9. The Gardner Formula for the Discrete Cube

9.1 Overview

This chapter continues the work of Chapter 2. We study the Hamiltonian

$$- H_{N,M}(\boldsymbol{\sigma}) = \sum_{k \le M} u(S_k) \; ; \quad S_k = S_k(\boldsymbol{\sigma}) = \frac{1}{\sqrt{N}} \sum_{i \le N} g_{i,k} \sigma_i \; . \tag{9.1}$$

We are concerned mainly with the case where $\exp u(x)$ is nearly $\mathbf{1}_{\{x \ge \tau\}}$ for a certain number τ (which is fixed once and for all). We assume the following

$$u \le 0 \; ; \quad x \ge \tau \quad \Rightarrow \quad u(x) = 0 \; . \tag{9.2}$$

Since it is very desirable that u be differentiable, we assume that $u^{(3)}$ exists, and that for a certain number D

$$1 \le \ell \le 3 \quad \Rightarrow \quad |u^{(\ell)}| \le D \; . \tag{9.3}$$

The difference between the work of Chapter 2 and the work we are going to present is that the dependence on D of our estimates will be much weaker; every occurrence of D in the estimates will now be multiplied by an exponentially small factor $\exp(-N/L)$. This will allow to have D depend on N. The overall content of the present chapter is that there exists $\alpha(\tau) > 0$ such that if $M/N \le \alpha(\tau)$ and (9.3) holds for $D = \exp(N/L)$ then we understand very well the system governed by the Hamiltonian (9.1). The very weak requirement (9.3) for $D = \exp(N/L)$ allows to find (given N and M) a function u satisfying this requirement and for which $\exp u(x)$ is a very good approximation of $\mathbf{1}_{\{x \ge \tau\}}$. It is worth repeating this. We will approximate the function $\mathbf{1}_{\{x \ge \tau\}}$ by a function u which varies with N. What makes the argument work is that condition (9.3) for $D = \exp(N/L)$ becomes very weak for large N.

At this point it is probably wise to make explicit a rather important difference between the way we look at spin glasses and traditional statistical mechanics. In spin glasses, there is no "limiting system" as $N \to \infty$, and the object under study is really the system considered for a given large value of N. With this in mind, it is quite natural to try to approximate the function $\mathbf{1}_{\{x \ge \tau\}}$ by a function u that depends on the situation under study, i.e. on N.

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The central difference between the situation of Chapter 8 and the present situation is that we no longer have a magic proof of the fact that $R_{1,2} \simeq \langle R_{1,2} \rangle$, and we will have to work very hard to prove that $R_{1,2} \simeq$ Const. (On the other hand, the fact that the spins are bounded removes several minor - yet irritating - sources of complications.)

The overall approach is the same as in Chapter 2, and it would be very helpful for the reader to have Sections 2.2 and 2.3 at hand while proceeding. We use the smart path of Section 2.1, and we attempt to show that the terms I and II of Proposition 2.2.2 nearly cancel out. This is done though the "cavity in M" method of Section 2.3; what we need is a better estimate than Lemma 2.3.2 provides. In the remainder of this section we try to outline the general strategy that will achieve this. Since we describe the overall structure of the approach, we do not recall the definitions of the various quantities involved in complete detail, as these details are irrelevant now and will be given in due time. For the time being, we recall that the average $\langle \cdot \rangle_{t,\sim}$ corresponds to the Hamiltonian (2.30) (i.e. when M has been replaced by $M-1$), while $\nu_{t,v}$ is given by the formula

$$\nu_{t,v}(f) = \mathsf{E} \frac{\left\langle f \exp \sum_{\ell \le n} u(S_v^\ell) \right\rangle_{t,\sim}}{\langle \mathsf{E}_\xi \exp u(S_v^1) \rangle_{t,\sim}^n} ,$$

where E_ξ denotes expectation "in all the r.v.s labeled ξ". At first sight the above formula differs from the formula (2.35). This is simply because in (2.35) we made the convention that the expectation E_ξ is built-in the bracket $\langle \cdot \rangle_{t,\sim}$, while in the present chapter we find it more economical to write explicitly this expectation instead of constantly reminding the reader of this convention.

Given a function f on Σ_N^4, and $B_v \equiv 1$ or $B_v \equiv u'(S_v^1)u'(S_v^2)$, we want to bound $\frac{\mathrm{d}}{\mathrm{d}v}\nu_{t,v}(B_v f)$. After differentiation and integration by parts, this quantity is a sum of terms of the type

$$\nu_{t,v}(f(R_{1,2}^t - q)A) \tag{9.4}$$

where A is a monomial in the quantities $u'(S_v^\ell)$, $u''(S_v^\ell)$, $u'''(S_v^\ell)$ and where $R_{1,2}^t = N^{-1}\left(\sum_{i<N} \sigma_i^1 \sigma_i^2 + t\sigma_N^1 \sigma_N^2\right)$. Of course it does not matter that we have $R_{1,2}^t$ rather than $R_{1,2}$. The problem is that A might take huge values, because the derivatives of u can be very large (which could not happen in Section 2.3) and we have to show that somehow these huge values cancel out. With the notation of Section 2.2 we have

$$\begin{aligned} \nu_{t,v}(f(R_{1,2}^t - q)A) &= \mathsf{E}\, \frac{\left\langle f(R_{1,2}^t - q)\mathsf{E}_\xi A \exp \sum_{\ell \le n} u(S_v^\ell) \right\rangle_{t,\sim}}{\langle \mathsf{E}_\xi \exp u(S_v^1) \rangle_{t,\sim}^n} \\ &= \mathsf{E}\,\mathsf{E}'\, \frac{\left\langle f(R_{1,2}^t - q)\mathsf{E}_\xi A \exp \sum_{\ell \le n} u(S_v^\ell) \right\rangle_{t,\sim}}{\langle \mathsf{E}_\xi \exp u(S_v^1) \rangle_{t,\sim}^n} , \end{aligned} \tag{9.5}$$

where E' denotes expectation only in the randomness of the S_v^ℓ. This randomness is independent of the randomness of $\langle \cdot \rangle_{t,\sim}$. We then separate the numerator and the denominator using the Cauchy-Schwarz inequality

$$\mathsf{E}' \left| \frac{\langle f(R^t_{\ell,\ell'} - q)\mathsf{E}_\xi A \exp \sum_{\ell \le n} u(S_v^\ell)\rangle_{t,\sim}}{\langle \mathsf{E}_\xi \exp u(S_v^1)\rangle^n_{t,\sim}} \right| \le \mathrm{I} \times \mathrm{II} \tag{9.6}$$

where

$$\mathrm{I} = \left(\mathsf{E}' \Big\langle f(R^t_{\ell,\ell'} - q)\mathsf{E}_\xi A \exp \sum_{\ell \le n} u(S_v^\ell) \Big\rangle^2_{t,\sim} \right)^{1/2} \tag{9.7}$$

and

$$\mathrm{II} = \left(\mathsf{E}' \frac{1}{\langle \mathsf{E}_\xi \exp u(S_v^1)\rangle^{2n}_{t,\sim}} \right)^{1/2} . \tag{9.8}$$

We will bound both terms separately. To bound the denominator in (9.8) from below we cannot do better than

$$\langle \mathsf{E}_\xi \exp u(S_v^1)\rangle_{t,\sim} \ge \langle \mathsf{E}_\xi \mathbf{1}_{\{S_v^1 \ge \tau\}}\rangle_{t,\sim} .$$

This quantity is closely connected (in particular when $v = 1$) to the quantity $\langle \mathbf{1}_{\{S_M \ge \tau\}}\rangle_{t,\sim}$, a random variable for which we have obtained the estimate (8.23). This estimate is however insufficient, even if we consider only the case $n = 6$. Indeed, given a random variable $X \ge 0$, to obtain the integrability of X^{-12}, it does not suffice to know that $\mathsf{P}(X \le \varepsilon) \le \varepsilon^{1/L}$, we need something like $\mathsf{P}(X \le \varepsilon) \le \varepsilon^a$ for $a > 12$. So we will have to improve on the estimate (8.23), and this will be the purpose of Section 9.3.

To control the term (9.7), if $A = A((S_v^\ell)_{\ell \le n})$, let $A' = A((S_v^{\ell+n})_{\ell \le n})$ and define a replicated version f' of f similarly. Then

$$\begin{aligned} &\mathsf{E}' \left\langle f(R^t_{1,2} - q)\mathsf{E}_\xi A \exp \sum_{\ell \le n} u(S_v^\ell) \right\rangle^2_{t,\sim} \\ &= \left\langle f f'(R^t_{1,2} - q)(R^t_{n+1,n+2} - q)\mathsf{E}'\mathsf{E}_\xi A A' \exp \sum_{\ell \le 2n} u(S_v^\ell) \right\rangle_{t,\sim} . \end{aligned} \tag{9.9}$$

To control this quantity, we will prove the following. There is an exponentially small set of configurations $(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^{2n})$ such that, outside this set, we have

$$\left| \mathsf{E}'\mathsf{E}_\xi A A' \exp \sum_{\ell \le 2n} u(S_v^\ell) \right| \le L .$$

The reason for this is simply that when there is enough independence among the r.v.s $(S_v^\ell)_{\ell \le 2n}$, one can eliminate the derivatives of u occurring in A and A' through integration by parts (and these were the cause for A to be large). On the exceptionally small set of configurations we use (9.3) to control $|A|$. In this manner we will prove that the quantity (9.9) is at most

$$\begin{aligned}&L\langle |f||f'||R^t_{1,2}-q||R^t_{\ell+n,\ell'+n}-q|\rangle_{t,\sim}+\mathcal{R}\\&=L\langle |f||R^t_{1,2}-q|\rangle^2_{t,\sim}+\mathcal{R}\end{aligned}$$

where $\mathcal{R}$ is exponentially small. Therefore

$$\mathrm{I}\le L\langle |f||R^t_{1,2}-q|\rangle_{t,\sim}+\mathcal{R}$$

which (modulo the fact that we have $\langle\cdot\rangle_{t,\sim}$ rather than $\langle\cdot\rangle_t$) is very much what we are looking for. We should also point out that it does not work to use the Cauchy-Schwarz inequality on the whole of E in (9.5); this would yield a bound $\mathsf{E}\langle |f|^2|R^t_{1,2}-q|^2\rangle^{1/2}$, which is useless.

Learning how to perform integration by parts will occupy Section 9.4.

There is a further complication. Each of the two bounds previously described needs the knowledge that the average $\langle\cdot\rangle_{t,\sim}$ is not pathological. We know how to prove this when $Z_{N,M}=\sum_{\boldsymbol{\sigma}}\exp(-H_{N,M}(\boldsymbol{\sigma}))$ is not too small. We will prove a priori that this is the case with overwhelming probability, provided $\alpha=M/N$ is not too large.

Once these obstacles are overcome, we can recover the results of Section 2.4 when $L\alpha\exp L\tau^2\le 1$, but this time under the much less stringent conditions (9.3) with $D=\exp(N/L)$. This will be done in Section 9.5. The main estimate is obtained in Proposition 9.5.5. Roughly speaking, this Proposition replaces the estimate

$$\left|\frac{\mathrm{d}}{\mathrm{d}v}\nu_{t,v}(B_vf)\right|\le K(D)\left(\nu_{t,v}(|f|^{\tau_1})^{1/\tau_1}\nu_{t,v}(|R_{1,2}-q|^{\tau_2})^{1/\tau_2}+\frac{1}{N}\nu_{t,v}(|f|)\right)$$

of Lemma 2.3.2 by the estimate

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}v}\nu_{t,v}(B_vf)\right|\le L\exp L\tau^2\Bigg(&\nu_{t,v}(|f|^{\tau_1})^{1/\tau_1}\nu_{t,v}(|R_{1,2}-q|^{\tau_2})^{1/\tau_2}\\&+\frac{1}{N}\nu_{t,v}(|f|)+\max|f|D^4\exp\left(-\frac{N}{L}\right)\Bigg)\,.\end{aligned}$$

From this, we will be able, in Section 9.6, to deduce the Gardner formula for the cube when $L\alpha\exp L\tau^2\le 1$ by repeating (in a simpler manner) the arguments of Section 8.3. We will also, in Section 9.8, show the surprising fact that, in the end, the differentiability of u is largely irrelevant. In the remainder of the chapter, we will prove a central limit theorem for the overlaps, and we will investigate the Bernoulli model, when the Gaussian randomness is replaced by coin-flipping randomness.

9.2 A Priori Estimates

We already have the tools to prove that $Z_{N,M}$ is typically not too small. This will be done in Theorem 9.2.3 below. We start by a simple observation.

Lemma 9.2.1. *Consider a probability measure G on Σ_N; assume that G has a density proportional to $W \le 1$ with respect to the uniform measure on Σ_N, and assume that for a certain number $t > 0$ we have*

$$Z := \sum_{\boldsymbol{\sigma}} W(\boldsymbol{\sigma}) \ge 2^N \exp\left(-\frac{Nt^2}{8}\right) . \tag{9.10}$$

Then we have

$$G^{\otimes 2}\big(\{(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)\ ;\ R_{1,2} \ge t\}\big) \le \exp\left(-\frac{Nt^2}{4}\right) . \tag{9.11}$$

Proof. Using that $W \le 1$ in the second line and (9.10) in the third line, we obtain

$$\begin{aligned} G^{\otimes 2}(\{(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)\ ;\ R_{1,2} \ge t\}) &= \frac{1}{Z^2} \sum_{R_{1,2} \ge t} W(\boldsymbol{\sigma}^1) W(\boldsymbol{\sigma}^2) \\ &\le \frac{1}{Z^2} \mathrm{card}\{(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)\ ;\ R_{1,2} \ge t\} \\ &\le 2^{-2N} \exp\left(\frac{Nt^2}{4}\right) \mathrm{card}\{(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)\ ;\ R_{1,2} \ge t\} . \end{aligned}$$

Now, if $(\eta_i)_{i \le N}$ are independent Bernoulli r.v.s,

$$\begin{aligned} 2^{-2N} \mathrm{card}\{(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)\ ;\ R_{1,2} > t\} &= \mathsf{P}\left(\sum_{i \le N} \eta_i \ge tN\right) \\ &\le \exp\left(-\frac{Nt^2}{2}\right) \end{aligned}$$

by the subgaussian inequality (A.16) used for $a_i = 1$. $\square$

Lemma 9.2.2. *There exists a number L and a number $\lambda_0 > 0$ with the following property. Consider a probability measure G on Σ_N; assume that G has a density proportional to $W \le 1$ with respect to the uniform measure on Σ_N, and assume that*

$$Z := \sum_{\boldsymbol{\sigma}} W(\boldsymbol{\sigma}) \ge 2^N \exp\left(-\frac{N}{32}\right) . \tag{9.12}$$

Then, for independent standard normal r.v.s $(g_i)_{i \le N}$ we have

$$\begin{aligned} & L \exp\left(-\frac{N}{L}\right) \le \varepsilon \le \frac{1}{L} \exp(-L\tau^2) \\ \Rightarrow\quad & \mathsf{P}\left(G\left(\left\{\boldsymbol{\sigma}\ ;\ \frac{1}{\sqrt{N}} \sum_{i \le N} g_i \sigma_i \ge \tau\right\}\right) \le \varepsilon\right) \le \varepsilon^{1/L} . \end{aligned} \tag{9.13}$$

Moreover the r.v.

$$V = \frac{1}{\max\Big(\exp(-N/32), G\big(\{\boldsymbol{\sigma}\ ;\ \frac{1}{\sqrt{N}}\sum_{i\le N} g_i\sigma_i \ge \tau\}\big)\Big)} \tag{9.14}$$

satisfies

$$\mathsf{E}V^{\lambda_0} \le L\exp L\tau^2\ . \tag{9.15}$$

Of course the value $1/32$ is just a convenient choice. We write

$$a = \frac{1}{32}\ ;\ b^* = \exp(-aN) = \exp(-N/32)$$

throughout this chapter.

Proof. From (9.12) and Lemma 9.2.1, we see that (9.11) holds for $t = 1/2$. Thus (9.13) follows from Proposition 8.2.6 used for $b = 1$ and $c = 1/2$, $d = 32$. The r.v. V satisfies

$$\begin{aligned} t > \exp aN \quad &\Rightarrow \quad \mathsf{P}(V > t) = 0 \\ L\exp(L\tau^2) \le t \le \frac{1}{L}\exp\left(\frac{N}{L}\right) \quad &\Rightarrow \quad \mathsf{P}(V > t) \le Lt^{-1/L}\ , \end{aligned}$$

and the conclusion follows from Lemma 8.3.8. □

We recall that S_k is defined in (9.1) and we state the main result of this section.

Theorem 9.2.3. *There exists a number L with the following property. If $b \ge b^* = \exp(-aN)$, then*

$$\mathsf{P}\big(\mathrm{card}\{\boldsymbol{\sigma}\ ;\ \forall k \le M\ ,\ S_k(\boldsymbol{\sigma}) \ge \tau\} \le b2^N\big) \le b^{1/L}\exp(LM(1+\tau^2))\ . \tag{9.16}$$

This inequality is of interest only for $b \le 1$ so the larger the value of L, the weaker the inequality. If we take $b = b^* = \exp(-N/32)$, the right-hand side is $\exp(L_1M(1+\tau^2) - N/L_1)$, which is exponentially small as soon as $2L_1^2\alpha(1+\tau^2) \le 1$. This might be the place to remind the reader that by L we always denote a number, that does not depend on any parameter whatsoever, but that need not be the same at each occurrence. With this convention, the shorthand way to write the previous claim is that “when $b = b^* = \exp(-N/32)$, the right-hand side of (9.16) is exponentially small when $L\alpha(1+\tau^2) \le 1$” The reader will then understand by herself that the constant L occurring in this inequality is a new number that depends only on the (different) number L occurring in the right-hand side of (9.16).

Since $u(x) = 0$ for $x \ge \tau$, we have

$$Z_{N,M} \ge \mathrm{card}\{\boldsymbol{\sigma}\ ;\ \forall k \le M\ ,\ S_k(\boldsymbol{\sigma}) \ge \tau\}\ ,$$

and the previous result shows that $Z_{N,M}$ is typically $\ge 2^N\exp(-aN)$ when $L\alpha(1+\tau^2) \le 1$. Therefore in that case the Gibbs measure typically satisfies (9.13).

Research Problem 9.2.4. (Level 2) Can the main results of this chapter be proved under the condition $L\alpha(1+\tau^2) \le 1$ rather than under the condition $L\alpha \exp L\tau^2 \le 1$?

Apparently solving this problem requires finding a different approach.

Research Problem 9.2.5. (Level 2) Extend the results of this section to the case of the Hamiltonian

$$H_{M,N}(\boldsymbol{\sigma}) = \sum_{k \le M} u(S_k) + h \sum_{i \le N} \sigma_i \tag{9.17}$$

where h is large.

The point of this problem is that the influence of a large external field will make $R_{1,2}$ typically close to one, while our arguments constantly require that "$R_{1,2}$ be typically small", so the solution of this problem is also likely to require a different approach. Also one often gets the feeling (but maybe this has no basis) that adding an external field can only improve matters.

Let us also note that it should be obvious to the reader, once she understands our arguments, that for $\tau \le 0$, the condition $L\alpha \le 1$ suffices.

Proof of Theorem 9.2.3. We set

$$V_M = 2^{-N} \mathrm{card}\{\boldsymbol{\sigma}\ ;\ \forall k \le M\ ,\ S_k(\boldsymbol{\sigma}) \ge \tau\}$$

so that

$$V_M \le V_{M-1} \le 1\ .$$

Let us denote by G the probability measure on Σ_N of density $W(\boldsymbol{\sigma}) = \mathbf{1}_{\bigcap_{k \le M-1} U_k}(\boldsymbol{\sigma})$ with respect to the uniform measure on Σ_N. It satisfies the condition (9.12) of Lemma 9.2.2 provided $V_{M-1} \ge b^*$. Also,

$$\frac{V_M}{V_{M-1}} = G(\{\boldsymbol{\sigma}\ ;\ S_M(\boldsymbol{\sigma}) \ge \tau\})\ .$$

It then follows from (9.15) that if E_M denotes expectation only with respect to the r.v.s $g_{i,M}$, we have

$$V_{M-1} \ge b^* \quad \Rightarrow \quad \mathsf{E}_M \frac{1}{\max\left(b^*, V_M/V_{M-1}\right)^{\lambda_0}} \le L \exp L\tau^2 \le \exp L(\tau^2+1)\ . \tag{9.18}$$

When $V_{M-1} \ge b^*$ we further have, since $V_{M-1} \le 1$,

$$\begin{aligned}\max(b^*, V_M) &= V_{M-1} \max\left(\frac{b^*}{V_{M-1}}, \frac{V_M}{V_{M-1}}\right) \\ &\ge \max(b^*, V_{M-1}) \max\left(b^*, \frac{V_M}{V_{M-1}}\right)\end{aligned}$$

and combining with (9.18) yields

$$\mathsf{E}_M \frac{1}{\max(b^*, V_M)^{\lambda_0}} \le \frac{1}{\max(b^*, V_{M-1})^{\lambda_0}} \exp L(1+\tau^2) \; .$$

This relation remains true when $V_{M-1} \le b^*$ because then the left-hand side is $\le b^{*-\lambda_0}$ while the right-hand side is $\ge b^{*-\lambda_0}$. Iteration of this relation yields

$$\mathsf{E} \frac{1}{\max(b^*, V_M)^{\lambda_0}} \le \exp LM(1+\tau^2)$$

so that, if $b \ge b^*$,

$$\mathsf{P}(V_M \le b) b^{-\lambda_0} \le \exp LM(1+\tau^2) \; . \qquad \square$$

Throughout the chapter we use the notation

$$U_k = \{S_k \ge \tau\} \; .$$

Later it will be of fundamental importance that the r.v.

$$\operatorname{card}\{\boldsymbol{\sigma} \; ; \; \forall k \le M \; , \; S_k(\boldsymbol{\sigma}) \ge \tau\} = \operatorname{card} \bigcap_{k \le M} U_k$$

has small fluctuations. Since the argument is close in spirit to the previous one, we present it now, but the result itself will not be used before Section 9.6. We recall that $a = 1/32$ and from Chapter 8 the notation

$$\log_A(x) = \max(-A, \log x) \; . \tag{9.19}$$

Proposition 9.2.6. *There exists a number L with the following property. Consider any function u satisfying (9.2) and let*

$$Z = Z(u) = Z_{N,M}(u) = \sum_{\boldsymbol{\sigma}} \exp \sum_{k \le M} u(S_k(\boldsymbol{\sigma})) \; . \tag{9.20}$$

Then for each $t > 0$ we have

$$\begin{aligned} &\mathsf{P}\left(\left|\frac{1}{N}\log_{aN}(2^{-N}Z) - \frac{1}{N}\mathsf{E}\log_{aN}(2^{-N}Z)\right| \ge t\right) \\ &\le 2\exp\left(-\frac{1}{L}\min\left(\frac{N^2t^2}{M(1+\tau^2)}, \frac{Nt}{1+\tau^2}\right)\right) \; . \end{aligned} \tag{9.21}$$

This result includes the case $\exp u(x) = \mathbf{1}_{\{x \ge \tau\}}$.

Proof. We follow the proof of Proposition 8.3.6. Denoting Ξ_m the σ-algebra generated by the r.v.s $(g_{i,k})$ for $i \le N$, $k \le m$ and by E^m the conditional expectation given Ξ_m, we write

$$\frac{1}{N} \log_{aN}(2^{-N} Z) - \frac{1}{N} \mathsf{E} \log_{aN}(2^{-N} Z) = \sum_{m=1}^{M} X_m$$

where

$$X_m = \frac{1}{N} \mathsf{E}^m \log_{aN}(2^{-N} Z) - \frac{1}{N} \mathsf{E}^{m-1} \log_{aN}(2^{-N} Z) . \tag{9.22}$$

Using Bernstein's inequality for martingale difference sequences (A.41), it suffices to prove that

$$\mathsf{E} \exp \frac{|X_m|}{L(1+\tau^2)} \leq 2 . \tag{9.23}$$

Let us define

$$Z_m = \sum_{\boldsymbol{\sigma}} \exp \sum_{k \neq m} u(S_k(\boldsymbol{\sigma}))$$

and

$$Y = \mathbf{1}_{\{Z_m \geq b^*\}} \log_{aN} \frac{Z_m}{Z} .$$

Denoting by E_m expectation in the r.v.s $(g_{i,m})_{i \leq N}$ only, it suffices (as in (8.58)) to prove that

$$\mathsf{E}_m \exp 2\lambda Y \leq 2 \tag{9.24}$$

for $\lambda = 1/L(1+\tau^2)$. To prove this we may assume $Z_m \geq b^*$, for otherwise $Y = 0$. The probability measure G on Σ_N with density proportional to $W = \exp\left(\sum_{k \neq m} u(S_k)\right)$ then satisfies the conditions of Lemma 9.2.2, and thus, by (9.15) we have

$$\mathsf{E}_m \frac{1}{\max(b^*, G(U_m))^{\lambda_0}} \leq L \exp L\tau^2 . \tag{9.25}$$

Now

$$Z_m G(U_m) = \sum_{\boldsymbol{\sigma} \in U_m} \exp\Bigg(\sum_{k \neq m} u(S_k(\boldsymbol{\sigma}))\Bigg) \leq \sum_{\boldsymbol{\sigma}} \exp\Bigg(\sum_{k \leq M} u(S_k(\boldsymbol{\sigma}))\Bigg) = Z$$

because $u(S_m) = 0$ on U_m. Thus, using in the last equality that $Y = \log_{aN} Z_m/Z = \max(aN, Z_m/Z)$, we get

$$\frac{1}{\max(b^*, G(U_m))} = \min\left(\exp aN, \frac{1}{G(U_m)}\right) \geq \min\left(\exp aN, \frac{Z_m}{Z}\right)$$
$$= \exp Y$$

and (9.25) implies

$$\mathsf{E}_m \exp \lambda_0 Y \leq L \exp L\tau^2 \leq \exp L_2(1+\tau^2) ,$$

from which (9.24) follows through Hölder's inequality for $\lambda = \lambda_0/2L_2(1+\tau^2)$.
□

9.3 Gaussian Processes

The goal of this section is to bound the quantity (9.8). It should help the reader to look again at Section 8.2, up to Proposition 8.2.6. The arguments here are similar, just a bit more elaborate.

Theorem 9.3.1. *There exists a number L with the following property. Consider $0 < c \le 1/2$ and a jointly Gaussian family $(w_\ell)_{\ell \le n}$; assume that $\mathsf{E} w_\ell^2 = 1$ and that*

$$\forall \ell \neq \ell' \,, \quad \mathsf{E} w_\ell w_{\ell'} \le c \,. \tag{9.26}$$

Then if $nc \ge 2$ and $L \le s \le \sqrt{\log(nc/2)}/L$ we have

$$\mathsf{P}\big(\mathrm{card}\{\ell \le n \,;\; w_\ell \ge s\} \le n \exp(-Ls^2)\big) \le L \exp\left(-\frac{s^2}{Lc}\right) \,. \tag{9.27}$$

The point of (9.27) is that if we set $\varepsilon = \exp(-Ls^2)$ the bound is $L\varepsilon^{1/Lc}$, and the exponent will be large for c small.

The proof relies on an elementary geometrical lemma.

Lemma 9.3.2. *Consider a number $c > 0$, and vectors $(\mathbf{x}_\ell)_{\ell \le n}$ in a Hilbert space. Assume that $\|\mathbf{x}_\ell\| \le 1$ and $\mathbf{x}_\ell \cdot \mathbf{x}_{\ell'} \le c$ whenever $\ell \neq \ell'$. Then, for any vector $\mathbf{x}$ we have*

$$\mathrm{card}\{\ell \,;\; \mathbf{x} \cdot \mathbf{x}_\ell \ge \|\mathbf{x}\|\sqrt{2c}\} \le \frac{1}{c} \,.$$

Proof. Assume that $\mathbf{x} \cdot \mathbf{x}_\ell \ge \|\mathbf{x}\|\sqrt{2c}$ for $\ell \le k$. Then

$$k\|\mathbf{x}\|\sqrt{2c} \le \mathbf{x} \cdot \left(\sum_{\ell \le k} \mathbf{x}_\ell\right) \le \|\mathbf{x}\| \left\|\sum_{\ell \le k} \mathbf{x}_\ell\right\| \,,$$

and

$$\left\|\sum_{\ell \le k} \mathbf{x}_\ell\right\|^2 = \sum_{\ell \le k} \|\mathbf{x}_\ell\|^2 + \sum_{\ell \neq \ell'} \mathbf{x}_\ell \cdot \mathbf{x}_{\ell'} \le k + ck(k-1) \,.$$

Thus

$$k\sqrt{2c} \le \sqrt{k + ck(k-1)} \le \sqrt{k}\sqrt{1+ck} \,,$$

so that $2ck \le 1 + ck$ i.e. $k \le 1/c$. □

The following useful fact is a consequence of Theorem 1.3.4. We denote by $\mathbf{g} = (g_i)_{i \le N}$ a standard Gaussian vector.

