Chapter 5
Inverse Problems for Linear Waves

5.1 Inverse Problems for Harmonic Waves

5.1.1 Hierarchical Equation

Let us consider the hierarchical equation in the linear case (4.2). Our aim is to reconstruct the triplet of coefficients b, β, γ in this equation. The simplest possibility is to use the measurements of the harmonic waves for this purpose. Since the number of unknowns is three, at least three different harmonic waves have to be measured. On the basis of such measurements we can formulate the following inverse problem.

IPh1 Given the wavenumbers $k_j,\ j = 1, 2, 3$, of three harmonic waves with the frequencies ω_j, such that ω_j^2, $j = 1, 2, 3$, are different, determine b, β and γ.

Evidently, instead of the wavenumbers, the wavelengths $l_j = \frac{1}{k_j}$ or the phase velocities $c_{ph,j} = \frac{\omega_j}{k_j}$ can be measured and used as the data of this problem. Various experimental techniques are available for phase velocity measurement, e.g., pulse-echo and the continuous wave resonance method [73]. Secondly, in practice the number of measured waves may be bigger than 3. Then the data of the inverse problem consist of pairs $(\omega_j, k_j),\ j = 1, \ldots, N$, where $N > 3$. But as we will see later on, the additional measurements do not bring along complementary information for the reconstruction. They may only reduce the statistical impact of the measurement errors in the solution.

The usage of the explicit functions $k(\omega)$ and $\omega(k)$ in the solution of IPh1 is somewhat cumbersome. The simplest method follows directly from the dispersion relation (4.5). Indeed, in view of this relation, IPh1 is reduced to the following 3×3 linear system:

$$\delta\omega_j^2 k_j^2 \beta - \delta k_j^4 \gamma - k_j^2 b = -\omega_j^2, \quad j = 1, 2, 3. \tag{5.1}$$

Clearly, in case more measured pairs (ω_j, k_j) are available, the related linear system contains more than 3 equations and can be solved by means of least squares.

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In the study of uniqueness of the solution of IPh1 and other inverse problems in this chapter, a method of vanishing polynomial coefficients will be used. In order to demonstrate this method, we present the proof of uniqueness of IPh1 here, in the main text. The proofs of uniqueness results of more complicated inverse problems below will be shifted to Sect. 5.5.

In the uniqueness proof we distinguish the dispersive case $b\beta - \gamma \neq 0$ and the nondispersive case $b\beta - \gamma = 0$ (cf. Lemma 4.1). We start with the dispersive case. Suppose that IPh1 has two solutions: b, β, λ and $\widetilde{b}, \widetilde{\beta}, \widetilde{\lambda}$. Then, in addition to (5.1), the following system is satisfied:

$$\delta\omega_j^2 k_j^2 \widetilde{\beta} - \delta k_j^4 \widetilde{\gamma} - k_j^2 \widetilde{b} = -\omega_j^2, \quad j = 1, 2, 3. \tag{5.2}$$

Let us eliminate the quantity ω_j from these relations. To this end, we multiply (5.1) by $\delta k_j^2 \widetilde{\beta} + 1$, (5.2) by $\delta k_j^2 \beta + 1$, and subtract. Then we obtain the following equations:

$$k_j^2\big(\delta k_j^2 \widetilde{\gamma} + \widetilde{b}\big)\big(\delta k_j^2 \beta + 1\big) - k_j^2\big(\delta k_j^2 \gamma + b\big)\big(\delta k_j^2 \widetilde{\beta} + 1\big) = 0, \quad j = 1, 2, 3.$$

Evidently, $k_j \neq 0$ because $\omega_j \neq 0$. Therefore, we can divide by k_j^2 the obtained equations. The resulting relations can be rewritten in the form

$$\delta^2(\widetilde{\gamma}\beta - \gamma\widetilde{\beta})k_j^4 + \delta(\widetilde{\gamma} - \gamma + \widetilde{b}\beta - b\widetilde{\beta})k_j^2 + \widetilde{b} - b = 0, \quad j = 1, 2, 3. \tag{5.3}$$

The latter relations show that the three numbers $z = k_j^2$, $j = 1, 2, 3$, are the roots of the following quadratic function:

$$\mathcal{P}_2(z) = \delta^2(\widetilde{\gamma}\beta - \gamma\widetilde{\beta})z^2 + \delta(\widetilde{\gamma} - \gamma + \widetilde{b}\beta - b\widetilde{\beta})z + \widetilde{b} - b. \tag{5.4}$$

We note that the numbers $k_j^2 = [k(\omega_j)]^2$, $j = 1, 2, 3$, are different because the function $k(\omega)$ is strictly increasing and ω_j^2, $j = 1, 2, 3$, are different. However, a nontrivial quadratic function may have maximally 2 different roots. Thus, $\mathcal{P}_2$ must be trivial, i.e., identically zero. This implies that the coefficients of $\mathcal{P}_2$ vanish and we get the equations

$$\widetilde{\gamma}\beta - \gamma\widetilde{\beta} = 0, \qquad \widetilde{\gamma} - \gamma + \widetilde{b}\beta - b\widetilde{\beta} = 0, \qquad \widetilde{b} - b = 0. \tag{5.5}$$

The third equation in (5.5) automatically gives $\widetilde{b} = b$. This means that the second equation in (5.5) is transformed to the form

$$b(\widetilde{\beta} - \beta) - \widetilde{\gamma} + \gamma = 0. \tag{5.6}$$

In addition, the first equation in (5.5) can be rewritten as

$$\gamma(\widetilde{\beta} - \beta) - \beta(\widetilde{\gamma} - \gamma) = 0. \tag{5.7}$$

Note that (5.6) and (5.7) form a 2×2 linear homogeneous system for $\widetilde{b} - b$ and $\widetilde{\gamma} - \gamma$. The determinant of this system equals

$$\begin{vmatrix} b & -1 \\ \gamma & -\beta \end{vmatrix} = -(b\beta - \gamma)$$

and is different from zero in the dispersive case. Therefore, the solution of the system (5.6), (5.7) is trivial, i.e., $\widetilde{\beta} - \beta = \widetilde{\gamma} - \gamma = 0$. This together with the previously shown relation $\widetilde{b} = b$ implies that the solution of IPh1 is unique.

In the nondispersive case $b\beta - \gamma = 0$ we have $k(\omega) = \frac{1}{\sqrt{b}}\omega$ (see Lemma 4.1). Therefore,

$$b = \frac{\gamma}{\beta} = \frac{\omega_j^2}{k_j^2}, \quad j = 1, 2, 3. \tag{5.8}$$

It is not possible to separate γ and β from the quotient $\frac{\gamma}{\beta}$.

Let us summarise the obtained results in the form of a theorem.

Theorem 5.1 *The following statements are valid for* IPh1.

(i) *In the dispersive case the solution is unique.*
(ii) *In the nondispersive case infinitely many solutions occur: only b and the quotient $\frac{\gamma}{\beta}$ can be uniquely reconstructed from the data by means of the formula* (5.8).

It is very easy to establish the dispersivity during the practical solution: in the dispersive case all moduli of phase velocities $|c_{ph,j}| = |\frac{\omega_j}{k_j}|$ are different, but in the nondispersive case they are equal to each other.

5.1.2 Coupled System

Now we deal with the coupled system in the linear case (4.11), (4.12). Again, we try to reconstruct the coefficients of this system from measurements of harmonic waves in macro-level. In the present case we may use both acoustic and optical waves. The optical waves occur if $|\omega| > \sqrt{\frac{\alpha}{\delta}}$, as we saw in Sect. 4.1.2. Note that the dispersion relation (4.14) with (4.15) and its solutions (4.16), (4.17) contain the parameters ϑ_0 and ϑ_1 only in the form of the product $\vartheta = \vartheta_0\vartheta_1$. Therefore, ϑ_0 and ϑ_1 cannot be separated from measurements of such waves. We may expect to determine the quadruple $a_0, a_1, \alpha, \vartheta$. Let us pose the following inverse problem:

IPh2 Given the wavenumbers k_j, $j = 1, \ldots, 4$, of four (acoustic or optical) harmonic waves with the frequencies ω_j, such that ω_j^2, $j = 1, \ldots, 4$, are different, determine a_0, a_1, α and ϑ.

Again, the wavelengths $l_j = \frac{1}{k_j}$ or the phase velocities $c_{ph,j} = \frac{\omega_j}{k_j}$ can be used as the data of the inverse problem, as well.

The problem IPh2 is decomposed into two subproblems (they are also steps in the practical solution of IPh2):

(1) determine the coefficients $\varkappa_1, \ldots, \varkappa_4$ of (4.14) by means of the given pairs (ω_j, k_j), $j = 1, \ldots, 4$;

(2) solve the system (4.15) for a_0, a_1, α and ϑ by means of the computed values of $\varkappa_1, \ldots, \varkappa_4$.

The first subproblem is the 4×4 linear system

$$\omega_j^2 k_j^2 \varkappa_1 + k_j^4 \varkappa_2 + \omega_j^2 \varkappa_3 + k_j^2 \varkappa_4 = -\omega_j^4, \quad j = 1, \ldots, 4, \tag{5.9}$$

for the unknowns $\varkappa_1, \ldots, \varkappa_4$. In case more than 4 harmonic waves are measured, the system (5.9) contains more equations and can be solved by means of least squares.

