Chapter 2
Expanding Random Maps

For the convenience of the reader, we first give some introductory examples. In the remaining part of this chapter we present the general framework of *expanding random maps*.

2.1 Introductory Examples

Before giving the formal definitions of expanding random maps, let us now consider some typical examples.

The first one is a known random version of the Sierpiński gasket (see, for example [15]). Let $\Delta = \Delta(A, B, C)$ be a triangle with vertexes A, B, C and choose $a \in (A, B)$, $b \in (B, C)$ and $c \in (C, A)$. Then we can associate to $x = (a, b, c)$ a map

$$f_x : \Delta(A, a, c) \cup \Delta(a, B, b) \cup \Delta(b, C, a) \to \Delta,$$

such that the restriction of f_x to each one of the three subtriangles is a affine map onto Δ. The map f_x is nothing else than the generator of a deterministic Sierpiński gasket. Note that this map can be made continuous by identifying the vertices A, B, C (Fig. 2.1).

Now, suppose $x_1 = (a_1, b_1, c_1)$, $x_2 = (a_2, b_2, c_2), \ldots$ are chosen randomly which, for example, may mean that they form sequences of three dimensional independent and identically distributed (i.i.d.) random variables. Then they generate compact sets

$$\mathcal{J}_{x_1,x_2,x_3,\ldots} = \bigcap_{n \geq 1} (f_{x_n} \circ \ldots \circ f_{x_1})^{-1}(\Delta)$$

called *random Sierpiński gaskets* having the invariance property $f_{x_1}^{-1}(\mathcal{J}_{x_2,x_3,\ldots}) = \mathcal{J}_{x_1,x_2,x_3,\ldots}$. For a little bit simpler example of random Cantor sets we refer the reader to Sect. 5.3. In that example we provide a more detailed analysis of such random sets.

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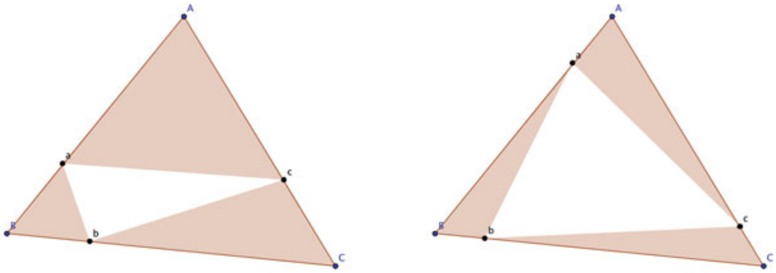

Fig. 2.1 Two different generators of Sierpiński gaskets

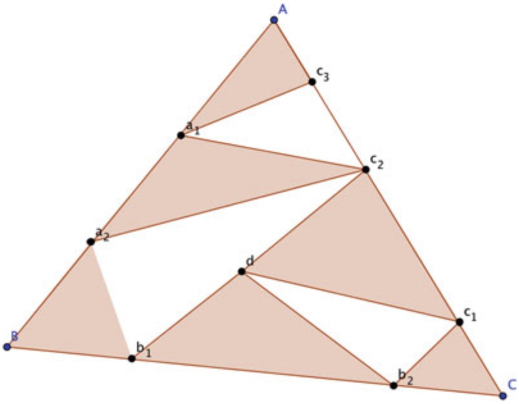

Fig. 2.2 A generator of degree 6

Such examples admit far going generalizations. First of all, we will consider much more general random choices than i.i.d. ones. We model randomness by taking a probability space $(X, \mathscr{B}, m)$ along with an invariant ergodic transformation $\theta : X \to X$. This point of view was up to our knowledge introduced by the Bremen group (see [1]).

Another point is that the maps f_x that generate the random Sierpiński gasket have degree 3. In the sequel of this manuscript, we will allow the degree d_x of all maps to be different (see Fig. 2.2) and only require that the function $x \mapsto \log(d_x)$ is measurable.

