

*... what person can dare say they cannot afford to take time for apartness
—indeed, who can afford not to take time for apartness?*

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Synopsis

We first introduce the notion of a (pre-)apartness between points and subsets in an abstract space X , and derive some elementary properties from our axioms. Each point-set pre-apartness gives rise to a topology—the apartness topology—on X , and to several constructively distinct continuity properties, which are explored in Section 2.3. Limits and the Hausdorff property are discussed in Section 2.4, and product pre-apartness spaces in Section 2.5. In the final section we discuss the role of impredicativity in our theory.

2.1 Pre-apartness

Throughout this chapter our basic structure will be an inhabited set X equipped with an inequality relation $\neq$, which in this context can also be called a *point-point apartness*. A subset S of X has two natural complementary subsets:

▷ the **logical complement**

$$\neg S \equiv \{x \in X : \forall y \in S \neg (x = y)\},$$

▷ and the **complement**

$$\sim S \equiv \{x \in X : \forall_{y \in S} (x \neq y)\}.$$

The properties of $\neq$ ensure that $\sim S \subset \neg S$. For each subset A of X we write

$$A \sim S \equiv A \cap \sim S.$$

For the record we note that although $S \subset \neg \neg S$ and $\neg S \cup \neg T \subset \neg(S \cap T)$, in neither case can we prove the reverse inclusions constructively; the same applies with $\neg$ replaced by $\sim$.

We require our inhabited set X to carry also a special relation, $\bowtie$, between points x and subsets S . If $x \bowtie S$, we say that x is **apart from** S . We shall write down the axioms for $\bowtie$ in a moment; but for convenience we introduce here the **apartness complement**

$$-S \equiv \{x \in X : x \bowtie S\}$$

of S , and the notation

$$A - S \equiv A \cap -S.$$

As we shall see, in a quasi-metric space (X, ρ) the apartness complement $-S$ is the set of those $x \in X$ that are bounded away from S .

The following are the axiomatic properties, holding for all $x \in X$ and all subsets A, B of X , that we require of a **pre-apartness** $\bowtie$:

$$\mathbf{A1} \quad x \bowtie \emptyset$$

$$\mathbf{A2} \quad -A \subset \sim A$$

$$\mathbf{A3} \quad x \bowtie (A \cup B) \Leftrightarrow x \bowtie A \wedge x \bowtie B$$

$$\mathbf{A4} \quad -A \subset \sim B \Rightarrow -A \subset -B.$$

We then call the pair $(X, \bowtie)$ —or, when the apartness relation is clearly understood, just the set X itself—a **pre-apartness space**, and the data defining the relations $\neq$ and $\bowtie$ the **pre-apartness structure** on X . When we want to emphasise that the pre-apartness is associated with a particular set X , we sometimes denote¹ it by $\bowtie_X$.

Axiom **A1** says that $X = -\emptyset$. Since $-X \subset \sim X$ (by axiom **A2**) and therefore $-X \subset \neg X = \emptyset$, we see that $\emptyset = -X$. As a special case of axiom **A2**, we have

$$\forall_{x, y \in X} (x \bowtie \{y\} \Rightarrow x \neq y).$$

¹The reader is warned that we apply subscripts with different meanings to an apartness. For example, if τ is a topology, then we denote an important related pre-apartness relation by $\bowtie_\tau$. In practice, it should be clear whether a subscript attached to the symbol $\bowtie$ signifies a set, a topology, or—in Chapter 3—a quasi-uniform structure.

Also, **A3** is equivalent to

$$-(A \cup B) = -A \cap -B.$$

Axiom **A4** may look a little mysterious, so we pause to show where it comes from. In the classical theory of proximity spaces [35], the primitive point-set notion is a binary relation δ of *proximity*, and what we call *apartness* is the negation of proximity. One of the conditions that can be imposed on a proximity is the **Lodato property**,

$$(x \delta B \wedge \forall_{y \in B} (y \delta A)) \Rightarrow x \delta A,$$

where, for example, $x \delta A$ is shorthand for $\{x\} \delta A$. Under classical logic, since $\bowtie$ is the denial of δ , this last implication is equivalent to

$$x \bowtie A \Rightarrow (x \bowtie B \vee \exists_{y \in B} (y \bowtie A))$$

and therefore to

$$(x \bowtie A \wedge \forall_y (y \bowtie A \Rightarrow y \notin B)) \Rightarrow x \bowtie B.$$

In the classical theory of nearness, where $\sim$ and $\neg$ would coincide, this last statement is just **A4** in disguise.

If, in addition to **A1–A4**, the pre-apartness satisfies

$$\mathbf{A5} \quad x \bowtie A \Rightarrow \forall_{y \in X} (x \neq y \vee y \bowtie A),$$

then we call it an **apartness**, and the space X an **apartness space**. The value of the additional axiom **A5** is that it provides us with (classically automatic) alternatives that facilitate many constructive proofs. Nevertheless, it makes good sense to work without **A5** where it is not needed.

The canonical example of a full-blooded apartness space is a quasi-metric space (X, ρ) with the pre-apartness defined by

$$x \bowtie A \Leftrightarrow \exists_{r>0} \forall_{y \in A} (\rho(x, y) \geq r). \quad (2.1)$$

In this case, the apartness complement of a subset A of X ,

$$-A \equiv \{x \in X : \exists_{r>0} \forall_{y \in A} (\rho(x, y) \geq r)\},$$

is also known as the **metric complement** of A . It is routine to verify that $\bowtie$ has properties **A1–A3**. For **A4**, let $x \bowtie A$ and $-A \subset \sim B$. Choose $r > 0$ such that $\rho(x, z) \geq r$ for all $z \in A$. If $y \in B$ and $\rho(x, y) < r/2$, then for all $z \in A$ we have

$$\rho(y, z) \geq \rho(x, z) - \rho(x, y) > r/2, \quad (2.2)$$

so $y \in -A$ and therefore $y \in \sim B$, which is absurd. Hence $\rho(x, y) \geq r/2$ for all $y \in B$, and therefore $x \bowtie B$. This completes the verification of **A4**. For

A5 we let x, A , and r be as before. Then for each $y \in X$, either $\rho(x, y) > 0$ and therefore $x \neq y$, or else $\rho(x, y) < r/2$. In the second case, for all $z \in A$ we have (2.2) and therefore $y \bowtie A$.

The apartness defined at (2.1) is called the **quasi-metric apartness** on X , and X , taken with this apartness, is called a **quasi-metric apartness space**. When ρ is actually a metric, we replace *quasi-metric* by *metric* in both places in the preceding sentence.

Classically, with the law of excluded middle at hand, it is easy to prove that every pre-apartness space satisfies **A5**. Is this the case constructively? First note that if X is any set with an inequality relation, then

$$\forall x \in X \forall A \subset X (x \bowtie A \Leftrightarrow x \in \sim A)$$

defines a pre-apartness on X . Now take $X = [0, 1]$ with the usual inequality relation and the pre-apartness just defined, and take $A = (0, 1]$. Then $0 \bowtie A$. Given a binary sequence $(a_n)_{n \geq 1}$ with at most one term equal to 1, define

$$y \equiv \sum_{n=1}^{\infty} \frac{a_n}{n} \in X.$$

If $y \neq 0$, then $a_n = 1$ for some n ; if $y \bowtie A$, then $a_n = 0$ for all n . Thus if axiom **A5** holds in the pre-apartness space X , we can derive **LPO**.

We conclude from this example that **A5** cannot be derived constructively from **A1–A4**. On the other hand, since **A5** holds in a metric apartness space, its negation cannot be derived from **A1–A4**. Thus **A5** is constructively independent of **A1–A4**.

In the remaining results of this section, except in some of the examples, we assume that X is a pre-apartness space, that $x, y, z, \dots$ are points of X , and that $A, B, \dots$ are subsets of X . If we want to specialise to an apartness space, we shall make it explicit that we are also assuming **A5**.

Proposition 2.1.1 *If $A \subset B$, then $\neg B \subset \neg A$.*

Proof. Since $A \cup B = B$, the conclusion follows from **A3**. ■

Proposition 2.1.2 *$A \subset \sim \neg A$.*

Proof. By **A2**, $\neg A \subset \sim A$; whence $A \subset \sim \sim A \subset \sim \neg A$. ■

Proposition 2.1.3 *If $\sim A \subset \sim B$, then $\neg A \subset \neg B$.*

Proof. Since $\neg A \subset \sim A$, this is a simple application of **A4**. ■

If Y is an inhabited subset of X , we have a natural inequality $\neq_Y$ and point-set relation $\bowtie_Y$ defined for $y, y' \in Y$ and $S \subset Y$ by

$$\begin{aligned} y \neq_Y y' &\Leftrightarrow y \neq y', \\ y \bowtie_Y S &\Leftrightarrow y \bowtie S, \end{aligned} \tag{2.3}$$

where on the right side, $\neq$ and $\bowtie$ are the original inequality and pre-apartness on X . We say that $\neq_Y$ and $\bowtie_Y$ are **induced** on Y by their counterparts on X . It is easy to show that $\bowtie_Y$ satisfies **A1–A3**. If, in addition, it satisfies **A4** in the form

$$(Y - A \subset Y \sim B) \Rightarrow (Y - A \subset Y - B),$$

then it is a pre-apartness on Y ; in that case, taken with the induced inequality and pre-apartness, Y is called a **pre-apartness subspace** of X ; if $\bowtie_Y$ satisfies **A5**, then we call Y an **apartness subspace** of X . We usually omit the subscript, and denote the induced inequality and apartness on Y simply by $\neq$ and $\bowtie$.

Proposition 2.1.4 *Let $(X, \bowtie)$ be a pre-apartness space satisfying the condition*

$$\forall x, y \in X \forall S \subset X ((x \in -S \wedge y \notin -S) \Rightarrow x \neq y). \quad (2.4)$$

Then every inhabited subset of X is a pre-apartness subspace.

Proof. Let Y be an inhabited subset of X , and let A, B be subsets of Y such that $Y - A \subset Y \sim B$. If $y \in B \subset Y$, then $y \notin X - A$, so, by (2.4), $y \neq x$ for each $x \in X - A$. Thus $X - A \subset X \sim B$. By **A4** in the space X , we have $X - A \subset X - B$; from which it follows that $Y - A \subset Y - B$. ■

We refer to condition (2.4) as the **reverse Kolmogorov property**. It is equivalent to the condition

$$\forall S \subset X (\neg -S = \sim -S). \quad (2.5)$$

If X is an apartness space and Y is an inhabited subset of X , then, since **A5** both implies condition (2.4) and is inherited from $\bowtie_X$ by $\bowtie_Y$, it follows, with reference to Proposition 2.1.4, that Y is also an apartness space.

The following results about double and triple complements may appear somewhat unattractive to the reader unused to working constructively, but are occasionally useful.

Although any subset A of a set with an inequality is contained in its double complement $\sim\sim A$, constructively the double complement looks potentially larger than the set A itself, and we cannot in general prove that these two sets are equal. However, it is an easy exercise to prove that

$$\sim A = \sim(\sim\sim A) = \sim\sim(\sim A) = \sim\sim\sim A.$$

If A is a subset of our pre-apartness space X we can say more.

Proposition 2.1.5 $-A = -\sim\sim A = -\sim - A$.

Proof. Since $-A \subset \sim\sim - A$, we see from **A4** that

$$-A \subset -\sim - A. \quad (2.6)$$

Also, by Proposition 2.1.1,

$$- \sim \sim A \subset -A. \quad (2.7)$$

On the other hand, since $-A \subset \sim A$, we have $\sim \sim A \subset \sim -A$ and therefore (again by Proposition 2.1.1)

$$- \sim - A \subset - \sim \sim A. \quad (2.8)$$

Putting together (2.6)–(2.8), we obtain the desired identities. ■

We say that the point $x \in X$ is **near** the set $A \subset X$, and we write $\text{near}(x, A)$, if

$$\forall_{S \subset X} (x \in -S \Rightarrow \exists_{y \in X} (y \in A - S)).$$

Although our focus throughout the book is primarily on apartness, nearness will arise from time to time—notably, to characterise the closure of a set relative to the pre-apartness, and in our discussions of limits and product apartness spaces.

Proposition 2.1.6 *If $\text{near}(x, A)$ in X , then A is inhabited.*

Proof. By **A1**, $x \in -\emptyset$. So if $\text{near}(x, A)$, then, by the definition of *near*, there exists y in $A - \emptyset$, which equals A . ■

Proposition 2.1.7 *In the case where X is a metric apartness space, $\text{near}(x, A)$ if and only if $A \cap B(x, r)$ is inhabited for each $r > 0$.*

Proof. Suppose that $\text{near}(x, A)$. Given $r > 0$, let

$$S \equiv \{y \in X : \rho(x, y) \geq r/2\}.$$

Then

$$x \in -S \subset \overline{B}(x, r/2) \subset B(x, r).$$

Hence, by definition of the nearness predicate, there exists $y \in A - S \subset A \cap B(x, r)$.

Now suppose, conversely, that $A \cap B(x, r)$ is inhabited for each $r > 0$, and that $x \in -S$. Then there exists $r > 0$ such that $\rho(x, s) \geq r$ for each $s \in S$; whence $B(x, r) \subset -S$ and therefore $A - S$ is inhabited. Since S is arbitrary, we conclude that $\text{near}(x, A)$. ■

Proposition 2.1.8 *If $x \in A$, then $\text{near}(x, A)$.*

Proof. For each B with $x \in -B$ we have $x \in A - B$. Hence, by definition, $\text{near}(x, A)$. ■

Corollary 2.1.9 *If $x = y$, then $\text{near}(x, \{y\})$.*

Proposition 2.1.10 *If $\text{near}(x, A)$ and $A \subset B$, then $\text{near}(x, B)$.*

Proof. This follows directly from the definition of the nearness predicate. ■

The next result generalises its predecessor and shows that a pre-apartness space satisfies the classical Lodato condition (page 21).

Proposition 2.1.11 *Suppose that $\text{near}(x, A)$, and that $\text{near}(y, B)$ for each $y \in A$. Then $\text{near}(x, B)$.*

Proof. Let $x \in -S$. Then, by the definition of *near*, there exists $y \in A - S$. By hypothesis, $\text{near}(y, B)$. Since also $y \in -S$, it follows from the definition of *near* that there exists $z \in B - S$. Thus

$$\forall_{S \subset X} (x \in -S \Rightarrow \exists_{z \in X} (z \in B - S))$$

—that is, $\text{near}(x, B)$. ■

Proposition 2.1.12 *If $\text{near}(x, A)$, then $\text{near}(x, A \cup B)$ for all $B \subset X$.*

Proof. Apply Proposition 2.1.10 with B replaced by $A \cup B$. ■

Proposition 2.1.13 *If $\text{near}(x, A)$ and $x \bowtie B$, then $\text{near}(x, A - B)$.*

Proof. Let $x \in -S$. We need to show that $(A - B) - S$ is inhabited. To this end, observe that $x \in -B$, so by **A3**, $x \in -(B \cup S)$. Since $\text{near}(x, A)$, there exists $y \in A - (B \cup S)$; but $A - (B \cup S) = (A - B) - S$, so we are through. ■

One of the axioms of the classical theory of proximity presented in [35] is

$$\text{near}(x, A \cup B) \Leftrightarrow \text{near}(x, A) \vee \text{near}(x, B). \quad (2.9)$$

The implication from right to left here is a consequence of Proposition 2.1.12. To see that the implication from left to right in (2.9) is essentially nonconstructive, consider the metric apartness space $\mathbf{R}$. Given an increasing binary sequence $(a_n)_{n \geq 1}$ with $a_1 = 0$, define

$$S \equiv \left\{ \frac{1}{n} : a_n = 0 \right\}, \quad T \equiv \left\{ \frac{1}{n} : a_n = 1 \right\}.$$

Then 0 is near $S \cup T$. But if 0 is near S , then $a_n = 0$ for all n ; while if 0 is near T , then there exists $x \in T$ such that $|x| < 1/2$, so we can find n with $a_n = 1$. Thus the left-to-right implication in (2.9) implies **LPO**.

