2

Matchings and 1-Factors

In this chapter we consider matchings and 1-factors of graphs, and these results will frequently be used in latter chapters. Of course, they are interesting on their own, and have many applications in various areas of mathematics and computer science. The reader is referred to two books [182] by Lovász and Plummer and [254] by Yu and Liu for a more detailed treatment of matchings and 1-factors.

2.1 Matchings in bipartite graphs

In this section we investigate matchings in bipartite graphs, and the results shown in this section will play an important role throughout this book.

Two edges of a general graph are said to be **independent** if they have no common end-point and none of them is a loop. A **matching** in a general graph G is a set of pairwise independent edges of G (Fig. 2.1). If M is a matching in a general graph G, then the **subgraph of G induced by M**, denoted by $\langle M\rangle_G$ or $\langle M\rangle$, is the subgraph of G whose edge set is M and whose vertex set consists of the vertices incident with some edge in M. Then every vertex of $\langle M\rangle$ has degree one. Thus it is possible to define a matching as a subgraph whose vertices all have degree one, and we often regard a matching M as its induced subgraph $\langle M\rangle$.

Let M be a matching in a graph G. Then a vertex of G is said to be **saturated** or **covered** by M if it is incident with an edge of M; otherwise, it is said to be **unsaturated** or **not covered** by M. If every vertex of a vertex subset U of G is saturated by M, then we say that U is **saturated** by M. A matching with maximum cardinality is called a **maximum matching**. A matching that saturates all the vertices of G is called a **perfect matching** or a **1-factor** of G (Fig. 2.1). It is easy to see that a **maximal matching**, which is a maximal set of independent edges, is not a maximum matching, and a maximum matching is not a perfect matching. On the other hand, if

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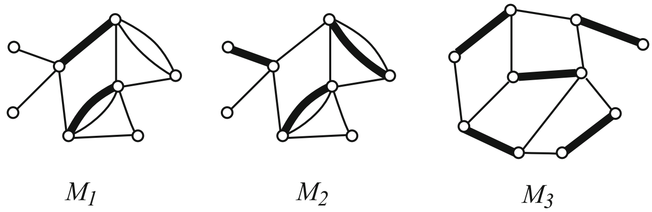


Fig. 2.1. A maximal matching M_1; a maximum matching M_2; and a perfect matching M_3.

a matching in a bipartite graph saturates one of its partite sets, then it is clearly a maximum matching.

For a matching M, we write

$$||M|| = \text{the size of } \langle M \rangle = \text{the number of edges in } M,$$
$$|M| = \text{the order of } \langle M \rangle = \text{the number of vertices saturated by } M.$$

We first give a criterion for a bipartite graph to have a matching that saturates one of its partite sets. Then we apply this criterion to some problems on matchings in bipartite graphs.

The criterion mentioned above is given in the following theorem, which was found by Hall [98] and is called the marriage theorem. This theorem appears throughout this book, and the proof given here is due to Halmos and Vaughan [97]. An algorithm for finding a maximum matching in a bipartite graph will be given in Algorithm 2.25.

Theorem 2.1 (The Marriage Theorem, Hall [98] (1935)**).** *Let G be a bipartite multigraph with bipartition (A, B). Then G has a matching that saturates A if and only if*

$$|N_G(S)| \geq |S| \qquad \textit{for all} \quad S \subseteq A. \tag{2.1}$$

Proof. We first construct a bipartite simple graph H from the given bipartite multigraph G by replacing all the multiple edges of G by single edges. Then it is obvious that G has the desired matching if and only if H has such a matching, and that G satisfies (2.1) if and only if H satisfies it. Therefore, we may assume that G itself has no multiple edges by considering H as the given bipartite graph.

Suppose that G has a matching M that saturates A (Fig. 2.2). Then for every subset $S \subseteq A$, we have

$$|N_G(S)| \geq |N_M(S)| = |S|.$$

We next prove sufficiency by induction on $|G|$. It is clear that we may assume $|A| \geq 2$. We consider the following two cases.

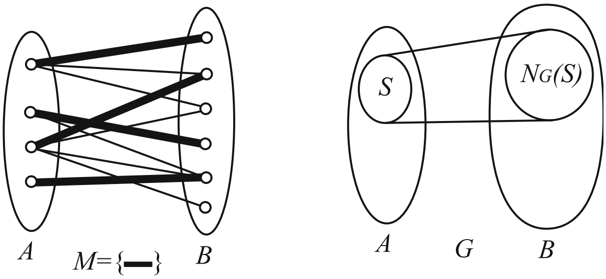


Fig. 2.2. A matching M that saturates A; S and $N_G(S)$.

Case 1. *There exists* $\emptyset \neq S \subset A$ *such that* $|N_G(S)| = |S|$.

Let $H = \langle S \cup N_G(S) \rangle_G$ and $K = \langle (A - S) \cup (B - N_G(S)) \rangle_G$ be induced subgraphs of G (Fig. 2.3). It is clear that H satisfies condition (2.1), and so H has a matching M_H that saturates S by induction. For every subset $X \subseteq A - S$, we have

$$|N_K(X)| = |N_G(X \cup S)| - |N_G(S)| \geq |X \cup S| - |S| = |X|.$$

Hence, by induction, K also has a matching M_K that saturates $A - S$. Therefore $M_H \cup M_K$ is the desired matching in G which saturates A.

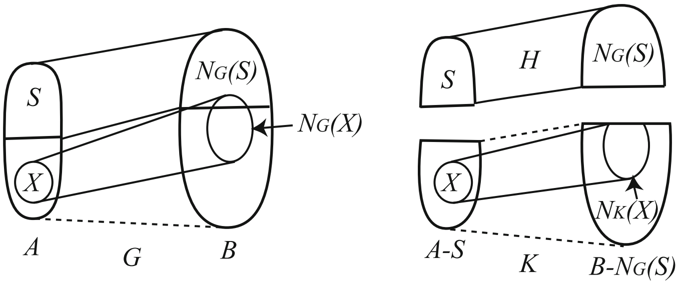


Fig. 2.3. The induced subgraphs H and K.

Case 2. $|N_G(S)| > |S|$ *for all* $\emptyset \neq S \subset A$.

Let $e = ab$ $(a \in A, b \in B)$ be an edge of G, and let $H = G - \{a, b\}$. Then for every subset $\emptyset \neq X \subseteq A - \{a\}$, by the assumption of this case, we have

$$|N_H(X)| \geq |N_G(X) \setminus \{b\}| > |X| - 1, \tag{2.2}$$

which implies $|N_H(X)| \geq |X|$. Therefore, H has a matching M' that saturates $A - \{a\}$ by induction. Then $M' + e$ is the desired matching in G. Consequently the theorem is proved. $\square$

If the edge set $E(G)$ of a multigraph G is partitioned into disjoint 1-factors $E(G) = F_1 \cup F_2 \cup \cdots \cup F_r$, where each $F_i = \langle F_i \rangle$ is a 1-factor of G, then we say that G is **1-factorable**, and call this partition a **1-factorization** of G. It is trivial that if G is 1-factorable, then G is regular.

We now give some results on matchings in bipartite graphs, most of which can be proved by making use of the marriage theorem. We begin with the following famous theorem, which was obtained by König in 1916 before the marriage theorem. However, our proof depends on the marriage theorem.

Theorem 2.2 (König [155]). *Every regular bipartite multigraph is 1-factorable, in particular, it has a 1-factor (Fig. 2.4).*

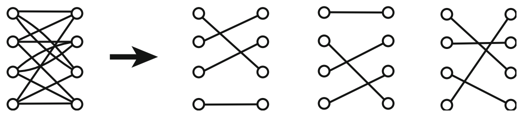


Fig. 2.4. A 3-regular bipartite multigraph and its 1-factorization.

Proof. Let G be an r-regular bipartite multigraph with bipartition (A, B). Then $|A| = |B|$ since $r|A| = e_G(A, B) = r|B|$. For every subset $X \subseteq A$, we have

$$r|X| = e_G(X, N_G(X)) \leq r|N_G(X)|,$$

and so $|X| \leq |N_G(X)|$. Hence by the marriage theorem, G has a matching M saturating A, which must saturate B since $|A| = |B|$. Thus M is a 1-factor of G.

It is obvious that $G - M$ is a $(r - 1)$-regular bipartite multigraph, and so it has a 1-factor by the same argument as above. By repeating this procedure, we can obtain a 1-factorization of G. $\square$

Lemma 2.3 (König [155]). *Let G be a bipartite multigraph with bipartition (A, B). If $|A| \geq |B|$ and the maximum degree of G is Δ, then there exists a Δ-regular bipartite multigraph which contains G as a subgraph and A as one of its partite sets.*

Proof. By adding $|A| - |B|$ new vertices to B if $|A| > |B|$, we obtain a bipartite graph with bipartition (A, B') such that $|A| = |B'|$. Add new edges joining vertices in A to vertices in B' whose degrees are less than Δ, one at a time until no new edge can be added. We show that by this procedure, we get the desired Δ-regular bipartite multigraph.

Suppose that the bipartite multigraph H obtained in this way has a vertex $a \in A$ such that $\deg_H(a) < \Delta$. Then there exists a vertex $b \in B'$ such that $\deg_H(b) < \Delta$ because

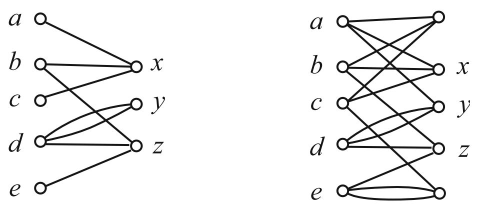


Fig. 2.5. A bipartite multigraph and a regular bipartite graph containing it.

$$\sum_{x \in B'} \deg_H(x) = e_H(B', A) = \sum_{x \in A} \deg_H(x) < \Delta|A| = \Delta|B'|.$$

Hence we can add a new edge ab to H, which is a contradiction. Therefore every vertex of A has degree Δ in H, which implies that every vertex of B' has degree Δ since $|A| = |B'|$. Consequently, H is the desired Δ-regular bipartite multigraph. □

The next theorem says that the chromatic index of a bipartite multigraph G is equal to the maximum degree of G.

Theorem 2.4 (König [155]). *Let G be a bipartite multigraph with maximum degree Δ. Then $E(G)$ can be partitioned into $E(G) = E_1 \cup E_2 \cup \cdots \cup E_\Delta$ so that each E_i $(1 \le i \le \Delta)$ is a matching in G.*

Proof. By Lemma 2.3, there exists a Δ-regular bipartite multigraph H which contains G as a subgraph. Then by Theorem 2.2, $E(H)$ can be partitioned into 1-factors $F_1 \cup F_2 \cup \cdots \cup F_\Delta$. It is obvious that $F_i \cap E(G)$ $(1 \le i \le \Delta)$ is a matching in G, and their union is equal to $E(G)$. Therefore the theorem follows. □

We now state various properties of matchings in bipartite graphs, some of which are generalizations of the marriage theorem, and some of which are properties that are specific to bipartite graphs.

Theorem 2.5. *A bipartite multigraph G has a matching that saturates all the vertices of degree $\Delta(G)$.*

Proof. Let $\Delta = \Delta(G)$. By Lemma 2.3, there exists a Δ-regular bipartite multigraph H that contains G as a subgraph. Then by Theorem 2.2, H has a 1-factor F. It is easy to see that $F \cap E(G)$ is a matching of G that saturates all the vertices v of G with degree Δ since every edge of H incident with v is an edge of G. □

Theorem 2.6. *Let G be a bipartite multigraph with bipartition (A, B). If $|N_G(S)| > |S|$ for all $\emptyset \ne S \subset A$, then for each edge e of G, G has a matching that saturates A and contains e.*

Proof. Let $e = ab$ ($a \in A, b \in B$) be any edge of G, and $H = G - \{a, b\}$. Then for every subset $\emptyset \neq X \subseteq A - \{a\}$, we have

$$|N_H(X)| \geq |N_G(X) \setminus \{b\}| > |X| - 1,$$

which implies $|N_H(X)| \geq |X|$. Hence by the marriage theorem, H has a matching M saturating $A - \{a\}$. Thus, $M + e$ is the desired matching in G that saturates A and contains e. □

Theorem 2.7 (Hetyei, Theorem 4.1.1 of [182]). *Let G be a bipartite multigraph with bipartition (A, B) such that $|A| = |B|$. Then the following three statements are equivalent.*
(i) G is connected, and for each edge e, G has a 1-factor containing e.
(ii) For every subset $\emptyset \neq X \subset A$, $|N_G(X)| > |X|$.
(iii) For every two vertices $a \in A$ and $b \in B$, $G - \{a, b\}$ has a 1-factor.

Proof. (i)⇒(ii) Suppose that $|N_G(Y)| \leq |Y|$ for some subset $\emptyset \neq Y \subset A$. Since G has a 1-factor F, we have $|N_G(Y)| = |Y|$ as $|N_G(Y)| \geq |N_F(Y)| = |Y|$. Since G is connected, G has an edge e joining a vertex in $A - Y$ to a vertex in $N_G(Y)$. However there exists no 1-factor in G containing e since $|N_G(Y)| - 1 < |Y|$. This contradicts (i). Thus (ii) holds.

(ii)⇒(iii) Let $H = G - \{a, b\}$ and $X \subseteq A - \{a\}$. Then

$$|N_H(X)| = |N_G(X) \setminus \{b\}| > |X| - 1,$$

which implies $|N_H(X)| \geq |X|$. Hence H has a matching saturating $A - \{a\}$, which is obviously a 1-factor of H as $|A - \{a\}| = |B - \{b\}|$.

(iii)⇒(i) Let $e = ab$ ($a \in A, b \in B$) be an edge of G. Since $G - \{a, b\}$ has a 1-factor F, G has a 1-factor $F + e$, which contains e. The proof of connectivity is left to the reader. □

Theorem 2.8 (Exercise 3.1.32 of [244]). *Suppose that a bipartite multigraph G with bipartition (A, B) has a matching saturating A. Then there exists a vertex $v \in A$ possessing the property that every edge incident with v is contained in a matching in G saturating A.*

Proof. We prove the theorem by induction on $|A|$. It is clear that we may assume $|A| \geq 2$.

If $|N_G(X)| > |X|$ for all $\emptyset \neq X \subset A$, then each vertex in A has the required property by Theorem 2.6. Hence we may assume that $|N_G(S)| \leq |S|$ for some $\emptyset \neq S \subset A$. Since G has a matching saturating A, it follows from the marriage theorem that $|N_G(S)| = |S|$.

By the inductive hypothesis, the induced subgraph $H = \langle S \cup N_G(S) \rangle_G$ has a vertex $v \in S$ such that every edge of H incident with v is contained in a matching in H saturating S (Fig. 2.6). Note that every edge of G incident with v is contained in H.

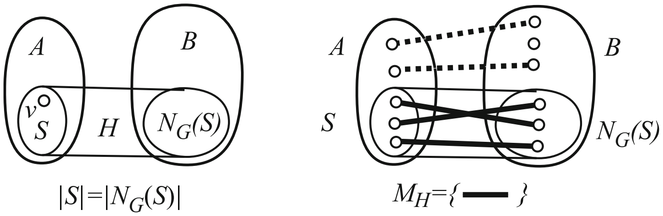


Fig. 2.6. A bipartite multigraph G and its subgraph H; A matching M in G saturating A.

Let M be a matching in G saturating A, and M_H be a matching in H saturating S. Then it is easy to see that $(M \setminus E(H)) \cup M_H$ is a matching in G that saturates A (see Fig. 2.6). Therefore the vertex v in S has the desired property. □

Theorem 2.9. *Let G be a bipartite multigraph with bipartition (A, B) such that $|N_G(X)| \geq |X|$ for every $X \subseteq A$. If two subsets S, $T \subseteq A$ satisfy $|N_G(S)| = |S|$ and $|N_G(T)| = |T|$, then*

$$|N_G(S \cup T)| = |S \cup T| \quad \textit{and} \quad |N_G(S \cap T)| = |S \cap T|.$$

In particular, if such subsets exist, there exists a unique maximum subset $A_0 \subseteq A$ such that $|N_G(A_0)| = |A_0|$

Proof. Since $N_G(S \cup T) = N_G(S) \cup N_G(T)$ and $N_G(S) \cap N_G(T) \supseteq N_G(S \cap T)$, we have

$$\begin{aligned} |N_G(S \cup T)| &= |N_G(S)| + |N_G(T)| - |N_G(S) \cap N_G(T)| \\ &\leq |S| + |T| - |N_G(S \cap T)| \\ &\leq |S| + |T| - |S \cap T| = |S \cup T|. \end{aligned}$$

On the other hand, $|N_G(S \cup T)| \geq |S \cup T|$ by the assumption. Hence $|N_G(S \cup T)| = |S \cup T|$, and also $|N_G(S \cap T)| = |S \cap T|$ by the above inequality. □

The following theorem is a generalization of the marriage theorem since the subgraph H with $f(x) = 1$ given in the following theorem is a matching.