Finally, the above examples are all expanding with an expanding constant

$$\gamma_x \geq \gamma > 1 .$$

As already explained in the introduction, the present monograph concerns random maps for which the expanding constants γ_x can be arbitrarily close to one. Furthermore, using an inducing procedure, we will even weaken this to the maps that are only expanding in the mean (see Chap. 7).

The example of random Sierpiński gasket is not conformal. Random iterations of rational functions or of holomorphic repellers are typical examples of conformal random dynamical systems. Random iterations of the quadratic family $f_c(z) = z^2 + c$ have been considered, for example, by Brück and Büger among others (see [8] and [9]). In this case, one chooses randomly a sequence of bounded parameters $c = (c_1, c_2, ...)$ and considers the dynamics of the family

$$F_{c_1,\ldots,c_n} = f_{c_n} \circ f_{c_{n_1}} \circ \ldots \circ f_{c_1}, \quad n \geq 1.$$

This leads to the dynamical invariant sets

$$\mathscr{K}_c = \{z \in \mathbb{C};\ F_{c_1,\ldots,c_n}(z) \not\to \infty\} \quad \text{and} \quad \mathscr{J}_c = \partial \mathscr{K}_c.$$

The set $\mathscr{K}_c$ is the filled in Julia set and $\mathscr{J}_c$ the Julia set associated to the sequence c.

The simplest case is certainly the one when we consider just two polynomials $z \mapsto z^2 + \lambda_1$ and $z \mapsto z^2 + \lambda_2$ and we build a random sequence out of them. Julia sets that come out of such a choice are presented in Fig. 2.3. Such random Julia sets are different objects as compared to the Julia sets for deterministic iteration of quadratic polynomials. But not only the pictures are different and intriguing, we

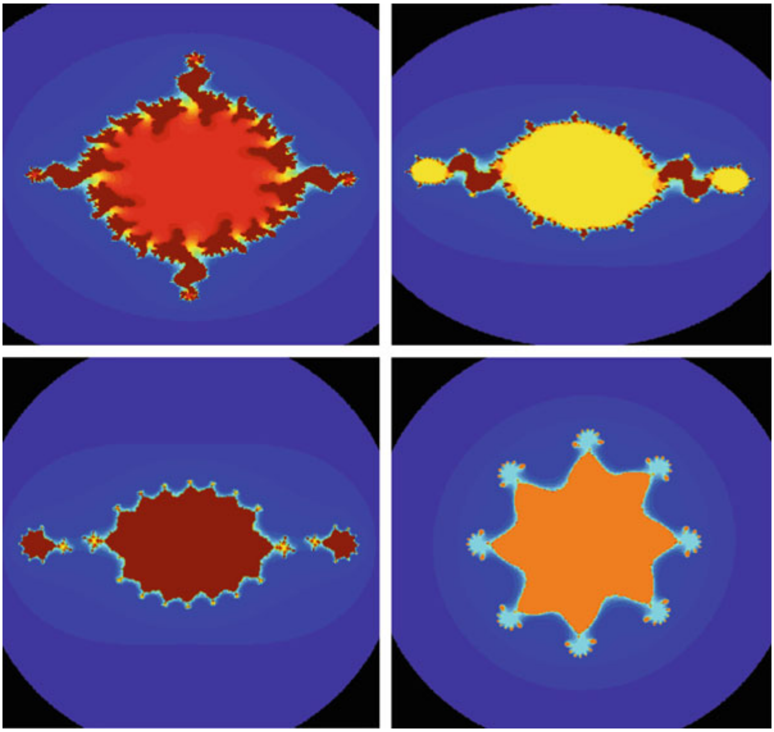

Fig. 2.3 Some quadratic random Julia sets

will see in Chap. 5 that also generically the fractal properties of such Julia sets are fairly different as compared with the deterministic case even if the dynamics are uniformly expanding. In Chap. 8 we present a more general class of examples and we explain their dynamical and fractal features.

