Chapter 2
Three Models and a Ground State

2.1 The Laughlin State and Its Parent Hamiltonian

Laughlin's theory [1–6] for a series of fractionally quantized Hall states is first and foremost the key to an explanation for the experimentally observed, fractionally quantized plateaus in the Hall resistivity of a spin-polarized, two-dimensional electron gas realized in semiconductor inversion layers [5, 7–10]. For our purposes here, however, we will view it primarily as an exact model, that is, a ground state which supports fractionally quantized excitations, and a model Hamiltonian for which this ground state is exact.

We will first review the theory in a planar geometry with open boundary conditions, and then turn to the spherical geometry, which will turn out to be the relevant geometry for the mapping of quantized Hall system onto a spin chain. We begin with a review of Landau level quantization in the plane.

2.1.1 Landau Level Quantization in the Planar Geometry

To describe the dynamics of charged particles (e.g. spin-polarized electrons) in a two-dimensional plane subject to a perpendicular magnetic field $\boldsymbol{B} = -B\boldsymbol{e}_z$, it is convenient to introduce complex particles coordinates $z = x + \mathrm{i}y$ and $\bar{z} = x - \mathrm{i}y$ [11, 12]. The associated derivative operators are

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right). \tag{2.1.1}$$

Note that hermitian conjugation yields a $-$ sign,

$$\left(\frac{\partial}{\partial z}\right)^{\dagger} = -\frac{\partial}{\partial \bar{z}}. \tag{2.1.2}$$

M. Greiter, *Mapping of Parent Hamiltonians*, Springer Tracts in Modern Physics 244,
DOI: 10.1007/978-3-642-24384-4_2, © Springer-Verlag Berlin Heidelberg 2011

We further define the complex momentum

$$p \equiv p_x + \mathrm{i}p_y = -2\mathrm{i}\hbar\frac{\partial}{\partial\bar{z}}, \quad \bar{p} = p_x - \mathrm{i}p_y = -2\mathrm{i}\hbar\frac{\partial}{\partial z}. \tag{2.1.3}$$

The single particle Hamilton operator is obtained by minimally coupling the gauge field to the canonical momentum,

$$H = \frac{1}{2M}\left(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}\right)^2, \tag{2.1.4}$$

where M is the mass of the particle and $e > 0$. In the symmetric gauge $\boldsymbol{A} = \frac{1}{2}B\boldsymbol{r}\times\boldsymbol{e}_z$, and with the definition of the magnetic length

$$l = \sqrt{\frac{\hbar c}{eB}}, \tag{2.1.5}$$

we write

$$\begin{aligned} H &= \frac{1}{2M}\left[\left(p_x + \frac{\hbar}{2l^2}y\right)^2 + \left(p_y - \frac{\hbar}{2l^2}x\right)^2\right] \\ &= \frac{1}{2M}\left[\Re^2\left(p - \frac{\mathrm{i}\hbar}{2l^2}z\right) + \Im^2\left(p - \frac{\mathrm{i}\hbar}{2l^2}z\right)\right] \\ &= \frac{1}{4M}\left\{p - \frac{\mathrm{i}\hbar}{2l^2}z, \bar{p} + \frac{\mathrm{i}\hbar}{2l^2}\bar{z}\right\}, \\ &= \frac{\hbar^2}{2Ml^2}\{a, a^\dagger\} \end{aligned} \tag{2.1.6}$$

where $\Re$ and $\Im$ denote the real and imaginary part, respectively. In the last line, we have introduced the ladder operators [12–14] [1]

$$a = \frac{l}{\sqrt{2}}\left(2\frac{\partial}{\partial\bar{z}} + \frac{1}{2l^2}z\right), \quad a^\dagger = \frac{l}{\sqrt{2}}\left(-2\frac{\partial}{\partial z} + \frac{1}{2l^2}\bar{z}\right), \tag{2.1.7}$$

which obey

$$\left[a, a^\dagger\right] = 1. \tag{2.1.8}$$

[1] We have not been able to find out who introduced the ladder operators for Landau levels in the plane. The energy eigenfunctions were known since Landau [11]. MacDonald [13] used the ladder operators in 1984, but neither gave nor took credit. Girvin and Jach [14] were aware of two independent ladders a year earlier, but neither spelled out the formalism, nor pointed to references. It appears that the community had been aware of them, but not aware of who introduced them. The clearest and most complete presentation we know of is due to Arovas [12].

With the cyclotron frequency $\omega_c = eB/Mc$ and (2.1.8) we finally obtain

$$H = \hbar\omega_c \left(a^\dagger a + \frac{1}{2}\right). \tag{2.1.9}$$

The kinetic energy of charged particles in a perpendicular magnetic field is hence quantized like a harmonic oscillator. The energy levels are called Landau levels.

It is convenient to write the ladder operators describing the cyclotron variables as

$$a = +\sqrt{2}l \exp\left(-\frac{1}{4l^2}\bar{z}z\right) \frac{\partial}{\partial \bar{z}} \exp\left(+\frac{1}{4l^2}\bar{z}z\right), \tag{2.1.10}$$

$$a^\dagger = -\sqrt{2}l \exp\left(+\frac{1}{4l^2}\bar{z}z\right) \frac{\partial}{\partial z} \exp\left(-\frac{1}{4l^2}\bar{z}z\right), \tag{2.1.11}$$

and introduce a second set of ladder operators for the guiding center variables,

$$b = +\sqrt{2}l \exp\left(-\frac{1}{4l^2}\bar{z}z\right) \frac{\partial}{\partial z} \exp\left(+\frac{1}{4l^2}\bar{z}z\right), \tag{2.1.12}$$

$$b^\dagger = -\sqrt{2}l \exp\left(+\frac{1}{4l^2}\bar{z}z\right) \frac{\partial}{\partial \bar{z}} \exp\left(-\frac{1}{4l^2}\bar{z}z\right). \tag{2.1.13}$$

They likewise obey

$$\left[b, b^\dagger\right] = 1, \tag{2.1.14}$$

and commute with the cyclotron ladder operators:

$$\left[a, b\right] = \left[a, b^\dagger\right] = 0 \tag{2.1.15}$$

A calculation similar to the one presented above for H yields

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \hbar\left(b^\dagger b - a^\dagger a\right)\boldsymbol{e}_z \tag{2.1.16}$$

for the *canonical* angular momentum around the origin. [The *kinematical* angular momentum is given by the $a^\dagger a$ term in (2.1.16).]

Since the angular momentum (2.1.16) commutes with the Hamiltonian (2.1.9), we can use it to classify the vastly degenerate states within each Landau level. Specifically, we introduce the basis states

$$|n, m\rangle = \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{m!}} (a^\dagger)^n (b^\dagger)^m |0, 0\rangle, \tag{2.1.17}$$

where the vacuum state is by definition annihilated by both destruction operators,

$$a|0, 0\rangle = b|0, 0\rangle = 0. \tag{2.1.18}$$

Solving (2.1.18) yields the real space representation

$$\phi_0(z) \equiv \phi_0(z, \bar{z}) = \langle \boldsymbol{r}|0, 0\rangle = \frac{1}{\sqrt{2\pi l^2}} \exp\left(-\frac{1}{4l^2}|z|^2\right). \tag{2.1.19}$$

(In the following, we omit $\bar{z}$ from the argument of wave functions as a choice of notation.) The basis states (2.1.17) are trivially eigenstates of both H and L_z,

$$\begin{aligned} H|n, m\rangle &= \hbar\omega_c\left(n + \frac{1}{2}\right) \\ L_z|n, m\rangle &= \hbar(m - n)|n, m\rangle \end{aligned} \tag{2.1.20}$$

The particle coordinate and momentum are given in terms of the ladder operators by

$$z = \sqrt{2}l\left(a + b^\dagger\right), \quad p = -\frac{\mathrm{i}\hbar}{\sqrt{2}l}\left(a - b^\dagger\right). \tag{2.1.21}$$

This implies that we can write a complete, orthonormal set of basis states in the lowest Landau level ($n = 0$) as

$$\begin{aligned} \phi_m(z) &= \langle \boldsymbol{r}|0, m\rangle \\ &= \frac{1}{\sqrt{m!}}(b^\dagger)^m \phi_0(z, \bar{z}) \\ &= \frac{1}{\sqrt{2\pi l^2 m!}}(a + b^\dagger)^m \exp\left(-\frac{1}{4l^2}|z|^2\right) \\ &= \frac{1}{\sqrt{2^{m+1}\pi m!}\, l^{m+1}}\, z^m \exp\left(-\frac{1}{4l^2}|z|^2\right). \end{aligned} \tag{2.1.22}$$

These states is describe narrow rings centered around the origin, with the radius determined by

$$\frac{\partial}{\partial r}\left|\phi_m(r)^2\right|\Big|_{r=r_m} \stackrel{!}{=} 0,$$

which yields $r_m = \sqrt{2m}l$. Since there are also m states inside the ring, the areal degeneracy is

$$\frac{\text{number of states}}{\text{area}} = \frac{m}{\pi r_m^2} = \frac{1}{2\pi l^2}, \tag{2.1.23}$$

The magnetic flux required for each state,

$$2\pi l^2 B = \frac{2\pi\hbar c}{e} = \Phi_0,$$

is hence given by the Dirac flux quantum. This implies that in each Landau level, there are as many single particle states in a given area as there are Dirac quanta of magnetic flux going through it. In the following, we set $l = 1$, and no longer keep track of wave function normalizations.

The N particle wave function for a filled lowest Landau level (LLL) on a circular disk is obtained by antisymmetrizing the basis states (2.1.22),

$$\begin{aligned}\psi(z_1, \ldots, z_N) &= \mathcal{A}\left\{z_1^0 z_2^1 \ldots z_N^{N-1}\right\} \cdot \prod_{i=1}^{N} e^{-\frac{1}{4}|z_i|^2} \\ &= \prod_{i<j}^{N}(z_i - z_j) \prod_{i=1}^{N} e^{-\frac{1}{4}|z_i|^2}. \end{aligned} \tag{2.1.24}$$

The most general form for the single particle wave function in the lowest Landau level is

$$\psi(z) = f(z)\, e^{-\frac{1}{4}|z|^2}, \tag{2.1.25}$$

where $f(z)$ is an analytic function of z. Since $\psi(z)$ is annihilated by the destruction operator a, the energy is trivially $\frac{1}{2}\hbar\omega_c$. The most general N particle state in the LLL is given by

$$\psi(z_1, \ldots, z_N) = f(z_1, \ldots, z_N) \prod_{i=1}^{N} e^{-\frac{1}{4}|z_i|^2}, \tag{2.1.26}$$

where $f(z_1, \ldots, z_N)$ is analytic in all the z's, and symmetric or antisymmetric for bosons or fermions, respectively. If we impose periodic boundary conditions [15], we find that $\psi(z_1, z_2, \ldots, z_N)$, when viewed as a function of z_1 while $z_2, \ldots, z_N$ are parameters, has exactly as many zeros as there states in the LLL, i.e. as there are Dirac flux quanta going through the unit cell or principal region. If $\psi(z_1, \ldots, z_N)$ describes fermions and is hence antisymmetric, there will be at least one zero seen by z_1 at each of the other particle positions. The most general wave function is hence

$$\psi(z_1, \ldots, z_N) = P(z_1, \ldots, z_N) \prod_{i<j}^{N}(z_i - z_j) \prod_{i=1}^{N} e^{-\frac{1}{4}|z_i|^2}, \tag{2.1.27}$$

where P is a symmetric polynomial in the z_i's. In the case of a completely filled Landau level, there are only as many zeros as there are particles, which implies that all except one of the zeros in z_1 will be located at the other particle positions $z_2, \ldots, z_N$. This yields (2.1.24) as the unique state for open boundary conditions. For periodic boundary conditions, there is one additional zero as there cannot be a zero seen by z_1 at z_1. The location of this zero, which Haldane and Rezayi [15] refer

to as the center-of-mass zero, encodes the information about the boundary phases a test particle acquires as it is taken around one of the meridians of the torus.

To elevate the most general LLL state (2.1.26) into the $(n+1)$-th Landau level, we only have to apply $\left(a^\dagger\right)^n$ to all the particles in the LLL,

$$\begin{aligned}\psi_n(z_1,\ldots,z_N) &= \prod_{i=1}^{N}\left(a_i^\dagger\right)^n \psi(z_1,\ldots,z_N)\\ &= \prod_{i=1}^{N} e^{-\frac{1}{4}|z_i|^2}\prod_{i=1}^{N}\left(2\frac{\partial}{\partial z_i}-\bar{z}_i\right)^n f(z_1,\ldots,z_N).\end{aligned} \tag{2.1.28}$$

The energy per particle in this state is $\hbar\omega_c\left(n+\frac{1}{2}\right)$.

2.1.2 The Laughlin State

The experimental observation which Laughlin's theory [1] explains is a plateau in the Hall resistivity of a two-dimensional electron gas at a Landau level filling fraction $\nu = 1/3$. The filling fraction denotes the number of particles divided by the number of number of states in each Landau level in the thermodynamic limit, and is defined through

$$\frac{1}{\nu} = \frac{\partial N_\Phi}{\partial N}, \tag{2.1.29}$$

where N_Φ is the number of Dirac flux quanta through the sample and N is the number of particles. For a wave function at $\nu = 1/3$, we consequently have three times as many zeros seen by z_1 as there are particles, and the polynomial $P(z_1, z_2, \ldots, z_N)$ in (2.1.27) has two zeros per particle. The experimental findings, as well as early numerical work by Yoshioka, Halperin, and Lee [16], are consistent with, if not indicative of, a quantum liquid state at a preferred filling fraction $\nu = 1/3$. Since the kinetic energy is degenerate in each Landau level, such a liquid has to be stabilized by the repulsive Coulomb interactions between the electrons. This implies that the wave function should be highly effective in suppressing configurations in which particles approach each other, as there is a significant potential energy cost associated with it. We may hence ask ourselves whether there is any particular way of efficiently distributing the zeros of $P(z_1, z_2, \ldots, z_N)$ in this regard.

Laughlin's wave function amounts to attaching the additional zeros onto the particles, such that each particle coordinate $z_2, \ldots, z_N$ becomes a triple zero of z_1 when $\psi(z_1, z_2, \ldots, z_N)$ is viewed as a function of z_1 with parameter $z_2, \ldots, z_N$. For filling fraction $\nu = 1/m$, where m is an odd integer if the particles are fermions and an even integer if they are bosons, he proposed the ground state wave function

$$\psi_m(z_1, \ldots, z_N) = \prod_{i<j}^{N} (z_i - z_j)^m \prod_{i=1}^{N} e^{-\frac{1}{4}|z_i|^2}. \tag{2.1.30}$$

There are hence no zeros wasted—all of them contribute in keeping the particles away from each other effectively, as ψ_m vanishes as the mth power of the distance when two particles approach each other. This is the uniquely defining property of Laughlin's state, and also the property which enabled Haldane [3] to identify a parent Hamiltonian, which singles out the state as its unique and exact ground state. We discuss the Hamiltonian in Sect. 2.1.6 below. The wave function (2.1.30) describes an incompressible quantum liquid, as the construction is only possible at filling fractions $\nu = 1/m$.

One of the assumptions of the theory is that we can neglect transitions into higher Landau levels, as the Landau level splitting $\hbar\omega_c$ is much larger then the potential energy per particle, a condition met by the systems amenable to experiment. Formally, the LLL limit requires $\omega_c \to \infty$ while keeping the magnetic length l^2 constant, which is achieved by taking $M \to 0$. The LLL limit is hence a zero mass limit.

Even within this limit, which we assume to hold in the following, the Laughlin state (2.1.30) is not the exact ground state for electrons with (screened) Coulomb interactions at filling fraction $\nu = 1/3$. It is, however, reasonably close in energy and has a significant overlap with the exact ground state for finite systems. The difference between the exact ground state and Laughlin's state is that in the exact ground state, the zeros of $P(z_1, z_2, \ldots, z_N)$ are attached to the particle coordinates, but do not coincide with them [2, 17]. At long distances, the physics described by both states is identical. In particular, the topological quantum numbers of both states, such as the charge and the statistics of the (fractionally) charged excitations, or the degeneracies on closed surfaces of genus one and higher, are identical.

The Laughlin state can be characterized through the notion of "superfermions" [18]. For fermions (bosons), the relative angular momentum is quantized as $\hbar l$, where l is an odd (even) integer, due to the antisymmetry (symmetry) of the wave function under interchange of particles. In the LLL, the relative angular momentum between pairs of fermions can only have components with $l = 1, 3, 5, \ldots$, but no negative values. If we interchange the particles through winding them counterclockwise around each other, these components acquire a phase factor $e^{i\pi l}$. The smallest component hence acquires a phase π, as required by Fermi statistics. For the Laughlin state (2.1.30), the smallest component of relative angular momentum is $l = m$, and the phase this component acquires upon interchange is $m\pi$, while only a phase π is required by Fermi statistics. In this sense, the particles are "superfermions" for m odd, $m > 1$. In the exact ground state for Coulomb interaction, the electrons are "approximate superfermions".

For completeness, we wish to mention that there is a variant of Haldane's parent Hamiltonian [3] for the planar geometry, due to Trugman and Kivelson [19]. They noted that since the Laughlin state (2.1.30) contains a term $(z_i - z_j)^m$ for each pair, it is annihilated by the short range potential interaction

$$V^{(m)} = \sum_{i<j}^{N} \left(\nabla_i^2\right)^{(m-1)/2} \delta^{(2)}(z_i - z_j) \tag{2.1.31}$$

for m odd, and

$$V^{(m)} = \sum_{i<j}^{N} \left(\nabla_i^2\right)^{(m-2)/2} \delta^{(2)}(z_i - z_j) \tag{2.1.32}$$

for m even, as well as by the same terms with any smaller power of the Laplacian. If we combine these terms with the kinetic terms (2.1.9), the resulting Hamiltonian will single out (2.1.30) as the exact and unique ground state.

2.1.3 Fractionally Charged Quasiparticle Excitations

Laughlin [1] created the elementary, charged excitations of the fractionally quantized Hall state (2.1.30) through a *Gedankenexperiment*. If one adiabatically inserts one Dirac quantum of magnetic flux through an infinitely thin solenoid at a position ξ, and then removes this flux quanta via a singular gauge transformation, the final Hamiltonian will be identical to the initial one. The final state will hence be an eigenstate of the initial Hamiltonian as well. The adiabatic insertion of the flux will induce an electric field

$$\oint \boldsymbol{E} \mathrm{d}\boldsymbol{s} = E_\varphi \cdot 2\pi r = \frac{1}{c} \frac{\partial \phi}{\partial t}, \tag{2.1.33}$$

which in turn will change the canonical angular momentum L_z around ξ by

$$\Delta L_z = \int F_\varphi \cdot r \mathrm{d}t = \frac{e}{2\pi c} \int \frac{\partial \phi}{\partial t} \mathrm{d}t = \frac{e}{2\pi c} \cdot \phi_0 = \hbar. \tag{2.1.34}$$

If we choose a basis of eigenstates of angular momentum around ξ, the basis states evolve according to

$$(z - \xi)^m e^{-\frac{1}{4}|z|^2} \to (z - \xi)^{m+1} e^{-\frac{1}{4}|z|^2}. \tag{2.1.35}$$

Note that the kinematical angular momentum, which is given by the second term in (2.1.16), has eigenvalue $-\hbar n$, where n labels the Landau level. In this process, it remains zero as the states remain in the lowest Landau level—as there are no states with positive kinematical angular momentum, the insertion of the flux just shifts the states within the LLL.

The Laughlin ground state (2.1.30) evolves in the process into

$$\psi_\xi^{\mathrm{QH}}(z_1, \ldots, z_N) = \prod_{i=1}^{N} (z_i - \xi) \prod_{i<j}^{N} (z_i - z_j)^m \prod_{i=1}^{N} e^{-\frac{1}{4}|z_i|^2}, \tag{2.1.36}$$

which describes a quasihole excitation at ξ. It is easy to see that if the electron charge is $-e$, the charge of the quasihole is $+e/m$. If we were to create m quasiholes at ξ by inserting m Dirac quanta, the final wave function would be

$$\psi_{\xi}^{m\text{QH's}}(z_1,\ldots,z_N)=\prod_{i=1}^{N}(z_i-\xi)^m\prod_{i<j}^{N}(z_i-z_j)^m\prod_{i=1}^{N}e^{-\frac{1}{4}|z_i|^2}, \tag{2.1.37}$$

i.e., we would have created a true hole in the liquid, which is screened as all the other particles. Since the hole has charge $+e$, the quasihole has charge $+e/m$. One may view the quasihole as a zero in the wave function which is not attached to any of the electrons.

The quasielectron, i.e. the antiparticle of the quasihole, has charge $-e/m$ and is created by inserting the flux adiabatically in the opposite direction, thus lowering the angular momentum around some position ξ by $\hbar$, or alternatively, by removing one of the zeros from the wave function. To accomplish this formally, we first rewrite (2.1.36) in terms of ladder operators:

$$\psi_{\xi}^{\text{QH}}(z_1,\ldots,z_N)=\prod_{i=1}^{N}\left(\sqrt{2}b_i^{\dagger}-\xi\right)\prod_{i<j}^{N}(z_i-z_j)^m\prod_{i=1}^{N}e^{-\frac{1}{4}|z_i|^2}. \tag{2.1.38}$$

The insertion of a flux quanta in the opposite direction, or the lowering of angular momentum around ξ, will then correspond to the Hermitian conjugate operation. Laughlin [4] hence proposed for the quasielectron wave function

$$\begin{aligned}\psi_{\bar{\xi}}^{\text{QE}}(z_1,\ldots,z_N)&=\prod_{i=1}^{N}\left(\sqrt{2}b_i-\bar{\xi}\right)\prod_{i<j}^{N}(z_i-z_j)^m\prod_{i=1}^{N}e^{-\frac{1}{4}|z_i|^2}\\&=\prod_{i=1}^{N}e^{-\frac{1}{4}|z_i|^2}\prod_{i=1}^{N}\left(2\frac{\partial}{\partial z_i}-\bar{\xi}\right)\prod_{i<j}^{N}(z_i-z_j)^m.\end{aligned} \tag{2.1.39}$$

While the quasihole excitation (2.1.36) is still an exact eigenstate of Haldane's parent Hamiltonian, this is not true for the quasielectron (2.1.39). The problem here is that while there is a clean and unique way of introducing an additional zero (we just put it somewhere), there is no such clean way of removing one. One can view the quasielectron as a region, in which n electrons nearby share $2n-1$ zeros attached to the particles. In other words, one zero is missing, but not from any specific electron—rather, the dearth is distributed among all the electrons nearby. The charge of the quasielectron is accordingly not as localized as it is for the quasihole.