Theorem 2.10 (Generalized Marriage Theorem). *Let G be a bipartite multigraph with bipartition (A, B), and let $f : A \to \mathbb{N}$ be a function. Then G has a subgraph H such that*

$$\deg_H(x) = f(x) \qquad \textit{for all} \quad x \in A, \quad \textit{and} \tag{2.3}$$

$$\deg_H(y) \leq 1 \qquad \textit{for all} \quad y \in B \tag{2.4}$$

if and only if

$$|N_G(S)| \geq \sum_{x \in S} f(x) \qquad \textit{for all} \quad S \subseteq A. \tag{2.5}$$

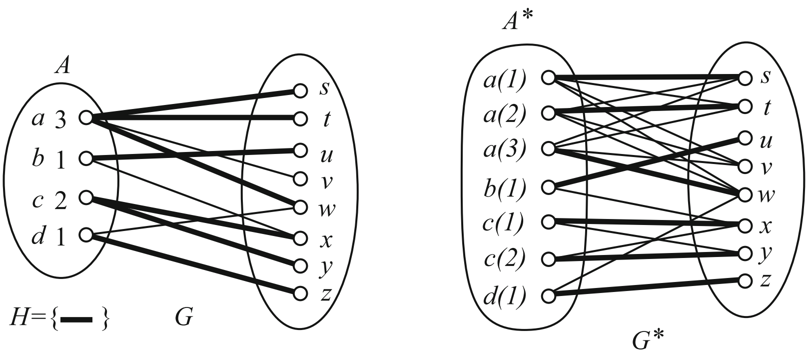


Fig. 2.7. A bipartite graph G and its subgraph H, where numbers denote $f(v)$; and G^*.

Proof. We first assume that G has a subgraph H that satisfies (2.3) and (2.4) (Fig. 2.7). Then for every $S \subseteq A$, we have

$$|N_G(S)| \geq |N_H(S)| = \sum_{x \in S} \deg_H(x) = \sum_{x \in S} f(x).$$

Hence (2.5) holds.

In order to prove the sufficiency, we construct a new bipartite simple graph G^* with bipartition $A^* \cup B$ as follows. For every vertex $v \in A$, define $f(v)$ vertices $v(1), v(2), \ldots, v(f(v))$ of A^*, and connect them to all the vertices of $N_G(v)$ by edges (Fig. 2.7). Then

$$|A^*| = \sum_{x \in A} f(x) \qquad \text{and} \qquad N_{G^*}(v(i)) = N_G(v) \quad \text{for all} \quad v(i) \in A^*.$$

For every subset $\emptyset \neq S^* \subseteq A^*$, let $S = \{x \in A : x(i) \in S^* \text{ for some } i\}$. Then we have by (2.5) that

$$|N_{G^*}(S^*)| = |N_G(S)| \geq \sum_{x \in S} f(x) \geq |S^*|.$$

Hence by the marriage theorem, G^* has a matching M^* that saturates A^*. It is clear that the subgraph H of G induced by M^* satisfies

$$\deg_H(x) = f(x) \quad \text{for all} \quad x \in A, \qquad \text{and} \quad \deg_H(y) \leq 1 \quad \text{for all} \quad y \in B.$$

Hence the theorem is proved. $\square$

The next theorem gives a formula for the size of a maximum matching in a bipartite graph. Moreover, this formula includes the marriage theorem since $|N_G(S)| \geq |S|$ for all $S \subseteq A$ implies $||M|| = |A|$ as $|\emptyset| - |N_G(\emptyset)| = 0$.

Theorem 2.11 (Ore [204]). *Let G be a bipartite multigraph with bipartition (A, B), and M a maximum matching in G. Then the size of M is given by*

$$||M|| = |A| - \max_{X \subseteq A}\{|X| - |N_G(X)|\}.$$

Proof. Let M be a maximum matching in G, $d = \max_{X\subseteq A}\{|X| - |N_G(X)|\}$, and $S \subseteq A$ such that $|S| - |N_G(S)| = d$. Then

$$\begin{aligned} ||M|| &= |N_M(A - S)| + |N_M(S)| \\ &\leq |A - S| + |N_G(S)| = |A| - (|S| - |N_G(S)|) \\ &= |A| - d. \end{aligned}$$

In order to prove the inverse inequality, we construct a new bipartite multigraph H with bipartition $(A, B \cup D)$ from G by adding a new vertex set D of d vertices and by joining every vertex of D to all the vertices of A (Fig. 2.8). Then for every $\emptyset \neq X \subseteq A$, since $d \geq |X| - |N_G(X)|$, we have

$$|N_H(X)| = |N_G(X)| + |D| = |N_G(X)| + d \geq |X|.$$

Hence, by the marriage theorem, H has a matching M_H saturating A. Then $M_H \cap E(G)$ is a matching in G that contains at least $|A| - d$ edges. Therefore $||M|| \geq |A| - d$, and the theorem is proved. $\square$

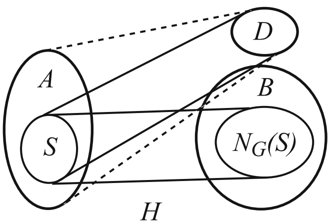


Fig. 2.8. A new bipartite multigraph H.

An **induced matching** in a general graph G is a matching that is an induced subgraph of G, that is, an induced matching can be expressed as $\langle U \rangle_G$ for some vertex set $U \subseteq V(G)$.

Theorem 2.12 (Liu and Zhou, [173]). *Let G be a connected bipartite simple graph with bipartition (A, B). Then the size of a maximum induced matching M in G is given by*

$$||M|| = \max\{|X| : X \subseteq A \textit{ such that } N_G(Y) \neq N_G(X) \textit{ for all } Y \subset X\}.$$

Proof. We say that a subset $X \subset A$ has the property $\mathcal{P}$ if $N_G(Y) \neq N_G(X)$ for all $Y \subset X$. Let

$$k = \max\{|X| : X \subseteq A,\ X \text{ has the property } \mathcal{P}\}.$$

Let M be a maximum induced matching in G, and let A_M and B_M be the sets of vertices of A and B, respectively, which are saturated by M. Since M is an induced matching, we have $E_G(A_M, B_M) = M$. Then for every $Y \subset A_M$, we have $N_G(Y) \neq N_G(A_M)$ since

$$|N_G(Y) \cap B_M| = |Y| < |A_M| = |N_G(A_M) \cap B_M|.$$

Hence A_M has the property $\mathcal{P}$, and thus $||M|| = |A_M| \leq k$.

We now show that $||M|| \geq k$. We may assume $k \geq 2$. Let $S = \{a_1, a_2, \ldots, a_k\}$ be a maximum subset of A that has the property $\mathcal{P}$. Then $N_G(Y) \neq N_G(S)$ for all $Y \subset S$. For every $1 \leq i \leq k$, we have

$$N_G(a_i) \not\subseteq \bigcup_{j \neq i} N_G(a_j) \qquad \text{by} \qquad N_G(S - \{a_i\}) \neq N_G(S),$$

and so we can choose

$$b_i \in N_G(a_i) \setminus \bigcup_{j \neq i} N_G(a_j).$$

Then $\{a_i b_i \mid 1 \leq i \leq k\}$ is an induced matching with k edges, which implies $||M|| \geq k$. Consequently the theorem is proved. □

It is known that there are polynomial time algorithms for many problems involving matchings in bipartite graphs, some of which will be shown in Section 2.3. On the other hand, it is remarkable that the following problem is NP-complete [41]: Is there an induced matching of size k in a bipartite graph?

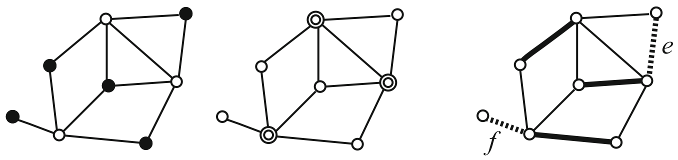
2.2 Covers and transversals

In this section we discuss covers of bipartite graphs and transversals of family of subsets. Some of the results are called **min-max theorems** because they say that the minimum value of some invariant is equal to the maximum value of another invariant.

When a vertex v is incident with an edge e, we say that v and e **cover** each other. A **vertex cover** of a graph G is a set of vertices that cover all the edges of G. A vertex cover of minimum cardinality is called a **minimum vertex cover**, and its cardinality is denoted by $\beta(G)$ (Fig. 2.9). Similarly, an **edge cover** of a graph G is defined to be a set of edges that cover all the vertices of G. An edge cover with minimum cardinality is called a **minimum edge cover**, and its cardinality is denoted by $\beta'(G)$.

Recall that a set of vertices of a graph G is said to be independent if no two of its vertices are adjacent, and a set of edges of G is said to be independent if no two of its edges have an end-point in common, that is, a set of edges is independent if and only if it forms a matching. We analogously define a **maximum independent vertex set** and a **maximum independent edge set** of G, and denote their cardinalities by $\alpha(G)$ and $\alpha'(G)$, respectively.

$\beta(G)$ = the minimum cardinality of a vertex cover
$\beta'(G)$ = the minimum cardinality of an edge cover
$\alpha(G)$ = the maximum cardinality of an independent set of vertices
$\alpha'(G)$ = the maximum cardinality of an independent set of edges



(1) $U=\{\bullet\}$ (2) $W=\{\circledcirc\}$ (3) $M=\{\text{—}\}$

Fig. 2.9. (1) A maximum independent set of vertices U ; (2) A minimum vertex cover $W = V(G) - U$; (3) A maximum matching M and a minimum edge cover $M + \{e, f\}$.

Lemma 2.13 (Gallai [91]). *A vertex set S of a simple graph G is independent if and only if $\overline{S} = V(G) - S$ is a vertex cover. Moreover,*

$$\alpha(G) + \beta(G) = |G|.$$

Proof. Suppose that S is independent. Then no edge of G joins two vertices of S, that is, for each edge e of G, at least one end-point of e is contained in $\overline{S}$. Hence $\overline{S}$ is a vertex cover.

Conversely, if $\overline{S}$ is a vertex cover, then every edge is incident with a vertex in $\overline{S}$, which implies that no edge joins two vertices in $S = V(G) - \overline{S}$, and thus S is independent.

Moreover, from the above arguments, it follows that S is a maximum independent set of vertices if and only if $\overline{S}$ is a minimum vertex cover. Thus $\alpha(G) + \beta(G) = |S| + |\overline{S}| = |G|$. $\square$

The next theorem shows that a similar equality also holds for edges.

Theorem 2.14 (Gallai [91]). *If a simple graph G has no isolated vertices, then $\alpha'(G) + \beta'(G) = |G|$.*

Proof. Let M be a maximum matching of G. For every vertex v of G unsaturated by M, by adding one edge incident with v to M, we can obtain an edge cover which contains the following number of edges:

$$||M|| + |G| - 2||M|| = |G| - ||M|| = |G| - \alpha'(G).$$

Hence $\beta'(G) \leq |G| - \alpha'(G)$, which implies $\alpha'(G) + \beta'(G) \leq |G|$.

Conversely, if L is a minimum edge cover of G, then $\langle L \rangle$ is a spanning subgraph of G and each component of $\langle L \rangle$ is a tree. Thus the number of components of $\langle L \rangle$ is $|G| - ||L||$ (see Problem 1.3). By choosing one edge from each component of $\langle L \rangle$, we can obtain a matching that contains $|G| - ||L||$ edges. Hence $\alpha'(G) \geq |G| - \beta'(G)$, and thus $\alpha'(G) + \beta'(G) \geq |G|$. Consequently, we have $\alpha'(G) + \beta'(G) = |G|$. □

It follows that for each edge e of a maximum matching in a graph G, at least one of the end-points of e must be contained in a vertex cover of G. Hence $\alpha'(G) \leq \beta(G)$. However, the equality $\alpha'(G) = \beta(G)$ does not hold in general except for bipartite graphs.

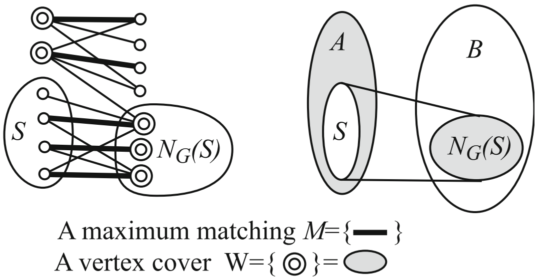


Fig. 2.10. A maximum matching and a minimum vertex cover in a bipartite graph.

Theorem 2.15 (König [156]). *In a bipartite simple graph G without isolated vertices, $\alpha'(G) = \beta(G)$. Moreover this implies $\alpha(G) = \beta'(G)$.*

Proof. Since $\alpha'(G) \leq \beta(G)$ as shown above, it suffices to prove $\alpha'(G) \geq \beta(G)$.

Let (A, B) be the bipartition of G, and let M be a maximum matching in G. Let S be a subset of A such that

$$|S| - |N_G(S)| = \max_{X \subseteq A}\{|X| - |N_G(X)|\},$$

and $W = (A - S) \cup N_G(S)$. Then W is a vertex cover of G since G has no edge joining S to $B - N_G(S)$ (see Fig. 2.10). By Theorem 2.11, we have

$$\beta(G) \leq |W| = |A| - |S| + |N_G(S)| = ||M|| = \alpha'(G).$$

Therefore the theorem is proved. □

$$M = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$|\{\bigcirc\}| = 3$
$\{$1st row, 4th row, 2nd column$\}$
contains all the 1-entries.

Fig. 2.11. A $(0,1)$-matrix M.

A $(0,1)$-matrix is a matrix whose entries are all 0 or 1. The preceding theorem leads to the following interesting property of $(0,1)$-matrices.

Theorem 2.16 (König-Egerváry [155], [74]). *Let $M = (m_{ij})$ be a $(0,1)$-matrix. Then the maximum number of 1-entries of M, such that no two of them lie on the same row or column, is equal to the minimum number of rows and columns that contain all the 1-entries of M (Fig. 2.11).*

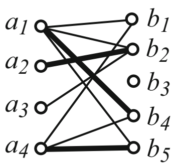


Fig. 2.12. The bipartite graph corresponding to the $(0,1)$-matrix in Fig. 2.11, where $\alpha'(G) = \#\{a_1b_4, a_2b_2, a_4b_5\} = 3$ and $\beta(G) = \#\{a_1, b_2, a_4\} = 3$.

Proof. From the given $n \times m$ $(0,1)$-matrix $M = (m_{ij})$, we construct a bipartite graph G with bipartite sets $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_m\}$ as follows:

$$a_ib_j \in E(G) \qquad \text{if and only if} \qquad m_{ij} = 1. \qquad \text{(see Fig. 2.12)}$$

Then a set of 1-entries, no two of which lie on the same row or column, corresponds to a matching of G, and a set of rows and columns that contains all the 1-entries corresponds to a vertex cover of G. Therefore the theorem follows from Theorem 2.15. $\square$

Let X be a finite set and $\mathcal{F} = \{S_1, S_2, \dots, S_n\}$ a family of subsets of X, where the S_i's are not necessarily distinct. We say that $\mathcal{F}$ has a **transversal** if there exists a set of n distinct elements of X, one from each S_i. For example, a family $\{\{a, b, e\}, \{b\}, \{a, c, d\}, \{b, c\}\}$ of subsets of $\{a, b, c, d, e\}$ has a transversal $\{a, b, d, c\}$. On the other hand, a family $\{\{a, b, e\}, \{b, c\}, \{c\}, \{b, c\}\}$ has no transversal.

Theorem 2.17 (Hall [98]). *A family $\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$ of subsets of X has a transversal if and only if*

$$|\bigcup_{i \in I} S_i| \geq |I| \qquad \textit{for all} \quad I \subseteq \{1, 2, \ldots, n\}. \tag{2.6}$$

Proof. We prove only the sufficiency since the necessity is immediate. We construct a bipartite graph G with partite sets $\{S_1, S_2, \ldots, S_n\}$ and X as follows: a vertex S_i is adjacent to a vertex $x \in X$ in G if and only if $x \in S_i$ (see Fig. 2.13). Then for every subset $\{S_i \mid i \in I\}$ of $\{S_1, S_2, \ldots, S_n\}$, we have by (2.6) that

$$|N_G(\{S_i \mid i \in I\})| = |\bigcup_{i \in I} S_i| \geq |I|.$$

Hence by the marriage theorem, G has a matching M saturating $\{S_1, S_2, \ldots, S_n\}$. Then we can obtain a transversal from M by taking the set of vertices of X saturated by M. $\square$

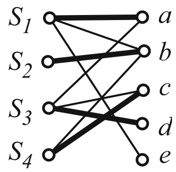


Fig. 2.13. The bipartite graph corresponding to $\{S_1 = \{a, b, e\}, S_2 = \{b\}, S_3 = \{a, c, d\}, S_4 = \{b, c\}\}$, its matching saturating $\{S_1, S_2, S_3, S_4\}$, and a transversal $\{a, b, c, d\}$ obtained from it.

Let X be a finite set and $\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$ and $\mathcal{H} = \{T_1, T_2, \ldots, T_n\}$ be two families of subsets of X. Then we say that $\mathcal{F}$ and $\mathcal{H}$ have a **common transversal** if there exists a set of n distinct elements of X that is a transversal of both $\mathcal{F}$ and $\mathcal{H}$, i.e., there exists a set $\{x_1, x_2, \ldots, x_n\}$ of n distinct elements of X such that $x_i \in S_i \cap T_{(i)}$ for all $1 \leq i \leq n$, where $\{T_{(1)}, T_{(2)}, \ldots, T_{(n)}\}$ is a rearrangement of $\mathcal{H} = \{T_1, T_2, \ldots, T_n\}$.

