Chapter 2
Basic Topological Properties of Finite Spaces

In this chapter we present some results concerning elementary topological aspects of finite spaces. The proofs use basic elements of Algebraic Topology and have a strong combinatorial flavour. We study further homotopical properties including classical homotopy invariants and finite analogues of well-known topological constructions.

2.1 Homotopy and Contiguity

Recall that two simplicial maps $\varphi, \psi : K \to L$ are said to be *contiguous* if for every simplex $\sigma \in K$, $\varphi(\sigma) \cup \psi(\sigma)$ is a simplex of L. Two simplicial maps $\varphi, \psi : K \to L$ lie in the same *contiguity class* if there exists a sequence $\varphi = \varphi_0, \varphi_1, \ldots, \varphi_n = \psi$ such that φ_i and φ_{i+1} are contiguous for every $0 \leq i < n$.

If $\varphi, \psi : K \to L$ lie in the same contiguity class, the induced maps in the geometric realizations $|\varphi|, |\psi| : |K| \to |L|$ are homotopic (see Corollary A.1.3 of the appendix).

In this section we study the relationship between contiguity classes of simplicial maps and homotopy classes of the associated maps between finite spaces. These results appear in [11].

Lemma 2.1.1. *Let $f, g : X \to Y$ be two homotopic maps between finite T_0-spaces. Then there exists a sequence $f = f_0, f_1, \ldots, f_n = g$ such that for every $0 \leq i < n$ there is a point $x_i \in X$ with the following properties:*

1. *f_i and f_{i+1} coincide in $X \smallsetminus \{x_i\}$, and*
2. *$f_i(x_i) \prec f_{i+1}(x_i)$ or $f_{i+1}(x_i) \prec f_i(x_i)$.*

Proof. Without loss of generality, we may assume that $f = f_0 \leq g$ by Corollary 1.2.6. Let $A = \{x \in X \mid f(x) \neq g(x)\}$. If $A = \emptyset$, $f = g$ and there is nothing to prove. Suppose $A \neq \emptyset$ and let $x = x_0$ be a maximal point

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of A. Let $y \in Y$ be such that $f(x) \prec y \leq g(x)$ and define $f_1 : X \to Y$ by $f_1|_{X \smallsetminus \{x\}} = f|_{X \smallsetminus \{x\}}$ and $f_1(x) = y$. Then f_1 is continuous for if $x' > x$, $x' \notin A$ and therefore

$$f_1(x') = f(x') = g(x') \geq g(x) \geq y = f_1(x).$$

Repeating this construction for f_i and g, we define f_{i+1}. By finiteness of X and Y this process ends. $\square$

Proposition 2.1.2. *Let $f, g : X \to Y$ be two homotopic maps between finite T_0-spaces. Then the simplicial maps $\mathcal{K}(f), \mathcal{K}(g) : \mathcal{K}(X) \to \mathcal{K}(Y)$ lie in the same contiguity class.*

Proof. By the previous lemma, we can assume that there exists $x \in X$ such that $f(y) = g(y)$ for every $y \neq x$ and $f(x) \prec g(x)$. Therefore, if C is a chain in X, $f(C) \cup g(C)$ is a chain on Y. In other words, if $\sigma \in \mathcal{K}(X)$ is a simplex, $\mathcal{K}(f)(\sigma) \cup \mathcal{K}(g)(\sigma)$ is a simplex in $\mathcal{K}(Y)$. $\square$

Proposition 2.1.3. *Let $\varphi, \psi : K \to L$ be simplicial maps which lie in the same contiguity class. Then $\mathcal{X}(\varphi) \simeq \mathcal{X}(\psi)$.*

Proof. Assume that φ and ψ are contiguous. Then the map $f : \mathcal{X}(K) \to \mathcal{X}(L)$, defined by $f(\sigma) = \varphi(\sigma) \cup \psi(\sigma)$ is well-defined and continuous. Moreover $\mathcal{X}(\varphi) \leq f \geq \mathcal{X}(\psi)$, and then $\mathcal{X}(\varphi) \simeq \mathcal{X}(\psi)$. $\square$

2.2 Minimal Pairs

In this section we generalize Stong's ideas on homotopy types to the case of pairs (X, A) of finite spaces (i.e. a finite space X and a subspace $A \subseteq X$). As a consequence, we will deduce that every core of a finite T_0-space can be obtained by removing beat points from X. Here we introduce the notion of *strong collapse* which plays a central role in Chap. 5. Most of the results of this section appear in [11].

Definition 2.2.1. A pair (X, A) of finite T_0-spaces is a *minimal pair* if all the beat points of X are in A.

The next result generalizes the result of Stong (the case $A = \emptyset$) studied in Sect. 1.3 and its proof is very similar to the original one.

Proposition 2.2.2. *Let (X, A) be a minimal pair and let $f : X \to X$ be a map such that $f \simeq 1_X$ rel A. Then $f = 1_X$.*

Proof. Suppose that $f \leq 1_X$ and $f|_A = 1_A$. Let $x \in X$. If $x \in X$ is minimal, $f(x) = x$. In general, suppose we have proved that $f|_{\hat{U}_x} = 1|_{\hat{U}_x}$. If $x \in A$, $f(x) = x$. If $x \notin A$, x is not a down beat point of X. However $y < x$ implies $y = f(y) \leq f(x) \leq x$. Therefore $f(x) = x$. The case $f \geq 1_X$ is similar, and the general case follows from Corollary 1.2.6. $\square$

Corollary 2.2.3. *Let (X, A) and (Y, B) be minimal pairs, $f : X \to Y$, $g : Y \to X$ such that $gf \simeq 1_X$ rel A, $gf \simeq 1_Y$ rel B. Then f and g are homeomorphisms.*

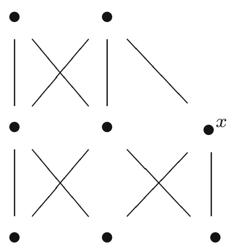
Definition 2.2.4. If x is a beat point of a finite T_0-space X, we say that there is an *elementary strong collapse* from X to $X \smallsetminus x$ and write $X \searrow\!\!\!\searrow^{e} X \smallsetminus x$. There is a *strong collapse* $X \searrow\!\!\!\searrow Y$ (or a *strong expansion* $Y \nearrow\!\!\!\nearrow X$) if there is a sequence of elementary strong collapses starting in X and ending in Y.

Stong's results show that two finite T_0-spaces are homotopy equivalent if and only if there exists a sequence of strong collapses and strong expansions from X to Y (since the later is true for homeomorphic spaces).

