Chapter 2
Perturbation Method: Lindstedt-Poincaré

We seek an expansion that is valid for small but finite amplitude motions. It is convenient to introduce a small, dimensionless parameter ε which is of the order of the amplitude of the motion and can be used as a crutch, or a bookkeeping device, in obtaining the approximate solution [22, 33–35]. For instance, the equation of motion of an autonomous weakly nonlinear system has the form:

$$\ddot{u} + \omega^2 u = \varepsilon f(u), \quad 0 < \varepsilon \ll 1 \tag{2.1}$$

where in general f is a nonlinear function and $\dot{u} = \frac{du}{dt}$.

The perturbation method aims at getting a *periodic* solution of Eq. 2.1 in the form of a power series with respect to ε. This method, introduced about 1830 by Poisson, was at first applied formally, without any theoretical justification. Nevertheless it has been successfully used to obtain some effective solutions especially in celestial mechanics. By the end of the nineteenth century, the method has been improved as far as calculation is concerned by Gylden, Lindstedt, Bohlin, and others. However, the main contribution to the perturbation method is due to Poincaré [1], who elaborated in 1892 its theoretical grounds and made possible its systematic application to various problems of nonlinear oscillations. We shall discuss in this section a variant of the perturbation method proposed by Lindstedt in 1883.

There is no real and rigorous connection between the way that ε is used in different terms of a nonlinear differential equation and the manner in which it can be applied to the damping, for example. Besides, it forces the assumption onto the problem that there is a genuine strength similarity between damping, excitation, or nonlinear restoring force. In reality this may be a rather vague supposition and almost impossible to prove unequivocally. The use of ε, also introduces the implicit assumption that occasionally certain term is of the second order in ε, whereas other term can be of the first order in ε. Again, such comparison is hard to contemplate accurately and definitively, but if it does turn out to be physically unacceptable, then the particular ordering scheme cannot be used. It is important to proceed with caution and to use one's experience judiciously. The parameter ε can be seen as a

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convenient universal scaling parameter for different, apparently unrelated quantities within the equation of motion such as damping, excitation amplitude, and coefficients of nonlinear terms. On that basis, one might construct a set of physically reasonable orderings which together convey the requirements for soft excitation, relatively weak damping and subordinate status for geometric nonlinearities when compared with their linear counterparts. The other possibility for introducing ε is based on formal nondimensionalization of the dependent variable (i.e. the chosen coordinate) and the independent variable.

Let us suppose that Eq. 2.1 has a periodic solution $u(t)$ of some period T. We cannot try a solution of the form:

$$u(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \ldots \tag{2.2}$$

because some functions $u_k(t)$ could be aperiodic. Such a situation occurs, i.e. for the expansion of the periodic function $\sin(1+\varepsilon)t$:

$$\sin(1+\varepsilon)t = \sin t + \varepsilon t \cos t - \frac{1}{2}\varepsilon^2 t^2 \sin t + \ldots.$$

whose coefficients are not periodic. Terms like $t\cos t$, $t^2 \sin t$, . . . in which the time t appears as "amplitude" are called *secular terms*. It is obvious that the existence of such terms, which grow beyond and bound as $t \to \infty$, destroys the periodicity of the expansion when only a finite number of its terms are considered, which is usually the case.

This difficulty may be avoided by developing the period $T(\varepsilon)$ in a power series with respect to ε:

$$T(\varepsilon) = \frac{2\pi}{\omega}\left(1 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots\right) \tag{2.3}$$

and introducing into Eq. 2.1 the new independent variable

$$\tau = \frac{2\pi t}{T} = \frac{\omega t}{1 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots} \tag{2.4}$$

Then, Eq. 2.1 becomes

$$\frac{d^2 u}{d\tau^2} + \left(1 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots\right)^2 u = \frac{\varepsilon}{\omega^2}\left(1 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots\right)^2 f(u) \tag{2.5}$$

and has a periodic solution $u(\tau)$ of constant period 2π. Consequently, we may assume for $u(\tau)$ in Eq. 2.5 the power series

$$u(\tau) = u_0(\tau) + \varepsilon u_1(\tau) + \varepsilon^2 u_2(\tau) + \ldots \tag{2.6}$$

whose coefficients $u_k(\tau)$ have to be periodic functions of τ of period 2π.

Now, we suppose that the initial conditions are

$$u(0) = a, \quad \dot{u}(0) = 0, \tag{2.7}$$

We satisfy the conditions (2.7) by requiring that

$$u_0(0) = a, \quad u_{k+1}(0) = 0, \quad \frac{du_k(0)}{d\tau}\left(= \frac{du_k(\tau)}{d\tau}|_{\tau=0}\right) = 0 \text{ for } k = 0, 1, 2... \tag{2.8}$$

By substituting Eq. 2.6 into Eq. 2.5 and taking into account that

$$(1 + \varepsilon h_1 + \varepsilon^2 h_2 + \varepsilon^3 h_3 + ...)^2 = 1 + 2\varepsilon h_1 + \varepsilon^2(h_1^2 + 2h_2) + 2\varepsilon^3(h_3 + h_1 h_2) + ... \tag{2.9}$$

and

$$\begin{aligned} f(u) = & f(u_0) + \left(\varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \ldots\right) f'(u_0) + \frac{1}{2}(\varepsilon u_1 + \varepsilon^2 u_2 + \\ & + \varepsilon^3 u_3 + \ldots)^2 f''(u_0) + \ldots . = f(u_0) + \varepsilon u_1 f'(u_0) + \varepsilon^2 [u_2 f'(u_0) + \\ & + \frac{1}{2} u_1^2 f''(u_0)] + \varepsilon^3 [u_3 f'(u_0) + u_1 u_2 f''(u_0)] + \ldots \end{aligned} \tag{2.10}$$

by equating coefficients of like powers of ε, we obtain the set of recursive linear differential equations:

$$\begin{aligned} & \frac{d^2u_0}{d\tau^2} + u_0 = 0 \\ & \frac{d^2u_1}{d\tau^2} + u_1 = -2h_1 u_0 + \frac{f(u_0)}{\omega^2} := -2h_1 u_0 + \varphi_1(\tau) \\ & \frac{d^2u_2}{d\tau^2} + u_2 = -2h_2 u_0 - 2h_1 u_1 - h_1^2 u_0 + \frac{u_1 f'(u_0) + 2h_1 f(u_0)}{\omega^2} := \\ & \qquad - \left(2h_2 + h_1^2\right) u_0 + \varphi_2(\tau) \\ & \frac{d^2u_3}{d\tau^2} + u_3 = -2(h_3 + h_1 h_2) u_0 - \left(h_1^2 + 2h_2\right) u_1 - 2h_1 u_2 + \\ & \qquad + \frac{u_2 f'(u_0) + \frac{1}{2} u_1^2 f''(u_0) + 2h_1 u_1 f'(u_0) + \left(h_1^2 + 2h_2\right) f(u_0)}{\omega^2} := \\ & \qquad - 2(h_3 + h_1 h_2) u_0 + \varphi_3(\tau) \end{aligned} \tag{2.11}$$

and so on, where the symbol := means "note".

The general solution of Eq. 2.11_1 satisfying Eq. 2.8 is

$$u_0(\tau) = a \cos \tau \tag{2.12}$$

In order to integrate Eq. 2.11$_2$, we develop the function $\varphi_1(\tau)$ in a Fourier series. We obtain

$$\varphi_1(\tau) = C_{10} + C_{11}\cos\tau + C_{12}\cos 2\tau + C_{13}\cos 3\tau + \ldots \tag{2.13}$$

and therefore, Eq. 2.11$_2$ becomes

$$\frac{d^2u_1}{d\tau^2} + u_1 = -2h_1 a\cos\tau + C_{10} + C_{11}\cos\tau + \sum_{k=2}^{\infty} C_{1k}\cos k\tau \tag{2.14}$$

The solution of this equation does not contain secular terms provided that there are no terms containing $\cos\tau$and $\sin\tau$in the right-hand side. This condition gives $-2\mathrm{h}_1\mathrm{a} + \mathrm{C}_{11} = 0$, whence

$$h_1 = \frac{C_{11}}{2a} \tag{2.15}$$

The general solution of Eq. 2.14 is then

$$u_1(\tau) = C_{10} + \sum_{k=2}^{\infty}\frac{C_{1k}}{1-k^2}\cos k\tau + \alpha_1\cos\tau + \beta_1\sin\tau \tag{2.16}$$

From initial conditions (2.8) we deduce that

$$\begin{aligned} \alpha_1 &= -C_{10} - \sum_{k=2}^{\infty}\frac{C_{1k}}{1-k^2} \\ \beta_1 &= 0 \end{aligned} \tag{2.17}$$

We now have

$$u_1(\tau) = C_{10}(1-\cos\tau) + \sum_{k=2}^{\infty}\frac{C_{1k}}{1-k^2}(\cos k\tau - \cos\tau) \tag{2.18}$$

Introducing Eqs. 2.12 and 2.18 into Eq. 2.11$_3$ yields $\varphi_2(\tau)$ and hence can be expanded in a cosine series, and so on.

Assume that we have determined in this way the functions $u_\mathrm{k}(\tau)$, and the constants h_k, for $k = 1,2,\ldots,$m-1, and that we may calculate $\varphi_\mathrm{m}(\tau)$. By expanding $\varphi_\mathrm{m}(\tau)$ in a Fourier series, we obtain

$$\varphi_m(\tau) = C_{m0} + C_{m1}\cos\tau + \sum_{k=2}^{\infty} C_{mk}\cos k\tau \tag{2.19}$$

and the m-th equation in Eq. 2.11 becomes

$$\frac{d^2u_m}{d\tau^2} + u_m = -(2h_m + 2h_1h_{m-1} + 2h_2h_{m-2} + \ldots)a\cos\tau + \\ + C_{m0} + C_{m1}\cos\tau + \sum_{k=2}^{\infty} C_{mk}\cos k\tau \tag{2.20}$$

Requiring the periodicity of $u_m(\tau)$ gives

$$h_m = \frac{C_{m1}}{2a} - h_1h_{m-1} - h_2h_{m-2}\ldots \tag{2.21}$$

and we deduce as before, by initial conditions (2.8) that $u_m(\tau)$ may be taken under the form

$$u_m(\tau) = C_{m0}(1 - \cos\tau) + \sum_{k=2}^{\infty} \frac{C_{mk}}{1 - k^2}(\cos k\tau - \cos\tau) \tag{2.22}$$

We may thus successfully determine the function $u_k(\tau)$ and the constants h_k. The solution of Eq. 2.5 is then Eq. 2.6, where τ is given by Eq. 2.4 and the period T is given by Eq. 2.3.

It is apparent from above that there are no special difficulties in applying the perturbation method up to any step. However, the second and following steps do not qualitatively change the approximate solution. They only introduce small quantitative corrections of order ε^2 or higher, which, usually, do not justify the amount of calculation involved.

Poincaré has shown by an example that, in general, the series (2.6) obtained by the method of Lindstedt-Poincaré may not converge. In the variant of the perturbation method devised by him, the solution is obtained once again as a power series with respect to ε, which uniformly converges to $u(t)$ if ε and the initial amplitude $|a|$ are sufficiently small, but which may contain secular terms. Poincaré was interested in astronomical problems, in which the presence of secular terms is harmless, because of the relatively slow motion of the planets.

However, for the study of nonlinear vibration with comparatively high frequencies, a casting-out of the secular terms, as the one of Lindstedt discussed above, seems to be better suited. Moreover, since the expansion (2.6) is practically limited to its first one or two terms, one is mainly interested in the asymptotic behaviour for $\varepsilon \to 0$ of this truncated expansion, and the possible divergence of the whole series is generally immaterial.

