Solving Nonlinear Partial Differential Equations with Maple and Mathematica

(Maple and Mathematica Scripts)

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Introduction

1.1 Basic Concepts

1.1.1 Types of Partial Differential Equations

Problem 1.1

Maple:

```
with(PDEtools): declare(u(x,y),v(x,t));
U,V:=diff_table(u(x,y)),diff_table(v(x,t));
Eq1:=U[x,x]+U[y,y]=0;    Eq2:=x*U[x]+y*U[y]=x^2+y^2;
Eq3:=V[t]+v(x,t)*V[x]=0; Eq4:=U[x]^2+U[y]^2=n(x,y)^2;
tr1:=(x,y,U)->{U[x,x]=p,U[y,y]=q}; tr2:=(x,y,U)->{U[x]=p,U[y]=q};
F1:=(p,q)->subs(tr1(x,y,U),Eq1); F2:=(p,q)->subs(tr2(x,y,U),Eq2);
F3:=(p,q)->subs(tr2(x,t,V),Eq3); F4:=(p,q)->subs(tr2(x,y,U),Eq4);
F1(p,q); F2(p,q); F3(p,q); F4(p,q);
```

Mathematica:

```
{eq1=D[u[x,y],{x,2}]+D[u[x,y],{y,2}]==0, eq2=x*D[u[x,y],x]
 +y*D[u[x,y],y]==x^2+y^2, eq3=D[v[x,t],t]+v[x,t]*D[v[x,t],x]==0,
 eq4=D[u[x,y],x]^2+D[u[x,y],y]^2==n[x,y]^2}
tr1[x_,y_,u_]:={D[u[x,y],{x,2}]->p,D[u[x,y],{y,2}]->q};
tr2[x_,y_,u_]:={D[u[x,y],x]->p,D[u[x,y],y]->q};
f1[p_,q_]:=eq1/.tr1[x,y,u]; f2[p_,q_]:=eq2/.tr2[x,y,u];
f3[p_,q_]:=eq3/.tr2[x,t,v]; f4[p_,q_]:=eq4/.tr2[x,y,u];
{f1[p,q], f2[p,q], f3[p,q], f4[p,q]}
```

□

Problem 1.2

Maple:

```
with(PDEtools): declare(u(x,y),F1(p,r,q)); U:=diff_table(u(x,y));
PDE1:=-2*y^2*U[x,x]+x^2*U[y,y]/2=0; show;
tr1:=(x,y,U)->{U[x,x]=p,U[y,y]=r,U[x,y]=q};
tr2:=F->{a=diff(lhs(F(p,q,r)),p),b=1/2*diff(lhs(F(p,q,r)),q),
         c=diff(lhs(F(p,q,r)),r)}; delta:=b^2-a*c;
F1:=(p,r,q)->subs(tr1(x,y,U),PDE1); F1(p,r,q); tr2(F1);
delta1:=subs(tr2(F1),delta)-rhs(F1(p,r,q));
is(delta1,'positive'); coulditbe(delta1,'positive');
```

Mathematica:

```
pde1=-2*y^2*D[u[x,y],{x,2}]+x^2*D[u[x,y],{y,2}]/2==0
tr1[x_,y_,u_]:={D[u[x,y],{x,2}]->p,D[u[x,y],{y,2}]->r,
 D[u[x,y],{x,y}]->q}; tr2[f_]:={a->D[f[p,q,r][[1]],p],
 b->1/2*D[f[p,q,r][[1]],q],c->D[f[p,q,r][[1]],r]};
f1[p_,r_,q_]:=pde1/.tr1[x,y,u]; delta=b^2-a*c
{f1[p,r,q], tr2[f1], delta1=delta/.tr2[f1]-f1[p,r,q][[2]]}
{Reduce[delta1>0], FindInstance[delta1>0,{x,y}]}
```

Maple:

```
interface(showassumed=0): assume(x<0 or x>0, y<0 or y>0);
with(LinearAlgebra): A1:=Matrix([[-2*y^2,0],[0,x^2/2]]);
D1:=Determinant(A1); is(D1,'negative'); coulditbe(D1,'negative');
```

Mathematica:

```
{a1={{-2*y^2,0},{0,x^2/2}},d1=Det[a1],Reduce[d1<0],
 FindInstance[d1<0,{x,y}]}
```

Maple:

```
with(LinearAlgebra): with(VectorCalculus): with(PDEtools):
declare(v(xi,eta)); interface(showassumed=0):
vars:=x,y; varsN:=xi,eta; assume(x<0 or x>0,y<0 or y>0);
Op1:=Expr->subs(y=y(x),Expr); Op2:=Expr->subs(y(x)=y,Expr);
A1:=Matrix([[-2*y^2,0],[0,x^2/2]]); D1:=Determinant(A1);
is(D1,'negative'); coulditbe(D1,'negative');
m1:=simplify((-A1[1,2]+sqrt(-D1))/A1[1,1],radical,symbolic);
```

```
m2:=simplify((-A1[1,2]-sqrt(-D1))/A1[1,1],radical,symbolic);
Eq1:=dsolve(diff(y(x),x)=-Op1(m1),y(x));
Eq11:=lhs(Eq1[1])^2=rhs(Eq1[1])^2; Eq12:=solve(Eq11,_C1);
g1:=Op2(Eq12); Eq2:=dsolve(diff(y(x),x)=-Op1(m2),y(x));
Eq21:=lhs(Eq2[1])^2=rhs(Eq2[1])^2; Eq22:=solve(Eq21,_C1);
g2:=Op2(Eq22); Jg:=Jacobian(Vector(2,[g1,g2]),[vars]);
dv:=Gradient(v(varsN),[varsN]); ddv:=Hessian(v(varsN),[varsN]);
ddu:=Jg^%T.ddv.Jg+add(dv[i]*Hessian(g||i,[vars]),i=1..2);
Eq3:=simplify(Trace(A1.ddu))=0;
tr1:={isolate(subs(isolate(g1=xi,x^2),g2=eta),y^2),
      isolate(subs(isolate(g1=xi,y^2),g2=eta),x^2)};
NormalForm:=collect(expand(subs(tr1,Eq3)),diff(v(varsN),varsN));
c1:=coeff(lhs(NormalForm),diff(v(varsN),varsN));
NormalFormF:=collect(NormalForm/c1,diff(v(varsN),varsN));
CanonicalForm:=expand(expand(dchange(
 {xi=lambda+mu,eta=mu-lambda},NormalFormF))*(-4));
```

Mathematica:

```
jacobianM[f_List?VectorQ,x_List]:=Outer[D,f,x]/;Equal@@(
 Dimensions/@{f,x}); hessianH[f_,x_List?VectorQ]:=D[f,{x,2}];
gradF[f_,x_List?VectorQ]:=D[f,{x}]; op1[expr_]:=expr/.y->y[x];
op2[expr_]:=expr/.y[x]->y; {vars=Sequence[x,y],
 varsN=Sequence[xi,eta], a1={{-2*y^2,0},{0,x^2/2}}, d1=Det[a1],
 Reduce[d1<0],FindInstance[d1<0,{x,y}], m1=Assuming[{x>0,y>0},
 Simplify[(-a1[[1,2]]+Sqrt[-d1])/a1[[1,1]]]], m2=Assuming[
 {x>0,y>0},Simplify[(-a1[[1,2]]-Sqrt[-d1])/a1[[1,1]]]]}
{eq1=DSolve[D[y[x],x]==-op1[m1],y[x],x], eq11=eq1[[1,1,1]]^2==
 eq1[[1,1,2]]^2,eq12=Solve[eq11,C[1]][[1,1,2]], g[1]=Expand[
 op2[eq12]*2], eq2=DSolve[D[y[x],x]==-op1[m2],y[x],x], eq21=
 eq2[[1,1,1]]^2==eq2[[1,1,2]]^2,eq22=Solve[eq21,C[1]][[1,1,2]],
 g[2]=Expand[op2[eq22]*2]}
{jg=jacobianM[{g[1],g[2]},{vars}], dv=gradF[v[varsN],{varsN}],
 ddv=hessianH[v[varsN],{varsN}]}
{ddu=Transpose[jg].ddv.jg+Sum[dv[[i]]*hessianH[g[i],{vars}],
 {i,1,2}], eq3=Simplify[Tr[a1.ddu]]==0, tr0={y^2->Y,x^2->X},
 tr01={Y->y^2,X->x^2}, tr1=Flatten[{Expand[Solve[First[g[2]==
 eta/.{Solve[g[1]==xi/.tr0,X]/.tr01}/.tr0],Y]/.tr01],Expand[
 Solve[First[g[1]==xi/.{Solve[g[2]==eta/.tr0,Y]/.tr01}/.tr0],
 X]/.tr01]}], nForm=Collect[Expand[eq3/.tr1],D[v[varsN],varsN]]}
c1=Coefficient[nForm[[1]],D[v[varsN],varsN]]
normalFormF=Collect[Thread[nForm/c1,Equal],D[v[varsN],varsN]]
```

```
nF[x_,t_]:=D[D[u[x,t],x],t]+(2*t*D[u[x,t],x]
          -2*x*D[u[x,t],t])/(4*t^2-4*x^2)==0; nF[xi,eta]
tr2={xi->lambda+mu,eta->mu-lambda}; nFT[v_]:=((Simplify[
 nF[xi,eta]/.u->Function[{xi,eta},u[(xi-eta)/2,(xi+eta)/2]]])
 /.tr2//ExpandAll)/.{u->v}; canonicalForm=nFT[v]
```

□

Problem 1.3

Maple:

```
with(LinearAlgebra): with(VectorCalculus): with(PDEtools):
declare(v(xi,eta)); Op1:=Expr->subs(y=y(x),Expr);
Op2:=Expr->subs(y(x)=y,Expr); vars:=x,y; varsN:=xi,eta;
A1:=Matrix([[x^2,x*y],[x*y,y^2]]); D1:=Determinant(A1);
m1:=simplify((-A1[1,2]+sqrt(-D1))/A1[1,1],radical,symbolic);
Eq1:=dsolve(diff(y(x),x)=-Op1(m1),y(x)); Eq11:=solve(Eq1,_C1);
g1:=Op2(Eq11); g2:=x; Jg:=Jacobian(Vector(2,[g1,g2]),[vars]);
dv:=Gradient(v(varsN),[varsN]); ddv:=Hessian(v(varsN),[varsN]);
ddu:=Jg^%T.ddv.Jg+add(dv[i]*Hessian(g||i,[vars]),i=1..2);
NorF:=simplify(Trace(A1.ddu))=0; CanF:=expand(NorF/x^2);
```

Mathematica:

```
jacobianM[f_List?VectorQ, x_List]:=Outer[D,f,x]/;Equal@@(
 Dimensions/@{f,x}); hessianH[f_,x_List?VectorQ]:=D[f,{x,2}];
gradF[f_,x_List?VectorQ]:=D[f,{x}]; op1[expr_]:=expr/.y->y[x];
op2[expr_]:=expr/.y[x]->y; {vars=Sequence[x,y],varsN=Sequence[
 xi,eta], a1={{x^2,x*y},{x*y,y^2}}, d1=Det[a1]}
m1=Assuming[{x>0,y>0},Simplify[(-a1[[1,2]]+Sqrt[-d1])/a1[[1,1]]]]
eq1=DSolve[D[y[x],x]==-op1[m1],y[x],x]/.Rule->Equal//First
{eq11=Solve[eq1,C[1]][[1,1,2]], g[1]=op2[Eq11], g[2]=x}
{jg=jacobianM[{g[1],g[2]},{vars}], dv=gradF[v[varsN],{varsN}],
 ddv=hessianH[v[varsN],{varsN}], ddu=Transpose[jg].ddv.jg
 +Sum[dv[[i]]*hessianH[g[i],{vars}],{i,1,2}]}
{norF=Simplify[Tr[a1.ddu]]==0,
 canF=Thread[norF/x^2,Equal]//Expand}
```

□

Problem 1.4

Maple:

```
with(PDEtools): declare((u,v)(x,y),(F1,F2)(p,r,q),G(u(x,t)));
U,V,GV:=diff_table(u(x,y)),diff_table(v(x,t)),
  diff_table(G(v(x,t))); PDE1:=U[x,y]^2-U[x,x]*U[y,y]=F(x,y);
tr1:=(x,y,U)->{U[x,x]=p,U[y,y]=r,U[x,y]=q};
tr2:=H->{a=diff(lhs(H(p,q,r)),p),b=1/2*diff(lhs(H(p,q,r)),q),
         c=diff(lhs(H(p,q,r)),r)}; delta:=b^2-a*c;
F1:=(p,r,q)->subs(tr1(x,y,U),PDE1); F1(p,r,q); tr2(F1);
delta1:=subs(tr2(F1),delta)=rhs(F1(p,r,q));
PDE2:=V[t,t]-G(v)*V[x,x]-GV[x]*V[x]=0;
F2:=(p,r,q)->subs(tr1(x,t,V),PDE2); F2(p,r,q); tr2(F2);
delta2:=subs(tr2(F2),delta)=rhs(F2(p,r,q));
```

Mathematica:

```
pde1=D[u[x,y],{x,y}]^2-D[u[x,y],{x,2}]*D[u[x,y],{y,2}]==f[x,y]
tr1[x_,y_,u_]:={D[u[x,y],{x,2}]->p, D[u[x,y],{y,2}]->r,
                D[u[x,y],{x,y}]->q};
f1[p_,r_,q_]:=pde1/.tr1[x,y,u]; f1[p,r,q]
tr2[f_]:={a->D[f[p,q,r][[1]],p], b->1/2*D[f[p,q,r][[1]],q],
          c->D[f[p,q,r][[1]],r]}; delta=b^2-a*c; tr2[f1]
delta1=(delta/.tr2[f1])==f1[p,r,q][[2]]
dgDv=D[g[v[x,t]],x]*D[v[x,t],x];
pde2=D[v[x,t],{t,2}]-g[v[x,t]]*D[v[x,t],{x,2}]-dgDv==0
f2[p_,r_,q_]:=pde2/.tr1[x,t,v]; {f2[p,r,q], tr2[f2]}
delta2=(delta/.tr2[f2])==f2[p,r,q][[2]]
```

□

Problem 1.5

Maple:

```
with(PDETools): with(LinearAlgebra): declare((F1,F2)(u,v));
B1:=<<F1(u,v)+u*diff(F1(u,v),u),v*diff(F1(u,v),u)>|
    <u*diff(F1(u,v),v),F1(u,v)+v*diff(F1(u,v),v)>>;
B2:=<<F2(u,v)+u*diff(F2(u,v),u),v*diff(F2(u,v),u)>|
    <u*diff(F2(u,v),v),F2(u,v)+v*diff(F2(u,v),v)>>;
Eq1:=beta1*B1+beta2*B2-lambda*Matrix(2,2,shape=identity)=0;
Eq2:=Determinant(lhs(Eq1))=0; Eq3:=factor(Eq2);
L1:=solve(op(1,lhs(Eq3)),lambda);
L2:=solve(op(2,lhs(Eq3)),lambda); L12:=L2-L1;
A1:=subs(u*diff(F1(u,v),v)=v*diff(F1(u,v),u),B1);
A2:=subs(u*diff(F2(u,v),v)=v*diff(F2(u,v),u),B2);
```

Mathematica:

```
b1={{f1[u,v]+u*D[f1[u,v],u],u*D[f1[u,v],v]},{v*D[f1[u,v],u],
 f1[u,v]+v*D[f1[u,v],v]}}; b2={{f2[u,v]+u*D[f2[u,v],u],
 u*D[f2[u,v],v]},{v*D[f2[u,v],u],f2[u,v]+v*D[f2[u,v],v]}};
Map[MatrixForm,{b1,b2}]
{eq1=beta1*b1+beta2*b2-lambda*IdentityMatrix[2]==0,
 eq2=Det[eq1[[1]]]==0, eq3=Factor[eq2]}
{l1=Solve[eq3[[1,1]]==0,lambda][[1,1,2]],
 l2=Solve[eq3[[1,2]]==0,lambda][[1,1,2]], l12=l2-l1}
a1=b1/.u*D[f1[u,v],v]->v*D[f1[u,v],u]
a2=b2/.u*D[f2[u,v],v]->v*D[f2[u,v],u]
```