It readily follows from Proposition 2.1.7 that in the context of a metric space, $\text{near}(x, A)$ implies $\neg(x \bowtie A)$. This holds in a general pre-apartness space.

Proposition 2.1.14 $\neg(\text{near}(x, A) \wedge x \bowtie A)$.

Proof. Assume that $\text{near}(x, A)$ and $x \bowtie A$. Then $x \in -A$, and so, by the definition of near , there exists $y \in A - A$. This contradicts axiom **A2**. ■

We observe that in the metric apartness space $\mathbf{R}$, the statement

$$\forall A \subset \mathbf{R} \ (\neg(1 \bowtie A) \Rightarrow \text{near}(1, A))$$

implies the law of excluded middle in the form

$$\neg\neg P \Rightarrow P.$$

Indeed, let P be any proposition satisfying $\neg\neg P$, and consider the subset

$$A \equiv \{x \in \mathbf{R} : x = 0 \vee (x = 1 \wedge P)\}$$

of $\mathbf{R}$. If $1 \bowtie A$, then $\neg P$, which contradicts our initial assumption; hence $\neg(1 \bowtie A)$. But if $\text{near}(1, A)$, then there exists $x \in A$ such that $x > 0$; whence $x = 1$, $1 \in A$, and therefore P holds.

Proposition 2.1.15 *If X is a pre-apartness space with the reverse Kolmogorov property, and if $\text{near}(x, A)$, and $y \bowtie A$, then $x \neq y$.*

Proof. Assume that $\text{near}(x, A)$ and $y \bowtie A$. Proposition 2.1.14 shows that $x \notin -A$; whence $x \neq y$, by (2.4). ■

Proposition 2.1.16 *If X satisfies **A5** and if $\neg(x \neq y)$, then $\text{near}(x, \{y\})$.*

Proof. By **A5**, for each S with $x \in -S$ we have either $x \neq y$ or else $y \in -S$; the former alternative is ruled out, so we must have $y \in -S$. It follows that $\text{near}(x, \{y\})$. ■

An immediate consequence of this proposition is

Corollary 2.1.17 *If X is an apartness space such that*

$$\forall_{x,y \in X} (\text{near}(x, \{y\}) \Rightarrow x = y), \quad (2.10)$$

then the inequality on X is tight.

A pre-apartness space X is said to be $\mathbf{T}_1$ if

$$\forall_{x,y \in X} (x \neq y \Rightarrow x \bowtie \{y\}). \quad (2.11)$$

Since the inequality relation is symmetric, a $\mathbf{T}_1$ pre-apartness space has symmetric point-point pre-apartness:

$$\forall_{x,y \in X} (x \bowtie \{y\} \Leftrightarrow y \bowtie \{x\}).$$

Every quasi-metric apartness space is $\mathbf{T}_1$.

Proposition 2.1.18 *If X is a $\mathbf{T}_1$ pre-apartness space and $\text{near}(x, \{y\})$ in X , then $\neg(x \neq y)$.*

Proof. Assume that $x \neq y$; then $x \bowtie \{y\}$, by (2.11). This contradicts Proposition 2.1.14. ■

Corollary 2.1.19 *If X is a $\mathbf{T}_1$ apartness space and $\text{near}(x, \{y\})$ in X , then $\text{near}(y, \{x\})$.*

Proof. Apply Propositions 2.1.18 and 2.1.16, noting that $\neg(x \neq y)$ if and only if $\neg(y \neq x)$. ■

Corollary 2.1.20 *If X is a $\mathbf{T}_1$ pre-apartness space with tight inequality, then (2.10) holds.*

Proof. Apply Proposition 2.1.18. ■

Proposition 2.1.21 *If X is a $\mathbf{T}_1$ apartness space, and a, b are points of X with $a \neq b$, then for each $x \in X$ either $x \neq a$ or $x \neq b$.*

Proof. By (2.11), $a \bowtie \{b\}$; whence, by **A5**, either $x \neq a$ or else $x \bowtie \{b\}$; in the latter event, **A2** shows that $x \neq b$. ■

Proposition 2.1.22 *If X is a $\mathbf{T}_1$ apartness space containing two distinct points, then for each $x \in X$ there exists $y \in X$ such that $x \neq y$.*

Proof. Let a, b be points of X with $a \neq b$. Then either $x \neq a$ or $x \neq b$, by the previous proposition. ■

Proposition 2.1.23 *If X is a $\mathbf{T}_1$ pre-apartness space, $\text{near}(x, A)$, and $x \neq y$, then there exists $z \in A$ such that $y \neq z$.*

Proof. In this case, $x \in -\{y\}$, so by the definition of *near*, there exists $z \in A$ with $z \bowtie \{y\}$. The desired conclusion follows from **A2**. ■

We end the section by establishing the extensionality of apartness and nearness.

Proposition 2.1.24 *If X is an apartness space, $x \bowtie A$, $x = x'$, and $A = A'$, then $x' \bowtie A'$.*

Proof. By **A5**, either $x \neq x'$ or else, as must be the case, $x' \bowtie A$. Since $A' = A$, we have $x' \in -A \subset \sim A = \sim A'$, by **A2**; whence $x' \bowtie A'$, by **A4**. ■

Proposition 2.1.25 *If X is an apartness space, $\text{near}(x, A)$, $x = x'$, and $A = A'$, then $\text{near}(x', A')$.*

Proof. Let $x' \in -S$. Then $x \in -S$, by the previous proposition. Since x is near A , there exists $y \in A - S$. But $A' = A$, so $y \in A' - S$. It follows from our definition of *near* that $\text{near}(x', A')$. ■

2.2 Apartness and Topology

We assume that every topological space (X, τ) comes equipped with an inequality $\neq$. For a point x and a subset A of such a space we define

$$x \bowtie_{\tau} A \Leftrightarrow \exists U \in \tau (x \in U \subset \sim A) \quad (2.12)$$

and, of course,

$$\text{near}(x, A) \Leftrightarrow \forall B \subset X (x \in -B \Rightarrow \exists y \in X (y \in A - B)).$$

Thus $x \bowtie_{\tau} A$ if and only if x belongs to $(\sim A)^{\circ}$, the interior of $\sim A$ relative to the topology τ . It is easy to show that $\bowtie_{\tau}$ satisfies **A1–A4** for a pre-apartness—the **topological pre-apartness** corresponding to τ . We call X , taken with this pre-apartness, a **topological pre-apartness space**, or, if $\bowtie_{\tau}$ also satisfies **A5**, a **topological apartness space**. It is easily shown that if a topological space (X, τ) has the **topological A5 property**

$$\forall x \in X \forall U \in \tau (x \in U \Rightarrow \forall y \in X (x \neq y \vee y \in U)), \quad (2.13)$$

then the corresponding pre-apartness satisfies **A5**. Note that on a quasi-metric apartness space the pre-apartness corresponding to the quasi-metric topology coincides with the quasi-metric apartness.

The alert reader may have asked: is a point-set pre-apartness necessarily symmetric—that is, do we have

$$\forall x, y \in X (x \bowtie \{y\} \Rightarrow y \bowtie \{x\})?$$

We give a topological example to show that it is not. Consider the set $X \equiv \{0, 1\}$ with the discrete equality and inequality, the topology

$$\tau \equiv \{\emptyset, \{1\}, \{0, 1\}\},$$

and the corresponding topological pre-apartness $\bowtie_{\tau}$. Since $1 \in \{1\} \subset \sim \{0\}$, we have $1 \bowtie_{\tau} \{0\}$; but since $0 \notin \{1\}$ and $\{0, 1\} \not\subset \sim \{1\}$, we cannot have $0 \bowtie_{\tau} \{1\}$.

Occasionally it is helpful to consider the following condition, linking the inequality and the topology, on a topological space (X, τ) :

$$\forall x, y \in X \forall U \in \tau ((x \in U \wedge y \notin U) \Rightarrow x \neq y). \quad (2.14)$$

This condition always holds classically for the denial inequality, and is a constructive consequence of the topological **A5** property. Thus it holds if the space is discrete, and in any quasi-metric space.

Proposition 2.2.1 *Let (X, τ) be a topological space in which condition (2.14) holds. Then $\bowtie_{\tau}$ has the reverse Kolmogorov property.*

Proof. Let $x \in -S$ and $y \notin -S$ in X . Then there exists $U \in \tau$ such that $x \in U \subset \sim S$. It follows from this and the definition of $\boxtimes_\tau$ that $U \subset -S$, so $y \notin U$; whence $x \neq y$, by (2.14). ■

In light of Proposition 2.2.1, we refer to condition (2.14) as the **topological reverse Kolmogorov property** (with *topological* omitted when it is clear that we are referring to (2.14) rather than its counterpart (2.4) for an apartness space).

The following proposition will re-appear in Chapter 3.

Proposition 2.2.2 *Let (X, τ) be a topological space with the topological reverse Kolmogorov property. Then $(\neg S)^\circ = (\sim S)^\circ$ for each subset S of X .*

Proof. It is trivial that $(\sim S)^\circ \subset (\neg S)^\circ$. For the reverse inclusion, let $x \in (\neg S)^\circ$ and $y \in S$. Clearly, $y \notin (\neg S)^\circ$. Since $(\neg S)^\circ \in \tau$, it follows from (2.14) that $x \neq y$. Thus $(\neg S)^\circ \subset \sim S$. Since also $(\neg S)^\circ$ is open, it follows that $(\neg S)^\circ \subset (\sim S)^\circ$. ■

Proposition 2.2.3 *Let (X, τ) be a topological space with the topological reverse Kolmogorov property, let Y be a subspace of X , and let τ_Y be the subspace topology induced on Y by τ . Then the topological subspace (Y, τ_Y) also has the topological reverse Kolmogorov property.*

Proof. Let $x, y \in Y$ and $V \in \tau_Y$ be such that $x \in V$ and $y \notin V$. By the definition of τ_Y , there exists $U \in \tau$ such that $V = Y \cap U$. If $y \in U$, then $y \in Y \cap U = V$, a contradiction; hence $y \notin U$. Since $x \in U$, it follows from condition (2.14) in X that $x \neq y$. ■

A topological pre-apartness is $\mathbf{T}_1$ (as defined at (2.11) on page 26) if and only if

$$\forall_{x,y \in X} (x \neq y \Rightarrow \exists_{U \in \tau} (x \in U \wedge y \in \sim U))$$

—in other words, its topology τ satisfies the classical **first axiom of separation**. A quasi-metric space is a $\mathbf{T}_1$ topological apartness space.

Another example of a $\mathbf{T}_1$ apartness space is that associated with a **locally convex space**—that is, a vector space X over $\mathbf{R}$ or $\mathbf{C}$, together with a family $(p_i)_{i \in I}$ of seminorms that satisfies the following condition:

$$\forall_{x,y \in X} (x \neq y \Leftrightarrow \exists_{i \in I} (p_i(x - y) > 0)).$$

Recall from page 15 that the inequality on a vector space is tight, by definition; whence we must have

$$\forall_{x,y \in X} (x = y \Leftrightarrow \forall_{i \in I} (p_i(x - y) = 0)).$$

The family $(p_i)_{i \in I}$ defines the corresponding **locally convex topology**, in which the basic open neighbourhoods of x are of the form

$$\{x' \in X : \forall_{i \in I} (p_i(x - x') < \varepsilon)\}$$

for some $\varepsilon > 0$ and some inhabited, finitely enumerable subset F of I . It is left as an exercise for the reader to show that this topology has the reverse Kolmogorov property (2.14). To establish **A5** for the corresponding topological pre-apartness $\bowtie$, we argue as follows. Let $x \bowtie A$; then there exist $\varepsilon > 0$ and elements $i_1, \dots, i_n$ of I such that

$$U \equiv \left\{ y \in X : \sum_{k=1}^n p_{i_k}(x - y) < \varepsilon \right\} \subset \sim A.$$

Given $y \in X$, we have either $\sum_{k=1}^n p_{i_k}(x - y) > 0$ or $\sum_{k=1}^n p_{i_k}(x - y) < \varepsilon$. In the first case, $p_{i_k}(x - y) > 0$ for some k and so $x \neq y$. In the second case, $y \in U \subset \sim A$ and so, by definition of the pre-apartness, $y \bowtie A$. This completes the proof of **A5**. It is easy to see that the topological pre-apartness relation $\bowtie$ is **T**₁.

If (X, τ) is a topological space, and Y an inhabited subset of X , then the subspace topology τ_Y induced on Y by τ has an associated pre-apartness $\bowtie_{\tau_Y}$. A natural question arises: under what conditions does this pre-apartness coincide with the restriction of the relation $\bowtie_\tau$ to points and subsets of Y ?

Proposition 2.2.4 *Let X be a topological space satisfying (2.13), and let $\bowtie_\tau$ be the corresponding topological apartness. Let Y be an inhabited subset of X , let τ_Y be the subspace topology induced on Y by τ , and let $\bowtie_{\tau_Y}$ be the corresponding topological apartness on Y . Then*

$$\forall y \in Y \forall S \subset Y (y \bowtie_{\tau_Y} S \Leftrightarrow y \bowtie_\tau S).$$

Proof. Let $y \in Y \subset X$ and $S \subset Y$. If $y \bowtie_\tau S$, then there exists $U \in \tau$ such that $y \in U \subset X \sim S$; whence

$$y \in Y \cap U \subset Y \cap (X \sim S) = Y \sim S,$$

where $Y \cap U \in \tau_Y$; so $y \bowtie_{\tau_Y} S$.

Conversely, if $y \bowtie_{\tau_Y} S$, then there exists $V \in \tau_Y$ such that $y \in V \subset Y \sim S$. By definition of the subspace topology τ_Y , there exists $U \in \tau$ such that $V = Y \cap U$. Hence

$$y \in Y \cap U \subset Y \sim S \subset X \sim S.$$

Consider any $x \in U$ and any $s \in S$. By (2.13), either $x \neq s$ or $s \in U$; in the latter case, $s \in Y \cap U$, so $s \in Y \sim S$, which is absurd. It follows that $y \in U \subset X \sim S$ and therefore $y \bowtie_\tau S$. ■

Corollary 2.2.5 *Let Y be an inhabited subset of a topological space (X, τ) that satisfies (2.13). Then the restriction of the relation $\bowtie_\tau$ to points and subsets of Y is an apartness on Y .*

Proof. According to Proposition 2.2.4, the restriction in question is precisely the apartness induced on Y by the subspace topology τ_Y . ■

Corollary 2.2.6 *Let Y be an inhabited subset of a quasi-metric space X . Then the apartness induced on Y by the quasi-metric on X coincides with the apartness induced on Y by the original quasi-metric apartness on X .*

We now introduce an important topology on a given pre-apartness space $(X, \bowtie)$. A subset S of X is said to be **nearly open** if it can be written as a union of apartness complements: that is, if there exists a family $(A_i)_{i \in I}$ of subsets of X such that $S = \bigcup_{i \in I} -A_i$. The empty subset of X is nearly open ($\emptyset = -X$), X is nearly open ($X = -\emptyset$), and a union of nearly open sets is nearly open. Since, by a simple induction argument using **A3**, the intersection of a finite number of apartness complements is an apartness complement, it can easily be shown that a finite intersection of nearly open sets is nearly open. Thus the nearly open sets form a topology—the **apartness topology**, denoted by $\tau_{\bowtie}$ —on X for which the apartness complements form a basis. It is straightforward to prove that if $(X, \bowtie)$ is an apartness space, then the corresponding apartness topology satisfies (2.13).

