on T, if $s=0$ at a vertex in T, then $s\equiv 0$ on T by the Minimum Principle. Hence, if $\min\{s(x_0),s(x_1),s(x_{-1})\}=a$, then $a>0$. For an integer $m>1$, let $h_m(x)=a\frac{m-|x|}{m-1}$, measuring distances from x_0. Then, $\Delta h_m(x)=0$ if $x\neq x_0$. Then, on the segment $[x_1,x_m]$,

$$s(x_1) \geq a \; = h_m(x_1) \text{ and}$$

$$s(x_m) > 0 = h_m(x_m).$$

Hence, by Corollary 1.4.3, $s(x) \geq h_m(x)$ on $[x_1, x_m]$. Take any vertex z, $|z| > 1$. Then, $s(z) \geq a\frac{m-|z|}{m-1}$ for any $m > |z|$. Allow $m \to \infty$ to conclude that $s(z) \geq a$. Consequently, $s(x) \geq a$ if x is any vertex in $\{x_0, x_1, x_2, ...\}$. Similarly for any vertex x in $\{..., x_{-2}, x_{-1}, x_0\}$ also, $s(x) \geq a$. This means that the superharmonic function s on T attains its minimum value a at a vertex in T. Hence $s(x) = a$ for all x in T.

Remark 1.5.2. The above results, concerning the existence of positive superharmonic functions on standard homogeneous trees, bring out some of the differences between standard homogeneous trees of degree $q+1$ when $q \geq 2$ and $q = 1$. These are akin to the differences in the study of classical potential theory on $\Re^n, n \geq 2$. For example, $|x|^{2-n}$ is a positive superharmonic function on $\Re^n, n \geq 3$. But any positive superharmonic function on $\Re^2$ is a constant; this assertion [27, p.28] is a generalised version of the classical Liouville theorem in the complex plane.

In the case of homogeneous trees, there are no terminal vertices. One simple example of a tree with terminal vertices is $T = \{x_0, x_1, x_2, ...\}$ where x_0 is a terminal vertex and every vertex $x_i, i \geq 1$, has only two neighbours x_{i-1} and x_{i+1}; if $i \geq 1$, then $p(x_i, x_{i+1}) = p_i$ and $p(x_i, x_{i-1}) = q_i$ such that $p_i + q_i = 1$ and $p(x_0, x_1) = 1$. In this tree, constants are the only harmonic functions on T. For, if h is harmonic on T and $h(x_0) = a$, then $0 = \Delta h(x_0) = h(x_1) - h(x_0)$ so that $h(x_1) = a$. The values of h at other vertices are calculated as in the previous example to show that $h(x) = a$ for all x.

Now, to obtain the Green's formulas on a network, we have used the fact that $t(x,y)$ is symmetric. To obtain similar formulas in a tree T, we shall introduce the following modifications. Fix a vertex e in T. For a vertex x, let $\{e, x_1, ..., x_n, x\}$ be the path joining e and x. Let $\varphi(x) = \frac{p(e,x_1)p(x_1,x_2)...p(x_n,x)}{p(x,x_n)p(x_n,x_{n-1})...p(x_1,e)}$; take $\varphi(e) = 1$. Note that if x and y are neighbours, then $\varphi(y) = \frac{p(x,y)}{p(y,x)}\varphi(x)$; that is, $\varphi(y)p(y,x) = \varphi(x)p(x,y)$. Clearly, this equality holds even when x and y are not neighbours. Thus, for any pair of vertices x and y in T, if we define $\psi(x,y) = \varphi(x)p(x,y)$, then $\psi(x,y) = \psi(y,x) \geq 0$, and $\psi(x,y) = 0$ if and only if x and y are not neighbours.

For any real function u on T, we use the expression

$$\Delta u(x) = \sum_{y \sim x} p(x,y)[u(y) - u(x)].$$

Let us write $\Delta^{\bullet} u(x) = \sum\limits_{y \sim x} \psi(x,y)[u(y) - u(x)]$. Then,

$$\Delta^{\bullet} u(x) = \sum_{y \sim x} \varphi(x)p(x,y)[u(y) - u(x)] = \varphi(x)\Delta u(x).$$

Similarly, if E is any proper subset of T and if $s \in \partial E$, then write

$$\frac{\partial^\bullet}{\partial n^-} u(s) = \sum_{y \sim s, y \in E} \psi(s, y)[u(y) - u(s)],$$

so that $\frac{\partial^\bullet}{\partial n^-} u(s) = \varphi(s) \frac{\partial}{\partial n^-} u(s)$. Write

$$(u, v)_E^\bullet = \frac{1}{2} \sum_{x, y \in E} \psi(x, y)[u(x) - u(y)][v(x) - v(y)].$$

Theorem 1.5.4. *(Green's formulas in trees) Let f and g be two real-valued functions on a tree T. Let E be a finite subset of T. Then,*

i. $\sum_{x \in \overset{0}{E}} \varphi(x) f(x) \Delta g(x) + (f, g)_E^\bullet = - \sum_{s \in \partial E} \varphi(s) f(s) \dfrac{\partial}{\partial n^-} g(s).$

ii. $\sum_{x \in \overset{0}{E}} \varphi(x)[f(x) \Delta g(x) - g(x) \Delta f(x)] = - \sum_{s \in \partial E} \varphi(s)[f(s) \dfrac{\partial}{\partial n^-} g(s) - g(s) \dfrac{\partial}{\partial n^-} f(s)].$

Proof. Let us consider T as an infinite network with the symmetric conductance $\psi(x, y)$. Then, by Theorem 1.3.2, $\sum_{x \in \overset{0}{E}} f(x) \Delta^\bullet g(x) + (f, g)^\bullet = - \sum_{s \in \partial E} f(s) \frac{\partial^\bullet}{\partial n^-} g(s)$. Rewriting this, we get *(i)* and then *(ii)* since $(f, g)_E^\bullet = (g, f)_E^\bullet$. □

Remark 1.5.3. Suppose T is a standard homogeneous tree of degree $q + 1$. Fix a vertex e and measure distances from e. Let $B_m = \{x : |x| \leq m\}$. Then, $\{x : |x| < m\} = \overset{\circ}{B}_m$ and $\partial B_m = \{x : |x| = m\}$. Since $p(x, y) = (q + 1)^{-1}$ for any pair of neighbours x and y, we find $\varphi \equiv 1$. Hence, for functions f and g defined on B_m, we can write $\sum_{|x| < m} [f(x) \Delta g(x) - g(x) \Delta f(x)] = - \sum_{|s| = m} [f(s) \dfrac{\partial}{\partial n^-} g(s) - g(s) \dfrac{\partial}{\partial n^-} f(s)]$.

http://www.springer.com/978-3-642-21398-4

Harmonic Functions and Potentials on Finite or Infinite Networks
Anandam, V.
2011, X, 141 p., Softcover
ISBN: 978-3-642-21398-4