The plateau in the observed Hall resistivity occurs because the current in the experiments is carried by edge states, which are sensitive only to the topological quantum numbers of the state. In the vicinity of one of the prefered filling fractions $\nu=1/m$, the excess density of electrons yields to a finite density of quasielectrons or holes, which get pinned by disorder and hence do not contribute to the transport properties.

Fig. 2.1 Fractional statistics in two dimensions. The many particle wave function acquires a statistical phase θ whenever we interchange two anyons conterclockwise

2.1.4 Fractional Statistics

Possibly the most interesting property of fractionally quantized Hall states is that the quasiparticle excitations obey fractional statistics [20–22]. The possibility of fractional statistics [23–31] arises in two space dimensions because the space of trajectories for two identical particles consists of an infinite number of topologically distinct sectors, corresponding to the number of times the particles wind around each other. The laws of quantum mechanics allow us to assign distinct phases to paths belonging these sectors, which only need to satisfy the composition principle.

In three or more dimensions, by contrast, there are only two topological distinct sectors, corresponding to interchanging the particles or not interchanging them. The group which classifies all the topologically distinct trajectories is hence the permutation group, and since amplitudes are complex numbers, the possibilities for the quantum statistics are limited to the one-dimensional representations of the permutation group. There are only two such representations, the symmetric and the antisymmetric representation. These correspond to the familiar choices of Bose and Fermi statistics.

In two dimensions, the group is the braid group. The one-dimensional representations are obtained by assigning an arbitrary phase $\tau(T_i) = e^{i\theta}$ for each counterclockwise interchange T_i of the two particles, with statistical parameter $\theta \in]-\pi, \pi]$. Particles interpolating between the familiar choices of bosons ($\theta = 0$) and fermions ($\theta = \pi$) are generically called anyons. We will see in Sect. 2.3.3 that non-Abelian generalizations exist, where successive interchanges of anyons do not commute.

The most direct physical manifestation of the fractional statistics is the quantization of the relative angular momentum of the anyons (see Fig. 2.1). In three dimensions, there are three generators of rotations, and the relative angular momentum is quantized as $\hbar l$, with l an even integer for bosons and an odd integer l odd for fermions. In two dimensions, the wave function may acquire a phase $\exp\left(\frac{i}{\pi}\theta\varphi\right)$ as two anyons wind counterclockwise around each other with winding angle φ, which implies that the relative angular momentum is quantized as

$$L_{\text{rel}} = \hbar\left(-\frac{\theta}{\pi} + 2n\right), \tag{2.1.40}$$

where n is an integer. Note that the possibility of fractional statistics exists only for particles which are strictly two-dimensional, like vortices in an (approximately) two-dimensional quantum fluid.

The only established realization of fractional statistics is provided by the quasiparticles in the fractionally quantized Hall effect [20–22]. When Laughlin introduced the quasiparticles, he introduced them as localized defects or more precisely, vortices in an otherwise uniform quantum liquid. To address the question of their statistics, however, it is propitious to view them as particles, with a Hilbert space spanned by the parent wave function for the electrons. We consider here a Laughlin state with two quasiholes in an eigenstate of relative angular momentum in an "orbit" centered at the origin. Since the quasiholes have charge $e^* = +e/m$, the effective flux quantum seen by them is

$$\Phi_0^* = \frac{2\pi\hbar c}{e^*} = m\Phi_0, \tag{2.1.41}$$

and the effective magnetic length is

$$l^* = \sqrt{\frac{\hbar c}{e^* B}} = \sqrt{m}l. \tag{2.1.42}$$

We expect the single quasihole wave function to describe a particle of charge e^* in the LLL, and hence be of the general form

$$\phi(\bar{\xi}) = f(\bar{\xi})e^{-\frac{1}{4m}|\xi|^2}. \tag{2.1.43}$$

The complex conjugation reflects that the sign of the quasihole charge is reversed relative to the electron charge $-e$.

The electron wave function for the state with two quasiholes in an eigenstate of relative angular momentum is given by

$$\psi(z_1, \ldots, z_N) = \int \mathrm{D}[\xi_1, \xi_2]\phi_{p,m}(\bar{\xi}_1, \bar{\xi}_2)\psi^{\mathrm{QHs}}_{\xi_1,\xi_2}(z_1, \ldots, z_N) \tag{2.1.44}$$

with

$$\phi_{p,m}(\bar{\xi}_1, \bar{\xi}_2) = (\bar{\xi}_1 - \bar{\xi}_2)^{p+\frac{1}{m}} \prod_{k=1,2} e^{-\frac{1}{4m}|\xi_k|^2}, \tag{2.1.45}$$

where p is an even integer, and

$$\psi^{\mathrm{QHs}}_{\xi_1,\xi_2}(z_1, \ldots, z_N) = (\xi_1 - \xi_2)^{\frac{1}{m}} \prod_{k=1,2} e^{-\frac{1}{4m}|\xi_k|^2} \cdot \prod_{i=1}^{N}(z_i - \xi_1)(z_i - \xi_2) \prod_{i<j}^{N}(z_i - z_j)^m \prod_{i=1}^{N} e^{-\frac{1}{4}|z_i|^2}. \tag{2.1.46}$$

The quasihole coordinate integration extends over the complex plane,

$$\int \mathrm{D}[\xi_1, \xi_2] \equiv \int\!\!\int \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2,$$

where $\xi_1 = x_1 + \mathrm{i}y_1$ and $\xi_2 = x_2 + \mathrm{i}y_2$.

This needs explanation. We see that both $\phi_{p,m}(\bar{\xi}_1, \bar{\xi}_2)$ and $\psi^{\mathrm{QHs}}_{\xi_1,\xi_2}(z_1, \ldots, z_N)$ contain multiple valued functions of $\bar{\xi}_1 - \bar{\xi}_2$ and $\xi_1 - \xi_2$, respectively, while the product of them is understood to be single valued. The reason for this is that the Hilbert space for the quasiholes at ξ_1 and ξ_2 spanned by $\psi^{\mathrm{QHs}}_{\xi_1,\xi_2}(z_1, \ldots, z_N)$ has to be normalized and is, apart from the exponential, supposed to be analytic in ξ_1 and ξ_2. At the same time, we expect $\phi_{p,m}(\bar{\xi}_1, \bar{\xi}_2)$ to be of the general form (2.1.41), i.e. to be an analytic function of $\bar{\xi}_1, \bar{\xi}_2$ times the exponential.

The form (2.1.45) of the quasihole wave function including its branch cut, is indicative of fractional statistics with statistical parameter $\theta = \pi/m$. This indication, however, is by itself not conclusive, as it is possible to change the representation of the wave function through singular gauge transformations [20, 21, 25], where one removes or adds flux tubes with a fraction of a Dirac flux quanta to the particles, and hence turn an anyonic representation into a bosonic or fermionic one and vice versa. The physically unambivalent quantity is the relative angular momentum of the quasiholes, which for (2.1.45) is given by

$$L_{\mathrm{rel}} = -\hbar\left(p + \frac{1}{m}\right). \tag{2.1.47}$$

Comparing this with (2.1.40) yields $\theta = \pi/m$. This result agrees with the results of Halperin [20, 21] and of Arovas, Schrieffer, and Wilczek [22], who calculated the statistical parameter directly using the adiabatic theorem [32–35].

2.1.5 Landau Level Quantization in the Spherical Geometry

The formalism for Landau level quantization in a spherical geometry, i.e., for the dynamics of a charged particle on the surface of a sphere with radius R, in a magnetic (monopole) field, was pioneered by Haldane for the lowest Landau level [3, 36], and only very recently generalized to higher Landau levels [37]. We will content ourselves here with a review of the formalism for the lowest Landau level.

Following Haldane [3], we assume a radial magnetic field of strength

$$B = \frac{\hbar c s_0}{e R^2} \quad (e > 0). \tag{2.1.48}$$

The number of magnetic Dirac flux quanta through the surface of the sphere is

$$\frac{\Phi_{\mathrm{tot}}}{\Phi_0} = \frac{4\pi R^2 B}{2\pi\hbar c/e} = 2s_0, \tag{2.1.49}$$

which must be integer due to Dirac's monopole quantization condition [38]. In the following, we take $\hbar = c = 1$.

The Hamiltonian is given by

$$H = \frac{\mathbf{\Lambda}^2}{2MR^2} = \frac{\omega_\mathrm{c}}{2s_0}\mathbf{\Lambda}^2, \tag{2.1.50}$$

where $\omega_\mathrm{c} = eB/M$ is the cyclotron frequency,

$$\mathbf{\Lambda} = \boldsymbol{r} \times (-\mathrm{i}\nabla + e\boldsymbol{A}(\boldsymbol{r})) \tag{2.1.51}$$

is the dynamical angular momentum, $\boldsymbol{r} = R\boldsymbol{e}_\mathrm{r}$, and $\nabla \times \boldsymbol{A} = B\boldsymbol{e}_\mathrm{r}$. With (A.4)–(A.6) from Appendix A we obtain

$$\mathbf{\Lambda} = -\mathrm{i}\left(\boldsymbol{e}_\varphi \frac{\partial}{\partial\theta} - \boldsymbol{e}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right) + eR(\boldsymbol{e}_\mathrm{r} \times \boldsymbol{A}(\boldsymbol{r})). \tag{2.1.52}$$

Note that

$$\boldsymbol{e}_\mathrm{r}\mathbf{\Lambda} = \mathbf{\Lambda}\boldsymbol{e}_\mathrm{r} = 0, \tag{2.1.53}$$

as one can easily verify with (A.5). The commutators of the Cartesian components of $\mathbf{\Lambda}$ with themselves and with $\boldsymbol{e}_\mathrm{r}$ can easily be evaluated using (2.1.52) and (A.3)–(A.5). This yields

$$\left[\Lambda^i, \Lambda^j\right] = \mathrm{i}\varepsilon^{ijk}(\Lambda^k - s_0\, e_\mathrm{r}^k), \tag{2.1.54}$$

$$\left[\Lambda^i, e_\mathrm{r}^j\right] = \mathrm{i}\epsilon^{ijk}e_\mathrm{r}^k, \tag{2.1.55}$$

where $i, j, k = \mathrm{x}, \mathrm{y}, \text{ or } \mathrm{z}$, and e_r^k is the kth Cartesian coordinate of $\boldsymbol{e}_\mathrm{r}$. From (2.1.53)–(2.1.55), we see that that the operator

$$\boldsymbol{L} = \mathbf{\Lambda} + s_0\boldsymbol{e}_\mathrm{r} \tag{2.1.56}$$

is the generator of rotations around the origin,

$$\left[L^i, X^j\right] = \mathrm{i}\epsilon^{ijk}X^k \quad \text{with} \quad \boldsymbol{X} = \mathbf{\Lambda}, \boldsymbol{e}_\mathrm{r}, \text{ or } \boldsymbol{L}, \tag{2.1.57}$$

and hence the angular momentum. As it satisfies the angular momentum algebra, it can be quantized accordingly. Note that $\boldsymbol{L}$ has a component in the $\boldsymbol{e}_\mathrm{r}$ direction:

$$\boldsymbol{L}\boldsymbol{e}_\mathrm{r} = \boldsymbol{e}_\mathrm{r}\boldsymbol{L} = s_0. \tag{2.1.58}$$

If we take the eigenvalue of $\boldsymbol{L}^2$ to be $s(s+1)$, this implies $s = s_0 + n$, where $n = 0, 1, 2, \ldots$ is a non-negative integer (while s and s_0 can be integer or half integer, according to number of Dirac flux quanta through the sphere).

With (2.1.56) and (2.1.53), we obtain

$$\boldsymbol{\Lambda}^2 = \boldsymbol{L}^2 - s_0^2. \tag{2.1.59}$$

The energy eigenvalues of (2.1.50) are hence

$$\begin{aligned} E_n &= \frac{\omega_c}{2s_0}\left[s(s+1) - s_0^2\right] \\ &= \frac{\omega_c}{2s_0}\left[(2n+1)s_0 + n(n+1)\right] \\ &= \omega_c\left[\left(n+\frac{1}{2}\right) + \frac{n(n+1)}{2s_0}\right]. \end{aligned} \tag{2.1.60}$$

The index n hence labels the Landau levels.

To obtain the eigenstates of (2.1.50), we have to choose a gauge and then explicitly solve the eigenvalue equation. We choose the latitudinal gauge

$$\boldsymbol{A} = -\boldsymbol{e}_\varphi \frac{s_0}{eR}\cot\theta. \tag{2.1.61}$$

The singularities of $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ at the poles are without physical significance. They describe infinitly thin solenoids admitting flux $s_0\Phi_0$ each and reflect our inability to formulate a true magnetic monopole.

The dynamical angular momentum (2.1.52) becomes

$$\boldsymbol{\Lambda} = -\mathrm{i}\left[\boldsymbol{e}_\varphi \frac{\partial}{\partial\theta} - \boldsymbol{e}_\theta \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\varphi} - \mathrm{i}s_0\cos\theta\right)\right]. \tag{2.1.62}$$

With (A.5) we obtain

$$\boldsymbol{\Lambda}^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial\varphi} - \mathrm{i}s_0\cos\theta\right)^2. \tag{2.1.63}$$

To formulate the eigenstates, Haldane [3] introduced spinor coordinates for the particle position,

$$u = \cos\frac{\theta}{2}\exp\left(\frac{i\varphi}{2}\right), v = \sin\frac{\theta}{2}\exp\left(-\frac{i\varphi}{2}\right), \tag{2.1.64}$$

such that

$$\boldsymbol{e}_r = \boldsymbol{\Omega}(u,v) \;\equiv\; (u,v)\boldsymbol{\sigma}\begin{pmatrix}\bar{u}\\ \bar{v}\end{pmatrix}, \tag{2.1.65}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector consisting of the three Pauli matrices

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{2.1.66}$$

In terms of these, a complete, orthogonal basis of the states spanning the lowest Landau level ($n = 0, s = s_0$) is given by

$$\psi^s_{m,0}(u,v) = u^{s+m} v^{s-m} \tag{2.1.67}$$

with

$$m = -s, s+1, \ldots, s.$$

For these states,

$$\begin{aligned} L^z \psi^s_{m,0} &= m \psi^s_{m,0}, \\ H \psi^s_{m,0} &= \frac{1}{2} \omega_c \psi^s_{m,0}. \end{aligned} \tag{2.1.68}$$

To verify (2.1.68), we consider the action of (2.1.63) on the more general basis states

$$\begin{aligned} \phi^s_{m,p}(u,v) &= \left(\cos\frac{\theta}{2}\right)^{s+m} \left(\sin\frac{\theta}{2}\right)^{s-m} e^{i(m-p)\varphi} \\ &= \begin{cases} \bar{v}^{-p} u^{s+m} v^{s-m+p}, & \text{for } p < 0, \\ \bar{u}^{p} u^{s+m-p} v^{s-m}, & \text{for } p \geq 0. \end{cases} \end{aligned} \tag{2.1.69}$$

This yields

$$\begin{aligned} \mathbf{\Lambda}^2 \phi^s_{m,p} &= \left[s - \left(\frac{s\cos\theta - m}{\sin\theta} \right)^2 + \left(\frac{s_0 \cos\theta - m + p}{\sin\theta} \right)^2 \right] \phi^s_{m,p} \\ &= \left[s + \frac{2(s\cos\theta - m + p)(p - n\cos\theta) - (p^2 - n^2\cos^2\theta)}{\sin^2\theta} \right] \phi^s_{m,p}, \end{aligned} \tag{2.1.70}$$

For $p = n = 0$, this clearly reduces to $\mathbf{\Lambda}^2 \psi^s_{m,0} = s \psi^s_{m,0}$, and hence (2.1.68). The normalization of (2.1.67) can easily be obtained with the integral

$$\frac{1}{4\pi} \int \mathrm{d}\Omega\, \bar{u}^{S+m'} \bar{v}^{S-m'} u^{s+m} v^{s-m} = \frac{(s+m)!(s-m)!}{(2s+1)!} \delta_{mm'}, \tag{2.1.71}$$

where $\mathrm{d}\Omega = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$.

To describe particles in the lowest Landau level which are localized at a point $\mathbf{\Omega}(\alpha, \beta)$ with spinor coordinates (α, β),

$$\mathbf{\Omega}(\alpha,\beta) = (\alpha,\beta)\boldsymbol{\sigma} \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix}, \tag{2.1.72}$$

Haldane [3] introduced "coherent states" defined by

$$\{\mathbf{\Omega}(\alpha,\beta)\boldsymbol{L}\} \psi^s_{(\alpha,\beta),0}(u,v) = s \psi^s_{(\alpha,\beta),0}(u,v). \tag{2.1.73}$$

In the lowest Landau level, the angular momentum $\boldsymbol{L}$ can be written

$$\boldsymbol{L} = \frac{1}{2}(u, v)\boldsymbol{\sigma} \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \end{pmatrix}. \tag{2.1.74}$$

Note that u, v may be viewed as Schwinger boson creation, and $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ the corresponding annihilation operators (see Sect. 2.4.3). The solutions of (2.1.73) are given by

$$\psi^{s}_{(\alpha,\beta),0}(u, v) = (\bar{\alpha}u + \bar{\beta}v)^{2s}, \tag{2.1.75}$$

as one can verify easily with the identity

$$(\underline{a}\,\boldsymbol{\sigma}\,\underline{b})(\underline{c}\,\boldsymbol{\sigma}\,\underline{d}) = 2(\underline{a}\,\underline{d})(\underline{c}\,\underline{b}) - (\underline{a}\,\underline{b})(\underline{c}\,\underline{d}). \tag{2.1.76}$$

where $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$ are two-component spinors.

Haldane [3] further introduced two-particle coherent lowest Landau level states defined by

$$\{\boldsymbol{\Omega}(\alpha, \beta)(\boldsymbol{L}_1 + \boldsymbol{L}_2)\}\psi^{s,j}_{(\alpha,\beta),0}[u, v] = j\psi^{s,j}_{(\alpha,\beta),0}[u, v], \tag{2.1.77}$$

where $[u, v] := (u_1, u_2, v_1, v_2)$ and j is the total angular momentum quantum number,

$$(\boldsymbol{L}_1 + \boldsymbol{L}_2)^2\psi^{s,j}_{(\alpha,\beta),0}[u, v] = j(j + 1)\psi^{s,j}_{(\alpha,\beta),0}[u, v]. \tag{2.1.78}$$

The solution of (2.1.77) is given by

$$\psi^{s,j}_{(\alpha,\beta),0}[u, v] = (u_1v_2 - u_2v_1)^{2s-j} \prod_{i=1,2} (\bar{\alpha}u_i + \bar{\beta}v_i)^{j}. \tag{2.1.79}$$

It describes two particles with relative momentum $2s - j$ precessing about their common center of mass at $\boldsymbol{\Omega}(\alpha, \beta)$.

Since $0 \leq j \leq 2s$, the relative momentum quantum number $l = 2s - j$ has to be a non-negative integer. The restriction to non-negative integers is a consequence of Landau level quantization, and exists in the plane as well, as we discussed in Sect. 2.1.2. For bosons or fermions, l has to be even or odd, respectively. This implies that the projection Π_0 into the lowest Landau level of any rotationally invariant operator $V(\boldsymbol{r}_1 \cdot \boldsymbol{r}_2)$, such as two particle interactions, can be expanded as

$$\Pi_0 V(\boldsymbol{r}_1 \cdot \boldsymbol{r}_2)\Pi_0 = \sum_{l}^{2s} V_l P_{2s-l}(\boldsymbol{L}_1 + \boldsymbol{L}_2), \tag{2.1.80}$$

where the sum over l is restricted to even (odd) integer for bosons (fermions), V_l denotes the so-called pseudopotential coefficients, and $P_j(\boldsymbol{L})$ is the projection operator on states with total momentum $\boldsymbol{L}^2 = j(j+1)$.

As mentioned, this formalism was very recently generalized to include higher Landau levels as well [37]. The key insight permitting this generalization was that there are two mutually commuting SU(2) algebras with spin s, one for the cyclotron variables and one for the guiding center variables. These algebras are analogous to the the two mutually commuting ladder algebras $a, a^\dagger$ and $b, b^\dagger$ in the plane, which we introduced in Sect. 2.1.1.

2.1.6 The Laughlin State and Its Parent Hamiltonian on the Sphere

In analogy to (2.1.30), Haldane [3] writes the Laughlin $\nu = 1/m$ state for N particles on a sphere with $2s_0 = m(N-1)$ as

$$\psi_m[u, v] = \prod_{i<j}^{N} (u_i v_j - u_j v_i)^m. \tag{2.1.81}$$

Since the factors $(u_i v_j - u_j v_i)$ commute with the total angular momentum

$$\boldsymbol{L}_{\text{tot}} = \sum_{i=1}^{N} \boldsymbol{L}_i, \tag{2.1.82}$$

(2.1.81) is obviously invariant under spacial rotations around the sphere:

$$\boldsymbol{L}_{\text{tot}} \psi_m = 0. \tag{2.1.83}$$

The Laughlin droplet wave function centered at $\boldsymbol{\Omega}(\alpha, \beta)$ can be recovered by multiplying $\psi_m[u, v]$ by a factor

$$\prod_{i=1}^{N} (\bar{\alpha} u_i + \bar{\beta} v_i)^n,$$

and then taking the limit $n \to \infty$, $R \to \infty$, while $4\pi R^2/n = 2\pi l^2 = \text{const.}$, where l^2 is the magnetic length (2.1.5).

