Chapter 2
Financial Markets and Asset Pricing

2.1 Asset Pricing Theory

2.1.1 No-Arbitrage and the Stochastic Discount Factor

Essentially all modern asset pricing models rely on a single fundamental pricing equation according to which the price of an asset follows the relationship

$$P_{i,t} = E_t[M_{t+1} X_{i,t+1}] \tag{2.1}$$

where $P_{i,t}$ is the price of an asset i at time t, M_{t+1} is the stochastic discount factor (SDF) and $X_{i,t+1}$ represents the payoff of asset i at $t+1$. Payoffs in general can be split up into a future price component ($P_{i,t+1}$) and an earning stream ($D_{i,t+1}$) such as coupon payments on coupon-bearing bonds or dividends on stocks. Since future payoffs are uncertain, the SDF is used to value the state-contingent possible payoffs of the asset in $t+1$ so that the SDF takes the uncertain payoff back to present. But even if the cash flows generated by asset i may be known with certainty, the discount factors and future interest rates respectively are uncertain numbers depending on the future state of the economy.

Uncertainty in financial markets is mainly modeled by a probability space $(\Omega, \mathscr{F}, \mathscr{P})$ where Ω units all possible events of the state of the world and $\mathscr{F}$ is a collection of events of the sample space (σ-algebra). Every event in $\mathscr{F}$ has a certain numerical probability where the function $\mathscr{P} : \mathscr{F} \rightarrow [0,1]$ assigns a probability between 0 and 1 to any event imbedded in $\mathscr{F}$. Since the valuation of assets has a time dimension, Ω can be interpreted as a set of increasing dimension of time. The filtration describes the evolution of information (events) through time with $\mathscr{I}\mathscr{F} = \{\mathscr{F}_0, \ldots, \mathscr{F}_T\}$. A filtration can be understood as an ever increasing stream of information (events) and agents do not forget implying that $\mathscr{F}_s \subset \mathscr{F}_T$ whenever $s \leq T$. Accordingly, if the process $X_t = \{X_0, X_1, \ldots, X_T\}$ is a random variable, it is said to be $\mathscr{F}$-measurable and adapted with respect to the measurable space $(\Omega, \mathscr{F})$. Such a stochastic process can be a simple random walk in discrete time

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$$W_t = W_{t-1} + \epsilon_t$$
$$W_0 = 0 \quad \epsilon_t \sim N(0, \sigma^2). \tag{2.2}$$

In order to introduce the concept of the SDF and its asset pricing implications, this section starts with two famous theorems which guarantee that, under weak conditions, a SDF (1) exists and (2) the SDF is strictly positive. The *law of one price* asserts that if two portfolios have the same payoffs in every state of nature, then both portfolios must have the same price. Otherwise, an immediate situation would arise in which the prospect of an arbitrage profit would lead investors to sell the expensive portfolio and to buy the cheaper portfolio instead. Cochrane (2001, 64) shows that the law of one price implies the existence of a discount factor. However, the law of one price is restrictive since it assumes that the two portfolios are perfect substitutes and trade at the same price.

The concept of *no-arbitrage* goes further and describes that it is not possible to come up with a portfolio with zero costs but whose payoff is positive (and never negative) with a positive probability. This line of argument is close to the concept of market efficiency. A market is said to be efficient if there are no unexploited arbitrage opportunities which requires that all new and relevant information is instantly imbedded in market prices (Fama 1970, 1991).[1] As it will be demonstrated later in this Section, market efficiency and the concept of no-arbitrage then imply that returns of any asset should be the same except for compensations of various forms of risk. The following baseline model illustrates the concept of no-arbitrage.[2]

The model consists of N assets, each of them adapted with a price P_{it}. All prices at time t are compressed in the vector $P_t = (P_{1t}, \dots, P_{it})^\top$. A *trading* or *portfolio strategy* is a process $H_t = (H_{1t}, \dots, H_{it})$ where H_{it} stands for the amount of asset i held in the portfolio within the time interval $(t, t+1)$. The value of the portfolio at time t is defined as $V_t = H_t P_t = \sum_{i=1}^{N} H_{it} P_{it}$. The corresponding gain process to the trading strategy H satisfies

$$\delta_t(H) = H_{t-1} P_t - H_t P_t \tag{2.3}$$
$$H_{-1} \quad \text{zero by convention}$$

The logic is that the investor holds a portfolio H_{t-1} from $t-1$ and at time t prices are revealed and the investor reallocates her asset holdings. If $\delta_t < 0$, the investor has to finance the new portfolio with capital from outside; if $\delta_t > 0$, the new portfolio is cheaper than the one in time $t-1$ and the difference is a gain and can be put aside. If $\delta_t = 0$, a trading or a portfolio strategy is said to be self-financing so that

[1]For a survey on the efficient market hypothesis and its critics the reader might be referred to Malkiel (2003). Various forms of financial market efficiency are typically categorized according to the information set. Thereby, the literature distinguishes between the weak, the semi-strong and the strong form of market efficiency.

[2]See Irle (2003) and Duffie (2003) for a detailed analysis of arbitrage in a n-period model.

rebalancing the amount of assets held in the portfolio does not require or produce additional funds. A trading strategy is called *arbitrage strategy* if $\delta_t(H) \geq 0$ for $t = 0, 1, \ldots, T$ and the gain process has a positive probability $\mathscr{P}(\delta_t(H) > 0) > 0$ for at least at one point in time t. To be concrete, a portfolio H constitutes an arbitrage strategy if it holds that the initial value of the portfolio in $t-1$ is negative $V_{t-1} < 0$ but the value of the portfolio in t is always non-negative with probability $\mathscr{P}((V_t \geq 0) = 1)$. An arbitrage opportunity, thus, arrives when any zero-net-investment guarantees a positive payoff in some future state with no possibility of a negative payoff in all other future states. To put it in jargon, arbitrage implies the existence of a *free lunch*. However, there is no such thing as a free lunch! The model is said to be arbitrage-free whenever such an arbitrage strategy is not possible. Indeed, if a risk-free chance to come up with a zero-cost portfolio that does not permit a loss possibility existed, smart investors would pick this trading strategy to earn risk-free returns. However, at the same time, they would alter the asset prices held in the portfolio to bring about a no-arbitrage equilibrium.

The existence of a positive stochastic discount factor establishes a necessary and sufficient condition for a market to be arbitrage-free (Irle 2003, 114). Vice versa, Harrison and Kreps (1979) formally state that assuming away arbitrage opportunities is equivalent to the existence of a positive discount factor within a particular arbitrage-free market. In addition, Cochrane (2001, 51) demonstrates that with a complete set of contingent claims and state prices, a positive discount factor exists.[3] The latter result is reviewed in the next paragraphs.

In a simple discrete state model, there are $s = 1 \ldots S$ states of the world. A complete market implies that for each state s, there is a contingent claim with price $p_c(s)$ that pays off one unit in state s and 0 otherwise.[4] In a next step, for $i = 1 \ldots N$ assets in the economy, the price of an asset i is defined as $P(i)$ with payoff $X(si)$ in state s. In matrix and vector notation, the $N \times 1$ vector P collects the assets prices, X is the $S \times N$ matrix of payoffs and Q is the gross return on each asset for each state so that $Q = X^\top / P$ with element $Q_{si} = 1 + R_{si} = X(si)/P_i$. The state price vector then satisfies $X^\top p_c = P$ where p_c is the $S \times 1$ vector including the number of state prices $p_c(s)$. The price of asset i then follows

$$P(i) = \sum_{s=1}^{S} p_c(s) X(si). \tag{2.4}$$

This relation states that the price of an asset is simply equal to the sum of the state price in a given state multiplied by the amount of the payoff of the asset in state s. It can be expressed as a bundle of these contingent claims with $X(1i)$ contingent

[3] An excellent review on neoclassical finance, no-arbitrage and market efficiency is the Princeton Lecture Series on Finance by Ross (2005).

[4] The necessary condition of complete markets does not mean that investors have to trade explicit contingent claims. They rather need enough securities to span or to synthesize all contingent claims.

claims to state 1, $X(2i)$ claims to state 2 etc. Another way of looking at this relation is to replace the sum over states with the probabilities $\pi(s)$ of the states. To do so, the state-density function is defined as

$$M(s) = \frac{p_c(s)}{\pi(s)}$$

according to which $M(s)$ is the state price divided by the probability of state s occurring. An asset's price now can be written as

$$\begin{aligned} P(i) &= \sum_{s=1}^{S} \pi(s) M(s) X(s) \\ &= E[MX] \end{aligned} \tag{2.5}$$

The interpretation is straightforward. In states of small M, the state s is cheap in the sense that investors are reluctant to pay a high price to receive the payoff in this state. Consequently, for each $\{t\}_{t=0}^{T}$ and each asset $\{i\}_{i=1}^{N}$ with information $\mathscr{F}$ available at time $t-1$ and $\mathscr{P}(M_t > 0) = 1$ it holds that

$$P_{i,t-1} = E_{t-1}[M_t X_{i,t} | \mathscr{F}_{t-1}]. \tag{2.6}$$

M_t is the stochastic discount factor, a random variable. Applying (2.6) to above trading strategy, gives $V_{t-1} = E(M_t V_t)$. If it is assumed that V_t is strictly positive, ruling out arbitrage opportunities implies that M_t must be likewise positive so that V_{t-1} is positive, too – the condition of no-arbitrage. It also results in the proposition that a positive state price only exists if there are no arbitrage opportunities. Furthermore, the SDF and the state price vector are unique only in case of complete markets. In case of incomplete markets, many $M's$ may exist (Campbell et al. 1997).

Equation (2.4) can be expressed in terms of a gross return of the asset price. In a contingent claim market, (2.4) can be divided by the price of the asset $P(i)$ where the gross return is defined as $(1 + R(si)) = X(si)/P(i)$ for all s. This gross return can be interpreted as a payoff $X(s)$ with price one so that

$$1 = \sum_{s=1}^{S} p_c(s)(1 + R(si)). \tag{2.7}$$

Finally, from (2.5) it follows that

$$\begin{aligned} 1 &= \sum_{s=1}^{S} \pi(s) M(s)(1 + R(si)) \\ 1 &= E[M\ (1 + R(i))] \end{aligned} \tag{2.8}$$

so that the product of the pricing kernel and the asset's expected return equals one.