Lemma 9.3.3. *Consider a closed subset B of $\mathbb{R}^N$, and*

$$d(\mathbf{x}, B) = \inf\{d(\mathbf{x}, \mathbf{y}) \,;\; \mathbf{y} \in B\} \,,$$

the Euclidean distance from $\mathbf{x}$ to B. Then for $t > 0$, we have

$$\mathsf{P}\left(d(\mathbf{g}, B) \ge t + 2\sqrt{\log \frac{2}{\mathsf{P}(\mathbf{g} \in B)}}\right) \le 2 \exp\left(-\frac{t^2}{4}\right) \,. \tag{9.28}$$

Proof. The function $F(\mathbf{x}) = d(\mathbf{x}, B)$ satisfies (1.45) with $A = 1$, so that for all $t > 0$ by (1.46) we have

$$\mathsf{P}(|d(\mathbf{g}, B) - \mathsf{E}d(\mathbf{g}, B)| \geq t) \leq 2\exp\left(-\frac{t^2}{4}\right) . \tag{9.29}$$

If $t = \mathsf{E}d(\mathbf{g}, B)$, then

$$\mathsf{P}(\mathbf{g} \in B) \leq \mathsf{P}(|d(\mathbf{g}, B) - \mathsf{E}d(\mathbf{g}, B)| \geq t) ,$$

and combining with (9.29) we get

$$t = \mathsf{E}d(\mathbf{g}, B) \Rightarrow \mathsf{P}(\mathbf{g} \in B) \leq 2\exp\left(-\frac{t^2}{4}\right) ,$$

so that

$$\mathsf{E}d(\mathbf{g}, B) \leq 2\sqrt{\log\frac{2}{\mathsf{P}(\mathbf{g} \in B)}} ,$$

and combining with (9.29) gives (9.28). □

Proof of Theorem 9.3.1. We consider vectors $\mathbf{x}_\ell$ in $\mathbb{R}^N$ such that the sequence $(w_\ell)_{\ell \leq n}$ has the same law as $(\mathbf{x}_\ell \cdot \mathbf{g})_{\ell \leq n}$. (The existence of these vectors is proved in Section A.2 but will be obvious in the situation where we will apply the lemma.) Using (8.13) with $b = 1$, $c = 1/2$, $d = 2$ (and changing s into sL) yields that for $L \leq s \leq \sqrt{\log n}/L$ we have

$$\mathsf{P}\big(\mathrm{card}\{\ell \leq n\ ;\ w_\ell \geq s\} > n\exp(-Ls^2)\big) \geq 1 - L\exp\left(-\frac{s^2}{L}\right)$$

i.e. if

$$B = \left\{\mathbf{x} \in \mathbb{R}^N\ ;\ \mathrm{card}\{\ell \leq n\ ;\ \mathbf{x} \cdot \mathbf{x}_\ell \geq s\} > n\exp(-Ls^2)\right\} , \tag{9.30}$$

then

$$\mathsf{P}(\mathbf{g} \in B) \geq 1 - L\exp\left(-\frac{s^2}{L}\right) .$$

Consequently, there exists a large enough constant L_3 such that for $s \geq L_3$ we have $\mathsf{P}(\mathbf{g} \in B) \geq 1/2$. (Of course, according to our conventions about the meaning of the symbol L, we should simply say that $s \geq L$ implies that $\mathsf{P}(\mathbf{g} \in B) \geq 1/2$.) It then follows from (9.28) that for $t > 0$ we have

$$\mathsf{P}(d(\mathbf{g}, B) \geq t + 4) \leq 2\exp\left(-\frac{t^2}{4}\right) ,$$

and setting $v = t + 4$, it follows that for $t > 0$, it holds

$$\mathsf{P}(d(\mathbf{g}, B) \geq v) \leq L\exp\left(-\frac{v^2}{8}\right) . \tag{9.31}$$

Let $B_v = \{\mathbf{x}\,;\, d(\mathbf{x}, B) \le v\}$, so that (9.31) implies

$$\mathsf{P}(\mathbf{g} \in B_v) \ge 1 - L\exp\left(-\frac{v^2}{8}\right)\,. \tag{9.32}$$

By definition of B_v, for $\mathbf{g} \in B_v$ we can find $\mathbf{g}' \in B$ with $\|\mathbf{g} - \mathbf{g}'\| \le v$. We note that $\mathbf{x}_\ell \cdot \mathbf{x}_{\ell'} = \mathsf{E} w_\ell w_{\ell'} \le c$ for $\ell \ne \ell'$ so that by Lemma 9.3.2 we have

$$\mathrm{card}\big\{\ell \le n\,;\, (\mathbf{g}' - \mathbf{g}) \cdot \mathbf{x}_\ell \ge v\sqrt{2c}\big\} \le \frac{1}{c}\,.$$

On the other hand, since $\mathbf{g}' \in B$, recalling the definition (9.30) of B we have

$$\mathrm{card}\{\ell \le n\,;\, \mathbf{g}' \cdot \mathbf{x}_\ell \ge s\} \ge n\exp(-Ls^2)$$

and thus

$$\mathrm{card}\{\ell \le n\,;\, \mathbf{g} \cdot \mathbf{x}_\ell \ge s - v\sqrt{2c}\} \ge n\exp(-Ls^2) - \frac{1}{c}$$

because $\mathbf{g}' \cdot \mathbf{x}_\ell \ge s$ and $(\mathbf{g}' - \mathbf{g}) \cdot \mathbf{x}_\ell < v\sqrt{2c}$ imply $\mathbf{g} \cdot \mathbf{x}_\ell \ge s - v\sqrt{2c}$. Taking $v = s/2\sqrt{2c}$, we have shown that

$$\mathbf{g} \in B_v \quad\Rightarrow\quad \mathrm{card}\left\{\ell \le n\,;\, \mathbf{g} \cdot \mathbf{x}_\ell \ge \frac{s}{2}\right\} \ge n\exp(-Ls^2) - \frac{1}{c} \ge n\exp(-Ls^2)$$

provided $s \le \sqrt{\log(nc/2)}/L$. Combining with (9.32) this completes the proof. □

Corollary 9.3.4. *There exists a number L and a number $\overline{c} > 0$ with the following property. For a jointly Gaussian family $(w_\ell)_{\ell \le n}$ with $\mathsf{E} w_\ell^2 = 1$ and*

$$\ell \ne \ell' \quad\Rightarrow\quad \mathsf{E} w_\ell w_{\ell'} \le \overline{c}\,,$$

then for any number $\tau \ge 0$ and

$$Ln^{-1/L} \le \varepsilon \le \frac{1}{L}\exp(-L\tau^2)$$

we have

$$\mathsf{P}\big(\mathrm{card}\{\ell \le n\,;\, w_\ell \ge \tau\} \le \varepsilon n\big) \le L\varepsilon^{24}\,. \tag{9.33}$$

Proof. From (9.27) we obtain

$$\mathsf{P}\big(\mathrm{card}\{\ell \le n\,;\, w_\ell \ge \tau\} < n\exp(-Ls^2)\big) \le L\exp\left(-\frac{s^2}{L\overline{c}}\right)$$

provided $s \ge \tau$, $s \ge L$, $s \le \sqrt{\log(n\overline{c}/2)}/L$. Letting $\varepsilon = \exp(-Ls^2)$ we have

$$L\exp\left(-\frac{s^2}{L\overline{c}}\right) = L\varepsilon^{1/L_4\overline{c}} = L\varepsilon^{24}$$

if $\overline{c} = 1/24L_4$. This completes the proof. □

The meaning of the quantity $\overline{c}$ remains as above in the remainder of this chapter.

Proposition 9.3.5. *There exists a number L with the following property. Consider a probability measure G on Σ_N, and a family $(w(\boldsymbol{\sigma}))_{\boldsymbol{\sigma}\in\Sigma_N}$ of jointly Gaussian r.v.s such that $\mathsf{E}w^2(\boldsymbol{\sigma})=1$ and*

$$G^{\otimes 2}\big(\{(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\ ;\ \mathsf{E}w(\boldsymbol{\sigma}^1)w(\boldsymbol{\sigma}^2)>\overline{c}\}\big)\leq 32\exp\left(-\frac{N}{d}\right) \tag{9.34}$$

for a certain number d. Then for any number $\tau\geq 0$ we have

$$L\exp\left(-\frac{N}{Ld}\right)\leq\varepsilon\leq\frac{1}{L}\exp(-L\tau^2)\ \ \Rightarrow\ \ \mathsf{P}\big(G(\{\boldsymbol{\sigma}\ ;\ w(\boldsymbol{\sigma})\geq\tau\})\leq\varepsilon\big)\leq L\varepsilon^{24}\ . \tag{9.35}$$

Proof. We copy the proof of Proposition 8.2.6, using now Corollary 9.3.4 instead of Corollary 8.2.5. Let

$$Q_n=\{(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n)\ ;\ \forall\ell\neq\ell'\ ,\ \mathsf{E}w(\boldsymbol{\sigma}^\ell)w(\boldsymbol{\sigma}^{\ell'})\leq\overline{c}\}\ ,$$

so that since there are at most $n(n-1)/2\leq n^2/2$ choices for ℓ and ℓ' it follows from (9.34) that we have

$$32n^2\exp\left(-\frac{N}{d}\right)\leq 1\quad\Rightarrow\quad G^{\otimes n}(Q_n)\geq\frac{1}{2}\ .$$

For $(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n)\in Q_n$, consider the event

$$\Omega(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n)=\{\mathrm{card}\{\ell\leq n\ ;\ w(\boldsymbol{\sigma}^\ell)\geq\tau\}\leq\varepsilon n\}\ ,$$

so that Corollary 9.3.4 implies

$$Ln^{-1/L}\leq\varepsilon\leq\frac{1}{L}\exp(-L\tau^2)\quad\Rightarrow\quad\mathsf{P}(\Omega(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n))\leq L\varepsilon^{24}\ .$$

Thus if we define

$$Y=\int_{Q_n}\mathbf{1}_{\{\Omega(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n)\}}\mathrm{d}G(\boldsymbol{\sigma}^1)\cdots\mathrm{d}G(\boldsymbol{\sigma}^n)\ ,$$

we have

$$\mathsf{E}Y=\int_{Q_n}\mathsf{P}(\Omega(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n))\mathrm{d}G(\boldsymbol{\sigma}^1)\cdots\mathrm{d}G(\boldsymbol{\sigma}^n)\leq L\varepsilon^{24}\ ,$$

and therefore by Markov's inequality $\mathsf{P}(Y\geq 1/4)\leq L\varepsilon^{24}$. Now

$$Y=G^{\otimes n}(\{(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n)\in Q_n\ ;\ \mathrm{card}\{\ell\leq n\ ;\ w(\boldsymbol{\sigma}^\ell)\geq\tau\}\leq n\varepsilon\})$$

so that

$$\begin{aligned} Y \le \frac{1}{4} \Rightarrow G^{\otimes n}(\{(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^n) \in Q_n \; ; \; \mathrm{card}\{\ell \le n \; ; \; w(\boldsymbol{\sigma}^\ell) \ge \tau\} > n\varepsilon\}) \\ = G^{\otimes n}(Q_n) - Y \ge \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \; . \end{aligned}$$

In that case,

$$\begin{aligned} nG(\{\boldsymbol{\sigma} \; ; \; w(\boldsymbol{\sigma}) \ge \tau\}) &= \int \mathrm{card}\{\ell \le n \; ; \; w(\boldsymbol{\sigma}^\ell) \ge \tau\} \mathrm{d}G(\boldsymbol{\sigma}^1) \cdots \mathrm{d}G(\boldsymbol{\sigma}^n) \\ &\ge n\varepsilon G^{\otimes n}\big(\{(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^n) \in Q_n \; ; \; \mathrm{card}\{\ell \le n \; ; \\ &\quad w(\boldsymbol{\sigma}^\ell) \ge \tau\} > n\varepsilon\}\big) \\ &\ge \frac{n\varepsilon}{4} \; , \end{aligned}$$

so that we have proved that

$$Y \le \frac{1}{4} \quad \Rightarrow \quad G(\{\boldsymbol{\sigma} \; ; \; w(\boldsymbol{\sigma}) \ge \tau\}) \ge \frac{\varepsilon}{4} \; ,$$

and therefore

$$G(\{\boldsymbol{\sigma} \; ; \; w(\boldsymbol{\sigma}) \ge \tau\}) < \frac{\varepsilon}{4} \quad \Rightarrow \quad Y > \frac{1}{4} \; .$$

In conclusion, if $32n^2 \le \exp(N/d)$ and ε satisfies

$$Ln^{-1/L} \le \varepsilon \le \frac{1}{L} \exp(-L\tau^2)$$

we have

$$\mathsf{P}(G(\{\boldsymbol{\sigma} \; ; \; w(\boldsymbol{\sigma}) \ge \tau\}) < \varepsilon/4) \le L\varepsilon^{24} \; .$$

We conclude by taking n as large as possible. □

Corollary 9.3.6. *There exists a constant L with the following property. Consider a probability measure G on Σ_N and a family $(w(\boldsymbol{\sigma}))$ as in Proposition 9.3.5. Then if $b = L\exp(-N/Ld)$, for any number $\tau \ge 0$ we have*

$$\mathsf{E} \frac{1}{\max(b, G(\{\boldsymbol{\sigma} \; ; \; w(\boldsymbol{\sigma}) \ge \tau\}))^{12}} \le L \exp L\tau^2 \; . \tag{9.36}$$

Proof. We define $b = L\exp(-N/Ld)$ where L is the constant of (9.35). Let $Y = \max(b, G(\{\boldsymbol{\sigma} \; ; \; w(\boldsymbol{\sigma}) \ge \tau\}))$, so that by (9.35) we have

$$\varepsilon \le \varepsilon_0 := \frac{1}{L} \exp(-L\tau^2) \quad \Rightarrow \quad \mathsf{P}(Y \le \varepsilon) \le L\varepsilon^{24} \; ,$$

because $\mathsf{P}(Y \le \varepsilon) = 0$ if $\varepsilon < c$. We use that

$$\begin{aligned} \mathsf{E} \frac{1}{Y^{12}} &= 12 \int_0^\infty \mathsf{P}(Y \le \varepsilon) \varepsilon^{-13} \mathrm{d}\varepsilon \\ &\le L \int_0^{\varepsilon_0} \varepsilon^{11} \mathrm{d}\varepsilon + 12 \int_{\varepsilon_0}^\infty \varepsilon^{-13} \mathrm{d}\varepsilon \\ &\le L\varepsilon_0^{12} + \varepsilon_0^{-12} \le L \exp L\tau^2 \; , \end{aligned}$$

and this completes the proof. □

We recall the definition of S_k given in (9.1). If we define $w(\boldsymbol{\sigma}) = S_k(\boldsymbol{\sigma})$, then $R_{1,2} = \mathsf{E} S_k(\boldsymbol{\sigma}^1) S_k(\boldsymbol{\sigma}^2) = \mathsf{E} w(\boldsymbol{\sigma}^1) w(\boldsymbol{\sigma}^2)$. Here is a simple situation where (9.34) is satisfied in this case.

Proposition 9.3.7. *Assume that*

$$\mathrm{card}\big\{\boldsymbol{\sigma} \in \Sigma_N \ ; \ \forall k \le M-1 \ , \ S_k \ge \tau\big\} \ge 2^N \exp\left(-\frac{N\overline{c}^2}{8}\right) . \tag{9.37}$$

Consider the Gibbs measure G with Hamiltonian $-\sum_{k\le M-1} u(S_k(\boldsymbol{\sigma}))$. Then

$$G^{\otimes 2}\big(\{(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) \ ; \ R_{1,2} > \overline{c}\}\big) \le \exp\left(-\frac{N\overline{c}^2}{4}\right) . \tag{9.38}$$

Proof. Use Lemma 9.2.1 with $t = \overline{c}$ and $W(\boldsymbol{\sigma}) = \exp\sum_{k\le M-1} u(S_k(\boldsymbol{\sigma}))$. □

We must now take care of some (tedious and unsurprising) details in order to be able to apply the above principles to our interpolating Hamiltonians. We recall the notation

$$S_k^0(\boldsymbol{\sigma}) = \frac{1}{\sqrt{N}} \sum_{i<N} g_{i,k}\sigma_i = S_k(\boldsymbol{\sigma}) - \frac{g_{N,k}}{\sqrt{N}}$$
$$S_{k,t}(\boldsymbol{\sigma}) = S_k^0(\boldsymbol{\sigma}) + \sqrt{\frac{t}{N}} g_{N,k}\sigma_N + \sqrt{\frac{1-t}{N}} \xi_k \sigma_N \ ,$$

where ξ_k are independent standard Gaussian random variables (independent of all the other r.v.s already introduced). It is of course almost certain that replacing S_k by $S_{k,t}$ in the interpolating Hamiltonian cannot really change anything, but we must nonetheless check this. This occupies the rest of this section.

Lemma 9.3.8. *If we have $L\alpha(1+tau^2) \le 1$ for a large enough constant L then the following two events*

$$\mathrm{card}\big\{\boldsymbol{\sigma}; \ \forall k < M \ , \ S_k(\boldsymbol{\sigma}) \ge \tau + 3\big\} \ge 2^N \exp\left(-\frac{N\overline{c}^2}{16}\right) ; \tag{9.39}$$

$$\forall k < M \ , \qquad |g_{N,k}| \le \sqrt{N} \ . \tag{9.40}$$

occur with probability $\ge 1 - L\exp(-N/L)$.

Proof. We use Theorem 9.2.3 with $\tau+3$ instead of τ and $b = \exp(-N\overline{c}^2/16)$. □

Lemma 9.3.9. *If $N \geq 10$, the following holds true. Let us assume that (9.39) and (9.40) hold true. Then, for any number y we have*

$$\mathsf{E}_\xi \sum_{\boldsymbol{\sigma}} \exp\Bigl(\sum_{k<M} u(S_{k,t}(\boldsymbol{\sigma})) + \sigma_N y\Bigr) \geq 2^{N-1} \exp\left(-\frac{N\overline{c}^2}{16}\right) \mathrm{ch} y \ , \tag{9.41}$$

where E_ξ denotes expectation in the r.v.s ξ_k.

Proof. Let

$$A = \bigl\{\boldsymbol{\sigma} \ ; \ \forall k < M \ , \ S_k^0(\boldsymbol{\sigma}) \geq \tau + 2\bigr\} \ .$$

Let us assume that

$$\forall k < M \ , \quad |\xi_k| \leq \sqrt{N} \ . \tag{9.42}$$

Then using that $|g_{N,k}| \leq \sqrt{N}$ in the first line and that $|g_{N,k}| \leq \sqrt{N}$ and $|\xi_k| \leq \sqrt{N}$ in the second line yields

$$S_k^0(\boldsymbol{\sigma}) \geq S_k(\boldsymbol{\sigma}) - 1 \tag{9.43}$$

$$S_{k,t}(\boldsymbol{\sigma}) \geq S_k^0(\boldsymbol{\sigma}) - (\sqrt{t} + \sqrt{1-t}) \geq S_k^0(\boldsymbol{\sigma}) - 2. \tag{9.44}$$

Since $u(x) = 0$ for $x \geq \tau$ we have $u(S_{k,t}(\boldsymbol{\sigma})) = 0$ if $S_k^0(\boldsymbol{\sigma}) \geq \tau + 2$. Consequently for $\boldsymbol{\sigma} \in A$ we have

$$\exp\Bigl(\sum_{k<M} u(S_{k,t}(\boldsymbol{\sigma})) + \sigma_N y\Bigr) = \exp \sigma_N y \ ,$$

so that

$$\begin{aligned} Z := \sum_{\boldsymbol{\sigma}} \exp\Bigl(\sum_{k<M} u(S_{k,t}(\boldsymbol{\sigma})) + \sigma_N y\Bigr) &\geq \sum_{A} \exp\Bigl(\sum_{k<M} u(S_{k,t}(\boldsymbol{\sigma})) + \sigma_N y\Bigr) \\ &= \sum_{A} \exp \sigma_N y = \mathrm{card} A \, \mathrm{ch} y \ , \end{aligned} \tag{9.45}$$

because the set A is invariant under the transformation $(\sigma_1, \ldots, \sigma_{N-1}, \sigma_N) \mapsto (\sigma_1, \ldots, \sigma_{N-1}, -\sigma_N)$. Also, (9.39) and (9.43) imply that

$$\mathrm{card} A \geq 2^N \exp(-N\overline{c}^2/16) \ ,$$

and thus by (9.45), under (9.42) we have $Z \geq 2^N \exp(-N\overline{c}^2/16)\mathrm{ch} y$. Since ξ_k is standard Gaussian, we have $\mathsf{P}(|\xi_k| \geq \sqrt{N}) \leq 2\exp(-N/2)$, so that for $N \geq 10$ the event Ω described by (9.42) occurs with probability $\geq 1/2$ and this completes the proof since

$$\mathsf{E}_\xi Z \geq \mathsf{E}_\xi(\mathbf{1}_\Omega Z) \geq \mathsf{P}(\Omega) 2^N \exp(-N\overline{c}^2/16)\mathrm{ch} y \ . \qquad \square$$

We recall the notation $R_{1,2}^t = N^{-1}\left(\sum_{i<N} \sigma_i^1 \sigma_i^2 + t\sigma_N^1 \sigma_N^2\right)$.

Proposition 9.3.10. *For $N \geq 10$ the following occurs. Assume (9.39) and (9.40), and consider the measure G on Σ_N given by*

$$\int f \mathrm{d}G = \frac{\sum_{\boldsymbol{\sigma}} f(\boldsymbol{\sigma}) \mathsf{E}_\xi \exp\left(\sum_{k<M} u(S_{k,t}(\boldsymbol{\sigma})) + \sigma_N y\right)}{\sum_{\boldsymbol{\sigma}} \mathsf{E}_\xi \exp\left(\sum_{k<M} u(S_{k,t}(\boldsymbol{\sigma})) + \sigma_N y\right)} \,. \tag{9.46}$$

Then

$$G^{\otimes 2}\left(\left\{(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)\ ;\ |R^t_{1,2}| \geq \left(1 - \frac{1-t}{N}\right)\overline{c}\right\}\right) \leq 32 \exp\left(-\frac{N\overline{c}^2}{8}\right) \,. \tag{9.47}$$

Proof. Let $R^-_{1,2} = N^{-1}\sum_{i<N} \sigma^1_i \sigma^2_i$, so that (for $N \geq 10$)

$$|R^t_{1,2}| > \left(1 - \frac{1-t}{N}\right)\overline{c} \ \ \Rightarrow\ \ |R^-_{1,2}| > \frac{3\overline{c}}{4} \,.$$

Therefore if Z is as in (9.45), we have

$$\begin{aligned}
&G^{\otimes 2}\left(\left\{(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)\ ;\ |R^t_{1,2}| \geq \left(1 - \frac{1-t}{N}\right)\overline{c}\right\}\right)\\
&= \frac{1}{(\mathsf{E}_\xi Z)^2} \sum_{|R^t_{1,2}| > (1-(1-t)/N)\overline{c}} \exp y(\sigma^1_N + \sigma^2_N)\\
&\leq \frac{1}{(\mathsf{E}_\xi Z)^2} \sum_{|R^-_{1,2}| > 3\overline{c}/4} \exp y(\sigma^1_N + \sigma^2_N)\\
&\leq \frac{1}{(\mathsf{E}_\xi Z)^2} \mathrm{ch}^2 y \,\mathrm{card}\left\{(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)\ ;\ |R^-_{1,2}| > \frac{3\overline{c}}{4}\right\} \,,
\end{aligned}$$

because the condition $|R^-_{1,2}| \geq 3\overline{c}/4$ does not depend on the value of σ^1_N and σ^2_N. Now (A.18) implies

$$\begin{aligned}
\mathrm{card}\left\{(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)\ ;\ |R^-_{1,2}| > \frac{3\overline{c}}{4}\right\} &\leq 2^{2N+1} \exp\left(-\frac{9N\overline{c}^2}{32}\right)\\
&\leq 2^{2N+1} \exp\left(-\frac{N\overline{c}^2}{4}\right) \,,
\end{aligned}$$

and we conclude using (9.41). □

The following will allow to control the term (9.8).

Proposition 9.3.11. *There exists a number $q_0 > 0$ and a number L with the following properties. Assume that (9.39) and (9.40) hold. Consider the probability measure G given by (9.46) and denote by $\langle\cdot\rangle$ an average for G. Consider any number $0 \leq q \leq q_0$. Consider independent standard Gaussian r.v.s z, ξ', and set*

$$w(\boldsymbol{\sigma}) = \sqrt{v} S_{M,t}(\boldsymbol{\sigma}) + \sqrt{1-v}(z\sqrt{q} + \xi' \sqrt{1-q}) \ . \tag{9.48}$$

Denote by E' *expectation in the r.v.s* $g_{i,M}$ *and* z*. Then, for* $\overline{b} = L\exp(-N/L)$ *we have*

$$\mathsf{E}' \frac{1}{\max(\overline{b}, \mathsf{E}_\xi \langle \exp u(w(\boldsymbol{\sigma}))\rangle)^{12}} \le L \exp L\tau^2 \ , \tag{9.49}$$

where E_ξ *denotes expectation in* ξ' *and* ξ_M*.*

Proof. Be begin the proof by a few observations. Let us denote by E_g expectation in the r.v.s $g_{i,M}$ only (given z). Let

$$w'(\boldsymbol{\sigma}) = \frac{1}{\sqrt{1-(1-t)/N}} \left(\frac{1}{\sqrt{N}} \sum_{i<N} g_{i,M}\sigma_i + \sqrt{\frac{t}{N}} g_{N,M}\sigma_N \right) \ .$$

The purpose of the factor $1/\sqrt{1-(1-t)/N}$ is to ensure that $\mathsf{E}_g w'(\boldsymbol{\sigma})^2 = 1$ in order to apply Proposition 9.3.5. We have

$$\mathsf{E}_g w'(\boldsymbol{\sigma}^1) w'(\boldsymbol{\sigma}^2) = \frac{1}{1-(1-t)/N} R^t_{1,2} \ .$$

It follows from Proposition 9.3.10 that the family $w'(\boldsymbol{\sigma})$ satisfies the conditions of Proposition 9.3.5 for the probability measure G given by (9.46) and for the value $d = 8/\overline{c}^2$. Since d is a universal constant, the number b in (9.36) is of the type $L\exp(-N/L)$ and will from now on be denoted by $\overline{b}$. Therefore by (9.36) for any number $\tau' \ge 0$ we have

$$\mathsf{E}_g \frac{1}{\max(b, G(\{\boldsymbol{\sigma} \ ; \ w'(\boldsymbol{\sigma}) \ge \tau'\}))^{12}} \le L \exp L\tau'^2 \ . \tag{9.50}$$

We now start the proof of (9.49). We observe that

$$w(\boldsymbol{\sigma}) = \sqrt{v}\sqrt{1 - \frac{1-t}{N}} w'(\boldsymbol{\sigma}) + \sqrt{v}\sqrt{\frac{1-t}{N}} \xi_M + \sqrt{1-v}(z\sqrt{q} + \xi'\sqrt{1-q}). \tag{9.51}$$

Case 1 We have $v \ge 1/2$. We set

$$d = \frac{1}{2}\mathcal{N}\left(-\frac{z\sqrt{q}}{\sqrt{1-q}}\right)$$

where $\mathcal{N}(s) = \mathsf{P}(\xi' \ge s)$. Let us define

$$\tau' = \frac{\tau}{\sqrt{v}\sqrt{1-(1-t)/N}} \ .$$

Then (9.51) implies

$$\xi' \ge -\frac{z\sqrt{q}}{\sqrt{1-q}} \ ; \ \ \xi_M \ge 0 \ , \ \ w'(\boldsymbol{\sigma}) \ge \tau' \ \ \Rightarrow \ \ w(\boldsymbol{\sigma}) \ge \tau \ \Rightarrow \ \exp u(w(\boldsymbol{\sigma})) = 1 \ .$$

Therefore, if $w'(\boldsymbol{\sigma}) \geq \tau'$, we have

$$\mathsf{E}_\xi \exp u(w(\boldsymbol{\sigma})) \geq \mathsf{P}\left(\xi' \geq -\frac{z\sqrt{q}}{\sqrt{1-q}}, \xi_M \geq 0\right) = d\ ,$$

and thus

$$\mathsf{E}_\xi \langle \exp u(w(\boldsymbol{\sigma}))\rangle = \langle \mathsf{E}_\xi \exp u(w(\boldsymbol{\sigma}))\rangle \geq dG(\{w'(\boldsymbol{\sigma}) \geq \tau'\})\ , \tag{9.52}$$

so that (9.52) yields, using (9.50),

$$\begin{aligned}\mathsf{E}_g \frac{1}{\max(\overline{b}, \mathsf{E}_\xi\langle \exp u(w(\boldsymbol{\sigma}))\rangle)^{12}} &\leq \frac{1}{d^{12}} \mathsf{E}_g \frac{1}{\max(\overline{b}, G(\{w'(\boldsymbol{\sigma}) \geq \tau'\}))^{12}} \\ &\leq \frac{L}{d^{12}} \exp L\tau'^2 \\ &\leq \frac{L}{d^{12}} \exp L\tau^2\ ,\end{aligned} \tag{9.53}$$

since $\tau' \leq 2\tau$. Now, using the rough estimate $\mathcal{N}(s) \geq \exp(-s^2)/L$, we have

$$\frac{1}{d^{12}} = 2^{12}\mathcal{N}\left(-\frac{z\sqrt{q}}{\sqrt{1-q}}\right)^{-12} \leq L \exp\left(\frac{12z^2 q}{1-q}\right)\ ,$$

so that if $q \leq q_0$ we have $\mathsf{E}d^{-12} \leq L$ and taking expectation in z in (9.53) the result follows.