2.2 Preliminaries

Suppose $(X, \mathscr{B}, m, \theta)$ is a measure preserving dynamical system with invertible and ergodic map $\theta : X \to X$ which is referred to as *the base map*. Assume further that $(\mathscr{J}_x, \rho_x)$, $x \in X$, are compact metric spaces normalized in size by $\mathrm{diam}_{\rho_x}(\mathscr{J}_x) \le 1$. Let

$$\mathscr{J} = \bigcup_{x \in X} \{x\} \times \mathscr{J}_x. \tag{2.1}$$

We will denote by $B_x(z, r)$ the ball in the space $(\mathscr{J}_x, \varrho_x)$ centered at $z \in \mathscr{J}_x$ and with radius r. Frequently, for ease of notation, we will write $B(y, r)$ for $B_x(z, r)$, where $y = (x, z)$. Let

$$T_x : \mathscr{J}_x \to \mathscr{J}_{\theta(x)}, \quad x \in X,$$

be continuous mappings and let $T : \mathscr{J} \to \mathscr{J}$ be the associated skew-product defined by

$$T(x, z) = (\theta(x), T_x(z)). \tag{2.2}$$

For every $n \ge 0$ we denote $T_x^n := T_{\theta^{n-1}(x)} \circ ... \circ T_x : \mathscr{J}_x \to \mathscr{J}_{\theta^n(x)}$. With this notation one has $T^n(x, y) = (\theta^n(x), T_x^n(y))$. We will frequently use the notation

$$x_n = \theta^n(x), \quad n \in \mathbb{Z}.$$

If it does not lead to misunderstanding we will identify $\mathscr{J}_x$ and $\{x\} \times \mathscr{J}_x$.

2.3 Expanding Random Maps

A map $T : \mathscr{J} \to \mathscr{J}$ is called a *expanding random map* if the mappings $T_x : \mathscr{J}_x \to \mathscr{J}_{\theta(x)}$ are continuous, open, and surjective, and if there exist a function $\eta : X \to \mathbb{R}_+$, $x \mapsto \eta_x$, and a real number $\xi > 0$ such that following conditions hold.

Uniform Openness. $T_x(B_x(z, \eta_x)) \supset B_{\theta(x)}\big(T_x(z), \xi\big)$ for every $(x, z) \in \mathscr{J}$.

Measurably Expanding. There exists a measurable function $\gamma : X \to (1, +\infty)$, $x \mapsto \gamma_x$ such that, for m-a.e. $x \in X$,

$$\varrho_{\theta(x)}(T_x(z_1), T_x(z_2)) \ge \gamma_x \varrho_x(z_1, z_2) \quad \textit{whenever} \quad \varrho(z_1, z_2) < \eta_x, z_1, z_2 \in \mathscr{J}_x .$$

Measurability of the Degree. The map $x \mapsto \deg(T_x) := \sup_{y \in \mathcal{J}_{\theta(x)}} \# T_x^{-1}(\{y\})$ is measurable.

Topological Exactness. There exists a measurable function $x \mapsto n_\xi(x)$ such that

$$T_x^{n_\xi(x)}(B_x(z,\xi)) = \mathcal{J}_{\theta^{n_\xi(x)}(x)} \quad \text{for every } z \in \mathcal{J}_x \text{ and a.e. } x \in X. \tag{2.3}$$

Note that the measurably expanding condition implies that $T_x|_{B(z,\eta_x)}$ is injective for every $(x,z) \in \mathcal{J}$. Together with the compactness of the spaces $\mathcal{J}_x$ it yields the numbers $\deg(T_x)$ to be finite. Therefore the supremum in the condition of measurability of the degree is in fact a maximum.

In this work we consider two other classes of random maps. The first one consists of the *uniform expanding* maps defined below. These are expanding random maps with uniform control of measurable "constants". The other class we consider is composed of maps that are only *expanding in the mean*. These maps are defined like the expanding random maps above excepted that the uniform openness and the measurable expanding conditions are replaced by the following weaker conditions (see Chap. 7 for detailed definition).