Since all assets are priced according to the stochastic discount factor approach, it applies also to risk-free assets. An asset is said to be risk-free if it delivers the same payoff in all states of the world so that the payoff $X(s)$ is independent of s and one can write $X(s) = \bar{X}$ for all s. Often, instantaneous maturing assets are risk-free assets, since they have a payoff of 1 in the next period with certainty and no risk of price fluctuations are eminent. The price of such an asset is then

$$\begin{aligned} P(f) &= \sum_{s=1}^{S} p_c(s)\bar{X} = \bar{X}\sum_{s=1}^{S} p_c(s) \\ &= \bar{X}\sum_{s=1}^{S} \pi_s M(s) \\ &= \bar{X}E[M] \end{aligned} \tag{2.9}$$

which implies no-arbitrage. As the payoff of the risk-free asset can be characterized as a zero-coupon bond with payoff 1 in all states, the price equation collapses to

$$P(f) = E[M]$$

or in terms of its gross return

$$\begin{aligned} 1 &= 1/P(f)E[M] \\ 1 &= (1 + R(f))E[M] \\ E[M] &= \frac{1}{1 + R(f)}. \end{aligned} \tag{2.10}$$

Having pinned down the risk-free interest rate, another convenient way of expressing asset prices is to use the risk-neutral valuation approach instead of using the subjective state probabilities in order to price assets. For that purpose, risk neutral probabilities for state s are defined as

$$\pi^Q(s) = (1 + R_f)M(s)\pi(s) \tag{2.11}$$

with $1 + R_f = 1/E[M]$ so that an asset's price stated in state probabilities can be converted into a price equation in terms of risk-neutral probabilities. The asset pricing formula can be rewritten as

$$P(i) = \sum_{s=1}^{S} \pi(s)M(s)X(s) = \frac{1}{1+R_f}\sum_{s=1}^{S} \pi^Q(s)X(s) = \frac{E^Q[X]}{1+R_f} = E[M]E^Q[X] \tag{2.12}$$

where asset i is priced as if investors are risk-neutral but apply π^Q instead of π to the states in the economy. Thus, risk-neutral probabilities align more weight on unpleasant states if investors are risk-averse (see on this account Chap. 3.5).

2.1.2 Individual Agent Optimality and Asset Pricing Equations

The stochastic discount factor so far offers no link to consumption decisions. Indeed, the use of the SDF to price all assets in an economy must not rely on an explicit model of intertemporal optimal consumption or the presence of a representative agent. Even the absence of complete markets would not alter the existence of a positive, but maybe not a unique SDF (Cochrane 2001). The pricing equation can hold individually for a single investor with a single portfolio. In the following, the model is extended by an investor who possess a utility function with utility derived from consumption. In this case, it can be modeled to describe the optimal allocation of wealth between individual current consumption and financial assets. This is reasonable to assume since typically an investor holds and reallocates the portfolio for some purpose, for instance to transfer wealth in different states and periods to finance expenditures in different states in the future.

This section starts with a simple evaluation of household optimality in a contingent claim model following the work of Duffie (2003).[5] The single investor's expected utility is described by a strictly increasing utility function[6] with preferences over consumption whereas utility takes an additive form. The investor tries to maximize current plus expected discounted utility where C is current consumption, e are current endowments and β is the subjective discount factor. The problem is to maximize

$$\max U(C) = u(C) + \beta \sum_{s=1}^{S} \pi_s u(C(s))$$

subject to

$$C + \sum_{s=1}^{S} p_c(s)C(s) = e + \sum_{s=1}^{S} p_c(s)e(s).$$

The budget constraint contains current consumption, current endowments, future consumption and endowments both unknown to the agent in the current period. State prices $p_c(s)$ are used to value future consumption and endowments. In the

[5]See also Semmler (2003) or Wickens (2008) for this result.

[6]It must hold that if $a > c$ then $u(a) > u(c)$.

second period, the investor may purchase contingent claims to each possible state. Using the Lagrange multiplier analysis and defining λ as the multiplier, the first-order conditions (FOCs) are

$$u'(C) - \lambda = 0$$
$$\beta\pi(s)u'(C(s)) - \lambda p_c(s) = 0 \quad \forall\, s = 1 \ldots S.$$

If both conditions are combined, it can be shown that state prices follow

$$p_c(s) = \beta\pi(s)\frac{u'(C(s))}{u'(C)} \quad \forall\, s = 1 \ldots S \tag{2.13}$$

or

$$M(s) = \frac{p_c(s)}{\pi(s)} = \beta\frac{u'(C(s))}{u'(C)} \tag{2.14}$$

so that state prices are determined by state probabilities and the intertemporal rate of substitution between current and future consumption. Alternatively, the first-order condition says that the marginal rate of substitution between states in the coming period equals the relevant price ratios. For instance, it represents the rate at which the investor is willing to give up consumption in state 1 for consumption in state 2 through purchases and sales of contingent claims

$$\frac{M(s1)}{M(s2)} = \frac{u'(C(s1))}{u'(C(s2))}.$$

The aim of the next step is to derive the optimal path of consumption of an investor for an infinite time horizon. In this respect, the investor can transfer wealth over time through asset holdings. In this setup, she has the opportunity to trade one asset i so that total portfolio holdings are the sum of the asset holding i. One can think of asset i as being a simple zero-coupon bond. At the beginning of each period, the investor arrives with an endowment stream y_t and portfolio holdings from last period H_{t-1} valued with the price P_t at time t for resale. Resources are then the sum of the endowment and the revenue of sales of the asset in the portfolio at the prevailing price. At the end of period t, the single investor can either reallocate her portfolio holdings at the cost of P_t or she can use them for consumption purposes. In a dynamic setting, the investor, therefore, wishes to choose a stream of consumption $\{C_t\}_{t=0}^{\infty}$ to maximize the expected discounted sum of utilities.

$$\max\ E_0\left[\sum_{t=0}^{\infty}\beta^t u(C_t)\right]$$

subject to

$$
\begin{aligned}
& H_t P_t + C_t = y_t + H_{t-1} P_t \quad \forall\, t \geq 0 \\
& H_{-1} \quad \text{given.}
\end{aligned} \tag{2.15}
$$

The amount of assets held in the portfolio H_{t-1} from period $t-1$ is the state variable and the portfolio H_t chosen to hold in period t together with current endowments y_t are the control variables whose level optimally depicted by the utility-maximizing investor affects the resources available in the next period $t+1$. The consumption level C_t could also be treated as a control variable, yet as it will be demonstrated below, the use of the portfolio holdings as the control variable has the suitable advantage that it is a more simple task to find the Euler equation.[7] The intertemporal separability of the objective function and the budget constraint allows to convert an infinite period problem into a two-period problem with the appropriate rewriting of the objective function (Adda and Cooper 2003). This is achieved by considering that the investor commits to optimality in a dynamically consistent way, i.e. today's optimal choice is made with the knowledge that it will likewise be optimal next period onwards.

As a first step, the objective function can be expressed in terms of a value function $V(H_{t-1}, y_t)$ which is the value of the objective function at the optimum when the control variable has been optimally chosen. By using the recursive structure of the value function, it is possible to derive the Bellman equation whose first derivative allows for the derivation of the Euler equation. Essentially, the optimization problem is to seek a policy function $H_t = h(H_{t-1}, y_t)$ which maps the state (H_{t-1}, y_t) into the control H_t. As soon as the household has chosen the portfolio (and thus the consumption) level, the transition function $H_t = (y_t + H_{t-1} P_t - C_t) 1/P_t$ determines next period's state H_t. Such a procedure is recursive. The value of the objective function then can be written according to

$$
\begin{aligned}
V(H_{t-1}, y_t) &= \max \left\{ \sum_{t=0}^{\infty} \beta^t u(C_t) \right\} \\
V(H_{t-1}, y_t) &= \max \left\{ u(C_t) + \beta E_t \left[\sum_{t=0}^{\infty} u(C_{t+1}) \right] \right\} \\
V(H_{t-1}, y_t) &= \max \left\{ u(C_t) + \beta E_t [V(H_t, y_{t+1})] \right\}.
\end{aligned} \tag{2.16}
$$

Equation (2.16) is called the Bellman equation. To solve this problem, the literature offers different solution methods. The most popular techniques are the application

[7] As it is shown in the Appendix A, if it is possible to write the transition equation in such a way so that next period's state does not depend upon last period's state, the first derivative of the Bellman equation is easy to solve.

of the Lagrangian or the Envelope theorem. Appendix A gives a more rigorous formal statement of the Envelope theorem which should be used in this Section. After writing down the Bellman equation, the budget constraint can be solved for C_t. Applying this to (2.16) gives

$$V(H_{t-1}, y_t) = \max_{H_t} \{u(y_t + H_{t-1} P_t - H_t P_t) + \beta E_t [V(H_t, y_{t+1})]\} \tag{2.17}$$

The first-order condition with respect to the control variable H_t set equal to zero is

$$\frac{\partial V(H_{t-1}, y_t)}{\partial H_t} = u'(C_t)(-P_t) + \beta E_t \frac{\partial V(H_t, y_{t+1})}{\partial H_t} = 0. \tag{2.18}$$

Ljungqvist and Sargent (2004) show that for an interior solution to hold, the value function $V(H_{t-1}, y_t)$ is also differentiable with respect to its state variable H_{t-1}. The solution is called Envelope Theorem or the Benveniste-Scheinkmann condition. Consequently, the first derivative of the Bellman equation (2.17) is

$$\frac{\partial V(H_{t-1}, y_t)}{\partial (H_{t-1})} = u'(C_t) P_t \tag{2.19}$$

Taking (2.19) and moving it one period forward gives

$$\frac{\partial V(H_t, y_{t+1})}{\partial (H_t)} = u'(C_{t+1}) P_{t+1} \tag{2.20}$$

In a last step, (2.18) and (2.20) are combined to get the Euler equation

$$u'(C_t)\ P_t = \beta\ E_t[u'(C_{t+1})\ P_{t+1}]. \tag{2.21}$$

The Euler equation – the first-order condition for optimal consumption and portfolio choices of an investor – can be used to link asset prices and consumption. The equation offers an intuitive economic interpretation. The left-hand side of (2.21) gives the marginal utility (here it is a loss in utility) of giving up a small amount of utility and using the additional resources to buy an amount of assets which is added on the portfolio holdings H_t. The right-hand side expresses the discounted marginal utility gain at time $t + 1$ when pursuing this strategy. It captures the increase of marginal utility due to an increase of prices. The household continues to buy or sell assets until the marginal loss equals the marginal gain. This process continues until the household is indifferent to consuming a small amount at date t or in transferring resources via asset allocation to $t + 1$ to gain discounted marginal utility due to price increases. To sum up, if the agent is indifferent about changing the amount of assets she demands, then she is demanding the optimal amount of assets.