As in the plane, the uniquely specifying property of the Laughlin state (2.1.81) is that the smallest component of relative angular momentum is m, which is even for bosons and odd for fermions. Haldane [3] constructed a model Hamiltonian, which, together with the kinetic Hamiltonian (2.1.50), singles out (2.1.81) as exact and unique zero energy ground state, by assigning a finite energy cost to the components

of angular momentum smaller than m. With the most general two-particle interaction Hamiltonian given by

$$H_{\text{int}} = \sum_{i<j}^{N} \left\{ \sum_{l}^{2s} V_l \mathrm{P}_{2s-l}(\boldsymbol{L}_i + \boldsymbol{L}_j) \right\}, \tag{2.1.84}$$

where the values of l are restricted to even (odd) integers for bosons (fermions) and P_{2s-l} is as defined in (2.1.80), Haldane's Hamiltonian amounts to taking

$$V_l = \begin{cases} 1 & \text{for } l < m, \\ 0 & \text{for } l \geq m. \end{cases} \tag{2.1.85}$$

For all practical purposes, we need to rewrite (2.1.84) in terms of boson or fermion creation or annihilation operators,

$$\begin{aligned} H_{\text{int}} = & \sum_{m_1=-s}^{s} \sum_{m_2=-s}^{s} \sum_{m_3=-s}^{s} \sum_{m_4=-s}^{s} a^{\dagger}_{m_1} a^{\dagger}_{m_2} a_{m_3} a_{m_4} \delta_{m_1+m_2, m_3+m_4} \\ & \cdot \sum_{l=0}^{2s} \langle s, m_1; s, m_2 | 2s-l, m_1+m_2 \rangle V_l \langle 2s-l, m_3+m_4 | s, m_3; s, m_4 \rangle, \end{aligned} \tag{2.1.86}$$

where a_m annihilates a boson or fermion in the properly normalized single particle state

$$\psi^{s}_{m,0}(u, v) = \sqrt{\frac{(2s+1)!}{4\pi (s+m)!(s-m)!}}\, u^{s+m} v^{s-m}, \tag{2.1.87}$$

and $\langle s, m_1; s, m_2 | j, m_1+m_2 \rangle$ etc. are Clebsch–Gordan coefficients [39]. Essentially, we take two particles with L_z eigenvalues m_3 and m_4, change the basis into one where $m_3 + m_4$ and the total two particle momentum $2s-l$ are replacing the quantum numbers m_3 and m_4, multiply each amplitude by V_l, and convert the two particles states back into a basis of L_z eigenvalues m_1 and m_2.

The fractionally charged quasihole and quasielectron excitations of the Laughlin state (2.1.81) localized at $\boldsymbol{\Omega}(\alpha, \beta)$ on the sphere are given by

$$\psi^{\text{QH}}_{(\alpha,\beta)}[u, v] = \prod_{i=1}^{N} (\beta u_i - \alpha v_i) \prod_{i<j}^{N} (u_i v_j - u_j v_i)^m \tag{2.1.88}$$

and

$$\psi^{\text{QE}}_{(\alpha,\beta)}[u, v] = \prod_{i=1}^{N} \left(\bar{\beta} \frac{\partial}{\partial u_i} - \bar{\alpha} \frac{\partial}{\partial v_i} \right) \prod_{i<j}^{N} (u_i v_j - u_j v_i)^m, \tag{2.1.89}$$

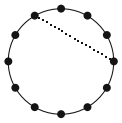
which increase or decrease the number of flux quanta $2s_0$ through the sphere by one, and decrease or increase $\boldsymbol{\Omega}(\alpha, \beta)\boldsymbol{L}_{\text{tot}}$ by $\frac{1}{2}N$.

Due to the formal simplicity, the sphere is particularly well suited to formulate the hierarchy of quantized Hall states, where all odd-denominator filling fractions can be obtained through successive condensation of quasiparticles into Laughlin-type fluids [3, 20, 21, 40].

2.2 The Haldane–Shastry Model

2.2.1 The $1/r^2$ Model of Haldane and Shastry

The Haldane–Shastry model [41–53] is one of the most important paradigms for a generic spin $\frac{1}{2}$ liquid on a chain. Consider a spin $\frac{1}{2}$ chain with periodic boundary conditions and an even number of sites N on a unit circle embedded in the complex plane:



N sites with spin $\frac{1}{2}$ on unit circle:

$\eta_\alpha = e^{\mathrm{i}\frac{2\pi}{N}\alpha}$ with $\alpha = 1, \ldots, N$

The $1/r^2$-Hamiltonian

$$H^{\mathrm{HS}} = \left(\frac{2\pi}{N}\right)^2 \sum_{\alpha<\beta}^{N} \frac{\boldsymbol{S}_\alpha \boldsymbol{S}_\beta}{|\eta_\alpha - \eta_\beta|^2}, \tag{2.2.1}$$

where $|\eta_\alpha - \eta_\beta|$ is the chord distance between the sites α and β, has the exact ground state

$$|\psi_0^{\mathrm{HS}}\rangle = \sum_{\{z_1,\ldots,z_M\}} \psi_0^{\mathrm{HS}}(z_1, \ldots, z_M) S_{z_1}^+ \cdot \ldots \cdot S_{z_M}^+ \Big| \underbrace{\downarrow\downarrow \ldots \ldots \downarrow}_{\text{all } N \text{ spins } \downarrow} \Big\rangle, \tag{2.2.2}$$

where the sum extends over all possible ways to distribute the $M = \frac{N}{2}$ $\uparrow$-spin coordinates z_i on the unit circle and

$$\psi_0^{\mathrm{HS}}(z_1, z_2, \ldots, z_M) = \prod_{i<i}^{M} (z_i - z_j)^2 \prod_{i=1}^{M} z_i. \tag{2.2.3}$$

The ground state has momentum

$$p_0 = -\frac{\pi}{2} N, \tag{2.2.4}$$

where we have adopted a convention according to which the "vacuum" state $|\downarrow\downarrow \ldots \downarrow\rangle$ has momentum $p = 0$ (and the empty state $|0\rangle$ has $p = \pi(N-1)$) and energy

$$E_0 = -\frac{\pi^2}{24}\left(N + \frac{5}{N}\right). \tag{2.2.5}$$

We will verify (2.2.4) and (2.2.5) in Sects. 2.2.3 and 2.2.4, respectively.

2.2.2 Symmetries and Integrability

The Haldane–Shastry Hamiltonian (2.2.1) is clearly invariant under space translations (rotations of the unit circle), time reversal, parity, and global SU(2) spin rotations generated by

$$\boldsymbol{S}_{\text{tot}} = \sum_{\alpha=1}^{N} \boldsymbol{S}_\alpha, \quad \left[H^{\text{HS}}, \boldsymbol{S}_{\text{tot}}\right] = 0. \tag{2.2.6}$$

The total spin trivially satisfies the standard commutation relations for angular momentum,

$$\left[S^i_{\text{tot}}, S^j_{\text{tot}}\right] = \mathrm{i}\, \varepsilon^{ijk} S^k_{\text{tot}}. \tag{2.2.7}$$

The model possesses an additional symmetry [46, 54] generated by the rapidity operator

$$\boldsymbol{\Lambda} = \frac{\mathrm{i}}{2} \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} \frac{\eta_\alpha + \eta_\beta}{\eta_\alpha - \eta_\beta} \boldsymbol{S}_\alpha \times \boldsymbol{S}_\beta, \quad \left[H^{\text{HS}}, \boldsymbol{\Lambda}\right] = 0, \tag{2.2.8}$$

which measures the spin current. It transforms as a vector under spin rotations,

$$\left[S^i_{\text{tot}}, \Lambda^j\right] = \mathrm{i}\, \varepsilon^{ijk} \Lambda^k. \tag{2.2.9}$$

Note that even though both $\boldsymbol{S}_{\text{tot}}$ and $\boldsymbol{\Lambda}$ commute with the Hamiltonian, they do not commute mutually, but generate an infinite dimensional associative algebra with certain defining relations and consistency conditions, the Yangian $\mathrm{Y}(\mathrm{sl}_2)$ [55, 56]. Since the commutator of the total spin squared with the rapidity operator does not vanish in general,

$$\left[\boldsymbol{S}^2_{\text{tot}}, \Lambda^i\right] = -\mathrm{i}\, \varepsilon^{ijk} \left\{S^j_{\text{tot}}, \Lambda^k\right\}, \tag{2.2.10}$$

elements of the Yangian algebra connect degenerate eigenstates with different total spins. With these elements, it is possible to generate all the eigenstates of the model from all the completely spin polarized eigenstates.

The Yangian symmetry of the model [46, 54] implies significant degeneracies in the spectrum and hence indicates integrability. The model is not integrable in the usual

sense, however, as the method of quantum inverse scattering [57] is not applicable to models with long-range interactions. Talstra and Haldane [58] have nonetheless succeeded in constructing an infinite set of mutually commuting integrals of motion for the model by using the determinant rather than the trace of the monodromy matrix. These integrals provide the framework for the model's integrability. The integrability is hence only indirectly related to the Yangian symmetry.

The model is further amenable to exact solution via the asymtotic Bethe Ansatz [44, 47, 54, 59–64], even though the application of this method to models with long-range interactions is likewise heuristic.

2.2.3 Ground State Properties

The ground state (2.2.3) is real (and hence both parity and time-reversal invariant), a spin singlet, and can equivalently be obtained by Gutzwiller projection [65–71], as we will verify now after evaluating the total momentum.

Ground state momentum. To determine the momentum p_0 (in units of inverse lattice spacings $1/a$) we translate the ground state (2.2.3) counterclockwise by one lattice spacing around the unit circle,

$$\boldsymbol{T}\left|\psi_0^{\mathrm{HS}}\right\rangle = e^{\mathrm{i}p_0}\left|\psi_0^{\mathrm{HS}}\right\rangle. \tag{2.2.11}$$

With $\boldsymbol{T}z_i = e^{\mathrm{i}\frac{2\pi}{N}}z_i$, we find

$$p_0 = \frac{2\pi}{N}\left(2\frac{M(M-1)}{2} + M\right) = \pi M,$$

and hence (2.2.4). Note that the sign of p_0 is irrelevant for (2.2.3), as N is always even, and p_0 is 0 or π. The sign will become significant only in Sects. 2.2.6 and 2.2.7 below, when we assign spinons momenta for states with N odd.

Singlet property. Since $S^{\mathrm{z}}_{\mathrm{tot}}\left|\psi_0^{\mathrm{HS}}\right\rangle = 0$, it suffices to show that $\left|\psi_0^{\mathrm{HS}}\right\rangle$ is annihilated by S^-_{tot}:

$$\begin{aligned} S^-_{\mathrm{tot}}\left|\psi_0^{\mathrm{HS}}\right\rangle &= \sum_{\alpha=1}^{N} S^-_\alpha \sum_{\{z_1,\ldots z_M\}} \psi_0^{\mathrm{HS}}(z_1, z_2, \ldots z_M) S^+_{z_1}\ldots S^+_{z_M}|{\downarrow\downarrow}\ldots{\downarrow}\rangle \\ &= \sum_{\{z_2,\ldots,z_M\}} \underbrace{\sum_{\alpha=1}^{N} \psi_0^{\mathrm{HS}}(\eta_\alpha, z_2, \ldots, z_M)}_{=0} S^+_{z_2}\ldots S^+_{z_M}|{\downarrow\downarrow}\ldots{\downarrow}\rangle, \end{aligned} \tag{2.2.12}$$

since $\psi_0^{\mathrm{HS}}(\eta_\alpha, z_2, \ldots, z_M)$ contains only powers $\eta_\alpha^1, \eta_\alpha^2, \ldots, \eta_\alpha^{N-1}$ and

$$\sum_{\alpha=1}^{N} \eta_\alpha^m = N\delta_{m,0} \mod N. \tag{2.2.13}$$

Parity and time reversal invariance. We begin by showing that ψ_0^{HS} is real. With $\bar{z}_i = 1/z_i$ and hence

$$(z_i - z_j)^2 = -z_i z_j |z_i - z_j|^2, \tag{2.2.14}$$

we write

$$\begin{aligned}\psi_0^{\mathrm{HS}}(z_1, z_2, \ldots, z_M) &= \pm \prod_{i<j}^{M} |z_i - z_j|^2 \prod_{i<j}^{M} z_i z_j \prod_{i=1}^{M} z_i \\ &= \pm \prod_{i<j}^{M} |z_i - z_j|^2 \prod_{i=1}^{M} G(z_i)\end{aligned} \tag{2.2.15}$$

where

$$G(\eta_\alpha) = (\eta_\alpha)^{\frac{N}{2}} = \begin{cases} +1 & \alpha \text{ even} \\ -1 & \alpha \text{ odd.} \end{cases} \tag{2.2.16}$$

The gauge factor $G(z_i)$ effects that the Marshall sign criteria [72] is fulfilled.

Since parity tranforms $\eta_\alpha \to \eta_{-\alpha} = \bar{\eta}_\alpha$ and hence $z_i \to \bar{z}_i$, the fact that ψ_0^{HS} is real implies that $|\psi_0^{\mathrm{HS}}\rangle$ is invariant under parity. Time reversal transforms [73]

$$\mathrm{i} \to -\mathrm{i}, \quad \boldsymbol{S}_\alpha \to -\boldsymbol{S}_\alpha, \quad |s, m\rangle \to \mathrm{i}^{2m} |s, -m\rangle,$$

which implies $z_i \to \bar{z}_i$, $S_\alpha^+ \to -S_\alpha^-$, and $|{\downarrow}{\downarrow} \ldots {\downarrow}\rangle \to (-\mathrm{i})^N |{\uparrow}{\uparrow} \ldots {\uparrow}\rangle$. The basis states in (2.2.2) hence transform according to

$$S_{z_1}^+ \cdot \ldots \cdot S_{z_M}^+ |{\downarrow}{\downarrow} \ldots {\downarrow}\rangle \to S_{z_1}^- \cdot \ldots \cdot S_{z_M}^- |{\uparrow}{\uparrow} \ldots {\uparrow}\rangle. \tag{2.2.17}$$

Together with the singlet property, this implies that $|\psi_0^{\mathrm{HS}}\rangle$ is invariant under time reversal.

Generation by Gutzwiller projection. The ground state of the model was first obtained by Gutzwiller projection from a completely filled one-dimensional band which in total contains as many spin $\frac{1}{2}$ fermions as there are lattice sites [65, 68–71]:

$$|\psi_0^{\mathrm{HS}}\rangle = \mathrm{P}_{\mathrm{GW}} |\psi_{\mathrm{SD}}^N\rangle, \quad |\psi_{\mathrm{SD}}^N\rangle \equiv \prod_{q \in \mathcal{I}} c_{q\uparrow}^\dagger c_{q\downarrow}^\dagger |0\rangle, \tag{2.2.18}$$

where the Gutzwiller projector

$$\mathrm{P}_{\mathrm{GW}} \equiv \prod_{i=1}^{N} \left(1 - c_{i\uparrow}^\dagger c_{i\uparrow} c_{i\downarrow}^\dagger c_{i\downarrow}\right) \tag{2.2.19}$$

eliminates configurations with more than one particle on any site and the interval $\mathcal{I}$ contains $M = \frac{N}{2}$ adjacent momenta. We will now show that (2.2.18) is equivalent

to (2.2.3). With lattice constant $a = \frac{2\pi}{N}$, the allowed momenta are given by integers, $q = 0, 1, \ldots, N-1$. With

$$c_q^\dagger = \sum_{\alpha=1}^{N} e^{\mathrm{i}\frac{2\pi}{N}\alpha q} c_\alpha^\dagger = \sum_{\alpha=1}^{N} \eta_\alpha^q c_\alpha^\dagger, \tag{2.2.20}$$

the (unnormalized) single particle momentum eigenstates are given by

$$\phi_q(z) = \langle z|q\rangle = \langle 0|c_z c_q^\dagger|0\rangle = z^q. \tag{2.2.21}$$

The many particle wave function for M fermions with adjacent momenta $q \in \mathcal{I} = [q_1, q_1 + M - 1]$ is hence given by

$$\phi_{\mathcal{I}}(z_1, z_2, \ldots, z_M) = \prod_{i=1}^{M} z_i^{q_1} \cdot \mathcal{A}\{z_1^0 z_2^1 \ldots z_M^{M-1}\} = \prod_{i=1}^{M} z_i^{q_1} \prod_{i<j}^{M} (z_i - z_j). \tag{2.2.22}$$

The Gutzwiller state (2.2.18) is given by

$$\begin{aligned} \left|\psi_0^{\mathrm{HS}}\right\rangle = & \sum_{\{z_1,\ldots,z_M;\, w_1,\ldots,w_M\}} \phi_{\mathcal{I}}(z_1, \ldots, z_M)\phi_{\mathcal{I}}(w_1, \ldots, w_M) \\ & \cdot c_{z_1\uparrow}^\dagger \ldots c_{z_M\uparrow}^\dagger c_{w_1\downarrow}^\dagger \ldots c_{w_M\downarrow}^\dagger |0\rangle, \end{aligned} \tag{2.2.23}$$

where the sum extends over all possible ways to distribute the coordinates z_i and w_k on mutually distinct lattice sites.

Let $\tilde{\mathcal{I}}$ contain all those M momenta not contained in $\mathcal{I}$, and $w_1, \ldots, w_M$ denote the sites which are not occupied by any of the z_i's. Then

$$\begin{aligned} \phi_{\mathcal{I}}(w_1, \ldots, w_M) &= \langle 0|c_{w_M} \ldots c_{w_1} \prod_{q\in\mathcal{I}} c_q^\dagger|0\rangle \\ &= \mathrm{sign}[z; w] \cdot \langle 0| \prod_{q\in\mathcal{I}} c_q \prod_{q\in\tilde{\mathcal{I}}} c_q \; c_{z_1}^\dagger \ldots c_{z_M}^\dagger \prod_{q\in\mathcal{I}} c_q^\dagger|0\rangle \\ &= \mathrm{sign}[z; w] \cdot \langle 0| \prod_{q\in\tilde{\mathcal{I}}} c_q \; c_{z_1}^\dagger \ldots c_{z_M}^\dagger|0\rangle \\ &= \mathrm{sign}[z; w] \cdot \phi_{\tilde{\mathcal{I}}}{}^*(z_1, \ldots, z_M) \\ &= \mathrm{sign}[z; w] \cdot \prod_{i=1}^{M} \bar{z_i}^M \cdot \phi_{\mathcal{I}}{}^*(z_1, \ldots, z_M), \end{aligned} \tag{2.2.24}$$

where

$$\mathrm{sign}[z; w] \equiv \langle 0|c_{w_M} \ldots c_{w_1} c_{z_M} \ldots c_{z_1} \prod_{q\in\tilde{\mathcal{I}}} c_q^\dagger \prod_{q\in\mathcal{I}} c_q^\dagger|0\rangle \tag{2.2.25}$$

is an overal sign associated with ordering the z's and w's according to the lattice sites indices α. Since

$$\mathrm{sign}[z; w] \cdot c_{z_1\uparrow}^\dagger \ldots c_{z_M\uparrow}^\dagger c_{w_1\downarrow}^\dagger \ldots c_{w_M\downarrow}^\dagger|0\rangle = S_{z_1}^+ \cdot \ldots \cdot S_{z_M}^+|\downarrow\downarrow \ldots \downarrow\rangle \tag{2.2.26}$$

we may write

$$\left|\psi_0^{\mathrm{HS}}\right\rangle = \sum_{\{z_1,\ldots,z_M\}} |\phi_{\mathcal{I}}(z_1,\ldots,z_M)|^2 \prod_{i=1}^{M} G(z_i) S_{z_1}^{+} \cdot \ldots \cdot S_{z_M}^{+} |\downarrow\downarrow \ldots \downarrow\rangle. \quad (2.2.27)$$

This is equivalent to (2.2.15).

As an aside, it is very easy to verify the singlet property in the Gutzwiller formulation (2.2.18) of the ground state. To begin with, filling the same single particle states with $\uparrow$ and $\downarrow$ spin fermions obviously yields a singlet,

$$\boldsymbol{S}_{\mathrm{tot}} \left|\psi_{\mathrm{SD}}^{N}\right\rangle = 0. \quad (2.2.28)$$

The Gutzwiller projector (2.2.19), however, commutes with the local spin operators and hence also with the total spin,

$$\left[\mathrm{P}_{\mathrm{GW}}, \boldsymbol{S}_{\alpha}\right] = \left[\mathrm{P}_{\mathrm{GW}}, \boldsymbol{S}_{\mathrm{tot}}\right] = 0. \quad (2.2.29)$$

Hence

$$\boldsymbol{S}_{\mathrm{tot}} \left|\psi_0^{\mathrm{HS}}\right\rangle = 0. \quad (2.2.30)$$

Norm. The norm of the ground state is [74]

$$\sum_{\{z_1,\ldots,z_M\}} \prod_{i<j}^{M} |z_i - z_j|^4 = \left(\frac{N}{2\pi \mathrm{i}}\right)^M \oint \frac{\mathrm{d}z_1}{z_1} \ldots \oint \frac{\mathrm{d}z_M}{z_M} \prod_{i\neq j}^{M} \left(1 - \frac{z_i}{z_j}\right)^2 = \frac{N^M (2M)!}{2^M}. \quad (2.2.31)$$

Relation to the chiral spin liquid. The Haldane–Shastry ground state may be viewed as the one-dimensional analog of the abelian or $S = \frac{1}{2}$ chiral spin liquid [75–82], which is essentially a Laughlin $m = 2$ quantized Hall state [1] for spin flips on a two dimensional lattice. The spinons in the chiral spin liquid were understood to obey half-Fermi statistics long before this was realized for the Haldane–Shastry model.