2.1 The Oscillator with Cubic Elastic Restoring Force

The oscillator with cubic elastic restoring force occupies an important place in the theory of nonlinear systems, since it is the simplest oscillator displaying specific nonlinear properties. On the other hand, it provides a first approximation for the behaviour of a much large class of oscillators. Indeed, for sufficiently small values of $|u|$, the cubic characteristic may approximate as well as we please an elastic characteristic given by an arbitrary analytic function of u. Finally, another argument in favour of a detailed study of this oscillator is the possibility that it gives to compare an exact solution with the approximate solutions obtained by various analytical methods.

Let us consider the equation of motion of a conservative oscillator with cubic elastic restoring force (well-known Duffing equation)

$$\ddot{u} + \omega^2 u\left(1 + \varepsilon u^2\right) = 0 \tag{2.23}$$

with the initial conditions

$$u(0) = a > 0, \quad \dot{u}(0) = 0 \tag{2.24}$$

where ω, ε and a are constants.

2.1.1 The Exact Solution of Duffing Equation

The potential energy per unit mass for Eq. 2.23 is

$$G(u) = \int_0^u \omega^2 u\left(1 + \varepsilon u^2\right) du = \frac{\omega^2 u^2}{2}\left(1 + \frac{\varepsilon}{2} u^2\right) \tag{2.25}$$

The energy equation reduces in this case to

$$E(t) = \frac{v^2}{2} + G(u) = E_0 \tag{2.26}$$

where $v^2/2$ is the kinetic energy per unit mass ($\dot{u}=v$) and

$$E_0 = \frac{v_0^2}{2} + G(u_0) \tag{2.27}$$

is the initial total energy of the system per unit mass. From initial condition (2.24) we obtain the energy equation

$$\frac{v^2}{2} + \frac{\omega^2 u^2}{2}\left(1 + \frac{\varepsilon}{2}u^2\right) = E_0 \tag{2.28}$$

where

$$E_0 = \frac{\omega^2 a^2}{2}\left(1 + \frac{\varepsilon}{2}a^2\right) \tag{2.29}$$

From Eqs. 2.28 and 2.29 we deduce the velocity as function of the displacement

$$v(u) = \pm\omega\sqrt{(a^2 - u^2)\left[1 + \frac{\varepsilon}{2}(a^2 + u^2)\right]} \tag{2.30}$$

from which the extreme values of the speed

$$v_{1,2} = \pm\omega a\sqrt{1 + \frac{\varepsilon}{2}a^2} \tag{2.31}$$

results for $u = 0$. The extreme values of the displacement may be obtained by putting $v = 0$ in Eq. 2.30. We find, as expected, $u_{1,2} = \pm a$. From Eq. 2.28 we see that the phase trajectories are symmetric with both coordinate axes.

Next, it follows from Eq. 2.30 that the time necessary for the representative point to move in the lower half-plane from $A(a,0)$ to the point of abscissa u is

$$t(u) = -\frac{1}{\omega}\int_a^u \frac{du}{\sqrt{(a^2 - u^2)\left[1 + \frac{\varepsilon}{2}(a^2 + u^2)\right]}} \tag{2.32}$$

By interchanging the limits of integration and putting $u = a\eta$, where η is a new variable, Eq. 2.32 becomes

$$t(u) = \frac{\sqrt{2}}{\varepsilon\omega a}\int_{\frac{u}{a}}^{1} \frac{d\eta}{\sqrt{(1 - \eta^2)\left(1 + \frac{2}{\varepsilon a^2} + \eta^2\right)}} \tag{2.33}$$

This expression may be further transformed by means of the elliptic integral of the first kind $F(\varphi.k)$, [36, 37]. We then obtain

$$t(u) = \frac{F\left(arc\ \cos\frac{u}{a}; k\right)}{\omega\sqrt{1 + \varepsilon a^2}} \tag{2.34}$$

where

$$k = \sqrt{\frac{\varepsilon a^2}{2(1 + \varepsilon a^2)}} \tag{2.35}$$

Finally, by inverting the function (2.34), we find the exact solution of the equation of motion

$$u = a\,cn\left(\omega t\sqrt{1+\varepsilon a^2}; k\right) \tag{2.36}$$

where cn is Jacobi's elliptic function (elliptic cosine).

We also notice that the quarter of the vibration period may be calculated by putting $u = 0$ into Eq. 2.36, and hence the period is given by

$$T(a) = \frac{4}{\omega\sqrt{1+\varepsilon a^2}} F\left(\frac{\pi}{2}; k\right) = \frac{4}{\omega\sqrt{1+\varepsilon a^2}} K(k) \tag{2.37}$$

where $K(k)$ is the complete elliptic integral of the first kind.

In the special case when ε is sufficiently small, then

$$k^2 = \frac{\varepsilon a^2}{2(1+\varepsilon a^2)} = \frac{\varepsilon a^2}{2}\left[1 - \varepsilon a^2 + \left(\varepsilon a^2\right)^2 - \left(\varepsilon a^2\right)^3 + \ldots\right] \tag{2.38}$$

$$\frac{1}{\sqrt{1+\varepsilon a^2}} = 1 - \frac{\varepsilon a^2}{2} + \frac{3\left(\varepsilon a^2\right)^2}{8} - \frac{5\left(\varepsilon a^2\right)^3}{16} + \frac{35\left(\varepsilon a^2\right)^4}{128} \ldots \tag{2.39}$$

$$K(k) = \frac{\pi}{2}\left(1 + \frac{k^2}{4} + \frac{9k^4}{64} + \frac{25k^6}{256} + \frac{1225k^8}{16384} + \ldots\right)$$

After some manipulation we obtain

$$T(a) = \frac{2\pi}{\omega}\left(1 - \frac{3\varepsilon a^2}{8} + \frac{57\varepsilon^2 a^4}{256} - \frac{315\varepsilon^3 a^6}{2048} + \ldots\right) \tag{2.40}$$

Using the series expansion of the elliptic cosine

$$cn(x,k) = \frac{2\pi}{kK(k)}\left(\frac{\sqrt{q(k)}}{1+q(k)}\cos\frac{\pi x}{2K(k)} + \frac{q(k)\sqrt{q(k)}}{1+q^3(k)}\cos\frac{3\pi x}{2K(k)} + \ldots\right)$$

where

$$q(k) = \exp\left[-\frac{\pi K\left(\sqrt{1-k^2}\right)}{K(k)}\right] \tag{2.41}$$

we obtain the solution (2.36) in the form

$$u(t) = \frac{2\pi a}{kK(k)}\left(\frac{\sqrt{q(k)}}{1+q(k)}\cos\frac{2\pi}{T}t + \frac{q(k)\sqrt{q(k)}}{1+q^3(k)}\cos 3\frac{2\pi}{T}t + \ldots\right) \tag{2.42}$$

We remark that Eqs. 2.40 and 2.42 may be very useful, especially when comparing the exact solution to approximate solutions obtained by other methods.