We make this decomposition of IPh2 only in the dispersive case. In the nondispersive case it is much easier to use the simple formulas of $k(\omega)$ and $k_2(\omega)$ from the item 3 of Lemma 4.2 for the solution.

In the dispersive case we study the uniqueness for the subproblems (5.9) and (4.15), respectively. For the first subproblem the following theorem holds.

Theorem 5.2 *In the dispersive case the solution of the system* (5.9) *is unique.*

The proof of this theorem is contained in Sect. 5.5.

Now let us consider the second subproblem. The first equations in (4.15) form an independent subsystem for a_0 and a_1. It has two pairs of solutions $(a_0, a_1) = (a_{0,1}, a_{1,1})$ and $(a_0, a_1) = (a_{0,2}, a_{1,2})$ where

$$\begin{aligned} a_{0,1} &= \frac{-\varkappa_1 + \sqrt{\varkappa_1^2 - 4\varkappa_2}}{2}, \qquad & a_{1,1} &= \frac{-\varkappa_1 - \sqrt{\varkappa_1^2 - 4\varkappa_2}}{2}, \\ a_{0,2} &= \frac{-\varkappa_1 - \sqrt{\varkappa_1^2 - 4\varkappa_2}}{2}, \qquad & a_{1,2} &= \frac{-\varkappa_1 + \sqrt{\varkappa_1^2 - 4\varkappa_2}}{2}. \end{aligned} \tag{5.10}$$

The third equation in (4.15) gives $\alpha = -\delta\varkappa_3$. From the fourth equation in (4.15) we get the formula for ϑ, namely $\vartheta = a_0\alpha - \delta\varkappa_4$. The quantity ϑ depends on the chosen value of a_0. Consequently, the second subproblem has two solutions:

$$a_0 = a_{0,1}, \qquad a_1 = a_{1,1}, \qquad \alpha = -\delta\varkappa_3, \qquad \vartheta = \vartheta_1 := a_{0,1}\alpha + \delta\varkappa_4, \tag{5.11}$$

$$a_0 = a_{0,2}, \qquad a_1 = a_{1,2}, \qquad \alpha = -\delta\varkappa_3, \qquad \vartheta = \vartheta_2 := a_{0,2}\alpha + \delta\varkappa_4. \tag{5.12}$$

Let us select the solutions that meet the physical restrictions (3.43) and (3.44). In view of the definitions of ϑ_1 and ϑ_2 the relation

$$\vartheta_2 = a_{0,2}\alpha - a_{0,1}\alpha + \vartheta_1 \tag{5.13}$$

holds. Since $a_{0,2} \le a_{0,1}$ (see (5.10)) and $\alpha > 0$ we have from (5.13)

$$\vartheta_1 \ge \vartheta_2.$$

Consequently, two different cases may occur:

$$\text{either} \quad \vartheta_1 > 0,\ \vartheta_2 \le 0 \quad \text{or} \quad \vartheta_1 > 0,\ \vartheta_2 > 0.$$

The third case $\vartheta \leq 0$, $\vartheta_2 \leq 0$ is impossible because then neither of the solutions (5.11) and (5.12) meets the physical condition $\vartheta > 0$.

In the case $\vartheta_1 > 0$, $\vartheta_2 \leq 0$ only the first solution (5.11) is physical. Then due to (5.13) and the relation $a_{0,2} = a_{1,1}$ (see (5.10)) we have

$$0 \geq \vartheta_2 = a_{0,2}\alpha - a_{0,1}\alpha + \vartheta_1 = a_{1,1}\alpha - a_{0,1}\alpha + \vartheta_1.$$

This implies that $a_{0,1}\alpha - a_{1,1}\alpha - \vartheta_1 \geq 0$. Thus, by virtue of the dispersivity assumption $a_0\alpha - a_1\alpha - \vartheta \neq 0$, we see that the material has the normal dispersion (cf. Lemma 4.2 for the types of dispersion).

In the case $\vartheta_1 > 0$, $\vartheta_2 > 0$ both solutions (5.11) and (5.12) are physical. Again, in view of (5.13) and the relations $a_{0,2} = a_{1,1}$, $a_{0,1} = a_{1,2}$ we obtain

$$0 < \vartheta_2 = a_{0,2}\alpha - a_{0,1}\alpha + \vartheta_1 = a_{1,1}\alpha - a_{0,1}\alpha + \vartheta_1,$$

$$0 < \vartheta_1 = a_{0,1}\alpha - a_{0,2}\alpha + \vartheta_2 = a_{1,2}\alpha - a_{0,2}\alpha + \vartheta_2.$$

This yields $a_{0,1}\alpha - a_{1,1}\alpha - \vartheta_1 < 0$ and $a_{0,2}\alpha - a_{1,2}\alpha - \vartheta_2 < 0$. Therefore, the material has the anomalous dispersion.

Let us compare the solutions (5.11) and (5.12). The component α is the same. Moreover, if $\varkappa_1^2 - 4\varkappa_2 > 0$ then by (5.10) $a_{0,1} > a_{0,2}$ and hence the components a_0, a_1 and ϑ of the solutions are different. But in case $\varkappa_1^2 - 4\varkappa_2 = 0$ the solutions (5.11) and (5.12) entirely coincide. Then $a_0 = a_{0,j} = a_{1,j} = a_1$, i.e., this is the midpoint of the anomalous dispersion.

Let us summarise the obtained results.

Lemma 5.1 *The following statements are valid for the second subproblem.*

(i) *In the case of the normal dispersion the solution is unique and has the form* (5.11).

(ii) *In the case of the anomalous dispersion two solutions occur: they are given by* (5.11) *and* (5.12), *contain the same value of* α, *but the other components* a_0, a_1 *and* ϑ *coincide only in the case of the midpoint of the anomaly.*

Putting Theorem 5.2 and Lemma 5.1 together, we have the next theorem.

Theorem 5.3 *The following statements are valid for* IPh2.

(i) *In the case of the normal dispersion the solution is unique.*

(ii) *In the case of the anomalous dispersion two solutions occur: they contain the same value of* α, *but the other components* a_0, a_1 *and* ϑ *of the solutions coincide only in the case of the midpoint of the anomaly.*

It remains to consider the nondispersive case when $a_0\alpha - a_1\alpha - \vartheta = 0$. Due to the assertion 3 of Lemma 4.2 any measurement pair (ω_j, k_j) gathered from an acoustic harmonic wave determines uniquely the parameter a_1:

$$a_1 = \frac{\omega_j^2}{k_j^2}. \tag{5.14}$$

But any two measurement pairs (ω_{j_i}, k_{j_i}), $i = 1, 2$, from optical harmonic waves give the following linear system for a_0 and α:

$$k_{j_i}^2 a_0 + \frac{1}{\delta}\alpha = \omega_{j_i}^2, \quad i = 1, 2. \tag{5.15}$$

The matrix of this system is regular because $k_{j_1}^2 \neq k_{j_2}^2$. This follows from the assumption $\omega_{j_1}^2 \neq \omega_{j_2}^2$ and the strict monotonicity of $k_2(\omega)$. Therefore, the solution of (5.15) is unique. The parameter ϑ is given in terms a_1, a_0 and α by $\vartheta = a_0\alpha - a_1\alpha$. Summing up, the determination of the full vector of coefficients is possible in case

the set of frequency-wavenumber pairs $\{(\omega_j, k_j),\ j = 1, \ldots, 4\}$

contains at least one pair from an acoustic wave and (5.16)

at least two pairs from optical waves

and we can formulate the next result.

Theorem 5.4 *The following statements are valid for* IPh2 *in the nondispersive case.*

(i) *If* (5.16) *holds then the solution is unique.*
(ii) *In the opposite case infinitely many solutions occur.*

Again, it is possible to determine whether the solution is unique or non-unique during the practical solving procedure. Firstly, if the data of IPh2 have the property (5.16) then the system (5.9) is always regular and the first step can be successfully performed to get the quantities $\varkappa_1, \ldots, \varkappa_4$. The second step is to be started with the computation of the first solution by the formula (5.11). In case this solution satisfies the condition $a_0\alpha - a_1\alpha - \vartheta \geq 0$, it is the unique solution of the inverse problem. But in case the condition $a_0\alpha - a_1\alpha - \vartheta < 0$ holds, the second solution exists too, and is to be computed by (5.12). These two solutions coincide when $\varkappa_1^2 - 4\varkappa_2 = 0$.

Secondly, if the property (5.16) is not valid then the system (5.9) may be singular or regular. In the singular case the material is nondispersive and IPh2 has infinitely many solutions that can be partially recovered by means of the formulas (5.14), (5.15) and $\vartheta = a_0\alpha - a_1\alpha$ depending on given data. But in case the system (5.9) is regular, it provides unique quantities $\varkappa_1, \ldots, \varkappa_4$ and the second step can be completed as above.