For example, $\{S'_1 = S'_2 = \{a, b\}, S'_3 = \{a, c, e\}\}$ and $\{T'_1 = \{b, c\}, T'_2 = \{e\}, T'_3 = \{a, b, e\}\}$ have a common transversal $\{b, a, e\}$, where $b \in S'_1 \cap T'_1$, $a \in S'_2 \cap T'_3$ and $e \in S'_3 \cap T'_2$. On the other hand, $\{S_1 = S_2 = \{a, b\}, S_3 = \{a, c, e\}\}$ and $\{T_1 = \{b, c\}, T_2 = \{b, d\}, T_3 = \{c, d, e\}\}$ have transversals but do not have a common transversal. Let $n = 3$, $I = \{1, 2\}$ and $J = \{1, 2, 3\}$, and substitute these into the left and right sides of (2.7) given in the following theorem. Then

$$|(S_1 \cup S_2) \cap (T_1 \cup T_2 \cup T_3)| = |\{b\}| = 1 < |I| + |J| - n = 2 + 3 - 3 = 2.$$

Hence inequality (2.7) does not hold.

Theorem 2.18. *Let $\mathcal{F}$ and $\mathcal{H}$ be two families of subsets of a set X. Then $\mathcal{F}$ and $\mathcal{H}$ have a common transversal if and only if for all subsets I and J of $\{1, 2, \ldots, n\}$, it follows that*

$$\Big|\Big(\bigcup_{i\in I} S_i\Big) \bigcap \Big(\bigcup_{j\in J} T_j\Big)\Big| \geq |I| + |J| - n. \tag{2.7}$$

Proof. Let $\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$ and $\mathcal{H} = \{T_1, T_2, \ldots, T_n\}$. Assume that $\mathcal{F}$ and $\mathcal{H}$ have a common transversal $\{x_1, x_2, \cdots, x_n\}$. Let I and J be subsets of $\{1, 2, \ldots, n\}$. We may assume that $|I| + |J| > n$ since otherwise (2.7) trivially holds. Then two sets $\{x_r \mid x_r \in S_i,\ i \in I\}$ and $\{x_r \mid x_r \in T_j,\ j \in J\}$ must have at least $|I| + |J| - n$ elements in common, and these elements are contained in

$$\Big(\bigcup_{i\in I} S_i\Big) \bigcap \Big(\bigcup_{j\in J} T_j\Big).$$

Hence (2.7) holds.

In order to prove sufficiency, we construct a bipartite graph G with bipartite sets

$$A = \{S_1, S_2, \ldots, S_n\} \cup X \qquad \text{and} \qquad B = \{T_1, T_2, \ldots, T_n\} \cup X'$$

as follows, where $X' = \{x' \mid x \in X\}$. Two vertices $S_i \in A$ and $x' \in X'$ are joined by an edge if $x \in S_i$. Similarly two vertices $x \in X$ and $T_j \in B$ are joined by an edge if $x \in T_j$. Moreover, $x \in X$ and $x' \in X'$ are joined by an edge, and G has no more edges.

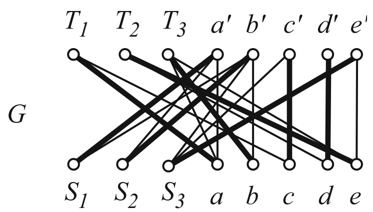


Fig. 2.14. The bipartite graph corresponding to two families of subsets.

For example, the bipartite graph G given in Fig. 2.14 corresponds to $X = \{a, b, c, d, e\}$, $\mathcal{F} = \{S_1 = S_2 = \{a, b\}, S_3 = \{b, c, e\}\}$ and $\mathcal{H} = \{T_1 = \{a, c\}, T_2 = \{d, e\}, T_3 = \{a, b, e\}\}$. G has a perfect matching, and these two families have a common transversal $\{a, b, e\}$ such that $a \in S_1 \cap T_1, b \in S_2 \cap T_3$ and $e \in S_3 \cap T_2$.

We first show that if the bipartite graph G has a perfect matching M, then $\mathcal{F}$ and $\mathcal{H}$ have a common transversal. If M contains an edge $S_i a'_k$ $(a'_k \in X')$, then M must contain an edge a_kT_j ($a_k \in X$) for some T_j, and thus $a_k \in S_i \cap T_j$. Hence it is easy to see that $\{a_k \mid S_i a'_k \in M\}$ forms a common transversal of $\mathcal{F}$ and $\mathcal{H}$. Note that if M contains an edge joining $a_t \in X$ to $a'_t \in X'$, then this fact implies that a_t is not chosen to be an element of the common transversal.

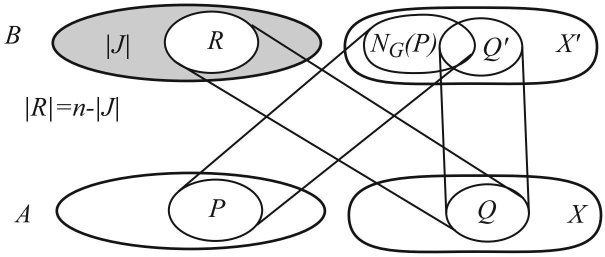


Fig. 2.15. Illustration of Theorem 2.18; $P = \{X_i \mid i \in I\}$, $J = \{j \mid T_j \cap Q = \emptyset\}$, and $R = \{T_j \mid T_j \cap Q \neq \emptyset\}$.

Let

$$P \subseteq \{S_1, S_2, \ldots, S_n\} \qquad \text{and} \qquad Q \subseteq X.$$

Then $P \cup X \subseteq A$. Put $I = \{i : S_i \in P\}$, $J = \{j : T_j \cap Q = \emptyset\}$ and $Q' = \{x' : x \in Q\} \subseteq X'$. Hence $Q' \subset B$ and

$$\Big(\bigcup_{i \in I} S_i\Big)' \setminus Q' = \Big(\bigcup_{i \in I} S_i\Big)' \cap (X' - Q') \supseteq \Big(\bigcup_{i \in I} S_i\Big)' \cap \Big(\bigcup_{j \in J} T_j\Big)',$$

where $(\bigcup_{i \in I} S_i)'$ and $(\bigcup_{j \in J} T_j)'$ denote the subsets of X' (Fig. 2.15). Therefore

$$\begin{aligned}
|N_G(P \cup Q)| &= |(N_G(P) \cup N_G(Q)) \cap X')| + |N_G(Q) \cap \{T_1, T_2, \ldots, T_n\}| \\
&= \Big|\Big(\bigcup_{i \in I} S_i\Big)' \cup Q'\Big| + \#\{T_j \ : T_j \cap Q \neq \emptyset\} \\
&\geq \Big|\Big(\bigcup_{i \in I} S_i\Big)' \setminus Q'\Big| + |Q'| + n - |J| \\
&\geq \Big|\Big(\bigcup_{i \in I} S_i\Big)' \cap \Big(\bigcup_{j \in J} T_j\Big)'\Big| + |Q'| + n - |J| \\
&\geq |I| + |J| - n + |Q'| + n - |J| \qquad \text{(by (2.7))} \\
&= |I| + |Q'| = |P| + |Q| = |P \cup Q|.
\end{aligned}$$

Consequently, G has a matching M saturating A by the marriage theorem. Since $|A| = |B|$, M is a perfect matching, and thus the proof is complete. □

2.3 Augmenting paths and algorithms

In this section we consider matchings in graphs by using alternating paths instead of neighborhoods, which is a new approach to matchings and useful for algorithms. For two sets X and Y of edges of a graph G, we define

$$X \triangle Y = (X \cup Y) - (X \cap Y).$$

Lemma 2.19. *Let M_1 and M_2 be matchings in a simple graph. Then each component of $\langle M_1 \triangle M_2 \rangle$ is either (i) a path whose edges are alternately in M_1 and M_2, or (ii) an even cycle whose edges are alternately in M_1 and M_2* (Fig. 2.16).

Proof. Let $H = \langle M_1 \triangle M_2 \rangle$ and v a vertex of H. Then $1 \le \deg_H(v) \le 2$, and $\deg_H(v) = 2$ implies that exactly one edge of M_1 and one edge of M_2 are incident with v. Hence each component C of H is a path or cycle, whose edges are alternately in M_1 and M_2. In particular, if C is a cycle, it must be an even cycle (see Fig. 2.16). □

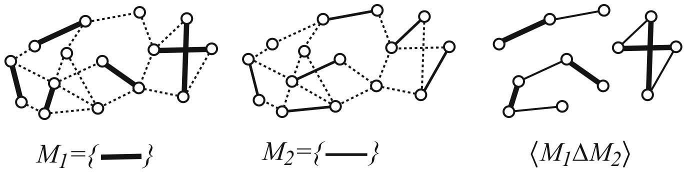


Fig. 2.16. Matchings M_1 and M_2; and $\langle M_1 \triangle M_2 \rangle$.

Let M be a matching in a simple graph G. Then a path of G is called an M**-alternating path** if its edges are alternately in M and not in M. If both end-vertices of an M-alternating path are unsaturated by M, then such an M-alternating path is called an M**-augmenting path**.

For example, in the graph G shown in Fig. 2.17, $M = \{a, b, c\}$ is a matching, (d, b, f, a) and (g, c, e) are M-alternating paths, and $P = (d, b, e, c, g)$ is an M-augmenting path, whose end-vertices u and v are not saturated by M. Furthermore, it is immediate that $M \triangle E(P) = \{a, d, e, g\}$ is a matching that contains $||M|| + 1$ edges and is larger than M. The next theorem states a characterization of maximum matchings by using augmenting path.

Theorem 2.20 (Berge [25]). *A matching M in a simple graph G is maximum if and only if there exists no M-augmenting path in G.*

Proof. The contraposition of the statement, which we shall prove, is the following. A matching M is not maximum if and only if there exists an M-augmenting path in G.

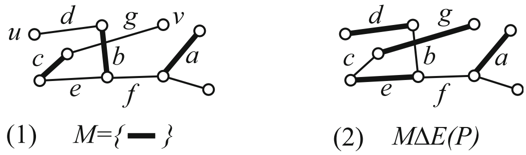


Fig. 2.17. (1) A matching $M = \{a, b, c\}$ and an M-augmenting path $P = (d, b, e, c, g)$; (2) $M \triangle E(P)$.

Suppose that there is an M-augmenting path P in G. Then $M \triangle E(P)$ is a matching which contains $||M|| + 1$ edges, and so M is not a maximum matching.

We next assume that M is not a maximum matching in G. Let M' be a maximum matching in G. Then by Lemma 2.19, $\langle M \triangle M' \rangle$ contains a path P in which the number of edges in M' is greater than the number of edges in M. Hence the two pendant edges of P are contained in M', which implies that P is an M-augmenting path of G. Therefore the contraposition is proved. □

By making use of M-alternating paths and M-augmenting paths, we can obtain some properties of matchings in a graph and also an algorithm for finding a maximum matching in a bipartite graph. We begin with one basic theorem and one result on a game played on a graph.

Theorem 2.21. *For any matching M in a simple graph G, G has a maximum matching that saturates all vertices saturated by M.*

Proof. Suppose M is not a maximum matching. By Theorem 2.20, G has an M-augmenting path P. Then $M \triangle E(P)$ is a matching which contains $||M||+1$ edges and saturates all the vertices saturated by M. Since a maximum matching can be obtained by repeating this procedure, we can find a maximum matching that saturates all the vertices saturated by M. □

Proposition 2.22 (Exercise 5.1.4 of [34]). *Two players play a game on a connected simple graph G by alternately selecting distinct vertices $v_1, v_2, v_3, \ldots$ so that $(v_1 v_2 v_3 \cdots)$ forms a path. The player who cannot select a vertex loses, that is, the player who selects the last vertex wins (Fig. 2.18). The second player has a winning strategy if G has a perfect matching; otherwise the first player has a winning strategy.*

Proof. Assume that G has a perfect matching M. If the first player selects a vertex x in his turn, then the second player selects a vertex that is joined to x by an edge of M. Since M is a perfect matching, the second player can always select such a vertex. Therefore, he wins since the game is over in a finite number of moves.

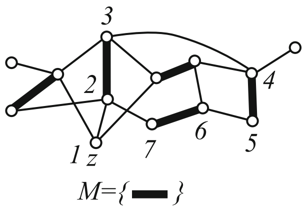


Fig. 2.18. A graph for which the first player can win, and its maximum matching M.

Next suppose that G has no perfect matching. Let M be a maximum matching and z be a vertex unsaturated by M. The first player selects z in the first turn, and if the second player selects a vertex y in his turn, then the first player selects a vertex that is joined to y by an edge of M. Then the vertices selected by the two players form an M-alternating path P starting at z. Since G has no M-augmenting path by Theorem 2.20, the path P does not pass through any other vertices unsaturated by M, which implies that the first player can always select a vertex. Therefore the first player wins. □

If a matching M in a graph G saturates no vertex in a subset $X \subset V(G)$, then we say that M **avoids** X.

Theorem 2.23 (Edmonds and Fulkerson [63]). *Let A and B be vertex subsets of a simple graph G such that $|A| < |B|$. Then*
(i) If there exist two matchings, one saturating A and the other saturating B, then there exists a matching that saturates A and at least one vertex in $B \setminus A$.
(ii) If there exist two maximum matchings, one avoiding A and the other avoiding B, then there exists a maximum matching that avoids A and at least one vertex in $B \setminus A$.

Proof. We first prove (i). Let M_A and M_B be matchings in G which saturate A and B respectively. If M_A saturates one vertex in $B \setminus A$, then M_A itself is the desired matching. Hence we may assume that M_A avoids $B \setminus A$.

Consider $\langle M_A \triangle M_B \rangle$. For every vertex b in $B \setminus A$, there exists a path in $\langle M_A \triangle M_B \rangle$ starting at b, which may end at a vertex in $A \setminus B$. Since $|A| < |B|$ and no path of $\langle M_A \triangle M_B \rangle$ ends at a vertex in $A \cap B$, there exists a path P in $\langle M_A \triangle M_B \rangle$ that starts at $b_1 \in B \setminus A$ and ends at $x \notin A$ (Fig. 2.19). Since P is an M_A-alternating path connecting b_1 and x, $M_A \triangle E(P)$ is the desired matching in G, which saturates A and $b_1 \in B \setminus A$.

We next prove (ii). Let N_A and N_B be maximum matchings in G which avoid A and B respectively. We may assume that N_A saturates $B \setminus A$ since otherwise N_A itself is the desired matching. Then by the same argument as above, there exists an N_A-alternating path P in $\langle N_A \triangle N_B \rangle$ starting at a vertex $b \in B \setminus A$ and ending at a vertex not contained in A. This implies that P does not pass through A. If the both pendant edges of P are contained in N_A, then

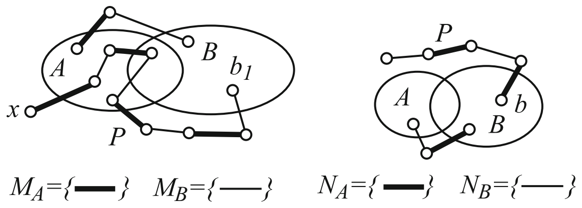


Fig. 2.19. M_A-alternating path P and N_A-alternating path P.

$N_B \triangle E(P)$ is a matching with $||N_B|| + 1$ edges, contrary to the maximality of N_B. Thus one pendant edge of P belongs to N_B, and thus $N_A \triangle E(P)$ is the desired maximum matching in G, which avoids A and $b \in B \setminus A$. $\square$

Theorem 2.24 (Mendelsohn and Dulmage [187]). *Let G be a bipartite simple graph with bipartition (A, B), and $X \subseteq A$ and $Y \subseteq B$. If X and Y are saturated by matchings in G, respectively, then G has a matching that saturates $X \cup Y$.*

Proof. Let M_X and M_Y be matchings in G saturating X and Y, respectively. We may assume that M_X does not saturate Y since otherwise M_X is the desired matching.

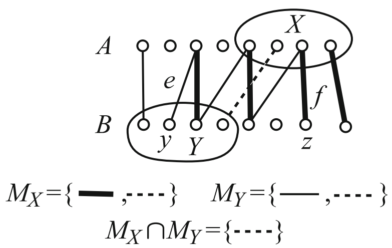


Fig. 2.20. A path $P = (y, e, \ldots, f, z)$ in $H = \langle M_X \triangle M_Y \rangle$.

Let $H = \langle M_X \triangle M_Y \rangle$, and choose a vertex $y \in Y$ unsaturated by M_X. Then there exists a path P in H starting at y, which can be expressed as

$$P = (y, e, \ldots, f, z), \quad y, \ldots, z \in V(H), \quad e, \cdots, f \in E(H),$$

where z is the other end-vertex of P and $e \in M_Y$. It is obvious that the vertices of P are alternately in B and A, and the edges of P are alternately in M_Y and M_X. If $z \in X$, then the pendant edge f of P must belong to M_Y, which contradicts the fact that X is saturated by M_X (Fig. 2.20). Hence

$z \notin X$. Similarly, if $z \in Y$, then $f \in M_X$, which contradicts the fact that Y is saturated by M_Y. Therefore $z \notin X \cup Y$.

Thus $M_X \triangle E(P)$ is a matching that saturates $V(\langle M_X \rangle)$ and $y \in Y \setminus V(M_X)$. By repeating this procedure, we can obtain the desired matching in G, which saturates $X \cup Y$. $\square$

We now give an algorithm for finding a maximum matching in a bipartite graph, which is often called the Hungarian Method.