2.1.2 Use of the Perturbation Method for Duffing Oscillator with Small Parameter

We shall apply the perturbation method to determine an approximate solution of Eq. 2.23 satisfying the initial conditions (2.24), under the assumption that the elastic nonlinearity is weak. We try a solution of the form (2.6) where the variable τ is given by Eq. 2.4 and the period T is given by Eq. 2.3. In this case, Eq. 2.5 becomes

$$\begin{aligned}
&u''_0 + \varepsilon u''_1 + \varepsilon^2 u''_2 + \varepsilon^3 u''_3 + \ldots + \\
&\quad + \left[1 + 2h_1\varepsilon + \left(h_1^2 + 2h_2\right)\varepsilon^2 + 2(h_3 + h_1h_2)\varepsilon^3 + \ldots\right](u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \\
&\quad + \varepsilon^3 u_3 + \ldots) = -\left[\varepsilon + 2h_1\varepsilon^2 + \left(h_1^2 + 2h_2\right)\varepsilon^2 + \ldots\right]\cdot\left[u_0^3 + 3u_0^2u_1\varepsilon + \right. \\
&\quad \left. + 3\left(u_0u_1^2 + u_0^2u_2\right)\varepsilon^2 + \ldots.\right]
\end{aligned} \tag{2.43}$$

where $u' = \frac{du}{d\tau}$.

Equating the coefficients of like powers of ε, we obtain the first three approximations from the following four equations:

$$u''_0 + u_0 = 0 \tag{2.44}$$

$$u''_1 + u_1 = -2h_1u_0 - u_0^3 \tag{2.45}$$

$$u''_2 + u_2 = -\left(h_1^2 + 2h_2\right)u_0 - 2h_1u_1 - \left(3u_0^2u_1 + 2h_1u_0^3\right) \tag{2.46}$$

$$\begin{aligned}
u''_3 + u_3 = {} & -2(h_3 + h_1h_2)u_0 - 2\left(h_1^2 + 2h_2\right)u_1 - 2h_1u_1 - \\
& - \left[\left(h_1^2 + 2h_2\right)u_0^3 + 6h_1u_0^2u_1 + 3\left(u_0u_1^2 + u_0^2u_2\right)\right]
\end{aligned} \tag{2.47}$$

From Eqs. 2.44 and 2.7 we obtain

$$u_0 = a\cos\tau \tag{2.48}$$

Taking into consideration Eq. 2.48, Eq. 2.45 may be also written in the form

$$u''_1 + u_1 = -\left(2ah_1 + \frac{3a^3}{4}\right)\cos\tau - \frac{a^3}{4}\cos 3\tau \tag{2.49}$$

from which, by Eqs. 2.15, 2.7 and 2.18, it follows that

$$h_1 = -\frac{3a^2}{8} \tag{2.50}$$

and

$$u_1(\tau) = \frac{a^3}{32}(\cos 3\tau - \cos\tau) \tag{2.51}$$

By substituting the last two solutions into Eq. 2.46, we obtain

$$u''_2 + u_2 = \left(\frac{57a^5}{128} - 2ah_2\right)\cos\tau + \frac{3a^5}{16}\cos 3\tau - \frac{3a^5}{128}\cos 5\tau \tag{2.52}$$

and we deduce in the same way as above that

$$h_2 = \frac{57a^4}{256} \tag{2.53}$$

and

$$u_2(\tau) = \frac{3a^5}{128}(\cos\tau - \cos 3\tau) + \frac{a^5}{1024}(\cos 5\tau - \cos\tau) \tag{2.54}$$

Finally, by substituting Eqs. 2.50, 2.51, 2.53 and 2.54 into Eq. 2.47, we have

$$\begin{aligned} u''_3 + u_3 = \left(-\frac{1107a^7}{4096} - 2ah_3\right)\cos\tau - \frac{705a^7}{4096}\cos 3\tau + \\ + \frac{13a^7}{512}\cos 5\tau - \frac{a^7}{512}\cos 7\tau \end{aligned} \tag{2.55}$$

from which it follows that

$$h_3 = -\frac{1107a^6}{8192} \tag{2.56}$$

and

$$u_3(\tau) = \frac{705a^7}{32768}(\cos 3\tau - \cos\tau) + \frac{13a^7}{12288}(\cos\tau - \cos 5\tau) + \\ + \frac{a^7}{24576}(\cos 7\tau - \cos\tau) \tag{2.57}$$

Taking into consideration Eqs. 2.51, 2.54 and 2.57, we obtain, from Eq. 2.6, the approximate solution satisfying Eqs. 2.23 and 2.24 to within an error of third-order in ε:

$$u(\tau) = a\cos\tau + \frac{\varepsilon a^3}{32}(\cos 3\tau - \cos\tau) + \frac{\varepsilon^2 a^5}{1024}(\cos 5\tau - 24\cos 3\tau + 23\cos\tau) + \\ + \frac{\varepsilon^3 a^7}{98304}(4\cos 7\tau - 104\cos 5\tau + 2115\cos 3\tau - 2015\cos\tau) \tag{2.58}$$

Next, by introducing Eqs. 2.50, 2.53 and 2.56 into Eq. 2.3, we derive the corresponding approximation for the period

$$T = \frac{2\pi}{\omega}\left(1 - \frac{3\varepsilon a^2}{8} + \frac{57\varepsilon^2 a^4}{256} - \frac{1107\varepsilon^3 a^6}{8192}\right) \tag{2.59}$$

The approximate expression (2.59) of the period, as obtained by the perturbation method, coincides with the exact expansion (2.40), as expected, to within terms of third-order in small parameter ε.