1. All local inverse branches do exist.
2. The function γ in the measurable expanding condition is allowed to have values in $(0,\infty)$ but subjects only the condition

$$\int_X \log \gamma_x \, dm > 0.$$

We employ an inducing procedure to expanding in the mean random maps in order to reduce then to the case of random expanding maps. This is the content of Chap. 7 and the conclusion is that all the results of the present work valid for expanding random maps do also hold for expanding in the mean random maps.

2.4 Uniformly Expanding Random Maps

Most of this paper and, in particular, the whole thermodynamical formalism is devoted to measurable expanding systems. The study of fractal and geometric properties (which starts with Chap. 5), somewhat against our general philosophy, but with agreement with the existing tradition (see for example [5,12,17]), we will work mostly with *uniform* and *conformal* systems (the later are introduced in Chap. 5).

A expanding random map $T : \mathcal{J} \to \mathcal{J}$ is called *uniformly expanding* if

- $\gamma_* := \inf_{x \in X} \gamma_x > 1$,
- $\deg(T) := \sup_{x \in X} \deg(T_x) < \infty$,
- $n_{\xi *} := \sup_{x \in X} n_\xi(x) < \infty$.

2.5 Remarks on Expanding Random Mappings

The conditions of uniform openness and measurably expanding imply that, for every $y = (x, z) \in \mathcal{J}$ there exists a unique continuous inverse branch

$$T_y^{-1} : B_{\theta(x)}(T(y), \xi) \to B_x(y, \eta_x)$$

of T_x sending $T_x(z)$ to z. By the measurably expanding property we have

$$\varrho(T_y^{-1}(z_1), T_y^{-1}(z_2)) \le \gamma_x^{-1} \varrho(z_1, z_2) \quad \text{for} \quad z_1, z_2 \in B_{\theta(x)}\big(T(y), \xi\big) \tag{2.4}$$

and

$$T_y^{-1}(B_{\theta(x)}(T(y), \xi)) \subset B_x(y, \gamma_x^{-1}\xi) \subset B_x(y, \xi).$$

Hence, for every $n \ge 0$, the composition

$$T_y^{-n} = T_y^{-1} \circ T_{T(y)}^{-1} \circ \ldots \circ T_{T^{n-1}(y)}^{-1} : B_{\theta^n(x)}(T^n(y), \xi) \to B_x(y, \xi) \tag{2.5}$$

is well defined and has the following properties:

$$T_y^{-n} : B_{\theta^n(x)}(T^n(y), \xi) \to B_x(y, \xi)$$

is continuous,

$$T^n \circ T_y^{-n} = \mathrm{Id}|_{B_{\theta^n(x)}(T^n(y), \xi)}, \quad T_y^{-n}(T_x^n(z)) = z$$

and, for every $z_1, z_2 \in B_{\theta^n(x)}\big(T^n(y), \xi\big)$,

$$\varrho(T_y^{-n}(z_1), T_y^{-n}(z_2)) \le (\gamma_x^n)^{-1} \varrho(z_1, z_2), \tag{2.6}$$

where $\gamma_x^n = \gamma_x \gamma_{\theta(x)} \cdots \gamma_{\theta^{n-1}(x)}$. Moreover,

$$T_x^{-n}(B_{\theta^n(x)}(T^n(y), \xi)) \subset B_x(y, (\gamma_x^n)^{-1}\xi) \subset B_x(y, \xi). \tag{2.7}$$

Lemma 2.1 *For every $r > 0$, there exists a measurable function $x \mapsto n_r(x)$ such that a.e.*

$$T_x^{n_r(x)}(B_x(z, r)) = \mathcal{J}_{\theta^{n_r(x)}(x)} \quad \textit{for every } z \in \mathcal{J}_x. \tag{2.8}$$

