

Chapter 2

Static Replication

A portfolio is said to be static if it is unmanaged, which means that the content is not changed through time. In this chapter we review some important situations where static replication can be used for pricing or for finding upper and lower bounds on prices.

We start with pricing forward contracts and general fixed-time payments. We then derive constraints on option prices in preparation for the next chapter, where options are priced with dynamic replication. Finally, the method of static replication is applied to more exotic contracts such as early exercisables and barrier options.

2.1 Forward Contracts

Under the specifications of a *forward contract*, the counterparties are obliged to exchange a certain underlying S for a *strike* price K at a given *maturity* T . The contract is therefore worth $S - K$ at T .

The pricing of a forward contract is trivial as we can immediately conclude that the time t value is $S - KP_{tT}$. Indeed, if we own this amount at t , by selling K bonds maturing at T , enough money is generated for a purchase of the underlying S . The strategy is worth $S - K$ at maturity, which is the same amount as that of the forward contract. The no-arbitrage principle implies that the forward contract must be worth $S - KP_{tT}$.

The cash amount K that is used in the initiation of a forward contract is by convention such that the contract *values to par*, i.e. the price equals to zero. This value of the strike is called the *forward*. It is usually denoted by F and is equal to $P_{0T}^{-1} S$.

2.2 European Options

A *European call option* gives the contract owner the right to buy the underlying S at time T for a given amount K . In contrast to forward contracts, the owner is not committed to the purchase but only does so if it is profitable. The option value at maturity is therefore $(S_T - K)_+ = \max(S_T - K, 0)$.

An option is said to be *in the money (ITM)* if $S > K$, *out of the money (OTM)* if $S < K$ and *at the money (ATM)* if $S = K$. Sometimes the forward is used in this classification, i.e. the conditions $F > K$, $F < K$ and $F = K$ are used to define whether an option is ITM, OTM or ATM. It is usually clear from the context which definition is used.

A *European put option* gives the owner the right to sell the underlying. The value at maturity is $(K - S_T)_+$. A *digital European call option* pays \$1 if $S_T > K$ and 0 otherwise. Thus, the value at maturity is $\theta(S_T - K)$, where θ is the *Heaviside function*. Similarly, a *digital European put option* is worth $\theta(K - S_T)$ at T .

Options belong to the type of contracts that cannot be priced with static replication. We postpone the pricing of these contracts to the next chapter in which dynamic replication is introduced. For the remainder of this chapter, we assume that option prices are known and use them for the static replication of more complex contracts.

2.3 Non-Linear Payoffs

We determine the present value of a contract V that pays $h(S)$ at T for a fixed, but arbitrary, function h with a well-defined second derivative. We have already covered the special case $h(S) = S - K$, for which the pricing can be done using the present values of the underlying S and the zero-coupon bond maturing at T . In the same manner, any linear payoff $h(S) = \alpha S - K$ can be priced. When $h(S)$ is non-linear, on the other hand, additional information is needed. The present values of European call options maturing at T turn out to provide sufficient information. This statement is made clear by the following computation:

$$\begin{aligned}
 h(S) &= \int_0^\infty h(K) \delta(S - K) dK \\
 &= \int_0^\infty \left(-\frac{d}{dK} (h(K) \theta(S - K)) + h'(K) \theta(S - K) \right) dK \\
 &= h(0) + \int_0^\infty h'(K) \theta(S - K) dK \\
 &= h(0) + \int_0^\infty \left(-\frac{d}{dK} (h'(K) (S - K)_+) + h''(K) (S - K)_+ \right) dK \\
 &= h(0) + h'(0)S + \int_0^\infty h''(K) (S - K)_+ dK
 \end{aligned}$$

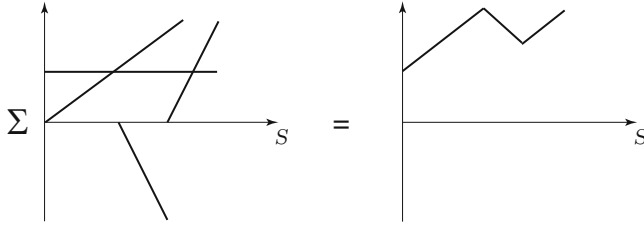


Fig. 2.1 Replication of an arbitrary payoff with zero-coupon bonds, the underlying and call options

Thus, if we at time $t = 0$ buy $h(0)$ number of bonds maturing at T , $h'(0)$ number of underlyings S and $h''(K)$ number of options with strikes in $[K, K + dK]$, for all K , then the contract is worth $h(S)$ at T (Fig. 2.1). The no-arbitrage principle implies that

$$V = h(0)P_{0T} + h'(0)S + \int_0^\infty h''(K)V^C(K)dK$$

where $V^C(K)$ is the present value of a call option with strike K .