So far, this section only analyzed the asset Euler equation for a single asset, i.e. a discount bond. The payoff is its price in the next period. It would be also interesting to see how a stock price in such a setting behaves. In this context, the available assets which can be used to transfer wealth have to be modified. As in the previous example, the investor can trade a discount bond with holdings H and, in addition, she can trade equity shares which entitles her to a (stochastic) stream of equity dividends $\{D\}_{t=0}^{\infty}$. Let $P_{eq,t}$ be the stock price of the amount of shares L the investor chooses to hold. The budget constraint of (2.15) is modified according to

$$H_t P_t + L_t(P_{eq,t} + D_t) + C_t = y_t + H_{t-1} P_t + L_{t-1} P_{eq,t} \qquad (2.22)$$
$$H_{-1}, N_{-1} \quad \text{given}$$

The investor again maximizes the value function

$$V(H_{t-1}, L_{t-1}, y_t) = \max \{u(c_t) + \beta E_t [V(H_t, L_t, y_{t+1})]\} \qquad (2.23)$$

At interior solutions, the first-order conditions are the Euler equations associated with the controls H_t and L_t.

$$u'(C_t)\ P_t = \beta\ E_t[u'(C_{t+1})\ P_{t+1}] \qquad (2.24)$$

$$u'(C_t)\ P_{eq,t} = \beta\ E_t[u'(C_{t+1})\ (P_{eq,t+1} + D_{t+1})]. \qquad (2.25)$$

The basic result is that for any asset $\{i\}_{i=1}^{N}$ in an economy which can be traded to transfer resources over time it holds that

$$u'(C_t)\ P_{i,t} = \beta\ E_t[u'(C_{t+1})\ X_{i,t+1}] \quad \forall\ i = 1, \ldots, N \qquad (2.26)$$

where $X_{i,t+1}$ is the payoff of the asset i in $t + 1$.

2.1.3 Representative Agent and Equilibrium Asset Pricing

The preceding Sect. 2.1.2 discusses the condition for optimal consumption. Indeed, it represents any equilibrium condition. An equilibrium holds if all agents maximize and the market clears. The introduction of a representative household does not change the fundamental asset Euler equation at all. It just opens the door to model general equilibrium considerations. Following Cochrane (2001), firstly, the model can ask what determines consumption at given price sequences and preferences. Then the task is to treat the price sequences and the periodical payoffs as exogenous variables and agents as price-takers. In addition, one can specify the generation of payoffs but they are not determined within the model. The finance literature often takes some linear production technologies as given to specify the real physical rate of interest to which the consumption stream adjusts.

Secondly, following the work of Lucas (1978) and Breeden (1979), one can solve the model at a given equilibrium consumption stream. Then, the model is given by identical, infinitely lived agents (or a representative agent) each of whom tries to maximize lifetime expected utility. Market clearing means that every agent who wants to buy one unit of assets at price $P_{i,t}$ must have a counterpart agent who wishes to sell the asset at price $P_{i,t}$. In a general equilibrium, this leads to a situation in which the sum of the total demand for assets must be zero. Essentially, any positive demand must be equalled by a corresponding negative demand. It holds that $\sum H_t = 0$.[8] However, since all agents are identical with identical preferences, this may appear as a pitfall. This is because if one agent wishes to buy an asset, all others wish to do so, equally. Therefore, to clear markets, prices must be exactly such that agents are indifferent of buying or selling assets. Such a situation occurs if one substitutes $H_t = H_{t-1} = 0$ in the budget constraint of (2.15) to see immediately that zero asset demand implies $C_t = y_t$. In equilibrium, agents consume their exogenous endowments at all dates. In this respect, the scope is to search for precisely those asset prices which make it optimal for the representative investor to consume her periodical endowments in each period. Taking (2.26), in equilibrium it holds that

$$u'(y_t)\ P_{i,t} = \beta\ E_t[u'(y_{t+1})\ X_{i,t+1}]. \tag{2.27}$$

It is also possible to rewrite the basic Euler equation to isolate the current price of an asset i

$$P_{i,t} = \beta\ E_t\left[\frac{u'(y_{t+1})}{u'(y_t)}\ X_{i,t+1}\right]. \tag{2.28}$$

The current equilibrium price depends positively on future expected payoffs. It is greater the higher time preference β appears to be. If β is high, then the representative agent is patient in consuming her endowments and is willing to transfer wealth via asset purchases to receive payoffs in $t+1$. Today's price also depends on the ratio of marginal utilities (marginal rate of substitution). The price of the asset i will be higher when marginal utility of consumption and income is high. Since it is assumed that the utility function is concave, the representative agent tries to smoothen consumption. That is, if the agent expects future consumption to be lower than today's consumption, then marginal utility derived from future consumption is higher than today's marginal utility. If the agent expects that income is very low in $t+1$, the agent tries to transfer wealth via asset purchases from time t to $t+1$. As long as the utility function is concave, the ratio of marginal utilities is inversely related to the change in consumption from date t to $t+1$ (i.e. the consumption growth rate). This is the explanation why a higher intertemporal rate of substitution evokes a high asset demand and pushes today's asset prices up.

It is also possible to inspect the dynamics of the fundamental asset pricing equation in a rational expectations equilibrium. Any rational asset pricing equilibrium is a pair of processes $\{P_t\ X_{i,t}\}$ that satisfy (2.28), $C_t = y_t$ and the transversality condition ($E_t[\lim_{T\to\infty} u'(C_{t+T})X_{i,t+T}] = 0$) given the exogenous process $\{y_t\}$.

[8] For simplification, in this Section, the amount of asset holdings in the portfolio is denoted by H.

The latter transversality condition implies that it is ensured that the agent does not overaccumulate assets so that a higher expected overall lifetime utility can be achieved by, for example, increasing consumption today. In a finite horizon, this would imply that the agent dies with positive asset holdings which is not optimal.

2.1.4 Asset Returns and a First Look at Risk

2.1.4.1 Returns, Pricing Kernel and Risk

This section sets the stage for how asset returns behave in an asset pricing equilibrium. To shed light on asset returns, there is no need to make references to an explicit consumption-based equilibrium framework with an intertemporal utility-maximizing investor in the first place. The basic asset pricing formula sets the current asset price equal to the expected product of the pricing kernel and the future payoff. Absence of arbitrage opportunities guarantees the existence of a unique and positive SDF. However, for the sake of a rigorous economic understanding, this section uses the consumption Euler equation as basic equilibrium condition to draw return and risk implications. Initially, recall that for any asset, the first-order condition satisfies

$$P_{i,t} = E_t\left[\beta \frac{u'(C_{t+1})}{u'(C_t)}\, X_{i,t+1}\right] \tag{2.29}$$

which is nothing else than a slight modification of the no-arbitrage pricing formula of (2.6). The SDF in a multi-period consumption-based asset pricing model is given by the expected ratios of marginal utilities

$$E_t[M_{t+1}] = E_t\left[\frac{u'(C_{t+1})}{u'(C_t)}\right]. \tag{2.30}$$

One-period gross holding returns which throughout this section are simply called returns take the general form of

$$1 + R_{i,t+1} = \frac{X_{i,t+1}}{P_{i,t}} \quad \forall\, i = 1, \ldots, N.$$

By dividing the left-hand side of (2.29) by the price and recalling that this operation yields an expression for a return, the most fundamental asset pricing equation takes the form of

$$1 = E_t\left[\beta \frac{u'(C_{t+1})}{u'(C_t)}\, \frac{X_{i,t+1}}{P_{i,t}}\right] = E_t\left[M_{t+1}(1 + R_{i,t+1})\right]. \tag{2.31}$$

This way of writing the model in discrete time goes back to Rubinstein (1976). It states that asset prices obey in such a way that their returns multiplied with the stochastic discount factor equal one. To understand the implications of (2.31), especially the joint dynamics of prices of asset i with the SDF, one can use the definition of the covariance to specify the expected product of the SDF and the return. For any independent random variables x, z, it holds that $E_t xz = E_t x E_t z + cov_t(x, z)$ where $cov_t = E_t(x - E_t x)(z - E_t z)$.[9] Then, the expectation of the product can be written as the product of expectations plus the covariance term.

$$E_t\left[M_{t+1}(1 + R_{i,t+1})\right] = E_t\left[M_{t+1}\right] E_t\left[1 + R_{i,t+1}\right] + cov_t\left(M_{t+1} R_{i,t+1}\right) \quad (2.32)$$

Substituting this expression into (2.31) and rearranging gives

$$1 + E_t[R_{i,t+1}] = \frac{1 - cov_t(R_{i,t+1} M_{t+1})}{E_t[M_{t+1}]}. \quad (2.33)$$

The conditional covariance is a measure for risk since it describes how random variables move together. Within the consumption-based setting, an asset's return and future consumption are random variables following stochastic processes. Equation (2.33) establishes a connection between the risk premium and a function for the consumption process. An asset that offers a low return when the stochastic discount factor moves in opposite direction, must bear a risk compensation. The economic intuition is as follows; a large positive pricing kernel corresponds to a state of low consumption in $t+1$ and high consumption in t. An asset characterized by a negative correlation with expected marginal utility is risky because it is not able to deliver wealth in a situation in which wealth is most valuable to investors. Consequently, risk-averse investors demand a risk compensation for holding this asset because it performs purely in a state where wealth is particularly important to investors (Campbell 2000). This means that the asset's return covaries positively with expected consumption growth but negatively with marginal utility. The covariance then can be interpreted as the risk premium of an asset. In contrast, if the return of an asset covaries positively with expected marginal utility and negatively with consumption growth, this will induce investors to demand a negative risk premium and lower returns. This asset precisely delivers wealth when high expected payoffs go hand in hand with expected states of high marginal utility.