Case 2 We have $v \leq 1/2$. Then (9.51) implies

$$\xi' \geq -\frac{z\sqrt{q}}{\sqrt{1-q}} + 2\tau\ ;\quad \xi_M \geq 0\ ,\quad w'(\boldsymbol{\sigma}) \geq 0 \quad\Rightarrow\quad w(\boldsymbol{\sigma}) \geq \tau\ ,$$

and thus (9.52) holds now for

$$\tau' = 0\ ,\quad d = \mathcal{N}\left(-\frac{z\sqrt{q}}{\sqrt{1-q}} + 2\tau\right)\ ,$$

and we proceed as before, using that

$$\mathsf{E}\exp 12\left(2\tau - \frac{z\sqrt{q}}{\sqrt{1-q}}\right)^2 \leq L\exp L\tau^2$$

if $q \leq q_0$. □

9.4 Integration by Parts

If g is a standard Gaussian r.v. and U is a smooth function (of moderate growth), the size of $\mathsf{E}U'(g)$ is governed by the size of U rather than by the size of U' because, by integration by parts,

$$|\mathsf{E}U'(g)| = |\mathsf{E}gU(g)| \leq \mathsf{E}|g||U(g)| \leq L\sup|U|\ .$$

More generally, we have the following elementary fact.

Lemma 9.4.1. *Consider independent standard Gaussian r.v.s* $h_1, \ldots, h_n$*, a smooth function* V *of* n *variables, and integers* $k_1, \ldots, k_n$*. Let* $k = \sum_{i \le n} k_i$*. Then*

$$\left| \mathsf{E} \frac{\partial^k V}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}(h_1, \ldots, h_n) \right| \le C(k_1, \ldots, k_n) \sup |V| \;, \tag{9.54}$$

where the number $C(k_1, \ldots, k_n)$ *depends only on* $k_1, \ldots, k_n$*. In fact, more generally, if* $\ell_1, \ldots, \ell_n$ *are integers* ≥ 0 *then*

$$\left| \mathsf{E} h_1^{\ell_1} \cdots h_n^{\ell_n} \frac{\partial^k V}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}(h_1, \ldots, h_n) \right| \le C(k_1, \ldots, k_n, \ell_1, \ldots, \ell_n) \sup |V| \;. \tag{9.55}$$

Proof. The proof goes by induction over k. For $k = 0$ the result is obvious. Assuming that (9.55) has been proved for $k - 1$ (and all values of $\ell_1, \ldots, \ell_n, k_1, \ldots, k_n$ such that $\sum_{i \le n} k_i = k - 1$) we prove it for k. we may and do assume that $k_1 \ge 1$, and we simply write, using integration by parts in h_1, that

$$\begin{aligned} &\mathsf{E} h_1^{\ell_1} \cdots h_n^{\ell_n} \frac{\partial^k V}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}(h_1, \ldots, h_n) \\ &= \mathsf{E} h_1^{\ell_1 + 1} \cdots h_n^{\ell_n} \frac{\partial^{k-1} V}{\partial x_1^{k_1 - 1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}}(h_1, \ldots, h_n) \\ &- \ell_1 \mathsf{E} h_1^{\ell_1 - 1} \cdots h_n^{\ell_n} \frac{\partial^{k-1} V}{\partial x_1^{k_1 - 1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}}(h_1, \ldots, h_n) \;, \end{aligned}$$

and the proof is complete. □

We proved (9.54) when $h_1, \ldots, h_n$ are independent. Certainly (9.54) will not hold without any condition on $h_1, \ldots, h_n$. For example, at the opposite from the independence situation consider the pair (h, h), a function f of one variable and $U(x_1, x_2) = f(x_1 - x_2)$, so that

$$\frac{\partial^2 U}{\partial x_1 \partial x_2}(x_1, x_2) = -f''(x_1 - x_2)$$

and

$$\mathsf{E} \frac{\partial^2 U}{\partial x_1 \partial x_2}(h, h) = -f''(0)$$

is certainly not controlled by $\sup |f|$. Still, it turns out that (9.54) will hold provided there is enough independence between the r.v.s $h_1, \ldots, h_n$. To see this, assume that there is a linear invertible operator $T : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$T(h_1, \ldots, h_n) = (w_1, \ldots, w_n) \;, \tag{9.56}$$

where the sequence $(w_1, \ldots, w_n)$ consists of independent standard Gaussian r.v.s. That is, T is given by a invertible matrix $(a_{\ell,\ell'})$, and (9.56) means that $w_\ell = \sum_{\ell' \le n} a_{\ell,\ell'} h_{\ell'}$. Consider the function $V = U \circ T^{-1}$ of n variables, so that $U = V \circ T$. Each term

$$\mathsf{E}\frac{\partial^k U}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}(h_1, \ldots, h_n)$$

is a linear combination of terms

$$\mathsf{E}\frac{\partial^k V}{\partial x_1^{\ell_1} \cdots \partial x_n^{\ell_n}}(T(h_1, \ldots, h_n)) \tag{9.57}$$

where $\ell_1 + \ldots + \ell_n = k$. The coefficients of this linear combination are determined by the coefficients of the matrix T. Using (9.56) and (9.54), each term (9.57) is controlled by $\sup|V| = \sup|U|$. Thus

$$\left|\mathsf{E}\frac{\partial^k U}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}(h_1, \ldots, h_n)\right| \le C(k_1, \ldots, k_n, \|T\|) \sup|U| \tag{9.58}$$

where $\|T\|$ is the size of the largest coefficient of the matrix T and the quantity $C(k_1, \ldots, k_n, \|T\|)$ depends only on $k_1, \ldots, k_n$ and $\|T\|$.

Here is a simple condition under which one can control $\|T\|$.

Definition 9.4.2. *A jointly Gaussian sequence* $(h_1, \ldots, h_n)$ *is* widely spread *if for each* $\ell \le n$ *we have* $\mathsf{E}h_\ell^2 \le 1$ *and there exists a Gaussian r.v.* z_ℓ *with* $\mathsf{E}z_\ell^2 \le 1$, $\mathsf{E}z_\ell h_\ell \ge 1/8$ *and* $\mathsf{E}z_\ell h_{\ell'} = 0$ *for* $\ell \ne \ell'$*.*

Of course here we assume that the whole family $(h_1, \ldots, h_n, z_1, \ldots, z_n)$ is jointly Gaussian. Equivalently, we may assume that the r.v.s z_ℓ belong to the linear span of $h_1, \ldots, h_n$. The choice of the constant $1/8$ is quite arbitrary.

It often helps to think in geometrical terms. This is the case here: consider the space W of linear combinations $h = \sum_{\ell \le n} a_\ell h_\ell$ provided with the scalar product $(h, h') = \mathsf{E}hh'$. Given $\ell \le n$, consider the linear span W_ℓ of $h_1, \ldots, h_{\ell-1}, h_{\ell+1}, \ldots, h_n$. Then

$$\sup\bigl\{(z, h_\ell)\ ;\ z \in W\ ;\ \|z\|^2 = 1\ ;\ \forall \ell \ne \ell'\ ,\ (z, h_{\ell'}) = 0\bigr\}$$

is the distance from h_ℓ to W_ℓ. So, the sequence $(h_1, \ldots, h_n)$ is widely spread if and only if for each ℓ this distance is $\ge 1/8$.

When

$$w = \sum_{\ell' \le n} a_{\ell'} h_{\ell'}\ ,$$

and if z_ℓ is as provided by the hypothesis of Definition 9.4.2, i.e. $\mathsf{E}z_\ell^2 \le 1$, $\mathsf{E}z_\ell h_\ell \ge 1/8$ and $\mathsf{E}z_\ell h_{\ell'} = 0$ for $\ell \ne \ell'$, we have

$$\|w\| \geq |(z_\ell, w)| = |a_\ell (z_\ell, h_\ell)| \geq \left|\frac{a_\ell}{8}\right| ,$$

so that $|a_\ell| \leq 8\|w\|$. It should be obvious that W is n-dimensional. Consider any orthonormal basis $w_1, \ldots, w_n$ of W, so that the sequence $w_1, \ldots, w_n$ is i.i.d. standard normal. We have just shown that the matrix of the map T such that (9.56) holds satisfies $\|T\| \leq 8$. Thus we have proved the following.

Proposition 9.4.3. *If the sequence* $(h_\ell)_{\ell \leq n}$ *is widely spread then*

$$\left| \mathsf{E} \frac{\partial^k U}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} (h_1, \ldots, h_n) \right| \leq C(k) \sup |U| , \tag{9.59}$$

where $C(k)$ *depends only on* $k = k_1 + \cdots + k_n$.

We now show that widely spread sequences occur naturally.

Proposition 9.4.4. *Consider a probability measure* G *on* $\Sigma_N = \{-1, 1\}^N$ *and assume that*

$$\forall \mathbf{x} \in \mathbb{R}^N , \quad G\Bigg(\bigg\{\boldsymbol{\sigma} \; ; \; \sum_{i \leq N-1} |\sigma_i - x_i|^2 \leq \frac{N}{16}\bigg\}\Bigg) \leq 4 \exp\left(-\frac{N}{32}\right) . \tag{9.60}$$

For $\boldsymbol{\sigma}$ *in* Σ_N*, let* $h(\boldsymbol{\sigma}) = N^{-1/2} \sum_{i<N} g_i \sigma_i$*. Then*

$$\begin{aligned} &G^{\otimes n}(\{(\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^n) \; ; \; (h(\boldsymbol{\sigma}^1), \ldots, h(\boldsymbol{\sigma}^n)) \,\text{is widely spread}\,\}) \\ &\geq 1 - L^n \exp\left(-\frac{N}{32}\right) . \end{aligned} \tag{9.61}$$

Proof. As a first step, given $\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^{n-1} \in \Sigma_N$ we show that $G(A) \geq 1 - L^n \exp(-N/32)$, where

$$A = \big\{\boldsymbol{\sigma} \; ; \; \exists z, \; \mathsf{E} z^2 = 1 \; , \; \mathsf{E} z h(\boldsymbol{\sigma}) \geq 1/8 \; ; \; \forall \ell \leq n-1 \; , \; \mathsf{E} z h(\boldsymbol{\sigma}^\ell) = 0\big\} ,$$

and where z is a Gaussian r.v. that belongs to the linear span of $(g_i)_{i \leq N}$. To prove this statement we consider the space $\mathbb{R}^{N-1}$ provided with the dot product $(\mathbf{x}, \mathbf{y}) = N^{-1} \sum_{i \leq N-1} x_i y_i$ and the associated distance. The condition $\boldsymbol{\sigma} \in A^c$ means that $\boldsymbol{\rho} = (\sigma_1, \ldots, \sigma_{N-1})$ is at distance $< 1/8$ from the linear span W of $\boldsymbol{\rho}^1, \ldots, \boldsymbol{\rho}^{n-1}$. According to Proposition A.7.1 we may find a subset F of W with card$F \leq L^n$ such that any point of the unit ball of W is within distance $1/8$ of F. Then if the distance of $\boldsymbol{\rho}$ to W is $\leq 1/8$, since $\boldsymbol{\rho}$ is of norm ≤ 1, $\boldsymbol{\rho}$ is within distance $1/8$ of the unit ball of W, so is within distance $\leq 1/4$ of F. Thus

$$G(A^c) \leq G\Bigg(\bigg\{\boldsymbol{\sigma} \; ; \; \exists \mathbf{x} \in F \; ; \; \sum_{i \leq N-1} |\sigma_i - x_i|^2 \leq \frac{N}{16}\bigg\}\Bigg) \leq L^n \exp\left(-\frac{N}{32}\right)$$

by (9.60) and this completes the proof that $G(A) \geq 1 - L^n \exp(-N/32)$. We then use Fubini Theorem to obtain that if

$$B_n = \left\{ \boldsymbol{\sigma}^1, \cdots \boldsymbol{\sigma}^n \; ; \; \exists z, \; \mathsf{E} z^2 = 1, \; \mathsf{E} z h(\boldsymbol{\sigma}^n) \geq 1/8, \; \forall \ell \leq n-1, \; \mathsf{E} z h(\boldsymbol{\sigma}^\ell) = 0 \right\} ,$$

then

$$G^{\otimes n}(B_n) \geq 1 - L^n \exp\left(-\frac{N}{32}\right) ,$$

and therefore

$$\begin{aligned} & G^{\otimes n}(\{(\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^n) \; ; \; (h(\boldsymbol{\sigma}^1), \ldots, h(\boldsymbol{\sigma}^n)) \, \text{is widely spread}\}) \\ & \geq 1 - nL^n \exp\left(-\frac{N}{32}\right) , \end{aligned}$$

which completes the proof. □

Condition (9.60) itself occurs naturally, as the following shows.

Proposition 9.4.5. *Assume (9.39) and (9.40), and consider G as in (9.46). Then G satisfies (9.60).*

Proof. If Z is as in (9.45), then

$$G\left(\left\{\boldsymbol{\sigma} \; ; \; \sum_{i \leq N-1} (\sigma_i - x_i)^2 \leq \frac{N}{16}\right\}\right) \leq \frac{1}{\mathsf{E}_\xi Z} \sum_B \exp y\sigma_N , \tag{9.62}$$

where the summation is over the set

$$B = \left\{ \boldsymbol{\sigma} \; ; \; \sum_{i \leq N-1} (\sigma_i - x_i)^2 \leq \frac{N}{16} \right\} .$$

Since B does not depend on the last coordinate, we have

$$\sum_B \exp y\sigma_N = \mathrm{ch} y \, \mathrm{card} B ,$$

and by (9.41) the right-hand side of (9.62) is $\leq \exp(N\overline{c}^2/16) 2^{-N+1} \mathrm{card} B$. Next we proceed to bound $\mathrm{card}B$. Consider $\lambda = 1/2$, so that $\exp(-\lambda) \leq 1 - \lambda/2$ and $1 + \exp(-\lambda) \leq 2(1 - \lambda/4)$. Since for each i either $|1 - x_i| \geq 1$ or $|1 + x_i| \geq 1$, we have

$$\begin{aligned} & \sum_{\boldsymbol{\sigma}} \exp\left(-\lambda \sum_{i \leq N-1} (x_i - \sigma_i)^2\right) \\ & = 2 \prod_{i \leq N-1} (\exp(-\lambda(1 + x_i)^2) + \exp(-\lambda(1 - x_i)^2)) \\ & \leq 2(1 + \exp(-\lambda))^{N-1} \\ & \leq 2^N \left(1 - \frac{\lambda}{4}\right)^{N-1} \\ & \leq 2^N \exp\left(-\frac{\lambda}{4}(N-1)\right) \end{aligned}$$

so that

$$\mathrm{card}B \exp\left(-\frac{\lambda N}{16}\right) \le \sum_{\boldsymbol{\sigma}\in B} \exp\Bigl(-\lambda \sum_{i\le N-1} (\sigma_i - x_i)^2\Bigr)$$
$$\le 2^N \exp\left(-\frac{\lambda}{4}(N-1)\right)$$

i.e., since $\lambda = 1/2$,

$$\mathrm{card}B \le 2^{N+1} \exp\left(-\frac{N}{16}\right) .$$

Since we may assume $\overline{c} \le 1/2$, we have

$$\exp(N\overline{c}^2/16)2^{-N+1}\mathrm{card}B \le L\exp(-N/32)$$

and the result follows. □

Our final technical result will allow us to deal with r.v.s such as in (9.48).

Proposition 9.4.6. *Assume that the sequence $(h_\ell)_{\ell\le n}$ is widely spread. Consider a number $q \le 1/2$ and Gaussian r.v.s h'_ℓ, z, ξ^ℓ. We assume that the r.v.s (h'_ℓ) are independent of the r.v.s (h_ℓ), and that the r.v.s z, ξ^ℓ are independent of the r.v.s h_ℓ and h'_ℓ. Then the sequence*

$$w_\ell = \sqrt{1-v}(z\sqrt{q} + \sqrt{1-q}\xi^\ell) + \sqrt{v}(h_\ell + h'_\ell)$$

is widely spread.

Proof. Since the sequence h_ℓ is widely spread, by definition, for $\ell \le n$ there exists a Gaussian r.v. z_ℓ with $\mathsf{E}z_\ell^2 = 1$, $\mathsf{E}z_\ell h_\ell \ge 1/8$, $\mathsf{E}z_\ell h_{\ell'} = 0$ if $\ell \ne \ell'$. we may assume that $\mathsf{E}z_\ell\xi^{\ell'} = \mathsf{E}z_\ell h'_{\ell'} = 0$ for each ℓ'. The Gaussian r.v.

$$g_\ell = \sqrt{1-v}\xi^\ell + \sqrt{v}z_\ell$$

satisfies $\mathsf{E}g_\ell^2 = 1$, $\mathsf{E}g_\ell w_\ell = (1-v)\sqrt{1-q} + v\mathsf{E}z_\ell h_\ell \ge 1/8$ and $\mathsf{E}g_\ell w_{\ell'} = 0$ if $\ell \ne \ell'$. □

9.5 The Replica Symmetric Solution

We have built the tools necessary to accomplish the program outlined in Section 9.1, and now we will perform the steps of this program in detail. We recall the number τ of (9.2).

Theorem 9.5.1. *There exists a number L with the following property. Consider a function u that satisfies (9.2), and assume that*

$$\forall \ell,\ 1 \le \ell \le 5\ , \quad |u^{(\ell)}| \le \exp\left(\frac{N}{L}\right)\ . \tag{9.63}$$

Consider α with

$$L\alpha \exp L\tau^2 \le 1\ . \tag{9.64}$$

Then, if z and ξ are independent standard normal r.v.s, the system of equations with unknown (r,p)

$$q = \mathsf{E}\mathrm{th}^2(z\sqrt{r})\ ; \qquad r = \frac{\alpha}{1-q}\mathsf{E}\left(\frac{\mathsf{E}_\xi \xi \exp u(\theta)}{\mathsf{E}_\xi \exp u(\theta)}\right)^2\ , \tag{9.65}$$

where $\theta = z\sqrt{q} + \xi\sqrt{1-q}$ has a unique solution. Moreover, consider the system with Hamiltonian (9.1). Then if $\alpha = M/N$ satisfies (9.64) and if q is as in (9.65) we have

$$\nu\big((R_{1,2} - q)^2\big) \le \frac{L}{N}\ . \tag{9.66}$$

The control of the first 5 derivatives in (9.63) is assumed as a blanket assumption for further use. The reader can check that to prove (9.66) it would suffice to control the first three derivatives.

Let us first study the system of equations (9.65). To compare with the equations (2.66) we recall that by integration by parts we have

$$\widehat{r}(q) := \frac{1}{1-q}\mathsf{E}\left(\frac{\mathsf{E}_\xi \xi \exp u(\theta)}{\mathsf{E}_\xi \exp u(\theta)}\right)^2 = \mathsf{E}\left(\frac{\mathsf{E}_\xi u'(\theta) \exp u(\theta)}{\mathsf{E}_\xi \exp u(\theta)}\right)^2\ . \tag{9.67}$$

Let us define

$$Y = \frac{\tau - z\sqrt{q}}{\sqrt{1-q}}\ .$$

We will prove first that

$$\left(\frac{\mathsf{E}_\xi \xi \exp u(\theta)}{\mathsf{E}_\xi \exp u(\theta)}\right)^2 \le L(Y^2+1)\ . \tag{9.68}$$

Since $u \le 0$ and

$$\xi \ge Y \Rightarrow \theta \ge \tau \Rightarrow u(\theta) \ge 0\ ,$$

we have

$$\mathsf{E}_\xi \exp u(\theta) \ge \mathsf{P}_\xi(\xi \ge Y)\ , \tag{9.69}$$

denoting by P_ξ the probability corresponding to E_ξ. Thus (9.68) is obvious when $Y \le 1$, since $|\mathsf{E}_\xi \exp u(\theta)| \le \mathsf{E}|\xi| \le L$ and $\mathsf{E}_\xi \exp u(\theta) \ge 1/L$. When $Y \ge 1$, it holds

$$|\mathsf{E}_\xi \xi \exp u(\theta)| \le \mathsf{E}_\xi \mathbf{1}_{\{|\xi|\le Y\}} |\xi| \exp u(\theta) + \mathsf{E}_\xi \mathbf{1}_{\{|\xi|>Y\}} |\xi| \exp u(\theta)$$
$$\le Y \mathsf{E}_\xi \exp u(\theta) + \mathsf{E}_\xi \mathbf{1}_{\{|\xi|>Y\}} |\xi| \,.$$

Therefore

$$\left| \frac{\mathsf{E}_\xi \xi \exp u(\theta)}{\mathsf{E}_\xi \exp u(\theta)} \right| \le Y + \frac{E_\xi \mathbf{1}_{\{|\xi|>Y\}} |\xi|}{\mathsf{E}_\xi \exp u(\theta)} \,. \tag{9.70}$$

We observe that

$$\mathsf{E}_\xi \mathbf{1}_{\{|\xi|>Y\}} |\xi| = \frac{2}{\sqrt{2\pi}} \int_Y^\infty x \exp\Big(-\frac{x^2}{2}\Big) \mathrm{d}x = \frac{2}{\sqrt{2\pi}} \exp\Big(-\frac{Y^2}{2}\Big) \,,$$

and that, by (3.136) and (9.69) we have

$$\mathsf{E}_\xi \exp u(\theta) \ge \frac{Y}{1+Y^2} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{Y^2}{2}\right) \,,$$

so combining with (9.70) we obtain

$$\left| \frac{\mathsf{E}_\xi \xi \exp u(\theta)}{\mathsf{E}_\xi \exp u(\theta)} \right| \le Y + L\frac{Y^2+1}{Y} \le LY + 1 \,.$$

This implies (9.68) and thus, going back to (9.67), we have

$$\widehat{r}(q) \le \frac{L(1+\tau^2)}{(1-q)^2} \,.$$

Therefore

$$q \le \frac{1}{2} \quad \Rightarrow \quad \widehat{r}(q) \le L(1+\tau^2) \,,$$

so that if the constant in (9.64) is large enough then

$$q \le \frac{1}{2} \quad \Rightarrow \quad \alpha\widehat{r}(q) \le \frac{1}{2}$$

and since $\mathsf{E}\mathrm{th}^2(z\sqrt{r}) \le \mathsf{E}z^2 r \le r$ the continuous function

$$q \mapsto \psi(q) = \mathsf{E}\mathrm{th}^2\big(z\sqrt{\alpha\widehat{r}(q)}\big) \tag{9.71}$$

maps the interval $[0,1/2]$ into itself; so the equation $q = \psi(q)$ has a solution.

To show that this solution is unique, one simply works harder along the same lines to prove that $|\psi'| < 1$. There is no point however to complete the details, since our argument will show that (9.66) holds for any solution of (9.65), and that therefore this solution is unique.

We turn to the proof of (9.66). We fix once and for all a solution (q,r) of the equations (9.65) and recalling (9.67) we set $\widehat{r} = \widehat{r}(q)$. As we explained in Section 9.1, the key to the results of the present chapter is a better estimate than (2.40) when using the "cavity in M" method; and we turn to this now. We think of t as fixed, and given a function f on 4 replicas, we recall that

$$\nu_{t,v}(fu'(S_v^\ell)u'(S_v^{\ell'})) = \mathsf{E}\frac{\left\langle f\mathsf{E}_\xi u'(S_v^\ell)u'(S_v^{\ell'})\exp\sum_{m\le 4}u(S_v^m)\right\rangle_{t,\sim}}{\langle \mathsf{E}_\xi \exp(S_v^1)\rangle_{t,\sim}^4}\;, \quad (9.72)$$

where E_ξ denotes expectation in the randomness of the variables ξ^ℓ, ξ_M^ℓ, where

$$S_v^\ell = S_v(\boldsymbol{\sigma}^\ell, \xi_M^\ell) = \sqrt{v}S_{M,t}(\boldsymbol{\sigma}^\ell, \xi_M^\ell) + \sqrt{1-v}(z\sqrt{q} + \xi^\ell\sqrt{1-q}) \quad (9.73)$$

$$S_{M,t}(\boldsymbol{\sigma}, \xi) = \sum_{i<N-1}\frac{1}{\sqrt{N}}g_{i,M}\sigma_i + \sqrt{\frac{t}{N}}g_{N,M}\sigma_N + \sqrt{\frac{1-t}{N}}\xi\;,$$

and where $\langle\cdot\rangle_{t,\sim}$ is the Gibbs average corresponding to the Hamiltonian (2.30).

Proposition 9.5.2. *Consider a function f on Σ_N^4, and*

$$\varphi(v) = \nu_{t,v}(fu'(S_v^\ell)u'(S_v^{\ell'})) \quad \text{or} \quad \varphi(v) = \nu_{t,v}(f)\;. \quad (9.74)$$

Assume that D is as in (9.3), and that (9.64) holds. Then

$$|\varphi'(v)| \le L\exp L\tau^2\Bigg(\sum_{\ell_1,\ell_2\le 6,\ell_1\ne\ell_2}\mathsf{E}\langle|f||R_{\ell_1,\ell_2}-q|\rangle_{t,\sim}$$
$$+\frac{1}{N}\mathsf{E}\langle|f|\rangle_\sim + \max|f|D^4\exp\left(-\frac{N}{L}\right)\Bigg)\;. \quad (9.75)$$

Proof. We observe the very important fact that, if $\ell\ne\ell'$, we have

$$u'(x_\ell)u'(x_{\ell'})\exp\sum_{m\le 4}u(x_m) = \frac{\partial^2}{\partial x_\ell\partial x_{\ell'}}\exp\sum_{m\le 4}u(x_m)\;. \quad (9.76)$$

To compute $\varphi'(v)$ we differentiate the relation (9.72) and we integrate by parts in all the Gaussian r.v.s occurring in S_v^ℓ. We recall the notation

$$R_{1,2}^t = \frac{1}{N}\sum_{i\le N-1}\sigma_i^1\sigma_i^2 + \frac{t}{N}\sigma_N^1\sigma_N^2\;.$$

Setting $S_v^{\ell\prime} = \partial S_v^\ell/\partial v$, (a quantity that should not be confused with $S_v^{\ell'}$) we see that $\mathsf{E}S_v^{\ell\prime}S_v^{\ell'} = (R_{\ell,\ell'}^t - q)/2$. We have explained in great detail in the proof of Lemma 2.3.2 how to compute $\varphi'(v)$ using integration by parts. This argument shows that $\varphi'(v)$ is a linear combination of terms

$$\mathsf{E}\frac{\langle f(R_{\ell_1,\ell_2}^t - q)\mathsf{E}_\xi V\rangle_{t,\sim}}{\langle\mathsf{E}_\xi\exp u(S_v^1)\rangle_{t,\sim}^6}\;, \quad (9.77)$$

where

$$V = V(S_v^1, \dots, S_v^6)$$

and

$$V(x_1, \dots, x_6) = \frac{\partial^k}{\partial x_1^{k_1} \cdots \partial x_6^{k_6}} \exp \sum_{m \le 6} u(x_m) \ ,$$

for integers $k_1, \dots, k_6$ with $k = \sum_{m \le 6} k_m \le 4$, and $k_m \le 3$. Specifically, $k = 2$ when $\varphi(v) = \nu_{t,v}(f)$ and $k = 4$ when $\varphi(v) = \nu_{t,v}(f u'(S_v^\ell) u'(S_v^{\ell'}))$.