Fixed-time payoffs can also be statically replicated with other option types. For example, according to the third line in the calculation above, digital calls options can be used. In this instance, the underlying S is not needed for the replication.

We move on to discuss which option type is preferable in the replication, call options or digital call options. From a theoretical point of view, the question is irrelevant as the two product types can be statically replicated from each other. For instance, using the third line in the above equation for $h(S) = (S - K)_+$ gives

$$(S - K)_+ = \int_0^\infty \theta(K' - K)\theta(S - K')dK' = \int_K^\infty \theta(S - K')dK'$$

Conversely, the relation $\theta(S - K) = -\frac{d}{dK}(S - K)_+$ shows that a digital call option can be approximated by a *call spread*, i.e. the difference between two call options, having the following payoff at T :

$$\frac{1}{\Delta K} ((S - K + \Delta K)_+ - (S - K)_+)$$

We conclude that a digital call option can be approximated by two call options while a large number of digitals are needed to approximate a call option. This suggests that call options should be used in static replication of contracts paying $h(S)$. The main reason for using call options, however, is that they are more liquid market instruments than digital call options.

The replicating formula is also applicable to payoffs with a discontinuous (mathematical) derivative. Consider, for example, a put option paying $h(S) =$

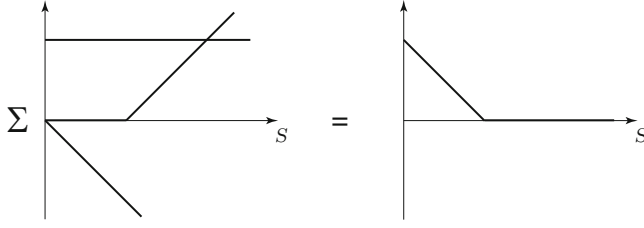


Fig. 2.2 Put-call parity

$(K - S)_+$ at T . Using $h'(S) = -\theta(K - S)$ and $h''(S) = \delta(K - S)$ in the replication formula gives:

$$V^P(K) = KP_{0T} - S + V^C(K)$$

This relation is called *put-call parity* and shows that a call and a put option only differ by a linear payoff (Fig. 2.2). The parity is obvious as the difference in payoff at maturity

$$(S - K)_+ - (K - S)_+ = S - K$$

is equal to the payoff of a forward contract.

Since a put equals a call up to a linear payoff, some of the calls in the replication formula can be replaced with puts. As puts are cheaper than calls when the strike is low, the cost of the options in the replication formula can be reduced by replacing low strike calls with puts. The details can be understood from the computation

$$\begin{aligned} & \int_0^{\bar{K}} h''(K)(K - S)_+ dK + \int_{\bar{K}}^{\infty} h''(K)(S - K)_+ dK \\ &= h'(\bar{K})(\bar{K} - S)_+ - \int_0^{\bar{K}} h'(K)\theta(K - S) dK \\ & \quad - h'(\bar{K})(S - \bar{K})_+ + \int_{\bar{K}}^{\infty} h'(K)\theta(S - K) dK \\ &= h'(\bar{K})(\bar{K} - S)_+ - h(\bar{K})\theta(\bar{K} - S) + \int_0^{\bar{K}} h(K)\delta(K - S) dK \\ & \quad - h'(\bar{K})(S - \bar{K})_+ - h(\bar{K})\theta(S - \bar{K}) + \int_{\bar{K}}^{\infty} h(K)\delta(S - K) dK \\ &= h'(\bar{K})(\bar{K} - S) - h(\bar{K}) + h(S) \end{aligned}$$

The no-arbitrage principle implies that

$$\begin{aligned} & \int_0^{\bar{K}} h''(K) V^P(K) dK + \int_{\bar{K}}^{\infty} h''(K) V^C(K) dK \\ &= (h'(\bar{K})\bar{K} - h(\bar{K})) P_{0T} - h'(\bar{K})S + V \end{aligned}$$

which shows how a fixed-time payoff can be replicated with low strike puts and high strike calls. Denoting the right-hand side with $g(\bar{K})$, we obtain

$$\begin{aligned} g'(\bar{K}) &= h''(\bar{K})(\bar{K} P_{0T} - S) \\ g''(\bar{K}) &= h'''(\bar{K})(\bar{K} P_{0T} - S) + h''(\bar{K}) P_{0T} \end{aligned}$$

Assume for a moment that the second derivative of h is positive. The only extreme point of $g(\bar{K})$ is then a minimum located at the forward $\bar{K} = P_{0T}^{-1}S = F$. We conclude that

$$V = (h(F) - h'(F)F) P_{0T} + h'(F)S + \int_0^F h''(K) V^P(K) dK + \int_F^{\infty} h''(K) V^C(K) dK$$

is the replication of V that has the cheapest option content. The same result is obtained if h has a negative second derivative.