Algorithm 2.25 (Hungarian Method) *Let G be a bipartite simple graph with bipartition (A, B). Then a maximum matching in G can be obtained by the following procedure. Let M be any matching in G, and $A_0 = A \setminus V(\langle M \rangle)$ be the set of vertices in A unsaturated by M. Let $B_1 = N_G(A_0) \subseteq B$ and define*

$$A_i = N_M(B_i), \quad B_{i+1} = N_G(A_i) \setminus (B_1 \cup B_2 \cup \cdots \cup B_i)$$

for every $1 \le i \le k$ until B_{k+1} contains an M-unsaturated vertex or $B_{k+1} = \emptyset$. If B_{k+1} contains an M-unsaturated vertex, say b_{k+1}, then we find an M-augmenting path P joining $b_{k+1} \in B_{k+1}$ to $a \in A_0$, and obtain a larger matching $M \triangle E(P)$ containing $||M||+1$ edges. We apply the above procedure to $M \triangle E(P)$. If $B_{k+1} = \emptyset$, then M is the desired maximum matching.

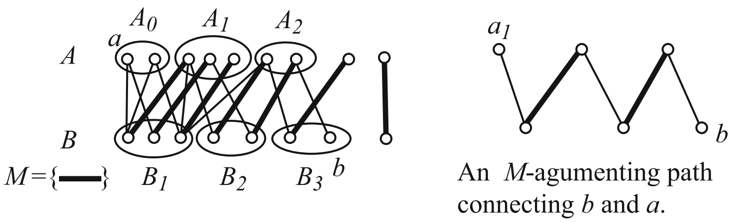


Fig. 2.21. An Algorithm for finding a maximum matching in a bipartite graph; and an improved algorithm.

Proof. It is easy to see that if B_{k+1} contains an M-unsaturated vertex, then we can find an M-augmenting path P, resulting in a larger matching $M \triangle E(P)$ (Fig. 2.21). Hence it suffices to show that if $B_{k+1} = \emptyset$, then M is a maximum matching.

Suppose $B_{k+1} = \emptyset$. Let $S = A_0 \cup A_1 \cup \cdots \cup A_k$ and $T = B_1 \cup B_2 \cup \cdots \cup B_k$. Then $N_G(S) = T$ and $N_M(T) = S - A_0$, and so $|T| = |N_M(T)| = |S - A_0| = |S| - |A_0|$. Hence

$$\begin{aligned} ||M|| &= |A| - |A_0| = |A| - (|S| - |T|) \\ &= |A| - (|S| - |N_G(S)|) \\ &\ge |A| - \max_{X \subseteq A}\{|X| - |N_G(X)|\}. \end{aligned}$$

By Theorem 2.11, this implies that M is a maximum matching in G. $\square$

Of course, we can apply the procedure given in Algorithm 2.25 for each vertex $a \in A_0$ individually, that is, put $B_1(a) = N_G(a) \subseteq B$ and obtain

$$A_i(a) = N_M(B_i(a)), \quad B_{i+1}(a) = N_G(A_i(a)) \setminus (B_1(a) \cup \cdots \cup B_i(a))$$

for every $1 \leq i \leq k$ until $B_{k+1}(a)$ contains an M-unsaturated vertex or $B_{k+1}(a) = \emptyset$. If $B_{k+1}(a)$ contains an M-unsaturated vertex, then we can get a larger matching than M. If $B_{k+1}(a) = \emptyset$, then we remove $S = \{a\} \cup A_1(a) \cup \cdots \cup A_k(a)$ and $T = B_1(a) \cup \cdots \cup B_k(a)$ from G, obtain $G - (S \cup T)$, and then apply the same procedure in $G - (S \cup T)$ for another vertex $a' \in A - \{a\}$.

2.4 1-factor theorems

In this section we investigate matchings and 1-factors of graphs. Since a 1-factor contains neither loops nor multiple edges, we can restrict ourselves to simple graphs when we consider 1-factors. So in this section, we mainly consider simple graphs. However, some results hold for multigraphs or general graphs, and these generalization are useful and interesting. For example, every 2-connected cubic simple graph has a 1-factor, but also every 2-connected cubic multigraph has a 1-factor. Thus we occasionally consider multigraphs and general graphs.

A criterion for a graph to have a 1-factor was obtained by Tutte [225] in 1947 and is one of the most important results in factor theory. We begin with this theorem, which is called the 1-factor theorem. The proof presented here is due to Anderson [16] and Mader [183]. Tutte's original proof uses the Pfaffian of a matrix. Other proofs of the 1-factor theorem are found in Hetyei [102] and Lovász [180].

After the 1-factor theorem, we discuss some other criteria for graphs to have 1-factors. For example, a criterion for a graph to have a 1-factor containing any given edge and a criterion for a tree to have a 1-factor, which is much simpler than the criterion of the 1-factor theorem, are shown.

For a vertex subset X of G and a component C of $G - X$, we simplify notation by denoting

$$E_G(C, X) = E_G(V(C), X) \quad \text{and} \quad e_G(C, X) = e_G(V(C), X).$$

A component of a graph is said to be **odd** or **even** according to whether its order is odd or even. For a graph G, $Odd(G)$ denotes the set of odd components of G, and $odd(G)$ denotes the number of odd components of G, that is,

$$odd(G) = |Odd(G)| = \text{the number of odd components of } G.$$

Lemma 2.26. *Let G be a general graph and $S \subseteq V(G)$. Then*

$$odd(G-S)+|S| \equiv odd(G-S)-|S| \equiv |G| \pmod 2. \tag{2.8}$$

In particular, if G is of even order, then

$$odd(G-S) \equiv |S| \pmod 2, \tag{2.9}$$

and $odd(G-v) \geq 1$ for every vertex v.

Proof. Let $C_1, C_2, \ldots, C_m$ be the odd components of $G-S$, and D_1, D_2, $\ldots, D_r$ the even components of $G-S$, where $m = odd(G-S)$. Then

$$|G| = |S| + |C_1| + \cdots + |C_m| + |D_1| + \cdots + |D_r| \equiv |S| + m \pmod 2.$$

Hence $odd(G-S)+|S| \equiv |G| \pmod 2$. It is obvious that $odd(G-S)+|S| \equiv odd(G-S)-|S| \pmod 2$. Therefore (2.8) holds. (2.9) is an immediate consequence of (2.8). $\square$

Before giving the 1-factor theorem, we make the following remark. A matching in a general graph contains no loops, and so does not contain a 1-factor. Then a 1-factor of a general graph is a spanning subgraph with all vertices degree one. Thus it is obvious that a general graph G has a 1-factor if and only if its underlying graph has a 1-factor, where the **underlying graph** of G is a simple graph obtained from G by removing all the loops and by replacing the multiple edges joining two vertices by a single edge joining them. Therefore, essential part of the following 1-factor theorem is that the theorem holds for simple graphs.

Theorem 2.27 (The 1-Factor Theorem, Tutte [225]**).** *A general graph G has a 1-factor if and only if*

$$odd(G-S) \leq |S| \qquad \text{for all} \quad S \subset V(G). \tag{2.10}$$

Proof. As we remarked above, we may assume that the given general graph G is a simple graph. Assume that G has a 1-factor F. Let $\emptyset \neq S \subseteq V(G)$, and $C_1, C_2, \ldots, C_m$ be the odd components of $G-S$, where $m = odd(G-S)$. Then for every odd component C_i of $G-S$, there exists at least one edge in F that joins C_i to S (Fig. 2.22). Hence

$$odd(G-S) = m \leq e_F(C_1 \cup C_2 \cup \cdots \cup C_m, S) \leq |S|.$$

We now prove the sufficiency by induction on $|G|$. By setting $S = \emptyset$, we have $odd(G) = 0$, which implies that each component of G is even. If G is not connected, then every component of G satisfies (2.10), and so it has a 1-factor by the inductive hypothesis. Hence G itself has a 1-factor. Therefore, we may assume that G is connected and has even order.

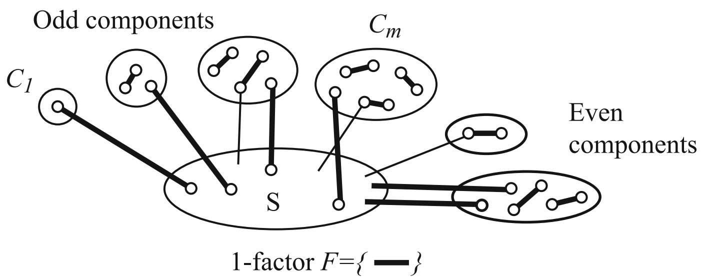


Fig. 2.22. For every odd component C_i of $G - S$, at least one edge of F joins C_i to S.

It follows from Lemma 2.26 and (2.10) that $odd(G - \{v\}) = |\{v\}| = 1$. Let S be a maximal subset of $V(G)$ with the property that $odd(G - S) = |S|$. Then $\emptyset \neq S \subset V(G)$ and

$$odd(G - X) < |X| \qquad \text{for all} \quad S \subset X \subseteq V(G). \tag{2.11}$$

Claim 1. *Every component of $G - S$ is of odd order.*

Suppose that $G - S$ has an even component D. Let v be any vertex of D. Then by Lemma 2.26, $D - v$ has at least one odd component, and so

$$odd(G - (S \cup \{v\})) = odd(G - S) + odd(D - v) \geq |S| + 1,$$

which implies $odd(G - (S \cup \{v\})) = |S \cup \{v\}|$ by (2.10). This contradicts the maximality of S. Hence Claim 1 holds.

Claim 2. *For any vertex v of each odd component C of $G - S$, $C - v$ has a 1-factor.*

Let C be an odd component of $G - S$, and v any vertex of C. Then for every subset $X \subseteq V(C - v)$, we have by (2.11) that

$$\begin{aligned} |S| + 1 + |X| &> odd(G - (S \cup \{v\} \cup X)) \\ &= odd(G - S) - 1 + odd(C - (\{v\} \cup X)) \\ &= |S| - 1 + odd((C - v) - X). \end{aligned}$$

Thus $odd((C - v) - X) < |X| + 2$, which implies $odd((C - v) - X) \leq |X|$ by (2.9). Hence $C - v$ has a 1-factor by the induction hypothesis, and thus Claim 2 is proved.

Let $C_1, C_2, \ldots, C_m$ be the odd components of $G - S$, where $m = odd(G - S) = |S|$. We construct a bipartite graph B with partite sets S and $\{C_1, C_2, \ldots, C_m\}$ as follows: a vertex x of S and C_i are joined by an edge of B if and only if x and C_i are joined by an edge of G (Fig. 2.23).

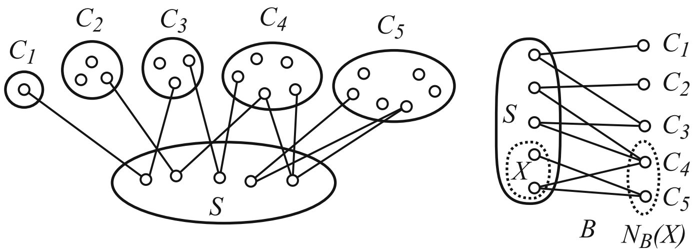


Fig. 2.23. $G - S$ and the bipartite graph B.

Claim 3. *The bipartite graph B has a 1-factor.*

It follows that $|N_B(S)| = |\{C_1, C_2, \dots, C_m\}| = |S|$ since G is connected. Assume that $|N_B(X)| < |X|$ for some $\emptyset \neq X \subset S$. Then every vertex $C_i \in \{C_1, C_2, \dots, C_m\} - N_B(X)$ is an isolated vertex of $B - (S - X)$, which implies C_i is an odd component of $G - (S - X)$, and thus we obtain

$$odd(G - (S - X)) \geq |\{C_1, C_2, \dots, C_m\} - N_B(X)| > m - |X| = |S - X|.$$

This contradicts (2.10). Therefore $|N_B(X)| \geq |X|$ for all $X \subseteq S$, and so by the marriage theorem, B has a matching K saturating S. Since $m = |S|$, K must saturate $\{C_1, C_2, \dots, C_m\}$, and thus K is a 1-factor of B.

For every edge $x_i C_i$ of K, choose a vertex $v_i \in V(C_i)$ that is adjacent to x_i in G, and take a 1-factor $F(C_i)$ of $C_i - v_i$, whose existence is guaranteed by Claim 2. Therefore, we obtain the following desired 1-factor of G:

$$\begin{aligned} &\Big(F(C_1) \cup \cdots \cup F(C_m)\Big) \\ &\cup \{x_i v_i \,:\, x_i C_i \in K,\ x_i \in S,\ v_i \in V(C_i),\ 1 \leq i \leq m\}. \end{aligned}$$

Consequently the theorem is proved. □

A graph G is said to be **factor-critical** if $G - v$ has a 1-factor for every vertex v of G (Fig. 2.24). It is easy to see that if G is factor-critical, then G is of odd order, connected and not a bipartite graph (Problem 2.8).

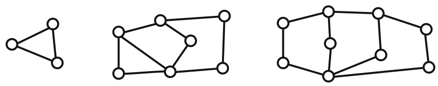


Fig. 2.24. Factor-critical graphs.

Theorem 2.28 (Edmonds [62]). *A simple graph G of even order has a 1-factor if and only if for every subset S of $V(G)$, the number of factor-critical components of $G - S$ is less than or equal to $|S|$.*

Proof. The necessity follows immediately from the 1-factor theorem since every factor-critical component of $G - S$ is an odd component of $G - S$.

We now prove the sufficiency. Suppose that G satisfies the condition in the theorem, but has no 1-factor. Then by the 1-factor theorem, there exists a subset $\emptyset \neq X \subseteq V(G)$ such that $odd(G - X) > |X|$, which implies $X \neq V(G)$. Take a maximal subset $\emptyset \neq S \subset V(G)$ such that $odd(G - S) > |S|$. Then

$$odd(G - Y) \leq |Y| \qquad \text{for all} \quad S \subset Y \subseteq V(G). \tag{2.12}$$

We first show that $G - S$ has no even component, since otherwise for a vertex u of an even component of $G - S$, we have by Lemma 2.26 that

$$odd(G - (S \cup \{u\})) \geq odd(G - S) + 1 > |S| + 1 = |S \cup \{u\}|,$$

contrary to (2.12). We shall next show that every odd component of $G - S$ is factor-critical. Let C be an odd component of $G - S$, and v any vertex of C. Then for every $X \subseteq V(C - v)$, (2.12) implies

$$\begin{aligned} |S| + 1 + |X| &\geq odd(G - (S \cup \{v\} \cup X)) \\ &= odd(G - S) - 1 + odd(C - (\{v\} \cup X)) \\ &> |S| - 1 + odd((C - v) - X). \end{aligned}$$

Hence $|X| + 2 > odd((C - v) - X)$, which implies $odd((C - v) - X) \leq |X|$ by (2.9). Therefore $C - v$ has a 1-factor by the 1-factor theorem, and so C is factor-critical. Consequently,

$$\text{the number of factor-critical components of } G - S = odd(G - S) > |S|.$$

This contradicts the assumption, and thus the theorem is proved. □

The next theorem gives a necessary and sufficient condition for a tree to have a 1-factor, and the proof presented here is due to Amahashi [14].

Theorem 2.29 (Chungphaisan). *A tree T of even order has a 1-factor if and only if $odd(T - v) = 1$ for every vertex v of T.*

Proof. Suppose that T has a 1-factor F. Then for every vertex v of T, let w be the vertex of T joined to v by an edge of F. It follows that the component of $T - v$ containing w is odd, and all the other components of $T - v$ are even (Fig. 2.25). Hence $odd(T - v) = 1$.

Suppose that $odd(T - v) = 1$ for every $v \in V(T)$. It is obvious that for each edge e of T, $T - e$ has exactly two components, and both of them are simultaneously odd or even. Define a set F of edges of T as follows:

$$F = \{e \in E(T) \, : \, odd(T - e) = 2\}.$$

For every vertex v of T, there exists exactly one edge e that is incident with v and satisfies $odd(T - e) = 2$ since $T - v$ has exactly one odd component, where e is the edge joining v to this odd component (Fig. 2.25). Therefore e is an edge of F, and thus F is a 1-factor of G. □

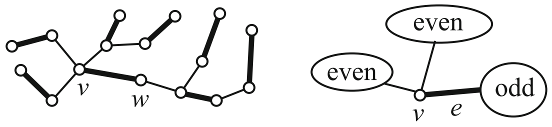


Fig. 2.25. A tree having a 1-factor, and a tree T satisfying $odd(T - v) = 1$.

We now give a variety of requirements for a graphs to have a 1-factor. A graph G is said to be **1-extendable** if for every edge e, G has a 1-factor containing e. More generally, for an integer $n \geq 1$, a graph G is said to be n**-extendable** if every matching of size n in G can be extended to a 1-factor of G.

Theorem 2.30 (Little, Grant and Holton, [168]). *A simple graph G is 1-extendable if and only if*

$$odd(G - S) \leq |S| - 2 \qquad \text{for all} \quad S \subset V(G) \tag{2.13}$$

such that $\langle S \rangle_G$ contains an edge.

Proof. We first prove the necessity. Let $S \subseteq V(G)$ such that $\langle S \rangle_G$ contains an edge, say $e = xy$ $(x, y \in S)$. Since G has a 1-factor F containing e, for each odd component C of $G - S$, there exists an edge in F joining C to a vertex in $S - \{x, y\}$, and hence

$$odd(G - S) \leq e_F(G - S, S - \{x, y\}) \leq |S| - 2.$$

We next prove the sufficiency. Let $e = xy$ $(x, y \in V(G))$ be any edge of G. We shall show that $G - \{x, y\}$ has a 1-factor, which obviously implies that G has a 1-factor containing e.