□

1.1.2 Nonlinear PDEs and Systems Arising in Applied Sciences

1.1.3 Types of Solutions of Nonlinear PDEs

1.2 Embedded Analytical Methods

1.2.1 Nonlinear PDEs

Problem 1.6

Maple:

```
with(PDEtools); declare(u(x,t)); PDE1:=diff(u(x,t),t$2)+
 diff(u(x,t)*diff(u(x,t),x),x)+diff(u(x,t),x$4)=0; PDE2:=expand(
 diff(u(x,t),t$2)-a*diff(u(x,t)*diff(u(x,t),x),x)-b*diff(u(x,t),
 x$4)); PDE3:=diff(u(x,t),t$2)-diff(u(x,t),x$2)-diff(3*u(x,t)^2,
 x$2)-diff(u(x,t),x$4); Sol1:=S->u(x,t)=-3*lambda^2*cos(lambda*
 (x+S*lambda*t)/2+C1)^(-2); Test11:=pdetest(Sol1(1),PDE1);
Test12:=pdetest(Sol1(-1),PDE1); Sol2:=S->u(x,t)=3*lambda^2/a*
 cosh(lambda*(x+S*lambda*t)/2/sqrt(b)+C1)^(-2);
Test21:=pdetest(Sol2(1),PDE2); Test22:=pdetest(Sol2(-1),PDE2);
f3:=S->1-A*exp(k*x+S*k*t*sqrt(1+k^2)); Sol3:=S->u(x,t)=2*diff(
 log(f3(S)),x$2); factor(Sol3(1)); factor(Sol3(-1));
Test31:=pdetest(Sol3(1),PDE3); Test32:=pdetest(Sol3(-1),PDE3);
```

Mathematica:

```
{pde1=D[u[x,t],{t,2}]+D[u[x,t]*D[u[x,t],x],x]+D[u[x,t],{x,4}]==0,
 pde2=Expand[D[u[x,t],{t,2}]-a*D[u[x,t]*D[u[x,t],x],x]
 -b*D[u[x,t],{x,4}]]==0, pde3=D[u[x,t],{t,2}]-D[u[x,t],{x,2}]
 -D[3*u[x,t]^2,{x,2}]-D[u[x,t],{x,4}]==0}
sol1[s_]:=u->Function[{x,t},-3*lambda^2*Cos[lambda*(x
 +s*lambda*t)/2+c1]^(-2)]; Map[FullSimplify,
 {test11=pde1/.sol1[1],test12=pde1/.sol1[-1]}]
sol2[s_]:=u->Function[{x,t},3*lambda^2/a*Cosh[lambda*(x
 +s*lambda*t)/2/Sqrt[b]+c1]^(-2)]; Map[FullSimplify,
 {test21=pde2/.sol2[1],test22=pde2/.sol2[-1]}]
f3[s_]:=1-a*Exp[k*x+s*k*t*Sqrt[1+k^2]]; sol3[s_]:=u->Function[
 {x,t},2*D[Log[f3[s]],{x,2}]]; Map[Factor,{sol3[1],sol3[-1]}]
Map[FullSimplify,{test31=pde3/.sol3[1],test32=pde3/.sol3[-1]}]
```

□

Problem 1.7

Maple:

```
with(PDEtools);  declare(u(x,t)); U:=diff_table(u(x,t));
interface(showassumed=0); assume(gamm>0);
PDE1:=I*U[t]+U[x,x]+gamm*abs(U[])^2*U[]=0;
Sol1:=u(x,t)=C1*exp(I*(C2*x+(gamm*C1^2-C2^2)*t+C3));
Test1:=pdetest(Sol1,PDE1); Test11:=simplify(evalc(Test1));
Sol2:=u(x,t)=A*sqrt(2/gamm)*(exp(I*B*x+I*(A^2-B^2)*t+I*C1))
      /(cosh(A*x-2*A*B*t+C2));
Test2:=pdetest(Sol2,PDE1); Test21:=simplify(evalc(Test2));
```

Mathematica:

```
pde1=I*D[u[x,t],t]+D[u[x,t],{x,2}]+gamm*Abs[u[x,t]]^2*u[x,t]==0
sol1=u->Function[{x,t},c1*Exp[I*(c2*x+(gamm*c1^2-c2^2)*t+c3)]]
test1=pde1/.sol1
test11=Assuming[{c1>0},test1//ComplexExpand//FullSimplify]
sol2=u->Function[{x,t},a*Sqrt[2/gamm]*(Exp[I*b*x+I*(a^2-b^2)*t
        +I*c1])/(Cosh[a*x-2*a*b*t+c2])]
test2=pde1/.sol2
test21=Assuming[{gamm>0,{a,b,c,c1,c2,x,t}\[Element] Reals},
       FullSimplify[test2]]
```

□

Problem 1.8

Maple:

```
with(PDEtools); declare(u(x,y));
PDE1:=y*diff(u(x,y),y)-x*diff(u(x,y),x)-f(x)/u(x,y)^n=0;
Sol1:=pdsolve(PDE1);
Test1:=pdetest(subs(n=9,Sol1),subs(n=9,PDE1));
Test2:=simplify(factor(expand(simplify(pdetest(Sol1,PDE1))
       assuming n>1)));
```

Mathematica:

```
Off[Solve::"ifun"];
pde1=y*D[u[x,y],y]-x*D[u[x,y],x]-f[x]/u[x,y]^n==0
sol1=DSolve[pde1,u,{x,y}]//FullSimplify//First
Print["sol1=",sol1]; HoldForm[sol1]==sol1
{test1=pde1/.sol1//FullSimplify, sol2=u[x,y]/.sol1}
Print["sol2=",sol2]; tr1={C[1][var_]:>f[var],K[1]->s}
sol3=(u[x,y]/.sol1)/.tr1//FullSimplify
```

□

Problem 1.9

Maple:

```
with(PDEtools);  declare(u(x,y));
PDE1:=u(x,y)*diff(u(x,y),x)=diff(u(x,y),y);
PDE2:=diff(u(x,y),x)^2+diff(u(x,y),y)=0;
sysCh:=charstrip(PDE1,u(x,y)); funcs:=indets(sysCh,Function);
Sol1:=dsolve(sysCh,funcs,explicit);
Sol2:=pdsolve(PDE2,HINT=strip);
```

□

Problem 1.10

Maple:

```
infolevel[pdsolve]:=5; with(PDEtools); declare(u(x,t));
U:=diff_table(u(x,t));
PDE1:=U[t]-nu*U[x,x]+a*U[]*U[x]=b*U[]*(1-U[])*(U[]-c);
casesplit(PDE1); separability(PDE1,u(x,t));
separability(PDE1,u(x,t),`*`); params1:={nu=1,a=1,b=1,c=1};
params2:={nu=1,a=1,b=1}; params3:={nu=1,a=-1,b=1};
Sol1:=pdsolve(subs(params1,PDE1),u(x,t));
```

```
Sol2:=pdsolve(subs(params2,PDE1),HINT='TWS');
Sol3:=pdsolve(subs(params3,PDE1),HINT='TWS(coth)');
for i from 1 to 3 do
      Test||i:=pdetest(Sol||i,subs(params||i,PDE1)) od;
pdsolve(subs(params1,PDE1),HINT=f(x)*g(t));
```

Mathematica:

```
Off[DSolve::"nlpde"];
pde1=D[u[x,t],t]-nu*D[u[x,t],{x,2}]+a*u[x,t]*D[u[x,t],x]==
 b*u[x,t]*(1-u[x,t])*(u[x,t]-c);
{params1={nu->1,a->1,b->1,c->1},params2={nu->1,a->1,b->1}}
sol1=DSolve[pde1/.params1,u,{x,t}]
sol2=DSolve[pde1/.params2,u,{x,t}]
{n1,n2}=Map[Length,{sol1,sol2}]
test1=Table[pde1/.params1/.sol1[[i]]//FullSimplify,{i,1,n1}]
test2=Table[pde1/.params2/.sol2[[i]]//FullSimplify,{i,1,n2}]
```

□

Problem 1.11

Maple:

```
with(PDEtools); declare(u(x,t));
PDE1:=diff(u(x,t),t$2)=a*exp(lambda*u(x,t))*diff(u(x,t),x$2);
params1:={a=1,lambda=1}; Sol1:=pdsolve(subs(params1,PDE1));
Sol2:=pdsolve(subs(params1,PDE1),build);
Sol3:=pdsolve(subs(params1,PDE1),HINT=`+`,build);
Sol4:=[SimilaritySolutions(subs(params1,PDE1))];
for i from 1 to 4 do pdetest(Sol||i,subs(params1,PDE1)); od;
```

□

Problem 1.12

Maple:

```
with(PDEtools); declare(u(x,t),G(u(x,t)));
KdVF:=diff(u(x,t),t)+G(u)*diff(u(x,t),x)+diff(u(x,t),x$3);
tr1:=G(u)=-6*u(x,t); tr2:=G(u)=6*u(x,t);
tr3:=G(u)=2*a*u(x,t)-3*b*u(x,t)^2;
for i from 1 to 3 do
 Sol||i:=pdsolve(subs(tr||i, KdVF),build);
 Test||i:=pdetest(Sol||i,subs(tr||i, KdVF)); od;
```

```
infolevel[TWSolutions]:=2;
Eq1:=subs(tr3,KdVF); Fs:=TWSolutions(functions_allowed);
Sol20:=pdsolve(Eq1,HINT='TWS');
Sol21:=TWSolutions(Eq1,singsol=false,functions=Fs):
N:=nops([Sol21]);
for i from 1 to N do op(i,[Sol21]) od;
Sol22:=TWSolutions(Eq1,singsol=false,parameters=[a,b],
       functions=[tan,cot],remove_redundant=true);
Sol23:=TWSolutions(Eq1,function=identity,output=ODE);
```

Mathematica:

```
Off[DSolve::"nlpde"];
kdVF=D[u[x,t],t]+g[u[x,t]]*D[u[x,t],x]+D[u[x,t],{x,3}]==0
{tr[1]=g[u[x,t]]->-6*u[x,t], tr[2]=g[u[x,t]]->6*u[x,t],
 tr[3]=g[u[x,t]]->2*a*u[x,t]-3*b*u[x,t]^2}
Do[sol[i]=DSolve[kdVF/.tr[i],u,{x,t}];
 test[i]=kdVF/.tr[i]/.sol[i]//FullSimplify;
 Print["sol[",i,"]=",sol[i]]; Print["test[",i,"]=",test[i]],
{i,1,3}];
```

□

Problem 1.13

Maple:

```
with(PDEtools); declare(u(x,t)); interface(showassumed=0):
assume(a>0,b>0,c>0); PDE1:=diff(u(x,t),t)-a*diff(u(x,t),x,x)-
 b*u(x,t)+c*abs(u(x,t))^2*u(x,t)=0; Sol1:=pdsolve(PDE1,
 HINT='TWS'); Sol2:=convert(Sol1,exp); Expr1:=expand(map(
 evalc,pdetest(Sol2,PDE1))); Test1:=simplify(Expr1);
```

□

Problem 1.14

Maple:

```
with(PDEtools): with(plots): declare(u(x,t));
PDE1:=diff(u(x,t),t)+a*diff(u(x,t),x)
     +b*u(x,t)*diff(u(x,t),x)-c*diff(u(x,t),x,x,t)=0;
Ops1:=frames=50,numpoints=200,color=blue,thickness=3;
Sol1:=pdsolve(PDE1); pdetest(Sol1,PDE1);
params:={a=1,b=1,c=1,_C1=-1,_C2=-1,_C3=1};
Sol2:=unapply(subs(params,Sol1),x,t);
animate(rhs(Sol2(x,t)),x=-10..10,t=0..5,Ops1);
```

Mathematica:

```
Off[DSolve::"nlpde"]; SetOptions[Plot,ImageSize->300,PlotStyle->
 {Hue[0.7],Thickness[0.01]},PlotPoints->100,PlotRange->All];
pde1=D[u[x,t],t]+a*D[u[x,t],x]+b*u[x,t]*D[u[x,t],x]-c*D[D[u[x,t],
 {x,2}],t]==0; sol1=DSolve[pde1,u,{x,t}]
test1=pde1/.sol1//FullSimplify
params={a->1,b->1,c->1,C[1]->1,C[2]->-1,C[3]->1}
f=(u/.sol1[[1]])/.params; f[x,t]
Animate[Plot[f[x,t],{x,-10,10},PlotRange->{-10,10}],
 {t,0,10,0.01},AnimationRate->0.4]
```

□

Problem 1.15

Maple:

```
with(PDEtools): declare(u(x,y,t)); Ops1:=axes=boxed,grid=[50,50],
 style=patchnogrid,shading=Z,orientation=[-40,50];
PDE1:=diff(diff(u(x,y,t),t)+a*u(x,y,t)*diff(u(x,y,t),x)
     +diff(u(x,y,t),x$3),x)+b*diff(u(x,y,t),y$2)=0;
Sol1:=pdsolve(PDE1); Test1:=pdetest(Sol1,PDE1);
params:={a=6,b=1,_C1=1,_C2=1,_C3=1,_C4=1};
Sol2:=subs(params,Sol1); xR:=-10..10; yR:=-10..10;
plot3d(subs(t=1,rhs(Sol2)),x=xR,y=yR,Ops1);
Sol3:=[SimilaritySolutions(PDE1,removeredundant=true)];
Test2:=map(pdetest,Sol3,PDE1); Sol4:=combine(subs(params,Sol3));
plot3d(subs(t=1,rhs(Sol4[3])),x=xR,y=yR,Ops1);
```

Mathematica:

```
Off[DSolve::nlpde]; pde1=D[D[u[x,y,t],t]+a*u[x,y,t]*
 D[u[x,y,t],x]+D[u[x,y,t],{x,3}],x]+b*D[u[x,y,t],{y,2}]==0;
{sol1=DSolve[pde1,u,{x,y,t}],test1=pde1/.sol1//FullSimplify}
params={a->6,b->1,C[1]->1,C[2]->1,C[3]->1,C[4]->1}
f=(u/.sol1[[1]])/.params; f[x,y,t]
Plot3D[f[x,y,1],{x,-10,10},{y,-10,10},BoxRatios->{1,1,1},
 Mesh->False,PlotPoints->{50,50},PlotRange->All,ViewPoint->
 {-60,90,60},ColorFunction->Function[{u},Hue[0.7+0.15*u]]]
```

□

1.2.2 Nonlinear PDEs with Initial and/or Boundary Conditions

Problem 1.16

Maple:

```
with(PDEtools): declare(f(x));
PDE1:=diff(u(x,t),t)-diff(u(x,t),x$2)+u(x,t)*diff(u(x,t),x)=0;
IC1:=u(x,0)=f(x); sys1:=[PDE1,IC1]; tr1:=f(x)=8*tan(4*x);
Sol:=[allvalues(pdsolve(sys1))]; map(simplify,Sol);
Sol1:=convert(simplify(algsubs(tr1,op(4,Sol))),tan);
pdetest(Sol1,[PDE1,subs(tr1,IC1)]);
```

□

Problem 1.17

Maple:

```
PDE1:=diff(u(x,t),t)+u(x,t)*diff(u(x,t),x$3)=0;
BC1:=u(0,t)=0,u(L,t)=0; Sol1:=pdsolve([PDE1,BC1]);
pdetest(Sol1,[PDE1,BC1]);

with(PDEtools): declare(u(x,t)); U:=diff_table(u(x,t)):
PDE1:=U[t]+U[]*U[x,x,x]=0;
BC1:=eval(U[],x=0)=0,eval(U[],x=L)=0;
Sol:=pdsolve([PDE1,BC1]);
```