Corollary 2.2.5. *Let X be a finite T_0-space and let $A \subseteq X$. Then, $X \searrow\!\!\!\searrow A$ if and only if A is a strong deformation retract of X.*

Proof. If $X \searrow\!\!\!\searrow A$, $A \subseteq X$ is a strong deformation retract. This was already proved by Stong (see Sect. 1.3). Conversely, suppose $A \subseteq X$ is a strong deformation retract. Perform arbitrary elementary strong collapses removing beat points which are not in A. Suppose $X \searrow\!\!\!\searrow Y \supseteq A$ and that all the beat points of Y lie in A. Then (Y, A) is a minimal pair. Since A and Y are strong deformation retracts of X, the minimal pairs (A, A) and (Y, A) are in the hypothesis of Corollary 2.2.3. Therefore A and Y are homeomorphic and so, $X \searrow\!\!\!\searrow Y = A$. □

Example 2.2.6. The space X



is contractible, but the point x is not a strong deformation retract of X, because $(X, \{x\})$ is a minimal pair.

Corollary 2.2.7. *Let (X, A) be a minimal pair such that A is a minimal finite space and $f \simeq 1_{(X,A)} : (X, A) \to (X, A)$. Then $f = 1_X$.*

If X and Y are homotopy equivalent finite T_0-spaces, the associated polyhedra $|\mathcal{K}(X)|$ and $|\mathcal{K}(Y)|$ also have the same homotopy type. However the converse is obviously false, since the associated polyhedra are homotopy equivalent if and only if the finite spaces are weak homotopy equivalent.

In Chap. 5 we will study the notion of *strong homotopy types* of simplicial complexes which have a very simple description and corresponds exactly to the concept of homotopy types of the associated finite spaces.

2.3 T_1-Spaces

We will prove that Hausdorff spaces do not have in general the homotopy type of any finite space. Recall that a topological space X satisfies the T_1-separation axiom if for any two distinct points $x, y \in X$ there exist open sets U and V such that $x \in U$, $y \in V$, $y \notin U$, $x \notin V$. This is equivalent to saying that the points are closed in X. All Hausdorff spaces are T_1, but the converse is false.

If a finite space is T_1, then every subset is closed and so, X is discrete.

Since the core X_c of a finite space X is the disjoint union of the cores of its connected components, we can deduce the following

Lemma 2.3.1. *Let X be a finite space such that X_c is discrete. Then X is a disjoint union of contractible spaces.*

Theorem 2.3.2. *Let X be a finite space and let Y be a T_1-space homotopy equivalent to X. Then X is a disjoint union of contractible spaces.*

Proof. Since $X \simeq Y$, $X_c \simeq Y$. Let $f : X_c \to Y$ be a homotopy equivalence with homotopy inverse g. Then $gf = 1_{X_c}$ by Theorem 1.3.6. Since f is a one to one map from X_c to a T_1-space, it follows that X_c is also T_1 and therefore discrete. Now the result follows from the previous lemma. □

Remark 2.3.3. The proof of the previous theorem can be done without using Theorem 1.3.6, showing that any map $f : X \to Y$ from a finite space to a T_1-space must be locally constant.

Corollary 2.3.4. *Let Y be a connected and non contractible T_1-space. Then Y does not have the same homotopy type as any finite space.*

Proof. Follows immediately from Theorem 2.3.2. □

For example, for any $n \geq 1$, the n-dimensional sphere S^n does not have the homotopy type of any finite space. However, S^n does have, as any finite polyhedron, the same weak homotopy type as some finite space.

2.4 Loops in the Hasse Diagram and the Fundamental Group

In this section we give a full description of the fundamental group of a finite T_0-space in terms of its Hasse diagram. This characterization is induced from the well known description of the fundamental group of a simplicial complex. The Hasse diagram of a finite T_0-space X will be denoted $\mathcal{H}(X)$, and $\mathtt{E}(\mathcal{H}(X))$ will denote the set of edges of the digraph $\mathcal{H}(X)$.

Recall that an *edge-path* in a simplicial complex K, is a sequence (v_0, v_1), $(v_1, v_2), \dots,$ (v_{r-1}, v_r) of ordered pairs of vertices in which $\{v_i, v_{i+1}\}$ is a simplex for every i. If an edge-path contains two consecutive pairs (v_i, v_{i+1}), (v_{i+1}, v_{i+2}) where $\{v_i, v_{i+1}, v_{i+2}\}$ is a simplex, we can replace the two pairs by a unique pair (v_i, v_{i+2}) to obtain an *equivalent* edge-path. The equivalence classes of edge-paths are the morphisms of a groupoid called the *edge-path groupoid* of K, which is denoted by $E(K)$. The full subcategory of edge-paths with origin and end v_0 is the *edge-path group* $E(K, v_0)$ which is isomorphic to the fundamental group $\pi_1(|K|, v_0)$ (see [75, Sect. 3.6] for more details).

Definition 2.4.1. Let (X, x_0) be a finite pointed T_0-space. An ordered pair of points $e = (x, y)$ is called an $\mathcal{H}$*-edge of* X if $(x, y) \in \mathtt{E}(\mathcal{H}(X))$ or $(y, x) \in \mathtt{E}(\mathcal{H}(X))$. The point x is called the *origin* of e and denoted $x = \mathfrak{o}(e)$, the point y is called the *end* of e and denoted $y = \mathfrak{e}(e)$. The *inverse* of an $\mathcal{H}$-edge $e = (x, y)$ is the $\mathcal{H}$-edge $e^{-1} = (y, x)$.

An $\mathcal{H}$*-path* in (X, x_0) is a finite sequence (possibly empty) of $\mathcal{H}$-edges $\xi = e_1 e_2 \dots e_n$ such that $\mathfrak{e}(e_i) = \mathfrak{o}(e_{i+1})$ for all $1 \leq i \leq n-1$. The *origin* of a non empty $\mathcal{H}$-path ξ is $\mathfrak{o}(\xi) = \mathfrak{o}(e_1)$ and its *end* is $\mathfrak{e}(\xi) = \mathfrak{e}(e_n)$. The origin and the end of the empty $\mathcal{H}$-path is $\mathfrak{o}(\emptyset) = \mathfrak{e}(\emptyset) = x_0$. If $\xi = e_1 e_2 \dots e_n$, we define $\overline{\xi} = e_n^{-1} e_{n-1}^{-1} \dots e_1^{-1}$. If ξ, ξ' are $\mathcal{H}$-paths such that $\mathfrak{e}(\xi) = \mathfrak{o}(\xi')$, we define the product $\mathcal{H}$-path $\xi\xi'$ as the concatenation of the sequence ξ followed by the sequence ξ'.

An $\mathcal{H}$-path $\xi = e_1 e_2 \dots e_n$ is said to be *monotonic* if $e_i \in \mathtt{E}(\mathcal{H}(X))$ for all $1 \leq i \leq n$ or $e_i^{-1} \in \mathtt{E}(\mathcal{H}(X))$ for all $1 \leq i \leq n$.

A *loop at* x_0 is an $\mathcal{H}$-path that starts and ends in x_0. Given two loops ξ, ξ' at x_0, we say that they are *close* if there exist $\mathcal{H}$-paths $\xi_1, \xi_2, \xi_3, \xi_4$ such that ξ_2 and ξ_3 are monotonic and the set $\{\xi, \xi'\}$ coincides with $\{\xi_1\xi_2\xi_3\xi_4, \xi_1\xi_4\}$.