Proposition 2.2.7 *In a topological pre-apartness space every nearly open set is open.*

Proof. Let (X, τ) be a topological pre-apartness space. It suffices to show that every apartness complement $-A$ in X is open. Let $x \in -A$ and choose $U \in \tau$ such that $x \in U \subset \sim A$. Then, by the definition of the apartness in X , $U \subset -A$. Hence $-A$ is open. ■

Corollary 2.2.8 *Let $(X, \bowtie)$ be a pre-apartness space, and let τ be the corresponding apartness topology. Then the topological pre-apartness structure corresponding to τ coincides with the original pre-apartness structure on X .*

Proof. Let $\bowtie_\tau$ denote the topological apartness corresponding to τ . If $x \bowtie S$, then $x \in -S \subset \sim S$, where $-S$ is the apartness complement of S relative to $\bowtie$; since $-S$ is open by Proposition 2.2.7, it follows from the definition of the topological pre-apartness that $x \bowtie_\tau S$.

Conversely, if $x \bowtie_\tau S$, then there exists a nearly open set U such that $x \in U \subset \sim S$, so (by definition of *nearly open*) there exists V such that $x \in -V \subset \sim S$. It follows from **A4** that $x \bowtie S$. ■

Corollary 2.2.8 provides the motivation for the definition (in Section 2.5 below) of the product of two pre-apartness spaces.

Proposition 2.2.9 *If X is a pre-apartness space with the reverse Kolmogorov property, then the corresponding apartness topology has the topological reverse Kolmogorov property.*

Proof. Consider $x, y \in X$ and $U \in \tau_{\bowtie}$ such that $x \in U$ and $y \notin U$. There exists $S \subset X$ such that $x \in -S \subset U$. Clearly, $y \notin -S$, so, by the reverse Kolmogorov property in $(X, \bowtie)$, we have $x \neq y$. ■

We say that a topological pre-apartness space is **topologically consistent** if every open subset of X is nearly open.

Proposition 2.2.10 *Every metric apartness space is topologically consistent.*

Proof. Let (X, ρ) be a metric space, and let $\bowtie$ be the corresponding metric apartness, which, as noted earlier, coincides with the topological apartness corresponding to the metric topology on X . In view of Proposition 2.2.7, it suffices to prove that every (metrically) open subset S of X is nearly open. Given $x \in S$, choose $r > 0$ such that $B(x, r) \subset S$. Then $x \in -\sim B(x, r/2) \subset B(x, r) \subset S$. It follows that x is an interior point of S in the apartness topology. Hence S is nearly open. ■

Proposition 2.2.11 *The following conditions are equivalent on a topologically consistent topological space (X, τ) .*

- (i) (X, τ) has the topological reverse Kolmogorov property (2.14).
- (ii) $(X, \bowtie_\tau)$ has the reverse Kolmogorov property (2.4).

Proof. Since τ coincides with the apartness topology arising from $\bowtie_\tau$, the desired conclusion follows from Propositions 2.2.1 and 2.2.9. ■

Classically, for every open set A of a topological pre-apartness space we have $A = \sim\sim A$ and therefore, by the definition of the topological pre-apartness, $A = -\sim A$, from which it follows that the space is topologically consistent. Constructively, although $A \subset -\sim A$ holds for an open set A , we cannot hope to prove the reverse inclusion even in the presence of (2.14). To see this, consider the metric subspace

$$X \equiv \{0, 1 - a, 2\}$$

of $\mathbf{R}$, where $a < 1$ and $\neg(a \leq 0)$. Let B be the open ball with centre 0 and radius 1 in X . Then $2 \in \sim B$. On the other hand, if $x \in \sim B$ and $x \neq 2$, then x must equal $1 - a$, so $\neg(1 - a < 1)$; whence $1 - a \geq 1$ and therefore $a \leq 0$, a contradiction. It follows that $\sim B = \{2\}$ and hence that $-\sim B = -\{2\}$. If $-\sim B \subset B$, then $1 - a$, which certainly belongs to $-\{2\}$, is in B ; whence $1 - a < 1$ and therefore $a > 0$. Thus, although (as observed in the proof of Proposition 2.2.10) any open ball B in a metric apartness space satisfies $B \subset -\sim B$, the proposition

Every open ball B in a metric space satisfies $B = -\sim B$

entails

$$\forall_{x \in \mathbf{R}} (\neg(x \leq 0) \Rightarrow x > 0),$$

a statement equivalent to Markov's principle. In fact, as is shown by an example which we now present, if every $\mathbf{T}_1$ topological apartness space with discrete inequality is topologically consistent, then the law of excluded middle holds.

Let A be the set of odd positive integers, B the set of even positive integers, and X the space $\mathbf{N}$, taken with the denial inequality. Given a statement P , for each positive integer n let

$$U_n \equiv \{0\} \cup \{k \in A : k > n\} \cup \{k \in B : k > n \wedge P\}.$$

The sets U_n , together with the singletons $\{n\}$ with $n \in \mathbf{N}^+$, form a countable basis for a **second-countable topology** τ (that is, a topology with a countable base of open sets) on X . Note that any neighbourhood of 0 in this topology must contain one of the sets U_n . Since X is discrete, it automatically satisfies **A5**; so τ induces a topological apartness structure on X . Suppose that X is topologically consistent. Since U_1 is open, it is nearly open and therefore there exists $V \subset X$ such that $0 \in -V \subset U_1$. By Proposition 2.2.7, $-V$ is τ -open, so there exists N such that $0 \in U_N \subset -V$. Assume that $\neg P$ holds. If $V \cap \{k \in B : k > N\}$ is inhabited, then since $U_N \subset -V$, we must have $\neg P$, a contradiction. Hence

$$\{k \in B : k > N\} \subset \neg V.$$

In view of the discreteness of X , we have $\neg V = \sim V$. On the other hand, given $m \in \sim V$, we have either $m = 0 \in -V$ (by our choice of V), or else $m \geq 1$, $m \in \{m\} \subset \sim V$, and therefore $m \bowtie V$, by definition of the topological pre-apartness. It follows from this and axiom **A2** that $\sim V = -V$; whence

$$\{k \in B : k > N\} \subset -V \subset U_1,$$

which is possible if and only if P holds. We conclude that if every topological apartness space is topologically consistent, then the law of excluded middle holds.

Note that in the foregoing example the space X is $\mathbf{T}_1$. Indeed, if $n \in X$ and $n \neq 0$, then $0 \in U_{n+1}$ and $n \in \sim U_{n+1}$; whereas if $m, n \in X$ and $n > m \geq 1$, then $n \in \{n\}, m \in \{m\}$, and $\{n\} \subset \sim \{m\}$.

As we shall see in a moment, the following property ensures topological consistency. We say that a topological space (X, τ) , or just the topology τ itself, is **topologically locally decomposable** if

$$\forall_{x \in X} \forall_{U \in \tau} (x \in U \Rightarrow \exists_{V \in \tau} (x \in V \wedge X = U \cup \sim V)).$$

This condition holds classically for any topological space with the denial inequality: just take $V = U$. Every metric space (X, ρ) is topologically locally decomposable: for if $x \in U$ and U is open in X , then, choosing $r > 0$ such that $B(x, r) \subset U$, we can take $V = B(x, r/2)$.

Proposition 2.2.12 *A topologically locally decomposable topological preapartness space (X, τ) is topologically consistent.*

Proof. Given $U \in \tau$ and $x \in U$, find $V \in \tau$ such that $x \in V$ and $X = U \cup \sim V$. Then $x \in V \subset \sim \sim V$, so $x \in -\sim V$. Since

$$-\sim V \subset \sim \sim V \subset U,$$

it follows that U is a union of apartness complements and so is nearly open. ■

We can see from Proposition 2.2.12 and the foregoing Brouwerian example that if every topological apartness space is topologically locally decomposable, then the law of excluded middle holds.

How do we produce a property like topological local decomposability but applicable to arbitrary pre-apartness spaces? We say that a pre-apartness space $(X, \bowtie)$, or just the pre-apartness $\bowtie$ itself, is **locally decomposable** if

$$\forall x \in X \forall S \subset X (x \in -S \Rightarrow \exists T \subset X (x \in -T \wedge X = -S \cup T)).$$

Local decomposability always holds classically: for if $x \in -S$, then, taking $T = \sim -S$, we have $X = -S \cup T$; also, by Proposition 2.1.5,

$$-S = -\sim -S = -T,$$

so $x \in -T$. Every metric space (X, ρ) is locally decomposable: for if $x \in -S$, then, choosing $r > 0$ such that $B(x, r) \subset -S$, we can take $T \equiv \sim B(x, r/2)$ to obtain $x \in -T$ and $X = -S \cup T$.

Proposition 2.2.13 *A locally decomposable pre-apartness space satisfies **A5**.*

Proof. Let $(X, \bowtie)$ be locally decomposable, let $x \in -S$, and choose T such that $x \in -T$ and $X = -S \cup T$. For each $y \in X$, either $y \in -S$ or $y \in T$; in the latter case, $x \bowtie \{y\}$ by Proposition 2.1.1, and so, by **A2**, $x \neq y$. ■

It will be clear as we develop the subject that local decomposability, which gives us not only **A5** but also stronger alternative conditions to play with, is a very powerful condition.

Proposition 2.2.14 *Let $(X, \bowtie)$ be a locally decomposable apartness space, and τ the corresponding apartness topology. Then (X, τ) is topologically locally decomposable.*

Proof. Let $U \in \tau$ and $x \in U$. Without loss of generality, we may assume that $U = -S$ for some $S \subset X$. Choosing $T \subset X$ such that $x \in -T$ and $X = -S \cup T$, set $V \equiv -T$; then $x \in V$. Moreover, for each $y \in X$ either $y \in -S = U$ or else

$$y \in T \subset \sim \sim T \subset \sim -T = \sim V.$$

Since x and U are arbitrary, we conclude that (X, τ) is topologically locally decomposable. ■

Proposition 2.2.15 *The induced pre-apartness on a topologically locally decomposable topological space is locally decomposable.*

Proof. Let (X, τ) be a topologically locally decomposable topological space. Let $\bowtie_\tau$ denote the corresponding topological apartness, and $-\tau$ the corresponding operation of apartness complementation. Given $x \in -\tau S$, and noting that $-S$ is in τ (by Proposition 2.2.7), construct $V \in \tau$ such that $x \in V$ and $X = -S \cup \sim V$. Let $T \equiv \sim V$. Then $X = -S \cup T$. Moreover, $x \in V \subset \sim T$, so $x \bowtie_\tau T$. Since x and S are arbitrary, it follows that $(X, \bowtie_\tau)$ is locally decomposable. ■

Corollary 2.2.16 *Let (X, τ) be a topological pre-apartness space. Then the following conditions are equivalent.*

- (i) X is topologically locally decomposable.
- (ii) X is topologically consistent, and $(X, \bowtie_\tau)$ is locally decomposable.

Proof. Assuming (i), we see from Proposition 2.2.12 that X is topologically consistent, and from Proposition 2.2.15 that it is locally decomposable. On the other hand, if (ii) holds, then Proposition 2.2.14 shows that $(X, \tau_{\bowtie})$ is topologically locally decomposable; so if X is also topologically consistent, then (i) holds. ■

In view of Corollary 2.2.16, for a topologically consistent topological pre-apartness space there should be no confusion if we use the phrases *topologically locally decomposable* and *locally decomposable* interchangeably.

Nearness now comes into play again, in our discussion of closed sets in the apartness topology.

Proposition 2.2.17 *Let A be a subset of the pre-apartness space X . Then*

$$\overline{A} = \{x \in X : \text{near}(x, A)\},$$

where the bar denotes closure in the apartness topology.

Proof. Let $\text{near}(x, A)$, and let $U \equiv \bigcup_{i \in I} -A_i$ be any nearly open set containing x . Choosing $i \in I$ such that $x \in -A_i$, we see from Proposition 2.1.13 that $\text{near}(x, A - A_i)$. So, by Proposition 2.1.6, there exists $y \in A - A_i \subset A \cap U$. Conversely, if A intersects each nearly open set containing x , then since $-B$ is nearly open for each $B \subset X$, we see immediately from the definition of *near* that $\text{near}(x, A)$. ■

Corollary 2.2.18 *A subset A of a pre-apartness space is closed in the apartness topology if and only if it is **nearly closed**, in the sense that*

$$A = \{x \in X : \text{near}(x, A)\}.$$

Proposition 2.2.19 *For each nearly open subset of a pre-apartness space X whose apartness topology has the reverse Kolmogorov property, the logical complement equals the complement and is nearly closed.*

Proof. Let $A \equiv \bigcup_{i \in I} -U_i$ be nearly open, and $x \in \overline{\neg A}$. Then $\text{near}(x, \neg A)$, by Proposition 2.2.17. For each $i \in I$ it follows that if $x \in -U_i$, then there exists $z \in (\neg A) - U_i \subset (\neg A) \cap A$, which is absurd. We conclude that $x \notin -U_i$; whence $x \in \sim - U_i$, by the reverse Kolmogorov property. Thus

$$x \in \bigcap_{i \in I} \sim - U_i = \sim A.$$

Hence

$$\neg A \subset \overline{\neg A} \subset \sim A \subset \neg A,$$

from which the desired conclusions now follow. ■

Having established the fundamental results connecting pre-apartness and topologies, we turn in the next section to an examination of types of continuity of mappings between pre-apartness spaces.

2.3 Apartness and Continuity

Intuitionistic logic enables us to distinguish between various classically equivalent types of continuity that we now introduce.

Let $f : X \rightarrow Y$ be a mapping between pre-apartness spaces. We say that f is

▷ **nearly continuous** if

$$\forall_{x \in X} \forall_{A \subset X} (\text{near}(x, A) \Rightarrow \text{near}(f(x), f(A)));$$

▷ **continuous** if

$$\forall_{x \in X} \forall_{A \subset X} (f(x) \bowtie f(A) \Rightarrow x \bowtie A)$$

—that is,

$$f^{-1}(-f(A)) \subset -A$$

for each $A \subset X$;

▷ **topologically continuous** if $f^{-1}(S)$ is nearly open in X for each nearly open $S \subset Y$.

It is almost trivial that the composition of continuous functions is continuous, and that the restriction of a continuous function to a pre-apartness subspace of its domain is continuous. Analogous remarks hold for nearly continuous functions and for topologically continuous ones.

For a mapping between quasi-metric spaces, continuity in our sense turns out to be equivalent to the standard one from elementary analysis courses.

Proposition 2.3.1 *The following are equivalent conditions on a mapping $f : X \rightarrow Y$ between quasi-metric spaces.*

- (i) f is continuous.
- (ii) For each $x \in X$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that if $x' \in X$ and $\rho(x, x') < \delta$, then $\rho(f(x), f(x')) < \varepsilon$.

Proof. Suppose that f is continuous. Given $x \in X$ and $\varepsilon > 0$, let

$$A \equiv \left\{ x' \in X : \rho(f(x), f(x')) > \frac{\varepsilon}{2} \right\}.$$

Then $f(x) \bowtie f(A)$ in Y , so $x \bowtie A$ in X . Pick $\delta > 0$ such that $\rho(x, x') \geq \delta$ for all $x' \in A$. If $x' \in X$ and $\rho(x, x') < \delta$, then $x' \notin A$ and therefore $\rho(f(x), f(x')) < \varepsilon$. Thus (i) implies (ii).

Now suppose that (ii) holds, and let $f(x) \bowtie f(A)$ in Y . Then there exists $\varepsilon > 0$ such that $\rho(f(x), f(x')) \geq \varepsilon$ for all $x' \in A$. Pick $\delta > 0$ as in (ii). If $x' \in A$ and $\rho(x, x') < \delta$, then $\rho(f(x), f(x')) < \varepsilon$, a contradiction. Hence $\rho(x, x') \geq \delta$ for each $x' \in A$; in other words, $x \bowtie A$. Hence (ii) implies (i). ■

A continuous mapping f of a pre-apartness space into a $\mathbf{T}_1$ pre-apartness space is strongly extensional: for if $f(x) \neq f(y)$, we have $f(x) \bowtie \{f(y)\}$; whence $x \bowtie \{y\}$ and therefore, by **A2**, $x \neq y$. Less trivial to establish is the strong extensionality of nearly continuous mappings.