Consider the (exceptional) event Ω that (9.39) or (9.40) fail. Using Lemma 9.3.8 we obtain that if the constant in (9.64) is large enough, then

$$\mathsf{P}(\Omega) \le L \exp\left(-\frac{N}{L}\right) \ .$$

On the other hand, since $V \le LD^4 \exp(\sum_{m \le 6} u(S_v^m))$, we have

$$\langle \mathsf{E}_\xi V \rangle_{t,\sim} \le D^4 \langle \mathsf{E}_\xi \exp \sum_{m \le 6} u(S_v^m) \rangle_{t,\sim} = \langle \mathsf{E}_\xi \exp u(S_v^1) \rangle_{t,\sim}^6 \ ,$$

so that

$$\left| \frac{\langle f(R^t_{\ell_1,\ell_2} - q) \mathsf{E}_\xi V \rangle_{t,\sim}}{\langle \mathsf{E}_\xi \exp u(S_v^1) \rangle_{t,\sim}^6} \right| \le 2 \max |f| \frac{\langle \mathsf{E}_\xi V \rangle_{t,\sim}}{\langle \mathsf{E}_\xi \exp u(S_v^1) \rangle_{t,\sim}^6} \le L \max |f| D^4 \ . \tag{9.78}$$

Therefore

$$\mathsf{E} \mathbf{1}_\Omega \left| \frac{\langle f(R^t_{\ell_1,\ell_2} - q) \mathsf{E}_\xi V \rangle_{t,\sim}}{\langle \mathsf{E}_\xi \exp u(S_v^1) \rangle_{t,\sim}^6} \right| \le L \max |f| D^4 \exp\left(-\frac{N}{L}\right) \ . \tag{9.79}$$

This controls what happens on the exceptional event Ω and we turn to the control of what happens on the "generic" event Ω^c. Let us denote by E' the expectation in the randomness of z and of the $g_{i,M}$, $i \le N$. This randomness is independent of Ω so that

$$\mathsf{E} \mathbf{1}_{\Omega^c} \left| \frac{\langle f(R^t_{\ell_1,\ell_2} - q) \mathsf{E}_\xi V \rangle_{t,\sim}}{\langle \mathsf{E}_\xi \exp u(S_v^1) \rangle_{t,\sim}^6} \right| = \mathsf{E} \mathbf{1}_{\Omega^c} \mathsf{E}' \left| \frac{\langle f(R^t_{\ell_1,\ell_2} - q) \mathsf{E}_\xi V \rangle_{t,\sim}}{\langle \mathsf{E}_\xi \exp u(S_v^1) \rangle_{t,\sim}^6} \right| . \tag{9.80}$$

Comparing (9.73) and (9.48) we obtain from (9.49) that

$$\mathbf{1}_{\Omega^c} \mathsf{E}' \frac{1}{\max(\overline{b}, \langle \mathsf{E}_\xi \exp u(S_v^1) \rangle_{t,\sim})^{12}} \le L \exp L\tau^2 \ . \tag{9.81}$$

In particular if $\Omega' = \Omega^c \cap \{\langle \mathsf{E}_\xi \exp u(S_v^1) \rangle_{t,\sim} \le \overline{b}\}$, we have (with the obvious notation that P' denotes the probability corresponding to E')

$$\mathsf{P}'(\Omega') \le L \overline{b}^{12} \exp L\tau^2 \ , \tag{9.82}$$

so that since $\bar{b}$ is exponentially small in N, Ω' is another exceptional event. We observe that, using (9.78) and (9.82), we have

$$\begin{aligned}\mathbf{1}_{\Omega^c}\mathsf{E}'\mathbf{1}_{\Omega'}\left|\frac{\langle f(R^t_{\ell_1,\ell_2}-q)\mathsf{E}_\xi V\rangle_{t,\sim}}{\langle \mathsf{E}_\xi \exp u(S_v^1)\rangle^6_{t,\sim}}\right| &\le L\,\max|f|\bar{b}^{12}D^4\exp L\tau^2\\ &\le \max|f|D^4\exp L\tau^2\exp\left(-\frac{N}{L}\right)\,.\end{aligned}$$

Having controlled what happens on the exceptional event Ω' we turn to the control of what happens on the generic event Ω'^c. We note that (9.81) implies

$$\begin{aligned}\mathbf{1}_{\Omega^c}\mathsf{E}'\mathbf{1}_{\Omega'^c}\frac{1}{\langle \mathsf{E}_\xi \exp u(S_v^1)\rangle^{12}_{t,\sim}} &\le \mathbf{1}_{\Omega^c}\mathsf{E}'\frac{1}{\max(\bar{b},\langle \mathsf{E}_\xi \exp u(S_v^1)\rangle_{t,\sim})^{12}}\\ &\le L\exp L\tau^2\,.\end{aligned}$$

Combining this with the Cauchy-Schwarz inequality we get

$$\begin{aligned}&\mathbf{1}_{\Omega^c}\mathsf{E}'\mathbf{1}_{\Omega'^c}\left|\frac{\langle f(R^t_{\ell_1,\ell_2}-q)\mathsf{E}_\xi V\rangle_{t,\sim}}{\langle \mathsf{E}_\xi \exp u(S_v^1)\rangle^6_{t,\sim}}\right|\\ &\le \mathbf{1}_{\Omega^c}L\exp L\tau^2(\mathbf{1}_{\Omega^c}\mathsf{E}'\langle f(R^t_{\ell_1,\ell_2}-q)\mathsf{E}_\xi V\rangle^2_{t,\sim})^{1/2}\,. \qquad (9.83)\end{aligned}$$

The remainder of the proof consists in controlling the expectation of the quantity (9.83). This is the main argument. We consider a replicated copy f' of f; that is, if $f=f(\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^6)$, we set $f'(\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^{12})=f(\boldsymbol{\sigma}^7,\dots,\boldsymbol{\sigma}^{12})$ and we consider

$$f^\sim = ff'(R^t_{\ell_1,\ell_2}-q)(R^t_{\ell_1+6,\ell_2+6}-q)\,.$$

Thus

$$\langle f(R^t_{\ell_1,\ell_2}-q)\mathsf{E}_\xi V\rangle^2_{t,\sim}=\langle f^\sim \mathsf{E}_\xi W\rangle_{t,\sim}\,, \qquad (9.84)$$

where

$$W(S_v^1,\dots,S_v^{12})=V(S_v^1,\dots,S_v^6)V(S_v^7,\dots,S_v^{12})\,.$$

In particular W is of the type

$$W(x^1,\dots,x^{12})=\frac{\partial^k}{\partial x_1^{k_1}\cdots\partial x_{12}^{k_{12}}}\exp\sum_{m\le 12}u(x_m)$$

for integers $k_1,\dots,k_{12}$ with $\sum_{m\le 12}k_m\le 8$ (and $k_m\le 3$). From (9.84) we have

$$\mathbf{1}_{\Omega^c}\mathsf{E}'\langle f(R^t_{\ell_1,\ell_2}-q)\mathsf{E}_\xi V\rangle^2_{t,\sim}=\mathbf{1}_{\Omega^c}\langle f^\sim \mathsf{E}'\mathsf{E}_\xi W\rangle_{t,\sim}=\mathrm{I}+\mathrm{II}\,, \qquad (9.85)$$

where I is the contribution to $\langle\cdot\rangle_{t,\sim}$ of all the configurations for which the sequence $S_v^1,\dots,S_v^{12}$ is widely spread, and II is the contribution of the other configurations. That is, if

$$A = \{(\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^{12}) \; ; \; S_v^1, \ldots, S_v^{12} \text{ is widely spread}\} \,,$$

$$\mathrm{I} = \mathbf{1}_{\Omega^c} \langle \mathbf{1}_A f^{\sim} \mathsf{E}' \mathsf{E}_\xi W \rangle_{t,\sim} \,,$$

$$\mathrm{II} = \mathbf{1}_{\Omega^c} \langle \mathbf{1}_{A^c} f^{\sim} \mathsf{E}' \mathsf{E}_\xi W \rangle_{t,\sim} \,.$$

We use Proposition 9.4.3 with $U(x_1, \ldots, x_{12}) = \exp \sum_{m \le 12} u(x_m)$, so that $0 \le U \le 1$ and we obtain

$$\mathrm{I} \le L \langle |f^{\sim}| \rangle_{t,\sim} \le L \langle |f| |R^t_{\ell_1,\ell_2} - q| \rangle^2_{t,\sim} \,.$$

The essential point of the proof is that only the bound for U, and not the much larger bound for W occurs here.

We recall that by definition of the event Ω, conditions (9.39) and (9.40) hold on Ω^c, and the probability G on Σ_N corresponding to the averages $\langle \cdot \rangle_{t,\sim}$ is of the type (9.46). Propositions 9.4.5 and 9.4.4 then imply that the set of configurations $(\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^{12})$ for which the sequence $h_1, \cdots, h_{12}$ is *not* widely spread is exponentially small for G, where

$$h_\ell = h(\boldsymbol{\sigma}^\ell) = \frac{1}{\sqrt{N}} \sum_{i<N} g_{i,M} \sigma_i^\ell \,.$$

We then use Proposition 9.4.6 with

$$h'_\ell = \sqrt{\frac{t}{N}} g_{N,M} \sigma_N^\ell + \sqrt{\frac{1-t}{N}} \xi_M^\ell$$

to obtain that the set A^c of configurations $(\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^{12})$ for which the sequence $S_v^1, \ldots, S_v^{12}$ is *not* widely spread is exponentially small, and since $|W| \le LD^8 \exp \sum_{m \le 12} u(x_m)$ we get

$$\mathrm{II} \le L \max |f| \exp\left(-\frac{N}{L}\right) D^8 \,,$$

so that (9.85) implies

$$\begin{aligned} \mathbf{1}_{\Omega^c} (\mathsf{E}' \langle f (R^t_{\ell_1,\ell_2} - q) \mathsf{E}_\xi V \rangle^2_{t,\sim})^{1/2} &\le L \langle |f| |R^t_{\ell_1,\ell_2} - q| \rangle_{t,\sim} \\ &\quad + L \max |f| \exp\left(-\frac{N}{L}\right) D^4 \,. \qquad (9.86) \end{aligned}$$

Finally, we write

$$\langle |f| |R^t_{\ell_1,\ell_2} - q| \rangle_{t,\sim} \le \langle |f| |R_{\ell_1,\ell_2} - q| \rangle_{t,\sim} + \frac{1}{N} \langle |f| \rangle_{t,\sim} \,,$$

and we combine this estimate with the previous ones to conclude the proof of (9.75). □

We can now fix the constant L of (9.63) once and for all so that (9.75) becomes

$$|\varphi'(v)| \le L \exp L\tau^2 \Bigg(\sum_{\ell_1,\ell_2 \le 6, \ell_1 \ne \ell_2} \mathsf{E}\langle |f||R_{\ell_1,\ell_2} - q|\rangle_{t,\sim}$$
$$+ \frac{1}{N}\mathsf{E}\langle |f|\rangle_{t,\sim} + \max|f| \exp\left(-\frac{N}{L}\right)\Bigg) . \tag{9.87}$$

To obtain the estimate (9.87) is the main effort in proving Theorem 9.5.1. However we would like however to have $\langle\cdot\rangle_t$ rather than $\langle\cdot\rangle_{t,\sim}$ occurring on the right-hand side, and we now learn how to compare these.

Lemma 9.5.3. *If $L\alpha(1+\tau^2) \le 1$ and $f \ge 0$ is a function on Σ_N^8, we have*

$$\mathsf{E}\langle f\rangle_{t,\sim} \le L \exp L\tau^2 \left(\nu_t(f) + (\max f)\exp\left(-\frac{N}{L}\right)\right) . \tag{9.88}$$

Proof. Let us denote by E' expectation in the r.v.s $g_{i,M}$, $i \le N$. Since $u \le 0$, we have

$$\nu_t(f) = \mathsf{E}\frac{\mathsf{E}_\xi\langle f \exp \sum_{\ell\le 8} u(S^\ell_{M,t})\rangle_{t,\sim}}{\mathsf{E}_\xi\langle \exp u(S^1_{M,t})\rangle^8_{t,\sim}} \ge \mathsf{E}\left\langle f\mathsf{E}'\mathsf{E}_\xi \exp \sum_{\ell\le 8} u(S^\ell_{M,t})\right\rangle_{t,\sim} . \tag{9.89}$$

Consider a number $d > 0$, to be determined later, and

$$A = \{(\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^8) \,;\, \forall \ell \ne \ell' \,,\, |R_{\ell,\ell'}| \le d\} .$$

In Lemma 9.5.4 below we show that we can choose d (which is a universal constant independent of any other parameter) so that

$$(\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^8) \in A \quad\Rightarrow\quad \mathsf{E}'\mathsf{E}_\xi \exp \sum_{\ell \le 8} u(S^\ell_{M,t}) \ge \frac{1}{L}\exp(-L\tau^2) . \tag{9.90}$$

Thus

$$\mathsf{E}\left\langle f\mathsf{E}'\mathsf{E}_\xi \exp\sum_{\ell\le 8} u(S^\ell_{M,t})\right\rangle_{t,\sim} \ge \frac{1}{L}\exp(-L\tau^2)\mathsf{E}\langle \mathbf{1}_A f\rangle_{t,\sim}$$
$$\ge \frac{1}{L}\exp(-L\tau^2)(\mathsf{E}\langle f\rangle_{t,\sim} - (\max f)\mathsf{E}\langle \mathbf{1}_{A^c}\rangle_{t,\sim}) .$$

Since d is a universal constant we may and do assume that $\overline{c} \le d$. It then follows from Lemma 9.3.8 and Proposition 9.3.10 that if $L\alpha(1+\tau^2) \le 1$ (and using (9.47) for $t = 0$) we have $\mathsf{E}\langle \mathbf{1}_{A^c}\rangle_{t,\sim} \le L\exp(-N/L)$. This concludes the proof, modulo the proof of (9.90), which is given in the next lemma. □

Lemma 9.5.4. *There exists a number $d > 0$ with the following property. If we consider Gaussian r.v.s $(w_\ell)_{\ell\le 8}$, such that $\mathsf{E}w_\ell^2 = 1$, $|\mathsf{E}w_\ell w_{\ell'}| \le d$ for $\ell \ne \ell'$, then for any value of τ we have*

$$\mathsf{P}(\forall \ell \le 8 \,,\, w_\ell \ge \tau) \ge \frac{1}{L}\exp(-L\tau^2) . \tag{9.91}$$

When applied to the case $w_\ell = S^\ell_{M,t}$, this proves (9.90) since $u(x) = 0$ for $x \geq \tau$.

Proof. It should be obvious that one can choose $d > 0$ so that the hypothesis on (w_ℓ) implies that we can find i.i.d. Gaussian r.v.s $(v_\ell)_{\ell \leq 8}$ with

$$w_\ell = \sum_{\ell' \leq 8} a_{\ell,\ell'} v_{\ell'}$$

where for each ℓ we have $|1 - a_{\ell,\ell}| + \sum_{\ell \neq \ell'} |a_{\ell,\ell'}| \leq 1/3$. Consequently,

$$w_\ell \geq v_\ell - \frac{1}{3} \max_{\ell'} |v_{\ell'}| \; . \tag{9.92}$$

To prove (9.91) we may and do assume that $\tau \geq 1$. Then on the event

$$\forall \ell \leq 8 \; , \quad 2\tau \leq v_\ell \leq 3\tau \; , \tag{9.93}$$

we have $w_\ell \geq \tau$ by (9.92); and the event (9.93) is of probability greater than or equal to $(1/L)\exp(-L\tau^2)$. □

Let us summarize what we have proved.

Proposition 9.5.5. *Under (9.63), and if $L\alpha(1+\tau^2) \leq 1$, for any function f on Σ_N^4 and if either $\varphi(v) = \nu_{t,v}(fu'(S^\ell_v)u'(S^{\ell'}_v))$ or $\varphi(v) = \nu_{t,v}(f)$ then whenever $1/\tau_1 + 1/\tau_2 = 1$ and $0 \leq v \leq 1$ we have*

$$|\varphi'(v)| \leq L \exp L\tau^2 \Bigg((\nu_t(|f|^{\tau_1})^{1/\tau_1} \nu_t(|R_{1,2} - q|^{\tau_2})^{1/\tau_2} + \frac{1}{N}\nu_t(|f|) + \max|f| \exp\left(-\frac{N}{L}\right)\Bigg) \; . \tag{9.94}$$

Proof. We combine (9.87) and (9.88) and we use Hölder's inequality. □

Research Problem 9.5.6. Is it true that (9.94) holds with a term $L(1+\tau^2)$ rather than $L\exp L\tau^2$?

Corollary 9.5.7. *Under (9.63) and if $L\alpha(1+\tau^2) \leq 1$, for any function f on Σ_N^4, whenever $1/\tau_1 + 1/\tau_2 = 1$ we have*

$$\begin{aligned} &|\nu_t(fu'(S^\ell_{M,t})u'(S^{\ell'}_{M,t})) - \widehat{r}\nu_t(f)| \\ &\leq L \exp L\tau^2 \Bigg((\nu_t(|f|^{\tau_1})^{1/\tau_1} \nu_t(|R_{1,2} - q|^{\tau_2})^{1/\tau_2} \\ &+ \frac{1}{N}\nu_t(|f|) + \max|f| \exp\left(-\frac{N}{L}\right)\Bigg) \; . \end{aligned} \tag{9.95}$$

Proof. If $\mathcal{B}$ denotes the right-hand side of (9.94) then

$$|\varphi(0) - \varphi(1)| \le \mathcal{B} \,. \tag{9.96}$$

Now $\varphi(1) = \nu_t(fu'(S^{\ell}_{M,t})u'(S^{\ell'}_{M,t}))$ and using (2.38) we see that $\varphi(0) = \widehat{r}\mathsf{E}\langle f\rangle_{t,\sim}$, so that

$$\left|\nu_t(fu'(S^{\ell}_{M,t})u'(S^{\ell'}_{M,t})) - \widehat{r}\mathsf{E}\langle f\rangle_{t,\sim}\right| \le \mathcal{B} \,. \tag{9.97}$$

Using again (9.96) in the case $\varphi(v) = \nu_{t,v}(f)$ yields $|\nu_t(f) - \mathsf{E}\langle f\rangle_{t,\sim}| \le \mathcal{B}$, and combining with (9.97) finishes the proof. □

Proposition 9.5.8. *Under (9.63), and if $L\alpha(1+\tau^2) \le 1$, for any function f on Σ_N^2, whenever $1/\tau_1 + 1/\tau_2 = 1$ we have*

$$\begin{aligned} |\nu_t'(f)| \le L\alpha \exp L\tau^2 \Bigg(&\nu_t(|f|^{\tau_1})^{1/\tau_1} \nu_t(|R_{1,2} - q|^{\tau_2})^{1/\tau_2} \\ &+ \frac{1}{N}\nu_t(|f|) + \max|f| \exp\left(-\frac{N}{L}\right)\Bigg) \,. \end{aligned} \tag{9.98}$$

Proof. Combine (9.95) with (2.23). □

Lemma 9.5.9. *Assume $L\alpha \exp L\tau^2 \le 1$. Consider a function f on Σ_N^2, $f \ge 0$. Then*

$$\forall t \,, \quad \nu_t(f) \le 2\nu(f) + L \max|f| \exp\left(-\frac{N}{L}\right) \,. \tag{9.99}$$

Proof. Using (9.98) for $\tau_1 = 1$, $\tau_2 = \infty$ we obtain

$$|\nu_t'(f)| \le L\alpha \exp L\tau^2 \left(\nu_t(f) + \max|f| \exp\left(-\frac{N}{L}\right)\right) \tag{9.100}$$

and we integrate using Lemma A.11.1. □

Now it is straightforward to check that one can prove (9.66) by following the steps of the proof of (2.67). Theorem 9.5.1 is proved.

Proposition 9.5.10. *Under the conditions of Theorem 9.5.1 we actually have*

$$\forall k \ge 1 \,, \quad \nu\big((R_{1,2} - q)^{2k}\big) \le \left(\frac{Lk}{N}\right)^k \,. \tag{9.101}$$

Proof. We copy the proof of Theorem 2.5.1. In (2.96)we get an extra term $L \max|f| \exp(-N/L) \le 2^{2k} \exp(-N/L)$. Now, for $x > 0$ we have

$$(ax)^x \ge \exp\left(-\frac{1}{ae}\right)$$

so that

$$\left(\frac{L_0 k}{4N}\right)^k \geq \exp\left(-\frac{4N}{L_0 e}\right)$$

and if L_0 is large enough we have

$$2^{2k}\exp\left(-\frac{N}{L}\right) \leq \left(\frac{L_0 k}{N}\right)^k$$

and the proof of Theorem 2.5.1 carries forward with no other changes. □

We recall that q and r are defined as in (9.65). We recall the notations (2.11) and (2.72):

$$p_{N,M}(u) = \frac{1}{N}\mathsf{E}\log\sum_{\boldsymbol{\sigma}}\exp(-H_{N,M}(\boldsymbol{\sigma}))$$

$$p(u) = -\frac{r}{2}(1-q) + \mathsf{E}\log(2\mathrm{ch}(z\sqrt{r})) + \alpha\mathsf{E}\log\mathsf{E}_\xi\exp u(z\sqrt{q}+\xi\sqrt{1-q}). \tag{9.102}$$

Theorem 9.5.11. *Under the conditions of Theorem 9.5.1 we have*

$$|p_{N,M}(u) - p(u)| \leq \frac{L}{N}\;. \tag{9.103}$$

The proof follows the approach of the second proof of Theorem 2.4.2. We recall the identity

$$p_{N,M+1}(u) - p_{N,M}(u) = \frac{1}{N}\mathsf{E}\log\left\langle\exp u\left(\frac{1}{\sqrt{N}}\sum_{i\leq N} g_{i,M+1}\sigma_i\right)\right\rangle\;.$$

We define

$$S_s^\ell = \sqrt{\frac{s}{N}}\sum_{i\leq N} g_{i,M+1}\sigma_i + \sqrt{1-s}\,\theta^\ell\;;\quad \theta^\ell = z\sqrt{q}+\xi^\ell\sqrt{1-q}\;,$$

$$\varphi(s) = \frac{1}{N}\mathsf{E}\log\mathsf{E}_\xi\langle\exp u(S_s^1)\rangle\;.$$

The excuse for using the same notation here and in (9.104) below as in (9.73) is of course that they serve the same purpose. By (2.89) we have

$$\varphi'(s) = -\frac{1}{2}\mathsf{E}\frac{\big\langle(R_{1,2}-q)u'(S_s^1)u'(S_s^2)\exp(u(S_s^1)+u(S_s^2))\big\rangle}{(\mathsf{E}_\xi\langle\exp u(S_s^1)\rangle)^2}\;. \tag{9.104}$$

As in the proof of Theorem 2.4.2 one needs to control $|\varphi'(0)|$ and $|\varphi''(s)|$.

Since $\varphi'(0) = -(\widehat{r}/2)\nu(R_{1,2}-q)$, we have $|\varphi'(0)| \leq L/N$ as a consequence of the next Lemma, that we will prove when we study central limit theorems in Section 9.7.

Lemma 9.5.12. *Under the conditions of Theorem 9.5.1 we have*

$$|\nu(R_{1,2} - q)| \leq \frac{L}{N} \, .$$

To control φ'', we compute it from (9.104) using integration by parts; this brings a new factor $(R_{\ell,\ell'} - q)$ in each resulting term. To bound the resulting quantity is not obvious a priori, because the denominator can be small, and the derivatives of u can be huge. But we simply repeat the steps of the proof of Proposition 9.5.2: we separate the numerator from the denominator using the Cauchy-Schwarz inequality, we integrate by parts, and so on. The proof is quite simpler than that of Proposition 9.5.2, because we do not have to be concerned with the pesky interpolating averages $\langle\cdot\rangle_{t,\sim}$. The reader who really likes to understand the previous techniques should carry out the detail of the proofs, as suggested by the following exercise.

Exercise 9.5.13. If $L\alpha \exp L\tau^2 \leq 1$, prove the inequality

$$|\varphi''(s)| \leq L \exp L\tau^2 \left(\nu((R_{1,2} - q)^2) + L \exp\left(-\frac{N}{L}\right)\right) \, . \tag{9.105}$$

Combining (9.105) and (9.66), we have

$$|\varphi''(s)| \leq \frac{L}{N} \exp L\tau^2 \, ,$$

and since $|\varphi'(0)| \leq L/N$ by Lemma 9.78 we have reached the bound

$$\left| p_{N,M+1}(u) - p_{N,M}(u) - \frac{1}{N} \mathsf{E} \log \mathsf{E}_\xi \exp u(\theta) \right| \leq \frac{L}{N} \exp L\tau^2 \, ,$$

and as in Theorem 2.4.2, summation over M (and the fact that $M \leq LN \exp L\tau^2$) yields (9.103).

9.6 The Gardner Formula for the Discrete Cube

We recall the notation

$$U_k = \{\boldsymbol{\sigma} \, ; \, S_k(\boldsymbol{\sigma}) \geq \tau\} \, ; \quad \mathcal{N}(x) = \mathsf{P}(\xi \geq x) \, ; \quad \mathcal{A}(x) = \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{\mathcal{N}(x)} \, .$$

In the case $\exp u(x) = \mathbf{1}_{\{x \geq \tau\}}$ the equations (9.65) become

$$q = \mathsf{E}\mathrm{th}^2(z\sqrt{r}) \, ; \qquad r = \frac{\alpha}{1-q} \mathsf{E}\mathcal{A}\left(\frac{\tau - z\sqrt{q}}{\sqrt{1-q}}\right)^2 \, , \tag{9.106}$$

where z is a standard Gaussian r.v.

Theorem 9.6.1. *There is a constant L with the following property. Consider τ and M with $L\alpha \exp L\tau^2 \le 1$. Then for $t \ge 0$ we have*

$$\mathsf{P}\left(\left|\frac{1}{N}\log\left(2^{-N}\mathrm{card}\bigcap_{k\le M} U_k\right) - \mathrm{RS}(\alpha)\right| \ge t\right) \le L\exp\left(-\frac{1}{L}\min\left(N, \frac{Nt}{1+\tau^2}, \frac{N^2t^2}{M(1+\tau^2)^2}\right)\right) , \tag{9.107}$$

where

$$\mathrm{RS}(\alpha) = -\frac{r}{2}(1-q) + \mathsf{E}\log\mathrm{ch}(z\sqrt{q}) + \alpha\mathsf{E}\log\mathcal{N}\left(\frac{\tau - z\sqrt{q}}{\sqrt{1-q}}\right) , \tag{9.108}$$

where $\alpha = M/N$ and q and r are solutions of the equations (9.106).

If one does not care about the dependence on τ^2 (which is unlikely to be sharp anyway) one can simplify (9.107) as

$$\mathsf{P}\left(\left|\frac{1}{N}\log\left(2^{-N}\mathrm{card}\bigcap_{k\le M} U_k\right) - \mathrm{RS}(\alpha)\right| \ge t\right) \le K\exp\left(-\frac{Nt^2}{K}\right)$$

for $t \le 1$, where K depends on τ only.

The existence of a solution to the equations (9.106) where $L\alpha\exp L\tau^2 \le 1$ was actually obtained in the proof of Theorem 9.5.1, because this part of the argument never used (9.63), so it remains valid in the case $\exp u(x) = \mathbf{1}_{\{x\ge\tau\}}$. A bit of extra work would show that these solutions are unique.

The following is a rather weak consequence of Theorem 9.6.1. The main motivation for proving it is that it was announced in Section 2.1.

Corollary 9.6.2. *When $\tau = 0$ there exists $\alpha_0 < 1$ such that*

$$\mathsf{P}\left(\bigcap_{k\le\alpha_0 N} U_k = \emptyset\right) \ge 1 - L\exp\left(-\frac{N}{L}\right) . \tag{9.109}$$

Proof. The difficulty is that Theorem 9.6.1 holds only for $\alpha \le 1/L$ while we are trying to prove something for α close to 1. As a first step we prove that for $\alpha > 0$ we have

$$\mathrm{RS}(\alpha) < -\alpha\log 2 . \tag{9.110}$$

Let us denote by $F(q,r,\alpha)$ the left-hand side of (9.108) when we consider q, r and α as unrelated variables. Then the conditions (9.106) mean that

$$\frac{\partial F}{\partial q}(q,r,\alpha) = \frac{\partial F}{\partial r}(q,r,\alpha) = 0 ,$$

so that

$$\mathrm{RS}'(\alpha) := \frac{\mathrm{d}}{\mathrm{d}\alpha}\mathrm{RS}(\alpha) = \mathsf{E}\log \mathcal{N}\left(\frac{-z\sqrt{q}}{\sqrt{1-q}}\right) .$$

Now, Jensen's inequality implies

$$\mathsf{E}\log \mathcal{N}\left(\frac{-z\sqrt{q}}{\sqrt{1-q}}\right) \le \log \mathsf{E}\mathcal{N}\left(\frac{-z\sqrt{q}}{\sqrt{1-q}}\right) , \tag{9.111}$$

and since $\mathcal{N}(x)+\mathcal{N}(-x) = 1$, the expectation in the right-hand side of (9.111) is equal to $1/2$. There cannot be equality in (9.111) unless $q = 0$, and this does not occur for $\alpha > 0$ as is apparent from the equations (9.108). Thus we have proved that $\mathrm{RS}'(\alpha) < -\log 2$ for $\alpha > 0$ and (9.110) follows.