Moreover, there exists a measurable function $j : X \to \mathbb{N}$ such that a.e. we have

$$T_{x_{-j(x)}}^{j(x)}(B_{x_{-j(x)}}(z, \xi)) = \mathcal{J}_x \quad \textit{for every } \ z \in \mathcal{J}_{x_{-j(x)}}. \tag{2.9}$$

Proof. In order to prove the first statement, consider $\gamma_0 > 1$ and let F be the set of $x \in X$ such that $\gamma_x \ge \gamma_0$. If γ_0 is sufficiently close to 1, then $m(F) > 0$. In the following section such a set will be called essential. In that section we also associate to such an essential set a set X'_{+F} (see (2.10)). Then for $x \in X'_{+F}$, the limit $\lim_{n\to\infty}(\gamma_x^n)^{-1} = 0$. Define

$$X_{+F,k} := \{x \in X'_{+F} : (\gamma_x^k)^{-1}\xi < r\}.$$

Then $X_{+F,k} \subset X_{+F,k+1}$ and $\bigcup_{k\in\mathbb{N}} X_{+F,k} = X'_{+F}$. By measurability of $x \mapsto \gamma_x$, $X_{+F,k}$ is a measurable set. Hence the function

$$X'_{+F} \ni x \mapsto n_r(x) := \min\{k : x \in X_{+F,k}\} + n_\xi(x)$$

is finite and measurable. By (2.7) and (2.3),

$$T_x^{n_r(x)}(B_x(z,r)) = \mathscr{J}_{\theta^{n_r(x)}(x)}.$$

In order to prove the existence of a measurable function $j : X \to \mathbb{N}$ define measurable sets

$$X_{\xi,n} := \{x \in X : n_\xi(x) \leq n\},\ X'_{\xi,n} := \theta^n(X_{\xi,n}) \text{ and } X'_\xi = \bigcup_{n\in\mathbb{N}} X'_{\xi,n}.$$

Then the map

$$X'_\xi \ni x \mapsto j(x) := \min\{n \in \mathbb{N} : x \in X'_{\xi,n}\}$$

satisfies (2.9) for $x \in X'_\xi$. Since $m(\theta^n(X_{\xi,n})) = m(X_{\xi,n}) \nearrow 1$ as n tends to ∞ we have $m(X'_\xi) = 1$. □

2.6 Visiting Sequences

Let $F \in \mathscr{F}$ be a set with a positive measure. Define the sets

$$V_{+F}(x) := \{n \in \mathbb{N} : \theta^n(x) \in F\} \quad \text{and} \quad V_{-F}(x) := \{n \in \mathbb{N} : \theta^{-n}(x) \in F\}.$$

The set $V_{+F}(x)$ is called *visiting sequence* (*of F at x*). Then the set $V_{-F}(x)$ is just a visiting sequence for θ^{-1} and we also call it *backward visiting sequence*. By $n_j(x)$ we denote the jth-visit in F at x. Since $m(F) > 0$, by Birkhoff's Ergodic Theorem we have that

$$m(X'_{+F}) = m(X'_{-F}) = 1,$$

where

$$X'_{+F} := \left\{x \in X : V_{+F}(x) \text{ is infinite and } \lim_{j\to\infty} \frac{n_{j+1}(x)}{n_j(x)} = 1\right\} \tag{2.10}$$

and X'_{-F} is defined analogously. The sets X'_{+F} and X'_{-F} are respectively called *forward* and *backward visiting for* F.

Let $\Psi(x,n)$ be a formula which depends on $x \in X$ and $n \in \mathbb{N}$. We say that $\Psi(x,n)$ holds *in a visiting way*, if there exists F with $m(F) > 0$ such that, for m-a.e. $x \in X'_{+F}$ and all $j \in \mathbb{N}$, the formula $\Psi(\theta^{n_j}(x), n_j(x))$ holds, where $(n_j(x))_{j=0}^{\infty}$ is the visiting sequence of F at x. We also say that $\Psi(x,n)$ holds *in a exhaustively visiting way*, if there exists a family $F_k \in \mathscr{F}$ with $\lim_{k\to\infty} m(F_k) = 1$ such that, for all k, m-a.e. $x \in X'_{+F_k}$, and all $j \in \mathbb{N}$, the formula $\Psi(\theta^{n_j}(x), n_j(x))$ holds, where $(n_j(x))_{j=0}^{\infty}$ is the visiting sequence of F_k at x.