2.2.4 Explict Solution

For the explict calculation presented here to be applicable to the one- and two-spinon eigenstates investigated in Sect. 2.2.6 below, we consider wavefunctions of the form [44, 50–52]

$$\psi(z_1,\ldots,z_M) = \phi(z_1,\ldots,z_M) \cdot \psi_0^{\mathrm{HS}}(z_1,\ldots,z_M), \quad (2.2.32)$$

where ψ_0^{HS} is given by (2.2.2) and $\phi[z] \equiv \phi(z_1, \ldots, z_M)$ a polynomial of degree strictly less than $N - 2M + 2$ in each of the z_i's. This implies that the degree of ψ^{HS} is strictly less than $N + 1$. N can be even or odd. This condition enables us to use a Taylor expansion when we calculate the action of the Hamiltonian (2.2.1) on the state. The result is that

$$H^{\mathrm{HS}}|\psi\rangle = \frac{2\pi^2}{N^2}\left(\lambda + \frac{N}{48}(N^2-1) + \frac{M}{6}(4M^2-1) - \frac{N}{2}M^2\right)|\psi\rangle, \tag{2.2.33}$$

provided that ϕ satisfies the eigenvalue equation

$$\sum_{j=1}^{M}\left(\frac{1}{2}z_j^2\frac{\partial^2}{\partial z_j^2} + \sum_{\substack{k=1\\k\neq j}}^{M}\frac{2z_j^2}{z_j - z_k}\frac{\partial}{\partial z_j} - \frac{N-3}{2}z_j\frac{\partial}{\partial z_j}\right)\phi[z] = \lambda\phi[z] \tag{2.2.34}$$

for λ. The derivative operators in (2.2.34) and below are understood to act on the analytic extension of $\phi(z_1, \ldots, z_M)$, in which the z_i's are allowed to take any value in the complex plane. For $\phi[z] = 1$, (2.2.33) shows that $\left|\psi_0^{\mathrm{HS}}\right\rangle$ is an eigenstate of H^{HS} with energy E_0 given by (2.2.5).

Derivation of (2.2.33) *and* (2.2.34). We first use $S^{\pm} = S^{\mathrm{x}} \pm i S^{\mathrm{y}}$ to rewrite (2.2.1) as the sum of a "kinetic" and a "potential" term,

$$H^{\mathrm{HS}} = \frac{2\pi^2}{N^2}\sum_{\alpha\neq\beta}^{N}\frac{1}{|\eta_\alpha - \eta_\beta|^2}\left(S_\alpha^+ S_\beta^- + S_\alpha^{\mathrm{z}} S_\beta^{\mathrm{z}}\right). \tag{2.2.35}$$

We first evaluate the action of the kinetic term on $|\psi\rangle$. Consider first

$$\begin{aligned} S_\alpha^+ S_\beta^-|\psi\rangle &= S_\alpha^+ S_\beta^- \sum_{\{z_2,\ldots,z_M\}}\psi(\eta_\beta, z_2, \ldots, z_M) S_\beta^+ S_{z_2}^+ \cdot \ldots \cdot S_{z_M}^+ |\downarrow\downarrow \ldots \downarrow\rangle \\ &= \sum_{\{z_2,\ldots,z_M\}}\psi(\eta_\beta, z_2, \ldots, z_M) S_\alpha^+ S_{z_2}^+ \cdot \ldots \cdot S_{z_M}^+ |\downarrow\downarrow \ldots \downarrow\rangle, \end{aligned} \tag{2.2.36}$$

where we have implicitly assumed that each spin configuration in the sum over $\{z_1, z_2, \ldots, z_M\}$ in (2.2.2) appears only once (and not $M!$ times due to permutations of the z_i's). We write this as

$$\left[S_\alpha^+ S_\beta^- \psi\right](\eta_\alpha, z_2, \ldots, z_M) = \psi(\eta_\beta, z_2, \ldots, z_M). \tag{2.2.37}$$

Note in particlular that $\left[S_\alpha^+ S_\beta^- \psi\right](z_1, z_2, \ldots, z_M)$ vanishes unless η_α equals one of the z_i's.

The action of the kinetic term on ψ is given by

$$T\psi[z] \equiv \left[\sum_{\alpha\neq\beta}^{N} \frac{S_\alpha^+ S_\beta^-}{|\eta_\alpha - \eta_\beta|^2}\psi\right](z_1,\ldots,z_M)$$
$$= \sum_{j=1}^{M} \sum_{\substack{\beta=1 \\ \eta_\beta\neq z_j}}^{N} \frac{\eta_\beta}{|z_j-\eta_\beta|^2} \frac{\psi(z_1,\ldots,z_{j-1},\eta_\beta,z_{j+1},\ldots,z_M)}{\eta_\beta}. \tag{2.2.38}$$

Since the last fraction is a polynomial of degree strictly less than N in β, we can Taylor expand it around z_j,

$$\frac{\psi(z_1,\ldots,\eta_\beta,\ldots,z_M)}{\eta_\beta} = \sum_{l=0}^{N-1} \frac{(\eta_\beta - z_j)^l}{l!} \frac{\partial^l}{\partial z_j^l} \frac{\Psi(z_1,\ldots,z_M)}{z_j}. \tag{2.2.39}$$

The sum over β yields

$$\sum_{\substack{\beta=1 \\ \eta_\beta\neq z_j}}^{N} \frac{\eta_\beta(\eta_\beta - z_j)^l}{|z_j-\eta_\beta|^2} = z_j^{l+1} A_l, \quad A_l = -\sum_{\alpha=1}^{N-1} \eta_\alpha^2(\eta_\alpha - 1)^{l-2}, \tag{2.2.40}$$

where A_0, A_1, and A_2 are evaluated with (B.14), (B.9), and (B.2) from Appendix B, respectively:

$$A_0 = -\sum_{\alpha=1}^{N-1} \frac{\eta_\alpha^2}{(\eta_\alpha-1)^2} = \frac{(N-1)(N-5)}{12},$$
$$A_1 = -\sum_{\alpha=1}^{N-1} \frac{\eta_\alpha^2}{\eta_\alpha - 1} = -\frac{N-3}{2},$$
$$A_2 = -\sum_{\alpha=1}^{N-1} \eta_\alpha^2 = 1,$$
$$A_l = -\sum_{\alpha=1}^{N} \eta_\alpha^2(\eta_\alpha-1)^{l-2} = 0 \quad \text{for } 2 < l \leq N-1.$$

In the last line, we have used that $\eta_\alpha^2(\eta_\alpha-1)^{l-2}$ vanishes for $\eta_\alpha = 1$ and contains only powers $\eta_\alpha^2, \ldots \eta_\alpha^{N-1}$ for $2 < l \leq N-1$. Substituion into (2.2.38) and (2.2.39) yields

$$
\begin{aligned}
T\psi[z] &= \sum_{j=1}^{M}\left(\frac{(N-1)(N-5)}{12}z_j - \frac{N-3}{2}z_j^2\frac{\partial}{\partial z_j} + \frac{1}{2}z_j^3\frac{\partial^2}{\partial z_j^2}\right)\frac{\psi[z]}{z_j} \\
&= \frac{M(N-1)(N-5)}{12}\psi[z] \; - \; \frac{N-3}{2}\underbrace{\sum_{j\neq k}^{M}\frac{2z_j}{z_j-z_k}}_{=M(M-1)}\psi[z] \\
&\quad + \sum_{j\neq k}^{M}\frac{z_j^2}{(z_j-z_k)^2}\psi[z] \; + \; \underbrace{\sum_{\substack{j,k,m=1\\ j\neq k\neq m\neq j}}^{M}\frac{2z_j^2}{(z_j-z_k)(z_j-z_m)}}_{=2M(M-1)(M-2)/3}\psi[z] \\
&\quad + \sum_{j=1}^{M}\psi_0^{\mathrm{HS}}[z]\left(\frac{1}{2}z_j^2\frac{\partial^2}{\partial z_j^2} + \sum_{k\neq j}^{M}\frac{2z_j^2}{z_j-z_k}\psi_0\frac{\partial}{\partial z_j} - \frac{N-3}{2}z_j\frac{\partial}{\partial z_j}\right)\phi[z],
\end{aligned}
$$

where we have used the algebraic identity (B.7) in the evaluation of the triple sum.

For the action of the potential term we write

$$
S_\alpha^{\mathrm{z}}S_\beta^{\mathrm{z}} = \left(S_\alpha^{\mathrm{z}}+\frac{1}{2}\right)\left(S_\beta^{\mathrm{z}}+\frac{1}{2}\right) - \frac{1}{2}(S_\alpha^{\mathrm{z}}+S_\beta^{\mathrm{z}}) - \frac{1}{4}.
$$

This yields

$$
\begin{aligned}
V\psi[z] &\equiv \left[\sum_{\alpha\neq\beta}^{N}\frac{S_\alpha^{\mathrm{z}}S_\beta^{\mathrm{z}}}{|\eta_\alpha-\eta_\beta|^2}\psi\right](z_1,\ldots,z_M) \\
&= \sum_{j\neq k}^{M}\frac{1}{|z_j-z_k|^2}\psi[z] - \sum_{\alpha\neq\beta}^{N}\frac{S_\alpha^{\mathrm{z}}+\frac{1}{2}}{|\eta_\alpha-\eta_\beta|^2}\psi[z] + \frac{1}{4}\underbrace{\sum_{\alpha\neq\beta}^{N}\frac{1}{|\eta_\alpha-\eta_\beta|^2}}_{=N(N^2-1)/12}\psi[z].
\end{aligned}
\tag{2.2.41}
$$

With

$$
\sum_{j\neq k}^{M}\frac{1}{|z_j-z_k|^2}\psi[z] + \sum_{j\neq k}^{M}\frac{z_j^2}{(z_j-z_k)^2}\psi[z] = \frac{1}{2}M(M-1)\psi[z]
$$

and

$$
\sum_{\alpha\neq\beta}^{N}\frac{S_\alpha^{\mathrm{z}}+\frac{1}{2}}{|\eta_\alpha-\eta_\beta|^2}\psi[z] = \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N-1}\frac{S_\alpha^{\mathrm{z}}+\frac{1}{2}}{|1-\eta_\beta|^2}\psi[z] = M\frac{N^2-1}{12}\psi[z],
$$

where we have substituted $\eta_\beta \to \eta_\beta\eta_\alpha$ and used (B.15), we obtain (2.2.33) and (2.2.34).

2.2.5 Factorization of the Hamiltonian

In Sect. 2.2.4 we have shown that $|\psi_0^{\mathrm{HS}}\rangle$ is an eigenstate of H^{HS} with energy E_0 given by (2.2.5). To show that $|\psi_0^{\mathrm{HS}}\rangle$ is the ground state (or at least one of several ground states), we factorize the Haldane–Shastry Hamiltonian [45, 50, 52]. For every site η_α, we define an auxiliary operator $\boldsymbol{D}_\alpha$ by

$$\boldsymbol{D}_\alpha = \frac{1}{2}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \frac{\eta_\alpha+\eta_\beta}{\eta_\alpha-\eta_\beta}\Big[\mathrm{i}(\boldsymbol{S}_\alpha\times\boldsymbol{S}_\beta)+\boldsymbol{S}_\beta\Big]. \tag{2.2.42}$$

The rapidity operator (2.2.8) is given in terms of these by

$$\sum_{\alpha=1}^{N}\boldsymbol{D}_\alpha = \boldsymbol{\Lambda}, \tag{2.2.43}$$

as one can easily see with (B.16).

We will show below that H^{HS} can be written as:

$$H^{\mathrm{HS}} = \frac{2\pi^2}{N}\left[\frac{2}{9}\sum_{\alpha=1}^{N}\boldsymbol{D}_\alpha^\dagger\boldsymbol{D}_\alpha + \frac{N+1}{12}\boldsymbol{S}_{\mathrm{tot}}^2\right] + E_0, \tag{2.2.44}$$

which consists of two positive semi-definite operators (i.e., operators with only non-negative eigenvalues) and a constant. The lowest energy eigenvalue of H^{HS} is therefore E_0, and $|\psi_0^{\mathrm{HS}}\rangle$ is a ground state.

Taking the ground state expectation value of (2.2.44) implies with

$$H^{\mathrm{HS}}|\psi_0^{\mathrm{HS}}\rangle = E_0|\psi_0^{\mathrm{HS}}\rangle \tag{2.2.45}$$

that

$$\boldsymbol{D}_\alpha|\psi_0^{\mathrm{HS}}\rangle = 0, \quad \forall\ \alpha = 1,\ldots,N \tag{2.2.46}$$

and $\boldsymbol{S}_{\mathrm{tot}}|\psi_0^{\mathrm{HS}}\rangle = 0$. This trivially implies

$$\boldsymbol{\Lambda}|\psi_0^{\mathrm{HS}}\rangle = 0, \tag{2.2.47}$$

i.e., there is no spin current in the ground state. Note that if other ground states were to exist, (2.2.44) shows that they would have to be singlets and likewise be annihilated by $\boldsymbol{D}_\alpha$. It is not very difficult to verify (2.2.46) directly, but since we have verified (2.2.45) in Sect. 2.2.4 and will verify (2.2.44) below, there is no need to do so.

Verification of (2.2.44). For convenience, we define the purely imaginary parameter

$$\theta_{\alpha\beta} \equiv \frac{\eta_\alpha+\eta_\beta}{\eta_\alpha-\eta_\beta}$$

and recall

$$\boldsymbol{D}_\alpha^\dagger = \frac{1}{2}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \theta_{\alpha\beta}\big[\mathrm{i}(\boldsymbol{S}_\alpha \times \boldsymbol{S}_\beta) - \boldsymbol{S}_\beta\big],$$

$$\boldsymbol{D}_\alpha = \frac{1}{2}\sum_{\substack{\gamma=1\\ \gamma\neq\alpha}}^{N} \theta_{\alpha\gamma}\big[\mathrm{i}(\boldsymbol{S}_\alpha \times \boldsymbol{S}_\gamma) + \boldsymbol{S}_\gamma\big].$$

For $S = \frac{1}{2}$ and $\alpha \neq \beta, \gamma$, we obtain

$$\begin{aligned}\mathrm{i}(\mathbf{S}_\alpha \times \mathbf{S}_\beta)\mathrm{i}(\mathbf{S}_\alpha \times \mathbf{S}_\gamma) &= \varepsilon^{ijk}\varepsilon^{ilm} S_\beta^j S_\alpha^k S_\alpha^l S_\gamma^m \\ &= \big(\delta^{jl}\delta^{km} - \delta^{jm}\delta^{kl}\big) S_\beta^j \left(\frac{1}{4}\delta^{kl} + \frac{\mathrm{i}}{2}\varepsilon^{kln} S_\alpha^n\right) S_\gamma^m \\ &= -\frac{1}{2}\mathbf{S}_\beta \mathbf{S}_\gamma - \frac{\mathrm{i}}{2}\mathbf{S}_\alpha(\mathbf{S}_\beta \times \mathbf{S}_\gamma), \end{aligned} \tag{2.2.48}$$

and therewith

$$\big[\mathrm{i}(\boldsymbol{S}_\alpha \times \boldsymbol{S}_\beta) - \boldsymbol{S}_\beta\big] \cdot \big[\mathrm{i}(\boldsymbol{S}_\alpha \times \boldsymbol{S}_\gamma) + \boldsymbol{S}_\gamma\big] = -\frac{3}{2}\big[\boldsymbol{S}_\beta \boldsymbol{S}_\gamma - \mathrm{i}\boldsymbol{S}_\alpha(\boldsymbol{S}_\beta \times \boldsymbol{S}_\gamma)\big].$$

This implies

$$\sum_{\alpha=1}^{N} \boldsymbol{D}_\alpha^\dagger \boldsymbol{D}_\alpha = -\frac{3}{8}\sum_{\alpha=1}^{N}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N}\sum_{\substack{\gamma=1\\ \gamma\neq\alpha}}^{N} \theta_{\alpha\beta}\theta_{\alpha\gamma}\left[\boldsymbol{S}_\beta \boldsymbol{S}_\gamma - \mathrm{i}\boldsymbol{S}_\alpha(\boldsymbol{S}_\beta \times \boldsymbol{S}_\gamma)\right].$$

For the terms with $\alpha \neq \beta = \gamma$, we use $\boldsymbol{S} \times \boldsymbol{S} = \mathrm{i}\boldsymbol{S}$ to write

$$\boldsymbol{S}_\beta \boldsymbol{S}_\beta - \mathrm{i}\boldsymbol{S}_\alpha(\boldsymbol{S}_\beta \times \boldsymbol{S}_\beta) = \frac{3}{4} + \boldsymbol{S}_\alpha \boldsymbol{S}_\beta,$$

and observe

$$\theta_{\alpha\beta}^2 = 1 - \frac{4}{|\eta_\alpha - \eta_\beta|^2}.$$

For the terms with $\alpha, \beta,$ and γ all distinct, the vector product term vanishes as it changes sign under interchange of the dummy indices β and γ. For these terms we rearrange the sums

$$\sum_{\alpha=1}^{N}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N}\sum_{\substack{\gamma=1\\ \gamma\neq\alpha}}^{N} = \sum_{\beta=1}^{N}\sum_{\gamma=1}^{N}\sum_{\substack{\alpha=1\\ \alpha\neq\beta,\gamma}}^{N}$$

and carry out the summation over α. With

$$\frac{1}{(\eta_\alpha-\eta_\beta)(\eta_\alpha-\eta_\gamma)}=\frac{1}{\eta_\beta-\eta_\gamma}\left(\frac{1}{\eta_\alpha-\eta_\beta}-\frac{1}{\eta_\alpha-\eta_\gamma}\right)$$

and

$$\sum_{\substack{\alpha=1\\ \alpha\neq\beta,\gamma}}^{N}\frac{\eta_\beta}{\eta_\alpha-\eta_\beta}=-\frac{N-1}{2}-\frac{\eta_\beta}{\eta_\gamma-\eta_\beta},$$

which follows directly from (B.12), we obtain

$$\sum_{\substack{\alpha=1\\ \alpha\neq\beta,\gamma}}^{N}\theta_{\alpha\beta}\theta_{\alpha\gamma}=\sum_{\substack{\alpha=1\\ \alpha\neq\beta,\gamma}}^{N}\left(1+\frac{2\eta_\beta}{\eta_\alpha-\eta_\beta}\right)\left(1+\frac{2\eta_\gamma}{\eta_\alpha-\eta_\gamma}\right)=N-\frac{8}{|\eta_\beta-\eta_\gamma|^2}.$$

Collecting all the terms yields

$$\begin{aligned}&\frac{8}{3}\sum_{\alpha=1}^{N}\boldsymbol{D}_\alpha^\dagger\boldsymbol{D}_\alpha\\&\quad=\sum_{\alpha\neq\beta}^{N}\left(\frac{4}{|\eta_\alpha-\eta_\beta|^2}-1\right)\left(\frac{3}{4}+\boldsymbol{S}_\alpha\boldsymbol{S}_\beta\right)+\sum_{\beta\neq\gamma}^{N}\left(\frac{8}{|\eta_\beta-\eta_\gamma|^2}-N\right)\boldsymbol{S}_\beta\boldsymbol{S}_\gamma\\&\quad=12\sum_{\alpha\neq\beta}^{N}\frac{\boldsymbol{S}_\alpha\boldsymbol{S}_\beta}{|\eta_\alpha-\eta_\beta|^2}-(N+1)\sum_{\alpha\neq\beta}^{N}\boldsymbol{S}_\alpha\boldsymbol{S}_\beta+\sum_{\alpha\neq\beta}^{N}\left(\frac{3}{|\eta_\alpha-\eta_\beta|^2}-\frac{3}{4}\right).\end{aligned}$$

With the identities

$$\sum_{\alpha\neq\beta}^{N}\boldsymbol{S}_\alpha\boldsymbol{S}_\beta=\boldsymbol{S}_{\mathrm{tot}}^2-\frac{3}{4}N$$

and

$$\sum_{\alpha\neq\beta}^{N}\left(\frac{3}{|\eta_\alpha-\eta_\beta|^2}-\frac{3}{4}\right)=\frac{1}{4}N(N^2-1)-\frac{3}{4}N(N-1),$$

where we have used (B.15), we obtain

$$\sum_{\alpha\neq\beta}^{N}\frac{\boldsymbol{S}_\alpha\boldsymbol{S}_\beta}{|\eta_\alpha-\eta_\beta|^2}=\frac{2}{9}\sum_{\alpha=1}^{N}\boldsymbol{D}_\alpha^\dagger\boldsymbol{D}_\alpha+\frac{N+1}{12}\boldsymbol{S}_{\mathrm{tot}}^2-\frac{N(N^2+5)}{48},$$

and hence (2.2.44). □

2.2.6 Spinon Excitations and Fractional Statistics

The elementary excitations for this model are free spinon excitations, which carry spin $\frac{1}{2}$ and no charge. They constitute an instance of fractional quantization, which is both conceptually and mathematically similar to the fractional quantization of charge in the fractional quantum Hall effect [1]. Their fractional quantum number is the spin, which takes the value $\frac{1}{2}$ in a Hilbert space (2.2.2) made out of spin flips S^+, which carry spin 1.

One-spinon states. To write the wave function for a $\downarrow$-spin spinon localized at site η_α, consider a chain with an odd number of sites N and let $M = \frac{N-1}{2}$ be the number of $\uparrow$ or $\downarrow$ spins condensed in the uniform liquid. The spinon wave function is then given by

$$\psi_{\alpha\downarrow}(z_1, z_2, \ldots, z_M) = \prod_{i=1}^{M}(\eta_\alpha - z_i)\psi_0^{\mathrm{HS}}(z_1, z_2, \ldots, z_M), \tag{2.2.49}$$

which we understand substituted into (2.2.2). It is easy to verify $S^{\mathrm{z}}_{\mathrm{tot}}\psi_{\alpha\downarrow} = -\frac{1}{2}\psi_{\alpha\downarrow}$ and $S^{-}_{\mathrm{tot}}\psi_{\alpha\downarrow} = 0$, which shows that the spinon transforms as a spinor under rotations.