2.1.3 Use of the Perturbation Method for Duffing Oscillators with Strong Parameter

Perturbation method provides the most versatile tool available in nonlinear dynamical systems and is constantly developed and applied to ever more complex problems. But, perturbation method has its own limitation: it is based on such assumption that a small parameter must exist in equation. This so called small parameter assumption greatly restricts applications of perturbation techniques, and as it is well-known, an overwhelming majority of nonlinear problems, especially those having strong nonlinearity, have no small parameters at all.

It is even more difficult to determine the so-called small parameter, which seems to be a special art requiring special techniques. An appropriate choice of small parameter may lead to ideal results; however an unsuitable choice of small parameter results in badly effects, sometimes seriously. Even if there exists suitable small parameter, the approximate solutions obtained by the perturbation method are valid, in most cases, only for small values for the parameter.

To overcome this limitation, many novel techniques have been proposed in recent years. For example Cheung et al. [38] presented a modified Lindstedt-Poincaré method. They expand ω^2 instead of ω:

$$\omega^2 = \omega_0^2 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots \tag{2.60}$$

and a new parameter is defined as

$$\alpha = \frac{\varepsilon h_1}{\omega_0^2 + \varepsilon h_1} \tag{2.61}$$

It is much better to expand u and ω^2 into a power series with respect to α:

$$u = u_0 + \alpha u_1 + \alpha^2 u_2 + \ldots \tag{2.62}$$

$$\omega^2 = \omega_0^2 + \alpha h_1 + \alpha^2 h_2 + \ldots \tag{2.63}$$

The essence of the proposed method is to expand the coefficient 1 from the equation $\ddot{u} + u = f(u)$ rather than the nonlinear frequency into a series of ε:

$$1 = \omega^2 + \varepsilon C_1 + \varepsilon^2 C_2 + \ldots \tag{2.64}$$

where the constants ω^2 and C_i can be identified by means of no secular terms.

Dai [39] introduced another transformation of the independent variable τ:

$$t = \tau + \varepsilon F(u(\tau)) + 0(\varepsilon^2)$$

where $F(u(\tau))$ is an unknown nonlinear functional further determined.

J.H. He [40–43] proposed some perturbation techniques (Taylor expansion, artificial parameter, linear perturbation method, parameterized perturbation method, bookkeeping artificial parameter perturbation method, iteration perturbation method, etc.) which are valid for large parameters. In what follows we present two versions of modified Lindstedt-Poincaré method applied for the Duffing nonlinear oscillator.

(a) In the first version, in 2004 H. Hu [44] presented a classical perturbation technique which is valid for large parameters. Hu considered the Duffing equations in the form

$$\ddot{u} + u + \varepsilon u^3 = 0 \tag{2.65}$$

and the initial condition

$$u(0) = a, \quad \dot{u}(0) = 0 \tag{2.66}$$

where ε is a parameter which does not need be small.

For Eq. 2.65, the solution is assumed to be in the form

$$u(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \ldots \tag{2.67}$$

and the fundamental frequency in the form [38], [42], [44]:

$$\Omega^2 = 1 + \varepsilon h_1 + \varepsilon h_2 + \ldots \tag{2.68}$$

where h_i are unknown constants.

By substituting Eqs. 2.67 and 2.68 into Eq. 2.65 we obtain

$$\begin{gathered}\left(\ddot{u}_0 + \varepsilon\ddot{u}_1 + \varepsilon^2\ddot{u}_2 + \ldots\right) + \left(\Omega^2 - \varepsilon h_1 - \varepsilon^2 h_2 \ldots\right)\left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots\right) + \\ + \varepsilon\left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots\right)^3 = 0\end{gathered} \tag{2.69}$$

By equating coefficients of like powers of ε, and processing as the standard perturbation method, we have

$$\ddot{u}_0 + \Omega^2 u_0 = 0 \tag{2.70}$$

$$\ddot{u}_1 + \Omega^2 u_1 = h_1 u_0 - u_0^3 \tag{2.71}$$

$$\ddot{u}_2 + \Omega^2 u_2 = h_2 u_0 + h_1 u_1 - 3u_0^3 u_1 \tag{2.72}$$

and so on.

The initial conditions are replaced by

$$u_0(0) = a, \quad \dot{u}_0(0) = 0, \quad u_i(0) = \dot{u}_i(0) = 0, \quad i = 1, 2, \ldots \tag{2.73}$$

Solving Eqs. 2.70 and 2.73, we have

$$u_0(t) = a\cos\Omega t \tag{2.74}$$

Substitution of u_0 into Eq. 2.71, results into

$$\ddot{u}_1 + \Omega^2 u_1 = \left(ah_1 - \frac{3}{4}a^3\right)\cos\Omega t - \frac{3}{4}a^3\cos 3\Omega t \tag{2.75}$$

Eliminating the secular term needs

$$h_1 = \frac{3}{4}a^2 \tag{2.76}$$

and thus the solution of Eq. 2.75 becomes

$$u_1(t) = \frac{a^3}{32\Omega^2}(\cos 3\Omega t - \cos \Omega t) \tag{2.77}$$

Now, by substituting Eqs. 2.76 and 2.77 into Eq. 2.72, we have the following equation

$$\ddot{u}_2 + \Omega^2 u_2 = \left(ah_2 + \frac{3a^5}{128\Omega^2}\right)\cos \omega t - \frac{3a^5}{128\Omega^2}\cos 5\Omega t \tag{2.78}$$

Avoiding the presence of secular terms requires

$$h_2 = -\frac{3a^4}{128\Omega^2} \tag{2.79}$$

The solution of Eq. 2.78 with the initial conditions (2.73) becomes

$$u_2(t) = \frac{a^5}{3072\Omega^4}(\cos 5\Omega t - \cos \Omega t) \tag{2.80}$$

If the second order approximation is sufficient, from relation (2.68) we have

$$\Omega^2 = 1 + \frac{3\varepsilon a^2}{4} - \frac{3\varepsilon^2 a^4}{128\Omega^2} \tag{2.81}$$

Solving for Ω gives

$$\Omega_{app} = \frac{1}{4}\sqrt{8 + 6\varepsilon a^2 + \sqrt{64 + 96\varepsilon a^2 + 30\varepsilon^2 a^4}} \tag{2.82}$$

The second approximate solution of Eq. 2.65 is obtained from Eqs. 2.67, 2.74, 2.77 and 2.80:

$$u(t) = a\cos \Omega t + \frac{\varepsilon a^3}{32\Omega^2}(\cos 3\Omega t - \cos \Omega t) + \frac{\varepsilon^2 a^5}{3072\Omega^4}(\cos 5\Omega t - \cos \Omega t) \tag{2.83}$$

where the frequency Ω is given by Eq. 2.82.