Consequently, we can find α_1 small enough so that Theorem 9.6.1 holds for α_1 while we have

$$\mathrm{RS}(\alpha_1) = -(\alpha_1 + 4\delta)\log 2 , \tag{9.112}$$

where $\delta > 0$. We are now going to prove that if $\alpha_0 = 1-\delta$ then

$$\mathsf{P}\left(\mathrm{card}\left(\bigcap_{k\le\alpha N} U_k\right) \le 4\times 2^{-\delta N}\right) \ge 1 - L\exp\left(-\frac{N}{L}\right) . \tag{9.113}$$

This implies (9.109) since

$$\mathrm{card}\left(\bigcap_{k\le\alpha_0 N} U_k\right) < 1 \Rightarrow \mathrm{card}\left(\bigcap_{k\le\alpha_0 N} U_k\right) = 0 .$$

To prove (9.113) we first observe that by Theorem 9.6.1, for any integer M with $LM\exp L\tau^2 \le N$ it holds

$$\mathsf{P}\left(\mathrm{card}\bigcap_{k\le M} U_k \le 2^{N(1+\delta/2)}\exp N\mathrm{RS}\left(\frac{M}{N}\right)\right) \ge 1 - L\exp\left(-\frac{N}{L}\right) .$$

Thus if $M = \lceil \alpha_1 N \rceil$ and

$$V = \bigcap_{k\le M} U_k ,$$

it follows from (9.112) that

$$\mathsf{P}(\mathrm{card}V \le 2^{N(1-\alpha_1-3\delta)}) \ge 1 - L\exp\left(-\frac{N}{L}\right) . \tag{9.114}$$

Moreover given any set V and any integer $M' > M$ we have

$$\mathsf{E}\mathrm{card}\left(V \cap \bigcap_{M<k\le M'} U_k\right) = 2^{M-M'}\mathrm{card}V ,$$

because any point of V has a 50% chance to belong to each set U_k. Therefore by Markov's inequality we have

$$\mathsf{P}\left(\operatorname{card}\left(V \cap \bigcap_{M<k\le M'} U_k\right) \le 2^{M-M'+\delta N}\operatorname{card}V\right) \ge 1 - L\exp\left(-\frac{N}{L}\right) ,$$

and if we combine with (9.114) we see that

$$\mathsf{P}\left(\operatorname{card}\left(\bigcap_{k\le M'} U_k\right) \le 2^{M-M'-(2\delta+\alpha_1)N}\right) \ge 1 - L\exp\left(-\frac{N}{L}\right) .$$

Taking $M' = \lfloor \alpha_0 N \rfloor$ this proves (9.113) since $M \le \alpha_1 N + 1$ and $M' \ge N\alpha_0 - 1$. □

Exercise 9.6.3. Prove from (9.108) that $\mathrm{RS}(0) = 0$ and $\mathrm{RS}'(0) = \mathcal{N}(\tau)$. Offer an intuitive explanation for this fact.

As in Chapter 8, the key to Theorem 9.6.1 is that the fluctuations of the random quantity $\log\left(2^{-N}\operatorname{card}\bigcap_{k\le M} U_k\right)$ are very small, so that it will be sufficient to compute its expectation (after suitable truncation), using Theorem 9.5.1.

Proof of Theorem 9.6.1. We will use a function u that satisfies (9.2), but such that $\exp u$ approximates well the function $\mathbf{1}_{\{x\ge\tau\}}$. We will require that, for a certain number τ' depending on N, with $\tau' < \tau$ we have

$$x \le \tau' \quad \Rightarrow \quad u(x) = -N .$$

In order to be able to use Theorem 9.5.1, we want u to satisfy (9.63), and yet $\tau - \tau'$ to be as small as possible. It is obvious from scaling arguments that u can be found with $|u^{(\ell)}| \le NL(\tau-\tau')^{-\ell}$ for $1 \le \ell \le 5$, so that we may achieve (9.63) with

$$\tau - \tau' \le L_1 \exp\left(-\frac{N}{L_1}\right) \tag{9.115}$$

for a certain number L_1.

Let us define

$$V = 2^{-N}\operatorname{card}\{\boldsymbol{\sigma} \in \Sigma_N \; ; \; \exists k \le M \; ; \; \tau' \le S_k(\boldsymbol{\sigma}) \le \tau\} .$$

This r.v. is very small, since, if g is standard Gaussian r.v.

$$\mathsf{E}V \le M\mathsf{P}(\tau' \le \; g \le \tau) \le M(\tau-\tau') \le L_2 \exp\left(-\frac{N}{L_2}\right)$$

since $M \le N$. In particular, if we consider the event

$$\Omega_1 = \left\{ \; V \le \exp\left(-\frac{N}{2L_2}\right)\right\} ,$$

Markov's inequality implies

$$\mathsf{P}(\Omega_1) \geq 1 - L_2 \exp\left(-\frac{N}{2L_2}\right) .$$

Consider also the event

$$\Omega_2 = \left\{2^{-N}\mathrm{card} \bigcap_{k \leq M} U_k \geq \exp\left(-\frac{N}{2L_2}\right)\right\} .$$

We use (9.16) with $b = \exp(-N/2L_2)$ to obtain that

$$L\alpha(1+\tau^2) \leq 1 \quad \Rightarrow \quad \mathsf{P}(\Omega_2) \geq 1 - L\exp\left(-\frac{N}{L}\right) . \tag{9.116}$$

Let us define

$$Z(u) = \sum_{\boldsymbol{\sigma}} \exp \sum_{k \leq M} u(S_k(\boldsymbol{\sigma})) ,$$

and prove the inequality

$$2^{-N}\mathrm{card} \bigcap_{k \leq M} U_k \leq 2^{-N} Z(u) \leq 2^{-N}\mathrm{card}\left(\bigcap_{k \leq M} U_k\right) + V + e^{-N} . \tag{9.117}$$

The left-hand side inequality is obvious since $u(x) = 0$ for $x \geq \tau$, so that

$$\boldsymbol{\sigma} \in \bigcap_{k \leq M} U_k \quad \Rightarrow \quad \exp \sum_{k \leq M} u(S_k(\boldsymbol{\sigma})) = 1 .$$

To prove the second inequality we consider the sets

$$A = \bigcap_{k \leq M} U_k \; ; \qquad B = \{\boldsymbol{\sigma} \; ; \; \exists k \leq M \; , \; \tau' \leq S_k(\boldsymbol{\sigma}) \leq \tau\}$$

$$C = \{\boldsymbol{\sigma} \; ; \; \exists k \leq M \; , \; S_k(\boldsymbol{\sigma}) < \tau'\}$$

so that $\Sigma_N \subset A \cup B \cup C$. For any $\boldsymbol{\sigma}$ we have that $\exp \sum_{k \leq M} u(S_k(\boldsymbol{\sigma})) \leq 1$ since $u \leq 0$; moreover if $\boldsymbol{\sigma} \in C$ we have $\exp \sum_{k \leq M} u(S_k(\boldsymbol{\sigma})) \leq e^{-N}$ because $u \leq 0$ and $u(x) = -N$ if $x < \tau'$. Therefore

$$\sum_{\boldsymbol{\sigma}} \exp \sum_{k \leq M} u(S_k(\boldsymbol{\sigma})) \leq \mathrm{card} A + \mathrm{card} B + 2^N e^{-N}$$

and this proves (9.117) since $V = 2^{-N}\mathrm{card} B$.

The point of introducing the events Ω_1 and Ω_2 is that on the event $\Omega_1 \cap \Omega_2$ we have

$$V \leq \exp\left(-\frac{N}{2L_2}\right) \leq 2^{-N}\mathrm{card} \bigcap_{k \leq M} U_k ,$$

and, since without loss of generality we may assume that $L_2 \geq 16$ the right-hand side inequality above implies that, recalling the notation $a = 1/32$ of (9.19),

$$e^{-N} \leq \exp(-aN) \leq 2^{-N} \mathrm{card} \bigcap_{k \leq M} U_k \, .$$

Therefore, using (9.117), on $\Omega_1 \cap \Omega_2$ we have

$$\exp(-aN) \leq 2^{-N} \mathrm{card} \bigcap_{k \leq M} U_k \leq 2^{-N} Z(u) \leq 3 \Big(2^{-N} \mathrm{card} \bigcap_{k \leq M} U_k \Big) \, , \tag{9.118}$$

and then

$$\Big| \log(2^{-N} Z(u)) - \log \Big(2^{-N} \mathrm{card} \bigcap_{k \leq M} U_k \Big) \Big| \leq 2 \, .$$

Recalling the notation (9.19), by (9.118) we also have $\log_{aN}(2^{-N} Z(u)) = \log(2^{-N} Z(u))$ on $\Omega_1 \cap \Omega_2$ and thus using (9.21) in the last line before (9.119) below,

$$\begin{aligned}
&\mathsf{P}\Big(\Big| \frac{1}{N} \log \Big(2^{-N} \mathrm{card} \bigcap_{k \leq M} U_k \Big) - \frac{1}{N} \mathsf{E} \log_{aN}(2^{-N} Z(u)) \Big| \geq t + \frac{2}{N} \Big) \\
&\leq \mathsf{P}\Big(\Big| \frac{1}{N} \log(2^{-N} Z(u)) - \frac{1}{N} \mathsf{E} \log_{aN}(2^{-N} Z(u)) \Big| \geq t \Big) \\
&\leq \mathsf{P}(\Omega_1^c \cup \Omega_2^c) + \mathsf{P}\Big(\Big| \frac{1}{N} \log_{aN}(2^{-N} Z(u)) - \frac{1}{N} \mathsf{E} \log_{aN}(2^{-N} Z(u)) \Big| \geq t \Big) \\
&\leq L \exp\Big(-\frac{N}{L} \Big) + 2 \exp\Big(-\frac{1}{L} \min\Big(\frac{N^2 t^2}{M(1+\tau^2)^2}, \frac{Nt}{1+\tau^2} \Big) \Big) \, .
\end{aligned} \tag{9.119}$$

Also, since $\sum_{k \leq M} u(S_k) \geq -NM$, we have $\log(2^{-N} Z(u)) \geq -MN$ and

$$\begin{aligned}
\Big| \frac{1}{N} \mathsf{E} \log_{aN}(2^{-N} Z(u)) - \frac{1}{N} \mathsf{E} \log(2^{-N} Z(u)) \Big| &\leq M \mathsf{P}(2^{-N} Z(u) \leq \exp(-Na)) \\
&\leq L \exp\Big(-\frac{N}{L} \Big) \leq \frac{L}{N}
\end{aligned}$$

using (9.16). Finally, Theorem 9.5.11 implies

$$\Big| \frac{1}{N} \mathsf{E} \log(2^{-N} Z(u)) - p(u) + \log 2 \Big| \leq \frac{L}{N} \, . \tag{9.120}$$

Therefore from (9.119) we get

$$\mathsf{P}\left(\left|\frac{1}{N}\log\left(2^{-N}\mathrm{card}\bigcap_{k\le N}U_k\right)-p(u)+\log 2\right|\ge t+\frac{L}{N}\right)$$
$$\le L\exp\left(-\frac{1}{L}\min\left(N,\frac{Nt}{1+\tau^2},\frac{N^2t^2}{M(1+\tau^2)^2}\right)\right)\ . \tag{9.121}$$

Recalling the definition (9.102) of $p(u)$ we observe that quantities $p(u)-\log 2$ and $\mathrm{RS}(\alpha)$ are computed by the same procedure, that is applied to the function u in the case of $p(u)$ and to the function $\mathbf{1}_{\{x\ge\tau\}}$ in the case of $\mathrm{RS}(\alpha)$. Therefore we expect that these quantities are exponentially close to each other. However proving this rigorously is no fun, one has to perform the tedious estimates required to prove that the function ψ of (9.71) satisfies $|\psi'|\le 1/2$, after which it is not so difficult to see that the unique solution of the equations (9.65) depends smoothly on the parameters. A simpler way to proceed is to fix a solution (q,r) of the equations (9.106) and to define

$$r'=\frac{\alpha}{1-q}\mathsf{E}\left(\frac{\mathsf{E}_\xi\xi\exp u(\theta)}{\mathsf{E}_\xi\exp u(\theta)}\right)^2$$

where $\theta=z\sqrt{q}+\xi\sqrt{1-q}$, so that $r'-r$ is very small, and $q-\mathsf{E}\mathrm{th}^2(z\sqrt{r'})$ is very small.

Then, nothing needs to be changed to the proof of (9.104) if one uses the values (q,r') rather than a solution of the equations (9.65), so that instead of (9.121) one obtains directly

$$\left|\frac{1}{N}\mathsf{E}\log(2^{-N}Z(u))-\mathrm{RS}(\alpha)\right|\le\frac{L}{N}\ ,$$

and then as in (9.121) one gets

$$\mathsf{P}\left(\left|\frac{1}{N}\log\left(2^{-N}\mathrm{card}\bigcap_{k\le N}U_k\right)-\mathrm{RS}(\alpha)\right|\ge t+\frac{L}{N}\right)$$
$$\le L\exp\left(-\frac{1}{L}\min\left(N,\frac{Nt}{1+\tau^2},\frac{N^2t^2}{M(1+\tau^2)^2}\right)\right)$$

from which (9.107) follows. □

Should the reader find none of the two above arguments above convincing, another possibility is to look for a refund of this book. One may also observe that the rate L/N in (9.120) is not critical since the right-hand side of (9.107) becomes small only for t about $1/\sqrt{N}$.

Theorem 9.6.1 is much more precise than Theorem 8.4.1. This suggests the following.

Research Problem 9.6.4. (Level 1) For α small, improve Theorem 8.4.1 to a statement as precise as (9.107).

Of course, (see Research Problem 8.3.5) the case of $\alpha\le\alpha_0<2$ is even more interesting, but it is no longer level 1.

9.7 Higher Order Expansion and Central Limit Theorems

The main result of this section is probably Theorem 9.7.12 below. The basic idea is simple, and has been used many times. If $\varphi(v)$ is given by (9.74) (reproduced in (9.123) just below) then (9.95) is a consequence of the inequality $|\varphi(1)-\varphi(0)| \le \sup|\varphi'(v)|$. Instead of the "first order expansion" we will use a "second order expansion", $|\varphi(1)-\varphi(0)-\varphi'(0)| \le \sup|\varphi''(v)|$. If we roughly describe the action of taking the derivatives of φ as "bringing out a factor $R_{1,2}-q$" then we expect that taking the second derivative "brings out another such factor" and increases accuracy by a factor $N^{-1/2}$.

We recall the notation S_v^ℓ of (9.73).

Proposition 9.7.1. *There exists a number L with the following property. Consider a function u that satisfies (9.2), as well as*

$$\forall \ell\,,\ 1 \le \ell \le 5\,,\quad |u^{(\ell)}| \le \exp\frac{N}{L}\,. \tag{9.122}$$

Consider a function f on Σ_N^4, consider $\ell \neq \ell' \le 4$ and consider either

$$\varphi(v) = \nu_{t,v}(fu'(S_v^\ell)u'(S_v^{\ell'})) \quad \text{or} \quad \varphi(v) = \nu_{t,v}(f)\,. \tag{9.123}$$

Then if $L\alpha(1+\tau^2) \le 1$, we have

$$|\varphi''(v)| \le (L\exp L\tau^2)\Big(\nu_t(f^2)^{1/2}\nu_t\big((R_{1,2}-q)^4\big)^{1/2} \tag{9.124}$$
$$+\frac{1}{N}\nu_t(|f|)+\max|f|\exp\left(-\frac{N}{L}\right)\Big)\,.$$

Proof. We differentiate twice the relation (9.72), which brings out a second factor of the type $(R^t_{\ell_1,\ell_2}-q)$ in each term. We then repeat the proof of Proposition 9.5.5 with the following small difference: we need to control a few more replicas, since $\varphi''(v)$ depends on 8 replicas while $\varphi'(v)$ depends only on 6 replicas. □

Corollary 9.7.2. *Under the conditions of Proposition 9.7.1, and if moreover $L\alpha\exp L\tau^2 \le 1$, we have*

$$|\varphi''(v)| \le L\exp L\tau^2\left(\frac{1}{N}\nu(f^2)^{1/2}+\max|f|\exp\left(-\frac{N}{L}\right)\right)\,. \tag{9.125}$$

Proof. This is a consequence of (9.124). Using (9.101), we have $\nu((R_{1,2}-q)^4)^{1/2} \le L/N$, and using (9.99) we have $\nu_t((R_{1,2}-q)^4)^{1/2} \le L/N$ and

$$\nu_t(f^2) \le L\nu(f^2)+L\max(f^2)\exp(-N/L)\,.$$ □

At this stage we realize that we are facing a nasty unsolved technical problem. While controlling "a few more replicas" (i.e. any finite number rather than 4 only) by increasing the value of the number L of Proposition 9.5.2 is easy, we do not know how to control all the replicas at the same time. Here is the precise version of the problem.

Research Problem 9.7.3. Given $\tau \geq 0$, does there exists a number L and a number $K_0(\tau)$ depending on τ only such that if u satisfies (9.2) and (9.122), then for $MK_0(\tau) \leq N$, any n and any function f on Σ_N^n, we have

$$|\varphi''(v)| \leq K(\tau, n)\left(\frac{1}{N}\nu(f^2)^{1/2} + \max(|f|)\exp\left(-\frac{N}{L}\right)\right) , \tag{9.126}$$

where $K(\tau, n)$ depends only on τ and n?

There is nothing specific about the second derivative here. It is probably the same problem to ask whether we have

$$|\varphi'(v)| \leq K(\tau, n)\left(\frac{1}{\sqrt{N}}\nu(f^2)^{1/2} + \max(|f|)\exp\left(-\frac{N}{L}\right)\right) . \tag{9.127}$$

The difficulty lies with our method of "separating the numerators from the denominators" using the Cauchy-Schwarz inequality. When we work with n replicas rather that 6 replicas, we have to replace the exponent 12 by $2n$ in (9.81). It is not difficult using Theorem 9.3.1 to see that given n, we can control $|\varphi''(v)|$ as in (9.126) for all f on Σ_N^n, provided $\alpha K(\tau, n) \leq 1$ but $K(\tau, n) \to \infty$ as $n \to \infty$. It is reasonable to think that the previous research problem is closely related to Research Problem 9.2.4. Here is a less technical question.

Research Problem 9.7.4. Can one prove a central limit theorem in the spirit of Theorem 1.10.1 under a condition of the type $\alpha K(\tau) \leq 1$?

More precisely, we would like to prove such a theorem where $O(k)$ denotes a quantity $A = A_{N,u}$ with $N^{k/2}A$ bounded independently of N *and* of the choice of u satisfying (9.2) (and maybe a mild condition such as (9.122)).

At present we know how to prove a central limit theorem for all overlaps only when assuming that u is "bounded from below independently of N".

Proposition 9.7.5. *There exists a constant L with the following property. Assume as usual that u satisfies (9.2) and (9.122), and moreover that*

$$-C \leq u \leq 0 . \tag{9.128}$$

Then for each n and each function f on Σ_N^n, we have, with the notation (9.123), that, whenever $1/\tau_1 + 1/\tau_2 = 1$, and $L\alpha \exp L\tau^2 \leq 1$

$$|\varphi'(v)| \le K(C,n,\tau)\biggl(\nu_t(|f|^{\tau_1})^{1/\tau_1}\nu_t(|R_{1,2}-q|^{\tau_2})^{1/\tau_2} + \frac{1}{N}\nu_t(|f|) + \max|f|\exp\left(-\frac{N}{L}\right)\biggr) \tag{9.129}$$

$$|\varphi''(v)| \le K(C,n,\tau)\left(\frac{1}{N}\nu_t(f^2)^{1/2} + \max|f|\exp\left(-\frac{N}{L}\right)\right)\ , \tag{9.130}$$

where of course $K(C,n,\tau)$ depends only on C, n and τ, and not N, u, or anything else.

Proof. The proof follows the lines of Proposition 9.5.2, but is much simpler. We use that $\exp u(S_v^1) \ge \exp(-C)$ rather than (9.81) and

$$\nu_t(f) \ge \mathsf{E}_\xi\left\langle f\exp\sum_{\ell\le n}u(S^\ell_{M,t})\right\rangle_{t,\sim} \ge \exp(-nC)\langle f\rangle_{t,\sim}$$

rather than (9.89). Only minor changes are needed for the remainder of the proof. □

Corollary 9.7.6. *Under the conditions of Proposition 9.7.5, for any function $f \ge 0$ on Σ_N^n we have*

$$\nu_t(f) \le K(C,n,\tau)\left(\nu(f) + \max|f|\exp\left(-\frac{N}{L}\right)\right)\ . \tag{9.131}$$

Proof. We copy the proof of (9.99), using now (9.129) instead of (9.94) to obtain

$$|\nu_t'(f)| \le K(C,n,\tau)\left(\nu(f) + \max|f|\exp\left(-\frac{N}{L}\right)\right)\ ,$$

and we integrate as usual. □

Proving central limit theorems requires the explicit computation of $\varphi'(0)$.

Lemma 9.7.7. *Assume that f is a function on Σ_N^n, and that, with our usual notation,*

$$\varphi(v) = \nu_{t,v}(fu'(S_v^\ell)u'(S_v^{\ell'}))\ , \tag{9.132}$$

where $\ell \ne \ell'$. Then

$$\varphi(0) = \widehat{r}\mathsf{E}\langle f\rangle_{t,\sim} \tag{9.133}$$

$$\begin{aligned}\varphi'(0) = &\sum_{1\le\ell_1<\ell_2\le n} c(\ell_1,\ell_2;\ell,\ell')\mathsf{E}\langle f(R^t_{\ell_1,\ell_2}-q)\rangle_{t,\sim} \\ &- n\sum_{\ell_1\le n} c(\ell_1,n+1;\ell,\ell')\mathsf{E}\langle f(R^t_{\ell_1,n+1}-q)\rangle_{t,\sim} \\ &+ \frac{n(n+1)}{2}c(n+1,n+2;\ell,\ell')\mathsf{E}\langle f(R^t_{n+1,n+2}-q)\rangle_{t,\sim}\end{aligned} \tag{9.134}$$

where

$$c(\ell_1,\ell_2;\ell,\ell') = c\big(\mathrm{card}(\{\ell_1,\ell_2\}\cap\{\ell,\ell'\})\big)\ ,$$

$$c(0) = \mathsf{E}\left(\frac{\mathsf{E}_\xi U'(\theta)}{\mathsf{E}_\xi U(\theta)}\right)^4 \ ;\ c(1) = \mathsf{E}\frac{\mathsf{E}_\xi U''(\theta)(\mathsf{E}_\xi U'(\theta))^2}{(\mathsf{E}_\xi U(\theta))^3}$$

$$c(2) = \mathsf{E}\left(\frac{\mathsf{E}_\xi U''(\theta)}{\mathsf{E}_\xi U(\theta)}\right)^2 \ ,$$

for $U(x) = \exp u(x)$ *and, as usual,* $\theta = z\sqrt{q} + \xi\sqrt{1-q}$.

Moreover if now

$$\varphi(v) = \nu_{t,v}(f) \tag{9.135}$$

then we obtain $\varphi(0) = \mathsf{E}\langle f\rangle_{t,\sim}$ *and*

$$\begin{aligned}\varphi'(0) = \widehat{r}\Big(&\sum_{1\le\ell_1<\ell_2\le n}\mathsf{E}\langle f(R^t_{\ell_1,\ell_2}-q)\rangle_{t,\sim}\\ &- n\sum_{\ell_1\le n}\mathsf{E}\langle f(R^t_{\ell_1,n+1}-q)\rangle_{t,\sim}\\ &+\frac{n(n+1)}{2}\mathsf{E}\langle f(R^t_{n+1,n+2}-q)\rangle_{t,\sim}\Big)\ .\end{aligned} \tag{9.136}$$

Proof. The proof of (9.133) is done in the course of the proof of Proposition 2.3.5. To prove (9.134) we proceed as follows. We observe that $U'(x) = u'(x)U(x)$, and as in (9.72) we write

$$\nu_{t,v}(fu'(S_v^\ell)u'(S_v^{\ell'})) = \mathsf{E}\frac{\big\langle f\mathsf{E}_\xi\prod_{r\le n}U^{(k(r))}(S_v^r)\big\rangle_{t,\sim}}{\langle\mathsf{E}_\xi\exp(S_v^1)\rangle^n_{t,\sim}}\ , \tag{9.137}$$

where $k(r)=1$ if either $r=\ell$ or $r=\ell'$, and $k(r)=0$ otherwise. Let us define

$$W_v(\ell_1,\ell_2) = \mathsf{E}\frac{\big\langle f(R^t_{\ell_1,\ell_2}-q)\mathsf{E}_\xi\prod_{r\le n+2}U^{(k(r,\ell_1,\ell_2))}(S_v^r)\big\rangle_{t,\sim}}{\langle\mathsf{E}_\xi\exp(S_v^1)\rangle^n_{t,\sim}}\ ,$$

were $k(r,\ell_1,\ell_2) = k(r)+1$ if $r=\ell_1$ or $r=\ell_2$, and $k(r,\ell_1,\ell_2)=k(r)$ otherwise. Then differentiation of (9.137) and integration by parts as we have learned to do in the proof of Lemma 2.3.2 yield the formula

$$\varphi'(v) = \sum_{1\le\ell_1<\ell_2\le n}W_v(\ell_1,\ell_2) - n\sum_{\ell_1\le n}W_v(\ell_1,n+1) + \frac{n(n+1)}{2}W_v(n+1,n+2)$$

Proceeding as in the proof of (9.133) we then obtain that

$$W_0(\ell_1,\ell_2) = c(\ell_1,\ell_2,\ell,\ell')\mathsf{E}\langle f(R^t_{\ell_1,\ell_2}-q)\rangle_{t,\sim}\ .$$

The proof of (9.136) is similar but simpler, taking now $k(r) = 0$ in (9.137). □

We do not really like to deal with the averages $\mathsf{E}\langle\cdot\rangle_{t,\sim}$ in (9.134), and would rather deal with ν_t instead. Relating these two averages is made easier by the small factor $R^t_{\ell_1,\ell_2} - q$. There, and everywhere else in this section, there are really two situations we can handle. Either a small number n of replicas is involved (say $n \le 8$), or else we assume that u bounded from below as in (9.128) and we can control any number of replicas. To lighten the exposition, we will state only the results in the second case, but we will remember when we need to prove Lemma 9.5.12 that it is then not needed to assume that u is bounded from below.

Lemma 9.7.8. *Under the conditions of Proposition 9.7.5, for a function f on Σ_N^n we have*

$$|\nu_t(f(R_{\ell_1,\ell_2} - q)) - \mathsf{E}\langle f(R^t_{\ell_1,\ell_2} - q)\rangle_{t,\sim}|$$
$$\le K\left(\frac{1}{N}\nu(f^2)^{1/2} + \max(|f|)\exp\left(-\frac{N}{L}\right)\right) . \tag{9.138}$$

Here and below, K **is permitted to depend on** C, n, τ**, but not on** N **nor** f**.**

Proof. Let $\varphi(v) = \nu_{t,v}(f(R^t_{\ell_1,\ell_2} - q))$, so that $\varphi(1) = \nu_t(f(R^t_{\ell_1,\ell_2} - q))$, $\varphi(0) = \mathsf{E}\langle f(R^t_{\ell_1,\ell_2} - q)\rangle_{t,\sim}$. Since

$$\left|\nu_t(f(R_{\ell_1,\ell_1} - q)) - \nu_t(f(R^t_{\ell_1,\ell_2} - q))\right| \le \frac{\nu_t(|f|)}{N} ,$$

we get

$$|\nu_t(f(R_{\ell_1,\ell_2} - q)) - \mathsf{E}\langle f(R^t_{\ell_1,\ell_2} - q)\rangle_{t,\sim}| \le \sup_v |\varphi'(v)| + \frac{1}{N}\nu_t(|f|) .$$

Next we use (9.129) for $\tau_2 = 4$, $\tau_1 = 4/3$, and $f(R^t_{\ell_1,\ell_2} - q)$ rather than f, to get

$$|\varphi'(v)| \le K\Big(\nu_t\big(|f(R^t_{\ell,\ell'} - q)|^{4/3}\big)^{3/4}\nu_t\big((R_{1,2} - q)^4\big)^{1/4}$$
$$+ \frac{1}{N}\nu_t(|f|) + \max|f|\exp\left(-\frac{N}{L}\right)\Big) .$$

Then we use again Hölder's inequality to get

$$\nu_t\big(|f|^{4/3}|R^t_{\ell_1,\ell_2} - q|^{4/3}\big) \le \nu_t(f^2)^{2/3}\nu_t\big((R^t_{\ell_1,\ell_2} - q)^4\big)^{1/3} ,$$

and the Cauchy-Schwarz inequality to obtain that $\nu_t(|f|) \le \nu_t(f^2)^{1/2}$. Finally we use Corollary 9.7.6 to replace ν_t by ν and (9.101) to see that $\nu((R_{\ell_1,\ell_2} -$

$q)^4) \le K/N^2$ and $\nu((R^t_{\ell_1,\ell_2} - q)^4) \le K/N^2$. Combining these estimates yields the result. □

Another nice feature is that we can change the value of t in the term $\nu_t(f(R_{\ell_1,\ell_2} - q))$ without creating a large error.