Let $S \subseteq V(G) - \{x, y\}$. Since $\langle S \cup \{x, y\} \rangle_G$ contains an edge e, we have

$$odd((G - \{x, y\}) - S) = odd(G - (S \cup \{x, y\})) \leq |S \cup \{x, y\}| - 2 = |S|.$$

Therefore $G - \{x, y\}$ has a 1-factor by the 1-factor theorem. $\square$

A criterion for a graph to be n-extendable is given in the following theorem. Since this theorem can be proved in a similar fashion to the proof of the above theorem, we omit the proof (Problem 2.10).

Theorem 2.31 (Chen [47]). *Let $n \geq 1$ be an integer, and G be a simple graph. Then G is n-extendable if and only if*

$$odd(G - S) \leq |S| - 2n \qquad \text{for all} \;\; S \subset V(G) \;\text{with}\; \alpha'(\langle S \rangle_G) \geq n, \tag{2.14}$$

where $\alpha'(\langle S \rangle_G)$ denotes the size of a maximum matching in $\langle S \rangle_G$.

Similarly, we consider the following problem. When does a graph G possess the property that for every edge e, G has a 1-factor excluding e? The answer to this question is given in the next theorem. An edge e of a connected graph G is called an **odd-bridge** if e is a bridge of G and $G-e$ consists of two odd components. In particular, such a graph G has even order. The next theorem was obtained by C. Chen.

Theorem 2.32 (Chen [45],[47]). *Let G be a connected simple graph. Then for every edge e of G, G has a 1-factor excluding e if and only if*

$$odd(G-S) \le |S| - \epsilon_2 \qquad \text{for all} \quad S \subset V(G), \tag{2.15}$$

where $\epsilon_2 = 2$ if $G-S$ has a component containing an odd-bridge; otherwise $\epsilon_2 = 0$.

Proof. We first prove the necessity. Let $S \subseteq V(G)$. Then $odd(G-S) \le |S|$ by the 1-factor theorem. Assume that $G-S$ has a component D containing an odd-bridge e. Consider a 1-factor F of G excluding e. Then for each odd component C of $G-S$, F contains at least one edge joining C to S. Furthermore, for each odd component C' of $D-e$, at least one edge of F joins C' to S. Hence

$$odd(G-S) + 2 \le e_F(G-S, S) \le |S|.$$

Consequently, (2.15) holds.

Conversely, assume that G satisfies (2.15). We shall show that for any edge e of G, $G-e$ has a 1-factor, which is of course a 1-factor of G excluding e. Let $S \subseteq V(G-e) = V(G)$. If e is an odd-bridge of an even component of $G-S$, then $odd(G-e-S) = odd(G-S)+2$; otherwise $odd(G-e-S) = odd(G-S)$. For example, if e is a bridge of an odd component C of $G-S$, then $C-e$ has exactly one odd component, and so $odd(G-S-e) = odd(G-S)$. Therefore

$$odd((G-e)-S) = odd(G-S-e) = odd(G-S) + \epsilon_2 \le |S|.$$

Hence $G-e$ has a 1-factor. $\square$

Theorem 2.33 (Corollary 1.6 of [32]). *Let G be a simple graph and W a vertex set of G. Then G has a matching that saturates W if and only if*

$$odd(G-S|W) \le |S| \qquad \text{for all} \quad S \subseteq V(G), \tag{2.16}$$

where $odd(G-S|W)$ denotes the number of those odd components of $G-S$ whose vertices are all contained in W (Fig. 2.26).

Proof. Assume that G has a matching M which saturates W. Then for every odd component C of $G-S$ such that $V(C) \subseteq W$, at least one edge of M joins C to S. Thus $odd(G-S|W) \le e_M(V(G)-S, S) \le |S|$.

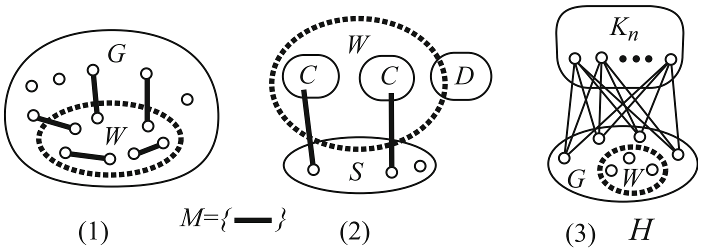


Fig. 2.26. (1) A matching M saturating W; (2) Each odd component C of $G-S$ is counted in $odd(G-S|W)$ but is not an odd component D of $G-S$; and (3) the graph H.

We next prove the sufficiency. By the 1-factor theorem, we may assume that W is a proper subset of $V(G)$, and so $V(G)-W \neq \emptyset$. Let $n = |G|$. We construct a new graph H from G by adding the complete graph K_n and by joining every vertex in $V(G)-W$ to every vertex of K_n (Fig. 2.26). Then H has even order, and it is easy to see that G has a matching saturating W if and only if H has a 1-factor.

Let $X \subseteq V(H)$. If $V(K_n) \subseteq X$, then $odd(H-X) \le |G| \le |X|$. If $V(K_n) \not\subseteq X$, then since $V(H)-(X\cup W)$ is contained in a component of $H-X$, we have by (2.16) that

$$odd(H-X) \le odd(G-V(G)\cap X|W)+1 \le |V(G)\cap X|+1 \le |X|+1,$$

which implies $odd(H-X) \le |X|$ by (2.9). Therefore H has a 1-factor, and thus G has the desired matching saturating W. □

For a graph G,

$$\mathrm{def}(G) = \max_{X\subset V(G)} \{odd(G-X)-|X|\}$$

is called the **deficiency** of G. Note that the deficiency is non-negative as $odd(G-\emptyset)-|\emptyset| \ge 0$. This concept is introduced in the next theorem.

Theorem 2.34 (Berge [26]). *Let M be a maximum matching in a simple graph G. Then the number $|M|$ of vertices saturated by M is given by*

$$|M| = |G| - \max_{X\subset V(G)} \{odd(G-X)-|X|\}. \tag{2.17}$$

Proof. The proof given here is due to Bollobás [32]. Let d be the deficiency of G and S a subset of $V(G)$ such that $odd(G-S)-|S| = d$. Then $odd(G-X)-|X| \le d$ for every $X \subset V(G)$.

We first show $|M| \le |G|-d$. Let M be a maximum matching of G. Then for any odd component C of $G-S$, if $V(C)$ is saturated by M, then at least

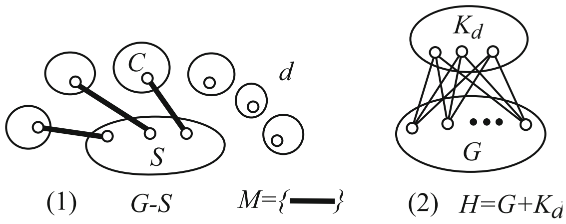


Fig. 2.27. (1) At least $odd(G-S)-|S|=d$ vertices are not saturated by M; (2) The graph $H=G+K_d$.

one edge of M joins C to S (Fig. 2.27). Thus at least $odd(G-S)-|S|=d$ odd components of $G-S$ are not saturated by M, which implies $|M|\le|G|-d$.

In order to prove the inverse inequality, we construct the join $H=G+K_d$, where K_d is the complete graph of order d. Then for every $\emptyset\ne Y\subseteq V(H)$, if $V(K_d)\not\subseteq Y$, then $odd(H-Y)\le 1\le|Y|$; and if $V(K_d)\subseteq Y$, then

$$odd(H-Y)=odd(G-V(G)\cap Y)\le|V(G)\cap Y|+d=|Y|.$$

Hence H has a 1-factor F by the 1-factor theorem. Then $F\cap E(G)=F-V(K_d)$ is a matching in G and saturates at least $|G|-d$ vertices. This implies that $|M|\ge|F\cap E(G)|\ge|G|-d$. Consequently the theorem is proved. □

2.5 Graphs having 1-factors

We shall show some classes of graphs that have 1-factors, and give some results on the sizes of maximum matchings. Among these results, the following is well-known: every $(r-1)$-edge connected r-regular graph of even order has a 1-factor that contains any given edge. The next lemma is useful when we prove the existence of 1-factors in regular graphs.

Lemma 2.35. *Let $r\ge 2$ be an integer. Let G be an r-regular general graph, and S a vertex subset of G. Then for every odd component C of $G-S$,*

$$e_G(C,S)\equiv r \pmod 2, \tag{2.18}$$

that is, $e_G(C,S)$ and r have the same parity. In particular, if G is an $(r-1)$-edge connected r-regular multigraph, then $e_G(C,S)\ge r$.

Proof. Since $|C|$ is odd, congruence (2.18) follows from

$$r\equiv r|C|=\sum_{x\in V(C)}\deg_G(x)=e_G(C,S)+2||C||\equiv e_G(C,S) \pmod 2,$$

where $||C||$ denotes the size of C. If G is an $(r-1)$-edge connected r-regular multigraph, then $e_G(C,S)\ge r-1$, and so by combining this inequality and (2.18), we have $e_G(C,S)\ge r$. □

The next theorem was first proved by Petersen in 1891 in a slightly weaker form. The stronger version, which we present here, is due to Errera and others (see Chapter 10 of [31]),

Theorem 2.36 (Petersen [209]). *Let G be a connected 3-regular general graph such that all the bridges of G, if any, are contained in a path of G. Then G has a 1-factor (Fig. 2.28). In particular, every 2-edge connected 3-regular multigraph has a 1-factor (Fig. 2.28).*

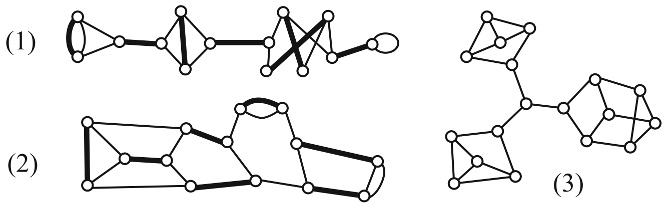


Fig. 2.28. (1) A 3-regular general graph having a 1-factor; (2) A 2-edge connected 3-regular multigraph having a 1-factor; and (3) A 3-regular simple graph having no 1-factor.

Proof. We prove only the first statement since the second statement follows immediately by noting that a 2-edge connected graph has no bridges. Let $\emptyset \neq S \subset V(G)$, and $C_1, C_2, \cdots, C_m$, $m = odd(G-S)$, be the odd components of $G-S$ such that

$$\begin{aligned} e_G(C_i, S) &= 1 \qquad \text{for all} \quad 1 \le i \le t, \quad \text{and} \\ e_G(C_j, S) &\ge 2 \qquad \text{for all} \quad t+1 \le j \le m. \end{aligned}$$

An edge joining C_i $(1 \le i \le t)$ to S is a bridge of G and is contained in a path of G, and thus $t \le 2$. By Lemma 2.35,

$$e_G(C_j, S) \ge 3 \qquad \text{for all} \qquad t+1 \le j \le m.$$

Therefore

$$3|S| \ge e_G(C_1 \cup C_2 \cup \cdots \cup C_m, S) \ge t + 3(m-t) = 3m - 2t \ge 3m - 4.$$

Hence $m \le |S| + 4/3 < |S| + 2$, which implies $m \le |S|$ by (2.9). Therefore G has a 1-factor by the 1-factor theorem. $\square$

The above theorem can be extended to r-regular graphs as follows. Note that an $(r-1)$-edge connected r-regular general graph contains no loops, and so it must be a multigraph.

Theorem 2.37 (Bäbler [22]). *Let $r \geq 2$ be an integer, and G be an $(r-1)$-edge connected r-regular multigraph of even order. Then for every edge e of G, G has a 1-factor containing e. In particular, G has a 1-factor.*

Proof. We use Theorem 2.30, which gives a necessary and sufficient condition for a multigraph to have a 1-factor containing any given edge. Let $\emptyset \neq S \subset V(G)$ such that $\langle S \rangle_G$ contains an edge. Let $C_1, C_2, \cdots, C_m$ be the odd components of $G - S$, where $m = odd(G-S)$. Then by Lemma 2.35, we have $e_G(C_i, S) \geq r$. Hence

$$r|S| = \sum_{x \in S} \deg_G(x) \geq e_G(C_1 \cup C_2 \cup \cdots \cup C_m, S) + 2 \geq rm + 2.$$

Hence $m \leq |S| - 2/r < |S|$, which implies $m \leq |S| - 2$ by (2.9). Therefore the theorem follows from Theorem 2.30. □

Theorem 2.38. *Let $r \geq 2$ be an even integer, and G be an $(r-1)$-edge connected r-regular multigraph of odd order. Then for every vertex v, $G - v$ has a 1-factor, that is, G is factor-critical.*

Proof. Let $\emptyset \neq S \subset V(G-v) = V(G) - v$. Let $C_1, C_2, \cdots, C_m$ be the odd components of $(G-v) - S = G - (S \cup \{v\})$, where $m = odd((G-v)-S)$. Then by Lemma 2.35, we have $e_G(C_i, S \cup \{v\}) \geq r$. Thus

$$r(|S|+1) = \sum_{x \in S \cup \{v\}} \deg_G(x) \geq e_G(C_1 \cup C_2 \cup \cdots \cup C_m, S \cup \{v\}) \geq rm.$$

Hence $m \leq |S| + 1$, which implies $m \leq |S|$ by (2.9). Consequently $G - v$ has a 1-factor by the 1-factor theorem. □

Theorem 2.39 (Plesník [211]). *Let $r \geq 2$ be an integer, and G an $(r-1)$-edge connected r-regular multigraph of even order. Then for any $r-1$ edges $e_1, e_2, \ldots, e_{r-1}$ of G, G has a 1-factor excluding $\{e_1, e_2, \ldots, e_{r-1}\}$.*

Proof. Let $H = G - \{e_1, \ldots, e_{r-1}\}$. It suffices to show that H has a 1-factor. Let $\emptyset \neq S \subset V(H) = V(G)$, and C_1, $C_2, \cdots, C_m$ be the odd components of $H - S$. Then by the same argument as in the proof of Lemma 2.35, we have

$$e_G(C_i, V(G) - V(C_i)) \geq r - 1 \quad \text{and} \quad e_G(C_i, V(G) - V(C_i)) \equiv r \pmod 2.$$

Hence

$$r \leq e_G(C_i, V(G) - V(C_i)) = e_G(C_i, S) + e_G(C_i, V(G) - S - V(C_i)). \tag{2.19}$$

Since C_i is a component of $H - S$, it follows that $E_G(C_i, V(G) - S - V(C_i)) \subseteq \{e_1, \ldots, e_{r-1}\}$, and it is clear that each e_j is contained in at most two such edge subsets, and thus

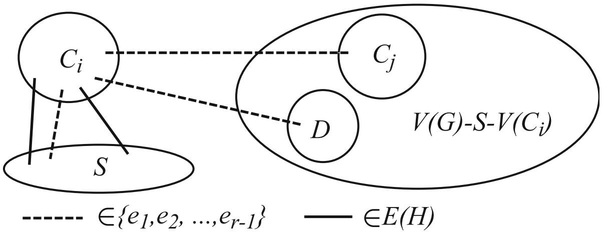


Fig. 2.29. $e_G(C_i, S)$ and $e_G(C_i, V(G) - S - V(C_i))$, where C_i and D denote an odd and even component of $H - S$, respectively.

$$\sum_{i=1}^{m} e_G(C_i, V(G) - S - V(C_i)) \leq 2\#\{e_1, e_2, \ldots, e_{r-1}\} = 2(r-1).$$

Therefore by (2.19), we have

$$\begin{aligned} rm &\leq \sum_{i=1}^{m} \big(e_G(C_i, S) + e_G(C_i, V(G) - S - V(C_i))\big) \\ &\leq \sum_{x \in S} \deg_G(x) + \sum_{i=1}^{m} e_G(C_i, V(G) - S - V(C_i)) \\ &\leq r|S| + 2(r-1). \end{aligned}$$

Hence $m \leq |S| + 2(1 - 1/r) < |S| + 2$, which implies $m \leq |S|$ by (2.9). Consequently H has a 1-factor, and the theorem follows. □

By Theorem 2.37, every $(r-1)$-edge connected r-regular multigraph of even order has a 1-factor. We can say that this result is the best in the sense that there exist infinitely many $(r-2)$-edge connected r-regular multigraphs of even order that have no 1-factor. An example is given below.

Example Let $r \geq 3$ be an odd integer. Let $\overline{K_{r-2}} = (r-2)K_1$ be the totally disconnected graph of order $r-2$ and R an $(r-2)$-edge connected simple graph of odd order such that $r-2$ vertices of R have degree $r-1$ and all the other vertices have degree r. Such an R can be obtained from the complete graph K_{r+2} by deleting one cycle with $r-2$ edges and two independent edges. It is easy to see that there exist infinitely many such graphs R for any given r.

We construct an $(r-2)$-edge connected r-regular simple graph G from $\overline{K_{r-2}}$ and r copies of R. Join each vertex of $\overline{K_{r-2}}$ to one vertex of degree $r-1$

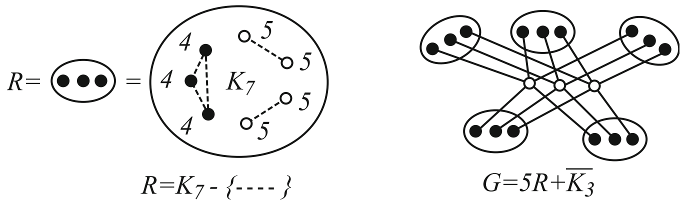


Fig. 2.30. A 3-edge connected 5-regular graph G having no 1-factors; numbers denote the degrees of vertices in R.

in every copy of R. The resulting graph is an $(r-2)$-edge connected r-regular simple graph G (Fig. 2.30). Then G has no 1-factor since

$$\begin{aligned} odd(G - V(\overline{K_{r-2}})) &= \text{the number of copies of } R \\ &= r > |V(\overline{K_{r-2}})| = r-2. \end{aligned}$$

We can similarly construct such graphs for even integers r.