Proposition 2.3.2 *A nearly continuous mapping of an apartness space into a $\mathbf{T}_1$ pre-apartness space is strongly extensional.*

Proof. Let $f : X \rightarrow Y$ be nearly continuous, where X is an apartness space and Y is a $\mathbf{T}_1$ pre-apartness space. Consider $x, x' \in X$ such that $f(x) \neq f(x')$. Define

$$A \equiv \{z \in X : z = x \vee (z = x' \wedge x \neq x')\}.$$

Note that $x \in A$. Consider any $U \subset X$ such that $x' \in -U$; by **A5**, either $x \neq x'$ and therefore $x' \in A - U$, or else $x \in -U$ and so $x \in A - U$. It follows that $\text{near}(x', A)$. Using the near continuity of f , we obtain $\text{near}(f(x'), f(A))$. Since also $f(x) \neq f(x')$, Proposition 2.1.23 shows there exists $z \in A$ such that $f(z) \neq f(x)$. Then $\neg(z = x)$, so we must have $z = x'$ and $x \neq x'$. ■

Proposition 2.3.3 *The following conditions are equivalent on a mapping $f : X \rightarrow Y$ between pre-apartness spaces.*

- (i) f is nearly continuous.
- (ii) For each nearly closed subset S of Y , $f^{-1}(S)$ is nearly closed.
- (iii) For each subset A of X , $f(\overline{A}) \subset \overline{f(A)}$.

Proof. Suppose that f is nearly continuous on X , and let S be a nearly closed subset of Y . If $x \in \overline{f^{-1}(S)}$, then $\text{near}(x, f^{-1}(S))$, by Proposition 2.2.17, and therefore $\text{near}(f(x), S)$. Since S is nearly closed, $f(x) \in S$; whence $x \in f^{-1}(S)$. Thus (i) implies (ii).

Now suppose that (ii) holds. Let $x \in X$ and $A \subset X$ be such that $\text{near}(x, A)$. Note that $A \subset f^{-1}(\overline{f(A)})$, so $\text{near}(x, f^{-1}(\overline{f(A)}))$, by Proposition 2.1.10. Since, by Corollary 2.2.18, $\overline{f(A)}$ is nearly closed, so is $f^{-1}(\overline{f(A)})$. Hence $x \in f^{-1}(\overline{f(A)})$, so $f(x) \in \overline{f(A)}$ and therefore $\text{near}(f(x), f(A))$, again by Proposition 2.2.17. Thus (ii) implies (i).

The equivalence of (i) and (iii) is a consequence of Proposition 2.2.17. ■

Proposition 2.3.4 *A topologically continuous mapping $f : X \rightarrow Y$ between pre-apartness spaces is nearly continuous.*

Proof. Consider $x \in X$ and $A \subset X$ such that $\text{near}(x, A)$. Let $B \subset Y$ and $f(x) \in -B$; then $x \in f^{-1}(-B)$. By the topological continuity of f , there exists a family $(A_i)_{i \in I}$ of subsets of X such that $f^{-1}(-B) = \bigcup_{i \in I} -A_i$. Choose i_0 with $x \in -A_{i_0}$. Since $\text{near}(x, A)$, there exists

$$y \in A - A_{i_0} \subset A \cap \left(\bigcup_{i \in I} -A_i \right);$$

whence

$$f(y) \in f(A) \cap f \left(\bigcup_{i \in I} -A_i \right) \subset f(A) - B.$$

Since B is arbitrary, we conclude that $\text{near}(f(x), f(A))$. ■

Corollary 2.3.5 *Every topologically continuous mapping of an apartness space into a $\mathbf{T}_1$ pre-apartness space is strongly extensional.*

Proof. Apply Propositions 2.3.4 and 2.3.2. ■

Proposition 2.3.6 *Let X be a pre-apartness space whose apartness topology has the reverse Kolmogorov property, and let f be a topologically continuous mapping of X into a pre-apartness space Y . Then f is continuous.*

Proof. Given $A \subset X$ and writing

$$B \equiv f^{-1}(-f(A)),$$

we see that $A \cap B = \emptyset$ and, by topological continuity, that $B = \bigcup_{i \in I} -V_i$ for some family $(V_i)_{i \in I}$ of subsets of X . It follows from Proposition 2.2.19 that $\neg B = \sim B$. For each i we therefore have

$$A \subset \neg B = \sim B \subset \sim -V_i$$

and therefore

$$-V_i \subset \sim\sim -V_i \subset \sim A.$$

Applying **A4**, we obtain $-V_i \subset -A$. Hence $B \subset -A$, and therefore f is continuous. ■

In order to obtain a partial converse to Proposition 2.3.6, we introduce the following **weak nested neighbourhoods property** for a pre-apartness space X .

$$\textbf{WNN: } x \in -A \Rightarrow \exists_{B \subset X} (x \in -B \wedge (\neg B \subset -A)).$$

This property is a simple consequence of local decomposability. It captures the idea that inside every basic neighbourhood of a point, relative to the apartness topology, there should be a strictly smaller neighbourhood of that point. In Chapter 3, when we have introduced a notion of apartness between sets, we shall deal with a stronger nested neighbourhoods property than this weak one. For our present purpose, though, the latter is certainly adequate.

Proposition 2.3.7 *Let X be a pre-apartness space, and Y a pre-apartness space with the weak nested neighbourhoods property. Then every continuous function $f : X \rightarrow Y$ is topologically continuous.*

Proof. Let $S \equiv \bigcup_{i \in I} -A_i$ be a nearly open subset of Y , and consider any $x \in f^{-1}(S)$. Choose $i \in I$ such that $f(x) \in -A_i$. By the weak nested neighbourhoods property, there exists $B \subset Y$ such that

$$f(x) \in -B \subset \neg B \subset -A_i.$$

It follows from this and the continuity of f that

$$x \in -f^{-1}(B) \subset f^{-1}(\neg B) \subset f^{-1}(-A_i) \subset f^{-1}(S).$$

Hence $f^{-1}(S)$ is a union of apartness complements in X and is therefore nearly open. ■

Propositions 2.3.6 and 2.3.7 now yield

Corollary 2.3.8 *Let X be f be a mapping of an apartness space X into a pre-apartness space Y with the weak nested neighbourhoods property. Then f is continuous if and only if it is topologically continuous.*

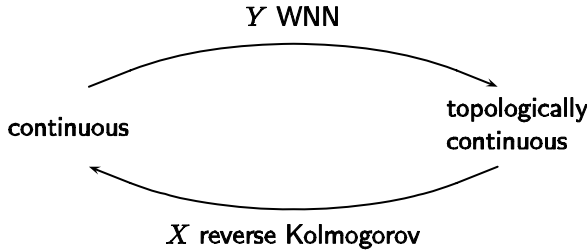
We end the section with a general type of topological space that has the weak nested neighbourhoods property. A pre-apartness space X is said to be **completely regular** if for each $x \in X$ and each $A \subset X$ with $x \bowtie A$, there exists a continuous function $\phi : X \rightarrow [0, 1]$ such that $\phi(x) = 0$ and $\phi(A) \subset \{1\}$. In that case, X is actually an apartness space. Every metric apartness space is completely regular.

Proposition 2.3.9 *A completely regular apartness space has the weak nested neighbourhoods property.*

Proof. Let X be completely regular, and let $x \in -A$ in X . There exists a continuous function $\phi : X \rightarrow [0,1]$ such that $\phi(x) = 0$ and $\phi(A) \subset \{1\}$. Let $B \equiv \phi^{-1}(\frac{1}{2}, 1]$. Since $\rho(\phi(x), \phi(y)) > 1/2$ for each $y \in B$, the continuity of ϕ ensures that $x \in -B$. Moreover, for each $z \in \neg B$ we have $\phi(z) \leq 1/2$, so $\phi(z) \bowtie \phi(A)$ and therefore, again by the continuity of ϕ , $z \in -A$. Hence $\neg B \subset -A$. ■

We see from this and Corollary 2.3.8 that for mappings from an apartness space into a completely regular space, continuity and topological continuity are equivalent.

The following diagram summarises the main connections between types of continuity for functions between pre-apartness spaces.



2.4 Limits

How do we fit convergence and limits into our framework? We first need to introduce nets as a generalisation of sequences.

Let $\mathfrak{D}$ be an inhabited set. By a **preorder** on $\mathfrak{D}$ we mean a binary relation on $\mathfrak{D}$ that is both reflexive,

$$\forall x \in \mathfrak{D} (x \succcurlyeq x),$$

and transitive,

$$\forall x, y, z \in \mathfrak{D} ((x \succcurlyeq y \wedge y \succcurlyeq z) \Rightarrow x \succcurlyeq z).$$

We call the pair $(\mathfrak{D}, \succcurlyeq)$ —or, when it is clear which partial order we are dealing with, $\mathfrak{D}$ itself—a **directed set** if for all $m, n \in \mathfrak{D}$, there exists $p \in \mathfrak{D}$ such that $p \succcurlyeq m$ and $p \succcurlyeq n$. A **net** in a set X is a mapping $n \rightsquigarrow x_n$ of such a set $\mathfrak{D}$ —the **index set**—into X , and is normally denoted by $(x_n)_{n \in \mathfrak{D}}$. If f is a mapping of X into a set Y , then $(f(x_n))_{n \in \mathfrak{D}}$ is a net in Y . A sequence in X is just a special case of a net in which the index set is the set $\mathbb{N}^+$.

To each x in a pre-apartness space X there correspond two special nets defined as follows. Let

$$\mathfrak{D}_x \equiv \{(\xi, U) : x \in -U \wedge \xi \in -U\},$$

with equality² defined by

$$(\xi, U) = (\xi', U') \Leftrightarrow (\xi = \xi' \wedge -U = -U').$$

For each $n \equiv (\xi, U)$ in $\mathfrak{D}_x$ define $x_n \equiv \xi$. It is easy to see that $\mathfrak{D}_x$ is a directed set under the **reverse inclusion preorder** defined by

$$(\xi, U) \succcurlyeq (\xi', U') \Leftrightarrow -U \subset -U',$$

so that $\mathcal{N}_x \equiv (x_n)_{n \in \mathfrak{D}_x}$ is a net—the **basic neighbourhood net** of x . Similarly,

$$\mathfrak{D}'_x \equiv \{(\xi, U) : x \in -U \wedge \xi \in -U \wedge \xi \neq x\}$$

is a directed set, and $\mathcal{N}'_x \equiv (x_n)_{n \in \mathfrak{D}'_x}$ is a net—the **basic punctured neighbourhood net** of x .

For convenience we introduce here a simple but valuable lemma.

Lemma 2.4.1 *Let X be a pre-apartness space, x a point of X , U a subset of X with $x \in -U$, and $\nu \equiv (\xi, U)$. Then*

$$-U = \{x_n : n \in \mathfrak{D}_x, n \succcurlyeq \nu\}. \quad (2.15)$$

If $\text{near}(x, X \sim \{x\})$, then

$$-U \cap \sim \{x\} = \{x_n : n \in \mathfrak{D}'_x, n \succcurlyeq \nu\}. \quad (2.16)$$

Proof. If $n \equiv (x_n, V) \succcurlyeq \nu$, then $x_n \in -V \subset -U$. Hence

$$\{x_n : n \succcurlyeq \nu\} \subset -U.$$

On the other hand, for each $y \in -U$ we have $(y, U) \succcurlyeq \nu$, so $y \in \{x_n : n \succcurlyeq \nu\}$. Both (2.15) and (2.16) readily follow from this. ■

In view of the standard elementary notion of convergence of sequences in a metric space and of the definition of the apartness topology, it makes sense to say that a net $(x_n)_{n \in \mathfrak{D}}$ in a pre-apartness space X **converges** to a **limit** x in X if

$$\forall_{U \subset X} (x \in -U \Rightarrow \exists_{N \in \mathfrak{D}} \forall_{n \succcurlyeq N} (x_n \in -U)).$$

We also say that the net is **apartness convergent** to x . It follows from Lemma 2.4.1 that the basic neighbourhood net of a point in a pre-apartness space converges to that point.

²We do not need to bother with the natural inequality on either $\mathfrak{D}_x$ or $\mathfrak{D}'_x$, so we leave the definition of this to the reader.

Proposition 2.4.2 *Let X be a pre-apartness space with the weak nested neighbourhoods property. Then the net $s \equiv (x_n)_{n \in \mathfrak{D}}$ converges to x in X if and only if*

$$\forall_{B \subset \mathfrak{D}} (x \bowtie s(B) \Rightarrow \exists_{N \in \mathfrak{D}} (B \subset \neg \{n : n \succcurlyeq N\})). \quad (2.17)$$

Proof. Suppose that s converges to x in X . Let $B \subset \mathfrak{D}$ and $x \bowtie s(B)$. By the weak nested neighbourhoods property, there exists $U \subset X$ such that $x \in -U$ and $\neg U \subset -s(B) \subset \neg s(B)$; whence

$$s(B) \subset \neg \neg s(B) \subset \neg \neg U \subset \neg -U.$$

Choosing N such that $x_n \in -U$ for all $n \succcurlyeq N$, we now see that

$$B \subset \neg \{n : n \succcurlyeq N\}.$$

Now suppose, conversely, that (2.17) holds. If $x \in -U$, we apply the weak nested neighbourhoods property, to obtain $V \subset X$ such that $x \in -V$ and $\neg V \subset -U$. With

$$B \equiv \{n \in \mathfrak{D} : x_n \in V\},$$

we see that $x \bowtie s(B)$; so there exists $N \in \mathfrak{D}$ such that

$$B \subset \neg \{n : n \succcurlyeq N\}.$$

If $n \succcurlyeq N$, then $x_n \in \neg V \subset -U$. ■

A topological space (X, τ) has an associated notion of topological convergence: a net $(x_n)_{n \in \mathfrak{D}}$ in X is said to **converge topologically** to $x \in X$ if for each neighbourhood U of x in X there exists $N \in \mathfrak{D}$ such that $x_n \in U$ for all $n \succcurlyeq N$. Since the $\bowtie_\tau$ -nearly open subsets of X are τ -open, we see that if a net in X converges topologically to a limit $x \in X$, then it converges to x relative to the topological pre-apartness on X .

Proposition 2.4.3 *The following are equivalent conditions on a topological pre-apartness space (X, τ) :*

- (i) *Apartness convergence is equivalent to topological convergence.*
- (ii) *Apartness convergence implies topological convergence.*
- (iii) *X is topologically consistent.*

Proof. In view of the remark immediately preceding this proposition, it is clear that (i) and (ii) are equivalent.

Supposing that (ii) holds, consider a τ -open set U and any $x \in U$. Let $(x_n)_{n \in \mathfrak{D}_x}$ be the basic neighbourhood net of x , which converges to x relative to the apartness $\bowtie$ associated with τ . By (ii), this net converges topologically to x , so there exists $N \equiv (\xi, V) \in \mathfrak{D}_x$ such that $x_n \in U$ for all $n \succcurlyeq N$.

For each $y \in -V$ we have $(y, V) \succ N$, by definition of the reverse inclusion preorder; so $y = x_{(y, V)} \in U$. Hence $x \in -V \subset U$. It follows that U is a union of apartness complements and is therefore nearly open. Hence (ii) implies (iii).

The proof that (iii) implies (ii) is left as an exercise. ■

Recall that when we refer to the closure $\overline{A}$ of a subset A of a pre-apartness space X , we mean the closure of A with respect to the apartness topology on X .

Proposition 2.4.4 *The closure of a subset A of a pre-apartness space X consists of all points of X that are limits of nets in A .*

Proof. If $(x_n)_{n \in \mathfrak{D}}$ is a net in A converging to an element x of X , then, clearly,

$$\text{near}(x, \{x_n : n \in \mathfrak{D}\})$$

and therefore $\text{near}(x, A)$; whence $x \in \overline{A}$, by Proposition 2.2.17.