In terms of prices, an asset characterized by a positive covariance of its payoff with the SDF ($cov_t(M_{t+1}, X_{i,t+1})$) must be traded at a higher price since the payoff is positively correlated with the SDF. This asset yields higher payoffs in a situation in which the investor has high marginal utility of transferring wealth into the next period. Buying this asset and the corresponding payoff may help the investor to smoothen consumption over time; it provides a hedge for the investor so that she is ready to pay a higher price. These kind of assets are countercyclical to current consumption and they are priced at a premium since the demand for these assets is high. In contrast, an asset whose payoff covaries negatively with the SDF is

[9] See, for instance, Wooldridge (2006).

valued at a discount which is nothing else than an additional risk compensation. It is negatively correlated with the marginal rate of substitution and it covaries positively with consumption growth. Buying this asset would even make the consumption stream more volatile. Therefore, the investor demands a risk premium which lowers the price of the asset and drives up required returns.

Since the realization of a risky asset return is uncertain in t, its realized return is not known until in period $t+1$ so that the timing convention $t+1$ as subscript for expected risky returns is appropriate. Moreover, (2.33) holds for any asset in the economy no matter whether it may be risky or risk-free. This link allows to derive a risk-free interest rate. A risk-free interest rate is the return of an asset f whose covariance with the pricing kernel is zero ($cov_t(R_{f,t} M_{t+1}) = 0$). It implies that the return is known in time t with certainty so that it can be denoted with the subscript t. The risk-free return obeys

$$1 + R_{f,t} = \frac{1}{E_t[M_{t+1}]}. \tag{2.34}$$

The definition for the risk-free rate can be substituted into (2.33) to obtain an expression for the expected excess return (risk premium) of asset i over the risk-free asset f.

$$E_t[XR_{i,t+1}] \equiv E_t[R_{i,t+1} - R_{f,t}] = -(1 + R_{f,t}) cov_t(R_{i,t+1} M_{t+1}). \tag{2.35}$$

The expected excess return of an asset i is positive if and only if its return is negatively correlated with the SDF – and in a consumption-based setting with the marginal rate of substitution.

Although the covariance between the SDF and the return of asset i determines its riskiness, it is straightforward to show that the volatility of the asset's underlying payoffs and the volatility of the SDF contribute to the nature of the risk premium since the covariation is implicitly driven by both volatility components. Thereby, the definition of the covariance can be refined as

$$\begin{aligned} cov_t(M_{t+1}, R_{i,t+1}) &= corr_t(M_{t+1}, R_{i,t+1}) \sqrt{var_t(M_{t+1})} \sqrt{var_t(R_{i,t+1})} \\ &= \rho_{Mi,t}\, \sigma_{M,t}\, \sigma_{i,t} \end{aligned} \tag{2.36}$$

with the volatilities σ_M, σ_i together with the correlation coefficient ρ_{Mi} determining the covariation.

Against these conditions, it is also possible to represent the basic no-arbitrage relation of (2.35) in an expected return-beta representation (Cochrane 2001). It holds that

$$\begin{aligned} E_t[1 + R_{i,t+1}] - (1 + R_{f,t}) &= -\left(\frac{cov_t(M_{t+1}, R_{i,t+1})}{std_t(M_{t+1})} \right) \left(\frac{std_t(M_{t+1})}{E_t[M_{t+1}]} \right) \\ &= \beta_{Mi,t} \lambda_{M,t}. \end{aligned} \tag{2.37}$$

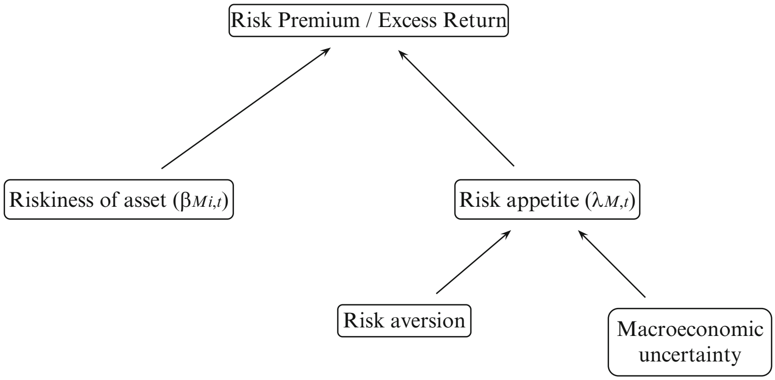

Fig. 2.1 Risk concepts (*Source*: Gai and Vause (2006))

where $\beta_{Mi,t}$ is the quantity of asset-specific risk and $\lambda_{M,t}$ is the market price of risk per unit of risk which should be equal across all assets traded in the economy.[10] The price of risk can therefore be defined as the reciprocal of investor's risk appetite; the latter measures the willingness of investors to bear units of risk. Thereby, the concepts of "risk premium," "risk appetite" and "risk aversion," though closely connected, are distinct concepts (Gai and Vause 2006). Figure 2.1 illustrates the risk premium decomposition of (2.37). First and foremost, risk appetite depends on investors' degree of reluctancy towards uncertainty; risk aversion is determined by investors' preferences over uncertain outcomes which should not change rapidly and much over time. The second component of risk appetite is the level of uncertainty. It relies on the overall macroeconomic environment and moves periodically in response to factors such as unemployment prospects, the macroeconomic policy set-up and the volatility of consumption streams. Most generally, changing risk appetite or market sentiment over the course of business cycles should then reflect shifts in macroeconomic uncertainty rather than altering preferences towards risk. By taking the price of risk and the quantity of risk together, the excess return (risk premium) for each asset can be determined.

The implications of no-arbitrage asset pricing theory can be confirmed in stylized facts on international business cycles. Stock and Watson (1999, 2003a) identify procyclical effects of macroeconomic dynamics and asset returns on financial markets. For example, in a recession period, both returns and expected economic growth (consumption) are low, indicating positive risk premia on assets. Moreover, during booms the two variables are high and trigger excess returns as well. This attributes to the overall finding that in an economy with procyclical returns, assets

[10] In this respect, (2.35) together with (2.36) can be restated as the Sharpe ratio (SR) $(E_t[1 + R_{i,t+1}] - R_{f,t})/\sigma_{i,t} = -\rho_{M,i,t}\sigma_{M,t}E_t[M_{t+1}]^{-1}$. The highest possible Sharpe ratio applies to a return that is perfectly negatively correlated with the SDF so that $SR = \sigma_M E_t[M_{t+1}]^{-1}$. The highest Sharpe ratio, thus, coincides with the market price of risk $\lambda_{M,t}$.

should incorporate positive risk premia. Research also shows that risk premia are not constant over time but vary depending on business conditions and the state of the economy.[11] Typically, in a recession episode, risk premia tend to be higher than in a boom phase due to higher macroeconomic uncertainty and volatility (Cochrane and Piazzesi 2005). This is plausible since in "bad times" investors demand higher returns to transfer resources to the next period. When increased uncertainty is reflected by asset markets, agents heavily discount expected future events in current prices through the stochastic discount factor. By way of contrast, a boom is often associated with lower macroeconomic uncertainty and lower (but maybe still positive) risk premia.

2.1.4.2 A Log-Normal Representation

The log-normal model of representing returns and the SDF has become the workhorse for pricing financial assets (Campbell et al. 1997). It offers convenient features for modeling interest rates and it is a reasonable approximation to historical asset prices, at least in the long-run. For that purpose, it is assumed that the SDF and gross returns are jointly log-normal distributed. A random variable is said to be log-normally distributed if $\log(X)$ follows a normal distribution with mean μ and variance σ^2. Conversely, if $\log(X) \sim N(\mu, \sigma^2)$, then the expected value of X is

$$E[X] = \exp\left(\mu + \frac{\sigma^2}{2}\right)$$

and

$$\log(E[X]) = \mu + \frac{\sigma^2}{2}.$$

Applying the joint distribution properties to (2.31), it follows

$$\begin{aligned} 1 &= E_t[M_{t+1}(1 + R_{i,t+1})] \\ &= \exp\left\{E_t[\log(M_{t+1}(1 + R_{i,t+1}))] + \frac{1}{2} var_t\left(\log(M_{t+1}(1 + R_{i,t+1}))\right)\right\}. \end{aligned} \tag{2.38}$$

Taking the logarithm gives

[11] See Pesando (1975), Fama (1984), Tzavalis and Wickens (1997), Hejazi and Li (2000) or Cochrane and Piazzesi (2005).

$$
\begin{aligned}
0 &= \log(E_t[M_{t+1}(1+R_{i,t+1})]) \\
&= E_t[\log(M_{t+1}(1+R_{i,t+1}))] + \frac{1}{2} var_t\,(\log(M_{t+1}(1+R_{i,t+1}))) \\
&= E_t[m_{t+1}] + E_t[r_{i,t+1}] + 0.5var_t(m_{t+1}) + 0.5var_t(r_{i,t+1}) + cov_t(m_{t+1}, r_{i,t+1})
\end{aligned}
\tag{2.39}
$$

where $\log M_{t+1} = m_{t+1}$ and $\log(1 + R_{i,t+1}) = r_{i,t+1}$.[12]

The risk-free rate with $cov_t(m_{t+1}, r_{f,t}) = 0$ and $var_t(r_{f,t}) = 0$ becomes

$$r_{f,t} = -E_t[m_{t+1}] - 0.5var_t(m_{t+1}). \tag{2.40}$$

Subtracting (2.40) from (2.39) gives the no-arbitrage condition for asset i

$$E_t[r_{i,t+1}] - r_{f,t} + 0.5var_t(r_{i,t+1}) = -cov_t(m_{t+1}, r_{i,t+1}) \tag{2.41}$$

where $0.5var_t(r_{i,t+1})$ stems from Jensen's Inequality. Finally, using the definition of the correlation coefficient the market price of risk takes the form of

$$\lambda_{M,t} = \sigma_{M,t}. \tag{2.42}$$

2.1.4.3 Pricing Nominal Returns

So far, the analysis above deals only with real assets according to which payoffs and asset prices are denominated in units of goods. It allows to derive a real risk-free interest rate which coincides with the reciprocal of the stochastic discount factor. Such a risk-free asset exactly delivers, say, one unit of good for the fixed delivery date. In practice, one hardly finds assets and returns denominated in consumption goods which may serve as a proxy for a real risk-free return. Index-linked bonds (TIPS) are the closest attempt to explicitly trade "real" assets on financial markets. They have part or all of their payoffs linked to a basket of weighted prices of consumption goods. However, even this type of asset may be not fully indexed since in the market for TIPS there are liquidity and technical factors that affect market prices (D'Amico et al. 2008a).