We say that two loops ξ, ξ' at x_0 are $\mathcal{H}$*-equivalent* if there exists a finite sequence of loops $\xi = \xi_1, \xi_2, \dots, \xi_n = \xi'$ such that any two consecutive are close. We denote by $\langle\xi\rangle$ the $\mathcal{H}$-equivalence class of a loop ξ and $\mathscr{H}(X, x_0)$ the set of these classes.

Theorem 2.4.2. *Let (X, x_0) be a pointed finite T_0-space. Then the product $\langle\xi\rangle\langle\xi'\rangle = \langle\xi\xi'\rangle$ is well defined and induces a group structure on $\mathscr{H}(X, x_0)$.*

Proof. It is easy to check that the product is well defined, associative and that $\langle\emptyset\rangle$ is the identity. In order to prove that the inverse of $\langle e_1 e_2 \dots e_n\rangle$ is $\langle e_n^{-1} e_{n-1}^{-1} \dots e_1^{-1}\rangle$ we need to show that for any composable $\mathcal{H}$-paths ξ, ξ' such that $\mathfrak{o}(\xi) = \mathfrak{e}(\xi') = x_0$ and for any $\mathcal{H}$-edge e, composable with ξ, one has that $\langle\xi e e^{-1} \xi'\rangle = \langle\xi\xi'\rangle$. But this follows immediately from the definition of close loops since e and e^{-1} are monotonic. $\square$

Theorem 2.4.3. *Let (X, x_0) be a pointed finite T_0-space. Then the edge-path group $E(\mathcal{K}(X), x_0)$ of $\mathcal{K}(X)$ with base vertex x_0 is isomorphic to $\mathscr{H}(X, x_0)$.*

Proof. Let us define

$$\begin{aligned}\varphi : \mathscr{H}(X, x_0) &\longrightarrow E(\mathcal{K}(X), x_0),\\ \langle e_1 e_2 \dots e_n \rangle &\longmapsto [e_1 e_2 \dots e_n],\\ \langle \emptyset \rangle &\longmapsto [(x_0, x_0)],\end{aligned}$$

where $[\xi]$ denotes the class of ξ in $E(\mathcal{K}(X), x_0)$.

To prove that φ is well defined, let us suppose that the loops $\xi_1\xi_2\xi_3\xi_4$ and $\xi_1\xi_4$ are close, where $\xi_2 = e_1 e_2 \dots e_n$, $\xi_3 = e'_1 e'_2 \dots e'_m$ are monotonic $\mathcal{H}$-paths. By induction, it can be proved that

$$[\xi_1\xi_2\xi_3\xi_4] = [\xi_1 e_1 e_2 \dots e_{n-j}(\mathfrak{o}(e_{n-j+1}), \mathfrak{e}(e_n))\xi_3\xi_4]$$

for $1 \le j \le n$. In particular $[\xi_1\xi_2\xi_3\xi_4] = [\xi_1(\mathfrak{e}(\xi_1), \mathfrak{e}(e_n))\xi_3\xi_4]$.

Analogously,

$$[\xi_1(\mathfrak{e}(\xi_1), \mathfrak{e}(e_n))\xi_3\xi_4] = [\xi_1(\mathfrak{e}(\xi_1), \mathfrak{e}(e_n))(\mathfrak{o}(e'_1), \mathfrak{o}(\xi_4))\xi_4]$$

and then

$$\begin{aligned}[\xi_1\xi_2\xi_3\xi_4] &= [\xi_1(\mathfrak{e}(\xi_1), \mathfrak{e}(e_n))(\mathfrak{o}(e'_1), \mathfrak{o}(\xi_4))\xi_4]\\ &= [\xi_1(\mathfrak{e}(\xi_1), \mathfrak{e}(e_n))(\mathfrak{e}(e_n), \mathfrak{e}(\xi_1))\xi_4] = [\xi_1(\mathfrak{e}(\xi_1), \mathfrak{e}(\xi_1))\xi_4] = [\xi_1\xi_4].\end{aligned}$$

If $\xi = (x_0, x_1)(x_1, x_2) \dots (x_{n-1}, x_n)$ is an edge-path in $\mathcal{K}(X)$ with $x_n = x_0$, then x_{i-1} and x_i are comparable for all $1 \le i \le n$. In this case, we can find monotonic $\mathcal{H}$-paths $\xi_1, \xi_2, \dots, \xi_n$ such that $\mathfrak{o}(\xi_i) = x_{i-1}$, $\mathfrak{e}(\xi_i) = x_i$ for all $1 \le i \le n$. Let us define

$$\begin{aligned}\psi : E(\mathcal{K}(X), x_0) &\longrightarrow \mathscr{H}(X, x_0),\\ [\xi] &\longmapsto \langle \xi_1\xi_2 \dots \xi_n \rangle.\end{aligned}$$

This definition does not depend on the choice of the $\mathcal{H}$-paths ξ_i since if two choices differ only for $i = k$ then $\xi_1 \dots \xi_k \dots \xi_n$ and $\xi_1 \dots \xi'_k \dots \xi_n$ are $\mathcal{H}$-equivalent because both of them are close to $\xi_1 \dots \xi_k \xi_k^{-1} \xi'_k \dots \xi_n$.

The definition of ψ does not depend on the representative. Suppose that $\xi'(x, y)(y, z)\xi''$ and $\xi'(x, z)\xi''$ are simply equivalent edge-paths in $\mathcal{K}(X)$ that start and end in x_0, where ξ and ξ' are edge-paths and x, y, z are comparable. In the case that y lies between x and z, we can choose the monotonic $\mathcal{H}$-path corresponding to (x, z) to be the juxtaposition of the corresponding to (x, y) and (y, z), and so ψ is equally defined in both edge-paths. In the case that $z \le x \le y$ we can choose monotonic $\mathcal{H}$-paths α, β from x to y and from z to x, and then α will be the corresponding $\mathcal{H}$-path to (x, y), $\overline{\alpha}\overline{\beta}$ that corresponding to (y, z) and $\overline{\beta}$ to (x, z). It only remains to prove that $\langle \gamma'\alpha\overline{\alpha}\overline{\beta}\gamma'' \rangle = \langle \gamma'\overline{\beta}\gamma'' \rangle$ for $\mathcal{H}$-paths γ' and γ'', which is trivial. The other cases are analogous to the last one.

It is clear that φ and ψ are mutually inverse. $\square$

Since $E(\mathcal{K}(X), x_0)$ is isomorphic to $\pi_1(|\mathcal{K}(X)|, x_0)$ (cf. [75, Corollary 3.6.17]), we obtain the following result.

Corollary 2.4.4. *Let (X, x_0) be a pointed finite T_0-space, then $\mathscr{H}(X, x_0) = \pi_1(X, x_0)$.*

Remark 2.4.5. Since every finite space is homotopy equivalent to a finite T_0-space, this computation of the fundamental group can be applied to any finite space.