Now, let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions. We write that

$$\operatorname*{s-lim}_{n\to\infty} f_n = f,$$

if that there exists a family $F_k \in \mathscr{F}$ with $\lim_{k\to\infty} m(F_n) = 1$ such that, for all k and m-a.e. $x \in X'_{+F_k}$ and all $j \in \mathbb{N}$,

$$\lim_{j\to\infty} f_{n_j}(x) = f(x),$$

where $(n_j)_{j=0}^{\infty}$ is the visiting sequence of F_k at x.

Suppose that $g_1, \dots, g_k : X \to \mathbb{R}$ is a finite collection of measurable functions and let $b_1, \dots, b_n$ be a collection of real numbers. Consider the set

$$F := \bigcap_{i=1}^{k} g_i^{-1}((-\infty, b_i]).$$

If $m(F) > 0$, then F is called *essential* for $g_1, \dots, g_k$ with constants $b_1, \dots, b_n$ (or just *essential*, if we do not want explicitly specify functions and numbers). Note that by measurability of the functions $g_1, \dots, g_k$, for every $\varepsilon > 0$ we can always find finite numbers $b_1, \dots, b_n$ such that the essential set F for $g_1, \dots, g_k$ with constants $b_1, \dots, b_n$ has the measure $m(F) \geq 1 - \varepsilon$.

2.7 Spaces of Continuous and Hölder Functions

We denote by $\mathscr{C}(\mathscr{J}_x)$ the space of continuous functions $g_x : \mathscr{J}_x \to \mathbb{R}$ and by $\mathscr{C}(\mathscr{J})$ the space of functions $g : \mathscr{J} \to \mathbb{R}$ such that, for a.e. $x \in X$, $x \mapsto g_x := g_{|\mathscr{J}_x} \in \mathscr{C}(\mathscr{J}_x)$. The set $\mathscr{C}(\mathscr{J})$ contains the subspaces $\mathscr{C}^0(\mathscr{J})$ of functions for which the function $x \mapsto \|g_x\|_\infty$ is measurable, and $\mathscr{C}^1(\mathscr{J})$ for which the integral

$$\|g\|_1 := \int_X \|g_x\|_\infty \, dm(x) < \infty.$$

Now, fix $\alpha \in (0, 1]$. By $\mathscr{H}^\alpha(\mathscr{J}_x)$ we denote the space of Hölder continuous functions on $\mathscr{J}_x$ with an exponent α. This means that $\varphi_x \in \mathscr{H}^\alpha(\mathscr{J}_x)$ if and only if $\varphi_x \in \mathscr{C}(\mathscr{J}_x)$ and $v(\varphi_x) < \infty$ where

$$v_\alpha(\varphi_x) := \inf\{H_x : |\varphi(z_1) - \varphi(z_2)| \le H_x \varrho_x^\alpha(z_1, z_2)\}, \tag{2.11}$$

where the infimum is taken over all $z_1, z_2 \in \mathscr{J}_x$ with $\varrho(z_1, z_2) \le \eta$.

A function $\varphi \in \mathscr{C}^1(\mathscr{J})$ is called *Hölder continuous with an exponent* α provided that there exists a measurable function $H : X \to [1, +\infty)$, $x \mapsto H_x$, such that $\log H \in L^1(m)$ and such that $v_\alpha(\varphi_x) \le H_x$ for a.e. $x \in X$. We denote the space of all Hölder functions with fixed α and H by $\mathscr{H}^\alpha(\mathscr{J}, H)$ and the space of all α-Hölder functions by $\mathscr{H}^\alpha(\mathscr{J}) = \bigcup_{H \ge 1} \mathscr{H}^\alpha(\mathscr{J}, H)$.