Still, on a modeling level, it is straightforward to introduce nominal assets and to adequately price these asset in nominal terms. Using the fact that assets and goods are priced in currency units, the budget constraint of an investor can be modified to incorporate nominal bonds in real terms. Let $P_{CPI,t}$ be the price index, then a nominal bond costs in nominal terms $P^{\$}_{i,t}$ and in units of goods $P^{\$}_{i,t}/P_{CPI,t}$;

[12] The variance term can be written as the sum of the variances of the random variables and two times their covariance since it must hold that $var_t(x + y) = var_t(x) + var_t(y) + 2cov_t(x, y)$ (Wooldridge 2006).

it pays \$1 or equivalently $\$1/P_{CPI,t+1}$ in units of goods. An investor is faced with a maximization problem according to (2.15) with a modified budget constraint

$$H_t P^{\$}_{i,t}/P_{CPI,t} + C_t = y_t + H_{t-1} P^{\$}_{i,t}/P_{CPI,t}.$$

By no-arbitrage, there exists a stochastic discount factor which discounts expected real cash flows, i.e. payoffs denominated in units of goods. The real SDF can be used to value nominal assets whereas the nominal payoffs can be denominated in real terms. The real price of any nominal asset then obeys

$$\frac{P^{\$}_{i,t}}{P_{CPI,t}} = E_t\left[M_{t+1}\,\frac{X^{\$}_{i,t+1}}{P_{CPI,t+1}}\right]$$

or in terms of returns

$$1 = E_t\left[M_{t+1}\,\frac{P_{CPI,t}}{P_{CPI,t+1}}\,(1 + R^n_{i,t+1})\right] \tag{2.43}$$

$$1 = E_t\left[M_{t+1}\,\frac{1}{(1 + \pi_{t+1})}\,(1 + R^n_{i,t+1})\right] \tag{2.44}$$

with $(1 + \pi_{t+1}) = P_{CPI,t+1}/P_{CPI,t}$ denoting the gross inflation rate and $R^n_{i,t+1}$ the nominal return of asset i. The product of the real stochastic discount factor M_{t+1} and the price deflators can now be interpreted as a nominal stochastic discount factor or nominal pricing kernel with $M^{\$}_{t+1} = M_{t+1}/(1 + \pi_{t+1})$.

The pricing of nominal bonds also allows a derivation of a "modern" version of the Fisher equation. Fisher's theory of nominal interest rate determination (1986) implies that the gross nominal interest rate is a function of the gross real rate of return (the increase in real income and consumption, respectively) and the rate of appreciation or depreciation of one commodity standard in terms of another which is the expected gross inflation or deflation rate. In a no-arbitrage equilibrium, the goods-denominated return on a nominal bond and the return on a capital asset must be the same. Under perfect foresight, goods price changes are accurately predicted and fully incorporated in nominal returns. As a result, these returns fully adjust to inflation and leave the real rate unchanged at the constant level of the given return on the capital asset. Fisher was well aware that uncertainty or risk about the return on a real bond and expected inflation distorts the equilibrium condition; however, he stopped to give a rigorous geometric or algebraic interpretation. He, thus, assumed in a first step perfect foresight so that the investor is faced with perfectly known income streams, a given (ex-ante) real rate of return and goods prices.

When relaxing some of the restrictive assumptions, the Fisher equation has to be modified. If expected inflation is said to be random and perfect foresight is not imposed, price changes will not be fully reflected by one-for-one changes in nominal returns. Moreover, inflation risk will be translated into the pricing of nominal assets.

This is so because the real gross return of the nominal bond will as well become an uncertain number. Any risk-averse investor will demand a compensation for that inflation risk in terms of higher returns. To put it in a more formal statement, (2.43) can be modified to give a "modern" version of the Fisher no-arbitrage condition. If a one-period real bond exists which perfectly matches the properties of the real stochastic discount factor, risk-averse investors consider inflationary risk, so that the Fisher generalization for a one-period nominal bond can be written according to

$$(1 + R^n_{1,t+1})^{-1} = (1 + R_{f,t})^{-1} E_t \left[\frac{P_{CPI,t}}{P_{CPI,t+1}} \right] + Cov_t \left(M_{t+1}, \frac{P_{CPI,t}}{P_{CPI,t+1}} \right) \tag{2.45}$$

or

$$(1 + R^n_{1,t+1}) = (1 + R_{f,t}) E_t [1 + \pi_{t+1}] \zeta^{-1} - 1/Cov_t \left(M_{t+1}, (1 + \pi_{t+1}) \right). \tag{2.46}$$

The nominal one-period interest rate equals the real one-period risk-free return adjusted for expected inflation and an inflation risk premium.[13] The covariance term, thereby, captures the effect of inflationary risk on the nominal bond. As it was stated, with random inflation, the real return of the nominal asset remains uncertain. Inflation risk can result either in an increase or decrease of the nominal one-period bond's real return depending on the sign of the conditional covariance between the real stochastic discount factor and the reciprocal of gross inflation. If the covariance term of (2.45) is negative, then investors demand a higher nominal return to compensate for inflationary risk. In particular, in a consumption-based setting, if high marginal utility tomorrow with low expected consumption growth coincides with high inflation, high inflation erodes the nominal bond's real return in states in which the investor suffers from low expected consumption (Ireland 1996; Sarte 1998). This could be the case for a positive demand shock where inflation is pushed upwards due to high current consumption. Expected consumption growth then turns out to be weak (for given consumption expectations) so that high marginal utility of transferring wealth into the next period coincides with a reduction of the nominal bond's real return. If, however, the covariance term is positive, low marginal utility and high expected consumption go hand in hand with high inflation delivering a lower real return in times when expected consumption is already high. This constellation would produce low or even negative inflation risk premia and it would reduce the one-period nominal return. For instance, a negative supply shock brings down current consumption but triggers an increase in inflation. Since the investor simultaneously experience a lower expected marginal rate of substitution, buying a nominal bond acts like a hedge to her.

[13] A n-period version for pricing nominal bonds is provided by Wolman (2006).

Another reason why the modern version of the Fisher equation does not correspond to the original one under perfect foresight is that even if the covariance term is zero, a Jensen's Inequality effect ζ exists. Equation (2.46) implies that in general $E_t[P_{CPI,t+1}/P_{CPI,t}]$ is not the same as $1/E_t[P_{CPI,t}/P_{CPI,t1}]$ so that the modern version has to be augmented by the variability of inflation (see in a log-normal setting Benninga and Protopapadakis 1983; Sarte 1998). Greater variability in goods prices expressed in a higher ζ likely increases the expected real return of the nominal asset. Therefore, it rises the price of nominal bonds and makes them more attractive compared to inflation-indexed or real bonds. The higher the price of the nominal asset, the lower the (required) nominal return will tend to be. As inflation becomes more variable, the nominal bond's real value will increase since it is a convex function of the price level. Hence, to maintain a no-arbitrage equilibrium, the nominal return has to decline.

2.1.4.4 Valuation of Stock Prices

Traditional valuation principles for equities rely on the discounted cash-flow or the present value method. This model relates the price of a stock to its expected future payoffs discounted to the present using either constant or time-varying discount factors (Campbell et al. 1997). Since all expected future dividend streams enter the present value formula, temporary movements in expected cash-flows effect stock prices far less than persistent swings. Similarly, persistent changes in the discount rate have much greater influence on the valuation of stocks. Since the general no-arbitrage equilibrium condition of (2.31) holds for any asset, the stock price V_t satisfies

$$V_t = E_t[M_{t+1}(D_{t+1} + V_{t+1})] \tag{2.47}$$

where the next period's payoff X_{t+1} is determined by next period's dividend stream D_{t+1} and the stock price V_{t+1}. This expectational difference equation can be solved forward by repeatedly substituting out future prices. Using the law of iterated expectations, future-dated expectations can be eliminated. The price of a stock is given as

$$V_t = E_t\left[\sum_{j=1}^{T} M_{t,t+j} D_{t+j}\right] + E_t[M_{t,t+T} V_{t+T}] \tag{2.48}$$

$$M_{t,t+j} = \prod_{i=1}^{j} M_{t+i}$$

where $M_{t,t+j}$ compounds the one-period discount factors M_{t+i}.

The traditional present-value model (PVM) states that in terms of the no-arbitrage relation, there is a constant discount factor implying that $M_{t+i} = 1/(1+R)$ for all i. Taking the limit $T \to \infty$, the price of an equity is

$$V_t = E_t\left[\sum_{j=1}^{\infty}\left(\frac{1}{1+R}\right)^j D_{t+j}\right] + E_t\left[\lim_{T\to\infty}\left(\left(\frac{1}{1+R}\right)^T V_{t+T}\right)\right].$$

In the previous Sect. 2.1.3, the Lucas model (1978) was presented in which the representative agent in each period consumes her periodical endowments. To explore the dynamics of a rational expectations equilibrium in this asset pricing economy it holds that

$$V_t = E_t\left[\sum_{j=1}^{\infty}\beta^j \frac{u'(y_{t+j})}{u'(y_t)} D_{t+j}\right] + E_t\left[\lim_{T\to\infty}\frac{\beta^T u'(y_T)}{u'(y_t)} V_{t+T}\right].$$

Here, the constant risk free rate is replaced by a sequence of expected ratios of marginal utilities in equilibrium. The transversality condition which imposes a no-bubble solution in equilibrium $E_t\left[\lim_{T\to\infty}\left(\beta^T u'(y_T)\right)/u'(y_t)V_{t+T}\right] = 0$ finally breaks the last part of the solution down to the fundamental stock price equation where the price solely depends on expected future dividends

$$V_t = E_t\left[\sum_{j=1}^{\infty}\beta^j \frac{u'(y_{t+j})}{u'(y_t)} D_{t+j}\right]. \tag{2.49}$$

In this respect, the Gordon model is a specification of the fundamental no-arbitrage pricing equation in which the expected rate of growth of dividends g^D is assumed to be constant and nonzero over time (Gordon 1962). Applying the present value formula with a constant risk-free rate, stock prices evolve according to

$$V_t = D_t \sum_{j=1}^{\infty}\left(\frac{(1+g^D)}{1+R}\right)^j.$$

The fundamental value of an equity is made up of expected dividend streams. Tests on the present value, however, consistently arrive at the conclusion that, if at all, the link between prices and dividends appears to vary over time. For example, the dividend-price ratios and the dividend yield are not constants nor do they fluctuate around these constants. Indeed, there may be a dividend-based fair value for stock prices, but basically it means that over time, current prices diverge from the fundamental value, though dividend-price ratios can serve as predictors for excess returns on stock prices (Fama and French 1988). Other macro variables forecast stock returns as well including the investment/capital ratio and the consumption/wealth ratio (Lettau and Ludvigson 2001). Still, the stylized relation between stock returns and macroeconomic forces is found to be unstable so that unambiguous links lack empirical support (Panetta 2001).