Lemma 9.7.9. *Under the conditions of Proposition 9.7.5, for a function f on Σ_N^n and $0 \le t \le 1$, we have*

$$\left|\nu_t'(f(R_{\ell_1,\ell_2} - q))\right| \le K\left(\frac{1}{N}\nu(f^2)^{1/2} + \max(|f|)\exp\left(-\frac{N}{L}\right)\right). \tag{9.139}$$

As a consequence, if $0 \le t, t' \le 1$ then

$$\begin{aligned}\left|\nu_t(f(R_{\ell_1,\ell_2} - q)) - \nu_{t'}(f(R_{\ell_1,\ell_2} - q))\right| &\le K\bigg(\frac{1}{N}\nu(f^2)^{1/2} \\ &\quad + \max(|f|)\exp\left(-\frac{N}{L}\right)\bigg).\end{aligned} \tag{9.140}$$

This will allow us in particular to replace the computation of $\nu_t(\cdot)$ by that of $\nu_0(\cdot)$, for which one can take advantage of the decoupling of the last spin.

Proof. Using (9.129) as in Corollary 9.5.7 implies that for a function f on Σ_N^n it holds

$$\begin{aligned}|\nu_t(fu'(S^\ell_{M,t})u'(S^{\ell'}_{M,t})) - \widehat{r}\nu_t(f)| &\le K\bigg(\nu_t(|f|^{\tau_1})^{1/\tau_2}\nu_t(|R_{1,2} - q|^{\tau_2})^{1/\tau_2} \\ &\quad + \frac{1}{N}\nu_t(|f|) + \max|f|\exp\left(-\frac{N}{L}\right)\bigg).\end{aligned} \tag{9.141}$$

Using this for $f(R_{\ell_1,\ell_2} - q)$ rather than f and using Hölder's inequality as in Lemma 9.7.8 we get

$$\begin{aligned}&\left|\nu_t\big(f(R_{\ell_1,\ell_2} - q)u'(S^\ell_{M,t})u'(S^{\ell'}_{M,t})\big) - \widehat{r}\nu_t(f(R_{\ell_1,\ell_2} - q))\right| \\ &\le K\left(\frac{1}{N}\nu_t(f^2) + \max|f|\exp\left(-\frac{N}{L}\right)\right).\end{aligned}$$

Combining with Proposition 2.2.2 and (9.131) we can bound $|\nu_t'(f(R_{\ell_1,\ell_2} - q))|$ as in (9.139). □

In the proof of a central limit theorem for the overlap, there is the aspect of controlling the error terms, and the matter of handling the algebra, which are rather distinct. In order to illustrate the basic procedure before we get into algebraic complications, we prove Lemma 9.5.12 (as was promised when this lemma was stated). The method of proof should certainly not come as a surprise. Throughout the proof $O(2)$ denotes quantity A such that $N|A|$ remains bounded independently of N.

Proof of Lemma 9.5.12. We have $\nu(R_{1,2}-q)=\nu(f)$ for $f=\varepsilon_1\varepsilon_2-q$. We have $\nu_0(f)=0$, and we compute $\nu_t'(f)$ using (2.23). For each term of the type $\nu_t(f\varepsilon_\ell\varepsilon_{\ell'}u'(S^\ell_{M,t})u'(S^{\ell'}_{M,t}))$ we consider the function

$$\varphi(v)=\nu_{t,v}\big(f\varepsilon_\ell\varepsilon_{\ell'}u'(S^\ell_v)u'(S^{\ell'}_v)\big)\ ,$$

so that by (9.130) we have

$$\varphi(1)=\varphi(0)+\varphi'(0)+O(2)\ . \tag{9.142}$$

Using (9.133) and (9.134) we see that $\varphi(0)=\widehat{r}\mathsf{E}\langle f\varepsilon_\ell\varepsilon_{\ell'}\rangle_{t,\sim}$ and that $\varphi'(0)$ is a linear combination of terms of the type $\mathsf{E}\langle\varepsilon_\ell\varepsilon_{\ell'}f(R^t_{\ell_1,\ell_2}-q)\rangle_{t,\sim}$. The coefficients of these terms are all the type $\pm c(j)$ for $j=0,1,2$. Using Lemmas 9.7.8 and 9.7.9 we get successively

$$\begin{aligned}\mathsf{E}\langle\varepsilon_\ell\varepsilon_{\ell'}f(R^t_{\ell_1,\ell_2}-q)\rangle_{t,\sim}&=\nu_t\big(\varepsilon_\ell\varepsilon_{\ell'}(\varepsilon_1\varepsilon_2-q)(R_{\ell_1,\ell_2}-q)\big)+O(2)\\&=\nu_0\big(\varepsilon_\ell\varepsilon_{\ell'}(\varepsilon_1\varepsilon_2-q)(R_{\ell_1,\ell_2}-q)\big)+O(2)\\&=\nu_0\big(\varepsilon_\ell\varepsilon_{\ell'}(\varepsilon_1\varepsilon_2-q)(R^-_{\ell_1,\ell_2}-q)\big)+O(2)\\&=\nu_0(\varepsilon_\ell\varepsilon_{\ell'}(\varepsilon_1\varepsilon_2-q))\nu_0(R^-_{\ell_1,\ell_2}-q)+O(2)\\&=\nu_0(\varepsilon_\ell\varepsilon_{\ell'}(\varepsilon_1\varepsilon_2-q))\nu_0(R_{\ell_1,\ell_2}-q)+O(2)\\&=\nu_0(\varepsilon_\ell\varepsilon_{\ell'}(\varepsilon_1\varepsilon_2-q))\nu(R_{1,2}-q)+O(2)\ .\end{aligned}\tag{9.143}$$

In this manner we obtain from (9.142) that

$$\alpha\nu_v\big(f\varepsilon_\ell\varepsilon_{\ell'}u'(S^\ell_v)u'(S^{\ell'}_v)\big)=\alpha\widehat{r}\mathsf{E}\langle f\varepsilon_\ell\varepsilon_{\ell'}\rangle_{t,\sim}+A\nu(R_{1,2}-q)+O(2)\ ,$$

where A is a sum of terms of the type $\pm\alpha c(j)\nu_0(\varepsilon_\ell\varepsilon_{\ell'}(\varepsilon_1\varepsilon_2-q))$.

We collect the terms in (2.23). We associate each quantity $\alpha\widehat{r}\mathsf{E}\langle f\varepsilon_\ell\varepsilon_{\ell'}\rangle_{t,\sim}$ with the corresponding term in (2.25), and we get

$$\nu_t'(f)=\mathrm{I}+\mathrm{II}+O(2)\ ,$$

where I is a sum of terms of the kind $\pm r(\mathsf{E}\langle\varepsilon_\ell\varepsilon_{\ell'}f\rangle_{t,\sim}-\nu_t(\varepsilon_\ell\varepsilon_{\ell'}f))$ and $\mathrm{II}=A\nu(R_{1,2}-q)$, with A a sum of terms of the type $\pm\alpha c(j)\nu_0(\varepsilon_\ell\varepsilon_{\ell'}(\varepsilon_1\varepsilon_2-q))$.

To control the term I we will control separately each difference

$$\mathsf{E}\langle\varepsilon_\ell\varepsilon_{\ell'}f\rangle_{t,\sim}-\nu_t(\varepsilon_\ell\varepsilon_{\ell'}f)\ .$$

For each such difference we consider the function $\varphi(v)=\nu_{t,v}(f\varepsilon_\ell\varepsilon_{\ell'})$ and we use (9.142) again, together with the second part of Lemma 9.7.7, so $\varphi(0)=\mathsf{E}\langle\varepsilon_\ell\varepsilon_{\ell'}f\rangle_{t,\sim}$ and $\varphi'(0)$ is given by (9.136). Proceeding as in (9.143) if follows from $\nu_0(\varepsilon_1\varepsilon_2-q)=0$ that

$$\mathsf{E}\langle f(R^t_{\ell_1,\ell_2}-q)\rangle_{t,\sim}=\nu_0(\varepsilon_1\varepsilon_2-q)\nu(R_{1,2}-q)+O(2)=O(2)\ ,$$

so that $\mathrm{I}=O(2)$.

In conclusion we have shown that

$$\nu_t'(f) = A\nu(R_{1,2} - q) + O(2) \ ,$$

and since $\nu_0(f) = 0$,

$$\nu_1(f) = \nu(R_{1,2} - q) = A\nu(R_{1,2} - q) + O(2) \ .$$

This implies that $\nu(R_{1,2} - q) = O(2)$ (the desired result) provided we can show that $|A| \leq 1/2$. To prove Lemma 9.5.12 we need to know this is the case under a condition of the form $L\alpha \exp L\tau^2 \leq 1$. It suffices to show that the quantities $\widehat{r}, c(0), c(1), c(2)$ are all $\leq \exp L\tau^2$. We have obtained this bound for $\widehat{r}$ in the proof of the uniqueness of the equations (9.65); the case of $c(j)$ is entirely similar. □

We now turn to the "algebra". We consider the numbers

$$d(0) = \widehat{q} - q^2 \ ; \quad d(1) = q - q^2 \ ; \quad d(2) = 1 - q^2 \ ,$$

where $q = \mathsf{E}\mathrm{th}^2(z\sqrt{r})$ and $\widehat{q} = \mathsf{E}\mathrm{th}^4(z\sqrt{r})$. We observe the formula

$$\nu_0((\varepsilon_1\varepsilon_2 - q)\varepsilon_1\varepsilon_2) = d(\mathrm{card}\{1,2\} \cap \{\ell, \ell'\}) \ . \tag{9.144}$$

We consider the numbers

$$b_0(2) = \ c(2)\, d(2) - 4\, c(1)\, d(1) + 3\, c(0)\, d(0) \tag{9.145}$$

$$\begin{aligned} b_0(1) = \ & c(1)\, d(2) + c(2)\, d(1) - 2\, c(1)\, d(1) - 3\, c(0)\, d(1) - 3\, c(1)\, d(0) \\ & + 6\, c(0)\, d(0) \end{aligned} \tag{9.146}$$

$$\begin{aligned} b_0(0) = \ & c(2)\, d(0) + c(0)\, d(2) + 4\, c(1)\, d(1) - 8\, c(1)\, d(0) - 8\, c(0)\, d(1) \\ & + 10\, c(0)\, d(0) \end{aligned} \tag{9.147}$$

and finally

$$b(2) = \alpha b_0(2) - \alpha\widehat{r}^2(d(2) - 4d(1) + 3d(0)) \tag{9.148}$$
$$b(1) = \alpha b_0(1) - \alpha\widehat{r}^2(d(2) - 4d(1) + 3d(0)) \tag{9.149}$$
$$b(0) = \alpha b_0(0) - \alpha\widehat{r}^2(d(2) - 4d(1) + 3d(0)) \ . \tag{9.150}$$

Despite the fact that the previous formulas look a bit complicated, there definitely exists some structure (that is not entirely elucidated). The next lemma seems to indicate that, somewhere, we take the product of two operators. Clarifying what really happens seems related to Research Problem 1.8.3.

Lemma 9.7.10. *We have*

$$b(2)-2\,b(1)+b(0)=\alpha\,(c(2)-2\,c(1)+c(0))(d(2)-2\,d(1)+d(0)) \quad (9.151)$$

$$b(2)-4\,b(1)+3\,b(0)=\alpha\,(c(2)-4\,c(1)+3\,c(0))(d(2)-4\,d(1)+3\,d(0))\,. \quad (9.152)$$

Proof. Straightforward algebra. □

One should also mention that $d(2)-4\,d(1)+3\,d(0)$ is the quantity $1-4\,q+3\,\widehat{q}$, that already occurred in Section 1.8 (see e.g. (1.235)).

Theorem 9.7.11. *There exists a number L with the following property. Assume that u satisfies (9.2), (9.122) and (9.128). Then if $L\alpha\exp L\tau^2\le 1$, given a function f^- on Σ^n_{N-1}, which is a product of k functions of the type $R^-_{\ell,\ell'}-q$, $\ell,\ell'\le n$, and given integers $x,y\le n$, we have*

$$\begin{aligned}\nu((\varepsilon_x\varepsilon_y-q)f^-)=&\sum_{1\le\ell<\ell'\le n} b(\ell,\ell';x,y)\nu(f^-(R_{\ell,\ell'}-q))\\&-n\sum_{\ell\le n} b(\ell,n+1;x,y)\nu(f^-(R_{\ell,n+1}-q))\\&+\frac{n(n+1)}{2}b(0)\nu(f^-(R_{n+1,n+2}-q))\\&+O(k+2)\,,\end{aligned} \quad (9.153)$$

where $b(\ell,\ell';x,y)=b(\mathrm{card}(\{\ell,\ell'\}\cap\{x,y\}))$ and $b(j)$, $j=0,1,2$ are given by (9.148) to (9.150).

Here, $O(k+2)$ is a quantity B such that $|B|\le K(C,n,\tau)N^{-(k+2)/2}$, when $K(C,n,\tau)$ is independent of n (and in fact also of the choice of u).

Once this theorem is proved, we can copy the proof of Theorem 1.10.1 in the present setting. Repeating the computations of Section 1.8, the values of A, B, and C are now given by

$$\begin{aligned}A^2&=\frac{1-2q+\widehat{q}}{N(1-(b(2)-2b(1)+b(0)))}\\B^2&=\frac{1}{1-(b(2)-4b(1)+3b(0))}\left(\frac{1}{N}(q-\widehat{q})+(b(1)-b(0))A^2\right)\\C^2&=\frac{1}{1-(b(2)-4b(1)+3b(0))}\left(\frac{1}{N}(\widehat{q}-q^2)+b(0)A^2+(4b(1)-6b(0))B^2\right).\end{aligned}$$

Theorem 9.7.12. *There exists a number L with the following property. Assume that u satisfies (9.2), (9.122) and (9.128). Then if $L\alpha\exp L\tau^2\le 1$, the following occurs with the values of A,B,C given above. Consider an integer n. For $1\le\ell<\ell'\le n$ consider integers $k(\ell,\ell')$. For $1\le\ell\le n$ consider* *integers* $k(\ell)$*. Set* $k_1 = \sum_{1\le\ell<\ell'\le n} k(\ell,\ell')$*,* $k_2 = \sum_{1\le\ell\le n} k(\ell)$*, consider an integer* k_3 *and set* $k = k_1 + k_2 + k_3$*. Then*

$$\nu\Bigl(\prod_{1\le\ell<\ell'\le n} T_{\ell,\ell'}^{k(\ell,\ell')}\prod_{1\le\ell\le n} T_\ell^{k(\ell)} T^{k_3}\Bigr)$$
$$= \prod_{1\le\ell<\ell'\le n} a(k(\ell,\ell'))\prod_{1\le\ell\le n} a(k(\ell))a(k_3)A^{k_1}B^{k_2}C^{k_3} + O(k+1)\,.$$

So, it remains only to prove (9.153). The scheme of proof is as follows. we may assume $x=1$, $y=2$, we use that $\nu_0((\varepsilon_1\varepsilon_2-q)f^-)=0$, so that

$$\nu((\varepsilon_1\varepsilon_2-q)f^-) = \int_0^1 \nu_t'((\varepsilon_1\varepsilon_2-q)f^-)\mathrm{d}t\,.$$

We compute $\nu_t'((\varepsilon_1\varepsilon_2-q)f^-)$ using (2.23). For each term of the type

$$\nu_t\bigl((\varepsilon_1\varepsilon_2-q)f^-\varepsilon_\ell\varepsilon_{\ell'}u'(S_{M,t}^\ell)u'(S_{M,t}^{\ell'})\bigr)\,,$$

we consider the function

$$\varphi(v) = \nu_{t,v}\bigl((\varepsilon_1\varepsilon_2-q)f^-\varepsilon_\ell\varepsilon_{\ell'}u'(S_v^\ell)u'(S_v^{\ell'})\bigr)\,.$$

We know from (9.101) that $\nu((R_{1,2}^- - q)^{2k}) = O(2k)$, so that from (9.99) we have $\nu_t((R_{1,2}^- - q)^{2k}) = O(2k)$ (uniformly in t) and we know through (9.130) that $\varphi''(v) = O(k+2)$, so

$$\varphi(1) = \varphi(0) + \varphi'(0) + O(k+2)\,. \tag{9.154}$$

We compute $\varphi'(0)$ using (9.134). According to Lemmas 9.7.8 and 9.7.9, within error $O(k+2)$ we may replace the terms

$$\mathsf{E}\langle(\varepsilon_1\varepsilon_2-q)f^-\varepsilon_\ell\varepsilon_{\ell'}(R_{\ell_1,\ell_2}^t - q)\rangle_{t,\sim}$$

by

$$\nu_0\bigl((\varepsilon_1\varepsilon_2-q)f^-\varepsilon_\ell\varepsilon_{\ell'}(R_{\ell_1,\ell_2}-q)\bigr)$$
$$= \nu_0(\varepsilon_\ell\varepsilon_{\ell'}(\varepsilon_1\varepsilon_2-q))\nu_0(f^-(R_{\ell_1,\ell_2}-q))$$

and we may in turn replace $\nu_0(f^-(R_{\ell_1,\ell_2}-q))$ by $\nu(f^-(R_{\ell_1,\ell_2}-q))$ using Lemma 9.7.9 again. The terms $\varphi(0) = \widehat{r}\mathsf{E}\langle(\varepsilon_1\varepsilon_2-q)\varepsilon_\ell\varepsilon_{\ell'}f^-\rangle_{t,\sim}$ regroup with the corresponding terms of the quantity II of (2.23). To compute the difference

$$\mathsf{E}\langle(\varepsilon_1\varepsilon_2-q)\varepsilon_\ell\varepsilon_{\ell'}f^-\rangle_{t,\sim} - \nu_t((\varepsilon_1\varepsilon_2-q)\varepsilon_\ell\varepsilon_{\ell'}f^-)\,,$$

we introduce the function $\varphi(v) = \nu_{t,v}((\varepsilon_1\varepsilon_2-q)\varepsilon_\ell\varepsilon_{\ell'}f^-)$, we use (9.154) and (9.136) to compute $\varphi'(0)$. Within an error $O(k+2)$ we reach as previously that this difference is a sum of terms

$$\pm\nu_0((\varepsilon_1\varepsilon_2 - q)\varepsilon_\ell\varepsilon_{\ell'})\nu(f^-(R_{\ell_1,\ell_2} - q)) .$$

By this procedure we have obtained that within an error $O(k+2)$, $\nu((\varepsilon_1\varepsilon_2 - q)f^-)$ is a certain sum of terms of the type $C_{\ell_1,\ell_2}\nu(f^-(R_{\ell_1,\ell_2} - q))$; and to complete the proof it remains to perform the algebra: the computation of these coefficients C_{ℓ_1,ℓ_2}. This computation will require carefully counting terms in certain formulas. This could look tedious, unless of course one keeps marveling, as one should, about why the relations we find can be at all true. For the computation of the terms C_{ℓ_1,ℓ_2}, it helps to use the quantities of (1.226), i.e.

$$T_{\ell,\ell'} = \frac{(\boldsymbol{\sigma}^\ell - \mathbf{b})\cdot(\boldsymbol{\sigma}^{\ell'} - \mathbf{b})}{N} \ , \quad T_\ell = \frac{(\boldsymbol{\sigma}^\ell - \mathbf{b})\cdot \mathbf{b}}{N} \ , \quad T = \frac{\mathbf{b}\cdot\mathbf{b}}{N} - q \ ,$$

for $\mathbf{b} = (\langle\sigma_i\rangle)_{i\le N}$. We start with a general identity.

Lemma 9.7.13. *Consider numbers $a(0), a(1)$ and $a(2)$. Given two integers $\ell, \ell' \le n$ we define*

$$a(\ell_1,\ell_2) = a(\mathrm{card}\{\ell_1,\ell_2\}\cap\{\ell,\ell'\}) .$$

Then for any function f on Σ_N^n we have the identity

$$\begin{aligned}
&\sum_{1\le\ell_1<\ell_2\le n} a(\ell_1,\ell_2)\nu(f(R_{\ell_1,\ell_2} - q))\\
&- n\sum_{\ell_1\le n} a(\ell_1, n+1)\nu(f(R_{\ell_1,n+1} - q))\\
&+ \frac{n(n+1)}{2} a(n+1,n+2)\nu(f(R_{n+1,n+2} - q))\\
&= \sum_{\ell_1<\ell_2} a(\ell_1,\ell_2)\nu(fT_{\ell_1,\ell_2}) + \sum_{\ell_1} a_1(\ell_1)\nu(fT_{\ell_1})\\
&+ (a(2) - 4a(1) + 3a(0))\nu(fT)
\end{aligned} \tag{9.155}$$

where

$$a_1(\ell_1) = \begin{cases} 2a(1) - 3a(0) & \text{if } \ell_1 \notin \{\ell,\ell'\} \\ a(2) - 2a(1) & \text{if } \ell_1 \in \{\ell,\ell'\}. \end{cases}$$

The reader observes that the range of summation need not be specified for ℓ_1 and ℓ_2 in the right hand side of (9.155), because $\nu(fT_{\ell_1})$ is zero if f does not depend on replica ℓ_1, and similarly for $\nu fT_{\ell_1,\ell_2}$.

Proof. We substitute the relation

$$R_{\ell_1,\ell_2} - q = T_{\ell_1,\ell_2} + T_{\ell_1} + T_{\ell_2} + T \tag{9.156}$$

in each of the terms in the left-hand side of the seeked relation (9.155), and we simply count how many times each term occurs in order to get the right-hand side of (9.155). This is straightforward but requires a bit of patience. The coefficient of $\nu(fT)$ is

$$\sum_{1\le \ell_1<\ell_2\le n} a(\ell_1,\ell_2) - n\sum_{\ell_1\le n} a(\ell_1, n+1) + \frac{n(n+1)}{2}a(0)\ . \tag{9.157}$$

In the first sum above, one term exactly is $a(2)$. There are $(n-2)(n-3)/2$ terms for which $\{\ell_1,\ell_2\}\cap\{\ell,\ell'\}=\emptyset$, and which are equal to $a(0)$. Since the sum has $n(n-1)/2$ terms, there are exactly

$$\frac{n(n-1)}{2} - \frac{(n-2)(n-3)}{2} - 1 = 2n-4$$

terms for which $\mathrm{card}(\{\ell_1,\ell_2\}\cap\{\ell,\ell'\})=1$, and which are equal to $a(1)$. The second sum of (9.157) is $2a(1)+(n-2)a(0)$ so that (9.157) is

$$\begin{aligned}&a(2)-4a(1)+a(0)\left(\frac{(n-2)(n-3)}{2} - n(n-2) + \frac{n(n+1)}{2}\right)\\ &= a(2)-4a(1)+3a(0)\ .\end{aligned}$$

To compute the coefficient of $\nu(fT_{\ell_1})$, we may assume $\ell_1\le n$, for otherwise $\nu(fT_{\ell_1})=0$ since $\langle fT_{\ell_1}\rangle=0$. This coefficient is then

$$\sum_{\ell_2\le n,\ell_2\ne\ell_1} a(\ell_1,\ell_2) - na(\ell_1,n+1)\ .$$

When $\ell_1\notin\{\ell,\ell'\}$, this is

$$2a(1)+(n-3)a(0)-na(0) = 2a(1)-3a(0) = a_1(\ell_1)\ ,$$

while if $\ell_1\in\{\ell,\ell'\}$ this is

$$a(2)+(n-2)a(1)-na(1) = a(2)-2a(1) = a_1(\ell_1)\ . \qquad \square$$

Lemma 9.7.14. *Under the conditions of Proposition 9.7.5, if f^- is a product of k functions of the type $R^-_{\ell,\ell'}-q$, $\ell,\ell'\le n$ and*

$$f' = \varepsilon_\ell\varepsilon_{\ell'}(\varepsilon_1\varepsilon_2-q)f^-\ ,$$

we have

$$\begin{aligned}\nu_t(f'u'(S^\ell_{M,t})u'(S^{\ell'}_{M,t})) = {}&\widehat{r}\mathsf{E}\langle f'\rangle_{t,\sim}\\ &+\sum_{\ell_1<\ell_2} c(\ell,\ell';\ell_1,\ell_2)\nu(f'T_{\ell_1,\ell_2})\\ &+\sum_{\ell_1} c(\ell_1;\ell,\ell')\nu(f'T_{\ell_1})\\ &+(c(2)-4c(1)+3c(0))\nu(f'T)\\ &+O(k+2)\end{aligned} \tag{9.158}$$

where

$$c(\ell_1;\ell,\ell') = \begin{cases} 2c(1) - 3c(0) \text{ if } \ell_1 \notin \{\ell,\ell'\} \\ c(2) - 2c(1) \text{ if } \ell_1 \in \{\ell,\ell'\}. \end{cases} \tag{9.159}$$

We also have

$$\nu_t(f'u'(S^{\ell}_{M,t})u'(S^{\ell'}_{M,t})) = \widehat{r}\mathsf{E}\langle f'\rangle_{t,\sim} + A_{\ell,\ell'} + O(k+2) \tag{9.160}$$

where

$$A_{\ell,\ell'} = d(\ell,\ell')\bigg(\sum_{\ell_1<\ell_2} c(\ell,\ell';\ell_1,\ell_2)\nu(f^-T_{\ell_1,\ell_2}) + \sum_{\ell_1} c(\ell_1;\ell,\ell')\nu(f^-T_{\ell_1}) + (c(2) - 4c(1) + 3c(0))\nu(f^-T)\bigg) \tag{9.161}$$

for $d(\ell,\ell') = d(1,2;\ell,\ell') = d(\mathrm{card}(\{1,2\}\cap\{\ell,\ell'\}))$.

Proof. We first show that (9.158) implies (9.160). For this we simply write (using Lemma 9.7.9) that

$$\begin{aligned}
\nu(f'T_{\ell_1,\ell_2}) &= \nu((\varepsilon_1\varepsilon_2 - q)\varepsilon_\ell\varepsilon_{\ell'}f^-T_{\ell_1,\ell_2}) \\
&= \nu_0((\varepsilon_1\varepsilon_2 - q)\varepsilon_\ell\varepsilon_{\ell'}f^-T_{\ell_1,\ell_2}) + O(k+2) \\
&= \nu_0((\varepsilon_1\varepsilon_2 - q)\varepsilon_\ell\varepsilon_{\ell'})\nu_0(f^-T_{\ell_1,\ell_2}) + O(k+2) \\
&= d(\ell,\ell')\nu_0(f^-T_{\ell_1,\ell_2}) + O(k+2) \\
&= d(\ell,\ell')\nu(f^-T_{\ell_1,\ell_2}) + O(k+2)\ .
\end{aligned}$$

Passing from the second to the third line goes via Lemma 2.2.1, using as an intermediate step if one wishes that $T_{\ell_1,\ell_2} = T^-_{\ell_1,\ell_2} + (\sigma^{\ell_1}_N - \langle\sigma_N\rangle)(\sigma^{\ell_2}_N - \langle\sigma_N\rangle)/N$, where $T^-_{\ell_1,\ell_2}$ does not depend on the last spins. In a similar manner we get

$$\nu(f'T_{\ell_1}) = d(\ell,\ell')\nu(f^-T_{\ell_1}) + O(k+2)\ ,$$

and

$$\nu(f'T) = d(\ell,\ell')\nu(f^-T) + O(k+2)\ .$$

Substituting these relations in (9.158) proves (9.160).