The exact frequency of the periodic motion of the Duffing equation is obtained from Eq. 2.37 with $\omega = 1$, because into Eqs. 2.65 and 2.24 ω is considered equal with 1:

$$\Omega_{ex} = \frac{\pi}{2}\sqrt{1+\varepsilon a^2}\left(\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-m^2\sin^2\theta}}\right)^{-1}, m = \frac{\varepsilon a^2}{2(1+\varepsilon a^2)} \tag{2.84}$$

For comparison, the exact frequency obtained by Eq. 2.84 and the approximate frequency given by Eq. 2.82 are listed in Table 2.1

Table 2.1 indicates that the formula (2.82) can give excellent approximate frequencies for both small and large parameters.

So, for large ε, the present approximate frequencies have the same feature as the exact one, even in case $\varepsilon a^2 \to \infty$. We have

$$\lim_{\varepsilon a^2\to\infty}\frac{\Omega_{app}}{\Omega_{ex}} = \frac{\sqrt{6+\sqrt{30}}}{2\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-0,5\sin^2\theta}} \approx 0,999691 \tag{2.85}$$

Therefore, for any value of ε, the maximum relative error of the second approximate frequency obtained for the Duffing equation by means of traditional perturbation technique, is less than 0.03%, with respect to the exact solution

It is at least strange that this traditional perturbation method work for large parameters.

If normal expansion is used for the frequency

$$\Omega = 1 + \varepsilon C_1 + \varepsilon^2 C_2 + \ldots \tag{2.86}$$

instead of expansion (2.68), the results are different in the cases of small and large parameters. For example, from Eq. 2.58 we derive the corresponding first approximate frequency (for $\omega = 1$) in the case that ε is small:

$$\Omega = 1 + \frac{3\varepsilon a^2}{8} \tag{2.87}$$

Table 2.1 Comparison of approximate frequencies with the corresponding exact frequency for the Duffing equation

εa^2	Ω_{app} (2.82)	Ω_{ex} (2.84)
0.2	1.07200	1.07200
0.6	1.20173	1.20173
1	1.31776	1.31778
5	2.15018	2.15042
10	2.86613	2.86664
100	8.53110	8.53359
1,000	26.8025	26.8107
10,000	84.7013	84.7245

On the other hand, the first approximate frequency, obtained from Eqs. 2.76 and 2.68 for any ε is

$$\Omega = \sqrt{1 + \frac{3}{4}\varepsilon a^2} \tag{2.88}$$

Formula (2.88) can give good approximate frequencies for both small and large parameters, but for large parameters, formula (2.87) is not valid, because

$$\sqrt{1 + \frac{3}{4}\varepsilon a^2} \neq 1 + \frac{3}{8}\varepsilon a^2.$$

We believe that the Duffing equation is one of rather equations in which the expansion (2.68) can be used instead of Eq. 2.86.

In Fig. 2.1 is presented a comparison between the numerical solution and analytic solution (2.83) of Eq. 2.65 in the case $a = \omega = 1$, $\varepsilon = 0.1$ while Fig. 2.2 presents a similar comparison for the analytic solution (2.83) of the Eq. 2.65 in the case $a = \omega = 1$, $\varepsilon = 1$. A very good agreement was found between the numerical and analytical results for Eq. 2.65 in both cases.

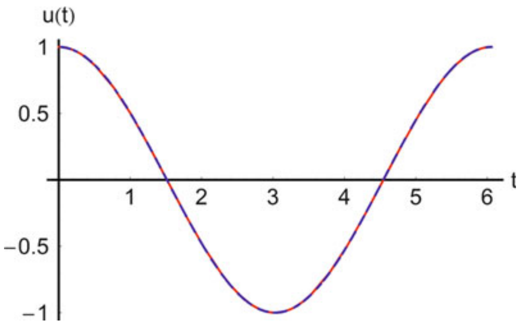

Fig. 2.1 Comparison between the results obtained for Eq. 2.65: ______numerical solution; _ _ _ _ analytic solution (2.83) for $a = \omega = 1$, $\varepsilon = 0.1$

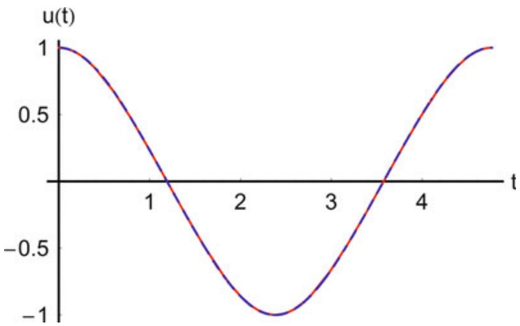

Fig. 2.2 Comparison between the results obtained for Eq. 2.65: _____numerical solution; _ _ _ _ analytic solution (2.83) for $a = \omega = 1$, $\varepsilon = 1$

It is clear that the Lindstedt-Poincaré procedure produces a periodic expression describing the motion of the system, a frequency-amplitude relationship which is a direct consequence of requiring the expression to be periodic and higher harmonics in the higher-order terms of the expression.