The fair value of a share is obtained by the expectational difference equation and solving this equation forward. This equation is stable if the expression $(1/1 + R)$ is smaller than one which is clearly the case; in addition, if the last term of the present value shrinks to zero with an increasing time horizon, then technically speaking the system has a unique solution. The market process guarantees that deviations from the present value are only a temporary event. Higher stock prices relative to the fundamental value reduce the dividend yield and trigger stock sales to finally reach the equilibrium solution again. However, rising stock prices and sudden crashes are the very nature of stock price dynamics. Ex-post, one would title such episodes as bubbles which have to burst some time in the future. The existence of a bubble is often associated with asset price dynamics that behave in a completely irrational way with what fundamentals would suggest.[14] However, it can be shown that bubbles may reflect rational expectations on behalf of investors.[15]

In general, there is another solution for the stock price $V_{B,t}$ that likewise satisfies (2.47) with $M_{t+i} = 1/(1+R)$ and that differs from the fundamental solution by the term B_t. Any solution including bubble solutions can be written as

$$\begin{aligned} V_{B,t} &= V_t + B_t \\ &= \frac{1}{1+R} E_t(V_{B,t+1} + D_{t+1}) \end{aligned} \tag{2.50}$$

where B_t stands for the non-fundamental solution. V_t still serves as the fundamental value of the underlying stock price. It can be shown that by subtracting (2.47) from (2.50), the law of motion for the bubble term is given by

$$B_t = \frac{1}{1+R} E_t B_{t+1}.$$

Following Brunnermeier (2008) and imposing rational expectations with $B_{t+1} = E_t[B_{t+1}] + \xi_{t+1}$ and $E_t[\xi_t] = 0$, the expression for the bubble term of the pricing equation becomes

$$B_{t+1} = (1+R)B_t + \xi_{t+1}$$

which is an explosive process due to $R > 0$. Since bubbles burst and suddenly disappear, rational bubbles rely on a mechanism which prohibits an ever increasing or decreasing path of price misalignments when the system is hit by an innovation ξ. To do so, Blanchard and Watson (1982) suggest a bubble which takes the form

[14] See Kindleberger (1995) for a treatment on episodes of bubbles, maniacs and panics.

[15] DeBondt and Thaler (1985) point out that with the help of experimental psychology, irrational market behavior can be observed when investors overreact to unexpected news.

$$B_{t+1} = \begin{cases} (\frac{1+R}{\pi})B_t + \xi_{t+1} & \text{with } \pi; \\ \xi_{t+1} & \text{with } 1-\pi. \end{cases}$$

The bubble can burst with probability $1 - \pi$ in any period and a probability of growing further with π. Since investors know that the bubble may collapse with a positive constant probability, the bubble grows faster than R to compensate for the likelihood of bursting.

2.1.4.5 Bond Prices and the SDF

The SDF model and its fundamental asset pricing equation is mostly used to price bonds and the term structure of interest rates. It allows linking prices of bonds with different maturities in a no-arbitrage framework. Compared to stock prices whose periodical cash flows, i.e. dividends, are stochastic, fixed-income securities have the convenient property that the underlying periodical payoffs are deterministic. For example, zero-coupon instruments always deliver a specified final payoff at the maturity date without any payoff during the life of the bond. Only if the investor sells the bond before maturity, she is faced with an uncertain price. The obvious risk solely arises from time variation in discount rates which is the basis for the valuation of the bond price. The variation itself is driven by the stochastic behavior of the discount rates. Any co-movements between discount rates and the underlying bond consequently stem from these time-series dynamics.

The stochastic discount factor can be defined either in real (M_{t+1}) or nominal terms ($M^{\$}_{t+1}$) depending on what kind of zero bonds are priced. For the sake of simplicity, in this Section, M_{t+1} stands for both factors. A zero-coupon bond trades at $P_{n,t}$ in period t and pays 1 \$ at maturity $t + n$. An one-period gross return of this asset is related to its price in the way that $(1 + R_{n,t+1}) = P_{n-1,t+1}/P_{n,t}$. The payoff of the zero bond in period $t + 1$ is, thus, $P_{n-1,t+1}$ which can be substituted into the no-arbitrage Equation of (2.31). Then, the two prices with different maturities are related recursively according to

$$P_{n,t} = E_t[M_{t+1}P_{n-1,t+1}] \tag{2.51}$$

where n denotes time to maturity. The price of a bond with one period to maturity can be expressed directly in terms of the stochastic discount factor of the next period conditional on time t. For $P_{0,t+1} = 1$ at the maturity date, the one-period discount price is

$$P_{1,t} = E_t[M_{t+1}]$$

and given the relation between one-period prices, returns $(1+R)$ and yields $(1+Y)$, the one-period short rate is given by

$$(1 + R_{f,t}) := (1 + Y_{1,t}) = \frac{1}{E_t[M_{t+1}]}$$

as the reciprocal of the SDF. By means of repeated substitution and solving forward by the law of iterated expectations the basic pricing equation (2.51) can be used to describe the determination of any bond price with maturity $n > 1$. For the two-period bond price, it holds that

$$\begin{aligned} P_{2,t} &= E_t[M_{t+1}P_{1,t+1}] \\ &= E_t[M_{t+1}E_{t+1}[M_{t+2}]] \\ &= E_t[E_{t+1}[M_{t+1}M_{t+2}]] \\ &= E_t[M_{t+1}M_{t+2}]. \end{aligned}$$

Long-term bonds in general can be expressed as

$$P_{n,t} = E_t[M_{t+1}, \ldots, M_{t+n}] = E_t\left[\prod_{i=1}^{n} M_{t+i}\right] = E_t M_{t,t+n} \tag{2.52}$$

where the price of a n-period bond is the product of successive expected one-period discount factors until maturity. In this context, pricing the term structure of interest rates is just a specification of a time series model for the expected dynamics of the stochastic discount factor. Indeed, as will be demonstrated later, most term structure models rely on a specification model of the stochastic properties of the discount factor which is firstly restricted by the absence of arbitrage opportunities in bonds markets and secondly, is used to price the whole spectrum of bond prices. Taking conditional expectations of the product of the SDFs given the information set at t is sufficient to describe the whole yield curve. A risk-adjustment to the price can be also included so that (2.52) becomes

$$P_{n,t} = \prod_{i=1}^{n} E_t[M_{t+i}] + \sum_{i=1}^{n-1} cov_t(M_{t+i+1}, M_{t+1}). \tag{2.53}$$

Furthermore, if the continuously compounded interest rate $i_{n,t}$ is defined as the negative of the logarithm of the pricing kernel, any bond price with maturity n can be written as

$$P_{n,t} = E_t\left[e^{-\sum_{i=0}^{n-1} i_{1,t+i}}\right] \tag{2.54}$$

so that the price is its expected present value discounted by the sum of one-period interest rates. The handling of risk components in the specification of (2.54) depends on the question whether actual probabilities $E^P[.]$ depart from risk-neutral probabilities $E^Q[.]$, i.e. whether the expectations operator $E[.]$ is specified as $E^P[.]$ or $E^Q[.]$. A formal representation of no-arbitrage term structure modeling is subject to Chap. 3.5.

2.2 Asset Pricing with Utility Specifications

2.2.1 Agents and Risk Aversion

Section 2.1.4 discussed asset pricing implications for a no-arbitrage equilibrium. In the consumption-based setting, in order to deduce empirical or dynamic results from these asset pricing relations, sufficient specifications on preferences and risk attitudes have to be made. Intertemporal asset pricing models rely on the assumptions about certain forms of utility functions. The von Neumann-Morgenstern utility function (vNM utility function) framework allows for the identification of expected utility and the identification of various forms of risk attitudes of inspected agents where the general probability space of Sect. 2.1.1 holds.

The probability space and the random variables makes it feasible to describe an agent's choices under risk.[16] Let the random variables defined on a finite set of monetary values be expressed as $X = \{x_1, x_2, \ldots, x_S\}$. Usually, x is regarded to be a monetary measure on income or wealth. Each of the random variables are combined with a probability $[\pi_1, \pi_2, \ldots, \pi_S]$ whose combinations with the variables are called a gamble a. The probabilities add up to one

$$a \equiv [x_1, x_2, \ldots, x_S; \pi_1, \pi_2, \ldots, \pi_S] \tag{2.55}$$

$$\sum_{s=1}^{S} \pi_s = 1.$$

If the preference relation $\mathscr{R}$ of an agent satisfies the von Neumann-Morgenstern axiomatic approach to expected utility, then the preference relation over the lottery is

$$U(a) = \sum_{s=1}^{S} \pi_s u(x_s) = EU(x) \tag{2.56}$$

where u is the utility function that represents $\mathscr{R}$ for money obtained with certainty in the different states of the world s (Barucci 2003). $U(a)$ can be expressed as the weighted sum of a function u in the different states of the world. $U(a)$ is then linear in the probabilities and is called expected utility function or felicity function.

In this setting, the agent faces choices characterized by risk according to which not only the utility associated with an amount of money has to be considered but also the probabilities of receiving that amount of money. What is likewise important, is the question about certain risk attitudes which is captured by the preference relation $\mathscr{R}$. When the preference relation can be expressed through the expected

[16] For the derivation of the following propositions see for example Hirshleifer (1975), Hammond (1987), Pratt (1992), Machina and Rothschild (1992), Barucci (2003), Rubinstein (2006) or Cowell (2006).

utility $u(x_s)$, it is possible to show the agent's risk preference with the help of the shape of the utility function.