2.5 Euler Characteristic

If the homology (with integer coefficients) of a topological space X is finitely generated as a graded abelian group, the Euler characteristic of X is defined by $\chi(X) = \sum\limits_{n \geq 0} (-1)^n rank(H_n(X))$. If Z is a compact CW-complex, its homology is finitely generated and $\chi(Z) = \sum\limits_{n \geq 0} (-1)^n \alpha_n$ where α_n is the number of n-cells of Z. A weak homotopy equivalence induces isomorphisms in homology groups and therefore weak homotopy equivalent spaces have the same Euler characteristic.

Since any finite T_0-space X is weak homotopy equivalent to the geometric realization of $\mathcal{K}(X)$, whose simplices are the non empty chains of X, the Euler characteristic of X is

$$\chi(X) = \sum_{C \in \mathcal{C}(X)} (-1)^{\#C+1}, \tag{2.1}$$

where $\mathcal{C}(X)$ is the set of nonempty chains of X and $\#C$ is the cardinality of C.

We will give a basic combinatorial proof of the fact that the Euler characteristic is a homotopy invariant in the setting of finite spaces, using only the formula 2.1 as definition.

Theorem 2.5.1. *Let X and Y be finite T_0-spaces with the same homotopy type. Then $\chi(X) = \chi(Y)$.*

Proof. Let X_c and Y_c be cores of X and Y. Then there exist two sequences of finite T_0-spaces $X = X_0 \supseteq \ldots \supseteq X_n = X_c$ and $Y = Y_0 \supseteq \ldots \supseteq Y_m = Y_c$, where X_{i+1} is constructed from X_i by removing a beat point and Y_{i+1} is constructed from Y_i, similarly. Since X and Y are homotopy equivalent, X_c and Y_c are homeomorphic. Thus, $\chi(X_c) = \chi(Y_c)$.

It suffices to show that the Euler characteristic does not change when a beat point is removed. Let P be a finite poset and let $p \in P$ be a beat point. Then there exists $q \in P$ such that if r is comparable with p then r is comparable with q.

Hence we have a bijection

$$\varphi : \{C \in \mathcal{C}(P) \mid p \in C,\ q \notin C\} \longrightarrow \{C \in \mathcal{C}(P) \mid p \in C,\ q \in C\},$$
$$C \longmapsto C \cup \{q\}.$$

Therefore

$$\chi(P) - \chi(P \smallsetminus \{p\}) = \sum_{p \in C \in \mathcal{C}P} (-1)^{\#C+1} = \sum_{q \notin C \ni p} (-1)^{\#C+1} + \sum_{q \in C \ni p} (-1)^{\#C+1}$$

$$= \sum_{q \notin C \ni p} (-1)^{\#C+1} + \sum_{q \notin C \ni p} (-1)^{\#\varphi(C)+1} = \sum_{q \notin C \ni p} (-1)^{\#C+1} + \sum_{q \notin C \ni p} (-1)^{\#C} = 0.$$

□

The Euler characteristic of finite T_0-spaces is intimately related to the Möbius function of posets, which is a generalization of the classical Möbius function of number theory. We will say just a few words about this. For proofs and applications we refer the reader to [29].

Given a finite poset P, we define the *incidence algebra* $\mathfrak{A}(P)$ of P as the set of functions $P \times P \to \mathbb{R}$ such that $f(x,y) = 0$ if $x \nleq y$ with the usual structure of $\mathbb{R}$-vector space and the product given by

$$fg(x,y) = \sum_{z \in P} f(x,z)g(z,y).$$

The element $\zeta_P \in \mathfrak{A}(P)$ defined by $\zeta_P(x,y) = 1$ if $x \leq y$ and 0 in other case, is invertible in $\mathfrak{A}(P)$. The *Möbius fuction* $\mu_P \in \mathfrak{A}(P)$ is the inverse of ζ_P.

The Theorem of Hall states that if P is a finite poset and $x, y \in P$, then $\mu_P(x,y) = \sum\limits_{n \geq 0} (-1)^{n+1} c_n$, where c_n is the number of chains of n-elements which start in x and end in y.

Given a finite poset P, $\hat{P} = P \cup \{0,1\}$ denotes the poset obtained when adjoining a minimum 0 and a maximum 1 to P. In particular, (2.1) and the Theorem of Hall, give the following

Corollary 2.5.2. *Let P be a finite poset. Then*

$$\widetilde{\chi}(P) = \mu_{\hat{P}}(0,1),$$

where $\widetilde{\chi}(P) = \chi(P) - 1$ denotes the reduced Euler characteristic of the finite space P.

One of the motivations of the Möbius function is the following inversion formula.

Theorem 2.5.3 (Möbius inversion formula). *Let P be a finite poset and let $f, g : P \to \mathbb{R}$. Then*

$$g(x) = \sum_{y \leq x} f(y) \text{ if and only if } f(x) = \sum_{y \leq x} \mu_P(y, x) g(y).$$

Analogously,

$$g(x) = \sum_{y \geq x} f(y) \text{ if and only if } f(x) = \sum_{y \geq x} \mu_P(y, x) g(y).$$

Beautiful applications of these formulae are: (1) the Möbius inversion of number theory which is obtained when applying Theorem 2.5.3 to the order given by divisibility of the integer numbers; (2) the inclusion–exclusion formula obtained from the power set of a set ordered by inclusion.

2.6 Automorphism Groups of Finite Posets

It is well known that any finite group G can be realized as the automorphism group of a finite poset. In 1946 Birkhoff [13] proved that if the order of G is n, G can be realized as the automorphisms of a poset with $n(n+1)$ points. In 1972 Thornton [78] improved slightly Birkhoff's result: He obtained a poset of $n(2r+1)$ points, when the group is generated by r elements.

We present here a result which appears in [10]. Following Birkhoff's and Thornton's ideas, we exhibit a simple proof of the following fact which improves their results

Theorem 2.6.1. *Given a group G of finite order n with r generators, there exists a poset X with $n(r+2)$ points such that $Aut(X) \simeq G$.*

Recall first that the *height* $ht(X)$ of a finite poset X is one less than the maximum number of elements in a chain of X. The *height* of a point x in a finite poset X is $ht(x) = ht(U_x)$.

Proof. Let $\{h_1, h_2, \ldots, h_r\}$ be a set of r generators of G. We define the poset $X = G \times \{-1, 0, \ldots, r\}$ with the following order

- $(g, i) \leq (g, j)$ if $-1 \leq i \leq j \leq r$
- $(gh_i, -1) \leq (g, j)$ if $1 \leq i \leq j \leq r$

Define $\phi : G \to Aut(X)$ by $\phi(g)(h, i) = (gh, i)$. It is easy to see that $\phi(g) : X \to X$ is order preserving and that it is an automorphism with inverse $\phi(g^{-1})$. Therefore ϕ is a well defined homomorphism. Clearly ϕ is a monomorphism since $\phi(g) = 1$ implies $(g, -1) = \phi(g)(e, -1) = (e, -1)$.