Finally, we remark that it is possible to separate ϑ_0 and ϑ_1 from the product $\vartheta = \vartheta_0\vartheta_1$ in case certain information about the microdeformation is available, too. More precisely, let us be given the amplitudes of the macro- and microdeformation A and $\widehat{A}$, respectively, of the first wave with $\omega = \omega_1$ and $k = k_1$. Then, due to (4.13) we get the parameter ϑ_0 by the formula

$$\vartheta_0 = \frac{A(\omega_1^2 - a_0 k_1^2)}{\widehat{A} k_1^2}. \tag{5.17}$$

5.1.3 General Consequences

Let us make some general conclusions from the results of the previous two subsections. It is natural to ask: is it possible to improve the nonuniqueness results of Theorems 5.1, 5.3 and 5.4 if we incorporate measurements of more harmonic waves or superpositions of such waves? Clearly, the nonuniqueness assertion Theorem 5.1(ii) remains valid in such generalisations. Every harmonic component of a nondispersive wave is governed by the simple linear relation $k = \frac{1}{\sqrt{b}}\omega$ and hence contains information about b only. Therefore, the following statement holds.

Corollary 5.1 *In the nondispersive case any superposition of harmonic wave solutions of the hierarchical equation does not contain enough information to recover all parameters*: *it is not possible to determine more than* $b = \frac{\beta}{\gamma}$.

Further, we ask: can we improve the assertion (ii) of Theorem 5.3 if we provide more measurements of harmonic wave packets? Again, the answer is no. The nonuniqueness in this theorem is caused by the properties of the nonlinear system (4.15) that is to be solved in the second step of the solution. Incorporating more harmonic components only overdetermines the system (5.9) which is to be solved in the first step and whose solution is already unique. Thus, we may formulate the following statement.

Corollary 5.2 *In the cases of weak and strong anomalous dispersion any superposition of harmonic wave solutions of the coupled system in macro-level does not contain enough information to recover all parameters*: *it is not possible to determine more than a single value for* α *and two different values for the other parameters*.

Theorem 5.4 shows that nondispersive waves in the coupled system contain enough information to recover uniquely all parameters only in case they contain at least one acoustic component and two different optical components. In the opposite case those waves are not sufficiently informative. For instance, concerning the acoustic waves packets the following statement holds.

Corollary 5.3 *In the nondispersive case any superposition of acoustic harmonic wave solutions of the coupled system in macro-level does not contain enough information to recover all parameters*: *it is not possible to determine more than* $a_1 = a_0 - \frac{\vartheta}{\alpha}$.

5.2 Inverse Problems for Gaussian Wave Packets

In this section we discuss the reconstruction of parameters from measurements of Gaussian wave packets. We focus ourselves on some problems that make use of the phase and group velocities and in some cases the dispersion parameter $d = \frac{k''(\omega)}{2}$, too. The parameter d can be extracted from measurements of the amplitude change and modulation dispersion of the phase shift solving one of (4.40)–(4.42). Probably it is most realistic to measure the amplitude change. However, the amplitude or modulation dispersion provide d^2, hence leave the sign of d open. The sign of d may also be determined from the additional observation of the sign of the phase shift, namely

$$\operatorname{sign} d = \operatorname{sign} \Phi(x).$$

Firstly, we pose and study some inverse problems for the hierarchical equation.

IPg1 Given the phase velocity c_{ph}, the group velocity c_g and d of a single wave packet with the central frequency ω_0, determine b, β and γ.

The data of this problem are related to first and second order derivatives of the dispersion function. From the basic dispersion equation (4.5), by differentiation we immediately deduce the following equations for $k' = k'(\omega)$ and $k'' = k''(\omega)$:

$$\begin{aligned}
&\delta\beta\omega_0 k\big(k + \omega k'\big) - 2\delta\gamma k^3 k' + \omega - bkk' = 0,\\
&\delta\beta\big[k^2 + 4\omega kk' + \omega^2\big(k'\big)^2 + \omega^2 kk''\big] - 2\delta\gamma k^2\big[3\big(k'\big)^2 + kk''\big]\\
&\quad + 1 - b\big[\big(k'\big)^2 + kk'\big] = 0.
\end{aligned}$$

Therefore, IPg1 is equivalent to the following 3×3 linear system for b, β, γ:

$$\left.\begin{aligned}
&\delta\omega_0^2 k_0^2\beta - \delta k_0^4\gamma - k_0^2 b = -\omega_0^2,\\
&\delta\omega_0 k_0\big(k_0 + \omega_0 k_0'\big)\beta - 2\delta k_0^3 k_0'\gamma - k_0 k_0' b = -\omega_0,\\
&\delta\big[k_0^2 + 4\omega_0 k_0 k_0' + \omega_0^2\big(k_0'\big)^2 + \omega_0^2 k_0 k_0''\big]\beta - 2\delta k_0^2\big[3\big(k_0'\big)^2 + k_0 k_0''\big]\gamma\\
&\quad - \big[\big(k_0'\big)^2 + k_0 k_0''\big]b = -1,
\end{aligned}\right\} \tag{5.18}$$

where $k_0 = \frac{\omega_0}{c_{ph}}$, $k_0' = \frac{1}{c_g}$ and $k_0'' = 2d$.

Another reconstruction procedure uses only phase and group velocities of the wave packets. In such a case at least two wave packets are to be incorporated. Let us pose the related inverse problem.

IPg2 Given the phase and group velocities $c_{ph,1}$ and $c_{g,1}$ of the first wave packet with the central frequency ω_1, and the phase velocity $c_{ph,2}$ of the second wave packet with the central frequency ω_2, such that $\omega_1^2 \neq \omega_2^2$, determine b, β and γ.

From the dispersion equation (4.5) and the corresponding equation for $k' = k'(\omega)$ we infer the following 3×3 linear system that is equivalent to IPg2:

$$\left.\begin{aligned}
&\delta\omega_j^2 k_j^2\beta - \delta k_j^4\gamma - k_j^2 b = -\omega_j^2, \quad j = 1, 2,\\
&\delta\omega_1 k_1\big(k_1 + \omega_1 k_1'\big)\beta - 2\delta k_1^3 k_1'\gamma - k_1 k_1' b = -\omega_1.
\end{aligned}\right\} \tag{5.19}$$

Here $k_j = \frac{\omega_j}{c_{ph,j}}$, $j = 1, 2$, and $k_1' = \frac{1}{c_{g,1}}$.

Theorem 5.5 *The following statements are valid for* IPg1 *and* IPg2.

(i) *In the dispersive case the solution is unique.*
(ii) *In the nondispersive case infinitely many solutions occur: only b and the quotient $\frac{\gamma}{\beta}$ can be uniquely reconstructed from the data by the formula*

$$b = \frac{\gamma}{\beta} = c_g^2. \tag{5.20}$$

The proof can be found in Sect. 5.5.

Clearly, in practice more information may be available, for instance, the phase and group velocities and the parameters d of several Gaussian wave packets. The formulated inverse problems incorporate minimum amounts of information sufficient for the unique reconstruction in the dispersive case.

Secondly, let us consider the determination of the parameters of the coupled system. We pose the following problem.

IPg3 Given the phase and group velocities $c_{ph,j}$, $c_{g,j}$, $j = 1, 2$, of two wave packets with the central frequencies ω_j, such that $\omega_1^2 \neq \omega_2^2$, determine a_0, a_1, α and ϑ.

The structure of this problem is similar to the structure of the inverse problem for harmonic waves IPh2 studied in Sect. 5.1.2. Namely, we decompose IPg3 into two subproblems:

(1) determine the coefficients $\varkappa_1, \ldots, \varkappa_4$ of (4.14) by means of the data $c_{ph,j}$, $c_{g,j}$, $j = 1, 2$;
(2) solve the system (4.15) for a_0, a_1, α and ϑ by means of the computed values of $\varkappa_1, \ldots, \varkappa_4$.

The first subproblem is again equivalent to a linear system. Indeed, let us differentiate (4.14) with respect to ω:

$$2\omega^3 + \varkappa_1\left(\omega k^2 + \omega^2 k k'\right) + 2\varkappa_2 k^3 k' + \varkappa_3\omega + \varkappa_4 k k' = 0.$$

Observing this expression and (4.14) we see that $\varkappa_1, \ldots, \varkappa_4$ is the solution vector of the following 4×4 system:

$$\left.\begin{aligned} & k_j^2\omega_j^2\varkappa_1 + k_j^4\varkappa_2 + \omega_j^2\varkappa_3 + k_j^2\varkappa_4 = -\omega_j^4, \quad j = 1, 2, \\ & \left(\omega_j k_j^2 + \omega_j^2 k_j k_j'\right)\varkappa_1 + 2k_j^3 k_j'\varkappa_2 + \omega_j\varkappa_3 + k_j k_j'\varkappa_4 \\ & \quad = -2\omega_j^3, \quad j = 1, 2. \end{aligned}\right\} \tag{5.21}$$

Theorem 5.6 *In the dispersive case the solution of the system* (5.21) *is unique.*

The proof is contained in Sect. 5.5.

To formulate a uniqueness result concerning IPg3, we combine Theorem 5.6 and Lemma 5.1 in the dispersive case and apply Corollary 5.3 in the nondispersive case. In the latter situation it is possible to use the formula $\frac{1}{c_g} = k'(\omega) = \frac{1}{\sqrt{a_1}}$ following from Lemma 4.2.

Theorem 5.7 *The following statements are valid for* IPg3.