The localized spinon (2.2.49) is not an eigenstate of the Hamiltonian (2.2.1). To obtain exact eigenstates, we construct momentum eigenstates according to

$$\psi_{m\downarrow}(z_1, z_2, \ldots, z_M) = \sum_{\alpha=1}^{N}(\bar{\eta}_\alpha)^m \psi_{\alpha\downarrow}(z_1, z_2, \ldots, z_M), \tag{2.2.50}$$

where the integer m corresponds to a momentum quantum number. Since $\psi_{\alpha\downarrow}(z_1, z_2, \ldots, z_M)$ contains only powers $\eta_\alpha^0, \eta_\alpha^1, \ldots, \eta_\alpha^M$ and

$$\sum_{\alpha=1}^{N}\bar{\eta}_\alpha^m \eta_\alpha^n = \delta_{mn} \mod N, \tag{2.2.51}$$

$\psi_{m\downarrow}(z_1, z_2, \ldots, z_M)$ will vanish unless $m = 0, 1, \ldots, M$. There are only roughly half as many spinon orbitals as there are sites. Spinons on neighboring sites hence cannot be orthogonal. With (2.2.33) and (2.2.34), we obtain

$$H^{\mathrm{HS}}|\psi_{m\downarrow}\rangle = \left[-\frac{\pi^2}{24}\left(N - \frac{1}{N}\right) + \frac{2\pi^2}{N^2}m(M-m)\right]|\psi_{m\downarrow}\rangle. \tag{2.2.52}$$

To make a correspondence between m and the spinon momentum p_m, we translate (2.2.50) counterclockwise by one lattice spacing (which we set to unity for present purposes) around the unit circle,

$$\boldsymbol{T}|\psi_{m\downarrow}\rangle = e^{\mathrm{i}(p_0+p_m)}|\psi_{m\downarrow}\rangle. \tag{2.2.53}$$

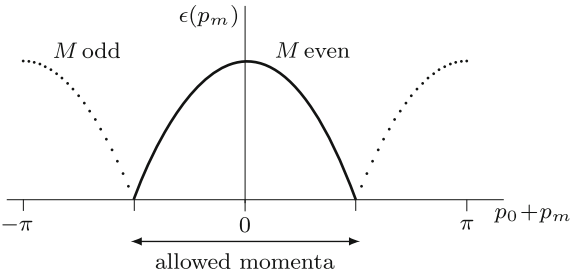


Fig. 2.2 Dispersion of a single spinon in a Haldane–Shastry chain

With ground state momentum $p_0 = -\frac{\pi}{2}N$, we find

$$p_m = \pi - \frac{2\pi}{N}\left(m + \frac{1}{4}\right). \tag{2.2.54}$$

The energy (2.2.52) can be written as $E = E_0 + \epsilon(p_m)$, with the spinon dispersion given by

$$\epsilon(p) = \frac{1}{2}p\,(\pi - p) + \frac{\pi^2}{8N^2}, \tag{2.2.55}$$

as depicted in Fig. 2.2. The interval of allowed spinon momenta spans only half of the Brillouin zone, and alternates with M even vs. M odd.

Two-spinon states. To write the wave function for two $\downarrow$-spin spinons localized at sites η_α and η_β, consider a chain with N even and $M = \frac{N-2}{2}$. The two-spinon state is then given by

$$\psi_{\alpha\beta}(z_1, z_2, \ldots, z_M) = \prod_{i=1}^{M}(\eta_\alpha - z_i)(\eta_\beta - z_i)\psi_0^{\mathrm{HS}}(z_1, z_2, \ldots, z_M). \tag{2.2.56}$$

A momentum basis for the two-spinon states is given by

$$\psi_{mn}(z_1, z_2, \ldots, z_M) = \sum_{\alpha,\beta=1}^{N}(\bar{\eta}_\alpha)^m(\bar{\eta}_\beta)^m\psi_{\alpha\beta}(z_1, z_2, \ldots, z_M), \tag{2.2.57}$$

where $M \geq m \geq n \geq 0$. For m or n outside this range, ψ_{mn} vanishes identically, reflecting the overcompleteness of the position space basis. With (2.2.33), (2.2.34), and the algebraic identity

$$\frac{x+y}{x-y}(x^m y^n - x^n y^m) = 2\sum_{l=0}^{m-n} x^{m-l}y^{n+l} - (x^m y^n + x^n y^m), \tag{2.2.58}$$

we obtain [44, 50–52]

$$H^{\mathrm{HS}}|\psi_{mn}\rangle = E_{mn}|\psi_{mn}\rangle + \sum_{l=1}^{l_{\max}} V_l^{mn}|\psi_{m+l,n-l}\rangle \tag{2.2.59}$$

with

$$\begin{aligned} E_{mn} &= -\frac{\pi^2}{24}\left(N - \frac{19}{N} + \frac{24}{N^2}\right) \\ &\quad + \frac{2\pi^2}{N^2}\left[m\left(\frac{N}{2} - 1 - m\right) + n\left(\frac{N}{2} - 1 - n\right) - \frac{m-n}{2}\right], \end{aligned} \tag{2.2.60}$$

$$V_l^{mn} = -\frac{2\pi^2}{N^2}(m - n + 2l), \tag{2.2.61}$$

and $l_{\max} = \min(M-m, n)$. Since the "scattering" of the non-orthogonal basis states $|\psi_{mn}\rangle$ in (2.2.59) only occurs in one direction, increasing $m-n$ while keeping $m+n$ fixed, the eigenstates of H^{HS} have energy eigenvalues E_{mn}, and are of the form

$$|\phi_{mn}\rangle = \sum_{l=0}^{l_M} a_l^{mn}|\psi_{m+l,n-l}\rangle. \tag{2.2.62}$$

A recursion relation for the coefficients a_l^{mn} is readily obtained from (2.2.59).

If we identify the single-spinon momenta for $m \geq n$ according to

$$p_m = \pi - \frac{2\pi}{N}\left(m + \frac{1}{2} + s\right), \quad p_n = \pi - \frac{2\pi}{N}\left(n + \frac{1}{2} - s\right), \tag{2.2.63}$$

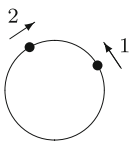
with a statistical shift $s = \frac{1}{4}$ [83, 84], we can write the energy

$$E_{mn} = E_0 + \epsilon(p_m) + \epsilon(p_n), \tag{2.2.64}$$

where E_0 is the ground state energy (2.2.5) and $\epsilon(p)$ the spinon dispersion (2.2.55).

Fractional statistics. The mutual half-fermi statistics of the spinons manifests itself in the fractional shift s in the single-spinon momenta (2.2.63), as we will elaborate now [85]. The Ansatz (2.2.57) unambiguously implies that the sum of the two-spinon momenta is given by $q_m + q_n = 2\pi - \frac{2\pi}{N}(m+n+1)$, and hence (2.2.63). The shift s is determined by demanding that the excitation energy (2.2.64) of the two-spinon state is a sum of single-spinon energies, which in turn is required for the explicit solution here to be consistent with the models solution via the asymptotic Bethe ansatz [54, 83, 86].

The shift decreases the momentum p_m of spinon 1 and increases momentum p_n of spinon 2. This may surprise at first as the basis states (2.2.57) are constructed symmetrically with regard to interchanges of m and n. To understand this asymmetry,



relative motion of one-dimensional anyons is unidirectional
(e.g. 2 moves clockwise relative to 1)

when anyons cross: $|\psi> \rightarrow e^{i\theta}|\psi>$

momentum spacing: $p_1 - p_2 = \Delta p \rightarrow \Delta p - \theta$

Fig. 2.3 Fractional statistics in one dimension. The crossings of the anyons are unidirectional, and the many particle wave function acquires a statistical phase θ whenever they cross

note that $M \geq m \geq n \geq 0$ implies $0 < p_m < p_n < \pi$. The dispersion (2.2.55) implies that the group velocity of the spinons is given by

$$v_g(p) = \partial_p \epsilon(p) = \frac{\pi}{2} - p, \tag{2.2.65}$$

which in turn implies that $v_g(p_m) > v_g(p_n)$. This means that the *relative motion* of spinon 1 (with q_m) with respect to spinon 2 (with q_n) is *always counterclockwise* on the unit circle (see Fig. 2.3). The shifts in the individual spinon momenta can hence be explained by assuming that the two-spinon state acquires a statistical phase $\theta = 2\pi s$ whenever the spinons pass through each other. This phase implies that q_m is shifted by $-\frac{2\pi}{N}s$ since we have to translate spinon 1 counterclockwise through spinon 2 and hence counterclockwise around the unit circle when obtaining the allowed values for q_m from the PBCs. Similarly, q_n is shifted by $+\frac{2\pi}{N}s$ since we have to translate spinon 2 clockwise through spinon 1 and hence clockwise around the unit circle when obtaining the quantization of q_n.

That the crossing of the spinons occurs only in one direction is a necessary requirement for fractional statistics to exist in one dimension. If the spinons could cross in both directions, the fact that paths interchanging them twice (i.e., once in each direction) are topologically equivalent to paths not interchanging them at all would imply $2\theta = 0 \mod 2\pi$ for the statistical phase, i.e., only allow for the familiar choices of bosons or fermions. With the scattering occurring in only one direction, arbitrary values for θ are possible. Note that the one-dimensional anyons break neither time-reversal symmetry (T) nor parity (P).

The fractional statistics of the spinons manifests itself further in the fractional exclusion (or generalized Pauli) principle introduced by Haldane [87]. If we consider a state with L spinons, we can easily see from (2.2.50), (2.2.51), and (2.2.57) that the number of orbitals available for further spinons we may wish to create is $M + 1$, where $M = \frac{N-L}{2}$ is the number of $\uparrow$ or $\downarrow$ spins in the remaining uniform liquid. (In this representation, the spinon wave functions are symmetric; two or more spinons can have the same value for m.) In other words, the creation of *two* spinons reduces the number of available single spinon states by *one*. They hence obey half-fermi statistics in the sense of Haldane's exclusion principle. (For fermions, the creation of two particles would decrease the number of available single particle by two, while this number would not change for bosons.)

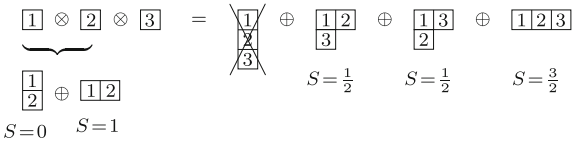


Fig. 2.4 Total spin representations of three $S = \frac{1}{2}$ spins with Young tableaux. For SU(n) with $n > 2$, the tableaux with three boxes on top of each other would exist as well

2.2.7 Young Tableaux and Many Spinon States

The easiest way to obtain the spectrum of the model is through the one-to-one correspondence between the Young tableaux classifying the total spin representations of N spins and the exact eigenstates of the the Haldane–Shastry model for a chain with N sites, which are classified by the total spins and the fractionally spaced single-particle momenta of the spinons [53].

This correspondence yields the allowed sequences of single-spinon momenta $p_1, \ldots, p_L$ as well as the allowed representations for the total spin of the states such that the eigenstates of the Haldane–Shastry model have momenta and energies

$$p = p_0 + \sum_{i=1}^{L} p_i, \quad E = E_0 + \sum_{i=1}^{L} \epsilon(p_i), \tag{2.2.66}$$

where p_0 and E_0 denote the ground state momentum and energy, respectively, and $\epsilon(p)$ is the single-spinon dispersion. The correspondence hence does not only provide the quantum numbers of all the states in the spectrum, but also shows that it is sensible to view the individual spinons as particles, rather than just as solitons or collective excitations in many body condensates. We now proceed by stating these rules without further motivating or even deriving them.

To begin with, the Hilbert space of a system of N identical SU(n) spins can be decomposed into representations of the total spin, which commutes with (2.2.1) and hence can be used to classify the eigenstates. These representations are compatible with the representations of the symmetric group S_N of N elements, which may be expressed in terms of Young tableaux [88, 89]. The general rule for obtaining Young tableaux is illustrated for three $S = \frac{1}{2}$ spins in Fig. 2.4. For each of the N spins, draw a box numbered consecutively from left to right. The representations of SU(n) are constructed by putting the boxes together such that the numbers assigned to them increase in each row from left to right and in each column from top to bottom. Each tableau indicates symmetrization over all boxes in the same row, and antisymmetrization over all boxes in the same column. This implies that we cannot have more than n boxes on top of each other for SU(n) spins. For SU(2), each tableau corresponds to a spin $S = \frac{1}{2}(\lambda_1 - \lambda_2)$ representation, with λ_i the number of boxes in the ith row, and stands for a multiplet $S^{\mathrm{z}} = -S, \ldots, S$.

The one-to-one correspondence between the Young tableaux and the non-interacting many-spinon eigenstates of the Haldane–shastry model is illustrated in

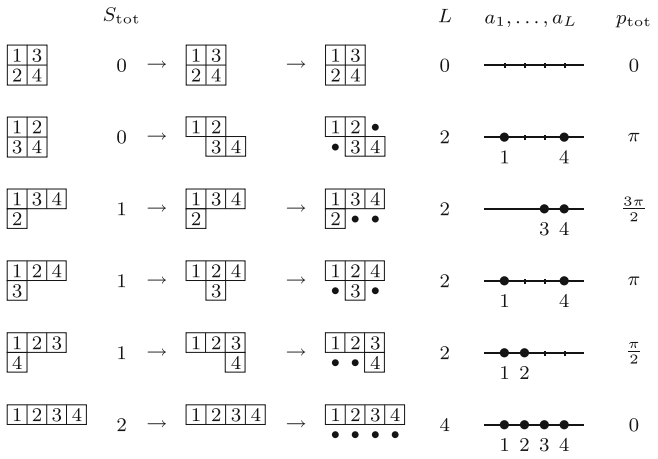


Fig. 2.5 Young tableau decomposition and the corresponding spinon states for an $S = \frac{1}{2}$ spin chain with $N = 4$ sites. The dots represent the spinons. The spinon momentum numbers a_i are given by the numbers in the boxes of the same column. Note that $\sum(2S_{\rm tot} + 1) = 2^N$

Fig. 2.5 for a chain with $N = 4$ sites. The rule is that in each Young tableau, we shift boxes to the right such that each box is below or in the column to the right of the box with the preceding number. Each missing box in the resulting, extended tableaux represents a spinon. The extended tableaux provide us with the total spin of each multiplet, which is given by the representation specified by the original Young tableau, as well as the number L of spinons present and the individual spinon momentum numbers a_i, which are just the numbers in the boxes above or below the dots representing the spinons. The single-spinon momenta are obtained from those via

$$p_i = \frac{\pi}{N}\left(a_i - \frac{1}{2}\right), \tag{2.2.67}$$

which implies $\delta \leq p_i \leq \pi - \delta$ with $\delta = \frac{\pi}{2N} \to 0$ for $N \to \infty$.

The total momentum and the total energies of the many-spinon states are given by (2.2.66) with

$$p_0 = -\frac{\pi}{2}N, \quad E_0 = -\frac{\pi^2}{24}\left(N + \frac{5}{N}\right), \tag{2.2.68}$$

and the single-spinon dispersion

$$\epsilon(p) = \frac{1}{2}p\,(\pi - p) + \frac{\pi^2}{8N^2}, \tag{2.2.69}$$

where we use a convention according to which the "vacuum" state $|\downarrow\downarrow \ldots \downarrow\rangle$ has momentum $p = 0$ (and the empty state $|0\rangle$ has $p = \pi(N-1)$).

This correspondence shows that spinons are non-interacting, with momentum spacings appropriate for half-fermions. We may interpret the Haldane–Shastry model as a reparameterization of a Hilbert space spanned by spin flips (2.2.2) into a basis which consists of the Haldane–Shastry ground state plus all possible many spinon states. The reward for such a reparameterization is that a highly non-trivial Hamiltonian in the original basis may be approximately or exactly diagonal in the new basis, as this basis is chosen in accordance with the quantum numbers of the elementary excitations.

2.3 The Moore–Read State and Its Parent Hamiltonian

2.3.1 The Pfaffian State and Its Parent Hamiltonian

The Pfaffian state at even denominator Landau level filling fractions was introduced independently by Moore and Read [90] as an example of a quantized Hall state which supports quasiparticle excitations which obey non-Abelian statistics, and by Wen, Wilczek, and ourselves [91, 92] as a candidate for the observed plateau in Hall resistivity at Landau level filling fraction $\nu = 5/2$, i.e., at $\nu = 1/2$ in the second Landau level [10, 93–96], a proposal which was subsequently strengthened [97–100] and which recently received experimental support through the direct measurement of the quasiparticle charge [101, 102].

The wave function first proposed by Moore and Read [90] is

$$\psi_0(z_1, z_2, \ldots, z_N) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j}^{N} (z_i - z_j)^m \prod_{i=1}^{N} e^{-\frac{1}{4}|z_i|^2}, \tag{2.3.1}$$

where the particle number N is even, m is even (odd) for fermions (bosons), and the Pfaffian is is given by the fully antisymmetrized sum over all possible pairings of the N particle coordinates,

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \equiv \mathcal{A}\left\{\frac{1}{z_1 - z_2} \cdot \ldots \cdot \frac{1}{z_{N-1} - z_N}\right\}. \tag{2.3.2}$$

The inverse Landau level filling fraction is given by

$$\frac{1}{\nu} = \frac{\partial N_\Phi}{\partial N} = \frac{\partial(m(N-1)-1)}{\partial N} = m. \tag{2.3.3}$$

The state describes a Laughlin state at $\nu = 1/m$ supplemented by a Pfaffian which implements p-wave pairing correlations. Since the Pfaffian is completely antisymmetric, it reverses the statistics from bosons to fermions or vice versa, but does not change the Landau level filling fraction.

The Pfaffian describes a BCS wave function [103–106] in position space, obtained by projecting on a definite number of particles [107, 108]. To see this, first rewrite the (unnormalized) BCS wave function as

$$\begin{aligned}|\psi_\phi\rangle &= \prod_{\boldsymbol{k}} \left(1 + e^{i\phi}\frac{v_{\boldsymbol{k}}}{u_{\boldsymbol{k}}} c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow}\right)|0\rangle \\ &= \prod_{\boldsymbol{k}} \exp\left(e^{i\phi}\frac{v_{\boldsymbol{k}}}{u_{\boldsymbol{k}}} c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow}\right)|0\rangle \\ &= \exp\left(e^{i\phi}\sum_{\boldsymbol{k}}\frac{v_{\boldsymbol{k}}}{u_{\boldsymbol{k}}} c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow}\right)|0\rangle \\ &= \exp\left(e^{i\phi} b^\dagger\right)|0\rangle, \end{aligned} \tag{2.3.4}$$

where the pair creation operator $b^\dagger$ is given by

$$\begin{aligned} b^\dagger &\equiv \sum_{\boldsymbol{k}} \frac{v_{\boldsymbol{k}}}{u_{\boldsymbol{k}}} c^\dagger_{\boldsymbol{k}\uparrow}\, c^\dagger_{-\boldsymbol{k}\downarrow} \\ &= \int \mathrm{d}^3\boldsymbol{x}_1 \mathrm{d}^3\boldsymbol{x}_2 \varphi(\boldsymbol{x}_1 - \boldsymbol{x}_2)\psi^\dagger_\uparrow(\boldsymbol{x}_1)\psi^\dagger_\uparrow(\boldsymbol{x}_2)|0\rangle. \end{aligned} \tag{2.3.5}$$

The wave function for each of the individual pairs, which only depends on the relative coordinate, is given by

$$\varphi(\boldsymbol{x}) = \frac{1}{V}\sum_{\boldsymbol{k}} \frac{v_{\boldsymbol{k}}}{u_{\boldsymbol{k}}} e^{i\boldsymbol{k}\boldsymbol{x}}. \tag{2.3.6}$$

If we now project out a state with $N/2$ pairs [107–109], we obtain

$$\begin{aligned}|\psi_N\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi e^{-iN\phi/2}|\psi_\phi\rangle \\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi e^{-iN\phi/2}\exp\left(e^{i\phi}b^\dagger\right)|0\rangle \\ &= \frac{1}{\left(\frac{N}{2}\right)!}\left(b^\dagger\right)^{N/2}|0\rangle, \end{aligned} \tag{2.3.7}$$

which is (up to a normalization) equivalent to

$$\begin{aligned}|\psi_N\rangle = \int &\mathrm{d}^3\boldsymbol{x}_1 \ldots \mathrm{d}^3\boldsymbol{x}_N \varphi(\boldsymbol{x}_1 - \boldsymbol{x}_2)\cdot\ldots\cdot\varphi(\boldsymbol{x}_{N-1} - \boldsymbol{x}_N) \\ &\cdot \psi^\dagger_\uparrow(\boldsymbol{x}_1)\psi^\dagger_\downarrow(\boldsymbol{x}_2)\ldots\psi^\dagger_\uparrow(\boldsymbol{x}_{N-1})\psi^\dagger_\downarrow(\boldsymbol{x}_N)|0\rangle. \end{aligned} \tag{2.3.8}$$

This implies that the many-particle wavefunction is given by a Pfaffian,

$$\psi(\boldsymbol{x}_1 \ldots \boldsymbol{x}_N) = \mathrm{Pf}\left(\varphi(\boldsymbol{x}_i - \boldsymbol{x}_j)\right). \tag{2.3.9}$$

This form nicely illustrates that all the pairs have condensed into the same state, which is the essence of superfluidity. For fermion pairings with even relative angular momentum of the pairs, such as s- or d-wave, the wave function $\varphi(\boldsymbol{x}_i - \boldsymbol{x}_j)$ of the pairs is symmetric in real space, and antisymmetric in spin space (i.e., a singlet), while for pairings with odd angular momentum, such as p-wave, $\varphi(\boldsymbol{x}_i - \boldsymbol{x}_j)$ is antisymmetric in real space and symmetric in spin space (i.e., a triplet).