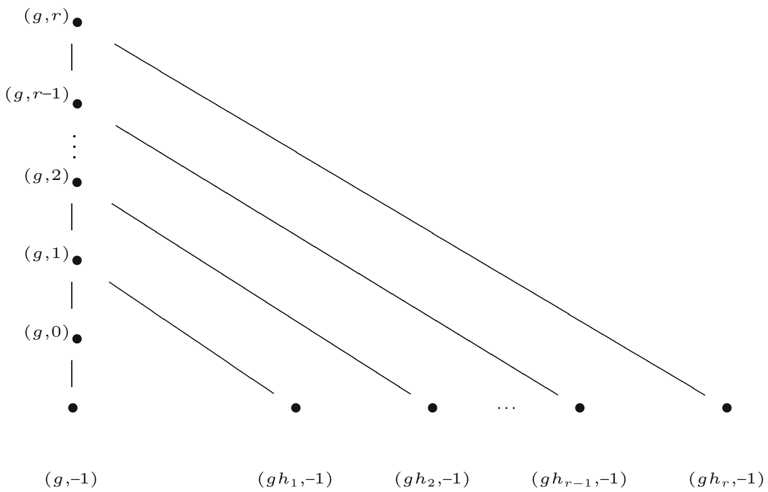


Fig. 2.1 $U_{(g,r)}$

It remains to show that ϕ is an epimorphism. Let $f : X \to X$ be an automorphism. Since $(e,-1)$ is minimal in X, so is $f(e,-1)$ and therefore $f(e,-1) = (g,-1)$ for some $g \in G$. We will prove that $f = \phi(g)$.

Let $Y = \{x \in X \mid f(x) = \phi(g)(x)\}$. Y is nonempty since $(e,-1) \in Y$. We prove first that Y is an open subspace of X. Suppose $x = (h,i) \in Y$. Then the restrictions

$$f|_{U_x}, \phi(g)|_{U_x} : U_x \to U_{f(x)}$$

are isomorphisms. On the other hand, there exists a unique automorphism $U_x \to U_x$ since the unique chain of $i+2$ elements must be fixed by any such automorphism. Thus, $f|_{U_x}^{-1}\phi(g)|_{U_x} = 1_{U_x}$, and then $f|_{U_x} = \phi(g)|_{U_x}$, which proves that $U_x \subseteq Y$. Similarly we see that $Y \subseteq X$ is closed. Assume $x = (h,i) \notin Y$. Since $f \in Aut(X)$, it preserves the height of any point. In particular $ht(f(x)) = ht(x) = i+1$ and therefore $f(x) = (k,i) = \phi(kh^{-1})(x)$ for some $k \in G$. Moreover $k \neq gh$ since $x \notin Y$. As above, $f|_{U_x} = \phi(kh^{-1})|_{U_x}$, and since $kh^{-1} \neq g$ we conclude that $U_x \cap Y = \emptyset$.

We prove now that X is connected. It suffices to prove that any two minimal elements of X are in the same connected component. Given $h,k \in G$, we have $h = kh_{i_1}h_{i_2}\ldots h_{i_m}$ for some $1 \leq i_1,\ i_2\ldots i_m \leq r$. On the other hand, $(kh_{i_1}h_{i_2}\ldots h_{i_s},-1)$ and $(kh_{i_1}h_{i_2}\ldots h_{i_{s+1}},-1)$ are connected via $(kh_{i_1}h_{i_2}\ldots h_{i_s},-1) < (kh_{i_1}h_{i_2}\ldots h_{i_s},r) > (kh_{i_1}h_{i_2}\ldots h_{i_{s+1}},-1)$. This implies that $(k,-1)$ and $(h,-1)$ are in the same connected component.

Finally, since X is connected and Y is closed, open and nonempty, $Y = X$, i.e. $f = \phi(g)$. Therefore ϕ is an epimorphism, and then $G \simeq Aut(X)$. □

If the generators $h_1, h_2, \dots, h_r$ are non-trivial, the open sets $U_{(g,r)}$ are as in Fig. 2.1. In that case it is not hard to prove that the finite space X constructed above is weak homotopy equivalent to a wedge of $n(r-1)+1$ circles, or in other words, that the order complex of X is homotopy equivalent to a wedge of $n(r-1)+1$ circles. The space X deformation retracts to the subspace $Y = G \times \{-1, r\}$ of its minimal and maximal points. A retraction is given by the map $f : X \to Y$, defined as $f(g,i) = (g,r)$ if $i \geq 0$ and $f(g,-1) = (g,-1)$. Now the order complex $\mathcal{K}(Y)$ of Y is a connected simplicial complex of dimension 1, so its homotopy type is completely determined by its Euler Characteristic. This complex has $2n$ vertices and $n(r+1)$ edges, which means that it has the homotopy type of a wedge of $1-\chi(\mathcal{K}(Y)) = n(r-1)+1$ circles.

On the other hand, note that in general the automorphism group of a finite space, does not say much about its homotopy type as we see in the following

Proposition 2.6.2. *Given a finite group G and a finite space X, there exists a finite space Y which is homotopy equivalent to X and such that $Aut(Y) \simeq G$.*

Proof. We make this construction in two steps. First, we find a finite T_0-space $\tilde{X}$ homotopy equivalent to X and such that $Aut(\tilde{X}) = 0$. To do this, assume that X is T_0 and consider a linear extension $x_1, x_2, \dots, x_n$ of the poset X (i.e. $X = \{x_1, x_2, \dots, x_n\}$ and $x_i \leq x_j$ implies $i \leq j$). Now, for each $1 \leq k \leq n$ attach a chain of length kn to X with minimum x_{n-k+1}. The resulting space $\tilde{X}$ deformation retracts to X and every automorphism $f : \tilde{X} \to \tilde{X}$ must fix the unique chain C_1 of length n^2 (with minimum x_1). Therefore f restricts to a homeomorphism $\tilde{X} \smallsetminus C_1 \to \tilde{X} \smallsetminus C_1$ which must fix the unique chain C_2 of length $n(n-1)$ of $\tilde{X} \smallsetminus C_1$ (with minimum x_2). Applying this reasoning repeatedly, we conclude that f fixes every point of $\tilde{X}$. On the other hand, we know that there exists a finite T_0-space Z such that $Aut(Z) = G$.

Now the space Y is constructed as follows. It contains one copy of $\tilde{X}$ and one of Z, and the additional relations $z \leq x$ for every $z \in Z$ and $x \geq x_1$ in $\tilde{X}$. So, all the elements of Z are smaller than $x_1 \in \tilde{X}$. Clearly Y deformation retracts to $\tilde{X}$. Moreover, if $f : Y \to Y$ is an automorphism, $f(x_1) \notin Z$ since $f(x_1)$ cannot be comparable with x_1 and distinct from it (cf. Lemma 8.1.1). Since there is only one chain of n^2 elements in $\tilde{X}$, it must be fixed by f. In particular $f(x_1) = x_1$, and then $f|_Z : Z \to Z$. Thus f restricts to automorphisms of $\tilde{X}$ and of Z and therefore $Aut(Y) \simeq Aut(Z) \simeq G$. □

2.7 Joins, Products, Quotients and Wedges

In this section we will study some basic constructions in the settings of finite spaces, simplicial complexes and general topological spaces. We will relate these constructions to each other and analyze them from the homotopical point of view.