□

Problem 1.18

Maple:

```
with(PDEtools): declare(u(x,t));
PDE1:=diff(u(x,y),x)^2+diff(u(x,y),y)^2=1; BC1:=u(x0,y0)=0;
Sol1:=pdsolve([PDE1,BC1]); pdetest(Sol1,[PDE1,BC1]);
Sol2:=subs(_C1=0,_C2=b,_c[2]=a,pdsolve(PDE1,explicit));
pdetest(Sol2,PDE1);
```

Mathematica:

```
Off[DSolve::"nlpde"]; {pde1=D[u[x,y],x]^2+D[u[x,y],y]^2==1,
 bc1=u[x0,y0]==0, sol1=DSolve[pde1,u,{x,y}],
 sol11=u[x,y]/.sol1[[1]], sol12=u[x,y]/.sol1[[2]]}
{tr0={x->x0,y->y0}, tr2={C[1]->b,C[2]->a}}
{eq1=(sol11/.tr0)==0, trC11=Solve[eq1,C[1]]}
sol11F[xN_,yN_]:=(sol11/.trC11//FullSimplify)/.{x->xN,y->yN}//
 First; {eq2=(sol12/.tr0)==0, trC12=Solve[eq2,C[1]]}
sol12F[xN_,yN_]:=(sol12/.trC12//FullSimplify)/.{x->xN,y->yN}//
 First; test11=Map[FullSimplify,{pde1/.{u->sol11F},
 bc1/.{u->sol11F}}]
test12=Map[FullSimplify,{pde1/.{u->sol12F},bc1/.{u->sol12F}}]
sol21F[xN_,yN_]:=sol11/.tr2/.{x->xN,y->yN};
sol22F[xN_,yN_]:=sol12/.tr2/.{x->xN,y->yN};
{test21=pde1/.u->sol21F, test22=pde1/.u->sol22F}
```
□

1.2.3 Nonlinear Systems

Problem 1.19

Maple:

```
with(PDEtools): declare((u,v)(x,t),F(u(x,t)),G(u(x,t)));
U,V:=diff_table(u(x,t)),diff_table(v(x,t)):
Sys1:={V[t]-F(u(x,t))*U[x]-G(u(x,t))=0,U[t]-U[x]=0};
Sol1:=pdsolve(Sys1,{u(x,t),v(x,t)}); pdetest(Sol1,Sys1);
```
□

Problem 1.20

Maple:

```
with(PDEtools): declare((u,v)(x,t));
U,V:=diff_table(u(x,t)),diff_table(v(x,t)):
sys1:={V[x]-2*U[]=0,V[t]-2*U[x]+U[]^2=0};
Sols:=pdsolve(sys1,{u(x,t),v(x,t)}); pdetest(Sols,sys1);
```
□

1.2.4 Nonlinear Systems with Initial and/or Boundary Conditions

Problem 1.21

Maple:

```
PDE1:=diff(u(x,t),t)=v(x,t)*diff(u(x,t),x)+u(x,t)+1;
PDE2:=diff(v(x,t),t)=-u(x,t)*diff(v(x,t),x)-v(x,t)+1;
IC1:=u(x,0)=exp(-x); IC2:=v(x,0)=exp(x);
sys1:={PDE1, PDE2}; Sols:=pdsolve(sys1); Sol1:=Sols[2];
sys11:=simplify(subs(IC1,IC2,subs(t=0,Sol1)));
C12:={_C1=-1,_C2=1}; sys12:=eval(subs(C12,sys11));
Sol12:=simplify(solve(sys12,{_C3,_C4}));
SolFin:=combine(subs(C12,Sol12,Sol1));
for i from 1 to 2 do
 simplify(expand(subs(SolFin,sys1[i]))); od;
```

□

Chapter 2
Algebraic Approach

2.1 Point Transformations

2.1.1 Transformations of Independent and/or Dependent Variables

Problem 2.1

Maple:

```
with(PDETools): declare(u(x,t),W(X,T)); U:=diff_table(u(x,t));
tr1:={X=a*x,T=t,W(X,T)=a*u(x,t)}; tr2:=solve(tr1,{x,t,u(x,t)});
Eq1:=U[t,t]=U[]^2*U[x,x]+U[]*U[x]^2;
Eq2:=dchange(tr2,Eq1,[X,T,W(X,T)]); Eq3:=expand(Eq2*a);
```

Mathematica:

```
{tr1={xN==a*x,tN==t}, tr2=w[xN,tN]==a*u[x,t],
 tr11=Solve[tr1,{x,t}], tr12=Solve[tr2,{u[x,t]}]}
eq1[x_,t_]:=D[u[x,t],{t,2}]==u[x,t]^2*D[u[x,t],{x,2}]+u[x,t]*
 D[u[x,t],x]^2; eq1T[v_]:=First[((eq1[x,t]/.u->Function[{x,t},
 u[a*x,t]/a])/.tr11)/.{u->v}]; eq1[x,t]
Expand[Thread[eq1T[w]*a,Equal]]
```

□

Problem 2.2

Maple:

```
with(PDETools): declare(u(x,t),w(X,T)); U:=diff_table(u(x,t));
tr1:={T=t-x,X=t+x}; tr2:=solve(tr1,{x,t}); Eq1:=U[x,t]=sin(U[]);
Eq2:=dchange(tr2,Eq1,[X,T]);
```

Mathematica:

```
{tr1={tN==t-x,xN==t+x}, tr2=Solve[tr1,{x,t}]}
eq1[x_,t_]:=D[D[u[x,t],x],t]==Sin[u[x,t]]; eq1[x,t]
eq1T[v_]:=First[((eq1[x,t]/.u->Function[{x,t},u[x+t,-x+t]])/.
 tr2)/.{u->v}]; Simplify[eq1T[u]]
```

□

Problem 2.3

Maple:

```
with(PDETools): declare((u,v)(x,t)); U,V:=diff_table(u(x,t)),
 diff_table(v(x,t)); tr11:={X=x+x0,T=t};
tr12:=solve(tr11,{x,t}); Eq1:=U[t]+U[]*U[x]+U[x,x,x];
Eq1T:=dchange(tr12,Eq1,[X,T],params=[x0]);
Eq2:=diff(u(x,t),t)-diff(F(u(x,t))*diff(u(x,t),x),x)=0;
tr21:={X=x,T=t+t0}; tr22:=solve(tr21,{x,t});
Eq2T:=dchange(tr22,Eq2,[X,T],params=[t0]);
Sys1:=[V[t]-F(u(x,t))*U[x]-G(u(x,t))=0, U[t]-V[x]=0];
tr3:={X=x,T=t+t0}; tr31:=solve(tr3,{x,t});
Sys1T:=dchange(tr31,Sys1,[X,T],params=[t0]);
```

Mathematica:

```
{tr11={xN==x+x0,tN==t}, tr12=Solve[tr11,{x,t}]}
eq1[x_,t_]:=D[u[x,t],t]+u[x,t]*D[u[x,t],x]+D[u[x,t],
 {x,3}]; eq1T[v_]:=First[((eq1[x,t]/.u->Function[{x,t},
 u[x+x0,t]])/.tr12)/.{u->v}];  {eq1[x,t], Simplify[eq1T[u]]}
{tr21={xN==x,tN==t+t0}, tr22=Solve[tr21,{x,t}]}
eq2[x_,t_]:=D[u[x,t],t]-D[f[u[x,t]]*D[u[x,t],x],x]==0; eq2[x,t]
eq2T[v_]:=First[((eq2[x,t]/.u->Function[{x,t},u[x,t+t0]])/.
 tr22)/.{u->v}]; Simplify[eq2T[u]]
{tr3={xN==x,tN==t+t0}, tr31=Solve[tr3,{x,t}]}
sys1[x_,t_]:={D[v[x,t],t]-f[u[x,t]]*D[u[x,t],x]-g[u[x,t]]==0,
 D[u[x,t],t]-D[v[x,t],x]==0}; sys1[x,t]//Simplify
sys1T[w1_,w2_]:=First[((sys1[x,t]/.u->Function[{x,t},
 u[x,t+t0]]/.v->Function[{x,t},v[x,t+t0]])/.tr31)/.
 {u->w1,v->w2}]; Simplify[sys1T[u,v]]
```

□

Problem 2.4

Maple:

```
with(PDETools): declare(u(x,t)); U:=diff_table(u(x,t));
tr1:={X=a*x,T=a*t}; tr2:=solve(tr1,{x,t});
Eq1:=U[t,t]=U[]^2*U[x,x]+U[]*U[x]^2;
Eq2:=dchange(tr2,Eq1,[X,T],params=[a]); Eq3:=expand(Eq2/a^2);
```

Mathematica:

```
{tr1={xN==a*x,tN==a*t}, tr2=Solve[tr1,{x,t}]}
eq1[x_,t_]:=D[u[x,t],{t,2}]==u[x,t]^2*D[u[x,t],
 {x,2}]+u[x,t]*D[u[x,t],x]^2; eq1[x,t]
eq1T[v_]:=First[((eq1[x,t]/.u->Function[{x,t},u[a*x,a*t]])/.
 tr2)/.{u->v}]; Expand[Thread[eq1T[u]/a^2,Equal]]
```

□

Problem 2.5

Maple:

```
with(PDETools): var:=x,y,t; varN:=X,Y,t; declare((u,v)(var),
 (F1,F2)(u(var),v(var)),(u,v)(varN),(G1,G2)(u(varN),v(varN)));
tr1:={X=x*cos(xi)+y*sin(xi),Y=-x*sin(xi)+y*cos(xi)};
tr2:=simplify(solve(tr1,{x,y})); f1:=u(var),v(var);
f2:=u(varN),v(varN); L1:=[F1(f2),F2(f2)]; L2:=[u(varN),v(varN)];
Sys1:=[diff(u(var),t)+diff(u(var)*F1(f1),x)+diff(u(var)*F2(f1),
 y)=0, diff(v(var),t)+diff(v(var)*F1(f1),x)+diff(v(var)*F2(f1),
 y)=0]; Sys2:=expand(dchange(tr2,Sys1,[X,Y],params=[xi]));
tr3:=[cos(xi)*F1(f2)+sin(xi)*F2(f2)=G1(f2),
     -sin(xi)*F1(f2)+cos(xi)*F2(f2)=G2(f2)];
for i to 2 do tr3||i:=rhs(tr3[i])=lhs(tr3[i]); od;
for i to 2 do Sys2Eq||i:=collect(simplify(Sys2[i],tr3),L1);
              Eq1||i:=diff(L2[i]*subs(tr31,G1(f2)),X);
              Eq2||i:=diff(L2[i]*subs(tr32,G2(f2)),Y);
        Eqs12||i:=simplify(diff(L2[i],t)+Eq1||i+Eq2||i,tr3); od;
Test1:=Eqs121-lhs(Sys2Eq1); Test2:=Eqs122-lhs(Sys2Eq2);
```

Mathematica:

```
{var=Sequence[x,y,t], varN=Sequence[xN,yN,t]}
{tr1={xN==x*Cos[xi]+y*Sin[xi],yN==-x*Sin[xi]+y*Cos[xi]},
 tr2=Simplify[Solve[tr1,{x,y}]], f1=Sequence[u[var],v[var]],
 f2=Sequence[u[varN],v[varN]], l1={fF1[f2],fF2[f2]},
 l2={u[varN],v[varN]}}
sys1[x_,y_,t_]:={D[u[var],t]+D[u[var]*fF1[f1],x]+D[
 u[var]*fF2[f1],y]==0, D[v[var],t]+D[v[var]*fF1[f1],x]+D[
 v[var]*fF2[f1],y]==0}; sys1[x,y,t]//Expand
sys2[w1_,w2_]:=First[((sys1[x,y,t]/.u->Function[{x,y,t},
 u[x*Cos[xi]+y*Sin[xi],-x*Sin[xi]+y*Cos[xi],t]]/.v->Function[
 {x,y,t},v[x*Cos[xi]+y*Sin[xi],-x*Sin[xi]+y*Cos[xi],t]])/.tr2)/.
 {u->w1,v->w2}]; sys2[u,v]//FullSimplify
tr3={Cos[xi]*fF1[f2]+Sin[xi]*fF2[f2]->gG1[f2],
    -Sin[xi]*fF1[f2]+Cos[xi]*fF2[f2]->gG2[f2]}
Do[tTr3[i]=tr3[[i,2]]->tr3[[i,1]]; Print[tTr3[i]],{i,1,2}];
Do[sys2Eq[i]=Collect[Simplify[sys2[u,v][[i]],tr3],l1];
 eq1[i]=D[l2[[i]]*(gG1[f2]/.tTr3[1]),xN];
 eq2[i]=D[l2[[i]]*(gG2[f2]/.tTr3[2]),yN];
 eqs12[i]=Simplify[D[l2[[i]],t]+eq1[i]+eq2[i],tr3];
 Print[sys2Eq[i]]; Print[Eq1[i]]; Print[eq2[i]];
 Print[eqs12[i]],{i,1,2}]; Map[Expand,{test1=
 eqs12[1]-sys2Eq[1][[1]], test2=eqs12[2]-sys2Eq[2][[1]]}]
```

□

Problem 2.6

Maple:

```
with(PDEtools): var:=x,t; declare((u,v,w1,w2)(x,t));
U,V:=diff_table(u(x,t)),diff_table(v(x,t)); T:=exp(-V[]/4);
tr1:={w1(var)=2*U[]*T,w2(var)=4*T}; tr2:=solve(tr1,{U[],V[]});
Sys1:={V[x]-2*U[]=0, V[t]-2*U[x]+U[]^2=0};
Eq1:=dchange(tr2,Sys1,[w1(var),w2(var)]);
Sys2:=convert(expand(map(`*`,Eq1,w2(var))),list);
tr3:=isolate(Sys2[1],w1(var));
Sys3:=[Sys2[1],subs(diff(w1(var),x)=W1,Sys2[2])];
Sys4:=simplify([Sys3[1],eval(Sys3[2],tr3)]/(-4));
SysFin:=eval(Sys4,W1=diff(w1(var),x));
```

Mathematica:

```
{var=Sequence[x,t], tN=Exp[-v[x,t]/4], tr1={w1[var]==2*u[x,t]*tN,
 w2[var]==4*tN}, tr2={Reduce[tr1,{u[x,t],v[x,t]}]/.C[1]->0},
 varsN=Sequence[tr2[[1,2,2]],tr2[[1,3,2]]]}
sys1={(D[#2,x]-2*(#1))&[varsN]==0, (D[#2,t]-2*D[#1,x]+
 (#1)^2)&[varsN]==0}; eq1=Expand[sys1]
sys2=Expand[Table[Thread[eq1[[i]]*w2[x,t],Equal],{i,1,2}]]
tr3=Solve[sys2[[1]],w1[var]]
sys3={sys2[[1]],sys2[[2]]/.D[w1[var],x]->wW1}
sys4=Simplify[{sys3[[1]],sys3[[2]]/.tr3}]
sysFin=Flatten[sys4/.wW1->D[w1[var],x]]
```

□

2.1.2 Hodograph Transformation

Problem 2.7

Maple:

```
with(PDEtools): declare(u(x,t),x(t,u)); U,X:=diff_table(u(x,t)),
 diff_table(x(t,u)); Eq1:=U[t]*U[x]^2=F(t,u)*U[x,x];
tr1:=diff(x,x)=X[u]*U[x]; tr2:=diff(x,t)=X[u]*U[t]+X[t];
tr3:=diff(lhs(tr1),x)=X[u,u]*U[x]^2+X[u]*U[x,x];
tr11:=U[x]=solve(tr1,U[x]); tr21:=U[t]=solve(tr2,U[t]);
tr31:=U[x,x]=solve(tr3,U[x,x]); tr32:=lhs(tr31)=subs(tr11,
 rhs(tr31)); HodographTr:={tr11,tr21,tr32};
Eq2:=subs(HodographTr,Eq1); Eq3:=Eq2*(-denom(lhs(Eq2)));
```