(i) *In the case of the normal dispersion the solution is unique.*
(ii) *In the case of the anomalous dispersion two solutions occur: they contain the same value of α, but the other components a_0, a_1 and ϑ of the solutions coincide only in the case of the midpoint of the anomaly.*
(iii) *In the nondispersive case infinitely many solutions occur: only a_1 and the quantity $a_0 - \frac{\vartheta}{\alpha}$ can be uniquely reconstructed from the data by the formula*

$$a_1 = a_0 - \frac{\vartheta}{\alpha} = c_g^2. \tag{5.22}$$

Finally, we mention that it is possible to pose and study inverse problems for the coupled system that involves the quantity d in the data set. But those problems are technically very complicated and require long computations in the proofs. Therefore, we do not present them in this book.

5.3 Reconstruction of Parameters from Spectra of Waves

It is possible to reconstruct the parameters in our models by means of more complex linear waves, too. The idea consists in extracting harmonic counterparts from the spectral decomposition of the wave, and reducing the problem to the inverse problem for harmonic waves discussed in Sect. 5.1. In practice, this means the determination of frequency-wavenumber pairs (ω_m, k_m) from the spectra and solution of either IPh1 or IPh2.

5.3.1 The Case of Deformation Boundary Condition

Let us consider right-propagating waves on the half-line $x > 0$ generated by the deformation boundary condition (4.23). As we saw in Sect. 4.2.2, the wave function is given by (4.24) and (4.25) with (4.26) in the cases of the hierarchical equation and the coupled system, respectively. This means that the Fourier transform of the wave function possesses the formula

$$v^F(x, \omega) = g^F(\omega) e^{ik(\omega)x}$$

(in the case of the coupled system this formula holds for lower frequencies $|\omega| < \sqrt{\frac{\alpha}{\delta}}$).

Suppose that the deformation function $v(x, t)$ is measured at some point $x_1 > 0$ over the time t. Upon computation of the Fourier transforms of the data, the equation

$$e^{ik(\omega)x_1} = \frac{v^F(x_1, \omega)}{g^F(\omega)} \tag{5.23}$$

can be solved for the function $k(\omega)$. By means of this function the frequency-wavenumber pairs (ω_m, k_m) for IPh1 or IPh2 can be computed.

At first sight, the solution of (5.23) is complicated because of the periodicity of the outer component e^{iz}. Nevertheless, the right solution can easily be extracted observing the qualitative behaviour of $e^{ik(\omega)x_1}$ over some frequency interval.

Let us make use of the real part of the quotient $\frac{v^F(x_1,\omega)}{g^F(\omega)}$ during the solution. Then we have to solve the equation

$$\cos\big[k(\omega)x_1\big] = \Re\frac{v^F(x_1,\omega)}{g^F(\omega)} \tag{5.24}$$

for $k(\omega)$. It is necessary to invert the cosine in a proper way. Let us think as follows. Since $k(\omega)$ is strictly increasing and the relations $k(0) = 0$, $\lim_{w\to\infty} k(\omega) = \infty$ hold, the function $\cos[k(\omega)x_1]$ oscillates between 1 and -1. More precisely, $\cos[k(\omega)x_1]$ decreases for $\omega \in \mathcal{I}_0 = (0, \zeta_1)$, increases for $\omega \in \mathcal{I}_1 = (\zeta_1, \zeta_2)$ and so on, where

$$0 < \zeta_1 < \zeta_2 < \cdots$$

is some increasing sequence of real numbers. Thus, for the right inversion of the cosine it is necessary to find the intervals of monotonicity $\mathcal{I}_j$ of the known right-hand side $\Re\frac{v^F(x_1,\omega)}{g^F(\omega)}$. Then the desired function $k(\omega)$ can be evaluated by the formula

$$k(\omega) = \frac{1}{x_1}\left[(-1)^n \arccos\Re\frac{v^F(x_1,\omega)}{g^F(\omega)} + \pi(n+\theta_n)\right] \quad \text{for } \omega \in \mathcal{I}_n. \tag{5.25}$$

Here $\theta_n = 0$ for even n and $\theta_n = 1$ for odd n.

Let's see how this method can be practically performed. Actually, we have at our disposal a time series of measured deformations

$$v_l = v(x_1, t_l), \quad l = 1, \ldots, N$$

in an interval $[T, T_1]$, where $t_l = T + l\eta$ and $\eta = \frac{T_1 - T}{N}$. To compute the Fourier transforms, different methods may be used. Let us choose the rectangular rule for truncated Fourier integrals, because this is compatible with the standard Discrete Fourier Transform available in mathematical softwares. Then the discrete spectra of the data are

$$g^F(\omega_m) \approx \hat{g}_m = \frac{e^{iT\omega_m}}{N}\sum_{l=1}^{N} e^{\frac{2\pi ib(l-1)(m-1)}{N}} g_l,$$

$$v^F(x_1,\omega_m) \approx \hat{v}_m = \frac{e^{iT\omega_m}}{N}\sum_{l=1}^{N} e^{\frac{2\pi ib(l-1)(m-1)}{N}} v_l$$

for $m = 1, \ldots, N$, where $\tau > 0$ is the stepsize in the frequency domain, $\omega_m = (m-1)\tau$, $g_l = g((l-1)\eta)$ and $b = \frac{\eta\tau}{2\pi}$. The discrete spectrum provides the oscillat-

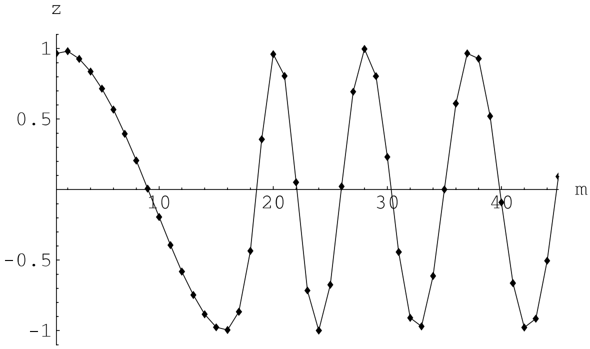

Fig. 5.1 Sequence z_m for $m = 1, \ldots, 45$

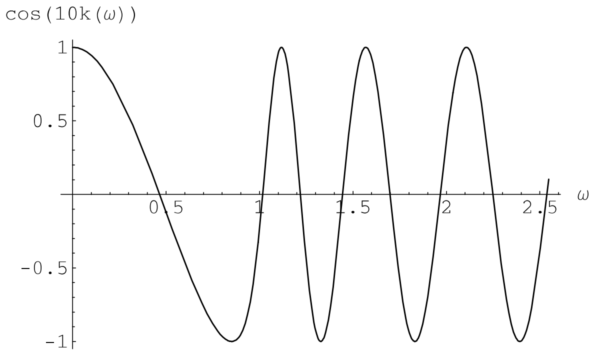

Fig. 5.2 Function $\cos[k(\omega)x_1]$ for $x_1 = 10$

ing sequence $z_m = \mathrm{Re}\,\frac{\hat{v}_m}{\hat{g}_m}$ of real numbers that decreases for $s_0 < m < s_1$, increases for $s_1 < m < s_2$ and so on, where

$$1 = s_0 < s_1 < s_2 < \cdots$$

are some numbers. To find frequency-wavenumber pairs, it is necessary to determine the critical numbers $s_1, s_2, \ldots$. Then the wavenumber k_m corresponding to the frequency ω_m can be evaluated by means of the following formula deduced from (5.25):

$$k_m = k(\omega_m) = \frac{1}{x_1}\big[(-1)^n \arccos z_m + \pi(n + \theta_n)\big] \quad \text{for } s_n < m < s_{n+1}. \tag{5.26}$$

The formula (5.26) is applicable for all ω_m except for the critical frequencies ω_{s_n}, because the discrete problem does not contain information about the intervals of monotonicity to which ω_{s_n} belong.

For example, Fig. 5.2 shows the periodic function $\cos(k(\omega)x_1)$ corresponding to the parameters $b = 10$, $\beta = \gamma = 10^4$, $\delta = 10^{-4}$ of the hierarchical equation and $x_1 = 10$. The intervals of monotonicity are $\mathcal{I}_1 = (0, 0.87)$, $\mathcal{I}_2 = (0.87, 1.12)$, $\mathcal{I}_3 = (1.12, 1.29), \ldots$. On the top picture Fig. 5.1 the real part of the ratio of spectra $z_m = \Re\frac{\hat{v}_m}{\hat{g}_m}$ is depicted for the discrete frequencies $\omega_m = (m-1)\tau$ where $\tau = \frac{\pi}{55}$.

The latter data were computed by the standard Discrete Fourier Transform applied to the solution corresponding to the boundary excitation $g(t) = e^{-\frac{t^2}{4}}$ at $x = 0$.

The sequence z_m is oscillating with critical numbers $s_0 = 0, s_1 = 16, s_2 = 20, s_3 = 24, \ldots$. The frequency-wavenumber pairs can be obtained from the sequence z_m by means of the formula (5.26). For instance, the wavenumbers corresponding to $\omega_m = (m-1)\tau$, $m = 2, \ldots, 15$, are $k_m = \frac{1}{10} \arccos z_m$, the wavenumbers corresponding to $\omega_m = (m-1)\tau$, $m = 17, \ldots, 19$, are $k_m = \frac{1}{10}[2\pi - \arccos z_m]$ and so on.