In the quantized Hall state, the requirement of analyticity in the complex coordinates constraints the possible form of the pair wave function decisively. Since the electrons are spin polarized, the only possible choice is the p-wave pairing described by the Pfaffian with $\varphi(z_i - z_j) = 1/(z_i - z_j)$. Note that this pair wave function would not be normalizable if it were not multiplied by at least an $m = 1$ Laughlin state.

One of the most important mathematical properties of the Pfaffian is that its square is equal to the determinant,

$$\mathrm{Pf}\left(\varphi(\boldsymbol{x}_i - \boldsymbol{x}_j)\right)^2 = \det(M_{ij}), \tag{2.3.10}$$

where

$$M_{ij} = \begin{cases} 0 & \text{for } i = j, \\ \varphi(\boldsymbol{x}_i - \boldsymbol{x}_j) & \text{for } i \neq j. \end{cases} \tag{2.3.11}$$

Another important identity, due to Frobenius [110], is given by (2.4.31) in Sect. 2.4.4 below.

The uniquely specifying property of the Pfaffian quantized Hall state (2.3.1) is that the wave function vanishes as the $(3m-1)$th power as *three* particles approach each other. This property simply reflects that there can be at most only one pair among each triplet of particles. This observation has led Wen, Wilczek, and ourselves [91, 92, 111] to propose the parent Hamiltonian

$$V^{(m)} = \sum_{i,j<k}^{N} \left(\nabla_i^2\right)^{(m-1)} \left(\delta^{(2)}(z_i - z_j)\delta^{(2)}(z_i - z_k)\right), \tag{2.3.12}$$

which, when supplemented with the kinetic Hamiltonian (2.1.9) as well as all similar terms with smaller powers of the Laplacian, singles out (2.3.1) as its unique ground state. For all practical purposes, however, it is best to formulate our parent Hamiltonian in terms of three-body pseudopotentials, as we will elaborate in Sect. 2.3.4.

2.3.2 Quasiparticle Excitations and the Internal Hilbert Space

One of the key properties of superconductors is that the magnetic vortices are quantized in units of one half of the Dirac flux quanta $\Phi_0 = 2\pi\hbar c/e$, in accordance to the charge $-2e$ of the Cooper pairs. The paring correlations in the Pfaffian Hall state have a similar effect on the vortices or quasiparticle excitations, which carry one half of the flux and charge they would carry without the pairing, i.e. they carry charge $e^* = e/2m$. The wave function for two flux $\frac{1}{2}$ quasiholes at positions ξ_1 and ξ_2 is easily formulated. We simply replace each factor in the Pfaffian in (2.3.1) by

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \to \mathrm{Pf}\left(\frac{(z_i - \xi_1)(z_j - \xi_2) + (z_i \leftrightarrow z_j)}{z_i - z_j}\right), \tag{2.3.13}$$

such that one member of each electron pair sees the additionally inserted zero at ξ_1 and the other member sees it at ξ_2. If we set $\xi_1 = \xi_2 = \xi$, we will recover a regular quasihole in the Laughlin fluid with charge $e^* = e/m$.

The internal Hilbert space spanned by the quasiparticle excitations only emerges as we consider the wave function for four charge $e^* = e/4$ quasiholes at positions $\xi_1, \ldots, \xi_4$, which is obtained by replacing the Pfaffian in (2.3.1) by

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \to \mathrm{Pf}\left(\frac{(z_i - \xi_1)(z_j - \xi_2)(z_i - \xi_3)(z_j - \xi_4) + (z_i \leftrightarrow z_j)}{z_i - z_j}\right). \tag{2.3.14}$$

We see that ξ_1 and ξ_3 belong to one group in that they constitute additional zeros seen by one member of each electron pair, while ξ_2 and ξ_4 belong to another group as they constitute zeros seen by the other members of each electron pair. The wave function is symmetric (or antisymmetric, depending on the number of electron pairs) under interchange of both groups. The state in the internal Hilbert space spanned by the quasihole affiliations with the two groups will change as we adiabatically interchange two quasiholes belonging to different groups, say ξ_3 and ξ_4. Naively, one might think that the dimension of the internal Hilbert space is given by the number of ways to partition the quasiholes at $\xi_1, \ldots, \xi_{2n}$ into two different groups, i.e., by $(2n-1)!!$ for $2n$ quasiholes. Note that the number of quasiholes has to be even on closed surfaces to satisfy the Dirac flux quantization condition [38]. The true dimension of the internal Hilbert space, however, is only 2^{n-1} [112]. The reason for this is that the internal Hilbert space is spanned by Majorana fermion states in the vortex cores [113], as we will elaborate in the following section.

The statistics is non-Abelian in the sense that the order according to which we interchange quasiholes matters. Let the matrix M_{ij} describe the rotation of the internal Hilbert space state vector which describes the adiabatic interchange two quasiholes at ξ_i and ξ_j:

$$|\psi\rangle \to M_{ij}|\psi\rangle.$$

The statistics is non-Abelian if the matrices associated with successive interchanges do not commute in general,

$$M_{ij}M_{jk} \neq M_{jk}M_{ij}.$$

Note that the internal state vector is protected in the sense that it is insensitive to local perturbations—it can *only* be manipulated through braiding of the vortices. For a sufficiently large number of vortices, on the other hand, any unitary transformation in this space can be approximated to arbitrary accuracy through successive braiding operations [114]. These properties together render non-Abelions preeminently suited for applications as protected qubits in quantum computation [115–119].

2.3.3 Majorana Fermions and Non-Abelian Statistics

The key to understanding the non-Abelian statistics [119] of the quasiparticle excitations of the Pfaffian state lies in the Majorana fermion modes in the vortices of p-wave superfluids [113, 120–122]. The p-wave pairing symmetry implies that the order parameter for the superfluid acquires a phase of 2π as we go around the Fermi surface,

$$\langle c^\dagger_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}} \rangle = \Delta_0(k) \cdot (k_\mathrm{x} + \mathrm{i}k_\mathrm{y}), \tag{2.3.15}$$

where $\Delta_0(k)$ can be chosen real. The Hamiltonian for a single vortex at the origin is given by

$$H = \int \mathrm{d}\boldsymbol{r} \left\{ \psi^\dagger \left(-\frac{\nabla^2}{2m} - \varepsilon_F \right) \psi + \psi^\dagger \left(e^{\mathrm{i}\varphi} \Delta_0(r) * (\partial_x - \mathrm{i}\partial_y) \right) \psi^\dagger + \mathrm{h.c.} \right\}, \tag{2.3.16}$$

where $A * B \equiv \frac{1}{2}\{A, B\}$ denotes the symmetrized product, and r and φ are polar coordinates. The order parameter $\Delta_0(r)$ vanishes inside the vortex core. We can obtain the energy eigenstates localized inside the vortex by solving the Bogoliubov–de Gennes equations [105] equations

$$\left[H, \gamma_n^\dagger(\boldsymbol{x}) \right] = E_n \gamma_n^\dagger(\boldsymbol{x}), \tag{2.3.17}$$

where n labels the modes and

$$\gamma_n^\dagger(\boldsymbol{x}) = u_n(\boldsymbol{x})\psi^\dagger(\boldsymbol{x}) + v_n(\boldsymbol{x})\psi(\boldsymbol{x}) \tag{2.3.18}$$

are the Bogoliubov quasiparticle operators. The low energy spectrum is given by [113, 120]

$$E_n = n\omega_0, \tag{2.3.19}$$

where n is an integer and $\omega_0 = \Delta^2/\varepsilon_F$ the level spacing. Note that while in an s-wave superfluid, the Bogoliubov operators

$$\gamma_{n\uparrow}(\boldsymbol{x}) = u_{n\uparrow}(\boldsymbol{x})\psi^\dagger(\boldsymbol{x}) + v_{n\downarrow}(\boldsymbol{x})\psi(\boldsymbol{x}) \tag{2.3.20}$$

combine ↑-spin electron creation operators with ↓-spin annihilation operators, in the p-wave superfluid, the operators (2.3.18) combine creation and annihilation operators of the *same* spinless (or spin-polarized) fermions. Since the Bogoliubov–de Gennes equations are not able to distinguish between particles and antiparticles, we obtain each physical solution twice: once with positive energy as a solution of the Bogoliubov–de Gennes equation (2.3.17) for the creation operators, and once with negative energy as a solution of the same equation for the annihilation operators,

$$\big[H, \gamma_n(\boldsymbol{x})\big] = -E_n\gamma_n(\boldsymbol{x}), \tag{2.3.21}$$

which is obtained from (2.3.17) by Hermitian conjugation. We resolve this technical artifact by discarding the negative energy solutions as unphysical. For the $n=0$ solution with at $E_0 = 0$, it implies that we get one fermion solution when we overcount by a factor of two. The physical solution at $E = 0$ is hence given by one half of a fermion, or a Majorana fermion, as

$$\gamma_0^\dagger(\boldsymbol{x}) = \gamma_0(\boldsymbol{x}). \tag{2.3.22}$$

In general, one fermion ψ, $\psi^\dagger$ consists of two Majorana fermions,

$$\psi = \frac{1}{2}(\gamma_1 + \mathrm{i}\gamma_2), \quad \psi^\dagger = \frac{1}{2}(\gamma_1 - \mathrm{i}\gamma_2), \tag{2.3.23}$$

which in turn are given by the real and imaginary part of the fermion operators,

$$\gamma_1 = \psi + \psi^\dagger, \quad \gamma_2 = -\mathrm{i}(\psi - \psi^\dagger). \tag{2.3.24}$$

They obey the anticommutation relations

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}, \tag{2.3.25}$$

as one may easily verify with (2.3.24). Majorana fermions are their own antiparticles, as $\gamma_i^\dagger = \gamma_i$. If we write the basis for a single fermion as $\{|0\rangle, \psi^\dagger|0\rangle\}$, we can write the fermion creation and annihilation operators as

$$\psi^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.3.26}$$

In this basis, the Majorana fermions are given by the first two Pauli matrices,

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_\mathrm{x}, \quad \gamma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \sigma_\mathrm{y}. \tag{2.3.27}$$

Returning to vortices in a p-wave superfluid, note that the order parameter acquires by definition a phase of 2π as we go around a vortex. This implies that the electron

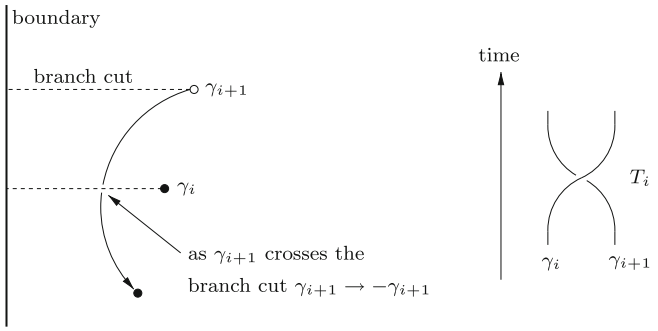


Fig. 2.6 The Majorana fermion γ_{i+1} acquires a $-$ sign as it crosses the branch cut from another vortex

creation and annihilation operators acquire a phase π, or a minus sign, which implies via (2.3.24) that the Majorana fermion states acquire likewise a minus sign,

$$\gamma_i \to -\gamma_i, \tag{2.3.28}$$

as we encircle a vortex. By choice of gauge, we can implement the phase change of 2π in the superconducting order parameter as a branch cut connecting the vortices to the left boundary of the system, and assume a convention according to which the Majorana fermion in each vortex crossing a branch cut acquires a minus sign, as illustrated in Fig. 2.6.

To obtain the non-Abelian statistics, Ivanov [121] considered permutations of $2n$ vortices by braiding, which form the braid group B_{2n} [123]. This group is generated by counterclockwise interchanges T_i of particles i and $i+1$, which are neighbors with regard to the positions of their branch cuts to the boundary. The algebra of the group is given by

$$\begin{aligned} T_i T_j &= T_j T_i \qquad \text{for } |i-j| > 1, \\ T_i T_j T_i &= T_j T_i T_j \qquad \text{for } |i-j| = 1, \end{aligned} \tag{2.3.29}$$

as illustrated in Fig. 2.7. Note that the braid group is different from the permutation group as

$$T_i^{-1} \neq T_i.$$

The convention for the minus signs acquired by the Majorana fermions defined in Fig. 2.6 implies the transformation rule

$$T_i(\gamma_j) = \begin{cases} \gamma_{j+1} & \text{for } i = j, \\ -\gamma_{j-1} & \text{for } i = j-1, \\ \gamma_j & \text{otherwise.} \end{cases} \tag{2.3.30}$$

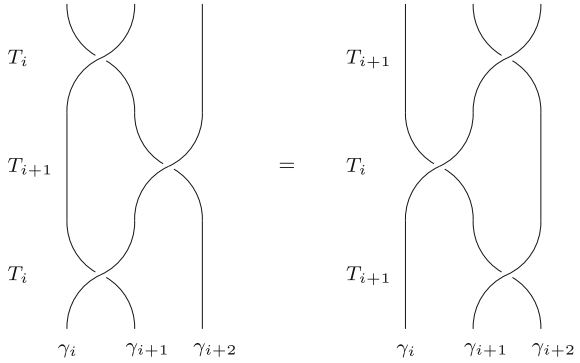


Fig. 2.7 Illustration of the defining algebra of the braid group B_{2n}: $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$

To describe the action of these transformations on the (internal) state vectors, we hence need to find a representation $\tau(T_i)$ of the braid group B_{2n} such that

$$\tau(T_i)\gamma_j \tau(T_i)^{-1} = T_i(\gamma_j) \tag{2.3.31}$$

with $T_i(\gamma_j)$ given by (2.3.30). The solution is [121]

$$\begin{aligned} \tau(T_i) &= \exp\left(\frac{\pi}{4}\gamma_{i+1}\gamma_i\right) \\ &= \cos\left(\frac{\pi}{4}\right) + \gamma_{i+1}\gamma_i \sin\left(\frac{\pi}{4}\right), \\ &= \frac{1}{\sqrt{2}}(1 + \gamma_{i+1}\gamma_i), \end{aligned} \tag{2.3.32}$$

as one can easily verify using $(\gamma_{i+1}\gamma_i)^2 = -1$. The inverse transformation is given by

$$\tau(T_i)^{-1} = \frac{1}{\sqrt{2}}(1 - \gamma_{i+1}\gamma_i). \tag{2.3.33}$$

A few steps of algebra yield

$$\tau(T_1) \left\{ \begin{array}{c} \gamma_1 \\ \gamma_2 \end{array} \right\} \tau(T_1)^{-1} = \left\{ \begin{array}{c} \gamma_2 \\ -\gamma_1 \end{array} \right\}.$$

This representation coincides with that of Nayak and Wilczek [112] for the statistics of the quasiholes in the Pfaffian state.

The simplest examples of this representation are the cases of two and four vortices [112, 121, 124], which we will elaborate now. In the case of two vortices, the two Majorana fermions γ_1 and γ_2 can be combined into a single fermion via (2.3.23), and the ground state is hence two-fold degenerate. The braid group B_2 has only one generator T_1 with representation

$$\begin{aligned}\tau(T_1) &= \exp\left(\frac{\pi}{4}\gamma_2\gamma_1\right)\\ &= \exp\left(-\mathrm{i}\frac{\pi}{4}(\psi-\psi^\dagger)(\psi+\psi^\dagger)\right)\\ &= \exp\left(\mathrm{i}\frac{\pi}{4}(2\psi^\dagger\psi-1)\right)\\ &= \exp\left(-\mathrm{i}\frac{\pi}{4}\sigma_z\right),\end{aligned} \tag{2.3.34}$$

where σ_z is the third Pauli matrix (2.1.66) in the basis $\{|0\rangle, \psi^\dagger|0\rangle\}$. The braiding is hence diagonal in this basis, and only gives an overall phase, which depends on whether the fermion state is occupied or not.

The non-Abelian statistics manifests itself only once we consider four vortices. Following Ivanov [121], we combine the four Majorana fermions into two fermions,

$$\psi_1 = \frac{1}{2}(\gamma_1 + \mathrm{i}\gamma_2), \quad \psi_2 = \frac{1}{2}(\gamma_3 + \mathrm{i}\gamma_4), \tag{2.3.35}$$

and accordingly for the fermion creation operators $\psi_1^\dagger$, $\psi_2^\dagger$. The braid group B_4 has three generators T_1, T_2, and T_3. Their representations in a basis of fermion occupation numbers

$$\{|0\rangle, \psi_1^\dagger|0\rangle, \psi_2^\dagger|0\rangle, \psi_1^\dagger\psi_2^\dagger|0\rangle\},$$

are given by two diagonal operators

$$\tau(T_1) = \exp\left(\frac{\pi}{4}\gamma_2\gamma_1\right) = \exp\left(-\mathrm{i}\frac{\pi}{4}\sigma_z^{(1)}\right) = \begin{pmatrix} e^{-\mathrm{i}\pi/4} & 0 & 0 & 0\\ 0 & e^{\mathrm{i}\pi/4} & 0 & 0\\ 0 & 0 & e^{-\mathrm{i}\pi/4} & 0\\ 0 & 0 & 0 & e^{\mathrm{i}\pi/4}\end{pmatrix},$$

$$\tau(T_3) = \exp\left(\frac{\pi}{4}\gamma_4\gamma_3\right) = \exp\left(-\mathrm{i}\frac{\pi}{4}\sigma_z^{(2)}\right) = \begin{pmatrix} e^{-\mathrm{i}\pi/4} & 0 & 0 & 0\\ 0 & e^{-\mathrm{i}\pi/4} & 0 & 0\\ 0 & 0 & e^{\mathrm{i}\pi/4} & 0\\ 0 & 0 & 0 & e^{\mathrm{i}\pi/4}\end{pmatrix},$$

and one off-diagonal operator,

$$\begin{aligned}\tau(T_2) &= \exp\left(\frac{\pi}{4}\gamma_3\gamma_2\right)\\ &= \frac{1}{\sqrt{2}}\left(1-\mathrm{i}(\psi_2+\psi_2^\dagger)(\psi_1-\psi_1^\dagger)\right) = \begin{pmatrix} 1 & 0 & 0 & -\mathrm{i}\\ 0 & 1 & -\mathrm{i} & 0\\ 0 & -\mathrm{i} & 1 & 0\\ -\mathrm{i} & 0 & 0 & 1\end{pmatrix}.\end{aligned}$$

Note that since the representations $\tau(T_i)$ given by (2.3.32) are even in the fermion operators, i.e., change the fermion numbers only by even integers, we may restrict

them to only even or odd sectors in the fermion numbers. For the example of four vortices, these sectors are given by $\{|0\rangle, \psi_1^\dagger \psi_2^\dagger |0\rangle\}$ and $\{\psi_1^\dagger |0\rangle, \psi_2^\dagger |0\rangle\}$. Each sector contains 2^{n-1} states, which is the degeneracy found for a Pfaffian state with an even number of electrons [112]. Physically, this reflects that while the number of fermions is not a good quantum number in a superfluid, the number of fermions modulo two, i.e., whether the number is even or odd, is a good quantum number.

Finally, note that the derivation of the non-Abelian statistics depends only on (a) the vortices possessing Majorana fermion modes, and (b) the Majorana fermions changing sign $\gamma_i \to -\gamma_i$ when the order parameter phase changes by 2π, as it does by definition when we go around a vortex.

2.3.4 The Pfaffian State and Its Parent Hamiltonian on the Sphere

The Pfaffian state is readily formulated in the spherical geometry [92]. The wave function for N particles at Landau level filling $\nu = 1/m$ on a sphere with $2s_0 = m(N-1) - 1$ magnetic flux quanta is given by

$$\psi_0[u, v] = \mathrm{Pf}\left(\frac{1}{u_i v_j - u_j v_i}\right) \prod_{i<j}^{N} (u_i v_j - u_j v_i)^m, \tag{2.3.36}$$

where m is even for fermions and odd for bosons. Note that the relation between flux and particle number implies that the states at $\nu = 1/2$ is not its own particle-hole conjugate [125, 126]. The formulation of quasihole excitations generalizes without incident from the planar geometry.

As mentioned in Sect. 2.3.1, the uniquely specifying property of the Pfaffian state (2.3.36) is that it vanishes as the $(3m-1)$-th power of the distance as *three* particles approach each other. For the spherical geometry, the corresponding parent Hamiltonian can be conveniently formulated using three-body pseudopotentials [127]. In analogy to the two-particle interaction Hamiltonian (2.1.84), we write the three-particle interaction Hamiltonian

$$H_{\mathrm{int}}^{(3)} = \sum_{i<j<k}^{N} \left\{ \sum_{l}^{2s} V_l^{(3)} \mathrm{P}_{3s-l}(\boldsymbol{L}_i + \boldsymbol{L}_j + \boldsymbol{L}_k) \right\}. \tag{2.3.37}$$

The three-body parent Hamiltonian proposed by Wen, Wilczek, and ourselves [91, 92] then amounts to taking

$$V_l^{(3)} = \begin{cases} 1 & \text{for } l < 3m-1, \\ 0 & \text{for } l \geq 3m-1. \end{cases} \tag{2.3.38}$$

The form (2.3.37) is not the most general one, as for $l \geq 6$ for bosons ($l \geq 9$ for fermions), the three particle state is no longer uniquely described by the three

body angular angular momentum l, and one may assign different pseudopotential coefficients to the different symmetric (antisymmetric), homogeneous, rotationally invariant polynomials of degree l describing the three body states [127]. This, however, should not concern us here as we are only interested in the case $m = 1$ for bosons and $m = 2$ for fermions. Furthermore, as in the case of two-body pseudopotentials, where l had to be even for bosons and odd for fermions, there exists a related restriction for the allowed values of l for three-body pseudopotentials. Specifically, we have no state with $l = 1 (l = 4)$ for bosons (fermions).