Mathematica:

```
eq1=D[u[x,t],t]*D[u[x,t],x]^2==f[t,u]*D[u[x,t],{x,2}]
{tr1=D[x,x]==D[x[t,u],u]*D[u[x,t],x], tr2=D[x,t]==D[x[t,u],u]*
 D[u[x,t],t]+D[x[t,u],t], tr3=D[tr1[[1]],x]==D[x[t,u],{u,2}]*
 D[u[x,t],x]^2+D[x[t,u],u]*D[u[x,t],{x,2}]}
Map[Flatten,{tr11=Solve[tr1,D[u[x,t],x]], tr21=Solve[tr2,
 D[u[x,t],t]],tr31=Solve[tr3,D[u[x,t],{x,2}]]}]
{tr32=tr31/.tr11, hodographTr={tr11,tr21,tr32}//Flatten,
 eq2=eq1/.hodographTr}
eq3=Thread[eq2*Denominator[eq2[[1]]],Equal]//Simplify
```

□

2.2 Contact Transformations

2.2.1 Legendre Transformation

Problem 2.8

Maple:

```
with(PDEtools): declare(u(x,t),w(xi,eta));
alias(u=u(x,t),w=w(xi,eta));
Eq1:=F(diff(u,x),diff(u,t))*diff(u,x$2)+G(diff(u,x),diff(u,t))
 *diff(u,x,t)+H(diff(u,x),diff(u,t))*diff(u,t$2)=0;
LegendreTr:={x=diff(w,xi),t=diff(w,eta),
             u=-w+diff(w,xi)*xi+diff(w,eta)*eta};
Eq2:=dchange(LegendreTr,Eq1,[xi,eta,w]);
Eq3:=simplify(Eq2); Eq4:=numer(lhs(Eq3))=rhs(Eq1);
```

Mathematica:

```
{var=Sequence[x,t],varN=Sequence[xi,eta]}
j1=D[u[var],{x,2}]*D[u[var],{t,2}]-D[D[u[var],x],t]^2
legTr={x->D[w[varN],xi],t->D[w[varN],eta],u[var]->
 -w[varN]+D[w[varN],xi]*xi+D[w[varN],eta]*eta}
legD1={D[u[var],x]->xi,D[u[var],t]->eta}
legD2={D[u[var],{x,2}]->j1*D[w[varN],{eta,2}],D[D[u[var],x],t]->
 -j1*D[D[w[varN],xi],eta],D[u[var],{t,2}]->j1*D[w[varN],{xi,2}]}
legendreTr={legTr,legD1,legD2}//Flatten
{eq1=f[D[u[var],x],D[u[var],t]]*D[u[var],{x,2}]+g[D[u[var],x],
 D[u[var],t]]*D[u[var],x,t]+h[D[u[var],x],D[u[var],t]]*
 D[u[var],{t,2}]==0, eq2=eq1/.legendreTr, eq3=eq2//FullSimplify}
eq4=eq3[[1,2]]==eq3[[2]]
```

□

2.2.2 Euler Transformation

Problem 2.9

Maple:

```
with(PDEtools): declare(u(x,t),w(xi,eta)); alias(u=u(x,t),
 w=w(xi,eta)); Eq1:=diff(u,t)*diff(u,x$2)=F(t,diff(u,x));
EulerTr:={u=-w+diff(w,xi)*xi,x=diff(w,xi),t=eta};
Eq2:=dchange(EulerTr,Eq1,[xi,eta,w]); Eq3:=simplify(Eq2);
```

Mathematica:

```
{var=Sequence[x,t],varN=Sequence[xi,eta]}
{eTr={u[var]->-w[varN]+D[w[varN],xi]*xi,x->D[w[varN],xi],t->eta},
 eD1={D[u[var],x]->xi, D[u[var],t]->-D[w[varN],eta]},
 eD2={D[u[var],{x,2}]->1/D[w[varN],{xi,2}], D[u[var],x,t]->
 -D[D[w[varN],xi],eta]/D[w[varN],{xi,2}], D[u[var],{t,2}]->
 (D[w[varN],xi,eta]^2-D[w[varN],{xi,2}]*D[w[varN],{eta,2}])/D[
 w[varN],{xi,2}]}, eulerTr={eTr,eD1,eD2}//Flatten}
eq1=D[u[var],t]*D[u[var],{x,2}]==f[t,D[u[var],x]]
eq2=eq1/.eulerTr
```
□

2.3 Transformations Relating Differential Equations

2.3.1 Bäcklund Transformations

Problem 2.10

Maple:

```
with(PDEtools): declare((u,w)(x,t)); alias(u=u(x,t),w=w(x,t));
BT:=[diff(w,x)=1/2*w*u,diff(w,t)=1/2*diff(w*u,x)];
Eq1:=isolate(BT[1],u); HEq:=expand(subs(Eq1,BT[2]));
Eq3:=expand(HEq/w); Eq4:=Diff(lhs(Eq3),x)=Diff(diff(w,x)/w,t);
Eq5:=Diff(rhs(Eq3),x); Eq6:=diff(w,x)=solve(BT[1],diff(w,x));
Eq7:=diff(w,x)/w=subs(Eq6,diff(w,x)/w);
Eq8:=subs(lhs(Eq7)^2=rhs(Eq7)^2,diff(Eq7,x));
Eq9:=isolate(Eq8,diff(w,x,x)/w); Eq10:=value(subs(Eq9,Eq5));
Eq11:=value(subs(Eq7,rhs(Eq4))); BurEq:=factor((Eq10=Eq11)*2);
```

Mathematica:

```
var=Sequence[x,t]; trBT={D[w[var],x]==1/2*w[var]*u[var],
                        D[w[var],t]==1/2*D[w[var]*u[var],x]}
{eq1=Solve[trBT[[1]],u[var]], eq11=D[eq1,x]//Flatten,
 heatEq=Simplify[trBT[[2]]/.eq1/.eq11]//First,
 eq3=Thread[heatEq/w[var],Equal]//Expand}
{eq4=HoldForm[D[eq3[[1]],x]==D[D[w[var],x]/w[var],t]],
 eq5=HoldForm[D[eq3[[2]],x]]}
```

```
{eq6=D[w[var],x]==Solve[trBT[[1]],D[w[var],x]][[1,1,2]],
 eq7=Thread[eq6/w[var],Equal],
 eq8=Thread[D[eq7,x],Equal]/.eq7[[1]]^2->eq7[[2]]^2,
 eq9=Thread[eq8-eq8[[1,1]],Equal]//ToRules,
 eq10=D[eq3[[2]]/.eq9,x],
 eq11=D[D[w[var],x]/w[var]/.ToRules[eq7],t]}
burgersEq=Factor[Thread[(eq10==eq11)*2,Equal]]
```
□

Problem 2.11

Maple:

```
interface(showassumed=0); assume(a<>0); with(PDEtools):
declare((u,v)(x,t)); alias(u=u(x,t),v=v(x,t));
SG:=diff(u,x,t)=sin(u); BT:=[diff(u-v,x)/2=beta*sin(1/2*(u+v)),
    diff(u+v,t)/2=sin(1/2*(u-v))/beta];
Eq1:=subs(BT,diff(BT[1],t)); Eq2:=subs(BT,diff(BT[2],x));
SGEqu:=combine(Eq2+Eq1,trig); SGEqv:=combine(Eq2-Eq1,trig);
```

Mathematica:

```
sGEq[u_]:=D[u,{x,t}]==Sin[u];
trBT={D[u[x,t]-v[x,t],x]/2->beta*Sin[1/2*(u[x,t]+v[x,t])],
      D[u[x,t]+v[x,t],t]/2->Sin[1/2*(u[x,t]-v[x,t])]/beta}
{eq1=D[trBT[[1]],t]/.trBT, eq2=D[trBT[[2]],x]/.trBT}
sGEqu=(eq2[[1]]+eq1[[1]]==eq2[[2]]+eq1[[2]])//FullSimplify
sGEqv=(eq2[[1]]-eq1[[1]]==eq2[[2]]-eq1[[2]])//FullSimplify
```

Maple:

```
with(plots): setoptions(plot,scaling=constrained,thickness=3);
Eq3:=subs(v=0,factor(BT*2)); I1:=int(1/sin(U/2),U);
Eq4:=int(2*beta,x)=convert(simplify(convert(I1,tan)),tan)+A(t);
Eq5:=int(2/beta,t)=convert(simplify(convert(I1,tan)),tan)+B(x);
t1:=tan(U/4); Eq6:=expand(isolate(Eq4,t1));
Eq7:=expand(isolate(Eq5,t1)); Eq8:=lhs(Eq6)=rhs(Eq6)*rhs(Eq7);
C:=select(has,rhs(Eq8),[A,B]); Eq9:=lhs(Eq8)=alpha*rhs(Eq8/C);
Eq10:=u=solve(Eq9,U); KinkSol:=unapply(rhs(Eq10),x,t,alpha,beta);
alpha1:=0.1; beta1:=10; xR:=-Pi..Pi; tR:=0..1;
plot3d(evalf(KinkSol(x,t,alpha1,beta1)),x=xR,t=tR,
 orientation=[71,73]); plot(evalf(KinkSol(x,0,alpha1,beta1)),
 x=xR,tickmarks=[spacing(Pi/2),spacing(Pi/2)]);
```

Mathematica:

```
Off[Solve::"ifun"]; SetOptions[Plot,ImageSize->500,PlotStyle->
 {Hue[0.9],Thickness[0.01]},PlotPoints->100,PlotRange->All];
SetOptions[Plot3D,BoxRatios->{1,1,1},PlotRange->All,ViewPoint->
 {-1,2,2}]; eq3=trBT/.{v[x,t]->0,D[v[x,t],x]->0,D[v[x,t],t]->0}
{eq4=Assuming[Cos[uN/4]>0,Simplify[Integrate[2*beta,x]==
 Integrate[1/Sin[uN/2],uN]+a[t]]], eq5=Assuming[Cos[uN/4]>0,
 Simplify[Integrate[2/beta,t]-Integrate[1/Sin[uN/2],uN]==b[x]]]}
{eq6=Solve[eq4,Tan[uN/4]], eq7=Solve[eq5,Tan[uN/4]]}
eq8=eq6[[1,1,1]]==eq6[[1,1,2]]*eq7[[1,1,2]]
const=Exp[Select[eq8[[2,2]],MemberQ[#1,a[t]]||MemberQ[#1,b[x]]&]]
eq9=eq8[[1]]==alpha*(Thread[eq8/const,Equal])[[2]]
eq10=Solve[eq9,uN]
kinkSol[xN_,tN_,alphaN_,betaN_]:=eq10[[1,1,2]]/.{x->xN,t->tN,
 alpha->alphaN,beta->betaN}; {alpha1=0.1, beta1=10}
Plot3D[N[kinkSol[x,t,alpha1,beta1]],{x,-Pi,Pi},{t,0,1}]
Plot[N[kinkSol[x,0,alpha1,beta1]],{x,-Pi,Pi}]
```

Maple:

```
c1:=1; Eq11:=simplify(subs({x=(X+c*T)/2,t=(X-c*T)/2},rhs(Eq10)));
tr1:=[beta=epsilon*sqrt((1-U)/(1+U)),c=U*m/(2*(beta-1/beta))];
Eq12:=simplify(subs(tr1[1],simplify(subs(tr1,Eq11))));
KinkSols:=unapply(Eq12,m,U,epsilon,T,X,alpha);
m1:=beta->1/2*(beta+1/beta);
U1:=(c,beta)->c*((beta^2-1)/(beta^2+1)); m1(beta1); U1(c1,beta1);
Kink:=plot(evalf(KinkSols(m1(beta1),U1(c1,beta1),1,0,X,alpha1)),
 X=xR,tickmarks=[spacing(Pi/2),spacing(Pi/2)]):
AntiKink:=plot(evalf(KinkSols(m1(beta1),U1(c1,beta1),-1,0,X,
 alpha1)),X=xR,color=blue,tickmarks=[spacing(Pi/2),
 spacing(Pi/2)]): display({Kink,AntiKink});
```

Mathematica:

```
c1=1; {eq11=eq10[[1,1,2]]/.{x->(xN+c*tN)/2,t->(xN-c*tN)/2}
 //FullSimplify, tr1={beta->\[Epsilon]*Sqrt[(1-uN)/(1+uN)],
 c->uN*m/(2*(beta-1/beta))}}
eq12=Assuming[uN>1,Simplify[(eq11/.tr1)/.tr1[[1]]]]
kinkSols[mN1_,uN1_,epsilonN1_,tN1_,xN1_,alphaN1_]:=eq12/.{m->mN1,
 uN->uN1,\[Epsilon]->epsilonN1,tN->tN1,xN->xN1,alpha->alphaN1};
m1[beta_]:=1/2*(beta+1/beta); v1[c_,beta_]:=
 c*((beta^2-1)/(beta^2+1)); {m1[beta1],v1[c1,beta1]}
```

```
kink=Plot[N[kinkSols[m1[beta1],v1[c1,beta1],1,0,xN,alpha1]],
 {xN,-Pi,Pi}]; antiKink=Plot[N[kinkSols[m1[beta1],v1[c1,beta1],
 -1,0,xN,alpha1]],{xN,-Pi,Pi},PlotStyle->{Hue[0.7],
 Thickness[0.01]}]; Show[{kink,antiKink}]
```
□

2.3.2 Miura Transformation

Problem 2.12

Maple:

```
with(PDEtools): declare((u,v)(x,t)); alias(u=u(x,t),v=v(x,t));
mKdVEq:=diff(v,t)-6*v^2*diff(v,x)+diff(v,x$3)=Mv;
KdVEq:=diff(u,t)-6*u*diff(u,x)+diff(u,x$3)=0;
trMiura:=u=v^2+diff(v,x); Eq1:=algsubs(trMiura,KdVEq);
subs(mKdVEq,Eq1); MvL:=lhs(mKdVEq); Eq2:=2*v*Mv +Diff(Mv,x)=0;
Eq3:=expand(2*v*MvL+diff(MvL,x))=0; evalb(Eq1=Eq3);
```

Mathematica:

```
var=Sequence[x,t]
mKdVEq=D[v[var],t]-6*v[var]^2*D[v[var],x]+D[v[var],{x,3}]->mv
kdVEq=D[u[var],t]-6*u[var]*D[u[var],x]+D[u[var],{x,3}]==0
trMiura=u[var]->v[var]^2+D[v[var],x]
trMD={D[trMiura,t],D[trMiura,x],D[trMiura,{x,3}]}
{eq1=kdVEq/.trMiura/.trMD//Expand,eq1/.mKdVEq,mvL=mKdVEq[[1]]}
{eq2=2*v[var]*mv+HoldForm[D[mv,x]]==0,
 eq3=Expand[2*v[var]*mvL+D[mvL,x]]==0, eq1===eq3}
```
□