5.3.2 The Case of Displacement Boundary Condition

Now we consider right-propagating waves on the half-line $x > 0$ generated by the displacement boundary condition (4.28). Then the wave function is given by (4.29) and (4.25) with (4.30) in the cases of the hierarchical equation and the coupled system, respectively. The Fourier transform of the wave function has the from

$$v^F(x, \omega) = ik(\omega)h^F(\omega)e^{ik(\omega)x}$$

(in the case of coupled system this holds for lower frequencies $|\omega| < \sqrt{\frac{\alpha}{\delta}}$).

Again, let the deformation function $v(x, t)$ be measured at some point $x_1 > 0$ over the time t. Having the Fourier transforms of the data, the equation

$$ik(\omega)e^{ik(\omega)x_1} = \frac{v^F(x_1, \omega)}{h^F(\omega)} \tag{5.27}$$

is to be solved for the function $k(\omega)$.

Note that in the present case the different physical quantities are given and measured (given displacement versus measured deformation). In turns out that such a feature reduces the periodicity problem. Indeed, taking the modulus of (5.27), we have

$$|k(\omega)| = \left| \frac{v^F(x_1, \omega)}{h^F(\omega)} \right|.$$

This means that for any positive frequency ω the corresponding wavenumber can be computed as $k = |\frac{v^F(x_1,\omega)}{h^F(\omega)}|$.

In the case of the discrete data

$$h^F(\omega_m) \approx \hat{h}_m = \frac{e^{iT\omega_m}}{N} \sum_{l=1}^{N} e^{\frac{2\pi ib(l-1)(m-1)}{N}} h_l,$$

$$v^F(x_1, \omega_m) \approx \hat{v}_m = \frac{e^{iT\omega_m}}{N} \sum_{l=1}^{N} e^{\frac{2\pi ib(l-1)(m-1)}{N}} v_l$$

with $m = 1, \dots, N$, where $\tau > 0$ is the stepsize in the frequency domain, $\omega_m = (m-1)\tau$, $h_l = h((l-1)\eta)$, $v_l = v(x_1, (l-1)\eta)$ and $b = \frac{\eta\tau}{2\pi}$, the wavenumber k_m that corresponds to the frequency ω_m is obtained by the formula

$$k_m = \left| \frac{\hat{v}_m}{\hat{h}_m} \right|.$$

5.4 Stability and Examples

5.4.1 Stability of Solutions

Now we ask the question: under what conditions are the solutions of the studied inverse problems stable, i.e., the errors of the solutions converge to zero provided that the errors of the data tend to zero? As we saw, our inverse problems are connected with certain linear system of algebraic equations. Namely, IPh1, IPg1, IPg2 and the first subproblems of IPh2 and IPg3 are equivalent to the linear systems (5.1), (5.18), (5.19), (5.9) and (5.21), respectively.

It is well-known that the stability of a solution of a linear system of algebraic equations automatically follows from the regularity of this system, i.e., the uniqueness of the solution. (The stability in the sense of convergence of sets can be considered for singular linear systems, too, but we omit such more complicated cases here.) Therefore, due to Theorems 5.1, 5.2, 5.5 and 5.6, the solutions of IPh1, IPg1, IPg2 and the first subproblems of IPh2 and IPg3 are stable in the dispersive case.

Further, the solutions of the second subproblem of IPh2 and IPg3 are given by the explicit formulas (5.10), (5.11) and (5.12) that contain continuous functions of $\varkappa_1 \dots, \varkappa_4$. By the continuity, the stability holds for these subproblems, too.

Summing up, we can formulate the following theorem.

Theorem 5.8 *The following statements are valid in the dispersive case.*

(i) *The unique solutions of* IPh1, IPg1 *and* IPg2 *are stable.*
(ii) *The solutions of* IPh2 *and* IPg3 (*one or two*, *depending on the type of the dispersion*) *are stable.*

5.4.2 Numerical Examples

We have tested the methods proposed in this chapter from the point of view of sensitivity with respect to the noise of the data. For both models (i.e. the hierarchical equation and the coupled system) the parameters were determined from the spectral composition of right-propagating waves corresponding to deformation boundary condition and Gaussian wave packets. As described above, the former problems contain as a sub-step the solution of inverse problems for harmonic waves.

Table 5.1 Relative errors in the spectral method for the hierarchical equation

ϵ	$\lvert\frac{b^\epsilon - b}{b}\rvert$	$\lvert\frac{\beta^\epsilon - \beta}{\beta}\rvert$	$\lvert\frac{\gamma^\epsilon - \gamma}{\gamma}\rvert$
0.01%	0.003%	0.011%	0.010%
0.1%	0.014%	0.12%	0.16%
1%	0.78%	2.6%	2.6%

Table 5.2 Relative errors in the spectral method for the coupled system

ϵ	$\lvert\frac{a_0^\epsilon - a_0}{a_0}\rvert$	$\lvert\frac{a_1^\epsilon - a_1}{a_1}\rvert$	$\lvert\frac{\alpha^\epsilon - \alpha}{\alpha}\rvert$	$\lvert\frac{\vartheta^\epsilon - \vartheta}{\vartheta}\rvert$
0.01%	0.072%	0.051%	0.058%	0.61%
0.1%	0.60%	0.47%	0.62%	5.2%
1%	3.6%	2.2%	3.3%	35%

The basic parameter choice for the coupled system was $a_0 = 100$, $a_1 = 1$, $\alpha = 10^{-4}$, $\vartheta = 0.002$ (the parameters α and ϑ contain the small quantity l^2, and hence it is natural to take them small). Then the corresponding parameters of the hierarchical equation are $b = 80$, $\beta = \gamma = 2 \times 10^5$ (computed by (3.45)). In all examples we took $\delta = 10^{-4}$.

The relative noise level of the data is denoted by ϵ and the computed parameters containing the noise are denoted by a_0^ϵ, a_1^ϵ, α^ϵ, ϑ^ϵ (coupled system) and b^ϵ, β^ϵ, γ^ϵ (hierarchical equation).

For the method of spectral decomposition the boundary impulse $g(t) = e^{\frac{-t^2}{4}}$ at $x = 0$ was chosen and the solution $v(x,t)$ corresponding to prescribed (or exact) parameters computed at $x_1 = 10$ for $t \in [0, 50]$. This solution was perturbed as follows

$$v^\epsilon(x_1, t_j) = v(x_1, t_j)(1 + R_j \epsilon)$$

where $t_j = j\tau$ are discrete time values with the step $\tau = 0.01$ and R_j is the uniformly distributed random number in the interval $[-1, 1]$. The time series $v^\epsilon(x_1, t_j)$ was used as the synthetic data for the reconstruction procedure.

All computations were repeated 100 times with different random vectors R_j and the biggest relative errors selected. Tables 5.1 and 5.2 show the relative errors in the hierarchical equation and coupled system, respectively.

Another reconstruction method consists in the usage of phase and group velocities c_{ph}, c_g and in IPg1 also the dispersion quantity d of Gaussian wave packets. We chose the packets with the initial amplitude $A = 100$, the Gaussian dispersion $\sigma = 0.1$ and the following central frequencies: $\omega_0 = 300$ for problem IPg1 and $\omega_1 = 100$, $\omega_2 = 400$ for problems IPg2 and IPg3. The exact velocities were computed by the formulas $c_{ph,j} = \frac{\omega_j}{k(\omega_j)}$, $c_{q,j} = \frac{1}{k'(\omega_j)}$, $j \in \{0; 1; 2\}$, where $k(\omega)$ is given by (4.8) or (4.16), and perturbed in the following manner:

$$c_{ph,j}^\epsilon = c_{ph,j}\bigl(1 + R_{ph}^j \epsilon\bigr), \qquad c_{g,j}^\epsilon = c_{g,j}\bigl(1 + R_g^j \epsilon\bigr)$$

Table 5.3 Relative errors in IPg1

ϵ	$\lvert\frac{b^\epsilon - b}{b}\rvert$	$\lvert\frac{\beta^\epsilon - \beta}{\beta}\rvert$	$\lvert\frac{\gamma^\epsilon - \gamma}{\gamma}\rvert$
0.01%	0.029%	0.037%	0.073%
0.1%	0.072%	0.095%	0.51%
1%	2.2%	3.4%	6.2%

Table 5.4 Relative errors in IPg2

ϵ	$\lvert\frac{b^\epsilon - b}{b}\rvert$	$\lvert\frac{\beta^\epsilon - \beta}{\beta}\rvert$	$\lvert\frac{\gamma^\epsilon - \gamma}{\gamma}\rvert$
0.01%	0.004%	0.007%	0.017%
0.1%	0.022%	0.045%	0.23%
1%	1.1%	2.1%	3.3%

Table 5.5 Relative errors in IPg3

ϵ	$\lvert\frac{a_0^\epsilon - a_0}{a_0}\rvert$	$\lvert\frac{a_1^\epsilon - a_1}{a_1}\rvert$	$\lvert\frac{\alpha^\epsilon - \alpha}{\alpha}\rvert$	$\lvert\frac{\vartheta^\epsilon - \vartheta}{\vartheta}\rvert$
0.01%	0.084%	0.008%	0.024%	0.46%
0.1%	0.81%	0.034%	0.19%	5.3%
1%	7.3%	0.86%	1.2%	73%

where R_{ph}^j and R_g^j are uniformly distributed random numbers in the interval $[-1, 1]$.