For all practical purposes, we once again need to rewrite (2.3.37) in terms of boson or fermion creation or annihilation operators,

$$\begin{aligned} H_{\text{int}}^{(3)} = & \sum_{m_1=-s}^{s} \sum_{m_2=-s}^{s} \sum_{m_3=-s}^{s} \sum_{m_4=-s}^{s} \sum_{m_5=-s}^{s} \sum_{m_6=-s}^{s} a_{m_1}^{\dagger} a_{m_2}^{\dagger} a_{m_3}^{\dagger} a_{m_4} a_{m_5} a_{m_6} \\ & \cdot \delta_{m_1+m_2+m_3, m_4+m_5+m_6} \\ & \cdot \sum_{j=0}^{2s} \sum_{l=3s-(j+s)}^{3s-|j-s|} V_l^{(3)} \langle s, m_1; s, m_2 | j, m_1 + m_2 \rangle \\ & \cdot \langle j, m_1 + m_2; s, m_3 | 3s - l, m_1 + m_2 + m_3 \rangle \\ & \cdot \langle 3s - l, m_4 + m_5 + m_6 | s, m_4; j, m_5 + m_6 \rangle \\ & \cdot \langle j, m_5 + m_6 | s, m_5; s, m_6 \rangle, \end{aligned} \tag{2.3.39}$$

where a_m annihilates a boson or fermion in the properly normalized single particle state

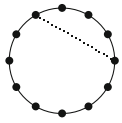
$$\psi_{m,0}^{s}(u, v) = \sqrt{\frac{(2s+1)!}{4\pi (s+m)!(s-m)!}}\, u^{s+m} v^{s-m}, \tag{2.3.40}$$

and $\langle s, m_1; s, m_2 | 2s - l, m_1 + m_2 \rangle$ etc. are Clebsch–Gordan coefficients [39].

2.4 An $S = 1$ Spin Liquid State Described by a Pfaffian

2.4.1 The Ground State

As for the Haldane–Shastry model, we consider a one-dimensional lattice with periodic boundary conditions and an even number of sites N on a unit circle embedded in the complex plane. The only difference is that now the spin on each site is $S = 1$:



N sites with spin 1 on unit circle:

$\eta_\alpha = e^{\mathrm{i}\frac{2\pi}{N}\alpha}$ with $\alpha = 1, \ldots, N$

The ground state wave function we consider here [128] is given by a bosonic Pfaffian state in the complex lattice coordinates z_i supplemented by a phase factor,

$$\psi_0^{S=1}(z_1, z_2, \ldots, z_N) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j}^{N} (z_i - z_j) \prod_{i=1}^{N} z_i . \tag{2.4.1}$$

The Pfaffian is given by the fully antisymmetrized sum over all possible pairings of the N particle coordinates,

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \equiv \mathcal{A}\left\{\frac{1}{z_1 - z_2} \cdot \ldots \cdot \frac{1}{z_{N-1} - z_N}\right\}. \tag{2.4.2}$$

The "particles" z_i represent re-normalized spin flips $\tilde{S}_\alpha^+$ acting on a vacuum with all spins in the $S^z = -1$ state,

$$\left|\psi_0^{S=1}\right\rangle = \sum_{\{z_1,\ldots,z_N\}} \psi_0^{S=1}(z_1, \ldots, z_N)\ \tilde{S}_{z_1}^+ \cdot \ldots \cdot \tilde{S}_{z_N}^+ |-1\rangle_N , \tag{2.4.3}$$

where the sum extends over all possibilities of distributing the N "particles" over the N lattice sites allowing for double occupation,

$$\tilde{S}_\alpha^+ \equiv \frac{S_\alpha^z + 1}{2} S_\alpha^+ , \tag{2.4.4}$$

and

$$|-1\rangle_N \equiv \otimes_{\alpha=1}^{N} |1, -1\rangle_\alpha . \tag{2.4.5}$$

This state may be viewed as the one-dimensional analog of the non-Abelian chiral spin liquid [129].

Like the ground state of the Haldane–Shastry model, the $S = 1$ state (2.4.1) describes a critical spin liquid in one dimension, with similarly algebraically decaying correlations. It does not, however, serve as a paradigm of the generic $S = 1$ spin state, as the generic state possesses a Haldane gap [130–133] in the spin excitation spectrum due to linearly confining forces between the spinons [128, 134–137].

One of the objectives of this work is to identify a parent Hamiltonian for which this state is the exact ground state, and hence accomplish what Haldane and Shastry have accomplished for the spin one-half Gutzwiller wave function.

2.4.2 Symmetries

Translational invariance. As for the Haldane–Shastry model, we obtain the ground state momentum p_0 (in units of inverse lattice spacings $1/a$) by translating the ground state by one lattice spacing around the unit circle,

$$\boldsymbol{T}\left|\psi_0^{S=1}\right\rangle = e^{\mathrm{i}p_0}\left|\psi_0^{S=1}\right\rangle. \tag{2.4.6}$$

With $\boldsymbol{T}z_i = \exp\left(\mathrm{i}\frac{2\pi}{N}\right) z_i$ we find

$$p_0 = \frac{2\pi}{N}\left(-\frac{N}{2} + \frac{N(N-1)}{2} + N\right) = \pi N, \tag{2.4.7}$$

which implies $p_0 = 0$ as N is even.

Invariance under SU(2) spin rotations. The proof of the singlet property is similar to the Haldane–Shastry model, but more instructive as it motivates the re-normalization of the spin-flip operators in (2.4.4).

Since $S^{\mathrm{z}}_{\mathrm{tot}}\left|\psi_0^{S=1}\right\rangle = 0$ by construction, it is sufficient to show $S^{-}_{\mathrm{tot}}\left|\psi_0^{S=1}\right\rangle = 0$. Note first that when we substitute (2.4.1) with (2.4.2) into (2.4.3), we may replace the antisymmetrization $\mathcal{A}$ in (2.4.2) by an overall normalization factor, as it is taken care by the commutativity of the bosonic operators $\tilde{S}_\alpha$. Let $\tilde{\psi}_0$ be $\psi_0^{S=1}$ without the antisymmetrization in (2.4.2),

$$\tilde{\psi}_0[z_i] = (N-1)!!\left\{\frac{1}{z_1 - z_2}\cdot\ldots\cdot\frac{1}{z_{N-1} - z_N}\right\}\cdot\prod_{i<j}^{N}(z_i - z_j)\prod_{i=1}^{N} z_i. \tag{2.4.8}$$

Since $\tilde{\psi}_0(z_1, z_2, \ldots, z_N)$ is still symmetric under interchange of pairs, we may assume that a spin flip operator S^-_α acting on $|\tilde{\psi}_0\rangle$ will act on the pair (z_1, z_2),

$$\begin{aligned} S^-_\alpha\left|\psi_0^{S=1}\right\rangle &= \sum_{\{z_3,\ldots,z_N\}}\Bigg\{\sum_{z_2(\neq\eta_\alpha)}\tilde{\psi}_0(\eta_\alpha, z_2, z_3, \ldots)S^-_\alpha\tilde{S}^+_\alpha\tilde{S}^+_{z_2} \\ &\qquad + \sum_{z_1(\neq\eta_\alpha)}\tilde{\psi}_0(z_1, \eta_\alpha, z_3, \ldots)S^-_\alpha\tilde{S}^+_{z_1}\tilde{S}^+_\alpha \\ &\qquad + \tilde{\psi}_0(\eta_\alpha, \eta_\alpha, z_3, \ldots)S^-_\alpha(\tilde{S}^+_\alpha)^2\Bigg\}\tilde{S}^+_{z_3}\ldots\tilde{S}^+_{z_N}|-1\rangle_N \\ &= \sum_{\{z_3,\ldots,z_N\}}\left\{\sum_{z_2} 2\tilde{\psi}_0(\eta_\alpha, z_2, z_3, \ldots)\tilde{S}^+_{z_2}\right\}\tilde{S}^+_{z_3}\ldots\tilde{S}^+_{z_N}|-1\rangle_N, \end{aligned} \tag{2.4.9}$$

where we have used

$$S^-_\alpha(\tilde{S}^+_\alpha)^n|1,-1\rangle_\alpha = n(\tilde{S}^+_\alpha)^{n-1}|1,-1\rangle_\alpha, \tag{2.4.10}$$

which follows directly form the definition (2.4.4).

This implies

$$S^-_{\rm tot}\left|\psi_0^{S=1}\right\rangle = \sum_{\alpha=1}^{N} S^-_\alpha \left|\psi_0^{S=1}\right\rangle$$
$$= 2 \sum_{\{z_2,\ldots,z_N\}} \underbrace{\sum_{\alpha=1}^{N} \tilde{\psi}_0(\eta_\alpha, z_2, \ldots, z_N)}_{=0} \tilde{S}^+_{z_2} \ldots \tilde{S}^+_{z_N} |-1\rangle_N, \qquad (2.4.11)$$

since $\tilde{\psi}_0(\eta_\alpha, z_2, \ldots, z_N)$ contains only powers $\eta_\alpha^1, \eta_\alpha^2, \ldots, \eta_\alpha^{N-1}$ in η_α, and

$$\sum_{\alpha=1}^{N} \eta_\alpha^m = N\delta_{m,0} \mod N.$$

Parity and time reversal invariance. To show that $\psi_0(z_1, \ldots, z_N)$ is real, and hence that $\left|\psi_0^{S=1}\right\rangle$ is invariant under parity, we calculate its complex conjugate,

$$\left(\psi_0^{S=1}[z]\right)^* = \mathrm{Pf}\left(\frac{1}{\frac{1}{z_i} - \frac{1}{z_j}}\right) \prod_{i<j}^{N} \left(\frac{1}{z_i} - \frac{1}{z_j}\right) \prod_{i=1}^{N} \frac{1}{z_i}$$
$$= (-1)^{\frac{N}{2}} \prod_{i=1}^{N} \frac{1}{z_i} \, (-1)^{\frac{N(N-1)}{2}} \prod_{i<j}^{N} \frac{1}{z_i z_j} \prod_{i=1}^{N} \frac{1}{z_i^2} \, \psi_0^{S=1}[z]$$
$$= \psi_0^{S=1}[z], \qquad (2.4.12)$$

as N is even and $z_i^N = 1$ for all i. Time reversal [73] transforms

$$\mathrm{i} \to -\mathrm{i}, \quad z_i \to \bar{z}_i, \quad \boldsymbol{S}_\alpha \to -\boldsymbol{S}_\alpha, \quad |s, m\rangle \to \mathrm{i}^{2m} |s, -m\rangle,$$

which implies that the basis states in (2.4.3) transform according to

$$\tilde{S}^+_{z_1} \cdot \cdots \cdot \tilde{S}^+_{z_N} |-1\rangle_N \to \tilde{S}^-_{z_1} \cdot \cdots \cdot \tilde{S}^-_{z_N} |+1\rangle_N, \qquad (2.4.13)$$

where

$$\tilde{S}^-_\alpha \equiv \frac{-S^{\rm z}_\alpha + 1}{2} S^-_\alpha, \quad |+1\rangle_N \equiv \otimes_{\alpha=1}^{N} |1, +1\rangle_\alpha. \qquad (2.4.14)$$

Together with the singlet property, this implies that $\left|\psi_0^{S=1}\right\rangle$ is invariant under time reversal.

All the symmetries properties discussed here will emerge almost trivially when we generate the state $\left|\psi_0^{S=1}\right\rangle$ through projection form Gutzwiller (or Haldane–Shastry ground) states in Sect. 2.4.4.

2.4.3 Schwinger Bosons

Schwinger bosons [138, 139] constitute a way to formulate spin-S representations of an SU(2) algebra (which can easily be generalized to SU(n), see e.g. [136]). The spin operators

$$\boldsymbol{S} = \frac{1}{2}\left(a^\dagger, b^\dagger\right)\boldsymbol{\sigma}\begin{pmatrix} a \\ b \end{pmatrix}, \tag{2.4.15}$$

where $\boldsymbol{\sigma} = (\sigma_\mathrm{x}, \sigma_\mathrm{y}, \sigma_\mathrm{z})$ is the vector consisting of the three Pauli matrices (2.1.66), are given in terms of boson creation and annihilation operators which obey the usual commutation relations

$$\begin{aligned} \left[a, a^\dagger\right] = \left[b, b^\dagger\right] = 1, \\ \left[a, b\right] = \left[a, b^\dagger\right] = \left[a^\dagger, b\right] = \left[a^\dagger, b^\dagger\right] = 0. \end{aligned} \tag{2.4.16}$$

It is readily verified with

$$\left[\sigma_i, \sigma_j\right] = 2\mathrm{i}\varepsilon^{ijk}\sigma_k \tag{2.4.17}$$

and (2.4.16), that S^x, S^y, and S^z satisfy the SU(2) algebra

$$\left[S^i, S^j\right] = \mathrm{i}\varepsilon^{ijk}S^k. \tag{2.4.18}$$

Written out in components we have

$$\begin{aligned} S^\mathrm{x} + \mathrm{i}S^\mathrm{y} = S^+ = a^\dagger b, \\ S^\mathrm{x} - \mathrm{i}S^\mathrm{y} = S^- = b^\dagger a, \\ S^\mathrm{z} = \frac{1}{2}(a^\dagger a - b^\dagger b). \end{aligned} \tag{2.4.19}$$

The spin quantum number S is given by half the number of bosons,

$$2S = a^\dagger a + b^\dagger b, \tag{2.4.20}$$

and the usual spin states (simultaneous eigenstates of $\boldsymbol{S}^2$ and S^z) are given by

$$|S, m\rangle = \frac{(a^\dagger)^{S+m}}{\sqrt{(S+m)!}}\frac{(b^\dagger)^{S-m}}{\sqrt{(S-m)!}}|0\rangle. \tag{2.4.21}$$

In particular, the spin-$\frac{1}{2}$ states are given by

$$|\uparrow\rangle = c_\uparrow^\dagger|0\rangle = a^\dagger|0\rangle, \quad |\downarrow\rangle = c_\downarrow^\dagger|0\rangle = b^\dagger|0\rangle, \tag{2.4.22}$$

i.e., $a^\dagger$ and $b^\dagger$ act just like the fermion creation operators $c_\uparrow^\dagger$ and $c_\downarrow^\dagger$ in this case. The difference shows up only when two (or more) creation operators act on the same

site or orbital. The fermion operators create an antisymmetric or singlet configuration (in accordance with the Pauli principle),

$$|0,0\rangle = c_{\uparrow}^{\dagger} c_{\downarrow}^{\dagger} |0\rangle, \tag{2.4.23}$$

while the Schwinger bosons create a totally symmetric or triplet (or higher spin if we create more than two-bosons) configuration,

$$\begin{aligned} |1,1\rangle &= \frac{1}{\sqrt{2}} (a^{\dagger})^2 |0\rangle, \\ |1,0\rangle &= a^{\dagger} b^{\dagger} |0\rangle, \\ |1,-1\rangle &= \frac{1}{\sqrt{2}} (b^{\dagger})^2 |0\rangle. \end{aligned} \tag{2.4.24}$$

Representations of spin $\frac{1}{2}$ states in terms of Schwinger bosons (rather than fermion creation operators or spin flips) are ideally suited for the construction of higher spin states through projection of $2S$ spin $\frac{1}{2}$'s onto the spin S representations (i.e., the symmetric representation) contained in

$$\underbrace{\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \otimes \ldots \otimes \mathbf{\frac{1}{2}}}_{2S} = \boldsymbol{S} \oplus (2S-1)\boldsymbol{S-1} \oplus \ldots \tag{2.4.25}$$

Classic examples include the formulation of the Affleck–Kennedy–Lieb–Tasaki (AKLT) model [134, 135] in terms of Schwinger bosons [139, 140] as well as the $S = 1$ chirality liquid [128].

2.4.4 Generation by Projection from Gutzwiller States

We will show now that the $S = 1$ ground state (2.4.1) can alternatively be generated by considering two (identical) Haldane–Shastry or Gutzwiller states (2.2.3) and projecting onto the triplet or $S = 1$ configuration contained in

$$\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \mathbf{0} \oplus \mathbf{1} \tag{2.4.26}$$

at each site [128, 129]. To begin with, we rewrite (2.2.2) in terms of Schwinger bosons,

$$\begin{aligned} \left|\psi_0^{\mathrm{HS}}\right\rangle &= \sum_{\{z_1, z_2, \ldots, z_M\}} \psi_0^{\mathrm{HS}}[z]\, S_{z_1}^{+} \cdot \ldots \cdot S_{z_M}^{+} \left|\downarrow\downarrow \ldots \downarrow\right\rangle \\ &= \sum_{\{z_1, \ldots, z_M; w_1, \ldots, w_M\}} \psi_0^{\mathrm{HS}}[z]\, a_{z_1}^{+} \ldots a_{z_M}^{\dagger} b_{w_1}^{+} \ldots b_{w_M}^{\dagger} |0\rangle \\ &\equiv \Psi_0^{\mathrm{HS}}\left[a^{\dagger}, b^{\dagger}\right] |0\rangle, \end{aligned} \tag{2.4.27}$$

where $M = \frac{N}{2}$ and the w_k's are those lattice sites which are not occupied by any of the z_i's. The $S = 1$ state (2.4.1) is then up to an overall normalization factor given by

$$\left|\psi_0^{S=1}\right\rangle = \left(\Psi_0^{\mathrm{HS}}\left[a^\dagger, b^\dagger\right]\right)^2 |0\rangle. \tag{2.4.28}$$

To verify (2.4.28), use the identity

$$\mathcal{S}\left\{ \prod_{\substack{i,j=1\\ i<j}}^{M} (z_i - z_j)^2 \prod_{\substack{i,j=M+1\\ i<j}}^{2M} (z_i - z_j)^2 \right\} = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j}^{2M} (z_i - z_j), \tag{2.4.29}$$

where $\mathcal{S}$ indicates symmetrization over all the variables in the curly brackets, and

$$\frac{1}{\sqrt{2}} (a^\dagger)^n (b^\dagger)^{(2-n)} |0\rangle = (\tilde{S}^+)^n |1, -1\rangle, \tag{2.4.30}$$

which is readily verified with (2.4.19), (2.4.24), and the definition (2.4.4). To proof (2.4.29), use the following identity due to Frobenius [110],

$$\det\left(\frac{1}{z_i - z_{M+j}}\right) = (-1)^{\frac{M(M+1)}{2}} \frac{\displaystyle\prod_{\substack{i,j=1\\ i<j}}^{M} (z_i - z_j) \prod_{\substack{i,j=M+1\\ i<j}}^{2M} (z_i - z_j)}{\displaystyle\prod_{i=1}^{M} \prod_{j=M+1}^{2M} (z_i - z_j)}. \tag{2.4.31}$$

The projective construction directly reveals several interesting features, which were not nearly as obvious in the previous formulation:

(a) Since the Haldane–Shastry ground state $\left|\psi_0^{\mathrm{HS}}\right\rangle$ is translationally invariant with ground state momentum $p_0 = 0$ or π (depending on whether $\frac{N}{2}$ is even or odd), the $S = 1$ state $\left|\psi_0^{S=1}\right\rangle$ is translationally invariant with $p_0 = 0$.

(b) Since $\left|\psi_0^{\mathrm{HS}}\right\rangle$ is a singlet, and the projection onto spin $S = 1$ on each site commutes with spin rotations, $\left|\psi_0^{S=1}\right\rangle$ has to be a singlet as well.

(c) Since $\psi_0^{\mathrm{HS}}(z_1, \ldots, z_M)$ is real with the sign of each spin configuration given by $\prod_{i=1}^{\mathrm{M}} G(z_i)$, the $S = 1$ wave function $\psi_0^{S=1}(z_1, \ldots, z_M)$ is likewise real with the sign given by $\prod_{i=1}^{N} G(z_i)$:

$$\psi_0^{S=1}(z_1, \ldots, z_N) = \left|\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j}^{N} (z_i - z_j)\right| \prod_{i=1}^{N} G(z_i), \tag{2.4.32}$$

with $G(\eta_\alpha) = \pm 1$ depending on whether α even or odd.

(d) Since $\left|\psi_0^{\mathrm{HS}}\right\rangle$ is invariant under parity and and time reversal, $\left|\psi_0^{S=1}\right\rangle$ is invariant as well.

2.4.5 Topological Degeneracies and Non-Abelian Statistics

We have seen in Sect. 2.3.3 that $2n$ spatially well separated quasiparticle excitations or vortices carrying half of a Dirac flux quanta each in the non-Abelian quantized Hall state described by the Pfaffian will span an internal or topological Hilbert space of dimensions 2^n (2^{n-1} for either even or odd fermion numbers), in accordance with the existence of one Majorana fermion state at each vortex core. The Majorana fermion states can only be manipulated through braiding of the vortices, with the interchanges being non-commutative or non-Abelian.

The question we wish to address in this section is whether there is any manifestation of this topological space of dimension 2^n, or the $2n$ Majorana fermion states, in the spinon excitation Hilbert space suggested by the $S = 1$ ground state (2.4.1). In Sect. 2.2.6, we have seen that the fractional statistics of the spinons in the Haldane–Shastry model, and presumably in any model supporting one-dimensional anyons, is encoded in the momentum spacings of the excitations. This is not too surprising, as there are no other suitable quantum numbers, like the relative angular momentum for two-dimensional anyons, available. We will propose now that the topological degeneracies, or the occupation numbers of the n fermions consisting of the $2n$ Majorana fermions, are once again encoded in the momentum spacings between single spinon states.