Liquid European options are found in the market only for a finite set of strikes. It means that the static replication strategy for non-linear payoff is not directly applicable in practice. Instead, European option prices for arbitrary strikes are typically inferred from the liquid market quotes by mathematical interpolation. Once this has been done, static replication can be used. As the outcome depends on the interpolation scheme, different market participants arrive at different conclusions regarding the price. This is particularly apparent when the payoff depends on strikes outside the liquid range, making extrapolation a necessity.

2.4 European Option Price Constraints

Before constructing option pricing models, it is useful to derive the asymptotic limits and the no-arbitrage conditions that a European call option price V has to satisfy. These conditions can be used to exclude inappropriate models. As the corresponding constraints for put options follow from put-call parity, it is sufficient to focus on European call options.

Consider first the asymptotic behavior of V : for very large values of K the call option is worthless, $V = 0$. In the limit of small values of K , the option certainly gets exercised at maturity. The option holder then needs the amount $K P_{0T}$ today to pay the strike price K at T in order to receive S . Today's value of the contract is therefore $S - K P_{0T}$.

We proceed to the no-arbitrage conditions and observe that the value of a contract paying $h(S) \geq 0$ at T is obviously positive. As $h(S) = \int_0^{\infty} h(K) \delta(S - K) dK$, this

requirement is equivalent with demanding positivity of a contract paying $\delta(S - K)$. Using

$$\delta(S - K) = \frac{d^2}{dK^2}(S - K)_+ = \frac{d^2}{dK^2}V_T$$

which is the $\Delta K \rightarrow 0$ limit of $(\Delta K)^{-2}(V_T(K + \Delta K) - 2V_T(K) + V_T(K - \Delta K))$, we obtain the constraint $\frac{d^2}{dK^2}V \geq 0$.

There is also a constraint for option combinations with different maturities: $V_2 - V_1$ is positive for V_1 and V_2 European call options with strike K and maturities $T_1 < T_2$. The statement can be verified by proving that a portfolio of one long unit of V_2 and one short unit of V_1 is always positive at T_1 . It is sufficient to consider the situation when V_1 is ITM at T_1 , giving a portfolio value of $V_2 - S + K$. From the conditions $V(K \rightarrow \infty) \rightarrow 0$, $V(K \rightarrow 0) \rightarrow S - KP_{0T}$ and $V''(K) \geq 0$ it follows that $V(K) \geq S - KP_{0T}$, which implies that $V_2(t = T_1) \geq S - KP_{T_1T_2} \geq S - K$, proving the statement. Observe that $V_2 \geq V_1$ is equivalent with the infinitesimal condition $\frac{dV}{dT} \geq 0$.

In summary, the following constraints must be satisfied by the European call option price (Fig. 2.3):

- $V(K \rightarrow 0) \rightarrow S - KP_{0T}$
- $V(K \rightarrow \infty) \rightarrow 0$
- $V(T \rightarrow 0) \rightarrow (S - K)_+$
- $\frac{d^2}{dK^2}V \geq 0$
- $\frac{d}{dT}V \geq 0$

From these fundamental constraints, it is possible to derive other interesting conditions on the option price. For instance, from conditions 1, 2 and 4, we obtain upper and lower bounds on the European call option price and its (mathematical) derivative with respect to the strike (which is the digital option price).

$$\begin{aligned} (S - KP_{0T})_+ &\leq V \leq S \\ -P_{0T} = \frac{dV}{dK}\Big|_{K=0} &\leq \frac{dV}{dK} \leq \frac{dV}{dK}\Big|_{K=\infty} = 0 \end{aligned}$$

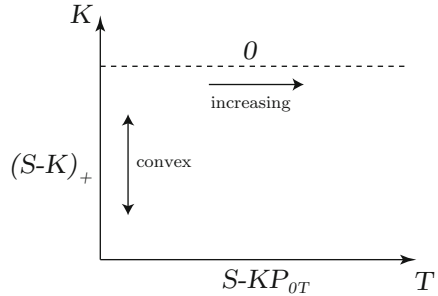


Fig. 2.3 No-arbitrage conditions and asymptotics for European call option prices

2.5 American and Bermudan Options

American options can be exercised any time up to the maturity T . For example, an American call option exercised at $t < T$ gives a payment $S - K$ at t . This is in contrast to European options which can only be exercised at maturity T . *Bermudan options* are something in between (just as Bermuda lies somewhere between Europe and America): they can only be exercised at certain prespecified dates. The extra optionality makes an American option more valuable than a Bermudan option, which in turn is worth more than a European option. There are, however, many instances where this extra optionality is worthless and all option types have equal value.