(b) In the second version we expand ω^2 in series (2.60) and define the parameter α by Eq. 2.61 such that

$$\varepsilon = \frac{\alpha}{h_1(1-\alpha)}, \quad (\omega_0 = 1) \tag{2.89}$$

With the transformation $\tau = \Omega t$, Eq. 2.65 becomes

$$\Omega^2 u'' + u + \varepsilon u^3 = 0 \tag{2.90}$$

where primes denotes differentiation with respect to τ. The fundamental frequency Ω given by (2.68) becomes

$$\begin{aligned}\Omega^2 &= \left(1 + \frac{\alpha}{1-\alpha}\right)\left(1 + \lambda_2\alpha^2 + \lambda^3\alpha^3 + \ldots\right)\\ &= \frac{1}{1-\alpha}\left(1 + \lambda_2\alpha^2 + \lambda^3\alpha^3 + \ldots\right)\end{aligned} \tag{2.91}$$

where h_1, λ_2, $\lambda_3 \ldots$ are unknown constants which will be determined in the later by perturbation steps successively.

Substituting Eqs. 2.89 and 2.91 into Eq. 2.90 yields:

$$\begin{aligned}&\left(1 + \lambda_2\alpha^2 + \lambda^3\alpha^3 + \ldots\right)\left(u''_0 + \alpha u''_1 + \alpha^2 u''_2 + \ldots\right) +\\ &\quad + (1-\alpha)\left(u_0 + \alpha u_1 + \alpha^2 u_2 + \ldots\right) + \frac{\alpha}{h_1}\left(u_0 + \alpha u_1 + \alpha^2 u_2 + \ldots\right)^3 = 0\end{aligned} \tag{2.92}$$

It can be seen from Eq. 2.61 that the value of α is always kept small regardless to magnitude of εh_1. It is observed that α is a new small parameter which is considered to be better than ε. It will enable a strongly nonlinear system corresponding to ε be transformed into a small parameter system with respect to α.

Equating the coefficients of like terms of α into Eq. 2.92 the following set of linear differential equation can be obtained

$$u''_0 + u_0 = 0, u_0(0) = a, u'_0(0) = 0 \tag{2.93}$$

$$u''_1 + u_1 = u_0 - \frac{u_0^3}{h_1}, u_1(0) = 0, u'_1(0) = 0 \tag{2.94}$$

$$u''_2 + u_2 = u_1 + \lambda_2 u_0 - \frac{3u_0^2 u_1}{h_1}, u_2(0) = 0, u'_2(0) = 0 \tag{2.95}$$

From Eq. 2.93 we obtain

$$u_0 = a\cos\tau \tag{2.96}$$

Equation 2.94 becomes

$$u''_1 + u_1 = a\left(1 - \frac{3a^2}{4h_1}\right)\cos\tau - \frac{a^3}{4h_1}\cos 3\tau, u_1(0) = 0, u'_1(0) = 0 \tag{2.97}$$

In order to ensure that no secular terms appear in the next iteration, resonance must be avoided, so that the coefficient of $\cos\tau$ in Eq. 2.97 requires being zero:

$$h_1 = \frac{3}{4}a^2 \tag{2.98}$$

So, from Eq. 2.97, we have the following solution

$$u_1 = \frac{a}{24}(\cos 3\tau - \cos\tau) \tag{2.99}$$

Substituting Eqs. 2.96 and 2.99 into Eq. 2.95, yields

$$u''_2 + u_2 = a\left(\lambda_2 + \frac{1}{24}\right)\cos\tau - \frac{1}{24}a\cos 5\tau \tag{2.100}$$

Avoiding the secular term into Eq. 2.100, we obtain

$$\lambda_2 = -\frac{1}{24} \tag{2.101}$$

and therefore Eq. 2.91 becomes

$$\Omega^2 = \frac{1}{1-\alpha}\left(1 + \lambda_2\alpha^2 + \ldots\right) = 1 + \frac{3}{4}\varepsilon a^2 - \frac{3\varepsilon^2 a^4}{128 + 96\varepsilon a^2} + \ldots \tag{2.102}$$

To illustrate the remarkable accuracy of the obtained results, we compare the approximate period

$$T_{app} = \frac{2\pi}{\Omega} = \frac{2\pi}{\sqrt{1 + \frac{3}{4}\varepsilon a^2 - \frac{3\varepsilon^2 a^4}{128+96\varepsilon a^2}}} \tag{2.103}$$

with the exact one obtained from Eq. 2.84

$$T_{ex} = \frac{4}{\sqrt{1+\varepsilon a^2}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}, \quad m = \frac{\varepsilon a^2}{2(1+\varepsilon a^2)} \tag{2.104}$$

What is rather surprising about the remarkable range of validity of Eq. 2.103 is that the actual asymptotic period is also of higher accuracy as $\varepsilon\alpha^2 \to \infty$

$$\lim_{\varepsilon a^2 \to \infty} \frac{T_{ex}}{T_{app}} = \frac{2\sqrt{\frac{17}{24}}}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - 0,5\sin^2\theta}} = 0,999271 \tag{2.105}$$

Therefore, the approximate analytical solution of the Duffing equation has an error which never exceeds 0.07% with respect to the exact solution.

These extensions of the Lindstedt-Poincaré method, which are simple and easy to use, are effective methods for dealing with strongly non-linear vibration of single degree of freedom systems which cannot be treated by the standard Lindstedt-Poincaré method.