Conversely, given x in $\overline{A}$, let

$$\mathfrak{D} \equiv \{(y, U) : x \in -U \wedge y \in A - U\}.$$

Then $\mathfrak{D}$ is directed by the usual reverse inclusion preorder $\succ$. Let $(y_n)_{n \in \mathfrak{D}}$ be the net in A defined by the mapping $(y, U) \rightsquigarrow y$, and let $U \subset X$ be such that $x \in -U$. Since $\text{near}(x, A)$ (by Proposition 2.2.17), there exists $y \in A - U$; let $n_0 \equiv (y, U) \in \mathfrak{D}$. For each $n \equiv (y_n, V)$ in $\mathfrak{D}$ with $n \succ n_0$ we have $x \in -V$ and $y_n \in A - V \subset -V \subset -U$. Thus $(y_n)_{n \in \mathfrak{D}}$ converges to x . ■

Since it is possible for a net in a pre-apartness space to have more than one limit, it is reasonable to look for a characterisation of those pre-apartness spaces in which a convergent net has a unique limit. With an eye on classical topology, we say that a topological space (X, τ) , or just its topology τ , is **(topologically) Hausdorff** if the following condition holds:

For all x, y in X with $x \neq y$, there exist U, V in τ such that $x \in U, y \in V$, and $U \subset \sim V$.

On the other hand, we say that a pre-apartness space $(X, \bowtie)$, or just its pre-apartness $\bowtie$, is **(apartness) Hausdorff** if it satisfies the following condition:

For all $x, y \in X$ with $x \neq y$, there exist $U, V \subset X$ such that $x \in -U, y \in -V$, and $-U \subset \sim -V$.

In that case, $-V \subset \sim -U$.

Note that for an apartness to be Hausdorff, it suffices that the following hold:

If $x, y \in X$ and $x \neq y$, then there exist $U, V \subset X$ such that $x \in -U, y \in -V$, and $-U \cap -V = \emptyset$.

For if we have such U and V , then for all $s \in -U$ and $t \in -V$, either $s \neq t$ or $s \in -V$, by **A5**; but the latter is ruled out, since $-U \cap -V = \emptyset$. Hence $-U \subset \sim -V$.

It should be clear that a pre-apartness is Hausdorff if and only if the corresponding apartness topology is Hausdorff.

Proposition 2.4.5 *Let (X, τ) be a topological space. If X is topologically Hausdorff, then $\bowtie_\tau$ is Hausdorff. Conversely, if $\bowtie_\tau$ is Hausdorff and X is topologically consistent, then X is topologically Hausdorff.*

Proof. Suppose first that X is topologically Hausdorff, and consider points $x, y \in X$ with $x \neq y$. Pick open subsets A, B of X with $x \in A, y \in B$, and $A \subset \sim B$. Since $x \in A \subset \sim \sim A$, we have $x \in -_\tau \sim A$, where $-_\tau$ denotes the apartness complement corresponding to $\bowtie_\tau$. Similarly, $y \in -_\tau \sim B$. Moreover,

$$-_\tau \sim B \subset -_\tau A \subset \sim A,$$

so

$$-_\tau \sim A \subset -_\tau -_\tau \sim B \subset \sim -_\tau \sim B.$$

Hence $\bowtie_\tau$ is Hausdorff.

Now suppose, conversely, that $\bowtie_\tau$ is Hausdorff. The remark preceding this proposition shows that the apartness topology corresponding to $\bowtie_\tau$ is Hausdorff. It follows that if (X, τ) is topologically consistent, then it is topologically Hausdorff. ■

Proposition 2.4.6 *A Hausdorff pre-apartness space is $\mathbf{T}_1$.*

Proof. Let X be a Hausdorff pre-apartness space, and let x, y be points of X with $x \neq y$. There exist $U, V \subset X$ such that $x \in -U \subset \sim -V$ and $y \in -V$. Applying **A4**, we see that $-U \subset - -V$, so $x \bowtie -V$. Since $\{y\} \subset -V$, it follows that $x \bowtie \{y\}$. ■

Classically, being Hausdorff is equivalent to having the **unique limits property**,

ULP: *If $(x_n)_{n \in \mathcal{D}}$ is a net converging to limits x and y in X , then $x = y$.*

From a constructive viewpoint, the unique limits property appears rather weak. To introduce a stronger constructive property, we say that a point y in X is **eventually bounded away from a net** $(x_n)_{n \in \mathcal{D}}$ in X if there exists $n_0 \in \mathcal{D}$ such that

$$y \in -\{x_n : n \succcurlyeq n_0\}.$$

We now state the **strong unique limits property** (classically equivalent to the unique limits property),

SULP: *If $(x_n)_{n \in \mathcal{D}}$ is a net in X that converges to a limit x , and if $x \neq y$ in X , then $(x_n)_{n \in \mathcal{D}}$ is eventually bounded away from y .*

It is straightforward to show that in the presence of a tight inequality, **SULP** implies **ULP**.

Not surprisingly (in view of classical topology), the Hausdorff condition is linked to our two types of uniqueness of limits.

Proposition 2.4.7 *A pre-apartness space is Hausdorff if and only if it has the strong unique limits property.*

Proof. Let X be a pre-apartness space. Assume first that X is Hausdorff, let $(x_n)_{n \in \mathfrak{D}}$ be a net converging to a limit x in X , and let $x \neq y$ in X . Choose U, V such that $x \in -U, y \in -V$, and $-U \subset \sim -V$. There exists n_0 such that $x_n \in -U$ for all $n \succcurlyeq n_0$. Then

$$y \in -V \subset \sim -U \subset \sim \{x_n : n \succcurlyeq n_0\},$$

so, by **A4**,

$$y \in -\{x_n : n \succcurlyeq n_0\}.$$

Hence **SULP** holds in X .

Now suppose, conversely, that X has the strong unique limits property. Let x, y be points of X with $x \neq y$. Since the net $\mathcal{N}_x$ converges to x , **SULP** ensures that there exists $n_0 \equiv (\xi, U) \in \mathfrak{D}_x$ such that

$$y \in -\{x_n : n \in \mathfrak{D}_x, n \succcurlyeq n_0\}.$$

By the definition of $\mathfrak{D}_x$ and Lemma 2.4.1,

$$x \in -U = \{x_n : n \in \mathfrak{D}_x, n \succcurlyeq n_0\}.$$

It follows that $y \in - -U \subset \sim -U$. Hence X is Hausdorff. ■

From this proposition and a remark just before it, we obtain

Corollary 2.4.8 *A Hausdorff pre-apartness space with tight inequality has the unique limits property.*

Corollary 2.4.8 has a noteworthy converse.

Proposition 2.4.9 *In an apartness space with the unique limits property the inequality is tight.*

Proof. Let X be such an apartness space, and let x, y be points of X with $\neg(x \neq y)$. Using this last property and **A5**, we see that

$$\forall_{U \subset X} (x \in -U \Leftrightarrow y \in -U).$$

It readily follows that the net $\mathcal{N}_x$ converges to both x and y ; whence, by **ULP**, $x = y$. ■

It is worth taking a little time out of the main development to show that the connections established in the preceding three results are the best possible within our constructive framework. We begin by showing that Hausdorff/**SULP** is not enough to establish tightness.

Proposition 2.4.10 *If every apartness space that has the strong unique limits property (or, equivalently, is Hausdorff) has tight inequality, then the law of excluded middle holds.*

Proof. Let P be any statement such that $\neg\neg P$ holds, and take $X \equiv \{0, 1, 2\}$ with equality satisfying

$$0 = 1 \Leftrightarrow P$$

and inequality given by

$$0 \neq 2, 1 \neq 2, \text{ and } (0 \neq 1 \Leftrightarrow \neg P).$$

Define a point-set pre-apartness $\bowtie$ on X by

$$x \bowtie A \Leftrightarrow x \in \sim A.$$

We show that $\bowtie$ is, in fact, an apartness. Consider all possible cases that arise when $x \in -A$. If $x = 0$, then $0 \in \sim A$. It follows that $A \subset \{2\}$: for if $y \in A$, then either $y = 1$ or $y = 2$; in the former case, $0 \neq 1$ (since $0 \in \sim A$) and therefore $\neg P$ holds, which contradicts our hypotheses. Since $1 \neq 2$, we have $1 \in \sim A$; since also $0 \neq 2$, we conclude that

$$\forall y \in X (0 \neq y \vee y \in \sim A = -A).$$

The case $x = 1$ is similar, and the case $x = 2$ is even easier to handle.

We claim that X is a Hausdorff apartness space and hence, by Proposition 2.4.7, has the strong unique limits property. If $x \neq y$, then without loss of generality, either $x = 0$ and $y = 2$, or else $x = 1$ and $y = 2$. Taking, for illustration, the former case, we have

$$0 \in \{0, 1\} = \sim \{2\} = -\{2\},$$

$$2 \in \{2\} = \sim \{0\} = -\{0\},$$

and

$$\sim \{2\} = \sim \sim \{0\}.$$

Thus there exist $U \equiv \{2\}$ and $V \equiv \{0\}$ such that $0 \in -U$, $2 \in -V$, and $-U \subset \sim -V$.

Finally, if the inequality on X is tight, then as $\neg(0 \neq 1)$, we have $0 = 1$ and therefore P . ■

Next we aim to show that even for a locally decomposable³ apartness space, the unique limits property does not entail being Hausdorff. For the proof we introduce a strange lemma and a general construction. The lemma may seem obvious, but we need it in order to avoid the axiom of choice.

³The reader should by now have realised that local decomposability is more than handy for tidying up loose ends in cases like this; but one should beware of putting too much trust in it, since there are situations for which even local decomposability is not strong enough to enable us to recover constructively the full form of a classical theorem about apartness.

Lemma 2.4.11 *Let $\mathcal{C}$ be a class of subsets of a set X , and let $(S_i)_{i \in I}$ be a family of subsets of X such that for each i , if S_i is inhabited, then it is a union of sets in $\mathcal{C}$. If $S \equiv \bigcup_{i \in I} S_i$ is inhabited, then it is also a union of sets in $\mathcal{C}$.*

Proof. It is straightforward to verify that

$$S = \bigcup \{U \in \mathcal{C} : \exists x \in X \exists i \in I (x \in U \subset S_i)\},$$

which gives exactly what we want. ■

Let X be a set with an inequality $\neq$ that is **cotransitive**, in the sense that

$$\forall x, y \in X (x \neq y \Rightarrow \forall z \in X (x \neq z \vee z \neq y)).$$

We say that a subset S of X is **cofinite** if it is the complement of a finitely enumerable subset. If X has at least two distinct points, then we define the **cofinite topology** on X to be

$$\tau_{\text{cof}} \equiv \{S \subset X : S \neq \emptyset \Rightarrow S \text{ is a union of cofinite sets}\}.$$

To see that this is a topology, first pick distinct points x, y of X . By cotransitivity,

$$X = \sim\{x\} \cup \sim\{y\},$$

so $X \in \tau_{\text{cof}}$. Also, by *ex falso quodlibet*, $\emptyset \in \tau_{\text{cof}}$. The unions axiom for a topology is an immediate consequence of Lemma 2.4.11 with $\mathcal{C} \equiv \tau_{\text{cof}}$. To verify the intersections axiom, let $(A_i)_{i \in I}$ and $(B_j)_{j \in J}$ be families of finitely enumerable subsets of X , and consider the elements

$$S \equiv \bigcup_{i \in I} \sim A_i, T \equiv \bigcup_{j \in J} \sim B_j$$

of τ_{cof} . We have

$$\begin{aligned} S \cap T &= \left(\bigcup_{i \in I} \sim A_i \right) \cap \left(\bigcup_{j \in J} \sim B_j \right) \\ &= \bigcup_{i \in I} \bigcup_{j \in J} (\sim A_i \cap \sim B_j) \\ &= \bigcup_{i \in I} \bigcup_{j \in J} \sim (A_i \cup B_j), \end{aligned}$$

where each of the sets $A_i \cup B_j$ is finitely enumerable. Hence $S \cap T \in \tau_{\text{cof}}$.

Proposition 2.4.12 *If every locally decomposable, $\mathbf{T}_1$ apartness space that contains two distinct points and has the unique limits property is Hausdorff, then Markov's principle holds.*

Proof. We take a specific case of the foregoing construction. Let $(a_n)_{n \geq 1}$ be a decreasing binary sequence such that

$$a_1 = 1 \wedge \neg \forall_n (a_n = 1).$$

Take

$$X \equiv \{0\} \cup \left\{ \frac{a_n}{n} : n = 1, 2, 3, \dots \right\}$$

with the discrete equality and inequality, let τ be the cofinite topology on X , and let $\bowtie$ be the corresponding pre-apartness. To show that X is topologically locally decomposable and hence (by Corollary 2.2.16 and Proposition 2.2.13) an apartness space, consider $x \in X$ and $U \in \tau$ with $x \in U$. Since X contains two distinct points, we may assume that $U = \sim A$ for some finitely enumerable (and hence, in this case, finite) set $A \subset X$; without loss of generality we may assume that A is inhabited. Consider first the case $x = 0$, in which

$$\emptyset \neq A \subset \left\{ \frac{1}{n} : n = 1, 2, 3, \dots \right\}.$$

Let

$$K \equiv \max \left\{ k : \frac{1}{k} \in A \right\} \quad (2.18)$$

and

$$V \equiv \{0\} \cup \left\{ \frac{a_k}{k} : k > K \right\} = \sim \left\{ \frac{a_k}{k} : k \leq K \right\}.$$

Then V is a neighbourhood of 0. For each $y \in X$, either $y = 0 \in U$ or else $y = 1/k$ for some k with $a_k = 1$. In the latter case, if $k > K$, then $y \in \sim A = U$; whereas if $k \leq K$, then $y \in \sim V$. This deals with the case $x = 0$. Now consider the case where $x = 1/m$ for some m with $a_m = 1$. If $a_{m+1} = 0$, then X is finite and hence topologically locally decomposable; so we may assume that $a_{m+1} = 1$. Since $x \neq 1/(m+1)$, we may further assume that $1/(m+1) \in A$. Thus $K > m$, where K is defined as at (2.18). Set

$$\begin{aligned} W &\equiv \left\{ \frac{a_k}{k} : (k > K \wedge a_k = 1) \vee k = m \right\} \\ &= \sim \left(\{0\} \cup \left\{ \frac{a_k}{k} : m \neq k \leq K \right\} \right). \end{aligned}$$

Then W is a neighbourhood of x . For each $y \in X$, either $y = 0$ and hence $y \in \sim W$, or else $y = a_k/k$ for some k with $a_k = 1$. If $k > K$, then $y \in \sim A = U$; if $k = m$, then $y = x \in U$; if $m \neq k \leq K$, then $y \in \sim W$. Thus, taking all the various cases together, we see that the pre-apartness space X is topologically locally decomposable.

We now confirm that the cofinite topology is $\mathbf{T}_1$. To this end, let $x \neq y$ in X . Either one of the points x and y is 0 or else both are nonzero. If, for example, $x = 0$, then $y = 1/n$ for some n with $a_n = 1$. Writing

$$U \equiv \{0\} \cup \left\{ \frac{a_k}{k} : k > n \right\} = \sim \left\{ \frac{a_k}{k} : k \leq n \right\},$$

we see that

$$U \in \tau \text{ and } x \in U \subset \sim \{y\}. \quad (2.19)$$

So we are left with the case where $x = 1/m$ and $y = 1/n$, with $a_m = a_n = 1$. In this case, without loss of generality taking $m > n$, we obtain (2.19) by defining

$$U \equiv \{0\} \cup \left\{ \frac{a_k}{k} : k \geq m \right\}.$$

Next, we prove that X has the unique limits property. Suppose that $(x_n)_{n \in \mathfrak{D}}$ is a net in X that converges to both x and y . Suppose also that $x \neq y$. For each n , if $a_n = 0$, then X is finite and so has the unique limits property; whence $x = y$, a contradiction. Thus $a_n = 1$ for all n , which is also a contradiction. We conclude that $\neg(x \neq y)$; since we are dealing with a discrete inequality, it follows that $x = y$.