In the Haldane–Shastry model, the spacings between neighboring momenta were always half integer, in accordance with half-fermi statistics, as the difference between consecutive spinon momentum numbers a_i was always an odd integer,

$$a_{i+1} - a_i = \text{odd}. \tag{2.4.33}$$

This follows directly from the construction of the extended Young tableaux illustrated in Fig. 2.5. When two spinons are in neighboring columns, the difference of the a_i is one and hence an odd integer; when we insert complete columns without spinons in between, the number of boxes we insert is always even.

We will now show that for the $S = 1$ chain with the Hilbert space parameterized by the ground state $\left|\psi_0^{S=1}\right\rangle$ and spinon excitations above it, the corresponding rule is

$$\begin{aligned} a_{i+1} - a_i &= \text{even or odd}, && \text{for } i \text{ odd}, \\ a_{i+1} - a_i &= \text{odd}, && \text{for } i \text{ even}. \end{aligned} \tag{2.4.34}$$

As $i = 1, 2, \ldots, 2n$, we have a total of n spacings which can be either even or odd, and another n spacings which are always odd. With the single spinon momenta given by

$$p_i = \frac{\pi}{N}\left(a_i - \frac{1}{2}\right), \tag{2.4.35}$$

this yields momentum spacings which can be either an integer or an half-integer times $\frac{2\pi}{N}$ for i odd. This is a topological distinction—for Abelian anyons, one choice

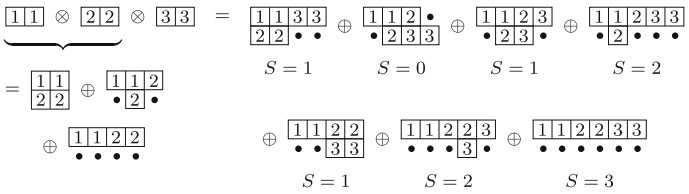


Fig. 2.8 Total spin representations of three $S = 1$ spins in terms of extended Young tableaux

corresponds to bosons or fermions (which are for many purposes equivalent in one dimension), and the other choice to half-fermions. For spinons which are well separated in momentum space, the states spanning this in total 2^n dimensional topological Hilbert space become degenerate as we approach the thermodynamic limit.

To derive (2.4.34), we introduce a second formalism of extended Young tableaux, this time for spin $S = 1$. The general rule we wish to propose for obtaining the tableaux is illustrated in Fig. 2.8 for three spins with $S = 1$. The construction is as follows. For each of the N spins, put a row of two adjacent boxes, which is equivalent to the Young tableau for a single spin without any numbers in the boxes. Put these N small tableaux on a line and number them consecutively from left to right, with the same number in each pair of boxes which represent a single spin. To obtain the product of some extended Young tableau representing spin S_0 on the left with a spin 1 tableau (i.e., a row of two boxes with the same number in it) on the right, we follow the rule

$$\boldsymbol{S}_0 \otimes \mathbf{1} = \begin{cases} \mathbf{1}, & \text{for } S_0 = 0, \\ \boldsymbol{S}_0 - \mathbf{1} \oplus \boldsymbol{S}_0 \oplus \boldsymbol{S}_0 + 1, & \text{for } S_0 = 1, 2, \ldots \end{cases} \tag{2.4.36}$$

i.e., we obtain only one new tableau with both boxes from the right added to the top row if the tableau on the left is a singlet, and three new tableaux if it is has spin one or higher. These three tableaux are constructed by adding both boxes to the bottom row (resulting in a representation $\boldsymbol{S}_0 - \mathbf{1}$), by adding the first box to the bottom row and the second box to the top row without stacking them on top of each other (resulting in a representation $\boldsymbol{S}_0$), and by adding both boxes to the top row (resulting in a representation $\boldsymbol{S}_0 + \mathbf{1}$). In each extended tableau, the boxes must be arranged such that the numbers are strictly increasing in each column from top to bottom, and that they are not decreasing from left to right in that the smallest number in each column cannot be smaller than the largest number in the column to the left of it. In analogy to the Haldane–Shastry model, the empty spaces in between the boxes are filled with dots representing spinons. The spinon momentum number a_i associated with each spinon is given by the number in the box in the same column. A complete table of all the extended Young tableaux for fours $S = 1$ spins is shown in Fig. 2.9. The assignment of physical single spinon momenta to the spinon momentum numbers (2.4.47) is identical to this assignment for the Haldane–Shastry model, as we can obtain the 3^N states of the $S = 1$ Hilbert space by Schwinger boson projection (i.e., by projecting on spin $S = 1$ on each site) from states contained in the $2^N \times 2^N$ dimensional Hilbert space of two $S = \frac{1}{2}$ models, a projection which commutes with

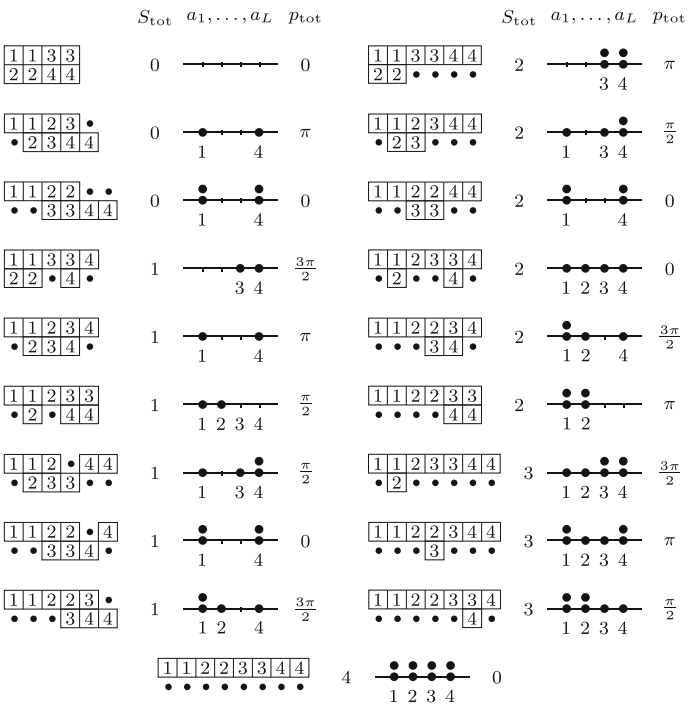
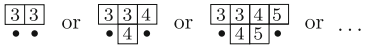


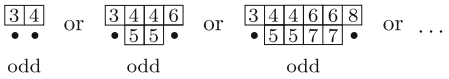
Fig. 2.9 Extended Young tableau decomposition for an $S = 1$ spin chain with $N = 4$ sites. The dots represent the spinons. The spinon momentum numbers a_i are given by the numbers in the boxes of the same column. Note that $\sum(2S_{\rm tot}+1) = 3^N$

the total momentum. The correctness of this assignment has further been verified numerically up to $N = 16$ sites [141].

With the tableau structure thus in place, all that is left to show is that the momentum spacings are according to (2.4.34). Looking at any of the tableaux in Fig. 2.9, we note that from left to right, the spinons alternate between being assigned to the first of the two boxes with a given number and being assigned to the second of such two boxes. This follows simply form the fact that the number of boxes in between the columns with the two neighboring spinons must be even. The first spinon momentum number a_1 is always odd, but all the other a_i's can be either even or odd. The rule is therefore that if i is odd, the ith spinon is assigned to the first of the two boxes with number a_i, and the momentum spacing $a_{i+1} - a_i$ can be either even or odd,



If i is even, however, the ith spinon is assigned to the second of the two boxes with number a_i, and the momentum spacing $a_{i+1} - a_i$ has to be odd, as we can insert only an even number of columns between the two spinons (recall that we cannot stack two boxes with the same number in it on top of each other):



The spacings between the single spinon momenta are hence as stated in (2.4.34). This result is consistent with the spinon basis proposed by Bouwknegt, Ludwig, and Schoutens [143] for the SU(2) level $k = 2$ Wess-Zumino-Witten model [144, 145].

2.4.6 Generalization to Arbitrary Spin S

The projective generation introduced in Sect. 2.4.4 can be generalized to arbitrary spin $S = s$:

$$\left|\psi_0^S\right\rangle = \left(\Psi_0^{\mathrm{HS}}\left[a^\dagger, b^\dagger\right]\right)^{2s}|0\rangle. \tag{2.4.37}$$

In order to write this state in a form similar to (2.4.1)–(2.4.5),

$$\left|\psi_0^S\right\rangle = \sum_{\{z_1,\ldots,z_{sN}\}} \psi_0^S(z_1,\ldots,z_{sN})\ \tilde{S}_{z_1}^+ \cdot \cdots \cdot \tilde{S}_{z_{sN}}^+ |-s\rangle_N, \tag{2.4.38}$$

where

$$|-s\rangle_N \equiv \otimes_{\alpha=1}^N |s,-s\rangle_\alpha \tag{2.4.39}$$

is the "vacuum" state in which all the spins are maximally polarized in the negative $\hat{z}$-direction, we introduce re-normalized spin flip operators $\tilde{S}^+$ which satisfy

$$\frac{1}{\sqrt{(2s)!}}(a^\dagger)^n(b^\dagger)^{(2s-n)}|0\rangle = (\tilde{S}^+)^n|s,-s\rangle. \tag{2.4.40}$$

If we assume a basis in which S^z is diagonal, we may write

$$\tilde{S}^+ \equiv \frac{1}{b^\dagger b + 1} a^\dagger b = \frac{1}{s - S^z + 1} S^+. \tag{2.4.41}$$

The wave function for the spin S state (2.4.37) is then with $M = \frac{N}{2}$ given by

$$\psi_0^S(z_1,\ldots,z_{sN}) = \prod_{m=1}^{2s}\left(\prod_{\substack{i,j=(m-1)M+1\\ i<j}}^{mM}(z_i - z_j)^2\right)\prod_{i=1}^{sN} z_i. \tag{2.4.42}$$

Note that these states are similar to the Read–Rezayi states [142] in the quantized Hall effect.

As for the $S = 1$ state discussed in Sect. 2.4.4, the projective construction (2.4.37) directly implies several symmetries. The state $|\psi_0^S\rangle$ is translationally invariant with ground state momentum $p_0 = -\pi Ns$, a spin singlet, and real:

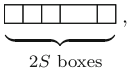
$$\psi_0^S(z_1, \ldots, z_{sN}) = \left|\psi_0^S(z_1, \ldots, z_{sN})\right| \prod_{i=1}^{sN} G(z_i), \tag{2.4.43}$$

with $G(z_i)$ given by (2.2.16).

2.4.7 *Momentum Spacings and Topological Degeneracies for Arbitrary Spin S*

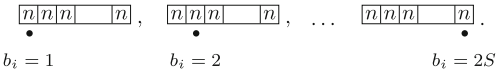
In Sect. 2.4.5, we have shown that the non-Abelian statistics of the Pfaffian state (2.3.1), and in particular the topological degeneracies associated with the Majorana fermion states in the vortex cores discussed in Sect. 2.3.3, manifests itself in topological choices for the (kinematical) momentum spacings of the spinon excitations above the $S = 1$ ground state (2.4.1). Specifically, we found that if we label the single spinon momenta in ascending order by $p_i < p_{i+1}$, the spacings $p_{i+1} - p_i$ can be either even or odd multiples of $\frac{\pi}{N}$ if i is odd, while it has to be an odd multiple if i is even.

In this Section, we formulate the corresponding restrictions for the general spin S chain with ground state (2.4.37). We will first state the rules and then motivate them. Recall that spinons are represented by dots placed in the empty spaces of extended Young tableaux, and that the momentum number a_i of spinon i is given by the number in the box it shares a column with. For general spin S, the tableau describing the representation on each site is given by



i.e., a horizontal array of $2S$ boxes indicating symmetrization, which all contain the same number.

If this number is n, the spinons we assign to any of these boxes will have momentum number $a_i = n$. Let us denote the number of the box a given spinon i with momentum number a_i is assigned to, by b_i, such that box number $b_i = 1$ corresponds to the first, and box number $b_i = 2S$ to the last box with number n in it:



We will see below that if a representation of a spin S chain with L spinons is written in terms of an extended Young tableau, the first spinon with momentum number a_1

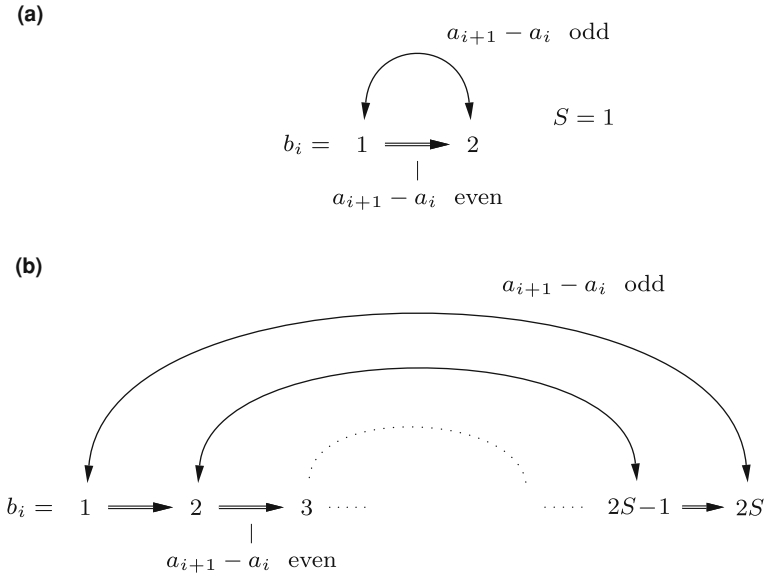


Fig. 2.10 Non-Abelian (SU(2) level $k = 2S$) statistics in one dimension: flow diagram for the (auxiliary) box numbers b_i, which serve to describe the restrictions for the spinon momentum number spacings $a_{i+1} - a_i$ for the critical models of spin chains introduced in Sects. 2.4.1 and 2.4.6 with **a** $S = 1$, and **b** general spin S. The unidirectional, horizontal arrows correspond to even integer momentum number spacings $a_{i+1} - a_i$, while the bidirectional, semicircle arrows correspond to odd integer spacings

will always have box number $b_1 = 1$, and the last spinon with a_L will have $b_L = 2S$. The restrictions corresponding to the non-abelian (SU(2) level $k = 2S$) statistics of the spinons are described by the flow diagram of the numbers b_i shown in Fig. 2.10.

Let us elaborate this diagram first for the case $S = 1$, which we have already studied in Sect. 2.4.5. In this case,

$$b_i = \begin{cases} 1, & \text{for } i \text{ odd}, \\ 2, & \text{for } i \text{ even}. \end{cases} \tag{2.4.44}$$

For i odd, we may move from $b_i = 1$ to $b_{i+1} = 2$ either via the horizontal arrow or via the semicircle in Fig. 2.10a, and $a_{i+1} - a_i$ may hence be either even or odd, respectively. For i even or $b_i = 2$, however, the semicircle is the only available continuation, which implies that the spacing $a_{i+1} - a_i$ must be odd.

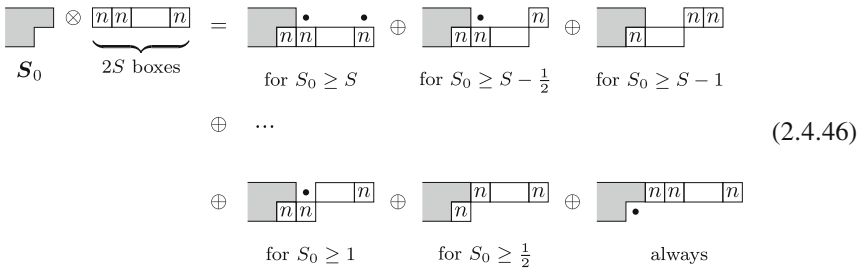
For general S, Fig. 2.10b implies that the spacings can be even or odd until $b_i = 2S$ is reached, which is then followed by an odd integer spacing $a_{i+1} - a_i$, as the semicircular arrow is the only possible continuation at this point. Note that for $S \geq 1$, the minimal number of spinons is two (these two spinons then have an odd integer spacing $a_2 - a_1$), and that we cannot have more than $2S$ spinons with the same momentum number $a_i = n$, as $a_{i+1} - a_i = 0$ is even.

We will now motivate this diagram. To begin with, we generalize the formalism of extended Young tableaux to arbitrary spin S. The construction is similar to the one

for $S = 1$ outlined in Sect. 2.4.5. For each of the N spins, put a row of $2S$ adjacent boxes. Put these N tableaux on a line and number them consecutively from left to right, with the same number in each row of $2S$ boxes representing a single spin. To obtain the product of some extended Young tableau representing spin S_0 on the left with a spin S tableau (i.e., a row of $2S$ boxes with the same number in it) on the right, we first recall

$$\boldsymbol{S}_0 \otimes \boldsymbol{S} = |\boldsymbol{S}_0 - \boldsymbol{S}| \oplus |\boldsymbol{S}_0 - \boldsymbol{S}| + 1 \oplus \ldots \oplus \boldsymbol{S}_0 + \boldsymbol{S}, \tag{2.4.45}$$

which implies that we obtain either $2S_0 + 1$ or $2S + 1$ new tableaux, depending on which number is smaller. In terms of extended Young tableaux, (2.4.44) translates into

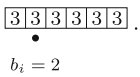


(2.4.46)

The first tableau on the right-hand side of (2.4.46) exists only for $S_0 \geq S$, the second only for $S_0 \geq S - \frac{1}{2}$, and so on. Note that the shape of the right boundary of the extended Young tableaux for $\boldsymbol{S}_0$ does not determine which tableaux are contained in the expansion of $\boldsymbol{S}_0 \otimes \boldsymbol{S}$, as this depends only on the number $S_0 - S$. In the expansion (2.4.46), the $2S$ boxes representing a single spin S always reside in adjacent columns. In an extended tableau, the numbers in the boxes are equal or increasing as we go from left to right, and strictly increasing from top to bottom. The empty spaces we obtain as we build up the tableaux via this method represent the spinons. Note that we cannot take a given tableau and just add a pair of spinons by inserting them somewhere, as the resulting tableau would not occur in the expansion. In Fig. 2.11, we illustrate the principle by writing out a few terms in the expansion for an $S = 2$ chain.

We now turn to the question what this construction implies for the momentum spacings of the spinons. It is very easy to see from Fig. 2.11 that $b_1 = 1$ and a_1 is odd, and that $b_2 = 2S$ and a_L is even (odd) for N even (odd).

Let us assume we have a spinon i with momentum number a_i and box number b_i. If we take $S = 3$, $a_i = 3$, and $b_i = 2$, this spinon would be represented by a dot which shares a column with the second box with number 3 in it,



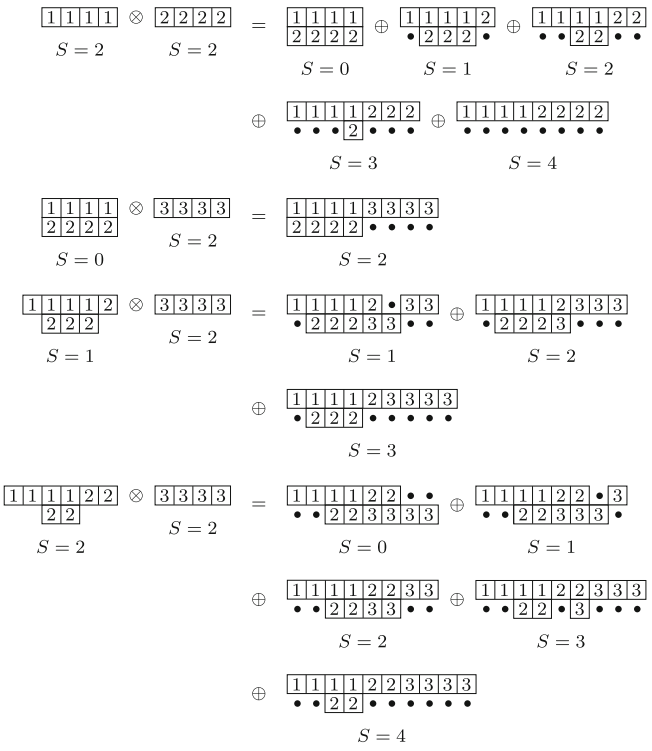
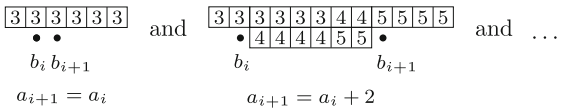


Fig. 2.11 Examples of products of extended tableaux for an $S = 2$ spin chain

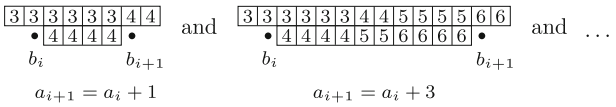
For the box number b_{i+1} of the next spinon, there are only two possibilities:

(i) $b_{i+1} = b_i + 1$, which implies that $a_{i+1} - a_i$ is even. The spinons either sit in neighboring columns with $a_{i+1} = a_i$, or contain an even number of spin S representations (with $2S$ boxes each) in between them. For our example, the corresponding tableaux are



This possibility produces the unidirectional, horizontal arrows in Fig. 2.10. If $b_i = 2S$, this possibility does not exist, and there are either no further spinons or $a_{i+1} - a_i$ has to be odd.

(ii) $b_{i+1} = 2S - b_i + 1$, which implies that $a_{i+1} - a_i$ is odd. For our example, the tableaux are



This possibility produces the bidirectional, semicircle arrows in Fig. 2.10.

This concludes the motivation of the flow diagram in Fig. 2.10b. As in Sects. 2.2.7 and 2.4.5, the single spinon momenta are given by

$$p_i = \frac{\pi}{N}\left(a_i - \frac{1}{2}\right). \tag{2.4.47}$$

This yields momentum spacings $p_{i+1} - p_i$ which can be either an integer or an half-integer times $\frac{2\pi}{N}$.

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From Abelian and non-Abelian Quantum Hall States to
Exact Models of Critical Spin Chains
Greiter, M.
2011, XIV, 194 p., Hardcover
ISBN: 978-3-642-24383-7