We saw in the previous section that $(S - KP_{0T})_+$ is a lower bound for the European call option price. As this amount is greater than the exercise value $S - K$, a call option should never be exercised early. We conclude that American, Bermudan and European call options have equal prices. Observe that if we permit the underlying to have cash flows such as dividend payments, there can be situations for which it is optimal to exercise early in order to obtain these cash flows.

For European put options, the lower bound $(KP_{0T} - S)_+$ is below $K - S$ for $S < K$ which means that there are instances when an early exercise is preferable. American and Bermudan put options are therefore worth strictly more than their European counterpart. To find an upper price bound, observe that, because of the possibility to exercise early, American and Bermudan put options with a time-dependent strike KP_{tT} must be worth more than a European put option with strike K . However, as $(KP_{tT} - S)_+$ is a lower bound for the European price, an early exercise is not feasible as it yields $KP_{tT} - S$. We conclude that the American, Bermudan and European put options have equal prices in this instance of a time-dependent strike. As put option prices increase with the strike, American and Bermudan put options with strike KP_{0T} are worth less than the corresponding options with strike P_{tT} , which in turn is worth as much as a European put option with strike K . Replacing K with KP_{0T}^{-1} , we conclude that American and Bermudan put options with strike K are bounded from below by the European put option with strike K and from above by the European put option with strike KP_{0T}^{-1} .

The argument leading to put-call parity for European options do not carry through to American and Bermudan options. Instead, the parity relation can be replaced with an upper and lower bound on the put option price when formulated in terms of the call option, or vice versa. Indeed, using put-call parity on the European put options in the above bounds together with the fact that American, Bermudan and European call options are worth equally much, we obtain

$$KP_{0T} - S + V^C \leq V^P \leq K - S + V^C$$

where we have used the common notation V for both American and Bermudan options. The right-hand side option has strike KP_{0T}^{-1} which because of decaying call prices with strike values can be replaced with strike K .

Early exercise decisions for call and put options with zero strike value are particularly simple to analyze. Consider first the trivial situation of a call option on an underlying that is restricted to being positive. As there is no cost in exercising the option, we definitely do that at some point in time and it does therefore not matter when the exercise is made. When the underlying can be negative as well as positive, it is suboptimal to exercise early in the zero strike case. Indeed, exercising early and holding the underlying to maturity is associated with the risk of the underlying becoming negative, which can be avoided by postponing the exercise till maturity. A swaption, i.e. an option on a swap, is an example of a zero strike option on an underlying that can be negative.

2.6 Barrier Options

We consider *barrier options* that are of call type, i.e. they pay $(S - K)_+$ at T if the underlying S breached (or did not breach) a barrier level B sometime between $t = 0$ and T . Options of put type can be treated in a parallel way. A barrier option is said to be of knock-out type if the payment occurs conditional on that the barrier was not breached. If the barrier needs to be breached for the payment to occur, the option is said to be of knock-in type. For instance, the payoff for a knock-out call option with a lower barrier can be written as $\mathbb{1}_{\min\{S_t \in [0, T]\} > B} (S_T - K)_+$.

As a barrier is either breached or not, the sum of a knock-out option and a knock-in option is equal to a standard option. This is known as the parity relation for barrier options. Assuming that we know how to price standard options, we can focus on the pricing of one of the option types. We choose to focus on knock-outs.

Knock-out call options can be classified into four different types depending on whether the barrier is above or below the strike and on whether the barrier is an upper or lower barrier. For an upper barrier that is below the strike, the call option is worthless, which means that there are only three non-trivial types of knock-outs.

Let us start with a lower barrier that is below the strike. The barrier lies in the out-of-the-money region and has relatively little effect on the option. Under certain modeling assumptions, we compute its price $V_{B,K}^C$ in the next section. The corresponding digital option, obtained from the K derivative of $V_{B,K}^C$, is denoted by $\bar{V}_{B,K}^C$. The corresponding put options are denoted by $V_{B,K}^P$ and $\bar{V}_{B,K}^P$ and have an upper barrier that lies above the strike. For now, we assume that these prices are given and use them to price the other two types of knock-out options.