2.3.3 Gardner Transformation

Problem 2.13

Maple:

```
with(PDEtools): declare((u,v,w)(x,t)); alias(u=u(x,t),v=v(x,t),
 w=w(x,t)); trMi:=u=v^2+diff(v,x);
KdV:=diff(u,t)-6*u*diff(u,x)+diff(u,x$3)=0;
trv:=v=1/(2*epsilon)+epsilon*w; Eq1:=expand(algsubs(trv,trMi));
trGar:=subs(1/epsilon^2=0,Eq1);
Eq2:=collect(expand(subs(trGar,KdV)),diff);
Eq3:=map(factor,lhs(Eq2)); tr1:=1+2*epsilon^2*w=G;
Eq4:=collect(subs(tr1,Eq3),G);
```

```
Eq5:=collect(expand((Eq4-op(1,Eq4))/epsilon),diff);
term1:=op(1,op(2,Eq5))=G1;
EqGar:=subs(G1=lhs(term1),map(int,subs(term1,Eq5),x))=0;
Eq6:=expand(subs(G=lhs(tr1),G*EqGar+epsilon*diff(EqGar,x)));
Test1:=expand(Eq6-Eq2);
Test2:=evalb(KdV=algsubs(w=u,subs(epsilon=0,EqGar)));
```

Mathematica:

```
{var=Sequence[x,t], trMi:=u[var]->v[var]^2+D[v[var],x]}
kdV=D[u[var],t]-6*u[var]*D[u[var],x]+D[u[var],{x,3}]==0
trv=v[var]->1/(2*epsilon)+epsilon*w[var]
{eq1=Expand[trMi/.trv/.D[trv,x]],trGar=eq1/.{1/epsilon^2->0},
 eq2=Expand[kdV/.trGar/.D[trGar,t]/.D[trGar,x]/.D[trGar,
 {x,3}]], tD1=Table[{D[w[x,t],{x,i}],D[w[x,t],{t,i}]},
 {i,1,3}]//Flatten, eq3=Collect[eq2[[1]],tD1]}
{eq31=Map[Factor,eq3], tr1=1+2*epsilon^2*w[var]->g}
{eq4=Collect[eq31/.tr1,g], eq5=Collect[Expand[(eq4-eq4[[3]])
 /epsilon],tD1], term1=eq5[[2,1]]->g1}
eqGar=(Integrate[eq5/.term1,x]/.{g1->term1[[1]]})==0
eq6=Thread[Thread[g*eqGar,Equal]+Thread[epsilon*Thread[
 D[eqGar,x],Equal],Equal],Equal]/.{g->tr1[[1]]}//Expand
test1=Thread[eq6-eq2,Equal]//Expand
test2=kdV===(eqGar/.{epsilon->0}/.w->u)
```
□

2.4 Linearizing and Bilinearizing Transformations

2.4.1 Hopf–Cole Transformation

Problem 2.14

Maple:

```
with(PDEtools): declare((u,psi,phi)(x,t)); alias(u=u(x,t),
 psi=psi(x,t),phi=phi(x,t));
BEq:=diff(u,t)+u*diff(u,x)=nu*diff(u,x$2);
ConservLaw:=diff(u,t)+Diff((u^2/2-nu*diff(u,x)),x)=0;
Eq1:=u=Diff(psi,x); Eq2:=-op(1,op(2,lhs(ConservLaw)))
 =Diff(psi,t); Eq3:=algsubs(Eq1,Eq2);
tr1:={psi=-2*nu*log(phi)}; tr2:=diff(psi,x)=value(subs(tr1,Eq1));
Eq4:=[lhs(tr2)=rhs(rhs(tr2)),diff(psi,x$2)=diff(rhs(rhs(tr2)),x),
 diff(op(tr1),t)]; Eq5:=expand(subs(Eq4,value(Eq3)));
HEq:=Eq5*(-1/2)*phi/nu;
```

Mathematica:

```
burgersEq=D[u[x,t],t]+u[x,t]*D[u[x,t],x]==nu*D[u[x,t],{x,2}]
{conservLaw=D[u[x,t],t]+Hold[D[(u[x,t])^2/2-nu*D[
 u[x,t],x],x]]==0, eq1=u[x,t]->Hold[D[psi[x,t],x]],
 eq2=conservLaw[[1,1]]==Hold[D[psi[x,t],t]]//Simplify}
{eq3=eq2/.eq1, l3=Level[eq3,{3}], eq31=-l3[[1]]==eq3[[2]]}
{tr1=psi[x,t]->-2*nu*Log[phi[x,t]],
 tr2=D[psi[x,t],x]==ReleaseHold[(eq1/.tr1)]}
{eq4={tr2[[1]]->(tr2[[2]])[[2]],D[psi[x,t],{x,2}]->
 D[(tr2[[2]])[[2]],x],D[tr1,t]}, eq5=(ReleaseHold[eq31]
 /.eq4)//Expand, heatEq=Thread[eq5*(-1/2)*phi[x,t]/nu,Equal]}
```

Maple:

```
with(PDEtools): declare((u,phi)(x,t)); alias(u=u(x,t),
 phi=phi(x,t)); BEq:=diff(u,t)+u*diff(u,x)-nu*diff(u,x$2)=0;
tr1:={u=-2*nu*diff(phi,x)/phi}; Eq1:=collect(expand(
 subs(tr1,BEq)),diff); Eq2:=expand(Eq1*phi^2/2);
Eq11:=expand(-diff(1/phi*(diff(phi,t)-nu*diff(phi,x$2)),x)=0);
Eq21:=expand(Eq11*phi^2*nu); Eq2-Eq21;
Eq3:=int(-Diff(1/phi*(diff(phi,t)-nu*diff(phi,x$2)),x),x)=f(t);
Eq4:=Eq3*phi; tr2:=phi=exp(int(-f(t),t))*F(x,t);
Eq5:=expand(algsubs(tr2,Eq4)/op(1,rhs(tr2))); Eq6:=Eq5-rhs(Eq5);
Sol1:=combine(pdsolve(Eq6,F(x,t),explicit));
Sol2:=subs(Sol1,tr2); SolFin:=simplify(subs(Sol2,tr1));
Test1:=pdetest(SolFin,BEq);
```

Mathematica:

```
burgersEq=D[u[x,t],t]+u[x,t]*D[u[x,t],x]-nu*D[u[x,t],{x,2}]==0
{tr1=u[x,t]->-2*nu*D[phi[x,t],x]/phi[x,t], tr11=D[tr1,t],
 tr12=Table[D[tr1,{x,i}],{i,1,2}]//Flatten}
{eq1=(burgersEq/.tr1/.tr11/.tr12)//Expand,
 eq2=Thread[eq1*phi[x,t]^2/2,Equal]//Expand,
 eq11=-D[1/phi[x,t]*(D[phi[x,t],t]-nu*D[phi[x,t],
 {x,2}]),x]==0, eq21=Thread[eq11*phi[x,t]^2*nu,Equal],
 eq22=eq2[[1]]-eq21[[1]]//Expand}
{eq3=Integrate[-D[1/phi[x,t]*(D[phi[x,t],t]-nu*D[phi[x,t],
 {x,2}]),x],x]==f[t], eq4=Thread[eq3*phi[x,t],Equal]//Expand}
tr2=phi[x,t]->Exp[Integrate[-f[t],t]]*fF[x,t]
{eq5=Expand[Thread[(eq4/.tr2/.D[tr2,t]/.D[tr2,{x,2}])
 /tr2[[2,1]],Equal]],eq6=Expand[Thread[eq5-eq5[[2]],Equal]]}
```

```
DSolve[eq6,fF,{x,t}]
{sol1=fF[x,t]->c[3]*c[1]*Exp[Sqrt[s[1]]*x+nu*s[1]*t]+
 c[3]*c[2]*Exp[-Sqrt[s[1]]*x+nu*s[1]*t], sol2=ToRules[
 tr2/.Rule->Equal/.sol1], solFin=tr1/.sol2/.D[sol2,x]}
test1=burgersEq/.solFin/.D[solFin,t]/.Table[D[solFin,{x,i}],
 {i,1,2}]//Simplify
```
□

2.4.2 Hopf–Cole-type Transformation

Problem 2.15

Maple:

```
with(PDEtools): declare((u,w,F,f,g)(x,t)); alias(u=u(x,t),
 w=w(x,t),F=F(x,t),f=f(x,t),g=g(x,t)); tr1:=u=diff(w,x$2);
PDE1:=u->diff(u,t)+6*u*diff(u,x)+diff(u,x$3)=0;
Eq1:=expand(PDE1(rhs(tr1))); Eq2:=map(int,Eq1,x);
tr2:=w=alpha*log(F);
Eq3:=collect(simplify(algsubs(tr2,Eq2))*F^4/alpha,[diff,F]);
Eq4:=expand(subs(alpha=2,Eq3)/F^2);
```

Mathematica:

```
var=Sequence[x,t]; tr1=u[var]->D[w[var],{x,2}]
pde1[u_]:=D[u,t]+6*u*D[u,x]+D[u,{x,3}]==0;
{eq1=pde1[tr1[[2]]]//Expand, eq2=Thread[Integrate[eq1,x],
 Equal], tr2=w[var]->alpha*Log[f[var]]}
eq3=eq2/.tr2/.D[D[tr2,x],t]/.Table[D[tr2,{x,i}],{i,1,4}]
eq31=Expand[Thread[eq3*f[var]^4/alpha,Equal]]
eq4=Thread[(eq3/.alpha->2)*f[x,t]^2/2,Equal]//Expand
```
□

2.5 Reductions of Nonlinear PDEs

2.5.1 Traveling Wave Reductions

Maple:

```
LinWaveEq:=diff(u(x,t),x$2)-(1/c^2)*diff(u(x,t),t$2)=0;
IC1:=u(x,0)=f(x); IC2:=D[2](u)(x,0)=g(x);
sysD:=[LinWaveEq,IC1,IC2]; pdsolve(sysD);
```

Problem 2.16

Maple:

```
with(PDEtools): declare(u(x,t),U(z)); interface(showassumed=0);
assume(nu>0,U(z)>0,u1>0,u2>0,u1>u2); A:=2*nu;
tr1:={lambda=1/2*(u1+u2),C=-1/2*u1*u2}; tr2:=U(z)-u1=u1-U(z);
tr3:=x-lambda*t=z; Eq1:=u->diff(u,t)+u*diff(u,x)-nu*diff(u,x$2);
Eq2:=Eq1(U(x-lambda*t)); Eq3:=convert(algsubs(tr3,Eq2),diff);
Eq4:=int(Eq3,z)=C; dU:=diff(U(z),z);
Eq41:=simplify(isolate(Eq4,diff(U(z),z)));
Sol1:=solve(Eq4,dU) assuming z<>0;
Sol2:=(dU=factor(subs(tr1,Sol1)))*A; Sol3:=Sol2/rhs(Sol2)/A*dz;
Sol3L:=lhs(Sol3); Sol3R:=rhs(Sol3); Sol4:=combine(int(Sol3L,z),
 symbolic)=int(coeff(Sol3R,dz),z); Sol5:=subs(tr2,Sol4);
Sol6:=solve(1=solve(Sol5,dz),U(z)) assuming z<>0;
SolU:=unapply(Sol6,z,u1,u2,nu); SolU(0,5,1,0.3);
plot(SolU(z,5,1,0.3),z=-5..5,0..6,thickness=3);
plot(SolU(z,5,1,0.001),z=-1..1,0..6,thickness=3);
```

Mathematica:

```
SetOptions[Plot,ImageSize->300,PlotPoints->100,PlotStyle->
 {Hue[0.9],Thickness[0.01]},PlotRange->{All,{0,6}}];
{a=2*nu, tr1={lambda->1/2*(u1+u2), c->-1/2*u1*u2},
 tr2=uN[z]-u1->u1-uN[z], tr3=x-lambda*t->z}
eq1[u_]:=D[u,t]+u*D[u,x]-nu*D[u,{x,2}]; eq2=eq1[uN[x-lambda*t]]
{eq3=eq2/.tr3, eq4=Integrate[eq3,z]==c, duN=D[uN[z],z],
 eq41=Reduce[eq4,D[uN[z],z]]//Simplify, sol1=Assuming[z!=0,
 Solve[eq4,duN]], sol2=Thread[(Factor[sol1/.tr1][[1,1]])*a,Rule]}
sol3=Thread[sol2/sol2[[2]]/a*dz,Rule]
{sol4=Integrate[sol3[[1]],z]==Integrate[Coefficient[
 sol3[[2]],dz],z], sol5=sol4/.tr2, sol6=Assuming[z!=0,
 Solve[1==Solve[sol5,dz][[1,1,2]],uN[z]]]//FullSimplify}
solU[zN_,uN1_,uN2_,nuN_]:=sol6[[1,1,2]]/.{z->zN,u1->uN1,
 u2->uN2,nu->nuN}; solU[0,5,1,0.3]
Show[GraphicsRow[{Plot[solU[zN,5,1,0.999],{zN,-Pi,Pi}],
 Plot[solU[zN,5,1,0.3],{zN,-Pi,Pi}],Plot[solU[zN,5,1,0.09],
 {zN,-Pi,Pi}]}]]
```

□

Problem 2.17

Maple:

```
with(PDEtools): declare(u(x,t),U(z)); tr1:=x+lambda*t=z;
Eq1:=u->-diff(u,t)+diff((F(u)*diff(u,x)),x);
Eq2:=expand(Eq1(U(lhs(tr1)))); Eq3:=algsubs(tr1,Eq2);
Eq31:=map(convert,Eq3,diff); Eq4:=diff(F(U(z))*diff(U(z),z),z);
Eq5:=convert(Eq4,diff); evalb(convert(op(2,Eq31)
 +op(3,Eq31),diff)=Eq5); Eq6:=-op(1,Eq31); Eq7:=int(Eq6,z);
Eq8:=int(diff(F(U(z))*diff(U(z),z),z),z); Eq9:=Eq7+C1=Eq8;
Eq10:=Eq9/(lhs(Eq9));
Eq11:=int(lhs(Eq10),z)+C2=Int(F(U(z))/denom(rhs(Eq10)),dU(z));
```

Mathematica:

```
eq1[u_]:=D[u,t]+D[f[u]*D[u,x],x]; tr1=x+lambda*t->z
{eq2=eq1[uN[tr1[[1]]]]//Expand, eq3=eq2/.tr1}
{eq4=D[f[uN[z]]*D[uN[z],z],z], eq3[[2]]+eq3[[3]]==eq4}
{eq6=-eq3[[1]], eq7=Integrate[eq6,z], eq8=Integrate[
 D[f[uN[z]]*D[uN[z],z],z],z], eq9=eq7+c1==eq8}
{eq10=Thread[eq9/eq9[[1]],Equal], eq11=Integrate[eq10[[1]],
 z]+c2==HoldForm[Integrate[f[uN[z]]/(c1-lambda*uN[z]),uN[z]]]}
```

□

Problem 2.18

Maple:

```
with(PDEtools): declare(u(x,t),U(z)); with(plots):
tr1:=x-c*t=z; tr2:={a=1,b=1,c=1}; A12:={A1=0,A2=0};
Eq1:=u->diff(u,t)+a*u*diff(u,x)+b*diff(u,x$3)=0;
Eq2:=expand(Eq1(U(lhs(tr1)))); Eq3:=algsubs(tr1,Eq2);
Eq4:=map(convert,Eq3,diff); Eq5:=map(int,lhs(Eq4),z)-A1=0;
Eq6:=expand(Eq5*2*diff(U(z),z)); Eq7:=map(int,Eq6,z);
Eq8:=lhs(Eq7)=A2; Eq9:=subs(A12,Eq8); Sol1:=[dsolve(Eq9,U(z))];
Sol11:=convert(subs(_C1=0,simplify(Sol1[2])),sech);
Sol2:=eval(subs(z=x-c*t,Sol11),tr2);
animate(plot,[rhs(Sol2),x=-20..20],t=0..20,numpoints=100,
 frames=50,thickness=2,color=blue);
Test1:=pdetest(u(x,t)=rhs(Sol2),subs(tr2,Eq1(u(x,t))));
```