2.8 Transfer Operator

For every function $g : \mathscr{J} \to \mathbb{C}$ and a.e. $x \in X$ let

$$S_n g_x = \sum_{j=0}^{n-1} g_x \circ T_x^j, \tag{2.12}$$

and, if $g : X \to \mathbb{C}$, then $S_n g = \sum_{j=0}^{n-1} g \circ \theta^j$. Let φ be a function in the Hölder space $\mathscr{H}^\alpha(\mathscr{J})$. For every $x \in X$, we consider the *transfer operator* $\mathscr{L}_x = \mathscr{L}_{\varphi,x} : \mathscr{C}(\mathscr{J}_x) \to \mathscr{C}(\mathscr{J}_{\theta(x)})$ given by the formula

$$\mathscr{L}_x g_x(w) = \sum_{T_x(z)=w} g_x(z) e^{\varphi_x(z)}, \quad w \in \mathscr{J}_{\theta(x)}. \tag{2.13}$$

It is obviously a positive linear operator and it is bounded with the norm bounded above by

$$\|\mathscr{L}_x\|_\infty \le \deg(T_x) \exp(\|\varphi\|_\infty). \tag{2.14}$$

This family of operators gives rise to the global operator $\mathscr{L} : \mathscr{C}(\mathscr{J}) \to \mathscr{C}(J)$ defined as follows:

$$(\mathscr{L} g)_x (w) = \mathscr{L}_{\theta^{-1}(x)} g_{\theta^{-1}(x)}(w).$$

For every $n > 1$ and a.e. $x \in X$, we denote

$$\mathscr{L}_x^n := \mathscr{L}_{\theta^{n-1}(x)} \circ \ldots \circ \mathscr{L}_x : \mathscr{C}(\mathscr{J}_x) \to \mathscr{C}(\mathscr{J}_{\theta^n(x)}).$$

Note that

$$\mathscr{L}_x^n g_x(w) = \sum_{z \in T_x^{-n}(w)} g_x(z) e^{S_n \varphi_x(z)}, \quad w \in \mathscr{J}_{\theta^n(x)}, \tag{2.15}$$

where $S_n \varphi_x(z)$ has been defined in (2.12). The dual operator $\mathscr{L}_x^*$ maps $C^*(\mathscr{J}_{\theta(x)})$ into $C^*(\mathscr{J}_x)$.

2.9 Distortion Properties

Lemma 2.2 *Let $\varphi \in \mathcal{H}^\alpha(\mathcal{J}, H)$, let $n \geq 1$ and let $y = (x, z) \in \mathcal{J}$. Then*

$$|S_n\varphi_x(T_y^{-n}(w_1)) - S_n\varphi_x(T_y^{-n}(w_2))| \leq \varrho^\alpha(w_1, w_2) \sum_{j=0}^{n-1} H_{\theta^j(x)}(\gamma_{\theta^j(x)}^{n-j})^{-\alpha}$$

for all $w_1, w_2 \in B(T_x^n(z), \xi)$.

Proof. We have by (2.6) and Hölder continuity of φ that

$$\begin{aligned}|S_n\varphi_x(T_y^{-n}(w_1)) - S_n\varphi_x(T_y^{-n}(w_2))| &\leq \sum_{j=0}^{n-1} |\varphi_x(T_x^j(T_y^{-n}(w_1))) - \varphi_x(T_x^j(T_y^{-n}(w_2)))| \\ &= \sum_{j=0}^{n-1} \left|\varphi_x(T_{T_x^j(y)}^{-(n-j)}(w_1)) - \varphi_x(T_{T_x^j(y)}^{-(n-j)}(w_2))\right| \\ &\leq \sum_{j=0}^{n-1} \varrho^\alpha\big(T_{T_x^j(x)}^{-(n-j)}(w_1), T_{T_x^j(x)}^{-(n-j)}(w_2)\big) H_{\theta^j(x)},\end{aligned}$$

hence $|S_n\varphi_x(T_y^{-n}(w_1)) - S_n\varphi_x(T_y^{-n}(w_2))| \leq \varrho^\alpha(w_1, w_2) \sum_{j=0}^{n-1} H_{\theta^j(x)}(\gamma_{\theta^j(x)}^{n-j})^{-\alpha}$. ◻