The next theorem shows that an n-edge connected r-regular simple graph with small order has a 1-factor even if it is not $(r-1)$-edge connected. This result with $n = 1$ was obtained by Wallis [240], and then generalized and extended by Zhao [255], Klinkenberg and Volkmann [154] and Volkmann [237].

Theorem 2.40 (Wallis [240]). *Let $r \geq 2$ be an integer, and G be an n-edge connected r-regular simple graph of even order. Define an integer $n' \in \{n, n+1\}$ so that $n' \equiv r \pmod 2$. Then the following two statements hold.*
(i) If r is odd and

$$\left\lfloor \frac{|G|-1}{r+2} \right\rfloor < \frac{2r}{r-n'},$$

then G has a 1-factor. In particular, every connected r-regular simple graph with order at most $3(r+2)$ has a 1-factor.
(ii) If r is even and

$$\left\lfloor \frac{|G|-1}{r+1} \right\rfloor < \frac{2r}{r-n'},$$

then G has a 1-factor. In particular, every connected r-regular simple graph with order at most $3(r+1)$ has a 1-factor. Furthermore, every connected 4-regular simple graph with order at most 20 has a 1-factor

Proof. We shall prove only (i) since (ii) can be proved in a similar way. Let $r \geq 3$ be an odd integer. Let $\emptyset \neq S \subset V(G)$, and $C_1, C_2, \ldots, C_m$ be the odd components of $G - S$, where $m = odd(G - S)$. If $|C_i| \leq r$, then

$$\begin{aligned} r|C_i| &= \sum_{x \in V(C_i)} \deg_G(x) = e_G(C_i, S) + 2||C_i|| \\ &\leq e_G(C_i, S) + |C_i|(|C_i|-1) \leq e_G(C_i, S) + r(|C_i|-1). \end{aligned}$$

Hence $e_G(S, C_i) \geq r$.

Since $r+1$ is an even integer, no C_i has order $r+1$. Let us define
$m_1 =$ the number of C_i with $|C_i| \leq r$; and
$m_2 =$ the number of C_i with $|C_i| \geq r+2$.
Then $m = m_1 + m_2$, and $e_G(S, C_i) \geq r$ if $|C_i| \leq r$, and by Lemma 2.35, $e_G(S, C_i) \geq n'$ if $|C_i| \geq r+2$. Hence we have

$$r|S| \geq e_G(S, C_1 \cup C_2 \cup \cdots \cup C_m) \geq rm_1 + n'm_2 = rm + (n' - r)m_2,$$

which implies $m \leq |S| + (r - n')m_2/r$. Therefore the inequality $m < |S| + 2$ holds if

$$m_2 < \frac{2r}{r - n'}. \tag{2.20}$$

On the other hand, it follows that

$$|G| \geq |S| + |C_1| + \cdots + |C_m| \geq |S| + (r+2)m_2,$$

which implies

$$m_2 \leq \left\lfloor \frac{|G| - |S|}{r+2} \right\rfloor \leq \left\lfloor \frac{|G| - 1}{r+2} \right\rfloor.$$

Consequently, if the following inequality holds, then inequality (2.20) holds, which implies $m < |S| + 2$ and G has a 1-factor by (2.9).

$$\left\lfloor \frac{|G| - 1}{r+2} \right\rfloor < \frac{2r}{r - n'}.$$

Therefore the first part of (i) is proved.

We next prove the second part of (i). If $|G| \leq 3(r+2)$ and $n' = 1$, then

$$\left\lfloor \frac{|G| - 1}{r+2} \right\rfloor = 2 < \frac{2r}{r-1} = \frac{2r}{r - n'}.$$

Hence G has a 1-factor by the first part of (i).

Notice that when r is even, every r-regular graph is 2-edge connected. Thus we can apply the result with $n' = 2$ and obtain the latter part of (ii). $\square$

We next identify some non-regular graphs which have 1-factors. The next theorem uses the binding number of a graph to give a sufficient condition for a graph to have a 1-factor.

Theorem 2.41 (Woodall [249]). *Let G be a connected simple graph of even order. If for every $\emptyset \neq S \subseteq V(G)$,*

$$N_G(S) = V(G) \qquad or \qquad |N_G(S)| > \frac{4}{3}|S| - 1, \tag{2.21}$$

then G has a 1-factor. In particular, a connected simple graph H with bind$(H) \geq 4/3$ has a 1-factor.

Proof. The proof is by contradiction. Assume that G has no 1-factor. Then by the 1-factor theorem and by (2.9), there exists a subset $\emptyset \neq S \subset V(G)$ such that $odd(G-S) \geq |S|+2$.

Let X be the set of isolated vertices of $G-S$, and $C_1, C_2, \ldots, C_k$ the odd components of $G-S$ with order at least three. Let $V = V(G)$ and $Y = V(C_1) \cup V(C_2) \cup \cdots \cup V(C_k)$ (Fig. 2.31). Then

$$\begin{aligned} &odd(G-S) = |X| + k \geq |S| + 2, \qquad |Y| \geq 3k \quad \text{and} \\ &|V| \geq |S| + |X| + |Y|. \end{aligned} \tag{2.22}$$

We consider two cases:

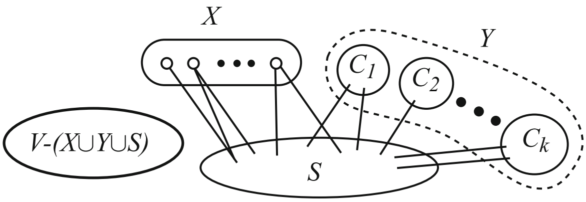


Fig. 2.31. The graph G with a subset S.

Case 1. $X \neq \emptyset$.

Since $N_G(V-S) \subseteq V - X \neq V$, it follows from (2.21) that

$$|N_G(V-S)| > \frac{4}{3}|V-S| - 1 = \frac{4|V|}{3} - \frac{4}{3}|S| - 1$$

and

$$|N_G(V-S)| \leq |V-X| = |V| - |X|.$$

By the previous two inequalities, we obtain

$$|V| < 4|S| + 3 - 3|X|. \tag{2.23}$$

On the other hand, it follows from (2.22) that

$$\begin{aligned} |V| &\geq |S| + |X| + |Y| \geq |S| + |X| + 3k \\ &\geq |S| + |X| + 3(|S| + 2 - |X|) \\ &= 4|S| - 2|X| + 6. \end{aligned}$$

Hence, by this inequality and (2.23), we have

$$4|S| - 2|X| + 6 \leq |V| < 4|S| + 3 - 3|X|,$$

which is a contradiction.

Case 2. $X = \emptyset$.

In this case $odd(G-S) = k$. Let $Z = V(C_2) \cup \cdots \cup V(C_k)$. Since $N_G(Z) \subseteq V(G) - V(C_1) \neq V(G)$, it follows from (2.21) that

$$\frac{4}{3}|Z| - 1 < |N_G(Z)| \leq |Z| + |S|.$$

Thus $|Z| < 3|S| + 3$. On the other hand, by (2.22) we have that

$$|Z| \geq 3(k-1) = 3(odd(G-S) - 1) \geq 3(|S| + 2 - 1).$$

Therefore

$$3(|S| + 1) \leq |Z| < 3|S| + 3.$$

This is again a contradiction. Consequently the theorem is proved. □

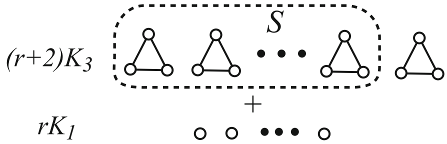


Fig. 2.32. A graph $G = (r+2)K_3 + rK_1$, which has no 1-factor and whose binding number is $(4/3) - (1/3r)$.

Consider a graph $G = (r+2)K_3 + rK_1$, where $r \geq 1$ is an integer (Fig. 2.32). Then we can easily show that G has no 1-factor, and setting $S = V((r+1)K_3)$, we have

$$bind(G) = \frac{|N_G(S)|}{|S|} = \frac{|V(G) - V(K_3)|}{|S|} = \frac{4(r+1)-1}{3(r+1)} = \frac{4}{3} - \frac{1}{3(r+1)}.$$

Hence the condition of Theorem 2.41 is sharp.

Recall that if G is t-tough, then $tough(G) \geq t$ and for a subset $S \subset V(G)$ with $\omega(G-S) \geq 2$, it follows that

$$\omega(G-S) \leq \frac{|S|}{tough(G)} \leq \frac{|S|}{t}.$$

The next theorem uses the idea of toughness to give a sufficient condition for a graph to have a 1-factor.

Theorem 2.42 (Chvátal [54], Exercise 3.4.11 of [182]). *Every 1-tough connected simple graph of even order has a 1-factor. Moreover, for every real number $\epsilon > 0$, there exist simple graphs G of even order that have no 1-factors and satisfy $tough(G) > 1 - \epsilon$.*

Proof. Let G be a 1-tough connected simple graph of even order, and $\emptyset \neq S \subset V(G)$. If $\omega(G-S) \geq 2$, then

$$odd(G-S) \leq \omega(G-S) \leq \frac{|S|}{tough(G)} \leq |S|.$$

If $\omega(G-S) = 1$, then $odd(G-S) \leq \omega(G-S) = 1 \leq |S|$. Hence $odd(G-S) \leq |S|$ always holds, and thus G has a 1-factor by the 1-factor theorem.

Let $G = K_m + tK_n$, which is a join of the complete graph K_m and the t copies of the complete graph K_n, where n is an odd integer (Fig. 2.33 (1)). Moreover, we can choose m, n and t so that

$$t > m, \quad \frac{m}{t} > 1 - \epsilon \quad \text{and} \quad m + nt \equiv 0 \pmod 2.$$

Then G has no 1-factor since $odd(G - V(K_m)) = t > |V(K_m)| = m$, and

$$tough(G) = \frac{|V(K_m)|}{\omega(G - V(K_m))} = \frac{m}{t} > 1 - \epsilon.$$

Therefore the theorem is proved. □

Recall that a graph G is said to be **claw-free** if G contains no induced subgraph isomorphic to the claw $K_{1,3}$ (Fig. 2.33 (2)).

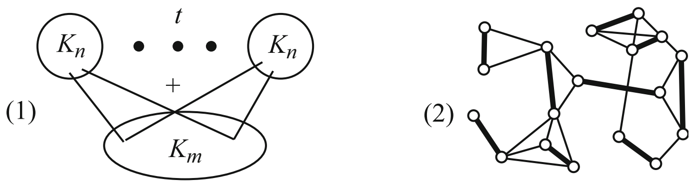


Fig. 2.33. (1) $K_m + tK_n$; (2) A claw-free graph and its 1-factor.

Theorem 2.43 (Sumner [218], Las Vergnas [164]). *Every connected claw-free simple graph of even order has a 1-factor.*

Proof. We prove the theorem by induction on $|G|$. We may assume $|G| \geq 3$. Let $P = (v_1, v_2, \ldots, v_k)$, $(v_i \in V(G))$ be a longest path of G. Then $k \geq 3$. It is immediate that $N_G(v_1) \subseteq V(P) - \{v_1\}$.

We shall show that $G - \{v_1, v_2\}$ is connected. If $\deg_G(v_2) = 2$, then $G - \{v_1, v_2\}$ is connected since $N_G(v_1) \subseteq V(P) - \{v_1\}$. Thus we may assume that $\deg_G(v_2) \geq 3$. For every $x \in N_G(v_2) - V(P)$, xv_3 must be an edge of G, since otherwise $\langle \{v_2, v_1, v_3, x\} \rangle_G \neq K_{1,3}$ implies that v_1x or v_1v_3 is an edge of G. Hence G contains a path longer than P, a contradiction. Therefore $G - \{v_1, v_2\}$ is a connected graph of even order, and is of course claw-free. By the induction hypothesis, $G - \{v_1, v_2\}$ has a 1-factor, and so does G. □

It is known that every 4-connected planar graph has a Hamiltonian cycle [229], which implies that it has a 1-factor if it is of even order, but, there are infinitely many 3-connected planar graphs of even order that have no 1-factors. The next theorem gives a lower bound for the order of a maximum matching of a planar graph. The proof given here seems to be different from that of [203].

Theorem 2.44 (Nishizeki and Baybars [203]). *Let G be a connected planar simple graph with $\delta(G) \geq 3$. Then the number $|M|$ of vertices saturated by a maximum matching M in G is*

$$|M| \geq \frac{2|G|+4}{3}. \tag{2.24}$$

If G is 2-connected, then

$$|M| \geq \frac{2|G|+8}{3}. \tag{2.25}$$

Proof. We first prove (2.25). Let G be a 2-connected planar simple graph. We may assume that G is drawn in the plane as a plane graph. Let $\emptyset \neq S \subset V(G)$. Let us denote by X the set of isolated vertices of $G-S$, and by $C_1, C_2, \ldots, C_m$ the odd components of $G-S$ of order at least three. Then

$$odd(G-S) = |X| + m, \qquad \text{and} \qquad |G| \geq |S| + |X| + 3m. \tag{2.26}$$

If $|S| = 1$, then $odd(G-S) \leq 1 = |S|$ since G is 2-connected. So $odd(G-S) - |S| \leq 0$. Assume $|S| \geq 2$. We construct a planar bipartite graph B with bipartite sets S and $X \cup \{v_1, v_2, \ldots, v_m\}$ from G by contracting $C_1, C_2, \ldots, C_m$ into single vertices $v_1, v_2, \ldots, v_m$ and by replacing all multiple edges in the resulting graph by single edges, that is, $s \in S$ and v_i are joined by an edge of B if and only if s and C_i are joined by an edge of G (Fig. 2.34).

Since G is a 2-connected graph with $\delta(G) \geq 3$, it follows that

$$\begin{aligned} \deg_B(x) &\geq 3 \qquad \text{for all} \quad x \in X, \qquad \text{and} \\ \deg_B(v_i) &\geq 2 \qquad \text{for all} \quad 1 \leq i \leq m. \end{aligned} \tag{2.27}$$

Since B is a planar bipartite graph, it follows from Theorem 1.10 that

$$||B|| \leq 2|B| - 4.$$

By combining this inequality and (2.27), we obtain

$$3|X| + 2m \leq ||B|| \leq 2|B| - 4 = 2(|S| + |X| + m) - 4.$$

Hence $|X| - 2|S| \leq -4$DBy this inequality and (2.26), we get

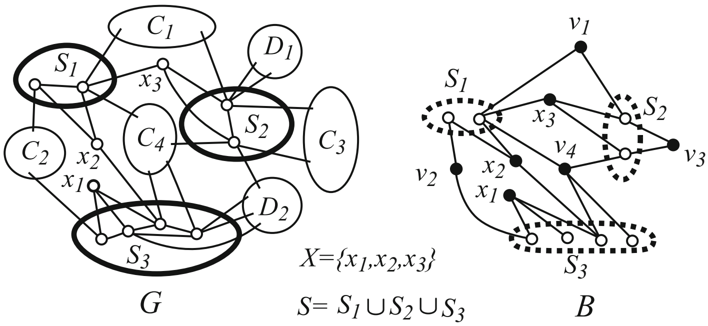


Fig. 2.34. G is a 2-connected plane graph, x_1, x_2, x_3 are the isolated vertices of $G - S$, $C_1, \ldots, C_4$ are the odd components of $G - S$ with order at least 3, and D_1, D_2 are the even components of $G - S$. B is the corresponding planar bipartite graph.

$$
\begin{aligned}
& odd(G-S) - |S| = |X| + m - |S| \\
& \leq |X| + \frac{|G| - |S| - |X|}{3} - |S| = \frac{|G| + 2(|X| - 2|S|)}{3} \\
& \leq \frac{|G| - 8}{3}.
\end{aligned}
$$

Consequently by (2.17) in Theorem 2.34, we have

$$
|M| = |G| - \max_S(odd(G-S) - |S|) \geq |G| - \frac{|G| - 8}{3} = \frac{2|G| + 8}{3},
$$

which implies the desired inequality (2.25).

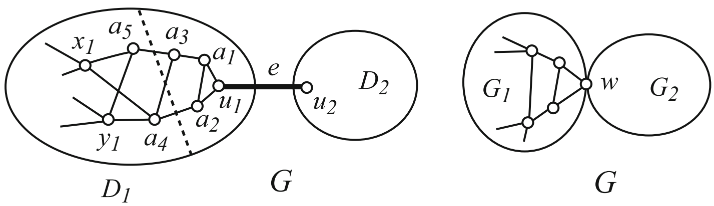


Fig. 2.35. A connected plane graph G with bridge e; and a connected plane graph G with cut vertex w.

We next prove (2.24) by induction on $|G|$, where G is a connected plane graph with $\delta(G) \geq 3$. Since (2.24) holds for a small graph G, we may assume that the order of G is not small (for example, $|G| \geq 10$). Suppose first that G has a bridge $e = u_1u_2$ (Fig. 2.35). Let D_1 and D_2 be the two components of $G - e$ containing u_1 and u_2, respectively. We shall later show that each D_i has a matching that covers at least $(2|D_i| + 2)/3$ vertices of D_i. If the above statement holds, then G has a matching that covers at least

$$\frac{2|D_1| + 2}{3} + \frac{2|D_2| + 2}{3} = \frac{2|G| + 4}{3} \quad \text{vertices.}$$

Hence (2.24) holds.