The quantity d involved in IPg1 is not directly measurable. But it is possible to compute it by means of the measurable amplitude change $A_1(x)$ at some point x, making use of the following formula deduced from (4.40):

$$d = \operatorname{sign} d \frac{\sigma^2}{x} \sqrt{\frac{A^4}{A_1^4(x)} - 1} \tag{5.28}$$

where $\operatorname{sign} d = \operatorname{sign} \Phi(x)$. A synthetic datum d was constructed in the following manner. Exact d and $A_1 = A_1(x)$ at $x = 10$ were evaluated by the formulas $d = \frac{k''(\omega_0)}{2}$ and (4.40). Thereupon A_1 was perturbed:

$$A_1^\epsilon = A_1\big(1 + R_{A_1}^j \epsilon\big)$$

where $R_{A_1}^j$ are again uniformly distributed random numbers in the interval $[-1, 1]$. Finally the perturbed d^ϵ was computed by inserting A_1^ϵ into (5.28).

Summing up, the quantities $c_{ph,j}^\epsilon, c_{g,j}^\epsilon$ and in IPg1 also d^ϵ formed the synthetic data for the inverse problems. The results for IPg1, IPg2 and IPg3 are presented in Tables 5.3, 5.4, 5.5.

First of all, the numerical results support the theoretical statements about the asymptotical stability: if ϵ tends to zero then the errors of the components of the solutions also approach zero.

The computations show that the inverse problems for the linear hierarchical equation are less sensitive with respect to the noise of the data than the inverse problems for the linear coupled system. The cause is that the condition number of the matrices of these problems is amplified by the increase of the dimension: from 3 in the hierarchical equation to 4 in the coupled system. Worst results are obtained for ϑ. But one cannot make a conclusion that the reconstruction of physical parameters from the hierarchical equation gives better results than the reconstruction from the coupled system, because the errors of the mathematical models have not been taken into account.

Another feature of the inverse problems for the linear models is that accuracy depends on the rate of dispersion of the waves. In almost nondispersive cases, i.e., when $b - \frac{\gamma}{\beta} \approx 0$ in the hierarchical equation or $a_0 - a_1 - \frac{\vartheta}{\alpha} \approx 0$ in the coupled system, results are very bad. For instance, in the case $a_0 = 2.1$, $a_1 = 1$, $\alpha = \vartheta = 10^{-4}$ the relative errors corresponding to $\epsilon = 10^{-3}$ in IPg3 are

$$\left|\frac{a_0^\epsilon - a_0}{a_0}\right| = 577\%, \qquad \left|\frac{a_1^\epsilon - a_1}{a_1}\right| = 0.32\%,$$
$$\left|\frac{\alpha^\epsilon - \alpha}{\alpha}\right| = 27\%, \qquad \left|\frac{\vartheta^\epsilon - \vartheta}{\vartheta}\right| = 820\%.$$

5.5 Proofs of Mathematical Statements

5.5.1 Proof of Theorem 5.2

As in the proof of Theorem 5.1 in Sect. 5.1.1, we can make use of the method of vanishing polynomial coefficients. However in the present case we cannot deduce polynomial equations for the single variables $z = k_j^2$ directly from a pair of systems of the form (5.9). The additional fourth order term ω_j^4 makes the immediate algebraic elimination of ω_j impossible. Nevertheless, it is possible to rewrite (5.9) in a form of an algebraic system containing $z = \frac{k_j}{\omega_j}$ and ω_j where the latter could be eliminated from a pair of systems.

Furthermore, in the present case a number k_j may be the value of either $k(\omega_j)$ or $k_2(\omega_j)$. Therefore, we take into consideration a general set of solutions of (4.14). Namely, for any $\omega \in \mathbb{C}$ we define

$$K(\omega) = \{k \in \mathbb{C} : k \text{ solves (4.14) for given } \omega\}.$$

Since (4.14) is a quartic equation, $K(\omega)$ contains maximally 4 elements for any $\omega \in \mathbb{C}$. We split the proof of Theorem 5.2 into lemmas.

Lemma 5.2 *Assume that* $a_0\alpha - a_1\alpha - \vartheta \neq 0$ *and let* $\varkappa_1, \ldots, \varkappa_4$ *be given by* (4.15) *in terms of* $a_0, a_1, \alpha, \vartheta$. *Moreover, let* $\omega_1, \omega_2 \in \mathbb{C}, \omega_1, \omega_2 \neq 0$, *and* $k_j \in K(\omega_j)$, $j = 1, 2$. *If* $\omega_1^2 \neq \omega_2^2$ *then the quotients* $s_j = \frac{k_j}{\omega_j}$ *satisfy* $s_1^2 \neq s_2^2$.

Proof Due to the choice of k_j, the equations

$$\omega_j^4 + \varkappa_1 \omega_j^2 k_j^2 + \varkappa_2 k_j^4 + \varkappa_3 \omega_j^2 + \varkappa_4 k_j^2 = 0 \tag{5.29}$$

hold for $j = 1, 2$. Let $\omega_1^2 \neq \omega_2^2$. Suppose on the contrary that $s_1^2 = s_2^2 =: s^2$. Then, dividing (5.29) by ω_j^4 we have

$$1 + \varkappa_1 s^2 + \varkappa_2 s^4 + \frac{1}{\omega_j^2}\left(\varkappa_3 + \varkappa_4 s^2\right) = 0, \quad j = 1, 2. \tag{5.30}$$

Subtracting these equations for $j = 1$ and $j = 2$ and observing that $\omega_1^2 \neq \omega_2^2$ we get

$$\varkappa_3 + \varkappa_4 s^2 = 0. \tag{5.31}$$

This together with (5.30) gives the equation

$$1 + \varkappa_1 s^2 + \varkappa_2 s^4 = 0. \tag{5.32}$$

Expressing s^2 from (5.31) and substituting into (5.32) we get

$$1 - \varkappa_1 \frac{\varkappa_3}{\varkappa_4} + \varkappa_2 \left(\frac{\varkappa_3}{\varkappa_4}\right)^2 = 0.$$

Using here the formulas (4.15) for $\varkappa_1, \ldots, \varkappa_4$ and simplifying we obtain

$$\frac{\vartheta}{(a_0\alpha - \vartheta)^2}(a_0\alpha - a_1\alpha - \vartheta) = 0.$$

But this relation cannot hold, because $\vartheta > 0$ and $a_0\alpha - a_1\alpha - \vartheta \neq 0$. Therefore, the supposition $s_1^2 = s_2^2$ was not right. We have $s_1^2 \neq s_2^2$ and the lemma is proved. □

We shall prove Theorem 5.2 in the following more general form.

Lemma 5.3 *Assume that* $a_0\alpha - a_1\alpha - \vartheta \neq 0$ *and let* $\omega_j \in \mathbb{C}$, $\omega_j \neq 0$, $j = 1, \ldots, 4$, *be such that* ω_j^2, $j = 1, \ldots, 4$, *are different. Moreover, let us choose some* $k_j \in K(\omega_j)$, $j = 1, \ldots, 4$. *Then the solution of* (5.9) *with the data* (ω_j, k_j), $j = 1, \ldots, 4$, *is unique.*

Proof To prove this assertion, we make use of the method of vanishing polynomial coefficients, again. Suppose that the system (5.9) has two solutions $\varkappa_1, \ldots, \varkappa_4$ and $\widetilde{\varkappa}_1, \ldots, \widetilde{\varkappa}_4$. We write this system up for these solutions and divide by ω_j^4 to get the following equations containing the quotients $s_j = \frac{k_j}{\omega_j}$:

$$1 + \varkappa_1 s_j^2 + \varkappa_2 s_j^4 + \frac{1}{\omega_j^2}\left(\varkappa_3 + \varkappa_4 s_j^2\right) = 0, \quad j = 1, \ldots, 4,$$

$$1+\widetilde{\varkappa}_1 s_j^2+\widetilde{\varkappa}_2 s_j^4+\frac{1}{\omega_j^2}\left(\widetilde{\varkappa}_3+\widetilde{\varkappa}_4 s_j^2\right)=0, \quad j=1,\ldots,4.$$

Let us eliminate ω_j from these relations. To this end we multiply the first equations by $\widetilde{\varkappa}_3+\widetilde{\varkappa}_4 s_j^2$ and the second equations by $\varkappa_3+\varkappa_4 s_j^2$ and subtract. Then we reach the following expressions:

$$\begin{aligned}&(\varkappa_4\widetilde{\varkappa}_2-\widetilde{\varkappa}_4\varkappa_2)s_j^6+(\varkappa_3\widetilde{\varkappa}_2-\widetilde{\varkappa}_3\varkappa_2+\varkappa_4\widetilde{\varkappa}_1-\widetilde{\varkappa}_4\varkappa_1)s_j^4\\&\quad+(\varkappa_4-\widetilde{\varkappa}_4+\varkappa_3\widetilde{\varkappa}_1-\widetilde{\varkappa}_3\varkappa_1)s_j^2+\varkappa_3-\widetilde{\varkappa}_3=0, \quad j=1,\ldots,4. \qquad (5.33)\end{aligned}$$