To prove (9.158) we deduce from Lemmas 9.7.7and 9.7.8 (handling the error terms as already explained) that

$$\begin{aligned}
\nu_t(f'u'(S^{\ell}_{M,t})u'(S^{\ell'}_{M,t})) &= \widehat{r}\mathsf{E}\langle f'\rangle_{t,\sim} \\
&\quad + \sum_{1\le\ell_1<\ell_2\le n} c(\ell_1,\ell_2;\ell,\ell')\nu(f'(R_{\ell_1,\ell_2} - q)) \\
&\quad - n\sum_{\ell_1\le n} c(\ell_1,n+1;\ell,\ell')\nu(f'(R_{\ell_1,n+1} - q)) \\
&\quad + \frac{n(n+1)}{2}c(n+1,n+2;\ell,\ell')\nu(f'(R_{n+1,n+2} - q)) \\
&\quad + O(k+2)\ .
\end{aligned}$$

We then use Lemma 9.7.13 with $a(j) = c(j)$ to get the result. □

Lemma 9.7.15. *Under the conditions of Proposition 9.7.5, if f^- is a product of k functions of the type $R^-_{\ell,\ell'} - q$, $\ell, \ell' \le n$, and if $f' = \varepsilon_\ell \varepsilon_{\ell'}(\varepsilon_1\varepsilon_2 - q)f^-$, we have*

$$\nu_t(f') = \mathsf{E}\langle f' \rangle_{t,\sim} + B_{\ell,\ell'} + O(k+2) \,, \tag{9.162}$$

where

$$B_{\ell,\ell'} = \widehat{r} d(\ell,\ell') \Bigg(\sum_{\ell_1 < \ell_2} \nu(f^- T_{\ell_1,\ell_2}) - \sum_{\ell_1} \nu(f^- T_{\ell_1}) \Bigg) \,. \tag{9.163}$$

Proof. This is entirely similar to Lemma 9.7.14, using (9.136) rather than (9.134). In fact, there is no need to reproduce the computation since the right-hand side of (9.136) is obtained from (9.134) simply by replacing everywhere each of the terms $c(0)$, $c(1)$ and $c(2)$ by $\widehat{r}$. □

Recalling the numbers $b_0(j)$ of (9.145) to (9.147), we define

$$b_0(\ell_1, \ell_2) = b_0(\mathrm{card}\{\ell_1, \ell_2\} \cap \{1,2\}) \,.$$

Lemma 9.7.16. *Let $A_{\ell,\ell'}$ and $B_{\ell,\ell'}$ be given by (9.161) and (9.163) respectively. Then the following identities hold:*

$$\begin{aligned}
&\sum_{1 \le \ell < \ell' \le n} A_{\ell,\ell'} - n \sum_{\ell \le n} A_{\ell,n+1} + \frac{n(n+1)}{2} A_{n+1,n+2} \\
&= \sum_{1 \le \ell_1 < \ell_2 \le n} b_0(\ell_1,\ell_2) \nu(f^-(R_{\ell_1,\ell_2} - q)) \\
&- n \sum_{\ell_1 \le n} b_0(\ell_1, n+1) \nu(f^-(R_{\ell_1,n+1} - q)) \\
&+ \frac{n(n+1)}{2} b_0(0) \nu(f^-(R_{n+1,n+2} - q))
\end{aligned} \tag{9.164}$$

and

$$\begin{aligned}
&\sum_{1 \le \ell < \ell' \le n} B_{\ell,\ell'} - n \sum_{\ell \le n} B_{\ell,n+1} + \frac{n(n+1)}{2} B_{n+1,n+2} \\
&= \widehat{r}(d(2) - 4d(1) + 3d(0)) \Bigg(\sum_{1 \le \ell_1 < \ell_2 \le n} \nu(f^-(R_{\ell_1,\ell_2} - q)) \\
&- \sum_{\ell_1 \le n} \nu(f^-(R_{\ell_1,n+1} - q)) + \frac{n(n+1)}{2} \nu(f^-(R_{n+1,n+2} - q)) \Bigg) .
\end{aligned} \tag{9.165}$$

Proof. We prove (9.164) first. We use Lemma 9.7.13 with $a(j) = b_0(j)$ to see that the right-hand side of this quantity is

$$\sum_{\ell_1<\ell_2} b_0(\ell_1,\ell_2)\nu(f^- T_{\ell_1,\ell_2}) + \sum_{\ell_1} b_1(\ell_1)\nu(f^- T_{\ell_1})$$
$$+ (b_0(2) - 4b_0(1) + 3b_0(0))\nu(f^- T) \ , \tag{9.166}$$

where

$$b_1(\ell) = \begin{cases} 2b_0(1) - 3b_0(0) \text{ if } \ell_1 \notin \{1,2\} \\ b_0(2) - 2b_0(1) \quad \text{if } \ell_1 \in \{1,2\}. \end{cases}$$

We will show that the coefficients of $\nu(f^- T_{\ell_1,\ell_2})$, $\nu(f^- T_{\ell_1})$ and $\nu(f^- T)$ are the same in (9.166) and in the left-hand side of (9.164). That is, recalling (9.161), that $d(\ell,\ell') = d(\mathrm{card}(\{\ell,\ell'\} \cap \{1,2\})$ and (9.159), we have to prove the relations

$$\sum_{1\le\ell<\ell'\le n} c(\ell,\ell';\ell_1,\ell_2)d(\ell,\ell') - n\sum_{\ell\le n} c(\ell,n+1;\ell_1,\ell_2)d(\ell,n+1)$$
$$+ \frac{n(n+1)}{2} c(n+1,n+2;\ell_1,\ell_2)d(n+1,n+2) = b_0(\ell_1,\ell_2) \ ; \tag{9.167}$$

$$\sum_{1\le\ell<\ell'\le n} c(\ell_1;\ell,\ell')d(\ell,\ell') - n\sum_{\ell\le n} c(\ell_1;\ell,n+1)d(\ell,n+1)$$
$$+ \frac{n(n+1)}{2} c(\ell_1;n+1,n+2)d(n+1,n+2) = b_1(\ell_1) \ ; \tag{9.168}$$

$$(c(2) - 4c(1) + 3c(0))\Bigg(\sum_{1\le\ell<\ell'\le n} d(\ell,\ell') - n\sum_{\ell\le n} d(\ell,n+1)$$
$$+ \frac{n(n+1)}{2} d(n+1,n+2)\Bigg) = b_0(2) - 4b_0(1) + 3b_0(0) \ . \tag{9.169}$$

To prove these relations we may assume that $n > \ell_1, \ell_2$. This is because in (2.23) we may increase n if we wish, since the extra terms this creates cancel out. The proof is completely straightforward, but it requires real patience. The impatient reader may jump ahead directly to the proof of Theorem 9.7.11.

Proof of (9.167). $\boxed{\textit{Case 1: } \{\ell_1,\ell_2\} = \{1,2\}.}$

There are respectively

$$1 \ ; \ \ 2n-4 \ ; \ \ \frac{(n-2)(n-3)}{2}$$

choices of $1 \le \ell < \ell' \le n$ for which $\mathrm{card}(\{\ell,\ell'\} \cap \{1,2\}) = 2$, 1, or 0. Therefore the term $\sum_{1\le\ell<\ell'\le n}$ in the left-hand side of (9.167) is

$$c(2)d(2) + c(1)d(1)(2n-4) + c(0)d(0)\left(\frac{(n-2)(n-3)}{2}\right) \ .$$

There are respectively 2 and $n-2$ choices of $\ell \le n$ for which $\operatorname{card}(\{\ell, n+1\} \cap \{1,2\}) = 1$ or 0, and the term $\sum_{\ell \le n}$ in the left-hand side of (9.167) is

$$2c(1)d(1) + (n-2)c(0)d(0) \; .$$

Therefore the left-hand side of (9.167) is

$$\begin{aligned}
&c(2)d(2) + c(1)d(1)(2n-4) + c(0)d(0)\left(\frac{(n-2)(n-3)}{2}\right) \\
&- n(2c(1)d(1) + (n-2)c(0)d(0)) + \frac{n(n+1)}{2}c(0)d(0) \\
&= c(2)d(2) - 4c(1)d(1) + 3c(0)d(0) = b_0(2) = b_0(1,2) = b_0(\ell_1, \ell_2) \; .
\end{aligned}$$

$\boxed{\textit{Case 2: } \operatorname{card}(\{1,2\} \cap \{\ell_1, \ell_2\}) = 1.}$

Without loss of generality we assume $\ell_1 = 1$, $\ell_2 = 3$. The sum $\sum_{1 \le \ell < \ell' \le n}$ in the left-hand side of (9.167) is best computed by first calculating the sum over ℓ' for $\ell = 1,2,3$. This sum is as follows.

If $\ell = 1$:

$$c(1)d(2) + c(2)d(1) + (n-3)c(1)d(1) \; ,$$

corresponding respectively to the case $\ell' = 2$, $\ell' = 3$, $\ell' \ge 4$.

If $\ell = 2$:

$$c(1)d(1) + (n-3)c(0)d(1) \; ,$$

corresponding respectively to $\ell' = 3$, $\ell' \ge 4$.

If $\ell = 3$:

$$(n-3)c(1)d(0) \; .$$

Moreover, the sum $\sum_{4 \le \ell < \ell' \le n}$ is

$$\frac{(n-3)(n-4)}{2}c(0)d(0) \; .$$

The sum $\sum_{\ell \le n}$ is

$$c(1)d(1) + c(0)d(1) + c(1)d(0) + (n-3)c(0)d(0) \; ,$$

the terms corresponding of course to the cases $\ell = 1$, $\ell = 2$, $\ell = 3$, $\ell \ge 4$. Collecting the terms and using that

$$\frac{(n-3)(n-4)}{2} - n(n-3) + \frac{n(n+1)}{2} = 6$$

to compute the coefficient of $c(0)\,d(0)$, we get a total contribution of

$$c(2)d(1) + c(1)d(2) - 2c(1)d(1) - 3c(0)d(1) - 3c(1)d(0) + 6c(0)d(0) \; ,$$

and this is $b_0(1) = b_0(\ell_1, \ell_2)$.

$\boxed{\textit{Case 3:}\ \ \{\ell_1, \ell_2\} \cap \{1, 2\} = \emptyset.}$

We first compute the sum $\sum_{1 \le \ell < \ell' \le n}$ in the left-hand side of (9.167). There are 6 pairs $1 \le \ell < \ell' \le n$ such that $\{\ell, \ell'\} \subset \{1, 2, \ell_1, \ell_2\}$, for a total contribution of

$$c(2)d(0) + c(0)d(2) + 4c(1)d(1) \; .$$

There are

$$\frac{n(n-1)}{2} - \frac{(n-4)(n-5)}{2} - 6 = 4n - 16$$

choices of $1 \le \ell < \ell' \le n$ for which $\operatorname{card}(\{\ell, \ell'\} \cap \{1, 2, \ell_1, \ell_2\}) = 1$, for a total contribution of

$$(2n-8)(c(0)d(1) + c(1)d(0)) \; .$$

Thus the sum $\sum_{1 \le \ell < \ell' \le n}$ in the left-hand side of (9.167) is

$$\begin{aligned} &c(2)d(0) + c(0)d(2) + 4c(1)d(1) + (2n-8)(c(0)d(1) + c(1)d(0)) \\ &+ \frac{(n-4)(n-5)}{2} c(0)d(0) \; . \end{aligned}$$

Next we compute the sum $\sum_{\ell \le n}$ in the left-hand side of (9.167). We distinguish the cases where $\ell = 1, 2, \ell_1, \ell_2$ to obtain that this sum is

$$2c(1)d(0) + 2c(0)d(1) + (n-4)c(0)d(0) \; .$$

Thus the left-hand side of (9.167) is

$$\begin{aligned} &c(2)d(0) + c(0)d(2) + 4c(1)d(1) + (2n-8)(c(0)d(1) + c(1)d(0)) \\ &+ \frac{(n-4)(n-5)}{2} c(0)d(0) - n(2c(1)d(0) + 2c(0)d(1) + (n-4)c(0)d(0)) \\ &+ \frac{n(n+1)}{2} c(0)d(0) \; . \end{aligned}$$

Using that

$$\frac{(n-4)(n-5)}{2} - n(n-4) + \frac{n(n+1)}{2} = 10 \; ,$$

this is

$$c(2)d(0) + c(0)d(2) + 4c(1)d(1) - 8(c(0)d(1) + c(1)d(0)) + 10c(0)d(0) \; ,$$

which is $b_0(0) = b_)(\ell_1, \ell_2)$, and we have proved (9.167).

Proof of (9.168). We set $c'(1) = c(2) - 2c(1)$, $c'(0) = 2c(1) - 3c(0)$, so that by (9.159) we have $c(\ell_1; \ell, \ell') = c'(\operatorname{card}(\{\ell_1\} \cap \{\ell, \ell'\}))$.

Case 1: $\boxed{\ell_1 \notin \{1, 2\}}$.

Without loss of generality we assume that $\ell_1 = 3$. To compute the sum $\sum_{1 \le \ell < \ell' \le n}$ of the left-hand side of (9.168), we again compute first the sum over ℓ' for $\ell = 1, 2, 3$.

If $\ell = 1$:

$$c'(0)d(2) + c'(1)d(1) + (n-3)c'(0)d(1) \ ,$$

corresponding respectively to the values $\ell' = 2$, $\ell' = 3$, $\ell' \ge 4$.

If $\ell = 2$:

$$c'(1)d(1) + (n-3)c'(0)d(1) \ ,$$

corresponding respectively to the values $\ell' = 3$ and $\ell' \ge 4$.

If $\ell = 3$:

$$(n-3)c'(1)d(0) \ .$$

Moreover the sum $\sum_{4 \le \ell < \ell' \le n}$ is

$$\frac{(n-3)(n-4)}{2} c'(0)d(0) \ .$$

The sum $\sum_{\ell \le n}$ is

$$2c'(0)d(1) + c'(1)d(0) + (n-3)c'(0)d(0).$$

The first term is the contribution of the values $\ell = 1, 2$, the second term is the contribution of the values $\ell = 3$ and the third term is the contribution of the values $\ell \ge 4$. Collecting the terms we find a total contribution of

$$2c'(1)d(1) + c'(0)d(2) - 3c'(1)d(0) - 6c'(0)d(1) + 6c'(0)d(0) \ .$$

Substituting the values of $c'(1)$ and $c'(0)$, algebra yields the following expression

$$\begin{aligned} &2(c(2)d(1) + c(1)d(2)) - 3(c(2)d(0) + c(0)d(2)) - 16c(1)d(1) \\ &+ 18(c(1)d(0) + c(0)d(1)) - 18c(0)d(0) \end{aligned}$$

and this is indeed $2b_0(1) - 3b_0(0) = b_1(\ell_1)$.

Case 2: $\boxed{\ell_1 \in \{1, 2\}}$.

Without loss of generality we assume that $\ell_1 = 1$. The contribution of the sum $\sum_{1 \le \ell < \ell' \le n}$ for the various values of ℓ is as follows.

If $\ell = 1$:

$$c'(1)d(2) + (n-2)c'(1)d(1) \ ,$$

corresponding to the terms $\ell' = 2$ and $\ell' \ge 3$.

If $\ell = 2$:

$$(n-2)c'(0)d(1) \ .$$

The sum of the contributions for $3 \le \ell \le n$ is

$$\frac{(n-2)(n-3)}{2} c'(0) d(0) \,.$$

The sum $\sum_{\ell \le n}$ is

$$c'(1)d(1) + c'(0)d(1) + (n-2)c'(0)d(0) \,,$$

corresponding to the cases $\ell = 1$, $\ell = 2$, and $\ell \ge 3$.

Collecting the terms, the total contribution is

$$c'(1)d(2) - 2c'(1)d(1) - 2c'(0)d(1) + 3c'(0)d(0) \,,$$

and substituting the values of $c'(1), c'(0)$ this is indeed $b_0(2) - 2b_0(1) = b_1(\ell_1)$. We have proved (9.168).

Proof of (9.169). It is simpler than the previous one. As in (9.151) we have

$$b_0(2) - 2\, b_0(1) + b_0(0) = (c(2) - 2\, c(1) + c(0))(d(2) - 2\, d(1) + d(0)) \,,$$

so it suffices to show that

$$\begin{aligned} &\sum_{1 \le \ell < \ell' \le n} d(\ell, \ell') - n \sum_{\ell \le n} d(\ell, n+1) + \frac{n(n+1)}{2} d(n+1, n+2) \\ &= d(2) - 4d(1) + 3d(0) \,. \end{aligned}$$

The computation has been done many times and is left to the reader.

We have proved (9.169), (9.168) and (9.167). Therefore we have proved (9.164).

To prove (9.165), we simply notice that we obtain $B_{\ell,\ell'}$ from $A_{\ell,\ell'}$ by replacing each of the quantities $c(2), c(1)$ and $c(0)$ by $\widehat{r}$: this replaces each of the quantities $b_0(2), b_0(1)$ and $b_0(0)$ by $\widehat{r}(d(2) - 4d(1) + 3d(0))$. □

Proof of Theorem 9.7.11. We apply (2.23) to the function $f = (\varepsilon_1\varepsilon_2 - q)f^-$. We apply (9.160) to each term $\nu_t(\varepsilon_\ell\varepsilon_{\ell'}(\varepsilon_1\varepsilon_2 - q)u'(S^\ell_{M,t})u'(S^{\ell'}_{M,t})f^-)$. Setting

$$D_{\ell,\ell'} = \mathsf{E}\langle(\varepsilon_1\varepsilon_2 - q)\varepsilon_\ell\varepsilon_{\ell'}f^-\rangle_{t,\sim} - \nu_t((\varepsilon_1\varepsilon_2 - q)\varepsilon_\ell\varepsilon_{\ell'}f^-) \,,$$

and, combining with the contribution of the term II of (2.23) we get

$$\begin{aligned} \nu'_t((\varepsilon_1\varepsilon_2 - q)f^-) = \alpha\Bigg(&\sum_{1 \le \ell < \ell' \le n} A_{\ell,\ell'} - n\sum_{\ell \le n} A_{\ell,n+1} + \frac{n(n+1)}{2} A_{n+1,n+2}\Bigg) \\ &+ r\Bigg(\sum_{1 \le \ell < \ell' \le n} D_{\ell,\ell'} - n\sum_{\ell \le n} D_{\ell,n+1} + \frac{n(n+1)}{2} D_{n+1,n+2}\Bigg) \\ &+ O(k+2) \,. \end{aligned}$$

By (9.162) we have $D_{\ell,\ell'} = -B_{\ell,\ell'} + O(k+2)$, and combining with (9.164) and (9.165) proves (9.153). □

9.8 An Approximation Procedure

In the previous sections we worked with functions u such that

$$\forall \ell\ ,\ 1 \le \ell \le 5\ ,\quad |u^{(\ell)}| \le \exp\left(\frac{N}{L}\right)\ . \tag{9.170}$$

In the present section we show that this condition is not really necessary. The method consists of showing that, given a function u (that does not take excessively large values) we can find a function v that satisfies (9.170), for which the Gibbs measures associated to u and v are nearly identical. This shows that in the end the differentiability of u is really irrelevant and it makes one wonder whether we have used the correct approach. Throughout the section we assume (9.2).

Let us give an example of what can be achieved.

Theorem 9.8.1. *There exists a number L with the following property. Assume that the measurable function u satisfies*

$$-\frac{N}{L} \le u \le 0\ ,$$

and consider the solution q of the equations (9.65). Then if $L\alpha \exp L\tau^2 \le 1$ we have

$$\forall k \ge 1\ ,\quad \nu\big((R_{1,2} - q)^{2k}\big) \le \left(\frac{Lk}{N}\right)^k\ .$$

Again, no differentiability of u whatsoever is necessary here.

We now describe the basic approximation procedure. We assume that

$$-D \le u \le 0\ , \tag{9.171}$$

and we will specify D later. Let us consider a (very small) number b. This parameter controls the quality of our approximation of u.

By scaling arguments, there exists a function ζ supported by the interval $[-b, b]$, with $\zeta \ge 0$, and such that

$$\int \zeta(x)\mathrm{d}x = 1$$

$$\forall \ell\ ,\ 0 \le \ell \le 5\ ,\quad |\zeta^{(\ell)}| \le \frac{L}{b^{\ell+1}}\ . \tag{9.172}$$

We define the function v by

$$\exp v(x) = \zeta * \exp u(x) = \int \exp u(t)\zeta(x-t)\mathrm{d}t\ , \tag{9.173}$$

so that $-D \le v \le 0$. Moreover, if $u(x) = 0$ for $x \ge \tau$, we have

$$x \geq \tau + b \quad \Rightarrow \quad v(x) = 0 \,. \tag{9.174}$$

We claim that

$$\forall \ell \,,\; 0 \leq \ell \leq 5 \,,\quad |v^{(\ell)}| \leq L \frac{\exp \ell D}{b^\ell} \,. \tag{9.175}$$

This follows from (9.173) and computation of the derivatives of v. For example, $v' \exp v = \zeta' * \exp u$, and since ζ' is supported by $[-b, b]$ and $|\zeta'| \leq L/b^2$,

$$|\zeta' * \exp u| \leq \int |\zeta'(x)| \mathrm{d}x \leq 2b \frac{L}{b^2} \,.$$

Lemma 9.8.2. *For any number x and any Gaussian r.v. g we have*

$$|\mathsf{E} \exp u(g+x) - \mathsf{E} \exp v(g+x)| \leq \frac{b}{\sqrt{\mathsf{E} g^2}} \,. \tag{9.176}$$

Proof. The function

$$W(x) = \mathsf{E} \exp u(g+x)$$

satisfies (using integration by parts in the second line),

$$W'(x) = \mathsf{E} u'(g+x) \exp u(g+x) = \frac{1}{\mathsf{E} g^2} \mathsf{E} g \exp u(g+x)$$

so that

$$|W'(x)| \leq \frac{\mathsf{E}|g|}{\mathsf{E} g^2} \leq \frac{1}{\sqrt{\mathsf{E} g^2}}$$

and thus $|W(x) - W(x-t)| \leq b/(\mathsf{E} g^2)^{1/2}$ for $|t| \leq b$. Now, the left-hand side of (9.176) is

$$\begin{aligned} |W(x) - \zeta * W(x)| &= \left| W(x) - \int W(x-t) \zeta(t) \mathrm{d}t \right| \\ &= \left| \int (W(x) - W(x-t)) \zeta(t) \mathrm{d}t \right| \\ &\leq \frac{b}{\sqrt{\mathsf{E} g^2}} \int \zeta(t) \mathrm{d}t = \frac{b}{\sqrt{\mathsf{E} g^2}} \,, \end{aligned}$$

and this completes the proof. □

Lemma 9.8.3. *For any subset A of Σ_N we have*

$$\mathsf{E} \left(\sum_{\boldsymbol{\sigma} \in A} \left(\exp \sum_{k \leq M} u(S_k(\boldsymbol{\sigma})) - \exp \sum_{k \leq M} v(S_k(\boldsymbol{\sigma})) \right) \right)^2 \\ \leq 2 \mathrm{card} A + b^2 N M^2 (\mathrm{card} A)^2 \,. \tag{9.177}$$

Proof. The left-hand side of (9.177) is

$$\sum_{\boldsymbol{\sigma},\boldsymbol{\sigma}'\in A} \mathsf{E}B(\boldsymbol{\sigma})B(\boldsymbol{\sigma}') \tag{9.178}$$

where

$$B(\boldsymbol{\sigma}) = \exp\sum_{k\le M} u(S_k(\boldsymbol{\sigma})) - \exp\sum_{k\le M} v(S_k(\boldsymbol{\sigma}))\,.$$

In the sum (9.178) we bound separately the terms for which $\boldsymbol{\sigma} = \pm\boldsymbol{\sigma}'$. For these, we use the trivial bound $|B(\boldsymbol{\sigma})| \le 2$. Next, consider a pair $(\boldsymbol{\sigma},\boldsymbol{\sigma}')$ with $\boldsymbol{\sigma} \ne \pm\boldsymbol{\sigma}'$. Since $\boldsymbol{\sigma} \ne -\boldsymbol{\sigma}'$, there exists a coordinate i with $\sigma_i \ne -\sigma'_i$, i.e. $\sigma_i = \sigma'_i$. Since $\boldsymbol{\sigma} \ne \boldsymbol{\sigma}'$, there exists a coordinate j with $\sigma_j = -\sigma'_j$. Without loss of generality, we assume that $\sigma_1 = \sigma'_1$, $\sigma_2 = -\sigma'_2$. Let us denote by E_0 integration in the variables $g_{i,k}$, $k \le M$, $i = 1,2$, all other r.v.s being fixed.

The key observation is that all the variables of the type

$$\sigma_1\, g_{1,k} + \sigma_2\, g_{2,k} \text{ and } \sigma'_1\, g_{1,k} + \sigma'_2\, g_{2,k} = \sigma_1\, g_{1,k} - \sigma_2\, g_{2,k}$$

are independent as $k \le M$, so that, under E_0, the r.v.s $B(\boldsymbol{\sigma})$ and $B(\boldsymbol{\sigma}')$ are independent, and

$$\mathsf{E}_0(B(\boldsymbol{\sigma})\,B(\boldsymbol{\sigma}')) = \mathsf{E}_0 B(\boldsymbol{\sigma})\,\mathsf{E}_0 B(\boldsymbol{\sigma}')\,.$$

Now

$$\mathsf{E}_0 B(\boldsymbol{\sigma}) = \prod_{k\le M} X_k - \prod_{k\le M} Y_k\,, \tag{9.179}$$

where

$$X_k = \mathsf{E}_0 \exp u(S_k(\boldsymbol{\sigma}))\,;\ Y_k = \mathsf{E}_0 \exp v(S_k(\boldsymbol{\sigma}))\,.$$

We use Lemma 9.8.2 with

$$g = \frac{1}{\sqrt{N}}(\sigma_1 g_{1,k} + \sigma_2 g_{2,k})$$

$$x = \frac{1}{\sqrt{N}}\sum_{i\ge 3}\sigma_i g_{i,k}$$

and we obtain

$$|X_k - Y_k| \le b\sqrt{N}\,. \tag{9.180}$$

Next, we use that for numbers $(x_k)_{k\le M}$, $(y_k)_{k\le M}$, if $|x_k|, |y_k| \le 1$, then

$$\left|\prod_{k\le M} x_k - \prod_{k\le M} y_k\right| \le \sum_{k\le M} |x_k - y_k|\,, \tag{9.181}$$

to deduce from (9.179) and (9.180) that

$$|\mathsf{E}_0 B(\boldsymbol{\sigma})| \le b\sqrt{N}M \ .$$

The same inequality holds for $\boldsymbol{\sigma}'$ rather than $\boldsymbol{\sigma}$, and thus

$$\sum \mathsf{E}(B(\boldsymbol{\sigma})B(\boldsymbol{\sigma}')) \le b^2 N M^2 (\mathrm{card} A)^2 \ ,$$

where the summation is over $\boldsymbol{\sigma} \ne \pm\boldsymbol{\sigma}'$, $\boldsymbol{\sigma}, \boldsymbol{\sigma}' \in A$. This finishes the proof. □

The Cauchy-Schwarz inequality and (9.177) imply the following.

Corollary 9.8.4. *We have*

$$\mathsf{E}\left(\left|\sum_{\boldsymbol{\sigma}\in A}\left(\exp\sum_{k\le M} u(S_k(\boldsymbol{\sigma})) - \exp\sum_{k\le M} v(S_k(\boldsymbol{\sigma}))\right)\right|\right)$$
$$\le 2\sqrt{\mathrm{card} A} + b\sqrt{N}M\mathrm{card} A \ . \tag{9.182}$$

We use the notation $\langle\cdot\rangle_u$ and $\langle\cdot\rangle_v$ to distinguish the Gibbs measures associated to u and v.

Corollary 9.8.5. *Assume that* $u(x) = 0$ *for* $x \ge \tau$. *Given any subset* I *of* $\{1,\ldots,N\}$ *and* $d \ge \exp(-N/32)$, *we have*

$$\mathsf{E}\left|\left\langle\prod_{i\in I}\sigma_i\right\rangle_u - \left\langle\prod_{i\in I}\sigma_i\right\rangle_v\right| \le L d^{1/L}\exp LM(1+\tau^2)$$
$$+ \frac{L}{d}2^{-N/2} + \frac{Lb}{d}\sqrt{N}M \ . \tag{9.183}$$

Proof. Consider the set

$$A = \left\{\boldsymbol{\sigma} \ ; \ \prod_{i\in I}\sigma_i = 1\right\}$$

so that

$$\left\langle\prod_{i\in I}\sigma_i\right\rangle_u = \langle\mathbf{1}_A\rangle_u - \langle\mathbf{1}_{A^c}\rangle_u = 2\langle\mathbf{1}_A\rangle_u - 1$$

and

$$\left|\left\langle\prod_{i\in I}\sigma_i\right\rangle_u - \left\langle\prod_{i\in I}\sigma_i\right\rangle_v\right| \le 2|\langle\mathbf{1}_A\rangle_u - \langle\mathbf{1}_A\rangle_v| \ .$$

We define

$$S_u = \sum_{\boldsymbol{\sigma}\in A}\exp\sum_{k\le M} u(S_k) \ ; \quad Z_u = \sum_{\boldsymbol{\sigma}}\exp\sum_{k\le M} u(S_k) \tag{9.184}$$

(and similarly we define S_v and Z_v). Thus (9.182) yields

$$\mathsf{E}|S_u - S_v| \le 2^{N/2+1} + b\sqrt{N}M2^N \tag{9.185}$$

$$\mathsf{E}|Z_u - Z_v| \le 2^{N/2+1} + b\sqrt{N}M2^N \,. \tag{9.186}$$

Since $\langle \mathbf{1}_A \rangle_u = S_u/Z_u$, we have $\langle \mathbf{1}_A \rangle_v = S_v/Z_v$, and

$$\begin{aligned} |\langle \mathbf{1}_A \rangle_u - \langle \mathbf{1}_A \rangle_v| &= \left| \frac{S_u}{Z_u} - \frac{S_v}{Z_v} \right| = \left| \frac{S_u - S_v}{Z_u} + \frac{S_v(Z_v - Z_u)}{Z_u Z_v} \right| \\ &\le \frac{|S_u - S_v|}{Z_u} + \frac{|Z_u - Z_v|}{Z_u} \end{aligned} \tag{9.187}$$

since $S_v \le Z_v$. Consider the event $\Omega = \{Z_u \le d2^N\}$. It follows from (9.16) that $\mathsf{P}(\Omega) \le d^{1/L} \exp LM(1+\tau^2)$. Then (9.187) implies

$$|\langle \mathbf{1}_A \rangle_u - \langle \mathbf{1}_A \rangle_v| \le \mathbf{1}_\Omega + \frac{2^{-N}}{d}(|S_u - S_v| + |Z_u - Z_v|) \,.$$

Taking expectations and using (9.185) and (9.186) completes the proof. □

Proposition 9.8.6. *There exists a constant L such that if $L\alpha \exp L\tau^2 \le 1$ and the function u satisfies $u(x) = 0$ for $x \ge \tau$ and*

$$-D = -\frac{N}{L} \le u \le 0 \,,$$

then given any subset I of $\{1, \ldots, N\}$ we have

$$\mathsf{E}\left| \left\langle \prod_{i \in I} \sigma_i \right\rangle_u - \left\langle \prod_{i \in I} \sigma_i \right\rangle_v \right| \le L \exp\left(-\frac{L}{N}\right) \,. \tag{9.188}$$

Proof. If L_1 denotes the constant in (9.170) we assume

$$D = \frac{N}{10L_1} \,, \tag{9.189}$$

so that, if $b = L_2 \exp(-N/10L_1)$ where L_2 is large enough, (9.175) proves that the function v satisfies (9.170). We then choose $d = \exp(-N/20L_1)$ so that if $L'M(1+\tau^2) \le N$ for L' large enough the bound in (9.183) is of the type $L\exp(-N/L)$. □

Lemma 9.8.7. *Under the conditions of Proposition 9.8.6, the following occurs. Consider for $1 \le \ell < \ell' < n$ integers $k(\ell, \ell')$. Then*

$$\mathsf{E}\left| \left\langle \prod_{1 \le \ell < \ell' \le n} R_{\ell,\ell'}^{k(\ell,\ell')} \right\rangle_u - \left\langle \prod_{1 \le \ell < \ell' \le n} R_{\ell,\ell'}^{k(\ell,\ell')} \right\rangle_v \right| \le Ln \exp\left(-\frac{N}{L}\right) \,. \tag{9.190}$$

The surprising part of this result is that here we study functions on n replicas; one would think that having to deal with the quantity Z_u of (9.184) occurring at power n in denominators will create trouble; the content of the lemma is that this is not the case.