Recall that the *simplicial join* $K * L$ (also denoted by KL) of two simplicial complexes K and L (with disjoint vertex sets) is the complex

$$K * L = K \cup L \cup \{\sigma \cup \tau |\ \sigma \in K, \tau \in L\}.$$

The simplicial cone aK with base K is the join of K with a vertex $a \notin K$. It is well known that for finite simplicial complexes K and L, the geometric realization $|K * L|$ is homeomorphic to the topological join $|K| * |L|$. If K is the 0-complex with two vertices, $|K * L| = |K| * |L| = S^0 * |L| = \Sigma|L|$ is the suspension of $|L|$. Here, S^0 denotes the discrete space on two points (0-sphere).

There is an analogous construction for finite spaces.

Definition 2.7.1. The *(non-Hausdorff) join* (also called the *ordinal sum*) $X \circledast Y$ of two finite T_0-spaces X and Y is the disjoint union $X \sqcup Y$ keeping the given ordering within X and Y and setting $x \leq y$ for every $x \in X$ and $y \in Y$.

Note that the join is associative and in general $X \circledast Y \neq Y \circledast X$. Special cases of joins are the *non-Hausdorff cone* $\mathbb{C}(X) = X \circledast D^0$ and the *non-Hausdorff suspension* $\mathbb{S}(X) = X \circledast S^0$ of any finite T_0-space X. Here $D^0 = *$ denotes the singleton (0-cell).

Remark 2.7.2. $\mathcal{K}(X \circledast Y) = \mathcal{K}(X) * \mathcal{K}(Y)$.

Given a point x in a finite T_0-space X, the *star* C_x of x consists of the points which are comparable with x, i.e. $C_x = U_x \cup F_x$. Note that C_x is always contractible since $1_{C_x} \leq f \geq g$ where $f : C_x \to C_x$ is the map which is the identity on F_x and the constant map x on U_x, and g is the constant map x. The *link* of x is the subspace $\hat{C}_x = C_x \smallsetminus \{x\}$. In case we need to specify the ambient space X, we will write $\hat{C}_x^X$. Note that $\hat{C}_x = \hat{U}_x \circledast \hat{F}_x$.

Proposition 2.7.3. *Let X and Y be finite T_0-spaces. Then $X \circledast Y$ is contractible if and only if X or Y is contractible.*

Proof. Assume X is contractible. Then there exists a sequence of spaces

$$X = X_n \supsetneq X_{n-1} \supsetneq \ldots \supsetneq X_1 = \{x_1\}$$

with $X_i = \{x_1, x_2, \ldots, x_i\}$ and such that x_i is a beat point of X_i for every $2 \leq i \leq n$. Then x_i is a beat point of $X_i \circledast Y$ for each $2 \leq i \leq n$ and therefore, $X \circledast Y$ deformation retracts to $\{x_1\} \circledast Y$ which is contractible. Analogously, if Y is contractible, so is $X \circledast Y$.

Now suppose $X \circledast Y$ is contractible. Then there exists a sequence

$$X \circledast Y = X_n \circledast Y_n \supsetneq X_{n-1} \circledast Y_{n-1} \supsetneq \ldots \supsetneq X_1 \circledast Y_1 = \{z_1\}$$

with $X_i \subseteq X$, $Y_i \subseteq Y$, $X_i \circledast Y_i = \{z_1, z_2 \ldots, z_i\}$ such that z_i is a beat point of $X_i \circledast Y_i$ for $i \geq 2$.

Let $i \geq 2$. If $z_i \in X_i$, z_i is a beat point of X_i unless it is a maximal point of X_i and Y_i has a minimum. In the same way, if $z_i \in Y_i$, z_i is a beat point of Y_i or X_i has a maximum. Therefore, for each $2 \leq i \leq n$, either $X_{i-1} \subseteq X_i$ and $Y_{i-1} \subseteq Y_i$ are deformation retracts (in fact, one inclusion is an identity and the other inclusion is strict), or one of them, X_i or Y_i, is contractible. This proves that X or Y is contractible. □

In Proposition 4.3.4 we will prove a result which is the analogue of Proposition 2.7.3 for collapsible finite spaces.

If X and Y are finite spaces, the preorder corresponding to the topological product $X \times Y$ is the product of the preorders of X and Y (Remark 1.1.2), i.e. $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$. If X and Y are two topological spaces, not necessarily finite, and A is strong deformation retract of a X, then $A \times Y$ is a strong deformation retract of $X \times Y$.

Proposition 2.7.4. *Let X_c and Y_c be cores of finite spaces X and Y. Then $X_c \times Y_c$ is a core of $X \times Y$.*

Proof. Since $X_c \subseteq X$ is a strong deformation retract, so is $X_c \times Y \subseteq X \times Y$. Analogously $X_c \times Y_c$ is a strong deformation retract of $X_c \times Y$ and then, so is $X_c \times Y_c \subseteq X \times Y$. We have to prove that the product of minimal finite spaces is also minimal. Let $(x, y) \in X_c \times Y_c$. If there exists $x' \in X_c$ with $x' \prec x$ and $y' \in Y_c$ with $y' \prec y$, (x, y) covers at least two elements (x', y) and (x, y'). If x is minimal in X_c, $\hat{U}_{(x,y)}$ is homeomorphic to $\hat{U}_y$. Analogously if y is minimal. Therefore, (x, y) is not a down beat point. Similarly, $X_c \times Y_c$ does not have up beat points. Thus, it is a minimal finite space. □

In particular $X \times Y$ is contractible if and only if each space X and Y is contractible. In fact this result holds in general, when X and Y are not necessarily finite.

Recall that the product of two nonempty spaces is T_0 if and only if each space is.

Proposition 2.7.5. *Let X and Y be finite T_0-spaces. Then $|\mathcal{K}(X \times Y)|$ is homeomorphic to $|\mathcal{K}(X)| \times |\mathcal{K}(Y)|$.*

Proof. Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the canonical projections. Define $f : |\mathcal{K}(X \times Y)| \to |\mathcal{K}(X)| \times |\mathcal{K}(Y)|$ by $f = |\mathcal{K}(p_X)| \times |\mathcal{K}(p_Y)|$. In other words, if $\alpha = \sum\limits_{i=0}^{k} t_i(x_i, y_i) \in |\mathcal{K}(X \times Y)|$ where $(x_0, y_0) < (x_1, y_1) < \ldots < (x_k, y_k)$ is a chain in $X \times Y$, $f(\alpha) = (\sum\limits_{i=0}^{k} t_i x_i, \sum\limits_{i=0}^{k} t_i y_i)$.