Mathematica:

```
SetOptions[Plot,ImageSize->500,PlotStyle->{Hue[0.9],
 Thickness[0.01]}]; {tr1=x-c*t->z, tr2={a->1,b->1,c->1}}
eq1[u_]:=D[u,t]+a*u*D[u,x]+b*D[u,{x,3}]==0;
{eq2=eq1[uN[tr1[[1]]]]//Expand, eq3=eq2/.tr1,
 eq4=eq3//TraditionalForm, eq5=Integrate[eq3[[1]],z]-c1==0}
{eq6=Thread[eq5*2*D[uN[z],z],Equal]//Expand,
 eq7=Thread[Integrate[eq6,z],Equal], eq8=eq7[[1]]==c2,
 eq9=eq8/.{c1->0,c2->0}, sol1=DSolve[eq9,uN[z],z]}
{sol11=Simplify[sol1[[2]]]/.{C[1]->0}, sol2=(sol11/.{z->x-c*t})/.
 tr2, test1=(eq1[u]/.tr2)/.{u->sol2[[1,2]]}}
sol3[xN_,tN_]:=sol2[[1,2]]/.{x->xN,t->tN}; Animate[Plot[
 sol3[x,t],{x,-20,20},PlotRange->{All,{0,Pi}}],{t,0,20}]
```
□

Problem 2.19

Maple:

```
with(PDEtools): declare(u(x,y),U(z)); tr1:=x+lambda*y=z;
Eq1L:=u->diff(u,x,y)^2;  Eq1R:=u->diff(u,x$2)*diff(u,y$2);
Eq2L:=Eq1L(U(lhs(tr1))); Eq3L:=algsubs(tr1,Eq2L);
Eq2R:=Eq1R(U(lhs(tr1))); Eq3R:=algsubs(tr1,Eq2R);
Eq4:=convert(Eq3L=Eq3R,diff); evalb(Eq4);
```

Mathematica:

```
eq1L[u_]:=D[D[u,x],y]^2; eq1R[u_]:=D[u,{x,2}]*D[u,{y,2}];
{tr1=x+lambda*y->z, eq2L=eq1L[uN[tr1[[1]]]], eq3L=eq2L/.tr1}
{eq2R=eq1R[uN[tr1[[1]]]], eq3R=eq2R/.tr1, eq4=eq3L==eq3R}
```
□

Problem 2.20

Maple:

```
with(PDEtools): with(plots): declare(u(x,t),phi(xi));
alias(u=u(x,t),phi=phi(xi));  interface(showassumed=0);
assume(Phi>Phi0); tr1:=x-lambda*t=xi; tr2:=lambda^2/c^2=U^2;
xR:=-2*Pi..2*Pi; SGEq:=u->diff(u,x$2)-1/c^2*diff(u,t$2)=sin(u);
Eq1:=expand(SGEq(phi(lhs(tr1)))); Eq2:=expand(subs(tr1,Eq1));
Eq3:=convert(algsubs(tr2,Eq2),diff);
Eq4:=expand(normal(Eq3*diff(phi(xi),xi)/(1-U^2)));
Eq5:=Diff(1/2*(Diff(phi,xi)^2)+cos(phi)/(1-U^2),xi)=0;
```

```
Eq41:=lhs(Eq4)-rhs(Eq4)=0; factor(Eq41-value(Eq5));
Eq6:=int(value(op(1,Eq5)),xi)=B; Eq7:=normal(convert(isolate(
 Eq6,diff(phi,xi)),radical)); Eq7R:=rhs(Eq7);
Eq80:=subs(phi=psi,1/sqrt(collect(-numer(op(1,op(2,Eq7R))),B)));
Eq8:=Int(Eq80,psi=Phi0..Phi); Eq90:=op(1,rhs(Eq7))/sqrt(-denom(
 op(1,op(2,rhs(Eq7))))); Eq9:=int(Eq90,eta=xi0..xi);
Eq10:=Eq8=Eq9; Eq11:=subs({B*(1-U^2)=1},Eq10);
Eq12:=simplify(subs(1-cos(psi)=2*sin(psi/2)^2,Eq11),symbolic);
Eq13:=normal(convert(value(Eq12),tan)); Eq13L:=lhs(Eq13);
Eq14:=expand(op(3,op(3,Eq13L))*op(1,Eq13L)*op(2,Eq13L)
     =rhs(Eq13)); Sol:=expand(isolate(Eq14,Phi));
tr3:={lambda=0.1,c=10.}; tr4:={tan(Phi0/4)=1,xi0=0}; tr5:=x-U*t;
Kink:=unapply(subs(xi=tr5,expand(subs(tr4,rhs(Sol)))),x,t,U);
AntiKink:=unapply(subs(xi=-tr5,eval(rhs(Sol),tr4)),x,t,U);
U1:=subs(tr3,lhs(tr2)); K:=plot(Kink(x,0,U1),x=xR):
AK:=plot(AntiKink(x,0,U1),x=xR,color=blue): display({K,AK});
```

Mathematica:

```
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&]; Off[Solve::"ifun"];
SetOptions[Plot,ImageSize->500,PlotStyle->{Hue[0.9],
 Thickness[0.01]},PlotRange->{All,{0,2*Pi}}];
sGEq[u_]:=D[u,{x,2}]-1/c^2*D[u,{t,2}]==Sin[u];
{tr1=x-lambda*t->xi, tr2=lambda^2/c^2->uN^2, eq1=sGEq[phi[
 tr1[[1]]]]//Expand, eq2=eq1/.tr1//Expand, eq3=eq2/.tr2,
 eq4=Thread[eq3*D[phi[xi],xi]/(1-uN^2),Equal]//Simplify,
 eq5=Hold[D[1/2*(D[phi[xi],xi]^2)+Cos[phi[xi]]/(1-uN^2),xi]==0],
 eq41=eq4[[1]]-eq4[[2]]==0, eq41==ReleaseHold[Eq5],
 eq51=D[Level[eq5,{3}][[1]],xi]}
{eq6=Integrate[eq51,xi]==b//FullSimplify, eq7=Flatten[Solve[
 eq6,D[phi[xi],xi]]], eq7R=eq7[[2,2]], term7=trS3[eq7R,Sqrt[2]],
 term70=Factor[term7^2], term71=Numerator[term70],
 term72=Denominator[term70], integrd=1/Sqrt[Collect[-term71,b]]/.
 {phi[xi]->psi}, integrd1=integrd/.{b*(1-uN^2)->1}}
{integrd2=Assuming[Sin[psi/2]>0,PowerExpand[integrd1/.
 {1-Cos[psi]->2*Sin[psi/2]^2}]], eq8=Assuming[phi0<phi &&
 (phi<=0||phi0>0) && (phi<=-2*Pi||phi0>=-2*Pi) &&
 (phi<=2*Pi||phi0>=2*Pi),Integrate[integrd2,{psi,phi0,phi}]]}
eq90=trS1[eq7R,Sqrt[2]]/Sqrt[-term72]
{eq9=Integrate[eq90,{eta,xi0,xi}], eq10=eq8==eq9,
 sol=First[Solve[eq10,phi]], tr3={lambda->0.1,c->10.},
 tr4={Cot[phi0/4]->1,xi0->0,C[1]->0}, tr5=x-uN*t}
```

```
kink[xN_,tN_,uNU_]:=Expand[(sol[[1,2]]/.tr4)/.{xi->tr5}]/.{x->xN,
 t->tN,uN->uNU}; antiKink[xN_,tN_,uNU_]:=Expand[(sol[[1,2]]/.
 tr4)/.{xi->-tr5}]/.{x->xN,t->tN,uN->uNU}; uN1=tr2[[1]]/.tr3
kS=Plot[kink[x,0,uN1],{x,-2*Pi,2*Pi}]; aKS=Plot[
 antiKink[x,0,uN1],{x,-2*Pi,2*Pi},PlotStyle->{Hue[0.7],
 Thickness[0.01]}]; Show[{kS,aKS}]
```

□

2.5.2 Ansatz Methods

Problem 2.21

Maple:

```
with(PDEtools): declare(u(x,t),U(z));
alias(u=u(x,t),U=U(z)); interface(showassumed=0);
assume(n>1); f:=tanh; tr1:=mu*(x+c*t)=z; tr2:=f(z)=Y;
PDE1:=u->diff(u,t$2)-diff(u,x$2)+u-u^3=0;
Eq1:=expand(PDE1(U(lhs(tr1)))); Eq2:=convert(expand(
 subs(tr1,Eq1)),diff); tr3:=U(z)=S(lhs(tr2));
Ansatz1:=S(Y)=Sum(a[i]*Y^i,i=0..M);
Eq3:=convert(algsubs(tr2,algsubs(tr3,Eq2)),diff);
Eq31:=collect(Eq3,Y); tr5:=isolate(3*M=M+2,M);
tr6:=value(subs(tr5,Ansatz1)); Eq4:=algsubs(tr6,Eq3);
Eq41:=collect(Eq4,Y); sys1:={}; SolsT:={}:
for i from 0 to 3 do sys1:=sys1 union {coeff(lhs(Eq4),Y,i)
 =0}; od: sys1; vars:=indets(sys1) minus {c};
Sols:=[allvalues([solve(sys1,vars)])]; NSols:=nops(Sols);
for i from 1 to NSols do Op||i:=op(i,Sols): Nops:=nops(Op||i):
 for j from 1 to Nops do SolsT:=SolsT union {u=subs(
  op(j,Op||i),subs(z=lhs(tr1),subs(Y=lhs(tr2),rhs(tr6))))};
od: od: SolsT; SolsTF:=select(has,SolsT,f);
for i from 1 to nops(SolsTF) do
 simplify(PDE1(rhs(SolsTF[1]))); od;
```

Mathematica:

```
Off[Solve::"svars"]; pde1[u_]:=D[u,{t,2}]-D[u,{x,2}]+u-u^3==0;
{tr1=mu*(x+c*t)->z,tr2=Tanh[z]->yN, sys1={}, solsT={}}
{eq1=pde1[uN[tr1[[1]]]]//Expand, eq2=eq1/.tr1//Expand,"eq2="eq2}
{tr3=uN[z]->s[tr2[[1]]], tr31=D[uN[z],{z,2}]->D[s[tr2[[1]]],
 {z,2}], ansatz1=s[yN]->Sum[a[i]*yN^i,{i,0,mN}],
 "ansatz1="ansatz1, eq3=(eq2/.tr3)/.tr31, tr32=Table[
 (Sech[z]^2)^i->(1-Tanh[z]^2)^i,{i,1,2}]}
```

```
{eq31=(eq3/.tr32)/.tr2, eq32=Collect[eq31,yN], tr5=Solve[
 3*mN==mN+2,mN]//First, tr6=ansatz1/.tr5, tr61=Table[D[ansatz1,
 {yN,i}]/.tr5,{i,1,2}], eq4=(eq32/.tr6)/.tr61,"eq4="eq4}
Do[sys1=Union[sys1,{Coefficient[eq4[[1]],yN,i]==0}],{i,0,3}];
{sys1, vars=Complement[Variables[Table[sys1[[i]]//First,
 {i,1,Length[sys1]}]],{c}], sols=Solve[sys1,vars],
 nSols=Length[sols]}
Do[solsT=Union[solsT,{u[x,t]->((tr6[[2]]/.yN->tr2[[1]])/.z->
 tr1[[1]])/.sols[[i]]}],{i,nSols-3,nSols}]; solsT
Table[pde1[solsT[[i]][[2]]]//FullSimplify,{i,1,Length[solsT]}]
```

□

Problem 2.22

Maple:

```
with(PDEtools): with(plots): declare(u(x,t),U(z));
alias(u=u(x,t),U=U(z)); interface(showassumed=0); f:=cos:
PDE1:=u->diff(u,t$2)-diff(u,x$2)+u-u^3=0; tr1:=x+c*t=z;
tr2:=f(z)=Y; Eq1:=expand(PDE1(U(lhs(tr1))));
Eq2:=convert(expand(subs(tr1,Eq1)),diff);
AnsatzC:=U(z)=lambda*cos(mu*z)^beta; AnsatzS:=U(z)=
 lambda*sin(mu*z)^beta; Eq3:=algsubs(AnsatzC,Eq2);
Eq31:=expand(simplify(Eq3,trig),cos); Eq32:=simplify(Eq31,
 power); Eq33:=combine(subs(cos(mu*z)=Y,lhs(Eq32)));
term11:=select(has,Eq33,Y^(3*beta)); term12:=select(has,
 term11,Y); term21:=select(has,Eq33,Y^(beta-2));
term22:=select(has,op(1,term21),Y); trbeta:=isolate(
 term12=term22,beta);
trmu:=sort(subs(trbeta,[solve(coeff(Eq33,Y^beta),mu)]))[1];
Eqlambda:=coeff(Eq33,Y^(3*beta))+coeff(Eq33,Y^(beta-2))=0;
Eqlambda1:=subs(mu=trmu,trbeta,Eqlambda);
Solslambda:=expand([solve(Eqlambda1,lambda)]);
trlambda:=sort(convert(Solslambda,set) minus {0})[1];
SolC:=simplify(subs(z=lhs(tr1),trbeta,lambda=trlambda,mu=trmu,
 u=rhs(AnsatzC))); SolS:=simplify(subs(z=lhs(tr1),trbeta,
 lambda=trlambda,mu=trmu,u=rhs(AnsatzS))); SolC:=convert(SolC,
 sec);  SolS:=convert(SolS,csc); simplify([PDE1(rhs(SolC)),
 PDE1(rhs(SolS))],trig); Sol1G:=simplify(subs(c=2,n=3,SolC));
Sol2G:=simplify(subs(c=1/2,n=3,SolC));
animate(rhs(Sol1G),x=0..Pi,t=0..5,view=[default,-200..200]);
animate(rhs(Sol2G),x=-3..3,t=0..1,frames=50);
```

Mathematica:

```
pde1[u_]:=D[u,{t,2}]-D[u,{x,2}]+u-u^3==0; {tr1=x+c*t->z,
 tr2=Cos[z]->yN, eq1=pde1[uN[tr1[[1]]]]//Expand, eq2=eq1/.tr1,
 "eq2="eq2, ansatzC=uN[z]->lambda*Cos[mu*z]^beta, ansatzS=
 uN[z]->lambda*Sin[mu*z]^beta, ansatzC1=Table[D[uN[z],{z,i}]->
 D[lambda*Cos[mu*z]^beta,{z,i}],{i,1,2}]}
{eq3=(eq2/.ansatzC)/.ansatzC1, eq31=eq3/.{Sin[x_]^2:>1-
 Cos[x]^2}//Factor, eq32=eq31[[1]]/.{Cos[mu*z]->yN}//Expand}
{y3b=yN^(3*beta), yb2=yN^(beta-2), term11=Select[eq32,MemberQ[
 #1,y3b]&], term12=Select[term11,MemberQ[#1,yN]&], term21=
 Select[eq32,MemberQ[#1,yb2]&], term22=Select[Factor[term21],
 MemberQ[#1,yN]&], trbeta=Solve[term12[[2]]==term22[[2]],beta]}
{eqmu=Reduce[Select[eq32,MemberQ[#1,yN^(beta)]&]==0,mu],
 eqmu1=eqmu[[3]], trmu=(eqmu1[[2,2]]//ToRules)/.trbeta}
{eqlambda=Select[eq32,MemberQ[#1,y3b]&]+Select[eq32,MemberQ[
 #1,yb2]&]==0, eqlambda1=(eqlambda/.trbeta)/.trmu//First}
{solslambda=Solve[eqlambda1,lambda]//Simplify,trlambda=
 solslambda[[3]], solC=(u[x,t]->ansatzC[[2]]/.trmu/.trlambda/.
 trbeta/.z->tr1[[1]]), solS=(u[x,t]->ansatzS[[2]]/.trmu/.
 trlambda/.trbeta/.z->tr1[[1]])}
Map[Simplify[pde1[#1[[1,1,2]]]]&,{solC,solS}]
{sol1G=solC/.{c->2,n->3}, sol2G=solC/.{c->1/2,n->3}}
f1[xN_,tN_]:=sol1G[[1,1,2]]/.x->xN/.t->tN; f2[xN_,tN_]:=
 sol2G[[1,1,2]]/.x->xN/.t->tN; Animate[Plot[f1[x,t],{x,0,Pi},
 PlotRange->{{0,Pi},{-200,200}},PlotStyle->Hue[0.7]],{t,0,5}]
Animate[Plot[f2[x,t],{x,-3,3},PlotRange->{{-3,3},{0,2}},
 PlotStyle->Hue[0.7]],{t,0,5}]
```