Set

$$Q_x := Q_x(H) = \sum_{j=1}^{\infty} H_{\theta^{-j}(x)}(\gamma_{\theta^{-j}(x)}^j)^{-\alpha}. \tag{2.16}$$

Lemma 2.3 *The function $x \mapsto Q_x$ is measurable and m-a.e. finite. Moreover, for every $\varphi \in \mathcal{H}^\alpha(\mathcal{J}, H)$,*

$$|S_n\varphi_x(T_y^{-n}(w_1)) - S_n\varphi_x(T_y^{-n}(w_2))| \leq Q_{\theta^n(x)}\varrho^\alpha(w_1, w_2)$$

for all $n \geq 1$, a.e. $x \in X$, every $z \in \mathcal{J}_x$ and $w_1, w_2 \in B(T^n(z), \xi)$ and where again $y = (x, z)$.

Proof. The measurability of Q_x follows directly from (2.16). Because of Lemma 2.2 we are only left to show that Q_x is m-a.e. finite. Let χ be a positive real number less or equal to $\int \log \gamma_x dm(x)$. Then, using Birkhoff's Ergodic Theorem for θ^{-1}, we get that

$$\liminf_{j\to\infty} \frac{1}{j} \sum_{k=0}^{j-1} \log \gamma_{\theta^{-j}(x)} \geq \chi$$

for m-a.e. $x \in X$. Therefore, there exists a measurable function $C_\gamma : X \to [1, +\infty)$ m-a.e. finite such that $C_\gamma^{-1}(x)e^{j\chi/2} \leq \gamma_{\theta^{-j+1}(x)}^j$ for all $j \geq 0$ and a.e. $x \in X$.

Moreover, since $\log H \in L^1(m)$ it follows again from Birkhoff's Ergodic Theorem that

$$\lim_{j\to\infty} \frac{1}{j} \log H_{\theta^{-j}(x)} = 0 \quad m\text{-}a.e.$$

There thus exists a measurable function $C_H : X \to [1, +\infty)$ such that

$$H_{\theta^j(x)} \le C_H(x) e^{j\alpha\chi/4} \quad \text{and} \quad H_{\theta^{-j}(x)} \le C_H(x) e^{j\alpha\chi/4} \tag{2.17}$$

for all $j \ge 0$ and a.e. $x \in X$. Then, for a.e. $x \in X$, all $n \ge 0$ and all $a \ge j \ge n-1$, we have

$$H_{\theta^j(x)} = H_{\theta^{-(n-j)}(\theta^n(x))} \le C_H(\theta^n(x)) e^{(n-j)\alpha\chi/4}.$$

Therefore, still with $x_n = \theta^n(x)$,

$$\begin{aligned} Q_{x_n} &= \sum_{j=0}^{n-1} H_{x_j} (\gamma_{x_j}^{n-j})^{-\alpha} \le \sum_{j=0}^{n-1} C_H(x_n) e^{(n-j)\alpha\chi/4} C_\gamma^\alpha(x_{n-1}) e^{-\alpha(n-j)\chi/2} \\ &\le C_\gamma^\alpha(x_{n-1}) C_H(x_n) \sum_{j=0}^{n-1} e^{-\alpha(n-j)\chi/4} \le C_\gamma^\alpha(x_{n-1}) C_H(x_n) (1 - e^{-\alpha\chi/4})^{-1}. \end{aligned}$$

Hence

$$Q_x \le C_\gamma^\alpha(\theta^{-1}(x)) C_H(x) (1 - e^{-\alpha\chi/4})^{-1} < +\infty.$$

$\square$

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