Let the random variables of a gamble only take on two values $\{\varepsilon_1, \varepsilon_2\}$ with probabilities $\{\pi, (1-\pi)\}$. Thus, one can construct a lottery $\tilde{x}$ with two possible wealth outcomes: $x_1 = x + \epsilon_1 < 0$ with probability π and $x_2 = x + \varepsilon_2 > 0$ with probability $1-\pi$; hereby, initial wealth is x. Further it as assumed that $\tilde{\varepsilon}$ represents a fair gamble, i.e. $E[\tilde{\varepsilon}] = \pi\varepsilon_1 + (1-\pi)\varepsilon_2 = 0$. Does a vNM-utility maximizer accept this lottery, i.e. does she put any positive value to this lottery? An agent is said to be risk averse if she is unwilling to accept this fair gamble but she strictly prefers another lottery that delivers the sure outcome x with certainty. Therefore, it must necessarily hold that she prefers x to $\tilde{x} = x + \tilde{\varepsilon}$. It represents the utility-decreasing aspect of pure risk-taking. If the first lottery is accepted, expected wealth is $E[\tilde{x}] = E[x + \tilde{\varepsilon}] = \pi_1 x_1 + (\pi_2)x_2 = x$ and expected utility takes the form of $E[U(x + \tilde{\varepsilon})] = \pi_1 u(x_1) + (1-\pi)u(x_2)$. If the first lottery is not accepted but instead the sure outcome is preferred, expected wealth is $E[x] = x$ with expected utility $E[U(x)] = U(x)$.

By exploiting the vNM utility function, an agent is said to be risk averse if $U(x) > E[U(\tilde{x})] = \pi(x+\varepsilon_1)+(1-\pi)(x+\varepsilon_2]$. Since $E[\tilde{\varepsilon}] = 0$, denying the lottery yields an expected utility of $U(E[\tilde{x}]) = U(x)$. It implies that $U(E[\tilde{x}]) > E[U(\tilde{x})]$. This is equivalent to assume a concave utility function with Jensen's Inequality to hold.[17] As Fig. 2.2 demonstrates, the utility function is concave, since drawing a line connecting the two points $u(x_1)$ and $u(x_2)$, the resulting utility $E[U(\tilde{x})]$ strictly lies below $U(E[\tilde{x}]) = U(x)$.[18]

Another way of looking at the lottery is to ask how much compensation a risk-averse agent would demand in order to accept taking the risky gamble. This compensation could be represented by an additional payment. A risk-averse agent is indifferent between the risky gamble $\tilde{x}$ and an amount of money CE which she can receive with certainty when both possibilities offer the same (expected) utility such that $U(CE) = E[U(\tilde{x})]$. This effect is called "certainty equivalence". It means that the expected wealth of the gamble $E[\tilde{x}] = x$ is always greater than the certainty equivalent CE. Stated differently, the certainty equivalent is the maximum price that she is willing to pay to receive the risk gamble. This is the "risk premium" associated with the risky outcome $\tilde{x}$, defined as

$$\rho_{\tilde{x}} = E[\tilde{x}] - CE = x - CE. \tag{2.57}$$

It is easy to consider the certainty equivalent as a risk-free asset which delivers a return R_f with certainty and yields wealth in case of investing an initial stock of

[17]Jensen Inequality states that for a concave function $h(x)$ where x is a random variable, it holds that $E[h(x)] \leq h(E[x])$ with $h''(x) < 0$.

[18]For the sake of completeness, an agent is said to be a risk lover, if the utility is convex so that $U(E[\tilde{x}]) < E[U(\tilde{x})]$ with $U''(\tilde{x}) > 0$. An agent is said to be risk-neutral if $U(E[\tilde{x}]) = E[U(\tilde{x})]$ with $U''(\tilde{x}) = 0$.

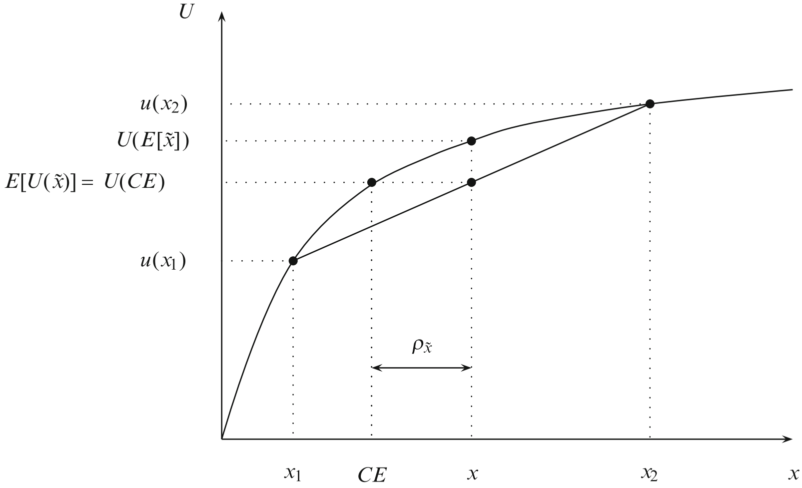

Fig. 2.2 Risk averse utility function

wealth in form of $x_{ce} = x_0(1 + R_f)$. In contrast, investing in a risky asset does not guarantee this wealth increase. Thus, a risk-averse investor demands the risk premium compensation $\rho_{\tilde{x}}$ in order to be indifferent between investing in the riskless asset with gross return R_f or in a stock of asset with expected gross return $E[R] = R_f + \rho_{\tilde{x}}$.

The magnitude of the risk aversion depends on the derivatives of the vNM utility function. By a second-order Taylor approximation of $U(\tilde{x})$ around x, one obtains

$$U(\tilde{x}) = U(x + \tilde{\varepsilon}) \approx U(x) + \tilde{\varepsilon}U'(x) + \frac{1}{2}U''(x)\tilde{\varepsilon}^2.$$

Expected utility then approximately takes the form of

$$E[U(\tilde{x})] \approx U(x) + \frac{1}{2}U''(x)\sigma^2$$

since $E[\tilde{\varepsilon}] = 0$ and $\sigma^2 = E[\tilde{\varepsilon}^2]$. Another Taylor expansion of the certainty equivalent centered at x up to the first order is

$$U(CE) = U(x - \rho_{\tilde{x}}) \approx U(x) - U'(x)\rho_{\tilde{x}}.$$

Since it must hold that $U(CE) = E[U(\tilde{x})]$ by definition, the risk premium can be calculated according to

$$\rho_{\tilde{x}} \approx -\frac{1}{2}\frac{U''(x)}{U'(x)}\sigma^2 \tag{2.58}$$

where

$$r_x^a = -\frac{U''(x)}{U'(x)}$$

is the Pratt-Arrow measure of absolute risk aversion (Barucci 2003). The risk premium is decomposed into two components, the variance of the lottery and the agent's risk aversion coefficient. If the curvature of an agent's utility function is greater when comparing it with another agent's utility function, then her risk coefficient is also greater which means that she demands a higher compensation for taking a risky lottery. By multiplying the coefficient of absolute risk aversion by x to get a percentage expression for the risk aversion, one finally obtains the coefficient of relative risk aversion.

$$r_x^r = \frac{U''(x)}{U'(x)}x. \tag{2.59}$$

2.2.2 Power Utility and General Equilibrium

Section 2.2.1 introduced some basic concepts for modeling expected utility. In order to shed further light on the dynamics of asset returns and consumption, one needs to define specific utility function. As a starting point, most of the finance and macroeconomic literature employs the easiest form of expected utility which is the time-separable power utility function. Following the work of Mehra and Prescott (1985), with power utility, the representative household maximizes the utility function of the form

$$U(C_t) = \frac{C_t^{1-\gamma} - 1}{1-\gamma} \tag{2.60}$$

where γ is the coefficient of relative risk aversion. As γ approaches one, the utility function approaches the log utility function $U(c_t) = \ln(C_t)$. As Campbell (2000, 2003) points out, the power utility function has several important properties. Firstly, it is scale-invariant to changes in aggregate wealth and consumption. This can be shown when calculating the first and second derivative to get a measure for the relative risk aversion. It holds that $-U''(C_t)/U'(C_t)C_t = C_t[-\gamma C_t^{-\gamma-1}]/[(1-\gamma)^{-1}(1-\gamma)C_t^{-\gamma}] = \gamma$ and therefore the power utility function has a constant relative risk aversion (CRRA) over all consumption levels. A second property of power utility is that the elasticity of intertemporal substitution ψ coincides with the reciprocal of the coefficient of risk aversion γ. This is in so far tricky as the elasticity of the intertemporal substitution is well defined in a world without uncertainty where a consumer moves consumption between time periods whereas the coefficient of relative risk aversion describes the willingness to transfer consumption between different states even in a one-period model. Finally, by taking the derivative of the utility function, the basic asset pricing equation of (2.31) is

$$1 = E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(1 + R_{i,t+1})\right] \tag{2.61}$$

Gross returns are high when investors are impatient, i.e. when β is low. Real returns also tend to be higher in times of high consumption growth and they are more sensitive to consumption if the risk aversion parameter is larger. A strongly shaped utility with high curvature reflects an agent's desire to smoothen consumption over time and states of the world and therefore, she is less willing to accept rearrangements of consumption streams in response to return changes.

