

PREFACE

The problem of finding a suitable constructive framework for general topology is important and elusive.

Errett Bishop²

In the mid-1970s, Douglas Bridges came across the article [35], advocating the use of nearness in the teaching of first courses in analysis. This led him to consider a constructive theory based on proximity—or, rather, the constructively more appropriate opposite notion of apartness—as a possible alternative approach to topology. Not, at that stage, being sufficiently mathematically experienced to make significant progress with this idea, he put it on one side until February 2000, when he and Luminița Vîță began the project on apartness spaces that is discussed in the present monograph. Clearly, two heads were better than one, as we were able to initiate a project that, in the intervening years, has grown substantially, with contributions from mathematicians in several countries and continents.

What do we mean by *constructive* in this context, and why might a constructive approach to topology be interesting or significant? The answer to the first question is simple: by *constructive mathematics* we mean, roughly, mathematics with intuitionistic logic and a set- or type-theoretic foundation that precludes the derivation of the law of excluded middle; in other words, we mean mathematics carried out in the style of the late Errett Bishop [9]. Every proof that is constructive in this sense embodies an algorithm that can be, and in several cases has been, extracted and then implemented; further, the original proof shows that the implementable algorithm (if correctly extracted)

²Letter to Douglas Bridges, dated 14 April 1975.

meets its specifications. One other important feature of Bishop's constructive mathematics is that it is completely consistent with classical mathematics—mathematics using classical logic.

In regard to the second question, we recall Bishop's deflationary remark [9] (page 63):

Very little is left of general topology after that vehicle of classical mathematics has been taken apart and reassembled constructively. With some regret, plus a large measure of relief, we see this flamboyant engine collapse to constructive size.

At the time he wrote [9], Bishop firmly believed that computation in analysis required uniform continuity—normally on compact sets—rather than the weaker notion of pointwise (let alone sequential) continuity:

The concept of a pointwise continuous function is not relevant. A continuous function [on the real line] is one that is uniformly continuous on compact intervals. ([9], pages ix–x)

Since the theorem asserting the uniform continuity of every pointwise continuous, real-valued mapping on the closed interval $[0, 1]$ cannot be proved within Bishop-style constructive mathematics,³ general topology, essentially a pointwise matter, would therefore have little constructive relevance; what was needed was a framework (such as metric-space theory) in which *uniform* notions could be expressed.

Bishop's remark, taken with his suggestion, in Appendix A to [9], that constructive theories like that of distributions might rely on ad hoc topological notions, may have served to deflect attention away from the problem of finding a suitable constructive framework for general topology. The advantage of solving this problem would presumably be that we would have a general, and hence almost certainly clarifying, framework for topological matters. Moreover, work of Ishihara [55] and others, 20 years after the publication of [9], has shown that even sequential continuity has an important role in Bishop-style mathematics; so there is definitely a place for a constructive theory of abstract spatial relationships that covers such highly non-uniform types of continuity.

One can certainly tackle topology by simply constructivising the classical development of the subject from the usual three axioms about open sets.⁴ However, the theory of apartness that we present in this book does much more, by encompassing both point-set topology and the theory of uniform spaces. This, we believe, enables us to see those subjects in a clearer light.

We now outline the contents of the three chapters in our book, which begins with one that introduces informal constructive (intuitionistic) logic and set

³The theorem is provable in one model of Bishop-style mathematics—namely, intuitionistic mathematics—but is false in the model in which everything is interpreted recursively.

⁴This has been done by Grayson [46, 47], Troelstra [87], and Waaldijk [91], the last two working within Brouwer's full intuitionistic framework.

theory, as well as—very briefly—the basic notions and notations for metric and topological spaces.

In Chapter 2 we introduce axioms for a point-set apartness, a more computationally informative notion than nearness.⁵ We then explore some of the elementary consequences of our axioms, and examine the associated topology and various types of continuity of mappings. One interesting feature of our constructive approach is that intuitionistic logic reveals distinctions that are invisible to the classical eye. For example, we present three natural notions of continuity for maps between point-set apartness spaces, notions that coalesce classically but are quite distinct constructively.