Remark. In general, perturbation methods work very well for weakly nonlinear dynamical systems and there exist cases when this procedure leads to inappropriate results. For instance, we consider the weakly nonlinear system

$$\begin{aligned} \dot{x} &= 0.5x - \varepsilon xy, & x(0) &= 4 \\ \dot{y} &= -0.3y + 2\varepsilon xy & y(0) &= 1 \end{aligned} \tag{2.106}$$

where ε is a small parameter and dot denotes derivative with respect to time t. For Eq. 2.106 we may suppose the power series

$$\begin{aligned} x(t) &= x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \varepsilon^3 x_3(t) + ... \\ y(t) &= y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \varepsilon^3 y_0(t) + ... \end{aligned} \tag{2.107}$$

By substituting Eqs. 2.107 into Eqs. (2.106) and equating the coefficients of like powers of ε, we obtain the following linear differential equations:

$$\begin{aligned} \dot{x}_0 &= 0.5x_0, & x_0(0) &= 4 \\ \dot{y}_0 &= -0.3y_0, & y_0(0) &= 1 \end{aligned} \tag{2.108}$$

$$\begin{aligned} \dot{x}_1 &= 0.5x_1 - x_0y_0, & x_1(0) &= 0 \\ \dot{y}_1 &= -0.3y_1 + 2x_0y_0, & y_1(0) &= 0 \end{aligned} \tag{2.109}$$

$$\begin{aligned} \dot{x}_2 &= 0.5x_2 - (x_0y_1 + x_1y_0), & x_2(0) &= 0 \\ \dot{y}_2 &= -0.3y_2 + 2(x_0y_1 + x_1y_0), & y_2(0) &= 0 \end{aligned} \tag{2.110}$$

$$\begin{aligned} \dot{x}_3 &= 0.5x_3 - (x_0y_2 + x_1y_1 + x_2y_0), & x_3(0) &= 0 \\ \dot{y}_3 &= -0.3y_3 + 2(x_0y_2 + x_1y_1 + x_2y_0), & y_3(0) &= 0 \end{aligned} \tag{2.111}$$

The Eqs. 2.108 yields

$$\begin{aligned} x_0 &= 4e^{0.5t} \\ y_0 &= e^{-0.3t} \end{aligned} \tag{2.112}$$

By substituting Eq. 2.112 into Eq. 2.109 and solving these equations, we obtain

$$\begin{aligned} x_1 &= \frac{40}{3}e^{0.2t} - \frac{40}{3}e^{0.5t} \\ y_1 &= 16e^{0.2t} - 16e^{-0.3t} \end{aligned} \tag{2.113}$$

Now, substituting Eqs. 2.112 and 2.113 into Eq. 2.110 and solving these equations, it results

$$\begin{aligned} x_2 &= \frac{5000}{9}e^{0.5t} - 320e^{0.7t} - \frac{2320}{9}e^{0.2t} + \frac{200}{9}e^{-0.1t} \\ y_2 &= 48e^{-0.3t} + 128e^{0.7t} - \frac{928}{3}e^{0.2t} + \frac{400}{3}e^{-0.1t} \end{aligned} \tag{2.114}$$

From Eqs. 2.111, 2.112, 2.113 and 2.114 we obtain

$$\begin{aligned} x_3 &= -\frac{7502000}{567}e^{0.5t} - \frac{5120}{7}e^{1.2t} + \frac{21760}{3}e^{0.7t} + \frac{12800}{3}e^{0.4t} \\ &\quad + \frac{86400}{27}e^{0.2t} - \frac{21200}{27}e^{-0.1t} + \frac{2000}{81}e^{-0.4t} \\ y_3 &= \frac{48544}{21}e^{-0.3t} + \frac{2048}{3}e^{1.2t} - \frac{8704}{3}e^{0.7t} + \frac{25600}{21}e^{0.4t} \\ &\quad + \frac{34592}{9}e^{0.2t} - \frac{42400}{9}e^{-0.1t} - \frac{4000}{9}e^{-0.4t} \end{aligned} \tag{2.115}$$

Finally, from Eqs. 2.107, 2.112, 2.113, 2.114 and 2.115, for $\varepsilon = 0.001$, we obtain an approximate solution of the fourth-order in the form

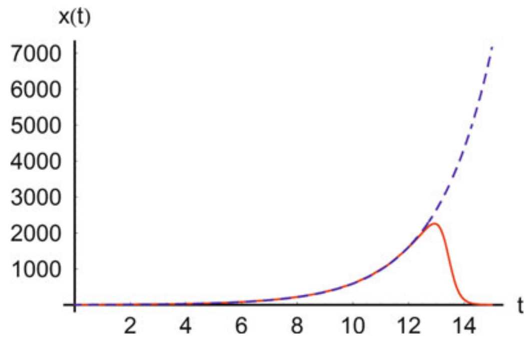

Fig. 2.3 Comparison between the numerical solution x(t) of Eq. 2.106 and approximate solution (2.116): _____numerical solution; _ _ _ _ approximate solution

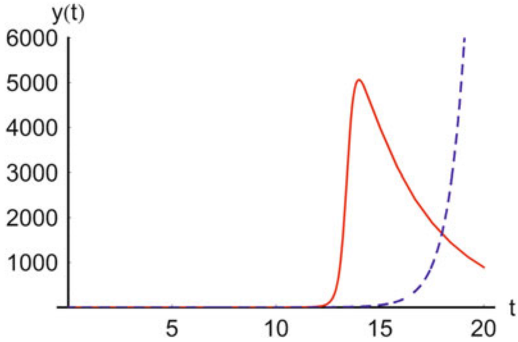

Fig. 2.4 Comparison between the numerical solution y(t) of Eq. 2.106 and approximate solution (2.116): _____numerical solution; _ _ _ _ approximate solution

$$\begin{aligned} x(t) &= -0.00000073e^{1.2t} - 0.00031274e^{0.71t} + 3.9872e^{0.5t} + \\ &+ 0.00000426e^{0.4t} + 0.01307e^{0.2t} + 0.00002143e^{-0.1t} + 2.46 \cdot 10^{-14}e^{-0.4t} \\ y(t) &= 0.0000006826e^{1.2t} + 0.00012509e^{0.71t} + 0.000001219e^{0.4t} + \\ &+ 0.01569e^{0.2t} + 0.0001286e^{-0.1t} + 0.98405e^{-0.3t} - 0.000000444e^{-0.4t} \end{aligned} \tag{2.116}$$

In Figs. 2.3 and 2.4 is presented a comparison between the numerical and approximate solutions (2.116) of the Eqs. (2.106) in the case of $\varepsilon = 0.001$, where an important discrepancy between these solutions can be observed.

In conclusion, the perturbation method does not work very well in all cases, even though in the differential equations is involved a small parameter. Therefore, this remark lead to the conclusion that more powerful methods should be developed to overcome these difficulties. Some suitable methods will be presented in Chaps. 6–9.

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