Finally, noting that $0 \neq 1$, suppose there exist $U, V \subset X$ such that $0 \in -U, 1 \in -V$, and $-U \cap -V = \emptyset$. Pick finitely enumerable sets $A, B \subset X$ such that $0 \in \sim A \subset -U$ and $1 \in \sim B \subset -V$. Let

$$N \equiv \max \left\{ n : \frac{1}{n} \in A \cup B \right\}.$$

If $a_{N+1} = 1$, then

$$\frac{1}{N+1} \in \sim A \cap \sim B \subset -U \cap -V,$$

a contradiction; hence $a_{N+1} = 0$. Recalling from page 43 the remark about the Hausdorff property in an *apartness* space, we see that if X is Hausdorff, then there exists N such that $a_n = 0$ for all $n > N$. ■

Having dealt with the convergence of nets, we now turn to the convergence of functions. Let X, Y be pre-apartness spaces, x a point of X such that near $(x, \sim \{x\})$, and f a mapping of $\sim \{x\}$ into Y . We say that $y \in Y$ is a **limit of f at x** , or a **limit of $f(t)$ as t tends to x** , if the net $f(\mathcal{N}'_x)$ converges to y in Y . We then write

$$f(t) \rightarrow y \text{ as } t \rightarrow x.$$

In the case where the limit of $f(\mathcal{N}'_x)$ is unique, we also write

$$\lim_{t \rightarrow x, t \in X} f(t) = y$$

or just

$$\lim_{t \rightarrow x} f(t) = y.$$

Proposition 2.4.13 *Let X and Y be pre-apartness spaces, x a point of X such that near $(x, \sim \{x\})$, y a point of Y , and f a mapping of $\sim \{x\}$ into Y . Then the following conditions are equivalent:*

- (i) y is a limit of f at x .

$$(ii) \quad \forall_{V \subset Y} (y \in -V \Rightarrow \exists_{U \subset X} (x \in -U \wedge f(-U \cap \sim \{x\}) \subset -V)).$$

Proof. Let $y \in -V \subset Y$. Suppose that y is a limit of f at x ; then there exists $n_0 \in \mathfrak{D}'_x$ such that $f(x_n) \in -V$ for all $n \succ n_0$. Writing $n_0 \equiv (\xi, U)$, we have $x \in -U$; moreover, by Lemma 2.4.1,

$$\{x_n : n \succ n_0\} = -U \cap \sim \{x\}.$$

It follows that

$$f(-U \cap \sim \{x\}) \subset -V. \quad (2.20)$$

Thus (i) implies (ii).

Conversely, assume (ii). With y and V as before, choose $U \subset X$ such that $x \in -U$ and (2.20) holds. Since near $(x, \sim \{x\})$, there exists $\xi \in -U \cap \sim \{x\}$. Let $\nu \equiv (\xi, U) \in \mathfrak{D}'_x$. By Lemma 2.4.1,

$$-U \cap \sim \{x\} = \{x_n : n \in \mathfrak{D}'_x, n \succ \nu\}.$$

It follows from (2.20) that $f(x_n) \in -V$ for all $n \succ \nu$. Hence y is a limit of f at x . ■

Proposition 2.4.14 *Let X, Y, Z be pre-apartness spaces such that Z has the weak nested neighbourhoods property. Let $x \in X$ and $y \in Y$, let f be a mapping of $\sim \{x\}$ into Y , and let g be a mapping of Y into Z . Suppose that y is a limit of f at x , and that g is continuous at y . Then $g(y)$ is a limit of $g \circ f$ at x .*

Proof. Let $g(y) \in -W \subset Z$. Using **WNN**, choose $A \subset Z$ such that $g(y) \in -A$ and $\neg A \subset -W$. Then, by continuity, $y \in -g^{-1}(A) \subset Y$, so, by Proposition 2.4.13, there exists $U \subset X$ such that $x \in -U$ and $f(-U \cap \sim \{x\}) \subset -g^{-1}(A)$. Since near $(x, \sim \{x\})$, there exists $\xi \in -U$ such that $\xi \neq x$. Then $n_0 \equiv (\xi, U)$ belongs to $\mathfrak{D}'_x$; also, by Lemma 2.4.1, for each $n \succ n_0$ in $\mathfrak{D}'_x$ we have $x_n \in -U \cap \sim \{x\}$ and therefore $f(x_n) \in -g^{-1}(A)$; whence $(g \circ f)(x_n) \notin A$ and therefore $(g \circ f)(x_n) \in -W$. ■

Proposition 2.4.15 *Let X, Y be pre-apartness spaces, $x \in X$, $y \in Y$, and f a mapping of $\sim \{x\}$ into Y such that y is a limit of f at x . Then f preserves the convergence of nets at x : that is, for each net $(x_n)_{n \in \mathfrak{D}}$ in $\sim \{x\}$ that converges to x in X , the net $(f(x_n))_{n \in \mathfrak{D}}$ converges to y .*

Proof. Let $(x_n)_{n \in \mathfrak{D}}$ be a net in $\sim \{x\}$ that converges to x in X , and let $y \in -V \subset Y$. By Proposition 2.4.13, there exists $U \subset X$ such that $x \in -U$ and $f(-U \cap \sim \{x\}) \subset -V$. Choose n_0 in $\mathfrak{D}$ such that $x_n \in -U$ for all $n \succ n_0$ in $\mathfrak{D}$. For such n we have $x_n \in -U \cap \sim \{x\}$ and therefore $f(x_n) \in -V$. ■

Proposition 2.4.16 *Let X, Y be pre-apartness spaces, f a topologically continuous mapping of X into Y , and $(x_n)_{n \in \mathfrak{D}}$ a net that converges to a limit x in X . Then the net $(f(x_n))_{n \in \mathfrak{D}}$ converges in Y to $f(x)$.*

Proof. Let $f(x) \in -V$ in Y ; then $f^{-1}(-V)$ is nearly open in V . Pick $U \subset X$ such that $x \in -U \subset f^{-1}(-V)$. There exists $n_0 \in \mathfrak{D}$ such that $x_n \in -U$, and therefore $f(x_n) \in -V$, for all $n \succ n_0$. ■

Corollary 2.4.17 *Let X be an apartness space, Y a pre-apartness space with the weak nested neighbourhoods property, $f : X \rightarrow Y$ a continuous mapping, and $(x_n)_{n \in \mathfrak{D}}$ a net that converges to a limit x in X . Then the net $(f(x_n))_{n \in \mathfrak{D}}$ converges in Y to $f(x)$.*

Proof. Apply Corollary 2.3.8 and Proposition 2.4.16. ■

Limits will re-appear in Chapter 3, in the context of complete pre-apartness spaces.

2.5 Product Pre-apartness Spaces

Let X_1 and X_2 be pre-apartness spaces, let $X \equiv X_1 \times X_2$, and recall our convention that, for example, $\mathbf{x}$ denotes the element (x_1, x_2) of X . We define the relation $\bowtie$ between points and subsets of X as follows:

$$\mathbf{x} \bowtie A \Leftrightarrow \exists U_1 \subset X_1 \exists U_2 \subset X_2 (\mathbf{x} \in -U_1 \times -U_2 \subset \sim A), \quad (2.21)$$

where for $k = 1, 2$ the set $-U_k$ is the apartness complement of U_k in the pre-apartness space X_k .

Proposition 2.5.1 *The relation $\bowtie$ defined at (2.21) is a pre-apartness on $X \equiv X_1 \times X_2$.*

Proof. For each $\mathbf{x} \in X$, since

$$\mathbf{x} \in X_1 \times X_2 = -\emptyset \times -\emptyset = \sim \emptyset,$$

we see that $\mathbf{x} \bowtie \emptyset$. It is clear from (2.21) that $-A \subset \sim A$ in $(X, \bowtie)$. If $\mathbf{x} \bowtie A \cup B$ in X , then there exist $U_k \subset X_k$ such that

$$\mathbf{x} \in -U_1 \times -U_2 \subset \sim (A \cup B) = \sim A \cap \sim B,$$

so $\mathbf{x} \bowtie A$ and $\mathbf{x} \bowtie B$. If, conversely, $\mathbf{x} \bowtie A$ and $\mathbf{x} \bowtie B$, then there exist $U_k, V_k \subset X_k$ such that

$$\mathbf{x} \in -U_1 \times -U_2 \subset \sim A,$$

$$\mathbf{x} \in -V_1 \times -V_2 \subset \sim B,$$

and therefore

$$\mathbf{x} \in (-U_1 \cap -V_1) \times (-U_2 \cap -V_2) \subset \sim A \cap \sim B.$$

Referring to **A3** in the space X_k , we now see that

$$\mathbf{x} \in -(U_1 \cup V_1) \times -(U_2 \cup V_2) \subset \sim(A \cup B).$$

Hence $\mathbf{x} \bowtie A \cup B$. This completes the verification of **A3** in X .

Finally, if $-U_1 \times -U_2 \subset \sim A$, then the definition (2.21) shows that $-U_1 \times -U_2 \subset -A$; so if also $-A \subset \sim B$, then $-U_1 \times -U_2 \subset \sim B$ and therefore $-U_1 \times -U_2 \subset -B$. It follows from this that **A4** holds in X . ■

We call the pre-apartness defined at (2.21) the **product pre-apartness** on X . Equipped with the usual inequality (see page 10) and the foregoing pre-apartness structure, X is known as the **product of the pre-apartness spaces** X_1 and X_2 . The corresponding nearness on X is then given by

$$\text{near}(\mathbf{x}, A) \Leftrightarrow \forall_{B \subset X} (\mathbf{x} \bowtie B \Rightarrow \exists_{y \in A} (y \bowtie B)).$$

Our next aim is to show that some of the most important properties hold in the product pre-apartness space if and only if they hold in each of its factors. We first prove two key lemmas, for which we note that (as is easily demonstrated) if $A_k \subset X_k$, then

$$\sim A_1 \times X_2 = \sim(A_1 \times X_2)$$

and

$$X_1 \times \sim A_2 = \sim(X_1 \times A_2).$$

Lemma 2.5.2 *Let $X \equiv X_1 \times X_2$ be the product of two pre-apartness spaces, and let $A_k \subset X_k$ ($k = 1, 2$). Then $-A_1 \times X_2 = -(A_1 \times X_2)$ and $X_1 \times -A_2 = -(X_1 \times A_2)$.*

Proof. Since $-A_1 \times X_2 = -A_1 \times -\emptyset$ and

$$-A_1 \times X_2 \subset \sim A_1 \times X_2 = \sim(A_1 \times X_2),$$

the definition of the product apartness shows that $-A_1 \times X_2 \subset -(A_1 \times X_2)$. Conversely, given (x_1, x_2) in $-(A_1 \times X_2)$, we can find subsets U_k of X_k such that

$$(x_1, x_2) \in -U_1 \times -U_2 \subset \sim(A_1 \times X_2) = \sim A_1 \times X_2$$

and therefore $x_1 \in -U_1 \subset \sim A_1$. It now follows from **A4** in the space X_1 that $x_1 \bowtie A_1$. Thus $-(A_1 \times X_2) \subset -A_1 \times X_2$. The other part of the lemma is proved similarly. ■

Lemma 2.5.3 *If X_k is a pre-apartness space and $U_k \subset X_k$ ($k = 1, 2$), then*

$$-U_1 \times -U_2 = -((U_1 \times X_2) \cup (X_1 \times U_2)).$$

Proof. By Lemma 2.5.2 and **A3**,

$$\begin{aligned} -U_1 \times -U_2 &= (-U_1 \times X_2) \cap (X_1 \times -U_2) \\ &= -(U_1 \times X_2) \cap -(X_1 \times U_2) \\ &= -((U_1 \times X_2) \cup (X_1 \times U_2)), \end{aligned}$$

as we required. ■

Proposition 2.5.4 *The product pre-apartness space $X \equiv X_1 \times X_2$ satisfies **A5** if and only if both X_1 and X_2 satisfy **A5**.*

Proof. Suppose first that X satisfies **A5**, fix $x_2 \in X_2$, and let $x_1 \bowtie A_1$ in X_1 . Then, by Lemma 2.5.2,

$$(x_1, x_2) \in -A_1 \times X_2 = -(A_1 \times X_2),$$

so $(x_1, x_2) \bowtie A_1 \times X_2$ in X . Applying **A5** in X , for each $x \in X_1$ we have either $(x_1, x_2) \neq (x, x_2)$ and therefore $x_1 \neq x$, or else $(x, x_2) \bowtie A_1 \times X_2$. In the latter case, by Lemma 2.5.2, $(x, x_2) \in -A_1 \times X_2$; so $x \in -A_1$, and therefore $x \bowtie A_1$, in X_1 . This completes the proof of **A5** in the pre-apartness space X_1 . The proof for X_2 is similar.

Now suppose, conversely, that **A5** holds in both X_1 and X_2 . Let $\mathbf{x} \bowtie A$ in $X_1 \times X_2$, and choose sets $U_k \subset X_k$ such that $\mathbf{x} \in -U_1 \times -U_2 \subset \sim A$; then $x_k \in -U_k$. Consider any $\mathbf{y} \in X$. Applying **A5** in the pre-apartness space X_1 , we have either $x_1 \neq y_1$ or $y_1 \in -U_1$. Since $\mathbf{x} \neq \mathbf{y}$ in the first case, we may assume that $y_1 \in -U_1$. Likewise, we may assume that $y_2 \in -U_2$. Hence $\mathbf{y} \in -U_1 \times -U_2 \subset \sim A$ and therefore $\mathbf{y} \bowtie A$. This completes the verification of **A5** in the space X . ■

Proposition 2.5.5 *The product pre-apartness space $X \equiv X_1 \times X_2$ is **T₁** if and only if both X_1 and X_2 are **T₁**.*

Proof. Suppose that X is **T₁**, fix $x_2 \in X_2$, and let $x \neq y$ in X_1 . Then $(x, x_2) \neq (y, x_2)$ in X ; so, by the **T₁** property in X , $(x, x_2) \bowtie \{(y, x_2)\}$ and there exist $U_k \subset X_k$ such that

$$(x, x_2) \in -U_1 \times -U_2 \subset \sim \{(y, x_2)\}.$$

It follows that $x \in -U_1$ and $-U_1 \times \{x_2\} \subset \sim \{(y, x_2)\}$; thus $x \in -U_1 \subset \sim \{y\}$, and therefore $x \bowtie \{y\}$, by **A4** in X_1 . Hence X_1 , and similarly X_2 , is a **T₁** pre-apartness space.

Now suppose, conversely, that both X_1 and X_2 are **T₁** pre-apartness spaces. Let $\mathbf{x} \neq \mathbf{y}$ in X . Then either $x_1 \neq y_1$ or else $x_2 \neq y_2$. Taking, for example, the first alternative and applying the **T₁** property in X_1 , we see that $x_1 \bowtie \{y_1\}$; whence $\mathbf{x} \in -\{y_1\} \times X_2 \subset \sim \{\mathbf{y}\}$. It follows that $\mathbf{x} \bowtie \{\mathbf{y}\}$. Hence the **T₁** property holds for X . ■

Proposition 2.5.6 *The product $X \equiv X_1 \times X_2$ of two pre-apartness spaces is locally decomposable if and only if both X_1 and X_2 are locally decomposable.*

Proof. Suppose first that X is locally decomposable. In order to prove that X_1 is locally decomposable, let $x_1 \in -U_1 \subset X_1$, and pick $x_2 \in X_2$. Then (note Lemma 2.5.2)

$$\mathbf{x} \equiv (x_1, x_2) \in -U_1 \times X_2 = -(U_1 \times X_2).$$

Hence, by the local decomposability of X , there exists $T \subset X$ such that

$$\mathbf{x} \in -T \wedge \forall_{\mathbf{y} \in X} (\mathbf{y} \in -(U_1 \times X_2) \vee \mathbf{y} \in T).$$

Let

$$V_1 \equiv \{\xi \in X_1 : (\xi, x_2) \in T\}.$$

It will suffice to show that $x_1 \in -V_1$ and $X_1 = -U_1 \cup V_1$. Since $\mathbf{x} \in -T$, we can find $W_k \subset X_k$ such that $\mathbf{x} \in -W_1 \times -W_2 \subset \sim T$. For any $x \in -W_1$ and $v \in V_1$ we have $(x, x_2) \in -W_1 \times -W_2 \subset \sim T$ and $(v, x_2) \in T$; whence $x \neq v$. Thus $-W_1 \subset \sim V_1$. Since $x_1 \in -W_1$, it follows from **A4** in X_1 that $x_1 \in -V_1$, as required. On the other hand, given x in X_1 , we have either $(x, x_2) \in -(U_1 \times X_2)$ and therefore (by Lemma 2.5.2) $x \in -U_1$, or else $(x, x_2) \in T$ and so $x \in V_1$. Thus X_1 , and similarly X_2 , is locally decomposable.