If G has no bridges but has a cut vertex w, then by letting $G = G_1 \cup G_2,\ V(G_1) \cap V(G_2) = \{w\}$ (see Fig. 2.35), we shall show that each G_i has a matching that covers at least $(2|G_i| + 4)/3$ vertices of G_i. Then G has a matching that covers at least the following number of vertices:

$$\frac{2|G_1| + 4}{3} + \frac{2|G_2| + 4}{3} - 2 = \frac{2(|G_1| + |G_2| - 1) + 4}{3} = \frac{2|G| + 4}{3}.$$

Hence (2.24) holds.

Let D_1 and D_2 be the two components of $G - e$, where e is a bridge of G. We shall show that each D_i has a matching that covers at least $(2|D_i| + 2)/3$ vertices of D_i by using the inductive hypothesis of the theorem. If $\delta(D_i) \geq 3$, then the above statement holds by induction. Without loss of generality, we may assume $\delta(D_1) = 2$, which implies $\deg_{D_1}(u_1) = 2$ as $\delta(G) \geq 3$ (see Fig. 2.35). Let a_1 and a_2 be the two vertices adjacent to u_1. If $\delta(D_1 - u_1) \geq 3$, then by induction, $D_1 - u_1$ has a matching that covers at least

$$\frac{2(|D_1| - 1) + 4}{3} = \frac{2|D_1| + 2}{3} \quad \text{vertices.}$$

Note that if $D_1 - u_1$ is disconnected, we apply the inductive hypothesis to each component and obtain the above desired matching of $D_1 - u_1$. Thus we may assume that $\delta(D_1 - u_1) = 2$. If a_1 and a_2 are not adjacent, then $D_1 - u_1 + a_1a_2$ has minimum degree three, and so by induction it has a matching M_1 covering at least $(2(|D_1| - 1) + 4)/3$ vertices. If M_1 contains an edge a_1a_2, then by considering $M_1 - a_1a_2 + a_1u_1$ we can get the desired matching of D_1. So we may assume that a_1 and a_2 are adjacent. If $\deg_{D_1 - u_1}(a_i) \geq 3$ for $i = 1, 2$, then we can apply the inductive hypothesis to $D_1 - u_1$ and obtain the desired statement as above. Thus we may assume that $\deg_{D_1 - u_1}(a_1) = 2$.

If $\delta(D_1 - u_1 - a_1) \geq 3$, then by induction it has a matching M_2 covering at least $(2(|D_1| - 2) + 4)/3$ vertices of $D_1 - u_1 - a_1$. Hence $M_2 + a_1u_1$ is the desired matching in D_1. By repeating this argument, we can finally find the desired matching of D_1. As an example, in Fig. 2.35, $\delta(D_1 - \{u_1, a_1, a_2, a_3\} + a_4a_5) \geq 3$ and M_3 is its matching covering at least $2(|D_1| - 4) + 4)/3$ vertices, and if $a_4a_5 \in M_3$, then $M_3 - a_4a_5 + a_5a_3 + a_1u_1 + a_4a_2$ is the desired matching in D_1.

Let $G = G_1 \cup G_2,\ V(G_1) \cap V(G_2) = \{w\}$, where w is a cut vertex of G. We shall show that each G_i has a matching that covers at least $(2|G_i| + 4)/3$ vertices of G_i by using the inductive hypothesis of the theorem. We can do this by a similar argument as above, Consequently the proof is complete. $\square$

A surface (a compact orientable 2-manifold) is a sphere on which a number of handles have been placed. The number of handles is referred to as the **genus** of the surface (Fig. 2.36). A surface with genus one is called a **torus**, and, of course, a plane is a surface with genus zero. A graph G is said to be embedded on a surface if G can be drawn on the surface in such a way that edges intersect only at their common end-points. The genus $\gamma(G)$ of a simple graph G is defined to be smallest genus of all surfaces on which G can be embedded.

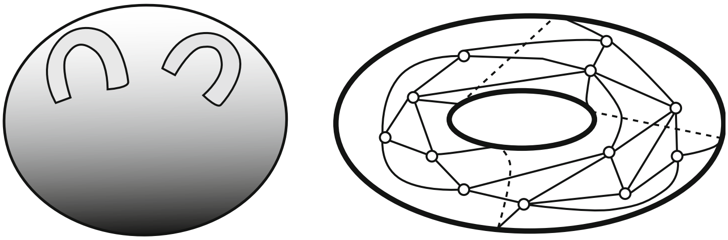


Fig. 2.36. A surface with genus two, and an embedding of a graph on the torus.

Theorem 2.45 (Nishizeki [202]). *Let G be an n-connected simple graph of even order. If $\gamma(G) < n(n-2)/4$, then G has a 1-factor. In particular, every 4-connected simple graph of even order which is embedded on the torus has a 1-factor.*

2.6 Structure theorem

We now consider the structure of a graph without 1-factors, and characterize such a graph by using vertex-decomposition together with certain properties. The resulting theorem is called the Gallai-Edmonds structure theorem and gives us much information about maximum matchings.

Let G be a graph. For a subset $X \subseteq V(G)$, $odd(G-X)-|X|$ is called the **deficiency** of X, and a subset $S \subseteq V(G)$ is called a **barrier** if

$$odd(G-S)-|S| = \mathrm{def}(G) = \max_{X \subseteq V(G)} \{odd(G-X)-|X|\}. \qquad (2.28)$$

That is, S is a barrier if its **deficiency** is equal to that of G. A barrier S is said to be **minimal** if no proper subset of S is a barrier.

Theorem 2.46. *Suppose that a connected simple graph G with even order has no 1-factor. Let S be a minimal barrier of G. Then every vertex $x \in S$ is joined to at least three odd components of $G-S$. In particular, x is the center of a certain induced claw subgraph of G.*

Proof. Since G has no 1-factor and has even order, we have $\mathrm{def}(G) \geq 2$ by the 1-factor theorem and Lemma 2.26. If $S = \{x\}$, then $odd(G - S) \geq 3$ since $odd(G - S) \equiv |S| \pmod 2$. Hence $x \in S$ is joined to at least three odd components of $G - S$ and the theorem holds. Thus we may assume that $|S| \geq 2$.

Suppose that a vertex $x \in S$ is joined to at most two odd components of $G - S$ (Fig. 2.37). Then $odd(G - (S - x)) \geq odd(G - S) - 2$, and thus

$$odd(G - (S - x)) - |S - x| \geq odd(G - S) - |S| - 1.$$

By Lemma 2.26, we have

$$odd(G - (S - x)) - |S - x| \equiv odd(G - S) - |S| \equiv |G| \pmod 2.$$

Hence

$$odd(G - (S - x)) - |S - x| \geq odd(G - S) - |S|,$$

which implies that $S - x$ is also a barrier. This contradicts the minimality of S. Therefore every $x \in S$ is joined to at least three odd components of $G - S$.

By taking three vertices adjacent to x from each of the three odd components of $G - S$, we can obtain an induced subgraph $K_{1,3}$ with center x. □

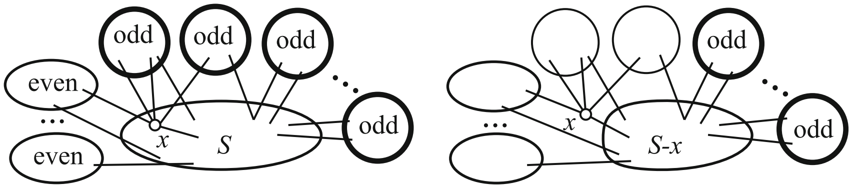


Fig. 2.37. Components of $G - S$ and those of $G - (S - x)$.

It is clear that Theorem 2.43, which says that every connected claw-free simple graph of even order has a 1-factor, is an immediate consequence of Theorem 2.46.

Consider a simple graph G. Let $D(G)$ denote the set of all vertices v of G such that v is not saturated by at least one maximum matching of G. Let $A(G)$ be the set of vertices of $V(G) - D(G)$ that are adjacent to at least one vertex in $D(G)$. Finally, define $C(G) = V(G) - D(G) - A(G)$. Then $V(G)$ is decomposed into three disjoint subsets

$$V(G) = D(G) \cup A(G) \cup C(G), \tag{2.29}$$

where

$$\begin{aligned}D(G) &= \{x \in V(G) \;:\; \text{There exists a maximum matching} \\ &\qquad\qquad\qquad\qquad \text{that does not saturate } x\}\\ A(G) &= N_G(D(G)) \setminus D(G)\\ C(G) &= V(G) - A(G) - D(G).\end{aligned}$$

Some properties of the above decomposition are given in the following Gallai-Edmonds structure theorem. This theorem was obtained by Gallai [92], [93] and Edmonds [62] independently and in different ways. The proof presented here is based on [120].

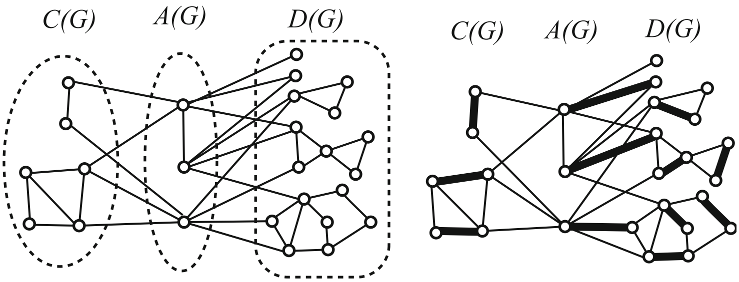


Fig. 2.38. The decomposition $V(G) = D(G) \cup A(G) \cup C(G)$ and a maximum matching of G.

Theorem 2.47 (Gallai-Edmonds Structure Theorem, [92], [93], [62] **).** *Let G be a simple graph, and $V(G) = D(G) \cup A(G) \cup C(G)$ the decomposition defined in (2.29). Then the following statements hold (Fig. 2.38):*
(i) Every component of $\langle D(G)\rangle_G$ is factor-critical.
(ii) $\langle C(G)\rangle_G$ has a 1-factor.
(iii) Every maximum matching M in G saturates $C(G) \cup A(G)$, and every edge of M incident with $A(G)$ joins a vertex in $A(G)$ to a vertex in $D(G)$.
(iv) The number $|M|$ of vertices saturated by a maximum matching M is given by

$$|M| = |G| + \omega(\langle D(G)\rangle_G) - |A(G)|, \tag{2.30}$$

where $\omega(\langle D(G)\rangle_G)$ denotes the number of components of $\langle D(G)\rangle_G$.

Proof. The proof of the theorem is by induction on $|G|$. We may assume that G is connected since otherwise each component of G satisfies the statements of the theorem and so does G. Moreover we may assume that G has no 1-factor since otherwise $D(G) = \emptyset$, $A(G) = \emptyset$ and $C(G) = V(G)$, and thus the theorem holds.

Let S be a maximal barrier of G, that is, S is a subset of $V(G)$ such that

$$odd(G - S) - |S| = \max_{X \subset V(G)} \{odd(G - X) - |X|\} = \mathrm{def}(G) > 0 \tag{2.31}$$

and

$$odd(G - Y) - |Y| < odd(G - S) - |S| \quad \text{for all} \quad S \subset Y \subseteq V(G). \tag{2.32}$$

The following claim can be proved in the same way as in the proof of the 1-factor theorem (see Problem 2.9).

Claim 1. *Every component of $G - S$ is factor-critical.*

Let $\{C_1, C_2, \ldots, C_m\}$ be the set of components of $G - S$, where $m = odd(G - S)$. We define a bipartite graph B with bipartition $S \cup \{C_1, C_2, \ldots, C_m\}$ as follows: a vertex $x \in S$ and C_i is joined by an edge of B if and only if x and C_i are joined by at least one edge of G (Fig. 2.39). Then B satisfies the following claim.

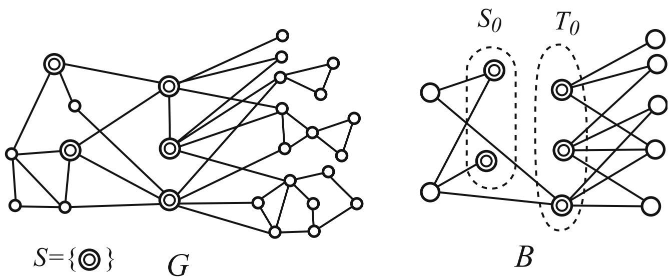


Fig. 2.39. A graph G with a maximal barrier S, and the bipartite graph B with subsets S_0 and T_0.

Claim 2. $|N_B(X)| \geq |X|$ *for all* $X \subseteq S$.

Assume that $|N_B(Y)| < |Y|$ for some $\emptyset \neq Y \subseteq S$. Then

$$\begin{aligned} & odd(G - (S - Y)) - |S - Y| \\ \geq\ & |\{C_1, C_2, \ldots, C_m\} - N_B(Y)| - |S - Y| \\ >\ & m - |Y| - |S - Y| = m - |S| = odd(G - S) - |S|, \end{aligned}$$

which contradicts (2.31). Hence Claim 2 holds.

Claim 3. *There exists a unique maximum proper subset $S_0 \subset S$ such that $|N_B(S_0)| = |S_0|$. Furthermore, $|N_B(Y) \setminus N_B(S_0)| > |Y|$ for every $\emptyset \neq Y \subseteq S - S_0$.*

By $|N_B(S)| = m > |S|$, Claim 2 and by Theorem 2.9, there exists a unique maximum proper subset $S_0 \subset S$ such that $|N_B(S_0)| = |S_0|$.

Let $\emptyset \neq Y \subseteq S - S_0$. Then it follows from the maximality of S_0 and $S_0 \subset Y \cup S_0$ that

$$|N_B(Y) \setminus N_B(S_0)| = |N_B(Y \cup S_0) - N_B(S_0)| > |Y \cup S_0| - |S_0| = |Y|.$$

Therefore the claim is proved.

Let $T_0 = S - S_0$. Then

$$\begin{aligned} odd(G - T_0) &= odd(G - S) - |N_B(S_0)| = odd(G - S) - |S_0| \\ &= odd(G - S) - (|S| - |T_0|), \end{aligned}$$

and so $odd(G - T_0) - |T_0| = odd(G - S) - |S| = \mathrm{def}(G)$.

Let $\{C_1', C_2', \dots, C_r'\}$ be the set of odd components of $G - S$ corresponding to $N_B(S_0)$, where $r = |N_B(S_0)| = |S_0|$, and let $\{C_1, C_2, \dots, C_k\}$ be the set of odd components of $G - T_0$. Then

$$Odd(G - S) = \{C_1', C_2', \dots, C_r'\} \cup \{C_1, C_2, \dots, C_k\},$$

and the following statements hold by the marriage theorem and Claim 3:

(i) *B has a matching saturating S;*

(ii) *every matching in B saturating S saturates $\{C_1', C_2', \dots, C_r'\}$; and*

(iii) *for each C_i $(1 \le i \le k)$, there exists a matching in B that saturates S but not C_i.*

Let H be a matching in B saturating S. Then for every odd component C_j' $(1 \le j \le r)$, there exists an edge in H joining C_j' to a vertex $x_j \in S_0$. Take an edge e_j of G joining x_j to a vertex v_j in C_j'. By Claim 4, $C_j' - v_j$ has a 1-factor R_j'.

Similarly, for an odd component C_i $(1 \le i \le k)$, if H has an edge joining C_i to $x_i \in S$, then $x_i \in T_0$ and we can find an edge e_i of G joining x_i to a vertex w_i of C_i and a 1-factor R_i of $C_i - w_i$. If H has no edge joining C_i to S, then take a maximum matching R_i in C_i. Define

$$M = \bigcup_{1 \le j \le r} (R_j' + e_j) \ + \bigcup_{1 \le i \le k} \{(R_i + e_i) \ \text{ or } \ R_i \ \}. \tag{2.33}$$

Then by Theorem 2.34, M is a maximum matching of G since the number of unsaturated vertices in M is $k - |T_0| = odd(G - T_0) - |T_0| = \mathrm{def}(G)$.

Conversely, every maximum matching in G is obtained in this way since every matching in G cannot saturate at least $k - |T_0|$ odd components in $\{C_1, C_2, \dots, C_k\}$, and every maximum matching M' does not saturate exactly $k - |T_0| = \mathrm{def}(G)$ components of $\{C_1, C_2, \dots, C_k\}$. Therefore M' induces a matching H' in B that saturates S and $\{C_1', C_2', \dots, C_r'\}$, and thus M' can be constructed from H' as above.

Claim 5. *$D(G) = V(C_1) \cup V(C_2) \cup \dots \cup V(C_k)$ and $A(G) = T_0$.*

It is clear that for every vertex v_i of any C_i $(1 \le i \le k)$, B has a matching that saturates S but not C_i, and $C_i - v_i$ has a 1-factor by Claim 1. Hence by (2.33), we can find a maximum matching in G that does not saturate v_i. Thus $V(C_1) \cup V(C_2) \cup \dots \cup V(C_k) \subseteq D(G)$. Since every maximum matching in G is obtained in the manner mentioned above, $D(G) \subseteq V(C_1) \cup V(C_2) \cup \cdots \cup V(C_k)$. Consequently, $D(G) = V(C_1) \cup V(C_2) \cup \cdots \cup V(C_k)$.