To derive further insights into the dynamics of asset prices, the literature often specifies the time-series distribution of consumption and returns. This step can of course be heavily disputed since it is questionable that return and consumption really follow the assumed specifications. However, for the sake of completeness and further implications, the log-normal distribution of random variables is likewise applied to returns and the consumption process. Following Sect. 2.1.4.1 and the log-normal representation of the SDF and returns, the basic asset pricing equation with power utility (2.61) is

$$0 = E_t[r_{i,t+1}] + \ln\beta - \gamma E_t[\Delta c_{t+1}] + \frac{1}{2}\left(\sigma_{i,t}^2 + \gamma^2\sigma_{c,t}^2 - 2\gamma\; cov_t(r_{i,t+1}, \Delta c_{t+1})\right) \tag{2.62}$$

where $r_{i,t+1} = \log(1 + R_{i,t+1})$, $\log C_{t+1} - \log C_t = \Delta c_{t+1}$, $\sigma_{i,t}^2$ denotes the conditional variance of the asset's return, $\sigma_{c,t}^2$ the conditional variance of consumption and $cov_t(r_{i,t+1}, \Delta c_{t+1})$ the conditional covariance of return i and the consumption process. Equation (2.62) was first derived by Hansen and Singleton (1983). It offers implications for both time-series ans cross-section return dynamics. In this context, the risk-free return obeys

$$r_{f,t+1} = -\ln\beta - \frac{\gamma^2\sigma_{c,t}^2}{2} + \gamma E_t(\Delta c_{t+1}). \tag{2.63}$$

The risk free rate is a linear function of expected consumption growth. The sensitivity of interest rate changes to consumption is captured by the slope parameter, i.e. the coefficient of relative risk aversion. The variance term reflects the precautionary savings motive. When consumption is uncertain and more volatile, investors with power utility value low consumption states higher than high consumption states (reflected by γ^2); this preference structure drives down the risk-free return. Cross-section implications are revealed by taking the difference of (2.62) and (2.63) to get a measure for the excess return and the risk premium of a risky asset i over the risk-free rate

$$E_t[r_{i,t+1} - r_{f,t+1}] + \frac{\sigma_{i,t}^2}{2} = \gamma cov[r_{i,t+1}, \Delta c_{t+1}]. \tag{2.64}$$

Finally, the market price of risk becomes

$$\lambda_{M,t} = \gamma\sigma_{c,t}. \tag{2.65}$$

2.2.3 Pitfalls and the CCAPM

Excess returns should obey in a way that the underlying asset i must be risky in the sense that it is positively correlated in terms of covariances with consumption growth and equivalently negatively correlated with the marginal rate of substitution. When taking the simple consumption-based asset pricing model to data, the outcome is both disappointing and puzzling. Table 2.1 reports summary statistics for US and German asset returns including the means and standard deviations of real stock returns and real short-term returns. Columns (1) to (4) report first and second moments for real stock and bond returns; columns (5) to (6) provide statistics on mean consumption and its standard deviation; finally, columns (7) and (8) display first and second moments of real dividend growth. The US stock market, on average, delivered real returns of 6.92% from 1970 to 1998 whereas German stock markets performed with an average real return of 9.84% over the sample period. In contrast, sample means for real bond returns ranged at 1.49% and 1.15% for the US and Germany, respectively. Volatility measured as the standard deviation of returns is

Table 2.1 Stylized facts on CCAPM data

Country	(1) $E[r_{eq}]$	(2) $\sigma_{r_{eq}}$	(3) $E[r_f]$	(4) σ_{r_f}	(5) $E[\Delta c]$	(6) $\sigma_{\Delta c}$	(7) $E[\Delta d_{eq}]$	(8) $\sigma_{\Delta d_{eq}}$
USA	6.92	17.56	1.49	1.69	1.81	0.91	0.61	16.80
GER	9.84	20.10	3.22	1.15	1.68	2.43	1.19	8.93

The equity premium puzzle	(9) $E[XR_{eq}] = E[r_{eq,t+1} - r_{f,t}] + \frac{\sigma_{eq}^2}{2}$	(10) $cov(\Delta c, r_{eq})$	(11) γ
USA	6.35	4.23	150.10
GER	8.67	1.45	599.47

The risk-free rate puzzle	(12) $[r_f]$	(13) γ	(14) β
USA	1.49	150.10	−175.92
GER	3.22	599.47	9,757

Note: Quarterly data are from Campbell (2003) and annualized accordingly. The U.S. sample period ranges from 1970:1 to 1998:4 and for Germany from 1978:4–1997:4. The calculated parameter of relative risk aversion is multiplied by 100 much higher for stock returns than for real bond returns. Average consumption growth over the sample period is curtly below 2.0% with low volatilities between 1.0 and 2.5%. Finally, dividend growth in Germany is slightly higher compared to US data; whereas US dividend growth volatility exceeded Germany's standard deviation.

Using (2.64) and interpreting asset i as a real stock price index, the model's implied parameters can be linked to historical stock market data. Calculating the equity premium as the difference between the average equity return and the short-term real interest rate, and taking the variance of stock returns and the covariance of stock returns and consumption growth as given, propositions on overall risk aversion become possible. Here γ is simply computed by dividing column (9) by column (10) since it holds that $E[XR] = \gamma cov(\Delta c, r_{eq})$. For instance, in the US, γ requires to take on the value of 150.10 and in Germany of 599.47 for (2.64) to fit. Thus, in order to explain the equity premium with the consumption-based model and power utility, for most countries the coefficient of relative risk aversion must be unreasonably high. This equity premium puzzle first described by Mehra and Prescott (1985) and heavily discussed in the literature is documented across most countries and sample periods.[19] The problem with the consumption-based model is that although the data suggest a fairly stable consumption stream over the sample period as documented by its low volatility, only an extreme high risk aversion coefficient can explain why investors demand such high equity premia over time and across countries. Indeed, even in a moot scenario in which the correlation coefficient $\rho(r_{eq}, \Delta c)$ is set to one so that the product of the standard deviations of equity returns and consumption growth equals its covariance, the coefficient of relative risk aversion would still take on an implausible value of 41.18.[20] A high equity premium, thus, arises from the smoothness of consumption rather than from the low correlation of returns and consumption (Campbell 2000).

Another pitfall with the CCAPM and power utility is that it misses to meet the historical data of the real risk-free rate. To see this, consider (2.63) and take unconditional expectations to get a measure of the mean consumption growth rate. The average real risk-free interest rate is affected by three factors. Firstly, the real rate is high if time preference, i.e. if $-\log \beta$ is high. Secondly, the rate is high if the average consumption growth is high whose effect on the level of the real rate depends on γ; finally, the rate tends to be lower with an increasing volatility of consumption due to precautionary savings. If the aim is to match the model's implied real return with the underlying sample and ignoring the precautionary savings motive, there are two ways to reconcile a positive consumption growth rate with a low real interest rate. Either agents prefer to spend resources for consumption later than sooner or they accept a high intertemporal substitution of consumption

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[19]For excellent reviews of the consumption-based puzzles see Campbell (2003), Cochrane (2008a) and Mehra and Prescott (2003).

[20]The correlation coefficient of any two random variables is defined as the covariance divided by the product of the standard deviation of the two variables. It holds that $\rho(x, y) = cov(x, y)/\sigma_x \sigma_y$.

between two periods without much compensation (Söderlind 2003). The former explanation is expressed in terms of a high β and the latter in terms of low risk aversion, i.e. a low γ. Surely, if one accounts for the quadratic term and the risk aversion coefficient is low enough, it presses the real rate down. For the sample period, with a given mean of the risk-free rate, the variance of consumption and the implied risk aversion coefficients of the equity premium calculation, the time preference parameters are reported in column (14). Obviously, they are at odds with sound economic reasoning which was firstly stressed by Weil (1989).

There are many suggested ways to cope with these puzzles within an equilibrium setting. One major strand of literature asks how to measure best implied consumption and return volatilities; while other approaches deal with changes in the specification of the utility function. Since one major explanation of the return puzzles stems from the low volatility of consumption growth, one resolution is the modeling of idiosyncratic shocks or time-variation in second moments (conditional variances) within GARCH models (Söderlind 2008). In addition, the composition of aggregate consumption might contribute to a better fit of the data with the consumption-based asset pricing model.

In continuing with the modification of the utility function, the goal of modeling utility functions that allow separating risk aversion from intertemporal substitution, delivers more promising results of replicating the stylized asset return facts. Such specification is important since in the power utility model both measures are jointly determined by the parameter γ though both concepts express different economic implications. Risk aversion, in general, represents a measure for an agent's sensitivity toward risk by means of substitution consumption across different states. Intertemporal substitution mirrors an agent's willingness to substitute consumption across time. The work of Epstein and Zin (1991) makes it possible to parameterize both dimensions separately though it covers many attractive features of a CES-based power utility function. Epstein-Zin preferences are said to be recursive, i.e. today's utility depends on tomorrow's expected utility. Thus, utility is – as opposed to the power utility framework – non-additive across states. In a log-normal representation, it also offers the convenient property that a high risk aversion coefficient does not imply a low average risk-free rate since the elasticity of intertemporal substitution and the coefficient of risk aversion may well diverge (Campbell et al. 1997). Another body of research focuses on non-separability in utility over time by allowing for habit formation, a positive effect of today's consumption on tomorrow's marginal utility (Constantinides 1990; Abel 1990; Campbell and Cochrane 1999).[21] The basic idea behind this concept is that current utility is determined by current consumption relative to some reference/habit level. This could be either the agent's past individual consumption (internal habit) or past aggregate consumption (external habit). Habit-forming agents, thus, dislike large and rapid cuts in consumption. As a consequence,

[21]When modeling the dynamics of inflation and output in macroeconomic models, the literature often refers to habit formation which allows for a lagged term in aggregate output in order to capture the empirical regularity of high persistence in output (Fuhrer 2000).

they demand a premium for holding risky assets that might force them to cut down rapidly on consumption. The result is a risk premium that is higher than the one implied by the time-separable power utility model.

Various other lines of explanations have been put forward to rationalize high equity returns relative to bond returns (DeLong and Magin 2009). Among them, the existence of transactions costs has been put forward that prevent investors to participate in equity markets in order to bear equity risk. Moreover, it has be argued that the data sample considered does not capture enough low-tail return risks coupled with consumption drops that rational investors should expect, either because the sample is too small or investors simply can not know the true lower-tail risk of equity returns and estimates are ad-hoc prior beliefs rather than derived from probability distributions. Finally, equipped with the behavioral finance approach to asset pricing, investors may pay too much attention to high short-term risks of equities and may value losses more seriously than gains so that they can not realize the very little long-term risk in equity returns relative to bond returns.

Though there are various attempts to cope with empirical asset-pricing regularities, it turns out to be difficult to approach these facts within a single theoretical framework. "Economists still do not have a complete explanation for the equity premium. Each of the explanations [...] has a well-developed research literature. Yet none of these explanations has achieved even a rough consensus: the plurality opinion is that the equity return premium remains a puzzle" (DeLong and Magin 2009, 203). To conclude, Mehra and Prescott (2003, p.982) claim that "over the long horizon the equity premium is likely to be similar to what it has been in the past and the returns to investment in equity will continue to substantially dominate that in T-bills for investors with a long planning horizon."

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