Now suppose, conversely, that each X_k is locally decomposable. Consider $\mathbf{x} \in X$ and $S \subset X$ such that $\mathbf{x} \in -S$. Choose $U_1 \subset X_1$ and $U_2 \subset X_2$ such that $\mathbf{x} \in -U_1 \times -U_2 \subset \sim S$. Since $x_k \in -U_k$, there exists $V_k \subset X_k$ such that

$$x_k \in -V_k \wedge \forall_{x \in X_k} (x \in -U_k \vee x \in V_k).$$

Let

$$T \equiv (V_1 \times X_2) \cup (X_1 \times V_2).$$

Then $\mathbf{x} \in -V_1 \times -V_2 \subset -T$, by Lemma 2.5.3. On the other hand, for each $\xi \in X$, either $\xi_1 \in -U_1$ and $\xi_2 \in -U_2$, in which case $\xi \in -S$; or else we have either $\xi_1 \in V_1$ or $\xi_2 \in V_2$, and so $\xi \in T$; thus $X = -S \cup T$. Hence X is locally decomposable. ■

Proposition 2.5.7 *The product $X \equiv X_1 \times X_2$ of two pre-apartness spaces is Hausdorff if and only if both X_1 and X_2 are Hausdorff.*

Proof. Suppose that X is Hausdorff, fix $x_2 \in X_2$, and let $x \neq y$ in X_1 . Then $(x, x_2) \neq (y, x_2)$ in X ; so there exist $U, V \subset X$ such that $(x, x_2) \in -U, (y, x_2) \in -V$, and $-U \subset \sim -V$. By definition of the product pre-apartness, there exist $U_k \subset X_k$ and $V_k \subset X_k$ such that $(x, x_2) \in -U_1 \times -U_2 \subset \sim U$ and $(y, x_2) \in -V_1 \times -V_2 \subset \sim V$. Then $x \in -U_1$ and $y \in -V_1$. Moreover,

$$-U_1 \times \{x_2\} \subset -U_1 \times -U_2 \subset -U \subset \sim -V \subset \sim (-V_1 \times \{x_2\}),$$

so $-U_1 \subset \sim -V_1$. Hence X_1 , and similarly X_2 , is Hausdorff.

Now suppose, conversely, that both X_1 and X_2 are Hausdorff. Let $\mathbf{x} \neq \mathbf{y}$ in X . Then either $x_1 \neq y_1$ or else $x_2 \neq y_2$. Taking, for example, the alternative $x_1 \neq y_1$ and applying the Hausdorff property in X_1 , we obtain $U_1, V_1 \subset X_1$ such that $x_1 \in -U_1, y_1 \in -V_1$, and $-U_1 \subset \sim -V_1$. Then, by Lemma 2.5.2 and the comment preceding it,

$$\begin{aligned} \mathbf{x} &\in -U_1 \times X_2 \subset (\sim -V_1) \times X_2 \\ &= \sim (-V_1 \times X_2) = \sim - (V_1 \times X_2). \end{aligned}$$

Since (again by Lemma 2.5.2) $\mathbf{y} \in - (V_1 \times X_2)$, we conclude that X is Hausdorff. ■

Classically, for the product proximity structure on X we have $\text{near}(\mathbf{x}, A)$ if and only if $\neg(\mathbf{x} \bowtie A)$, which is equivalent to the condition

$$\forall_{U_1 \subset X_1} \forall_{U_2 \subset X_2} (\mathbf{x} \in -U_1 \times -U_2 \Rightarrow \exists_{\mathbf{y}} (\mathbf{y} \in (-U_1 \times -U_2) \cap A)). \quad (2.22)$$

Constructively, we have

Proposition 2.5.8 *Let $X \equiv X_1 \times X_2$ be a product of two pre-apartness spaces, let $\mathbf{x}$ be a point of X , and let A be a subset of X . Then $\mathbf{x}$ is near A if and only if condition (2.22) holds.*

Proof. Suppose that $\text{near}(\mathbf{x}, A)$, and consider sets $U_k \subset X_k$ such that $\mathbf{x} \in -U_1 \times -U_2$. By Lemma 2.5.3,

$$\mathbf{x} \in -((U_1 \times X_2) \cup (X_1 \times U_2)),$$

so there exists $\mathbf{y}$ in

$$A - ((U_1 \times X_2) \cup (X_1 \times U_2)).$$

Again applying Lemma 2.5.3, we see that this last set equals $(-U_1 \times -U_2) \cap A$.

Conversely, suppose that (2.22) holds, and consider any subset B of X such that $\mathbf{x} \bowtie B$. There exist sets $U_k \subset X_k$ such that $\mathbf{x} \in -U_1 \times -U_2 \subset \sim B$. Then $-U_1 \times -U_2 \subset -B$, so

$$A - B \supset (-U_1 \times -U_2) \cap A,$$

which, by our hypothesis (2.22), is inhabited. Hence

$$\forall_{B \subset X} (\mathbf{x} \bowtie B \Rightarrow \exists_{\mathbf{y}} (\mathbf{y} \in A - B))$$

—that is, $\text{near}(\mathbf{x}, A)$. ■

The following is another way of looking constructively at the classical equivalence of $\text{near}(\mathbf{x}, A)$ and $\neg(\mathbf{x} \bowtie A)$.

Proposition 2.5.9 *Let $X \equiv X_1 \times X_2$ be the product of two pre-apartness spaces, let $\mathbf{x} \in X$, and let $A \subset X$. Suppose that the following condition holds:*

(*) *There exist $V_1 \subset X_1$ and $V_2 \subset X_2$ such that $\mathbf{x} \in -V_1 \times -V_2$ and $A \subset (V_1 \times X_2) \cup (X_1 \times V_2)$.*

Then $\mathbf{x} \bowtie A$. Conversely, if the spaces X_1, X_2 are locally decomposable and $\mathbf{x} \bowtie A$, then condition () holds.*

Proof. First assume (*) and construct V_1, V_2 with the stated properties. We have

$$\begin{aligned} -V_1 \times -V_2 &\subset \sim V_1 \times \sim V_2 \\ &= (\sim V_1 \times X_2) \cap (X_1 \times \sim V_2) \\ &= \sim (V_1 \times X_2) \cap \sim (X_1 \times V_2) \\ &= \sim ((V_1 \times X_2) \cup (X_1 \times V_2)). \end{aligned}$$

Hence

$$\mathbf{x} \in -V_1 \times -V_2 \subset \sim A$$

and therefore $\mathbf{x} \bowtie A$.

Now assume, conversely, that the spaces X_1, X_2 are locally decomposable and that $\mathbf{x} \bowtie A$. Choose $U_k \subset X_k$ such that

$$\mathbf{x} \in -U_1 \times -U_2 \subset \sim A.$$

For each k , use the local decomposability of X_k to find $V_k \subset X_k$ such that $x_k \in -V_k$ and $X_k = -U_k \cup V_k$. Then

$$A \subset (V_1 \times X_2) \cup (X_1 \times V_2).$$

For if $(a_1, a_2) \in A$, then either $a_1 \in -U_1$ and $a_2 \in -U_2$, which is impossible, or else, as must be the case, $a_1 \in V_1$ or $a_2 \in V_2$. Thus (*) holds. ■

We now look at the continuity of mappings into and out of product spaces. It is natural to begin with the continuity of the projection mappings on a product pre-apartness space. For this we need

Proposition 2.5.10 *Let $X \equiv X_1 \times X_2$ be a product of pre-apartness spaces, and let $A_k \subset X_k$ ($k = 1, 2$). Then $A_1 \times A_2$ is nearly open in X if and only if, for each k , A_k is nearly open in X_k .*

Proof. Suppose first that $A_1 \times A_2$ is nearly open in X , and let $\mathbf{x} \in A_1 \times A_2$. Then there exists $S \subset X$ such that $\mathbf{x} \in -S \subset A_1 \times A_2$. Construct $U_k \subset X_k$ such that $\mathbf{x} \in -U_1 \times -U_2 \subset \sim S$; then

$$\mathbf{x} \in -U_1 \times -U_2 \subset -S \subset A_1 \times A_2,$$

so $x_1 \in -U_1 \subset A_1$ and $x_2 \in -U_2 \subset A_2$. It readily follows that A_k is nearly open in X_k .

The converse is a straightforward application of Lemma 2.5.3. ■

Corollary 2.5.11 *Let $X \equiv X_1 \times X_2$ be a product of pre-apartness spaces. Then the apartness topology on X is the product of the apartness topologies on X_1 and X_2 .*

Proposition 2.5.12 *Let $X \equiv X_1 \times X_2$ be a product of pre-apartness spaces. Then for each k , the projection mapping $\text{pr}_k : X \rightarrow X_k$ is topologically continuous. If the apartness topology on X has the reverse Kolmogorov property, then pr_k is continuous.*

Proof. If $U \subset X_1$, then

$$\text{pr}_1^{-1}(-U) = -U \times X_2 = -U \times -\emptyset,$$

which, by Proposition 2.5.10, is nearly open in X . Hence pr_1 , and similarly pr_2 , is topologically continuous. For the second conclusion, we need only apply Proposition 2.3.6. ■

Proposition 2.5.13 *Let $X \equiv X_1 \times X_2$ be the product of two pre-apartness spaces, let $s \equiv (\mathbf{x}_n)_{n \in \mathfrak{D}}$ be a net in X , and let $\xi \in X$. Then s converges to ξ in X if and only if, for $k \in \{1, 2\}$, the net $\text{pr}_k \circ s$ converges to ξ_k in X_k .*

Proof. Only if follows from Propositions 2.5.12 and 2.4.16. To prove *if*, suppose that, for $k \in \{1, 2\}$, the net $\text{pr}_k \circ s$ converges in X_k to ξ_k . Let $\xi \in -U$ in X , and construct $U_k \subset X_k$ such that $\xi \in -U_1 \times -U_2 \subset -U$. Then $\xi_k \in -U_k$, so there exists $n_k \in \mathfrak{D}$ such that $\text{pr}_k(\mathbf{x}_n) \in -U_k$ for all $n \succcurlyeq n_k$. Compute $N \in \mathfrak{D}$ such that $N \succcurlyeq n_1$ and $N \succcurlyeq n_2$. Then for all $n \succcurlyeq N$ we have $\mathbf{x}_n \in -U_1 \times -U_2 \subset -U$. Hence s converges in X to ξ . ■

Proposition 2.5.14 *Let $X \equiv X_1 \times X_2$ be a product of pre-apartness spaces, and f a continuous mapping of X into a pre-apartness space Y . Then for each $x_2 \in X_2$ the mapping $x \rightsquigarrow f(x, x_2)$ is continuous on X_1 , and for each $x_1 \in X_1$ the mapping $x \rightsquigarrow f(x_1, x)$ is continuous on X_2 .*

Proof. Fixing x_2 in X_2 , define $g(x) \equiv f(x, x_2)$. Let $x \in X_1$ and $A \subset X_1$ satisfy $g(x) \bowtie g(A)$ —that is,

$$f(x, x_2) \bowtie f(A \times \{x_2\}).$$

Since f is continuous, we have $(x, x_2) \bowtie A \times \{x_2\}$, so there exist $U_k \subset X_k$ such that

$$(x, x_2) \in -U_1 \times -U_2 \subset \sim (A \times \{x_2\}).$$

Hence $x \in -U_1 \subset \sim A$, and therefore $x \bowtie A$, by **A4** in X_1 . Thus g is continuous. A similar argument shows that $x \rightsquigarrow f(x_1, x)$ is continuous on X_2 for each $x_1 \in X_1$. ■

The foregoing results enable us now to show that the product of two locally decomposable apartness structures has a desirable categorical property.

Proposition 2.5.15 *Let $X \equiv X_1 \times X_2$ be the product of two locally decomposable apartness spaces, and f a mapping of a pre-apartness space Y into X . Then f is continuous if and only if $\text{pr}_k \circ f$ is continuous for each k .*

Proof. Assume that $\text{pr}_k \circ f$ is continuous for each k . Let $y \in Y$ and $T \subset Y$ satisfy $f(y) \bowtie f(T)$. By Propositions 2.5.6 and 2.5.9, there exist $V_1 \subset X_1$ and $V_2 \subset X_2$ such that $f(y) \in -V_1 \times -V_2$ and

$$f(T) \subset (V_1 \times X_2) \cup (X_1 \times V_2).$$

Setting

$$\begin{aligned} T_1 &\equiv f^{-1}(f(T) \cap (V_1 \times X_2)), \\ T_2 &\equiv f^{-1}(f(T) \cap (X_1 \times V_2)), \end{aligned}$$

we have $T \subset T_1 \cup T_2$. Moreover, for each k , $\text{pr}_k \circ f(T_k) \subset V_k$, so

$$\text{pr}_k \circ f(y) \in -V_k \subset -\text{pr}_k \circ f(T_k).$$

Our continuity hypothesis now ensures that $y \in -T_1 \cap -T_2 \subset -T$.

The converse is simple, in view of Proposition 2.5.12. ■

Since **A5**, and hence the reverse Kolmogorov property, is a consequence of local decomposability, why did we not simply require that every apartness space be locally decomposable? We could have done this, but we chose not to in order to base our development on the weakest possible principles, with the application of stronger ones, such as **A5** or local decomposability, clearly signalled when it is needed. Nevertheless, local decomposability will play a significant part in what follows in Chapter 3—namely, the theory of apartness between sets.

The insomniac reader will have noticed that, although we have discussed the flow of several properties between a product space and its factors, we have not said anything in this connection about the reverse Kolmogorov property.

Proposition 2.5.16 *If the product $X \equiv X_1 \times X_2$ of two pre-apartness spaces has the reverse Kolmogorov property, then so do the spaces X_k . Conversely, if one of the spaces X_k has the property **A5** and both have the reverse Kolmogorov property, then X has the reverse Kolmogorov property.*

Proof. Suppose that X has the reverse Kolmogorov property. Consider $x, y \in X_1$ and $U \subset X_1$ such that $x \in -U$ and $y \notin -U$. Fix $b \in X_2$. Then, by Lemma 2.5.2,

$$(x, b) \in -U \times X_2 = -(U \times X_2).$$

On the other hand, if $(y, b) \in -(U \times X_2)$, then $y \in -U$, a contradiction; so $(y, b) \notin -(U \times X_2)$ and therefore, by the reverse Kolmogorov property in X , $(x, b) \neq (y, b)$. It now follows from the definition of the inequality on the

product space that $x \neq y$ in X_1 . Thus X_1 , and similarly X_2 , has the reverse Kolmogorov property.