Since $|\mathcal{K}(p_X)|$ and $|\mathcal{K}(p_Y)|$ are continuous, so is f. $|\mathcal{K}(X \times Y)|$ is compact and $|\mathcal{K}(X)| \times |\mathcal{K}(Y)|$ is Hausdorff, so we only need to show that f is a bijection. Details will be left to the reader. An explicit formula for $g = f^{-1}$ is given by

$$g(\sum_{i=0}^{k} u_i x_i, \sum_{i=0}^{l} v_i y_i) = \sum_{i,j} t_{ij}(x_i, y_j),$$

where $t_{ij} = \max\{0, \min\{u_0 + u_1 + \ldots + u_i, v_0 + v_1 + \ldots v_j\} - \max\{u_0 + u_1 + \ldots + u_{i-1}, v_0 + v_1 + \ldots v_{j-1}\}\}$. The idea is very simple. Consider the segments $U_0, U_1, \ldots, U_k \subseteq I = [0,1]$, each U_i of length u_i, $U_i = [u_0 + u_1 + \ldots + u_{i-1}, u_0 + u_1 + \ldots + u_i]$. Analogously, define $V_j = [v_0 + v_1 + \ldots + v_{j-1}, v_0 + v_1 + \ldots + v_j] \subseteq I$ for $0 \leq j \leq l$. Then t_{ij} is the length of the segment $U_i \cap V_j$. It is not hard to see that $g : |\mathcal{K}(X)| \times |\mathcal{K}(Y)| \to |\mathcal{K}(X \times Y)|$ is well defined since $support(\sum\limits_{i,j} t_{ij}(x_i, y_j))$ is a chain and $\sum t_{ij} = \sum\limits_{i,j} length(U_i \cap V_j) = \sum\limits_{i} length(U_i) = 1$. Moreover, the compositions gf and fg are the corresponding identities. □

A similar proof of the last result can be found in [81, Proposition 4.1].

If X is a finite T_0-space, and $A \subseteq X$ is a subspace, the quotient X/A need not be T_0. For example, if X is the chain of three elements $0 < 1 < 2$ and $A = \{0, 2\}$, X/A is the indiscrete space of two elements. We will exhibit a necessary and sufficient condition for X/A to be T_0.

Let X be a finite space and $A \subseteq X$ a subspace. We will denote by $q : X \to X/A$ the quotient map and by qx the class in the quotient of an element $x \in X$. Recall that $\overline{A} = \{x \in X \mid \exists\, a \in A \text{ with } x \geq a\}$ denotes the closure of A. We will denote by $\underline{A} = \{x \in X \mid \exists\, a \in A \text{ with } x \leq a\} = \bigcup\limits_{a \in A} U_a \subseteq X$, the open hull of A.

Lemma 2.7.6. *Let $x \in X$. If $x \in \overline{A}$, $U_{qx} = q(U_x \cup \underline{A})$. If $x \notin \overline{A}$, $U_{qx} = q(U_x)$.*

Proof. Suppose $x \in \overline{A}$. A subset U of X/A is open if and only if $q^{-1}(U)$ is open in X. Since $q^{-1}(q(U_x \cup \underline{A})) = U_x \cup \underline{A} \subseteq X$ is open, $q(U_x \cup \underline{A}) \subseteq X/A$ is an open set containing qx. Therefore $U_{qx} \subseteq q(U_x \cup \underline{A})$. The other inclusion follows from the continuity of q since $x \in \overline{A}$: if $y \in \underline{A}$, there exist $a, b \in A$ such that $y \leq a$ and $b \leq x$ and therefore $qy \leq qa = qb \leq qx$.

If $x \notin \overline{A}$, $q^{-1}(q(U_x)) = U_x$, so $q(U_x)$ is open and therefore $U_{qx} \subseteq q(U_x)$. The other inclusion is trivial. □

Proposition 2.7.7. *Let X be a finite space and $A \subseteq X$ a subspace. Let $x, y \in X$, then $qx \leq qy$ in the quotient X/A if and only if $x \leq y$ or there exist $a, b \in A$ such that $x \leq a$ and $b \leq y$.*

Proof. Assume $qx \leq qy$. If $y \in \overline{A}$, there exists $b \in A$ with $b \leq y$ and by the previous lemma $qx \in U_{qy} = q(U_y \cup \underline{A})$. Therefore $x \in U_y \cup \underline{A}$ and then $x \leq y$ or $x \leq a$ for some $a \in A$. If $y \notin \overline{A}$, $qx \in U_{qy} = q(U_y)$. Hence, $x \in U_y$.

Conversely if $x \leq y$ or there are some $a, b \in A$ such that $x \leq a$ and $b \leq y$, then $qx \leq qy$ or $qx \leq qa = qb \leq qy$. □

Proposition 2.7.8. *Let X be a finite T_0-space and $A \subseteq X$. The quotient X/A is not T_0 if and only if there exists a triple $a < x < b$ with $a, b \in A$ and $x \notin A$.*

Proof. Suppose there is not such triple and that $qx \leq qy$, $qy \leq qx$. Then $x \leq y$ or there exist $a, b \in A$ with $x \leq a$, $b \leq y$, and, on the other hand, $y \leq x$ or there are some $a', b' \in A$ such that $y \leq a'$, $b' \leq x$. If $x \leq y$ and $y \leq x$, then $x = y$. In other case, both x and y are in A. Therefore, $qx = qy$. This proves that X/A is T_0. Conversely, if there exists a triple $a < x < b$ as above, $qa \leq qx \leq qb = qa$, but $qa \neq qx$. Therefore, X/A is not T_0. □

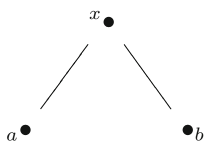
The non-existence of a triple as above is equivalent to saying that $A = \overline{A} \cap \underline{A}$, i.e.

$$X/A \text{ is } T_0 \text{ if and only if } A = \overline{A} \cap \underline{A}.$$

For example open or closed subsets satisfy this condition.

Now we want to study how the functors $\mathcal{X}$ and $\mathcal{K}$ behave with respect to quotients. Recall that $\mathcal{K}(\mathcal{X}(K))$ is the barycentric subdivision K' of K. Following [80] and [35], the *barycentric subdivision* of a finite T_0-space X is defined by $X' = \mathcal{X}(\mathcal{K}(X))$. Explicitly, X' consists of the nonempty chains of X ordered by inclusion. This notion will be important in the development of the simple homotopy theory for finite spaces studied in Chap. 4.

Example 2.7.9. Let $X = \mathbb{C}D_2 = \{x, a, b\}$ and let $A = \{a, b\}$ be the subspace of minimal elements.