After dealing with continuity, the chapter continues with a discussion of convergence for nets in an apartness space, followed by a study of the apartness structure on the product of two apartness spaces. It ends with some remarks about impredicativity and how that phenomenon might be avoided.

The theory really gets interesting—and a lot harder—when, in Chapter 3, we consider apartness between subsets. The five axioms on which this part of the theory is based enable us to bring most of the work on point-set apartness across to the set-set environment. Every set-set apartness gives rise to a point-set apartness in the obvious way: a point x is apart from a set S if the singleton subset $\{x\}$ is apart from S .

The canonical example of a set-set apartness arises from a quasi-uniform structure (which, if a certain symmetry condition holds, becomes a uniform structure). Having introduced both apartness and quasi-uniform spaces, we are well placed to examine the connection between, on the one hand, the uniform continuity of a mapping $f : X \rightarrow Y$ between quasi-uniform spaces and, on the other, strong continuity, in which if the two subsets of Y are apart, then their pre-images under f are apart in X . It is straightforward to show that uniform continuity entails strong continuity; but the converse direction of entailment is much harder to deal with. Similar difficulties are found in our discussion of the relation between Cauchy nets in the usual uniform-space sense, and nets that have the property of *total Cauchy*ness relative to the apartness structure associated with a given uniform structure. (The corresponding property of *total completeness* of an apartness space X , whereby every totally Cauchy sequence in X has a limit in X , is classically equivalent to the compactness of the underlying apartness topology on X .)

The difficulties, alluded to above, in the proofs of certain results are unified by our introduction of a general proof technique which has several applications in the theory and which resembles—but is considerably more complex than—a well-known technique introduced into constructive analysis by Ishihara [55]. Not only does our technique apply to the study of strong and uniform continuity, but also it leads to a neat proof that, under certain conditions, the uniform

⁵Actually, our fundamental notion is that of a *pre-apartness*. An apartness is a pre-apartness with an extra disjunction property that enables us to split arguments into cases.

convergence of a sequence of mappings between uniform spaces is equivalent to their being convergent in an apartness-space sense. (In order to provide a sound framework for the study of convergence of functions, we introduce structures, constructively weaker than apartness and quasi-uniform ones, on the space Y^X of mappings from a set X into an apartness space Y .)

After a discussion of totally Cauchy nets, the chapter continues by introducing notions of *locatedness* that lift that fundamental property of subsets of a metric space into the wider context of a uniform space. In particular, we connect locatedness and another important property, *total boundedness* (one that lifts more easily from the metric to the uniform context). We then deal with product apartness spaces, paying, as in the case of point-set apartness, particular attention to the flow of properties, such as total completeness, between a product apartness space and its two “factors”.

In the next section of the chapter we discuss a form of connectedness for apartness spaces. Then we introduce a special structure, denoted by $\mathcal{B}_w$, on an apartness space X . We use this to discuss uniform structures that are compatible with a given apartness structure on a set. Although we cannot generally prove that there is such a uniform structure, we can show that there is at most one totally bounded uniform structure of this sort, which is then the smallest compatible uniform structure. In addition, the properties of $\mathcal{B}_w$ lead us to a notion of *nearness* between subsets of an apartness space. If the apartness space has a strong separation property, then the point-set restriction of this nearness coincides with the notion of nearness associated, in Chapter 2, with the point-set apartness on X . Finally, we discuss Diener’s alternative approach to compactness in apartness spaces, based on his notions of *neat locatedness* and *neat compactness*.

There are other, successful and important, constructive approaches to topology that we should mention: the ones based on the notions of *frame*, *locale*, and *formal topology* [60, 61, 79, 80, 89]. Some work has been done towards clarifying the relation between these “point-free” approaches and our theory of apartness spaces [76]; this work is the subject of the Postlude at the end of the book. We believe that constructive mathematics has room for both the theory of apartness and the point-free approaches to topology, and that these should be regarded as equally valid and viable.

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