Now suppose that both X_1 and X_2 have the reverse Kolmogorov property and that X_1 is an apartness space. Let $\mathbf{x} \in -S$ and $\mathbf{y} \notin -S$ in X . Pick $U_k \subset X_k$ such that $\mathbf{x} \in -U_1 \times -U_2 \subset \sim S$. Since $x_1 \in -U_1$, by **A5** in X_1 we have either $y_1 \neq x_1$ or $y_1 \in -U_1$. In the latter case, since $\mathbf{y} \notin -U_1 \times -U_2$, we have $y_2 \notin -U_2$; but $x_2 \in -U_2$, so, by the reverse Kolmogorov property in X_2 , $x_2 \neq y_2$. Thus in either case, $\mathbf{x} \neq \mathbf{y}$. A similar argument covers the case where X_2 has the property **A5**. ■

The remarkable thing here is that, in the second part of Proposition 2.5.16, we cannot drop with impunity the hypothesis that one of the factor spaces X_k has the property **A5**; if we do so for even two-point spaces X_k , then we get entangled with the **weak disjunctive version of Markov's principle** ($\mathbf{MP}_{\text{or}}$):

If $(a_n)_{n \geq 1}$ is a binary sequence for which it is impossible that all terms equal 0, then

$$\neg \forall_n (a_{2n} = 0) \vee \neg \forall_n (a_{2n-1} = 0). \quad (2.23)$$

This statement is a simple consequence of Markov's principle.

Lemma 2.5.17 *The following are equivalent.*

- (i) $\mathbf{MP}_{\text{or}}$.
- (ii) *If x_1, x_2 are real numbers such that*

$$\neg (x_1 = 0 \wedge x_2 = 0), \quad (2.24)$$

then

$$\neg (x_1 = 0) \vee \neg (x_2 = 0).$$

Proof. Assuming $\mathbf{MP}_{\text{or}}$, let x_1, x_2 be real numbers such that (2.24) holds. Compute an increasing binary sequence $(\lambda_n)_{n \geq 1}$ such that

$$\begin{aligned} \lambda_n = 0 &\Rightarrow \max\{|x_1|, |x_2|\} < \frac{1}{n}, \\ \lambda_n = 1 - \lambda_{n-1} &\Rightarrow \max\{|x_1|, |x_2|\} > \frac{1}{n+1}. \end{aligned}$$

We may assume that $\lambda_1 = 0$. If $\lambda_n = 0$ or $\lambda_{n-1} = 1$, set $a_{2n} = a_{2n-1} \equiv 0$. If $\lambda_n = 1 - \lambda_{n-1}$, pick $k \in \{1, 2\}$ such that $|x_k| > 1/(n+1)$; if $k = 1$, set $a_{2n} \equiv 0, a_{2n-1} \equiv 1$; and if $k = 2$, set $a_{2n} \equiv 1, a_{2n-1} \equiv 0$. Consider the resulting sequence $(a_n)_{n \geq 1}$. If $a_n = 0$ for all n , then $\lambda_n = 0$ for all n , so $x_1 = 0 = x_2$, which contradicts (2.24). Thus we can apply $\mathbf{MP}_{\text{or}}$, to obtain (2.23). If the first alternative in (2.23) holds, suppose that $x_2 = 0$. If

$\lambda_k = 1 - \lambda_{k-1}$, then we must have $|x_1| > 1/(k+1)$ and $a_{2n} = 0$ for all n , a contradiction. Hence $\lambda_n = 0$ for all n , a further contradiction, from which we conclude that $\neg(x_2 = 0)$. Likewise, if the second alternative in (2.23) holds, then $\neg(x_1 = 0)$. Thus (i) implies (ii).

To prove the converse, assume (ii) and consider any binary sequence $(a_n)_{n \geq 1}$ with not all terms equal to 0. The real numbers

$$x_1 \equiv \sum_{n=1}^{\infty} 2^{-n} a_{2n}, \quad x_2 \equiv \sum_{n=1}^{\infty} 2^{-n} a_{2n-1} \quad (2.25)$$

satisfy (2.24). Hence either $\neg(x_1 = 0)$ or $\neg(x_2 = 0)$, from which we obtain (2.23). ■

By the **denial topology** on an inhabited set X we mean the topology τ' for which the logical complements of subsets of X form a base of open sets.

Proposition 2.5.18 *The following statement is equivalent to $\mathbf{MP}_{\text{or}}$:*

(†) *For each pair (x_1, x_2) of real numbers, if $X_k \equiv \{0, x_k\}$ is given the denial inequality and topology, then the product topological space $X_1 \times X_2$ has the reverse Kolmogorov property.*

Proof. Assuming (†), let x_1, x_2 be real numbers such that (2.24) holds, and let $X_k \equiv \{0, x_k\}$ be given the denial inequality and topology. Define

$$U_k \equiv \{x \in X_k : \neg(x = x_k)\}.$$

Then $x_k \in \neg U_k$, so (x_1, x_2) belongs to the open subset $\neg U_1 \times \neg U_2$ of the product topological space $X \equiv X_1 \times X_2$. On the other hand, if $(0, 0) \in \neg U_1 \times \neg U_2$, then $\neg\neg(0 = x_1)$ and $\neg\neg(0 = x_2)$, which implies that

$$\neg\neg(x_1 = 0 \wedge x_2 = 0),$$

a contradiction. Thus $(0, 0) \notin \neg U_1 \times \neg U_2$. Since X has the reverse Kolmogorov property, it follows that $(x_1, x_2) \neq (0, 0)$; whence

$$\neg(x_1 = 0) \vee \neg(x_2 = 0).$$

Thus (†) implies statement (ii) of Lemma 2.5.17 and therefore, by that lemma, $\mathbf{MP}_{\text{or}}$.

Now suppose, conversely, that $\mathbf{MP}_{\text{or}}$ holds. Let $x_1, x_2 \in \mathbf{R}$, let $X_k \equiv \{0, x_k\}$ be given the denial inequality and topology, and let $X \equiv X_1 \times X_2$ be the corresponding product topological space. Consider $\xi, \eta \in X$ and an open set $U \subset X$ such that $\xi \in U$ and $\eta \notin U$. We want to prove that $\xi \neq \eta$ in X —in other words, that there exists $k \in \{1, 2\}$ such that $\neg(\xi_k = \eta_k)$. Construct sets $U_k \subset X_k$ such that $\xi \in \neg U_1 \times \neg U_2 \subset U$; then $\eta \notin \neg U_1 \times \neg U_2$. Hence

$$\neg(\xi_1 - \eta_1 = 0 \wedge \xi_2 - \eta_2 = 0);$$

so, by Lemma 2.5.17, there exists $k \in \{1, 2\}$ such that $\neg(\xi_k - \eta_k = 0)$ and therefore $\neg(\xi_k = \eta_k)$. Hence $\xi \neq \eta$ in X , and we have proved that $\mathbf{MP}_{or}$ implies $(\dagger)$. ■

For any inhabited set with the denial inequality, the pre-apartness induced by the denial topology τ'_k is just the **denial pre-apartness**, given by

$$x \bowtie' S \Leftrightarrow x \in \neg S,$$

and the space (X, τ') is topologically consistent.

Corollary 2.5.19 $\mathbf{MP}_{or}$ is equivalent to the statement: For each pair (x_1, x_2) of real numbers, if $X_k \equiv \{0, x_k\}$ is given the denial inequality and pre-apartness, then the product pre-apartness space has the reverse Kolmogorov property.

Proof. This follows from Propositions 2.5.18 and 2.2.11, with reference to the remark immediately preceding this corollary. ■

2.6 Concluding Remarks on Impredicativity

There is one serious issue that needs to be addressed before we conclude this chapter: namely, the quantification over subsets of X in the definition of the expression $\text{near}(x, A)$. Such quantification appears to allow impredicativity, a notion viewed with horror by many of the pioneers of constructivism. Indeed, Beeson ([6], page 19) and others have suspected that the power set axiom—implicit in a second-order theory like ours as it stands—is inherently nonconstructive; to quote Myhill [72],

Power set seems especially nonconstructive and impredicative ... it does not involve ... putting together or taking apart sets that one has already constructed but rather selecting, out of the totality of all sets, all those that stand in the relation of inclusion to a given set.

Is it, then, the case that our theory really does depend on the full power-set axiom? The answer, we believe, is a firm “no”.

To justify this belief, we first note that the quantification in our definition of $\text{near}(x, A)$ takes place, not over all subsets of X , but *over subsets of X that are apart from x* :

$$\text{near}(x, A) \Leftrightarrow \forall_{S \subset X} (x \bowtie S \Rightarrow \exists_{y \in A} (y \bowtie S)).$$

Therefore, in order to avoid impredicativity, we need only that for each x in X the set of all subsets of X that are apart from x be well defined constructively. This can be accomplished by prescribing, for each inhabited set X with an inequality relation, a suitable family $\mathcal{A}(X)$ of subsets to which, and to no others, the relation $\bowtie$ may be applied. Such a family would need to satisfy at least (and maybe only) the following conditions:

- $\{x\} \in \mathcal{A}(X)$ for each $x \in X$.
- Finite unions of sets in $\mathcal{A}(X)$ belong to $\mathcal{A}(X)$.
- For any map $f : X \rightarrow Y$ between sets with inequality relations, if $S \in \mathcal{A}(X)$, then $f(S) \in \mathcal{A}(Y)$.
- For any map $f : X \rightarrow Y$ between sets with inequality relations, if $T \in \mathcal{A}(Y)$, then $f^{-1}(T) \in \mathcal{A}(X)$.
- Any interval belongs to $\mathcal{A}(\mathbf{R})$.

We would then impose upon the relation $\bowtie$ the requirement that $x \bowtie A$ only if $A \in \mathcal{A}(X)$. So, for example, axiom **A4** would then be written more carefully as

$$\forall_{A \in \mathcal{A}(X)} \forall_{B \in \mathcal{A}(X)} (-A \subset \sim B \Rightarrow -A \subset -B).$$

Another way of dealing with the spectre of impredicativity is based on the observation that some authors—for example Myhill [72]—while rejecting the power-set axiom, are prepared to accept that the set $\mathbf{R}^X$ of strongly extensional mappings from a set X (with an inequality) to $\mathbf{R}$ is constructively well defined. With this in mind, we could adopt the following procedure. First, introduce the axioms **A1–A5** for the apartness on X and define the notion of continuity of a mapping between apartness spaces. Next, having defined the canonical apartness on $\mathbf{R}$, introduce an extra axiom which entails that all apartness spaces are completely regular. Then $\text{near}(x, A)$ translates as

$$\begin{aligned} \forall_{f \in [0,1]^X} ((f \text{ is continuous} \wedge f(x) = 0 \wedge f(A) \subset \{1\}) \\ \Rightarrow \exists_{y \in A} (f(y) < 1)), \end{aligned}$$

where the universal quantification is taken over a genuine set of mappings.

The disadvantage of the approach suggested in the preceding paragraph is that it confines us to completely regular spaces. On the other hand, complete regularity applies widely in the classical theory of proximity spaces [75], and, as is easily shown, implies local decomposability in our theory.

There is yet another way in which we might avoid impredicativity: namely, by working with an informal notion of *well-constructed subset* of X , intended to capture the idea of a subset of X built up from some basic collection of subsets by purely predicative means. Our definition of $\text{near}(x, A)$ would then be re-cast as

$$\forall_S ((S \text{ is a well-constructed subset of } X \wedge x \bowtie S) \Rightarrow \exists_y (y \in A - S)).$$

This idea is not as imprecise as may at first appear: there is a constructive formalisation of Morse set theory [14] with a universal class (*not* a set) U , in which objects appear to be constructed predicatively if and only if they can be proved to belong to U . In the framework of that set theory, we could solve our impredicativity problem by introducing the following extra axiom for an apartness relation on X :

A0 $x \bowtie S \Rightarrow (x \in X \wedge S \subset X \wedge S \in U)$.

As a last word on impredicativity, we add that, in practice, nearness does not have much of a role to play in our theory; it is apartness that carries the significant computational information about points and sets. Thus little might be lost were we to exclude nearness from our constructive deliberations altogether.

Notes on Chapter 2

The relation between pre-apartness spaces and those that also satisfy **A5** is analogous to that between groups in general and abelian groups.

If Y is an inhabited subset of a pre-apartness space X , then classically the relation $\bowtie_Y$ always satisfies **A4**. For if $A, B \subset Y$ and $Y - A \subset Y \sim B$, then for any $x \in X - A$ we have either $x \in Y - A$ or $x \notin Y$; in either case this entails $x \in X \sim B$. Thus $X - A \subset X \sim B$, and therefore, by **A4** in X , $X - A \subset X - B$; whence $Y - A \subset Y \sim B$.

In the case where $\sim A$ and $\neg A$ coincide, we can prove classically, by contraposition, that if $\neg(x \bowtie A)$, then $\text{near}(x, A)$. For if $\neg\text{near}(x, A)$, then there exists $S \subset X$ such that $x \in -S$ and $A - S = \emptyset$; whence $x \in -S \subset \neg A = \sim A$, and therefore, by **A4**, $x \bowtie A$. Referring to Proposition 2.1.14, we see that in this case, $\text{near}(x, A)$ if and only if $\neg(x \bowtie A)$.

The reverse Kolmogorov property was introduced by Ishihara, who gave it the label $\mathbf{T}_0^\neq$. It looks rather like a reversal of the standard Kolmogorov property for a topological space—hence its name. For more on that and other properties of an inequality on pre-apartness and related spaces, see [48]. If every pre-apartness space with the reverse Kolmogorov property has the property **A5**, then we can prove **WLPO**. To see this, equip $X \equiv [0, 1]$ with the denial inequality and pre-apartness. Then X has the reverse Kolmogorov property. Let $(a_n)_{n \geq 1}$ be any binary sequence, let

$$y \equiv \sum_{n=1}^{\infty} 2^{-n} a_n \in X,$$

and let $S \equiv (0, 1]$. Then $0 \in -S$; but if we have either $0 \neq y$ or $y \bowtie S$, then

$$\neg \forall_n (a_n = 0) \vee \forall_n (a_n = 0).$$

Among the many merits of local decomposability are these: in its presence, continuity and topological continuity coalesce, a topological apartness space is topologically consistent, and, according to Proposition 2.5.15, product apartness spaces have precisely the categorical property that one would wish for.

We are grateful to Jeremy Clarke for the example showing that if every $\mathbf{T}_1$ topological apartness space is topologically consistent, then the law of excluded

middle holds. In the absence of topological consistency, there is an interesting range of topologies that can lie between a given one and its related apartness topology [59].

Proposition 2.2.12 and the comments preceding it give us another proof that every metric space (X, ρ) is topologically consistent.

It is an interesting exercise to show that even in the apartness space **R** we cannot prove that, in every case, the union of two nearly closed sets is nearly closed.

We do not know whether Proposition 2.3.7 holds without the hypothesis that Y have the weak nested neighbourhoods property (it certainly does under classical logic).

The classical theory of nets requires a partial order where we use a preorder. A partial order is a preorder R with the additional property of antisymmetry:

$$\forall_{x,y} ((xRy \wedge yRx) \Rightarrow x = y).$$

If we used a partial order in our constructive theory of nets, we would run into difficulties which the classical theory avoids by applications of the axiom of choice.

A Kripke model based on that on pages 137–138 of [24] shows that **MP**_{or} is independent of **BISH**.

In the first part of the proof of Proposition 2.5.18 we used the logical theorem

$$\neg\neg P \wedge \neg\neg Q \Rightarrow \neg\neg (P \wedge Q),$$

which the reader is invited to prove informally.

When we began the investigations that eventually led us to apartness spaces, we worked with primitive notions of both nearness and apartness [28]; to a large extent, those ideas have been superseded by our current theory of apartness, based on a relatively succinct set of axioms.

We have not attempted a rigorous justification of the statement that, in constructive Morse set theory, objects are constructed impredicatively if and only if they belong to the universe U , but we have little doubt that such a proof could be produced after a tedious examination of the many cases that would arise.

Hedin [51] has discussed means of developing a purely predicative theory of apartness within Martin-Löf type theory.



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