These relations show that $z=s_j^2$, $j=1,\ldots,4$, are the roots of the following cubic function:

$$\begin{aligned}f(z)=&(\varkappa_4\widetilde{\varkappa}_2-\widetilde{\varkappa}_4\varkappa_2)z^3+(\varkappa_3\widetilde{\varkappa}_2-\widetilde{\varkappa}_3\varkappa_2+\varkappa_4\widetilde{\varkappa}_1-\widetilde{\varkappa}_4\varkappa_1)z^2\\&+(\varkappa_4-\widetilde{\varkappa}_4+\varkappa_3\widetilde{\varkappa}_1-\widetilde{\varkappa}_3\varkappa_1)z+\varkappa_3-\widetilde{\varkappa}_3. \qquad (5.34)\end{aligned}$$

Since ω_j^2, $j=1,\ldots,4$, are different, by Lemma 5.2 the quantities s_j^2, $j=1,\ldots,4$, are also different. Consequently, the cubic function (5.34) has four different roots. Thus, it is trivial. Setting the coefficients of (5.34) equal to zero, after some transformations we arrive at the following 4×4 system for the vector $(\widetilde{\varkappa}_1-\varkappa_1, \widetilde{\varkappa}_2-\varkappa_2, \widetilde{\varkappa}_3-\varkappa_3, \widetilde{\varkappa}_4-\varkappa_4)$:

$$\begin{aligned}&&&\widetilde{\varkappa}_3-\varkappa_3&&=0\\&\varkappa_3(\widetilde{\varkappa}_1-\varkappa_1)-&&\varkappa_1(\widetilde{\varkappa}_3-\varkappa_3)-&&(\widetilde{\varkappa}_4-\varkappa_4)=0\\&\varkappa_4(\widetilde{\varkappa}_1-\varkappa_1)+\varkappa_3(\widetilde{\varkappa}_2-\varkappa_2)-&&\varkappa_2(\widetilde{\varkappa}_3-\varkappa_3)-&&\varkappa_1(\widetilde{\varkappa}_4-\varkappa_4)=0\\&&&\varkappa_4(\widetilde{\varkappa}_2-\varkappa_2)-&&\varkappa_2(\widetilde{\varkappa}_4-\varkappa_4)=0.\end{aligned}$$

For the determinant of this system we have

$$-\varkappa_2\varkappa_3^2-\varkappa_4^2+\varkappa_1\varkappa_3\varkappa_4=\frac{(a_0\alpha-a_1\alpha-\vartheta)\vartheta}{\delta^2}\neq 0,$$

because $\vartheta>0$ and $a_0\alpha-a_1\alpha-\vartheta\neq 0$. This implies that the system under consideration has only the trivial solution. Hence, $\widetilde{\varkappa}_1=\varkappa_1$, $\widetilde{\varkappa}_2=\varkappa_2$, $\widetilde{\varkappa}_3=\varkappa_3$, $\widetilde{\varkappa}_4=\varkappa_4$. The lemma is proved. □

Theorem 5.2 follows from Lemma 5.3 because wavenumbers k_j contained in the data of IPh2 belong to $K(\omega_j)$ for any $j=1,\ldots,4$.

5.5.2 *Proofs of Sect. 5.2*

Proof of Theorem 5.5 The assertion (ii) immediately follows from Corollary 5.1 and the formula $\frac{1}{c_g}=k'(\omega)=\frac{1}{\sqrt{b}}$ that is valid in the nondispersive case $b\beta-\gamma=0$ (see Lemma 4.1). Therefore, let us study in detail the dispersive case.

Firstly, we prove the uniqueness for IPg2. Suppose that IPg2 has two solutions: b, β, γ and $\widetilde{b}, \widetilde{\beta}, \widetilde{\gamma}$. As in the proof of Theorem 5.1, from the first two equations of (5.19) we deduce (5.3) for $j = 1, 2$. This means that $z = k_j^2$, $j = 1, 2$, are the roots of the quadratic function $\mathcal{P}_2(z)$ given by (5.4). Further, the third equation of (5.19) in the cases of these two solutions can be written

$$\omega_1\big(\delta\beta k_1^2 + 1\big) + \big(\delta\beta\omega_1^2 - 2\delta\gamma k_1^2 - b\big)k_1 k_1' = 0, \tag{5.35}$$

$$\omega_1\big(\delta\widetilde{\beta} k_1^2 + 1\big) + \big(\delta\widetilde{\beta}\omega_1^2 - 2\delta\widetilde{\gamma} k_1^2 - \widetilde{b}\big)k_1 k_1' = 0. \tag{5.36}$$

To eliminate k_1', we multiply (5.35) by $\delta\widetilde{\beta}\omega_1^2 - 2\delta\widetilde{\gamma}k_1^2 - \widetilde{b}$, (5.36) by $\delta\beta\omega_1^2 - 2\delta\gamma k_1^2 - b$, subtract and divide by $\omega_1 \neq 0$:

$$\big(\delta\beta k_1^2 + 1\big)\big(\delta\widetilde{\beta}\omega_1^2 - 2\delta\widetilde{\gamma}k_1^2 - \widetilde{b}\big) - \big(\delta\widetilde{\beta}k_1^2 + 1\big)\big(\delta\beta\omega_1^2 - 2\delta\gamma k_1^2 - b\big) = 0. \tag{5.37}$$

The next step is the elimination ω_1 from this equation. To this end, we use the first equation in (5.19) in the cases of both solutions:

$$\omega_1^2\big(\delta\beta k_1^2 + 1\big) = \delta\gamma k_1^4 + b k_1^2,$$

$$\omega_1^2\big(\delta\widetilde{\beta} k_1^2 + 1\big) = \delta\widetilde{\gamma} k_1^4 + \widetilde{b} k_1^2.$$

Applying these relations to ω_1-dependent terms in (5.37) we deduce that

$$\delta\widetilde{\beta}\big(\delta\gamma k_1^4 + b k_1^2\big) - \big(\delta\beta k_1^2 + 1\big)\big(2\delta\widetilde{\gamma}k_1^2 + \widetilde{b}\big)$$

$$- \delta\beta\big(\delta\widetilde{\gamma} k_1^4 + \widetilde{b} k_1^2\big) + \big(\delta\widetilde{\beta} k_1^2 + 1\big)\big(2\delta\gamma k_1^2 + b\big) = 0.$$

The latter relation can be rewritten as follows:

$$3\delta^2(\widetilde{\gamma}\beta - \gamma\widetilde{\beta})k_1^4 + 2\delta(\widetilde{\gamma} - \gamma + \widetilde{b}\beta - b\widetilde{\beta})k_1^2 + \widetilde{b} - b = 0. \tag{5.38}$$

Now we subtract from (5.38) the equation (5.3) for $j = 1$ and divide by $k_1^2 \neq 0$. We obtain the following equation:

$$2\delta^2(\widetilde{\gamma}\beta - \gamma\widetilde{\beta})k_1^2 + \delta(\widetilde{\gamma} - \gamma + \widetilde{b}\beta - b\widetilde{\beta}) = 0.$$

This shows that $\mathcal{P}_2'(k_1^2) = 0$. Hence, the number $z = k_1^2$ is a double root of the polynomial $\mathcal{P}_2(z)$. Since $k_1^2 \neq k_2^2$ (this follows from the strict monotonicity of $k(\omega)$ and the inequality $\omega_1^2 \neq \omega_2^2$) we see that the quadratic polynomial $\mathcal{P}_2$ has two different roots k_1^2 and k_2^2, where k_1^2 has the multiplicity 2. This is possible only in case $\mathcal{P}_2$ is the trivial polynomial. Setting the coefficients of $\mathcal{P}_2$ equal to zero, we prove the equalities $\widetilde{b} = b$, $\widetilde{\beta} = \beta$ and $\widetilde{\gamma} = \gamma$ as in the proof of Theorem 5.1. This completes the proof of the uniqueness for IPg2.

The uniqueness for IPg1 can be proved by the same method, i.e., showing that k_0 is a triple root of $\mathcal{P}_2$. However, this is somewhat complicated and involves long computations, because it is necessary to eliminate k_0', k_0'' and ω_0 from related equations. It is easier to use the explicit formula (4.6) for $\omega(k)$ for this purpose, because we have to apply it at a single argument k_0. From (4.6) we have

$$\frac{b+\delta\gamma k^2}{1+\delta\beta k^2} = \left[\frac{\omega(k)}{k}\right]^2. \tag{5.39}$$

By differentiation we deduce that

$$\frac{\gamma - b\beta}{(1+\delta\beta k^2)^2} = \frac{1}{2\delta k}\left\{\left[\frac{\omega(k)}{k}\right]^2\right\}'. \tag{5.40}$$

Differentiating once again we obtain

$$\frac{\beta(\gamma - b\beta)}{(1+\delta\beta k^2)^3} = -\frac{1}{4\delta k}\left[\frac{1}{2\delta k}\left\{\left[\frac{\omega(k)}{k}\right]^2\right\}'\right]'. \tag{5.41}$$