Since $N_B(S_0) = \{C'_1, C'_2, \ldots, C'_r\}$, it follows that $N_B(\{C_1, C_2, \ldots, C_k\}) = S - S_0 = T_0$. Therefore

$$A(D(G)) = N_G(D(G)) \setminus D(G) = T_0.$$

It is easy to see that $\langle V(C'_1) \cup \cdots \cup V(C'_r) \cup S_0 \rangle$ has a 1-factor and forms the even components of $G - T_0$. Consequently the proof is complete. $\square$

We mention one application of the Gallai-Edmonds structure theorem. Since every component of $\langle D(G) \rangle_G$ is factor-critical, Theorem 2.28 is an easy consequence of the Gallai-Edmonds structure theorem.

The following lemma is interesting its own right, but it is also useful for proving the Gallai-Edmonds structure theorem; we can first prove the following stability lemma without using Gallai-Edmonds structure theorem, and then apply the lemma to prove the structure theorem ([182] section 3.2). However, we shall prove the lemma using Gallai-Edmonds structure theorem since it is shorter.

Lemma 2.48 (The Stability Lemma). *Let G be a simple graph and $V(G) = C(G) \cup A(G) \cup D(G)$. Then for every vertex $u \in A(G)$, we have $A(G - u) = A(G) - u$, $C(G - u) = C(G)$ and $D(G - u) = D(G)$.*

Proof. Let $u \in A(G)$. Then every maximum matching M in G has an edge e incident with u. Thus $M - e$ is a maximum matching in $G - u$ since the size of a maximum matching in $G - u$ must be less than or equal to $||M|| - 1$ and $||M - e|| = ||M|| - 1$. Therefore $D(G) \subseteq D(G - u)$, and the size of a maximum matching in $G - u$ is $||M|| - 1$.

Assume that a maximum matching H in $G - u$ does not saturate a vertex $x \in A(G) - u$. Then H is a matching in G, and H does not saturate at least $\omega(\langle D(G) \rangle_G) - (|A(G)| - 2)$ vertices in $D(G)$ and two more vertices $x, u \in A(G)$. Therefore $|V(H)| \leq |V(M)| - 4$, and so $||H|| \leq ||M|| - 2$. This contradicts the fact that a maximum matching in $G - u$ has size $||M|| - 1$. Hence $(A(G) - u) \cap D(G - u) = \emptyset$. We can similarly show that H saturates $C(G)$, which implies $C(G) \cap D(G - u) = \emptyset$. Consequently, $D(G - u) = D(G)$.

The other equalities $A(G - u) = A(G) - u$ and $C(G - u) = C(G)$ follow immediately from $D(G - u) = D(G)$. $\square$

Since a factor-critical graph with order at least three is not a bipartite graph (Problem 2.8), if G is a bipartite graph, then every component of $\langle D(G) \rangle_G$ must be a single vertex. Thus the following theorem holds.

Theorem 2.49. *Let G be a connected bipartite simple graph with bipartition (X, Y), and let $C_X = C(G) \cap X$ and $C_Y = C(G) \cap Y$. Then the following statements hold (Fig. 2.40):*

(i) $\langle D(G) \rangle_G$ is a set of independent vertices of G.

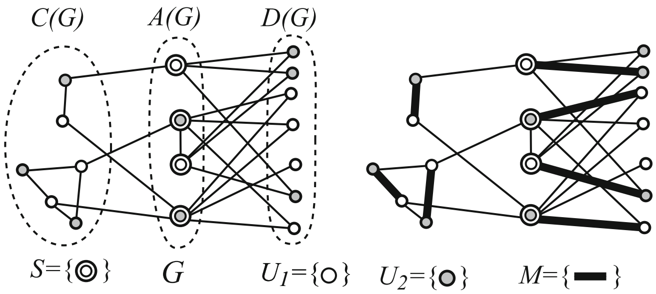


Fig. 2.40. The decomposition $V(G) = D(G) \cup A(G) \cup C(G)$ of a bipartite graph G with bipartition $U_1 \cup U_2$; and it maximum matching Mx.

(ii) $\langle C(G)\rangle_G$ *has a 1-factor, and* $|C_X| = |C_Y|$*.*
(iii) Every maximum matching M *in* G *consists of a 1-factor of* $\langle C(G)\rangle_G$ *and a matching in* $\langle A(G) \cup D(G)\rangle_G$ *saturating* $A(G)$*.*
(iv) Both $A(G) \cup C_X$ *and* $A(G) \cup C_Y$ *are minimum vertex covers of* G*.*
(v) Both $D(G) \cup C_X$ *and* $D(G) \cup C_Y$ *are maximum independent vertex subsets of* G*.*

2.7 Algorithms for maximum matchings

We gave an algorithm for finding a maximum matching in a bipartite graph in Section 2.3. In this section we shall propose an algorithm for finding a maximum matching in a simple graph, which was obtained by Edmonds [62].

Before stating the algorithm, let us recall Theorem 2.20, which says that "a matching M in a graph G is maximum if and only if G has no M-augmenting path". Therefore, to find a maximum matching, we should find M-augmenting paths or determine the non-existence of such paths. In order to effectively explore M-augmenting paths in a graph, we introduce some new concepts and notation.

We shall first explain the algorithm and new definitions by using examples. Let G be a graph and M a matching in G, and let v be a vertex unsaturated by M. We call v a **root**, and explore all the M-alternating paths starting with v. If $P = (v, x_1, x_2, x_3, x_4, \dots, x_k)$ is an M-alternating path, where each x_i is a vertex of G, then we call $x_1,\ x_3, \dots$ **inner vertices** and $v, x_2, x_4, \dots$ **outer vertices** (Fig. 2.41)

Consider the graph G and the matching M given in Fig. 2.41. We try to find all the M-alternating paths starting with v as follows:

$$\{v\}\ \ \{a, g\},\ \ \{b, h\},\ \ \{c, e, i\},\ \ \{d, f, j\}\ \text{ and }\ \{f, k\}.$$

In the last step, we find that f is simultaneously an outer and inner vertex since (u, a, b, e, f) and (u, a, b, c, d, f) are both M-alternating paths. Then we

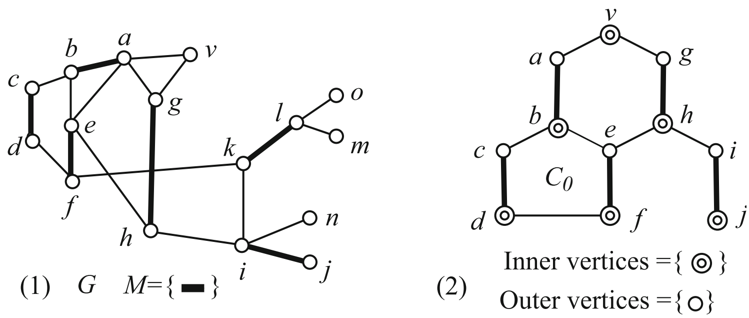


Fig. 2.41. (1) A matching M and an M-unsaturated vertex v; (2) The root v, inner vertices and outer vertices.

find an odd cycle $C_0 = (b, c, d, f, e, b)$ containing f, i.e., if we find a vertex that is simultaneously outer and inner, then there exists an odd cycle containing it, which can be easily found.

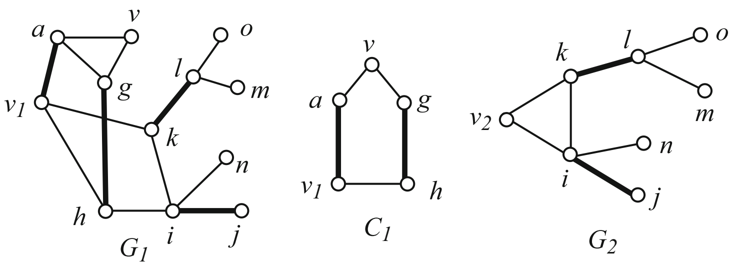


Fig. 2.42. The graph $G_1 = G/C_0$ with the matching M_1; an odd cycle $C_1 = (v, a, v_1, h, g, a)$; and the graph $G_2 = G_1/C_1$ with the matching M_2.

Next we contract C_0 into a single vertex v_1, which also implies that we delete loops and replace every multiple edge by a single edge, and denote the resulting graph by G_1. The matching of G_1 corresponding to M is obtained by deleting the edges in $C_0 \cap M$ (Fig. 2.42), i.e., we obtain

$$G_1 = G/C_0, \qquad M_1 = M - (E(C_0) \cap M) = M \cap E(G_1), \qquad v_1 = C_0.$$

In Fig. 2.42, we find an odd cycle $C_1 = (v, a, v_1, h, g)$ since h is simultaneously an outer and inner vertex in G_1. Then we obtain the new graph G_2 from G_1 by contracting C_1, where v_2 is the new root since C_1 contains the root v. In general, if the odd cycle C_i contains the root, then C_i corresponds to the new root in G_i/C_i. Otherwise, C_i corresponds to a saturated vertex in G_i/C_i.

$$G_2 = G_1/C_1, \qquad M_2 = M_1 \cap E(G_2), \qquad v_2 = C_1.$$

Next, we find an M_2-augmenting path

$$P_2 = (v_2, k, l, m) \qquad \text{in} \quad G_2.$$

From this path P_2, we can obtain an M_1-augmenting path

$$P_1 = (v, a, v_1, k, l, m) \quad \text{in} \quad G_1.$$

Since v_2 corresponds to the odd cycle C_1 in G_1, k and $v_1 \in V(C_1)$ are joined by an edge in G_1. There are two alternating paths in C_1 joining v_1 to v and one of them can be added to (k, l, m). Since v_1 corresponds to C_0 in G and v_1 can be replaced by the alternating path (a, b, e, f) in C_0, we obtain the desired M-augmenting path

$$P = (v, a, b, e, f, k, l, m) \qquad \text{in} \quad G.$$

Therefore we obtain a larger matching $M' = M \triangle E(P)$ in G (Fig. 2.43).

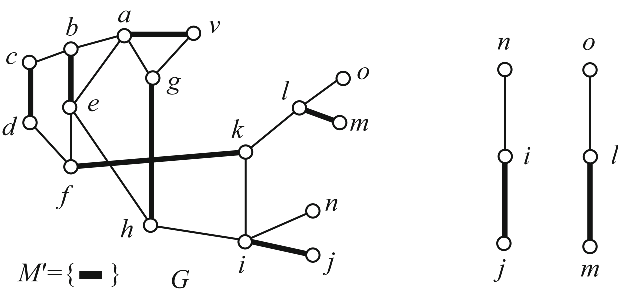


Fig. 2.43. A matching M' in G; and non-existence of M'-augmenting path in G.

There are two M'-unsaturated vertices n and o, and by the same argument as above, we can easily determine that G has no M'-augmenting paths starting with n or o, which implies that M' is a maximum matching in G.

We conclude this section with an algorithm for finding a maximum matching in a graph.

Algorithm 2.50 (Algorithm for maximum matchings) *Let G be a connected simple graph. Then a maximum matching of G can be obtained by repeating the following procedure. Let $i = 0$, $G_0 = G$, M_0 be any matching of G_0 and let v_0 be any M_0-unsaturated vertex of G_0. Initially, v_0 is the root.*

We explore all the M_i-alternating paths starting with v_i as explained above. If we find a vertex x_i that is both inner and outer, then we can find an odd cycle C_i containing x_i, and obtain a graph G_{i+1} from G_i by contracting C_i.

The contraction of C_i *is the vertex* v_{i+1} *of* G_{i+1}*. If* C_i *contains the root of* G_i*, then* v_{i+1} *is the new root of* G_{i+1} *and is unsaturated by a matching* $M_{i+1} = M_i - (E(C_i) \cap M_i)$*. Otherwise, the vertex* v_{i+1} *of* G_{i+1} *is a vertex saturated by* M_{i+1}*. Set* $i = i + 1$*, and repeat the procedure.*

If we find an M_i*-augmenting path in* G_i *connecting the root and another* M_i*-unsaturated vertex* y*, then we can find an* M_{i-1}*-augmenting path in* G_{i-1} *connecting the root of* G_{i-1} *and another* M_{i-1}*-unsaturated vertex. Set* $i = i-1$*, and repeating the above procedure until* $i = 1$*, we get the desired* M*-augmenting path starting with* v_0*. Moreover, if* G_i *has no* M_i*-augmenting path starting with the root, then* G *has no* M*-augmenting path starting with the root* v_0*.*

Some improvements on Algorithm 2.50 results in an algorithm that finds a maximum matching in $O(|G|^3)$ time ([182], Section 9).

2.8 Perfect matchings in cubic graphs

We conclude this chapter with some problems on perfect matchings in cubic graphs. Notice that matchings of a graph are considered as edge subsets of the graph, and that for a cubic simple graph, the edge connectivity is equal to the connectivity. The following conjecture due to Berge and Fulkerson was appeared first in [89] ([217]).

Conjecture 2.51 (Berge and Fulkerson [89]). Every 2-connected cubic simple graph G has six perfect matchings with the property that every edge of G is contained in precisely two of these perfect matchings.

Notice that if the edge set of a cubic graph G is decomposed into three perfect matchings M_1, M_2 and M_3, then every edge is contained in precisely one of these perfect matchings, and so by letting $M_4 = M_1$, $M_5 = M_2$ and $M_6 = M_3$, these six perfect matchings $\{M_i\}$ satisfy the above conjecture. On the other hand, Petersen graph of order 10 has perfect matchings but its edge set cannot be decomposed into three perfect matchings. However, Petersen graph has six perfect matchings having the property given in Conjecture 2.51 (see Fig. 2.44). Thus the conjecture says that every 2-connected cubic graph possesses a certain property closed to the decomposition of edge set into three perfect matchings.

If a cubic graph has six perfect matchings given in Conjecture 2.51, then any three of these perfect matchings have empty intersection. Thus the next conjecture holds if the above conjecture is true.

Conjecture 2.52 (Fan and Raspaud [83]). Every 2-connected cubic simple graph has three perfect matchings M_1, M_2, M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$.

Related problems concerning the above conjecture are found in [217]. The following conjecture says that the number of 1-factors of 2-connected cubic graph is large.

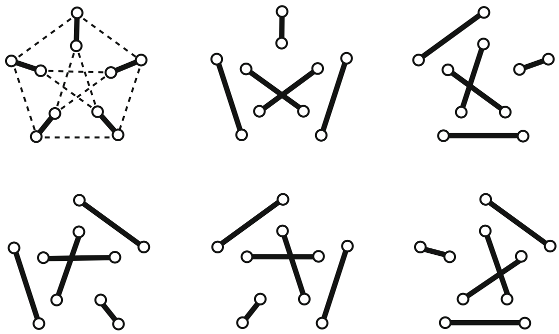


Fig. 2.44. Petersen graph has six perfect matchings such that every edge is contained in precisely two of these perfect matchings.

Conjecture 2.53 (Lovász and Plummer [182] Conjecture 8.1.8). The number of 1-factors of a 2-connected cubic simple graph is exponential in the number of vertices.

Voorhoeve [239] showned that the conjecture holds for bipartite graphs, and Chudnovsky and Seymour showed that it holds for planar graphs. Recently it is announced by Esperet, Kardos, King, Kral and Norine that the conjecture is settled.

Problems

2.1. Prove the marriage theorem by using Theorem 2.15.

2.2. Prove that if a tree has a 1-factor, then it has the unique 1-factor.

2.3. Prove that for every bipartite simple graph G with maximum degree Δ, there exists a Δ-regular bipartite simple graph H which contains G as a subgraph. Note that the two bipartite sets of H might be bigger than those of G.

2.4. Prove the following theorem: Let G be a bipartite simple graph with bipartition (A, B), and let $k \geq 1$ be an integer. Then G has a spanning subgraph H such that

$$\begin{aligned} \deg_H(x) &= 1 \qquad \text{for all} \quad x \in A, \quad \text{and} \\ \deg_H(y) &\leq k \qquad \text{for all} \quad y \in B \end{aligned}$$

if and only if

$$|N_G(S)| \geq \frac{|S|}{k} \qquad \text{for all} \quad S \subseteq A.$$

2.5. Let G be a bipartite simple graph with bipartition (A, B). Prove that if $|A| = |B|$ and $\delta(G) \geq |G|/4$, then G has a 1-factor.

2.6. Verify Theorem 2.16 for the following matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

2.7. Describe many graphs that have no 1-factors and satisfy $|N_G(S)| \geq |S|$ for all $S \subset V(G)$.

2.8. Prove that a factor-critical graph is connected, of odd order and is not a bipartite graph.

2.9. Let S be a maximal barrier of a graph G. Prove that every component of $G - S$ is factor-critical.

2.10. Prove Theorem 2.31

2.11. For every even integer $r \geq 4$, find an $(r-2)$-edge connected r-regular simple graph of even order that has no 1-factor.

2.12. Prove statement (ii) of Theorem 2.40.

2.13. Let G be a simple graph, v a vertex of G and M a maximum matching in G, and let M' be a maximum matching in $G - v$. Prove that $||M|| - 1 \leq ||M'|| \leq ||M||$ and $||M'|| = ||M||$ holds if and only if $v \in D(G)$.

2.14. Prove statements (iv) and (v) in Theorem 2.49.

2.15. Prove the following part of Algorithm 2.50: if G_1 has an M_1-augmenting path starting at the root in G_1, then G has an M-augmenting path starting at the root v.

2.16. Let M be a matching in a connected simple graph G and C be an odd cycle of G containing $(|C| - 1)/2$ edges of M, and let $G' = G/C$ be the graph obtained from G by contracting C. Prove that M is a maximum matching in G if and only if $M' = M \cap E(G')$ is a maximum matching in G'.

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