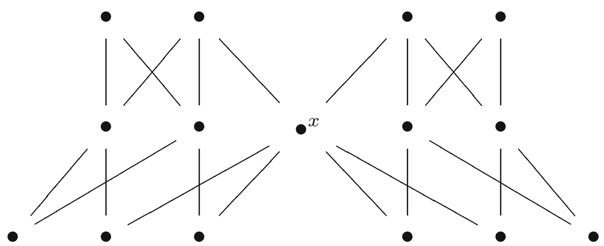
Then X/A is the Sierpinski space $\mathfrak{S}$ (the finite T_0-space with two points $0 < 1$) and $|\mathcal{K}(X)|/|\mathcal{K}(A)|$ is homeomorphic to S^1. Therefore $|\mathcal{K}(X)|/|\mathcal{K}(A)|$ and $|\mathcal{K}(X/A)|$ are not homotopy equivalent. However $X'/A' = S^0 \circledast S^0$ and then $|\mathcal{K}(X')|/|\mathcal{K}(A')|$ and $|\mathcal{K}(X'/A')|$ are both homeomorphic to a circle. The application $\mathcal{K}$ does not preserve quotients in general. In Corollary 7.2.2 we prove that if A is a subspace of a finite T_0-space X, $|\mathcal{K}(X')|/|\mathcal{K}(A')|$ and $|\mathcal{K}(X'/A')|$ are homotopy equivalent.

A particular case of a quotient X/A is the one-point union or wedge. If X and Y are topological spaces with base points $x_0 \in X$, $y_0 \in Y$, then the wedge $X \vee Y$ is the quotient $X \sqcup Y/A$ with $A = \{x_0, y_0\}$. Clearly, if X and Y are finite T_0-spaces, $A = \{x_0, y_0\} \subseteq X \sqcup Y$ satisfies $A = \overline{A} \cap \underline{A}$ and then $X \vee Y$ is also T_0. Moreover, if $x, x' \in X$, then x covers x' in X if and only if x covers x' in $X \vee Y$. The same holds for Y, and if $x \in X \smallsetminus \{x_0\}$, $y \in Y \smallsetminus \{y_0\}$ then x does not cover y in $X \vee Y$ and y does not cover x. Thus, the Hasse diagram of $X \vee Y$ is the union of the Hasse diagrams of X and Y, identifying x_0 and y_0.

If $X \vee Y$ is contractible, then X and Y are contractible. This holds for general topological spaces. Let $i : X \to X \vee Y$ denote the canonical inclusion and $r : X \vee Y \to X$ the retraction which sends all of Y to x_0. If H :

$(X \vee Y) \times I \to X \vee Y$ is a homotopy between the identity and a constant, then $rH(i \times 1_I) : X \times I \to X$ shows that X is contractible. The following example shows that the converse is not true for finite spaces.

Example 2.7.10. The space X of Example 2.2.6 is contractible, but the union at x of two copies of X is a minimal finite space, and in particular it is not contractible.



However, from Corollary 4.3.11 we will deduce that $X \vee X$ is homotopically trivial, or in other words, it is weak homotopy equivalent to a point. This is the first example we exhibit of a finite space which is homotopically trivial but which is not contractible. These spaces play a fundamental role in the theory of finite spaces.

In Proposition 4.3.10 we will prove that if X and Y are finite T_0-spaces, there is a weak homotopy equivalence $|\mathcal{K}(X)| \vee |\mathcal{K}(Y)| \to X \vee Y$.

2.8 A Finite Analogue of the Mapping Cylinder

The mapping cylinder of a map $f : X \to Y$ between topological spaces is the space Z_f obtained from $(X \times I) \sqcup Y$ by identifying each point $(x, 1) \in X \times I$ with $f(x) \in Y$. Both X and Y are subspaces of Z_f. We denote by $j : Y \hookrightarrow Z_f$ and $i : X \hookrightarrow Z_f$ the canonical inclusions where i is defined by $i(x) = (x, 0)$. The space Y is in fact a strong deformation retract of Z_f. Moreover, there exists a retraction $r : Z_f \to Y$ with $jr \simeq 1_{Z_f}$ rel Z_f which satisfies that $ri = f$ [75, Theorem 1.4.12].

We introduce a finite analogue of the classical mapping cylinder which will become important in Chap. 4. This construction was first studied in [8].

Definition 2.8.1. Let $f : X \to Y$ be a map between finite T_0-spaces. We define the *non-Hausdorff mapping cylinder* $B(f)$ as the following finite T_0-space. The underlying set is the disjoint union $X \sqcup Y$. We keep the given ordering within X and Y and for $x \in X$, $y \in Y$ we set $x \leq y$ in $B(f)$ if $f(x) \leq y$ in Y.

It can be proved that $B(f)$ is isomorphic to $(X \times \mathfrak{S}) \sqcup Y/_{(x,1)\sim f(x)}$ where $\mathfrak{S}$ denotes the Sierpinski space. However, we will omit the proof because this fact will not be used in the applications.

We will denote by $i : X \hookrightarrow B(f)$ and $j : Y \hookrightarrow B(f)$ the canonical inclusions of X and Y into the non-Hausdorff mapping cylinder.

Lemma 2.8.2. *Let $f : X \to Y$ be a map between finite T_0-spaces. Then Y is a strong deformation retract of $B(f)$.*

Proof. Define the retraction $r : B(f) \to Y$ of j by $r(x) = f(x)$ for every $x \in X$. Clearly r is order preserving. Moreover, $jr \geq 1_{B(f)}$ and then $jr \simeq 1_{B(f)}$ rel Y. □

By Corollary 2.2.5, for any map $f : X \to Y$ there is a strong collapse $B(f) \searrow\!\!\searrow Y$.

Since $ri = f$, any map between finite T_0-spaces can be factorized as a composition of an inclusion and a homotopy equivalence.

$$\begin{tikzcd} & B(f) \arrow[dr, "r"] & \\ X \arrow[ur, hook, "i"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

As in the classical setting, the non-Hausdorff mapping cylinder can be used to reduce many proofs concerning general maps to the case of inclusions. For example, f satisfies one of the following properties if and only if the inclusion i does: being a homotopy equivalence, a weak homotopy equivalence or a nullhomotopic map.

If X and Y are any two homotopy equivalent spaces there exists a third space Z containing both X and Y as strong deformation retracts. This space can be taken as the mapping cylinder of any homotopy equivalence $X \to Y$ (see [38, Corollary 0.21]). If $f : X \to Y$ is now a homotopy equivalence between finite T_0-spaces, Y is a strong deformation retract of $B(f)$ but X in general is just a (weak) deformation retract. Consider the space X and the point $x \in X$ of Example 2.2.6. The map $f : * \to X^{op}$ that maps $*$ into x is a homotopy equivalence. However $*$ is not a strong deformation retract of $B(f)$ by Corollary 2.2.5 because $(B(f), *)$ is a minimal pair. Although X is not in general a strong deformation retract of $B(f)$ for a homotopy equivalence $f : X \to Y$, we will see that if two finite T_0-spaces are homotopy equivalent, there exists a third finite T_0-space containing both as strong deformation retracts. This is stated in Proposition 4.6.6.

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