Setting $k = k_0 = \frac{\omega_0}{c_{ph}}$, the right-hand sides of (5.39)–(5.41) can be evaluated in terms of the data of IPg1. More precisely, since $\omega(k_0) = \omega_0$, $\omega'(k_0) = c_g$ and $\omega''(k_0) = -k''(\omega_0)[\omega'(k_0)]^3 = -2dc_g^3$, we obtain the following system:

$$\frac{b+\delta\gamma k_0^2}{1+\delta\beta k_0^2} = c_{ph}^2, \tag{5.42}$$

$$\frac{\gamma - b\beta}{(1+\delta\beta k_0^2)^2} = r_1 \tag{5.43}$$

$$\frac{\beta(\gamma - b\beta)}{(1+\delta\beta k_0^2)^3} = r_2 \tag{5.44}$$

where

$$r_1 = \frac{c_{ph}}{\delta k_0^2}(c_g - c_{ph}), \qquad r_2 = -\frac{1}{4\delta^2 k_0^4}\big[(c_g - 4c_{ph})(c_g - c_{ph}) - 2dc_g^3 c_{ph} k_0\big].$$

Dividing (5.43) by (5.44) we evaluate $\beta = [\frac{r_1}{r_2} - \delta k_0^2]^{-1}$. Once β is known, from (5.42) and (5.43) a 2×2 linear system for b and γ can be constructed:

$$b + \delta k_0^2 \gamma = c_{ph}^2\big(1+\delta\beta k_0^2\big),$$
$$-\beta b + \gamma = r_1\big(1+\delta\beta k_0^2\big)^2.$$

The determinant of this system is $1+\delta\beta k_0^2$ and it differs from zero because $\delta, \beta > 0$ (see (3.37)). Thus, the solution the linear system is unique. Summing up, the solution b, β, γ of IPg1 is unique. The theorem is proved. □

Proof of Theorem 5.6 Suppose that (5.21) has two solutions $\varkappa_1, \dots, \varkappa_4$ and $\widetilde{\varkappa}_1, \dots, \widetilde{\varkappa}_4$. This means that the following equalities hold:

$$
\begin{aligned}
&k_j^2\omega_j^2\varkappa_1 + k_j^4\varkappa_2 + \omega_j^2\varkappa_3 + k_j^2\varkappa_4 = -\omega_j^4, \quad j = 1, 2,\\
&k_j^2\omega_j^2\widetilde{\varkappa}_1 + k_j^4\widetilde{\varkappa}_2 + \omega_j^2\widetilde{\varkappa}_3 + k_j^2\widetilde{\varkappa}_4 = -\omega_j^4, \quad j = 1, 2,\\
&\left(\omega_j k_j^2 + \omega_j^2 k_j k_j'\right)\varkappa_1 + 2k_j^3 k_j'\varkappa_2 + \omega_j\varkappa_3 + k_j k_j'\varkappa_4 = -2\omega_j^3, \quad j = 1, 2,\\
&\left(\omega_j k_j^2 + \omega_j^2 k_j k_j'\right)\widetilde{\varkappa}_1 + 2k_j^3 k_j'\widetilde{\varkappa}_2 + \omega_j\widetilde{\varkappa}_3 + k_j k_j'\widetilde{\varkappa}_4 = -2\omega_j^3, \quad j = 1, 2.
\end{aligned}
$$

Dividing the first two equalities by ω_j^4 and the last two equalities by ω_j^3 and denoting $s_j = \frac{k_j}{\omega_j}$ we obtain

$$
\begin{aligned}
&1 + \varkappa_1 s_j^2 + \varkappa_2 s_j^4 + \frac{1}{\omega_j^2}\left(\varkappa_3 + \varkappa_4 s_j^2\right) = 0,\\
&1 + \widetilde{\varkappa}_1 s_j^2 + \widetilde{\varkappa}_2 s_j^4 + \frac{1}{\omega_j^2}\left(\widetilde{\varkappa}_3 + \widetilde{\varkappa}_4 s_j^2\right) = 0,\\
&2 + \varkappa_1 s_j^2 + \frac{\varkappa_3}{\omega_j^2} + \left(\varkappa_1 + 2\varkappa_2 s_j^2 + \frac{\varkappa_4}{\omega_j^2}\right)s_j k_j' = 0,\\
&2 + \widetilde{\varkappa}_1 s_j^2 + \frac{\widetilde{\varkappa}_3}{\omega_j^2} + \left(\widetilde{\varkappa}_1 + 2\widetilde{\varkappa}_2 s_j^2 + \frac{\widetilde{\varkappa}_4}{\omega_j^2}\right)s_j k_j' = 0,
\end{aligned}
\tag{5.45}
$$

where $j = 1, 2$. As in the proof of Lemma 5.3, the elimination of ω_j from the first two equations in (5.45) leads to expression (5.33). This shows that s_1^2 and s_2^2 are roots of the cubic function $f(z)$ defined by (5.34).

There is another possibility for eliminating k_j' and ω_j from (5.45), too. Namely, let us multiply the fourth equation by $\varkappa_1 + 2\varkappa_2 s_j^2 + \frac{\varkappa_4}{\omega_j^2}$, the third equation by $\widetilde{\varkappa}_1 + 2\widetilde{\varkappa}_2 s_j^2 + \frac{\widetilde{\varkappa}_4}{\omega_j^2}$ and subtract to get rid of k_j':

$$
\begin{aligned}
&\left(2 + \varkappa_1 s_j^2 + \frac{\varkappa_3}{\omega_j^2}\right)\left(\widetilde{\varkappa}_1 + 2\widetilde{\varkappa}_2 s_j^2 + \frac{\widetilde{\varkappa}_4}{\omega_j^2}\right)\\
&\quad - \left(2 + \widetilde{\varkappa}_1 s_j^2 + \frac{\widetilde{\varkappa}_3}{\omega_j^2}\right)\left(\varkappa_1 + 2\varkappa_2 s_j^2 + \frac{\varkappa_4}{\omega_j^2}\right) = 0, \quad j = 1, 2.
\end{aligned}
\tag{5.46}
$$

Further, we multiply the second equation in (5.45) by $2\varkappa_1 + 4\varkappa_1 s_j^2 + \frac{\varkappa_4}{\omega_j^2}$, the first equation by $2\widetilde{\varkappa}_1 + 4\widetilde{\varkappa}_1 s_j^2 + \frac{\widetilde{\varkappa}_4}{\omega_j^2}$ and subtract again. The result is

$$\left[1+\varkappa_1 s_j^2+\varkappa_2 s_j^4+\frac{1}{\omega_j^2}\big(\varkappa_3+\varkappa_4 s_j^2\big)\right]\left(2\widetilde{\varkappa}_1+4\widetilde{\varkappa}_1 s_j^2+\frac{\widetilde{\varkappa}_4}{\omega_j^2}\right)$$
$$-\left[1+\widetilde{\varkappa}_1 s_j^2+\widetilde{\varkappa}_2 s_j^4+\frac{1}{\omega_j^2}\big(\widetilde{\varkappa}_3+\widetilde{\varkappa}_4 s_j^2\big)\right]\left(2\varkappa_1+4\varkappa_1 s_j^2+\frac{\varkappa_4}{\omega_j^2}\right)=0,$$
$$j=1,2. \tag{5.47}$$

Finally, subtracting (5.46) from (5.47), only terms with the factor $\frac{1}{\omega_j^2}$ remain:

$$\begin{aligned}
&\frac{1}{\omega_j^2}\big[\big(\varkappa_3+\varkappa_4 s_j^2\big)\big(2\widetilde{\varkappa}_1+4\widetilde{\varkappa}_2 s_j^2\big)+\widetilde{\varkappa}_4\big(1+\varkappa_1 s_j^2+\varkappa_2 s_j^4\big)\\
&\quad-\big(\widetilde{\varkappa}_3+\widetilde{\varkappa}_4 s_j^2\big)\big(2\varkappa_1+4\varkappa_2 s_j^2\big)-\varkappa_4\big(1+\widetilde{\varkappa}_1 s_j^2+\widetilde{\varkappa}_2 s_j^4\big)\big]\\
&\quad-\frac{1}{\omega_j^2}\big[\varkappa_3\big(\widetilde{\varkappa}_1+2\widetilde{\varkappa}_2 s_j^2\big)+\widetilde{\varkappa}_4\big(2+\varkappa_1 s_j^2\big)\\
&\quad-\widetilde{\varkappa}_3\big(\varkappa_1+2\varkappa_2 s_j^2\big)-\varkappa_4\big(2+\widetilde{\varkappa}_1 s_j^2\big)\big]=0,\quad j=1,2.
\end{aligned}$$

Multiplying by $\omega_j^2\neq 0$ and simplifying we obtain

$$\begin{aligned}
&3(\varkappa_4\widetilde{\varkappa}_2-\widetilde{\varkappa}_4\varkappa_2)s_j^4+2(\varkappa_3\widetilde{\varkappa}_2-\widetilde{\varkappa}_3\varkappa_2+\varkappa_4\widetilde{\varkappa}_1-\widetilde{\varkappa}_4\varkappa_1)s_j^2\\
&\quad+\varkappa_4-\widetilde{\varkappa}_4+\varkappa_3\widetilde{\varkappa}_1-\widetilde{\varkappa}_3\varkappa_1=0,\quad j=1,2.
\end{aligned}$$

From these relations we have $f'(s_j^2)=0$ for $j=1,2$. This means that s_j^2, $j=1,2$, are double roots of the cubic function $f(\sigma)$. Since ω_j^2, $j=1,2$, are different, by Lemma 5.2 the quantities s_j^2, $j=1,2$, are also different. Therefore, the cubic function $f(\sigma)$ has two different double roots and hence it is trivial. The rest of the proof is identical to that of Lemma 5.3. □

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