Proof. We write

$$R_{\ell,\ell'} = \frac{1}{N}\sum_{i\le N}\sigma_i^\ell\,\sigma_i^{\ell'}$$

so that

$$R_{\ell,\ell'}^{k(\ell,\ell')} = \frac{1}{N^{k(\ell,\ell')}}\sum\prod_{k\le k(\ell,\ell')}\sigma^\ell_{i(\ell,\ell',k)}\,\sigma^{\ell'}_{i(\ell,\ell',k)}$$

where the summation is over all choices of $i(\ell,\ell',1),\dots,i(\ell,\ell',k(\ell,\ell'))$ of integers smaller than or equal to N. Thus

$$\prod_{\ell<\ell'} R_{\ell,\ell'}^{k(\ell,\ell')} = \frac{1}{N^{\overline{k}}}\sum\prod_{\ell<\ell'}\prod_{k\le k(\ell,\ell')}\sigma^\ell_{i(\ell,\ell',k)}\,\sigma^{\ell'}_{i(\ell,\ell',k)} \tag{9.191}$$

where $\overline{k}=\sum_{1\le\ell<\ell'\le n}k(\ell,\ell')$ and where the sum is over the $N^{\overline{k}}$ choices of indices $1\le i(\ell,\ell',k)\le N$ for $1\le\ell<\ell'\le n$, $k\le k(\ell,\ell')$. A product of any collection of spins σ_i^ℓ, $\ell\le n$, $i\le N$ (each of them occurring possibly several times) is of the type $\prod_{\ell\le n}\prod_{i\in I(\ell)}\sigma_i^\ell$ (where $I(\ell)$ is a certain subset of $\{1,\dots,N\}$). This is simply because $(\sigma_i^\ell)^2=1$. This is in particular the case of the double product in (9.191). Therefore

$$\Big\langle\prod_{\ell<\ell'}R_{\ell,\ell'}^{k(\ell,\ell')}\Big\rangle_u = \frac{1}{N^{\overline{k}}}\sum\prod_{\ell\le n}\Big\langle\prod_{i\in I(\ell)}\sigma_i^\ell\Big\rangle_u\,,$$

where the sum contains $N^{\overline{k}}$ terms. Thus to obtain (9.190) it suffices to bound the quantities

$$\mathsf{E}\Big|\prod_{\ell\le n}\Big\langle\prod_{i\in I(\ell)}\sigma_i^\ell\Big\rangle_u-\prod_{\ell\le n}\Big\langle\prod_{i\in I(\ell)}\sigma_i^\ell\Big\rangle_v\Big|\,.$$

This follows from (9.181) and (9.183). □

Proof of Theorem 9.8.1. Let (q_u,r_u) be a solution of the equations (9.65) for u. First, we show that the pair (q_u,r), where

$$r=\alpha\mathsf{E}\left(\frac{\mathsf{E}_\xi v'(\theta)\exp v(\theta)}{\mathsf{E}_\xi\exp v(\theta)}\right)^2$$

(and where $\theta=z\sqrt{q}+\xi\sqrt{1-q}$) is very close to be a solution of the equations (9.65) for v, and thus $\nu_v((R_{1,2}-q)^{2k})\le(Lk/N)^k$ by (9.101), from which the result is deduced for ν_u by expanding the power $(R_{1,2}-q)^{2k}$ and using (9.190) on each term. The details are straightforward. □

9.9 The Bernoulli Model

In the Bernoulli model the Gaussian r.v.s $g_{i,k}$ are replaced by independent random signs $\eta_{i,k}$ and the Hamiltonian is now

$$-H_{N,M}(\boldsymbol{\sigma}) = \sum_{k \le M} u\left(\frac{1}{\sqrt{N}} \sum_{i \le N} \eta_{i,k}\sigma_i\right) . \tag{9.192}$$

Throughout this section ν refers to this Hamiltonian. This model is harder to study than the Gaussian model, because we cannot use special Gaussian tools, such as integration by parts or more importantly Lemma 9.3.3. We must replace integration by parts by "approximate integration by parts" (as defined in Section 4.6, equation (4.198)). The error terms introduced by approximate integration by parts depend on the size of the derivatives of u. In order to be able to say anything at all about the structure of the Gibbs measure we essentially need to control the size of these derivatives uniformly over N. On the other hand, the problem of approximating $p_{N,M}(u)$ is easier: one can expect that it will suffices to approximate u by a smooth function (independent of N), for which one can understand the system for large N.

Theorem 9.9.1. *Assume that the function u of (9.192) satisfies (9.2), and, moreover*

$$\forall \ell\ ,\ 1 \le \ell \le 5\ ,\quad |u^{(\ell)}| \le D\ .$$

Then there is a number $N(D)$ and a number L such that if $L\alpha \exp L\tau^2 \le 1$ and $N \ge N(D)$ we have (9.66), i.e.

$$\nu\big((R_{1,2} - q)^2\big) \le \frac{L}{N}$$

and (9.103), where q is solution of the equations (9.65).

On the other hand, if u is not differentiable, we know very little about Gibbs' measure.

Research Problem 9.9.2. (Level 2) Assume that $u(x) = -\beta \mathbf{1}_{\{x \le 0\}}$. If β and α are small enough, is it true that

$$\lim_{N\to\infty} \lim_{M/N\to\alpha} \nu\big((R_{1,2} - q)^2\big) = 0 \tag{9.193}$$

where q is given by the equations (9.65)? And is it true that

$$\sup_N N\nu\big((R_{1,2} - q)^2\big) < \infty$$

if $\alpha = M/N \le \alpha_0$ (small enough)?

This problem, and (9.193) in particular, is a very good case for the "what else could happen?" argument. It illustrates well the substantial gap between heuristic arguments (however convincing) and mathematical proofs.

We turn to the proof of Theorem 9.9.1. This proof is obtained by suitably modifying the proof of Theorem 9.5.1, so that this theorem must be mastered first before attempting to read the next two pages. The reader interested only in Gardner's formula should skip to Theorem 9.9.4 below.

Proposition 9.9.3. *The conditions of Proposition 2.2.2 imply*

$$\nu_t'(f) = \mathrm{I} + \mathrm{II} + \mathcal{R} \, , \tag{9.194}$$

where I *and* II *are as in Proposition 2.2.2 and*

$$|\mathcal{R}| \le \frac{\alpha}{N} K(n,D) \nu_t(|f|) \, . \tag{9.195}$$

Proof. We repeat the proof of Proposition 2.2.2, replacing integration by parts by "approximate integration by parts" as defined in (4.197). The main terms are the same as in the Gaussian case, and create the term $\mathrm{I} + \mathrm{II}$. The issue is to prove that the error term satisfies (9.195). This error term occurs when performing "approximate integration by parts" in the term III of (2.28) (with $\eta_{N,M}$ instead of g_M). This is as simple as can be, and we have already dealt with a more complicated situation in Chapter 4 when we introduced this method of approximate integration by parts. Still, there is no harm in repeating the argument in the case of the typical term

$$\nu_t(\eta_M \varepsilon_\ell u'(S^\ell_{M,t}) f) = \mathsf{E} \eta_M \langle \varepsilon_\ell u'(S^\ell_{M,t}) f \rangle_t \, , \tag{9.196}$$

where $\eta_M = \eta_{N,M}$. We consider the function $v_\ell(x)$, obtained by replacing each occurrence of η_M in the explicit expression of $\langle \varepsilon_\ell \, u'(S^\ell_{M,t}) \, f \rangle_t$ by x, and assuming all the other r.v.s $\eta_{i,k}$ fixed. Since each occurrence of η_M in the explicit expression of $\langle \varepsilon_\ell u'(S^\ell_{M,t}) f \rangle_t$ is multiplied by a factor $\sqrt{t/N}$, each derivation brings out this factor, and it should be obvious that

$$|v_\ell'''(x)| \le \frac{t^{3/2}}{N^{3/2}} K(n) D^4 \langle |f| \rangle_{t,x} \, , \tag{9.197}$$

where $\langle \cdot \rangle_{t,x}$ means that in the explicit expression of $\langle \cdot \rangle_t$ each occurrence of η_M is replaced by x (so that $v_\ell(x) = \langle \varepsilon_\ell \, u'(S^\ell_{M,t}) \, f \rangle_{t,x}$). We want to relate $\langle |f| \rangle_{t,x}$ with $\langle |f| \rangle_t$. For this we simply observe that

$$\left| \frac{\partial}{\partial x} \langle |f| \rangle_{t,x} \right| \le \frac{K(n)D}{\sqrt{N}} \, \langle |f| \rangle_{t,x} \, ,$$

so that, by integration

$$\sup_{|x| \le 1} \langle |f| \rangle_{t,x} \le K(n,D) \langle |f| \rangle_{t,1} \, .$$

Since

$$\langle |f| \rangle_{t,1} \le 2 \frac{\langle |f| \rangle_{t,1} + \langle |f| \rangle_{t,-1}}{2} \ ,$$

and since expectation averages η_M over ± 1, we get

$$\mathsf{E} \frac{\langle |f| \rangle_{t,1} + \langle |f| \rangle_{t,-1}}{2} = \nu_t(|f|) \ ,$$

so that $\mathsf{E}\langle |f| \rangle_{t,1} \le 2\nu_t(|f|)$ and therefore

$$\mathsf{E} \sup_{|x| \le 1} \langle |f| \rangle_{t,x} \le K(n,D) \nu_t(|f|) \ . \tag{9.198}$$

Combining with (9.197) we get

$$\mathsf{E} \sup_{|x| \le 1} |v_\ell'''(x)| \le \frac{t^{3/2}}{N^{3/2}} K(n,D) \nu_t(|f|) \ .$$

It follows from (4.197) and (4.199) that the error term occurring in the approximate integration by parts of the quantity (9.196) is at most

$$\frac{t^{3/2}}{N^{3/2}} K(n,D) \nu_t(|f|) \ .$$

Despite the coefficient $\sqrt{N/t}$ in the definition of the term III of (2.28) this implies that the error term created while performing approximate integration by parts in this term satisfies (9.195), and this completes the proof. □

Proof of Theorem 9.9.1. The proof of (9.66) is based on the computation of $\nu_t'(f)$ using Corollary 9.5.7. We face the problem to evaluate

$$\nu_t(f u'(S_{M,t}^\ell) u'(S_{M,t}^{\ell'})) = \mathsf{E} \frac{\left\langle f \mathsf{E}_\xi u'(S_{M,t}^\ell) u'(S_{M,t}^{\ell'}) \exp \sum_{m \le 4} u(S_{M,t}^m) \right\rangle_{t,\sim}}{\langle \mathsf{E}_\xi \exp u(S_{M,t}^1) \rangle_{t,\sim}^4} \ , \tag{9.199}$$

where now

$$S_{M,t}^\ell = \frac{1}{\sqrt{N}} \sum_{i<N} \eta_i \sigma_i^\ell + \frac{\sqrt{t}}{\sqrt{N}} \eta_N \sigma_N^\ell + \frac{\sqrt{1-t}}{\sqrt{N}} \xi_M$$

for $\eta_i = \eta_{i,M}$. For this purpose we simply compare the quantity (9.199) with the similar quantity where $S_{M,t}^\ell$ is replaced by its "Gaussian version"

$$S_{M,t}^{g,\ell} = \frac{1}{\sqrt{N}} \sum_{i<N} g_i \sigma_i^\ell + \frac{\sqrt{t}}{\sqrt{N}} g_N \sigma_N^\ell + \frac{\sqrt{1-t}}{\sqrt{N}} \xi_M \ , \tag{9.200}$$

with $(g_i)_{i<N}$ independent standard Gaussian r.v.s. For this we use Trotter's method described in the proof of Theorem 8.5.2. If we fix the randomness in $\langle \cdot \rangle_{t,\sim}$ and think of the quantity

$$\frac{\left\langle f \mathsf{E}_\xi u'(S^\ell_{M,t}) u'(S^{\ell'}_{M,t}) \exp \sum_{m\le 4} u(S^m_{M,t}) \right\rangle_{t,\sim}}{\langle \mathsf{E}_\xi \exp u(S^1_{M,t})\rangle^4_{t,\sim}}$$

as a function $U(\eta_1, \dots, \eta_N)$, it is immediate that the fourth order partial derivatives of U are bounded by a quantity of the type

$$\frac{K(D)}{N^2}\langle |f|\rangle_{t,\sim}\ ,$$

so the error made while replacing $S^\ell_{M,t}$ by $S^{g,\ell}_{M,t}$ is at most $K(D)\langle|f|\rangle_{t,\sim}/N$. Thus, within this error term, it suffices to study the right-hand side of (9.199) when the quantities $S^\ell_{M,t}$ are replaced by the quantities $S^{g,\ell}_{M,t}$. This study has been done in Sections 9.3 to 9.5, and we leave it to the reader to check that the conclusion of Corollary 9.5.7 remains valid with the extra error term $K(D)\langle|f|\rangle_{t,\sim}/N$, and that the proof of (9.66) carries through. □

We now turn to the proof of Gardner's formula.

Theorem 9.9.4. *There exists a constant L with the following property. Consider a number τ and $\varepsilon > 0$. Then there is a number $N(\varepsilon,\tau)$ such that if $N > N(\varepsilon,\tau)$ and $LM \exp L\tau^2 \le N$, we have*

$$\mathsf{P}\left(\left|\frac{1}{N}\log\left(2^{-N}\mathrm{card}\bigcap_{k\le M} U_k\right) - \mathrm{RS}(\alpha)\right| \ge \varepsilon\right)$$
$$\le L\exp\left(-\frac{1}{L}\min\left(\frac{N^2\varepsilon^2}{M(1+\tau^2)}, \frac{N\varepsilon}{1+\tau^2}\right)\right)\ . \tag{9.201}$$

Here $U_k = \{S_k \ge \tau\} = \left\{\sum_{i\le N}\eta_{i,k}\sigma_i \ge \tau\sqrt{N}\right\}$, $\alpha = M/N$ and $\mathrm{RS}(\alpha)$ is given by (9.108).

In words, as expected, the Gardner formula is the same as in the Gaussian case, but we do not know how to prove as good a convergence rate.

Research Problem 9.9.5. (Level 2) Find the rate at which the convergence takes place in (9.201). For example, how fast does the median of $N^{-1}\log\left(2^{-N}\mathrm{card}\left(\bigcap_{k\le M} U_k\right)\right)$ converge to its limit?

The first major ingredient to the proof of Theorem 9.9.4 is the following.

Proposition 9.9.6. *There exists a constant L with the following property. Consider a probability measure G on Σ_N, and assume that*

$$G^{\otimes 2}\left(\left\{(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\ ;\ R_{1,2}\le\frac{1}{2}\right\}\right) \ge 1-\exp\left(-\frac{N}{16}\right)\ . \tag{9.202}$$

Then for any $\tau \ge 0$, we have

$$L\exp\left(-\frac{N}{L}\right)\le\varepsilon\le\frac{1}{L}\exp(-L\tau^2)\quad\Rightarrow$$
$$\mathsf{P}\left(G\left(\left\{\boldsymbol{\sigma}\,;\,\frac{1}{\sqrt{N}}\sum_{i\le N}\sigma_i\eta_i\ge\tau\right\}\right)\le\varepsilon\right)\le\varepsilon^{1/L}\,. \tag{9.203}$$

Proof. This is a special case of Proposition 8.5.7, when $\mathbf{b}=\mathbf{0}$, $a=1$, so $\psi(\sigma_i)=\sigma_i$ for $\sigma_i=\pm1$. ☐

Let us point out that the special case of Proposition 8.5.7 used above is much easier than the general case. It does not require in particular Propositions 8.5.3 or 8.5.6.

The second major ingredient to the proof of Theorem 9.9.4 is a weak form of (9.103). There is a simple approach to this result, that does not require a detailed study of the system with Hamiltonian (9.106), and in particular does not require Theorem 9.9.1. It is to use Trotter's method directly on the quantity

$$p^b_{N,M}(u)=\frac{1}{N}\mathsf{E}\log\sum_{\boldsymbol{\sigma}}\exp\sum_{k\le M}u(S_k)\,.$$

If we think of the right-hand side as a function $U(\eta_{i,k})$ of the variables $\eta_{i,k}$, all forth order derivatives of U are bounded by $K(D)N^{-3}$ (each differentiation brings an extra factor $N^{-1/2}$) and Trotter's method immediately implies that (with obvious notation)

$$|p^b_{N,M}(u)-p^g_{N,M}(u)|\le\frac{M}{N^2}K(D)\,, \tag{9.204}$$

which is not as good as (9.103) but will be sufficient for our purposes. The third major ingredient of the proof of Theorem 9.9.4 is the following, where $a=1/32$.

Proposition 9.9.7. *Consider $\varepsilon>0$. Then there is a number $\varepsilon'>0$ with the following property. Consider a function u satisfying (9.2) and*

$$\exp u(\tau-\varepsilon')\le\varepsilon'\,. \tag{9.205}$$

Then, for N large enough and any M with $LM\exp L\tau^2\le N$, we have

$$\left|\mathsf{E}\frac{1}{N}\log_{aN}\left(2^{-N}\sum_{\boldsymbol{\sigma}}\exp\sum_{k\le M}u(S_k)\right)-\mathsf{E}\frac{1}{N}\log_{aN}\left(2^{-N}\mathrm{card}\bigcap_{k\le M}U_k\right)\right|\le\varepsilon\,. \tag{9.206}$$

Proof. The proof is very similar to that of Proposition 8.3.9, but slightly simpler. We repeat the argument for the convenience of the reader. Let $C_m=\bigcap_{k<m}U_k$ and

$$V_m = 2^{-N} \sum_{\boldsymbol{\sigma} \in C_m} \exp\Bigl(\sum_{m \le k \le M} u(S_k) \Bigr)$$

so that the left-hand side of (9.206) is

$$\begin{aligned} &\frac{1}{N} |\mathsf{E} \log_{aN} V_1 - \log_{aN} V_{M+1}| \\ &\le \frac{1}{N} \sum_{1 \le m \le M} |\mathsf{E} \log_{aN} V_{m+1} - \mathsf{E} \log_{aN} V_m| \,. \end{aligned}$$

Let us fix m and denote by E_m expectation only on the r.v.s $(\eta_{i,m})_{i \le N}$. We are going to bound

$$|\mathsf{E}_m \log_{aN} V_{m+1} - \mathsf{E}_m \log_{aN} V_m| \,. \tag{9.207}$$

Let

$$Z_m = \sum_{\boldsymbol{\sigma} \in C_m} \exp\Bigl(\sum_{m < k \le M} u(S_k) \Bigr) \,.$$

Let us consider the probability measure G on Σ_N given by

$$G(B) = \frac{1}{Z_m} \sum_{\boldsymbol{\sigma} \in B \cap C_m} \exp\Bigl(\sum_{m < k \le M} u(S_k) \Bigr) \,.$$

Denoting by $\langle \cdot \rangle$ an average for G, we observe the identities

$$\begin{aligned} V_m &= 2^{-N} Z_m \langle \exp u(S_m) \rangle \\ V_{m+1} &= 2^{-N} Z_m \langle \mathbf{1}_{\{S_m \ge \tau\}} \rangle = 2^{-N} Z_m G(\{S_m \ge \tau\}) \,. \end{aligned}$$

Since $u(x) = 0$ for $x \ge \tau$, we have

$$Y := \langle \mathbf{1}_{U_m} \rangle = G(\{S_m \ge \tau\}) \le X := \langle \exp u(S_m) \rangle \,,$$

and using Lemmas 8.3.10 and 8.3.11 we see that for any $c > 0$ we have

$$|\log_{aN} V_{m+1} - \log_{aN} V_m| \le |\log_{aN} Y| \mathbf{1}_{\{Y \le c\}} + \frac{1}{c} |X - Y| \,. \tag{9.208}$$

Since $V_m, V_{m+1} \le 2^{-N} Z_m$, the left-hand side is zero unless $2^{-N} Z_m \ge \exp(-aN)$, so we may assume that this is the case. Then (9.202) holds by Lemma 9.2.1, so Proposition 9.9.6 implies

$$L \exp\left(-\frac{N}{L}\right) \le t \le \frac{1}{L} \exp(-L\tau^2) \quad \Rightarrow \quad \mathsf{P}(Y \le t) \le L t^{1/L} \,,$$

where P is the probability in the r.v.s $\eta_{i,k}$ only. It is then straightforward to see that one can choose $c > 0$ (depending on ε and τ only) with

$$\mathsf{E}_m|\log_{aN} Y|\mathbf{1}_{\{Y\le c\}} \le \frac{\varepsilon}{2} \, .$$

All that remains to prove is that $\mathsf{E}_m|X-Y|$ can be made $\le \varepsilon c/2$ if ε' in (9.205) is small enough. Now we observe that $X=Y=1$ if $S_M \ge \tau$, while $0 \le X \le \varepsilon'$ and $Y=0$ if $S_M \le \tau-\varepsilon'$, and thus

$$0 \le X - Y \le \varepsilon' + \langle \mathbf{1}_{\{\tau-\varepsilon'\le S_M\le\tau\}}\rangle \, ,$$

so that

$$\mathsf{E}_m|X-Y| \le \varepsilon' + \mathsf{P}\Big(\tau-\varepsilon' \le \frac{1}{\sqrt{N}}\sum_{i\le N}\eta_i \le \tau\Big)$$

where $(\eta_i)_{i\le N}$ are random signs. For large N, by the central limit theorem, the right-hand side is $\le \varepsilon' + 2\mathsf{P}(\tau-\varepsilon'\le g\le\tau) \le 3\varepsilon'$. □

Proof of Theorem 9.9.4. Without loss of generality we may assume that $\varepsilon \le a/10$. We consider ε' given by Proposition 9.9.7, and we find a function u that satisfies (9.2), (9.205), and

$$|\mathrm{RS}(\alpha) - (p(u) - \log 2)| \le \varepsilon \tag{9.209}$$

(where $\mathrm{RS}(\alpha)$ is given by (9.108) and $p(u)$ by (9.102)), and such that for a certain number D we have $|u^{(\ell)}| \le D$ for $0 \le \ell \le 5$. It follows from (9.204) and Theorem 2.4.2 that for $N \ge N(D,\tau)$ and $L\alpha\exp L\tau^2 \le 1$ we have

$$\left|\frac{1}{N}\mathsf{E}\log(2^{-N}Z) - (p(u)-\log 2)\right| \le \varepsilon \tag{9.210}$$

where $Z = \sum_{\boldsymbol{\sigma}} \exp \sum_{k\le M} u(S_k)$. Thus

$$\frac{1}{N}\mathsf{E}\log_{aN}(2^{-N}Z) \ge \frac{1}{N}\mathsf{E}\log(2^{-N}Z) \ge p(u) - \log 2 - \varepsilon \, .$$

Since $\mathrm{RS}(0) = 0$, without loss of generality we may assume that

$$\mathrm{RS}(\alpha) \ge -a/4 \tag{9.211}$$

because this is the case for α small, and since we assume $L\alpha\exp L\tau^2 \le 1$ for a large enough constant L. Since $\varepsilon \le a/10$, (9.209) implies that $p(u) - \log 2 - \varepsilon \ge -a/2$, and therefore

$$\frac{1}{N}\mathsf{E}\log_{aN}(2^{-N}Z) \ge -\frac{a}{2} \, .$$

Since $\log_{aN}(x) = -aN$ for $x \le \exp(-aN)$, we deduce from (9.21) by taking $t = a/2$ that

$$\begin{aligned}\mathsf{P}(2^{-N}Z \le \exp(-aN)) &\le \mathsf{P}\Big(\frac{1}{N}\log_{aN}(2^{-N}Z) \le -a\Big)\\ &\le \mathsf{P}\Big(\Big|\frac{1}{N}\log_{aN}(2^{-N}Z) - \mathsf{E}\frac{1}{N}\log_{aN}(2^{-N}Z)\Big| \ge \frac{a}{2}\Big)\\ &\le L\exp\Big(-\frac{N}{L(1+\tau^2)}\Big) \, .\end{aligned}$$

Since $2^{-N}Z \geq \exp(-MD)$ we obtain that for N large enough

$$\frac{1}{N}\mathsf{E}(\mathbf{1}_{\{2^{-N}Z\leq\exp(-aN)\}}\log_{aN}(2^{-N}Z)) \leq \frac{LMD}{N}\exp\left(-\frac{N}{L(1+\tau^2)}\right) \\ \leq \varepsilon$$

and thus

$$\left|\frac{1}{N}\mathsf{E}\log(2^{-N}Z) - \frac{1}{N}\mathsf{E}\log_{aN}(2^{-N}Z)\right| \leq \varepsilon\ .$$

Combining with (9.210) and (9.209) yields

$$\left|\frac{1}{N}\mathsf{E}\log_{aN}(2^{-N}Z) - \mathrm{RS}(\alpha)\right| \leq 3\varepsilon$$

and using (9.206) implies

$$\left|\frac{1}{N}\mathsf{E}\log_{aN}\left(2^{-N}\mathrm{card}\bigcap_{k\leq M}U_k\right) - \mathrm{RS}(\alpha)\right| \leq 4\varepsilon\ . \tag{9.212}$$

Recalling (9.211) and since $5\varepsilon \leq a/2$ we then get

$$\begin{aligned}
&\left|\frac{1}{N}\log_{aN}\left(2^{-N}\mathrm{card}\bigcap_{k\leq M}U_k\right) - \mathrm{RS}(\alpha)\right| < 5\varepsilon \\
\Rightarrow\ &\frac{1}{N}\log_{aN}\left(2^{-N}\mathrm{card}\bigcap_{k\leq M}U_k\right) > -a \\
\Rightarrow\ &\frac{1}{N}\log_{aN}\left(2^{-N}\mathrm{card}\bigcap_{k\leq M}U_k\right) = \frac{1}{N}\log\left(2^{-N}\mathrm{card}\bigcap_{k\leq M}U_k\right) \\
\Rightarrow\ &\left|\frac{1}{N}\log\left(2^{-N}\mathrm{card}\bigcap_{k\leq M}U_k\right) - \mathrm{RS}(\alpha)\right| < 5\varepsilon
\end{aligned}$$

and therefore, using (9.212) in the third line,

$$\begin{aligned}
&\left|\frac{1}{N}\log\left(2^{-N}\mathrm{card}\bigcap_{k\leq M}U_k\right) - \mathrm{RS}(\alpha)\right| \geq 5\varepsilon \\
\Rightarrow\ &\left|\frac{1}{N}\log_{aN}\left(2^{-N}\mathrm{card}\bigcap_{k\leq M}U_k\right) - \mathrm{RS}(\alpha)\right| \geq 5\varepsilon \\
\Rightarrow\ &\left|\frac{1}{N}\log_{aN}\left(2^{-N}\mathrm{card}\bigcap_{k\leq M}U_k\right) - \frac{1}{N}\mathsf{E}\log_{aN}\left(2^{-N}\mathrm{card}\bigcap_{k\leq M}U_k\right)\right| \geq \varepsilon\ .
\end{aligned}$$

The probability of the event above is exponentially small by Proposition 9.2.6, which remains valid in the present case, because this is the case of (9.13), as is proved by (9.203). □

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