□

Problem 2.23

Maple:

```
with(PDEtools):  declare(u(x,t),U(z)); alias(U=U(z),u=u(x,t));
PDE1:=u->diff(u,t$2)-diff(u,x$2)+u-u^3=0; tr1:=mu*(x+c*t)=z;
Eq1:=expand(PDE1(U(lhs(tr1)))); Eq2:=convert(expand(subs(tr1,
 Eq1)),diff); tr3:=U(z)=Sum(a[k]*exp(k*z),k=-r..s)/Sum(
 b[j]*exp(j*z),j=-p..q); trpc:=isolate(-3*r-3*p=-r-5*p,p);
trqd:=isolate(3*s+3*q=s+5*q,q); params:={p=1,r=1,q=1,s=1};
tr4:=value(subs(params,tr3)); Eq3:=factor(value(algsubs(
 subs(params,tr3),Eq2))); Eq31:=simplify(op(2,lhs(Eq3)));
```

```
for i from 1 to 3 do E||i:=coeff(Eq31,exp(i*z));
 E||(i+3):=coeff(Eq31,exp(-i*z)); od;
E7:=remove(has,Eq31,exp); E8:=c=c0; sys1:={seq(E||i,i=1..8)};
vars:={mu,c,seq(a[i],i=-1..1),seq(b[j],j=-1..1)};
Sols:=[allvalues([solve(sys1,vars)])]; NSols:=nops(Sols);
Sol1:=op(11,op(7,Sols)); Sol2:=op(15,op(7,Sols));
SolF1:=U=subs(c0=c,subs(Sol1,subs(z=lhs(tr1),rhs(tr4))));
SolF2:=U=subs(c0=c,subs(Sol2,subs(z=lhs(tr1),rhs(tr4))));
SolF12:=simplify(convert(convert(SolF1,trig),tanh),tanh);
SolF22:=simplify(convert(convert(SolF2,trig),tanh),tanh);
SolF13:=collect(subs(b[1]=-1,b[-1]=-1,SolF12),tanh);
SolF23:=collect(subs(b[1]=-1,b[-1]=-1,SolF22),tanh);
factor(PDE1(rhs(SolF13))); factor(PDE1(rhs(SolF23)));
```

Mathematica:

```
pde1[u_]:=D[u,{t,2}]-D[u,{x,2}]+u-u^3==0; tr1=mu*(x+c*t)->z
{eq1=pde1[uN[tr1[[1]]]]//Expand, eq2=eq1/.tr1}
{tr3=uN[z]->Sum[a[k]*Exp[k*z],{k,-r,s}]/Sum[b[j]*Exp[j*z],
 {j,-p,q}], trpc=Solve[-3*r-3*p==-r-5*p,p], trqd=Solve[
 3*s+3*q==s+5*q,q], params={p->1,r->1,q->1,s->1},
 tr4=tr3/.params, tr41=Table[D[tr4[[1]],{z,i}]->D[tr4[[2]],
 {z,i}],{i,1,2}]//Expand, eq3=((eq2/.tr4)/.tr41)//Factor}
eq31=eq3[[1,3]]
Do[eq[i]=Coefficient[eq31,Exp[i*z]]==0;Print[eq[i]],{i,1,6}];
s1=0; nL=Length[eq31];
Do[If[D[eq31[[i]],z]==0,s1=s1+eq31[[i]],s1=s1+0],{i,1,nL}];
{eq[7]=s1==0, eq[8]=c==c0}; sys1=Table[eq[i],{i,1,8}]
vars={mu,c,Table[a[i],{i,-1,1}],Table[b[j],{j,-1,1}]}//Flatten
sols=Reduce[sys1//FullSimplify,vars]
sol12[s_]:=Map[ToRules,{sols[[s,2,1]],sols[[s,3]],sols[[s,5]],
 sols[[s,6]],sols[[s,7]],sols[[s,8]]}]//Flatten;
{nSols=Length[sols], sol1=sol12[45], sol2=sol12[46]}
solF1=u[x,t]->((tr4[[2]]/.{z->tr1[[1]]})/.sol1)/.{c0->c}
solF2=u[x,t]->((tr4[[2]]/.{z->tr1[[1]]})/.sol2)/.{c0->c}
ruleSCH={Sinh[x_]:>Tanh[x]/(Sqrt[1-Tanh[x]^2]),Cosh[x_]:>1/
 (Sqrt[1-Tanh[x]^2])}; solF12=(solF1//ExpToTrig)/.ruleSCH//Factor
{solF22=(solF2//ExpToTrig)/.ruleSCH//Factor, tr5={a[1]->-1,
 a[-1]->-1}, tr6={a[1]->b[1],a[-1]->-b[-1]}, tr7={a[1]->-b[1],
 a[-1]->b[-1]}, tr8={b[1]->-1,b[-1]->-1}, solF13=solF12/.tr5,
 solF23=solF22/.tr5, solF14=solF12/.tr6, solF24=solF22/.tr7,
 solF15=solF14/.tr8, solF25=solF24/.tr8}
{pde1[solF15[[2]]],pde1[solF25[[2]]]}//FullSimplify
```

Maple:

```
tr31:=U=(a[-c]*exp(-c*z)+a[d]*exp(d*z))/(a[-p]*exp(-p*z)
 +a[q]*exp(q*z)); Ex1:=subs(tr31,U^3); NEx1:=nops(Ex1);
N1:=simplify(expand(numer(Ex1))); D1:=simplify(expand(
 denom(Ex1))); N1/D1; Ex11:=combine(factor(expand(N1*D1)))
 /combine(factor(expand(D1*D1)));
Ex2:=algsubs(tr31,diff(U,z$2)); NEx2:=nops(Ex2);
for i from 1 to NEx2 do
  N2||i:=simplify(expand(numer(op(i,Ex2))));
  D2||i:=simplify(expand(denom(op(i,Ex2)))); N2||i/D2||i;
  Ex2||i:=combine(factor(expand(N2||i*D2||i)))/combine(factor(
   expand(D2||i*D2||i))); od;
E1L:=expand(op(1,select(has,numer(Ex11),p))/z);
E1R:=expand(op(1,select(has,numer(Ex23),-c*z))/z);
isolate(E1L=E1R,c); Ex12:=combine(expand(N1*D1)/expand(D1*D1));
NumEx12:=numer(Ex12); NNumEx12:=nops(NumEx12);
E2L:=expand(op(1,select(has,op(NNumEx12,NumEx12),3*q*z))/z);
for i from 1 to NEx2 do
  N2||i:=simplify(expand(numer(op(i,Ex2))));
  D2||i:=simplify(expand(denom(op(i,Ex2)))); N2||i/D2||i;
  Ex22||i:=combine(expand(N2||i*D2||i)/expand(D2||i*D2||i)); od;
NumEx223:=numer(Ex223); NNumEx223:=nops(NumEx223);
E2R:=expand(op(1,select(has,op(NNumEx223,NumEx223),5*q*z))/z);
isolate(E2L=E2R,d);
```

Mathematica:

```
{tr31=uN[z]->(a[-c]*Exp[-c*z]+a[d]*Exp[d*z])/(a[-p]*Exp[-p*z]+
 a[q]*Exp[q*z]), ex1=uN[z]^3/.tr31, nEx1=Length[ex1]}
{n1=Expand[Numerator[ex1]], d1=Expand[Denominator[ex1]], n1/d1}
ex11=Expand[n1*d1]/Expand[d1*d1]
{ex20=D[uN[z],{z,2}]/.tr31/.D[tr31,{z,1}]/.D[tr31,{z,2}],
 ex2=ex20[[1]]+ex20[[2]]+ex20[[3,1]]*ex20[[3,2,1]]+ex20[[3,1]]*
 ex20[[3,2,2]], nEx2=Length[ex2]}
Do[n2[i]=Simplify[Expand[Numerator[ex2[[i]]]]]; d2[i]=Simplify[
 Expand[Denominator[ex2[[i]]]]]; n2[i]/d2[i]; ex2N[i]=Expand[
 n2[i]*d2[i]]/Expand[d2[i]*d2[i]]; Print["ex2N",i," ",ex2N[i]],
 {i,1,nEx2}];
{e1L=Expand[Factor[Numerator[ex11]][[1,2]]/z],
 e1R=Expand[Factor[Numerator[ex2N[3]]][[2,2]]/z],
 Solve[e1L==e1R,c]}
```

```
{ex12=Expand[n1*d1]/Expand[d1*d1], numEx12=Numerator[ex12],
 nnumEx12=Length[numEx12], e2L=Expand[numEx12[[nnumEx12,1,2]]/z]}
Do[n2[i]=Expand[Numerator[ex2[[i]]]]; d2[i]=Expand[Denominator[
 ex2[[i]]]]; n2[i]/d2[i]; ex22N[i]=Expand[n2[i]*d2[i]]/Expand[
 d2[i]*d2[i]]; Print["ex22N",i," ",ex22N[i]],{i,1,nEx2}];
{numEx22N3=Numerator[ex22N[3]], nnumEx223=Length[numEx22N3],
 e2R=Expand[numEx22N3[[nnumEx223,2,2]]/z],
 Solve[e2L==e2R,d]//Expand}
```

□

2.5.3 Self-Similar Reductions

Problem 2.24

Maple:

```
with(PDEtools): declare(u(x,t),W(X,T),U(xi));
alias(u=u(x,t),W=W(X,T),U=U(xi)); interface(showassumed=0);
assume(k>0,m>0,n>0,C>0,t>0);
DiffusEq:=(t,x,u)->diff(u,t)=a*diff(u,x$2)+b*u^n;
tr1:={t=T*C,x=X*C^k,u=C^m*W};
Eq1:=combine(dchange(tr1,DiffusEq(t,x,u),[T,X,W]));
Ex21:=select(has,lhs(Eq1),C); Ex22:=select(has,rhs(Eq1),k);
Ex23:=select(has,expand(rhs(Eq1)),n); Ex31:=op(2,Ex21);
Ex32:=op(2,select(has,Ex22,k));
Ex33:=op(2,select(has,Ex23,C)); Eqs:={Ex31=Ex32,Ex32=Ex33};
tr2:=convert((solve(Eqs,{k,m}) assuming n<>1),list);
alpha:=rhs(tr2[2]); beta:=-rhs(tr2[1]);
tr3:={xi=x*(t^beta),u=U(xi)*t^(alpha)}; tr31:=x*(t^beta)=xi;
Eq21:=DiffusEq(t,x,U(lhs(tr31))*t^alpha);
ODE1:=convert(algsubs(tr31,Eq21),diff);
c1:=select(has,op(2,lhs(ODE1)),[t,n]);
ODE11:=expand(simplify(ODE1/c1*t));
ODEFin:=map(factor,lhs(collect(factor(ODE11-rhs(ODE11)),xi)))=0;
```

Mathematica:

```
diffusEq[x_,t_]:=D[u[x,t],t]==a*D[u[x,t],{x,2}]+b*u[x,t]^n;
{tr11={x->xN*c^k,t->tN*c}, tr12=u->w*c^m, tr13={(c^m*w)[xN,tN]->
 c^m*w[xN,tN], D[(c^m*w)[xN,tN],{xN,2}]->c^m*D[w[xN,tN],{xN,2}],
 D[(c^m*w)[xN,tN],tN]->c^m*D[w[xN,tN],tN]}}
eq1T[v_]:=((Simplify[diffusEq[x,t]/.u->Function[{x,t},
 u[x/c^k,t/c]]])/.tr11//ExpandAll)/.{u->v};
```

```
{eq1=eq1T[w*c^m]/.tr13//PowerExpand, ex21=Select[eq1[[2]],
 MemberQ[#1,c]&], ex22=Select[eq1[[1]],MemberQ[#1,a]&],
 ex23=Select[eq1[[1]],MemberQ[#1,b]&]}
{ex31=ex21[[2]], ex32=ex22[[2,2]], ex33=ex23[[2,2]]}
tr2=Assuming[n!=1,Solve[{ex31==ex32,ex32==ex33},{m,k}]]
{alpha=tr2[[1,1,2]], beta=-tr2[[1,2,2]]}
{tr3={xi->x*t^(beta),u->uN[xi]*t^(alpha)},tr31=x*t^(beta)->xi}
dEq[x_,t_,u_]:=D[u,t]==a*D[u,{x,2}]+b*u^n;
eq21=dEq[x,t,uN[tr31[[1]]]*t^(alpha)]//PowerExpand
{ode1=eq21/.tr31,c1=Select[ode1[[2,1]],MemberQ[#1,t]&]}
ode11=Thread[ode1/c1,Equal]/.tr31//FullSimplify
odeFin=ode11/.tr31
```
□

Problem 2.25

Maple:

```
with(PDEtools): declare(u(x,t),W(X,T),U(xi));
alias(u=u(x,t),W=W(X,T),U=U(xi)); interface(showassumed=0);
assume(k>0,m>0,n>0,C>0,t>0);
NPDE:=(t,x,u)->diff(u,t$2)=a*diff(u^n*diff(u,x),x);
tr1:={t=T*C, x=X*C^k, u=C^m*W};
Eq1:=factor(combine(expand(dchange(tr1,NPDE(t,x,u),[T,X,W]))));
Ex21:=select(has,lhs(Eq1),[C]); Ex22:=select(has,rhs(Eq1),[k]);
Ex31:=op(2,Ex21); Ex32:=op(2,Ex22);
tr2:=solve(Ex31=Ex32,k) assuming k<>0; alpha:=m; beta:=-tr2;
tr3:={xi=x*(t^beta),u=U(xi)*t^alpha};
NPDE1:=u->diff(u,t$2)=a*diff(u^n*diff(u,x),x);
tr31:=x*(t^beta)=xi; Eq21:=NPDE1(U(lhs(tr31))*t^alpha);
ODE1:=convert(algsubs(tr31,Eq21),diff);
ODE12:=combine(expand((lhs(ODE1)-rhs(ODE1))*4*U*t^2/(t^m*U)));
ODE13:=map(factor,collect(factor(ODE12),[xi^2,U]));
ODEFin:=collect(ODE13,diff);
```

Mathematica:

```
npde[x_,t_]:=D[u[x,t],{t,2}]==a*D[u[x,t]^n*D[u[x,t],x],x];
{tr11={t->tN*c,x->xN*c^k},tr12={u->c^m*w},
 tr13={(c^m*w)[xN,tN]->c^m*w[xN,tN], Table[D[(c^m*w)[xN,tN],
 {xN,i}]->c^m*D[w[xN,tN],{xN,i}],{i,1,2}], D[(c^m*w)[xN,tN],
 {tN,2}]->c^m*D[w[xN,tN],{tN,2}]}//Flatten}
eq1T[v_]:=((Simplify[npde[x,t]/.u->Function[{x,t},
 u[x/c^k,t/c]]])/.tr11//ExpandAll)/.{u->v};
```

```
eq1=eq1T[w*c^m]/.tr13//PowerExpand//Simplify
{ex21=Select[eq1[[1]],MemberQ[#1,c]&], ex22=Select[eq1[[2]],
 MemberQ[#1,c]&], ex31=ex21[[2]], ex32=ex22[[2]]}
{tr2=Assuming[k!=0,Solve[ex31==ex32,k]]//Expand, alpha=m,
 beta=-tr2[[1,1,2]], tr3={xi->x*(t^beta), u->uN[xi]*t^alpha}}
npde1[u_]:=D[u,{t,2}]==a*D[u^n*D[u,x],x];
{tr31=x*(t^beta)->xi, tr32=x->xi/t^beta}
eq21=npde1[uN[tr31[[1]]]*t^alpha]
{ode1=eq21/.tr31/.tr32, ode12=Assuming[{m>0,n>0,t>0},
 (ode1[[1]]-ode1[[2]])*4*uN[xi]*t^2/(t^m*uN[xi])//
 PowerExpand]//ExpandAll}
ode13=Map[Factor,Collect[Factor[ode12],{xi^2,uN[xi]}]]
odeFin=Collect[ode13/.{m*n->q-2},{uN'[xi],uN''[xi]}]
```
□