When the lower barrier lies above the strike, we see in Fig. 2.4 that the price of the knock-out option equals

$$V_{B,B}^C + (B - K) \bar{V}_{B,B}^C$$

In the instance of an upper barrier above the strike, we see in Fig. 2.5 that this contract can be written as a sum of a spread put and a digital put, all with an upper

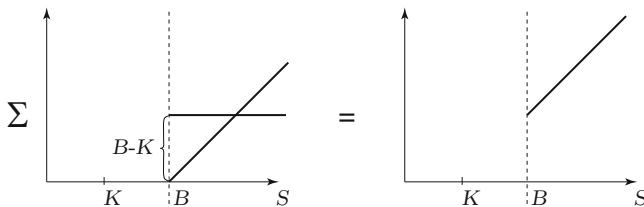


Fig. 2.4 Replication of a barrier option with strike < barrier < spot

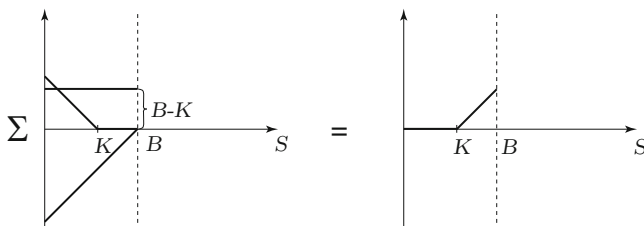


Fig. 2.5 Replication of a barrier option with spot, strike < barrier

knock-out barrier. The price is equal to

$$V_{B,K}^P - V_{B,B}^P + (B - K)\bar{V}_{B,B}^P$$

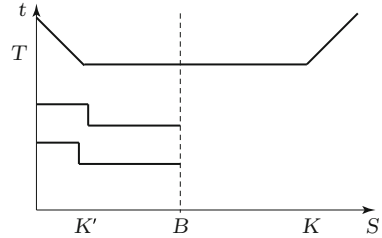
2.7 Model-Dependent Pricing

The relations derived so far have been independent of the process followed by the underlying and can be viewed as model independent results. We now show how static replication can be applied in more complex situations by using modeling assumptions. We illustrate the technique by showing how barrier options can be statically replicated by European options.

We price a knock-in option that matures at T and has strike K and barrier $B < K$. The corresponding knock-out option can be priced using the parity relation for barrier options. An alternative pricing method for barrier options is presented in Sect. 9.1.

We initially assume that the underlying value S equals the barrier value which means that the option has knocked in. It is clear graphically from Fig. 2.6 that if K is large enough, there exists a put option with strike $K' < B$ that is worth as much as the call option. We temporarily assume that the call and the put are equal at all times as long as $S = B$, i.e. along the dotted line in the figure. Under this assumption the knock-in call option is worth as much as the put option. Indeed, their prices are equal if the barrier is touched while they are both worthless otherwise. Observe that the method is model dependent as a model must be used to find the strike for the put option.

Fig. 2.6 Relication of a knock-in call by a put option and a strip of digital put options



It is possible to relax the assumption that the two options should have equal prices along the whole barrier. To prove this statement, assume first that their prices are equal along the barrier when close to maturity. The further away we come from the maturity, the more the prices start to deviate. When far enough away from the maturity, the prices will differ more than a specified tolerance level. The reason is that a different put strike should have been used for the two options to have equal values at the barrier. The incorrectness in the payoff profile used at maturity is therefore equal to the difference between two put option, with similar strikes, which can be approximated by a digital put option. The price difference at the current time can therefore approximately be adjusted by adding a digital put option. When going further back in time along the barrier the price difference will increase again, which can be periodically reset by adding more digital put options, see Fig. 2.6.

The reflection in the barrier can be done more efficiently by not using a single put options but several put options with the same strike K' . The result is a payoff profile with a steeper slope. This reflects the payoff of the call for longer time periods away from T , reducing the number of digital put options along the boundary. An example of when more than one put option is necessary is within the lognormal model that will be discussed thoroughly in the book.

The replication becomes particularly simple when the strike equals the barrier and interest rates are assumed to be zero. The price of a knock-out call option is then $S_0 - K$. The payoff can be replicated by holding this amount of cash and by entering at zero cost a forward contract to purchase the underlying for S_0 at T . If the barrier is not touched, the replicating strategy is worth $(S_T - S_0) + S_0 - K = S_T - K$ which is as much as the barrier option value. On the other hand, the forward position can be liquidated should the option be knocked out, yielding $(B - S_0) + S_0 - K = 0$ and showing that the replication is successful.



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