Problem 2.26

Maple:

```
with(PDEtools): declare(u(x,t),W(X,T),U(xi));
alias(u=u(x,t),W=W(X,T),U=U(xi)); interface(showassumed=0);
assume(C>0,t>0); SGEq:=(t,x,u)->diff(u,x,t)=sin(u);
tr1:={t=T*C^n,x=X*C^m,u=C^k*W};
Eq1:=combine(dchange(tr1,SGEq(t,x,u),[T,X,W]));
Ex21:=select(has,lhs(Eq1),C);
Ex22:=select(has,op(rhs(Eq1)),C); Ex31:=op(2,Ex21);
Ex32:=op(2,Ex22); tr2:=solve(Ex31=Ex32,{n,k});
alpha:=subs(k=0,-k/(2*n)); beta:=subs(tr2,n/m);
tr3:={xi=x/(t^beta),u=U(xi)*t^(alpha)}; tr31:=x/(t^beta)=xi;
Eq21:=SGEq(t,x,U(lhs(tr31))*t^alpha);
ODE1:=convert(algsubs(tr31,Eq21),diff);
```

Mathematica:

```
sGEq[x_,t_]:=D[u[x,t],x,t]==Sin[u[x,t]];
{tr11={t->tN*c^n,x->xN*c^m},tr12=u->c^k*w,
tr13={(c^k*w)[xN,tN]->c^k*w[xN,tN], D[(c^k*w)[xN,tN],xN,tN]->
 c^k*D[w[xN,tN],xN,tN]}}
eq1T[v_]:=((Simplify[sGEq[x,t]/.u->Function[{x,t},u[x/c^m,
 t/c^n]]])/.tr11//ExpandAll)/.{u->v};
{eq1=eq1T[w*c^k]/.tr13//PowerExpand, ex21=Select[eq1[[2]],
 MemberQ[#1,c]&], ex22=Select[eq1[[1,1]],MemberQ[#1,c]&],
 ex31=ex21[[2]]==0, ex32=ex22[[2]]==0}
tr2=Solve[{ex31,ex32},{n,k}]
```

```
{alpha=(-k/(2*n)/.tr2)[[1]], beta=(n/m/.tr2)[[1]]}
{tr3={xi->x/t^(beta),u->uN[xi]*t^(alpha)},tr31=x/t^(beta)->xi}
sGEq1[x_,t_,u_]:=D[u,x,t]==Sin[u];
ode1=sGEq1[x,t,uN[tr31[[1]]]*t^alpha]//.tr31
```
□

2.6 Separation of Variables

2.6.1 Ordinary Separation of Variables

Problem 2.27

Maple:

```
with(PDEtools): declare((u,W)(x,t),phi(x),psi(t));
interface(showassumed=0): assume(k>0,phi(x)>0,psi(t)>0):
tr1:=phi(x)*psi(t);
PDE1:=u->diff(u(x,t),t)=a*diff(u(x,t)^k*diff(u(x,t),x),x);
Eq2:=expand(PDE1(W)); Eq3:=subs(W(x,t)=tr1,Eq2);
Eq4:=simplify(collect(Eq3,[phi(x),psi(t)]));
Eq5:=Eq4/(phi(x)*psi(t)^(k+1)); Eq6:=map(simplify,Eq5);
Sol:=[dsolve(lhs(Eq6)=C,psi(t)),dsolve(rhs(Eq6)=C,phi(x))];
```

Mathematica:

```
tr1=w[x,t]->phi[x]*psi[t]
pde1[u_]=D[u[x,t],t]==a*D[u[x,t]^k*D[u[x,t],x],x];
{eq2=Expand[pde1[w]], eq3=eq2/.tr1/.D[tr1,t]/.Table[
 D[tr1,{x,i}],{i,1,2}], eq4=PowerExpand[eq3]}
eq5=Thread[eq4/(phi[x]*psi[t]^(k+1)),Equal]//Expand
eq6=Map[Simplify,eq5]
sol={DSolve[eq6[[1]]==c,psi,t],DSolve[eq6[[2]]==c,phi,x]}
```
□

Problem 2.28

Maple:

```
with(PDEtools): declare((u,W)(x,t),phi(x),psi(t));
interface(showassumed=0): assume(lambda>0):
tr1:=phi(x)+psi(t); PDE1:=u->diff(u(x,t),t$2)=
          a*diff(exp(lambda*u(x,t))*diff(u(x,t),x),x);
Eq2:=expand(PDE1(W)); Eq3:=expand(subs(W(x,t)=tr1,Eq2));
Eq4:=factor(Eq3/(exp(lambda*psi(t)))); Eq5:=map(simplify,Eq4);
Sol:=[dsolve(lhs(Eq5)=C,psi(t)),dsolve(rhs(Eq5)=C,phi(x))];
```

```
SolFin:=u(x,t)=factor(rhs(Sol[1])+rhs(Sol[2]));
SolFin1:=map(combine,simplify(combine(SolFin)
         assuming t>0,C>0,a>0,_C1>0,_C2>0));
Test1:=pdetest(SolFin1,PDE1(u));
```

Mathematica:

```
Off[Solve::"ifun"]; tr1=w[x,t]->phi[x]+psi[t]
tr1D[v_]:=Table[D[tr1,{v,i}],{i,1,2}];
pde1[u_]:=D[u,{t,2}]==a*D[Exp[lambda*u]*D[u,x],x];
{eq2=Expand[pde1[w[x,t]]], eq3=PowerExpand[eq2/.tr1/.tr1D[t]
 /.tr1D[x]], eq4=Factor[Thread[eq3/(Exp[lambda*psi[t]]),
 Equal]], eq5=Map[Simplify,eq4]}
{sol1=DSolve[eq5[[1]]==c,psi,t], sol2=DSolve[eq5[[2]]==c,phi,x]}
solFin=(psi[t]/.sol1)+(phi[x]/.sol2)//Simplify//First
{solFin1=Assuming[{t>0,c>0,x>0,a>0,C[1]>0,C[2]>0},FullSimplify[
 Map[ExpandAll,solFin]]], pde1[solFin1]//FullSimplify}
```

□

2.6.2 Partial Separation of Variables

Problem 2.29

Maple:

```
with(PDEtools): declare((u,W)(x,t),phi(x),psi(t));
interface(showassumed=0): assume(a>0): tr1:=phi(x)*psi(t);
PDE1:=u->diff(u(x,t),t)=F(t)*diff(u(x,t),x$2)
     +u(x,t)*diff(u(x,t),x)^2-a*u(x,t)^3;
Eq2:=expand(PDE1(W)); Eq3:=expand(subs(W(x,t)=tr1,Eq2));
Eq4:=collect(Eq3,[phi,psi]); Eq5:=expand(Eq4/F(t)/psi(t)/phi(x));
Eq6:=map(expand,Eq5); Eq7:=collect(Eq6,psi(t)^2);
Sol1:=[dsolve(op(1,op(1,rhs(Eq7)))=0,phi(x))];
Sol11:=subs(exp(_C1*sqrt(a))=C,Sol1);
Eq8:=expand(subs(Sol11[1],Eq7)); Sol2:=dsolve(Eq8,psi(t));
SolFin1:=algsubs(_C1/C=C,u(x,t)=factor(rhs(Sol11[1])*rhs(Sol2)));
SolFin2:=algsubs(_C1*C=C,u(x,t)=factor(rhs(Sol11[2])*rhs(Sol2)));
T1:=pdetest(SolFin1, PDE1(u)); T2:=pdetest(SolFin2, PDE1(u));
Sol3:=dsolve(op(2,rhs(Eq7))=a,phi(x));
K:=expand(algsubs(Sol3,F(t)*op(1,op(1,rhs(Eq7)))));
Eq9:=subs({op(1,op(1,rhs(Eq7)))=K/F(t),op(2,rhs(Eq7))=a},Eq7);
Eq10:=expand(Eq9*F(t)*psi(t)); Sol4:=[dsolve(Eq10,psi(t))];
Sol41:=combine(algsubs(a*(Int(F(t),t))=G,Sol4[1]) assuming G>0);
Sol42:=combine(algsubs(a*(Int(F(t),t))=G,Sol4[2]) assuming G>0);
```

```
SolFin3:=u(x,t)=rhs(Sol3)*rhs(Sol41);
SolFin4:=u(x,t)=rhs(Sol3)*rhs(Sol42);
SolFin31:=subs(G=a*(Int(F(t),t)),SolFin3);
SolFin41:=subs(G=a*(Int(F(t),t)),SolFin3);
T3:=pdetest(SolFin31,PDE1(u)); T4:=pdetest(SolFin41,PDE1(u));
```

Mathematica:

```
tr1=w[x,t]->phi[x]*psi[t]; trD[u_,var_]:=Table[D[u,{var,i}],
 {i,1,2}]//Flatten; pde1[u_]:=D[u[x,t],t]==f[t]*D[u[x,t],{x,2}]+
 u[x,t]*D[u[x,t],x]^2-a*u[x,t]^3; {eq2=Expand[pde1[w]],
 eq3=Expand[eq2/.tr1/.trD[tr1,t]/.trD[tr1,x]], eq4=Collect[eq3,
 {phi,psi}], eq5=Expand[Thread[eq4/f[t]/psi[t]/phi[x],Equal]]}
{eq6=Map[Expand,eq5], eq7=Collect[eq6,psi[t]^2]}
sol1=DSolve[eq7[[2,1,2]]==0,phi[x],x]
eq8=Expand[eq7/.sol1[[1]]]/.trD[sol1[[1]],x]
{sol2=DSolve[eq8,psi[t],t], solFin12=Table[u[x,t]->
 sol1[[i,1,2]]*sol2[[1,1,2]]/.C[1]^2->C[1],{i,1,2}]}
test12=Table[pde1[u]/.solFin12[[i]]/.trD[solFin12[[i]],x]/.
 trD[solFin12[[i]],t],{i,1,2}]
sol3=DSolve[eq7[[2,2]]==a,phi[x],x]
k=Expand[eq7[[2,1,2]]*f[t]/.sol3/.trD[sol3,x]]//First
eq9=eq7/.{eq7[[2,1,2]]->k/f[t],eq7[[2,2]]->a}
eq10=Expand[Thread[eq9*f[t]*psi[t],Equal]]
{sols4=DSolve[eq10,psi[t],t], solsFin34=Table[u[x,t]->
 sol3[[1,1,2]]*sols4[[i,1,2]],{i,1,2}],
test34=Table[pde1[u]/.solsFin34[[i]]/.trD[solsFin34[[i]],x]/.
 trD[solsFin34[[i]],t]//FullSimplify,{i,1,2}]}
```

□

Problem 2.30

Maple:

```
with(PDEtools): declare((u,W)(x,t),phi(x),psi(t));
interface(showassumed=0): assume(a>0,b>0,c>0):
tr1:=phi(x)+psi(t); PDE1:=u->diff(u(x,t),t)*diff(u(x,t),x$2)
  +a*diff(u(x,t),x)*diff(u(x,t),t$2)=b*diff(u(x,t),x$3)
  +c*diff(u(x,t),t$3); Eq2:=expand(PDE1(W));
Eq3:=expand(subs(W(x,t)=tr1,Eq2));
Sol1:=dsolve(op(1,op(1,lhs(Eq3)))=_C1,psi(t));
Eq4:=algsubs(Sol1,Eq3); Sol2:=dsolve(Eq4,phi(x));
SolFin1:=u(x,t)=subs(2*_C2=_C2,rhs(Sol1)+rhs(Sol2));
Sol3:=dsolve(op(2,op(2,lhs(Eq3)))=_C1,phi(x));
```

```
Eq5:=algsubs(Sol3,Eq3); Sol4:=dsolve(Eq5,psi(t));
SolFin2:=u(x,t)=subs(2*_C2=_C2,rhs(Sol3)+rhs(Sol4));
T1:=pdetest(SolFin1,PDE1(u)); T2:=pdetest(SolFin2,PDE1(u));
tr2:=[phi(x)=_C1*exp(-A1*lambda*x)+A2*lambda*x,
      psi(t)=_C2*exp( A3*lambda*t)+A4*lambda*t+_C3];
Eq6:=expand(algsubs(tr2[2],algsubs(tr2[1],Eq3)));
Eq61:=expand(Eq6/lambda^3); Eq62:=collect(Eq61,
 [exp(A1*lambda*x),exp(A3*lambda*t)]); C3:=A3=1;
Eq63:=subs(C3,Eq62); C2:=isolate(op(2,rhs(Eq63))/
 op(1,lhs(Eq63))=1,A2); Eq64:=expand(subs(C2,Eq63));
C1:=A1=a; C4:=subs(C1,expand(isolate(Eq64,A4)));
tr21:=subs({C1,C2,C3,C4},tr2); SolFin3:=u(x,t)=
 rhs(tr21[1])+rhs(tr21[2]); T3:=pdetest(SolFin3,PDE1(u));
```

Mathematica:

```
tr1=w[x,t]->phi[x]+psi[t]; trD[u_,var_]:=Table[D[u,{var,i}],
 {i,1,3}]//Flatten; pde1[u_]:=D[u[x,t],t]*D[u[x,t],{x,2}]+
  a*D[u[x,t],x]*D[u[x,t],{t,2}]==b*D[u[x,t],{x,3}]+
  c*D[u[x,t],{t,3}]; {eq2=Expand[pde1[w]], eq3=Expand[
  eq2/.tr1/.trD[tr1,x]/.trD[tr1,t]]}
sol1=DSolve[eq3[[1,1,1]]==C[1],psi[t],t]//First
{eq4=eq3/.sol1/.trD[sol1,t], sol2=DSolve[eq4,phi[x],x]//First}
{solFin1=u[x,t]->tr1[[2]]/.sol1/.sol2, sol3=DSolve[eq3[[
 1,2,2]]==C[1],phi[x],x]//First, eq5=eq3/.sol3/.trD[sol3,x]}
{sol4=DSolve[eq5,psi[t],t]//First, solFin2=u[x,t]->tr1[[2]]/.
 sol3/.sol4, solFin12={solFin1,solFin2}}
test12=Table[pde1[u]/.solFin12[[i]]/.trD[solFin12[[i]],x]/.
 trD[solFin12[[i]],t],{i,1,2}]
tr2={phi[x]->C[1]*Exp[-a1*lambda*x]+a2*lambda*x,
     psi[t]->C[2]*Exp[a3*lambda*t]+a4*lambda*t+C[3]}
eq6=Expand[eq3/.tr2/.trD[tr2,x]/.trD[tr2,t]]
eq61=Expand[Thread[eq6/lambda^3,Equal]]
eq62=Collect[eq61,{Exp[a1*lambda*x],Exp[a3*lambda*t]}]
{c3=a3->1, eq63=eq62/.c3, c2=Solve[eq63[[2,2]]/eq63[[1,1]]==1,
 a2], eq64=Expand[eq63/.c2], c1=a1->a, c4=Expand[Solve[
 eq64,a4]]/.c1, tr21=tr2/.c1/.c2/.c3/.c4//Flatten}
{solFin3=u[x,t]->tr1[[2]]/.tr21, test3=pde1[u]/.solFin3/.
 trD[solFin3,x]/.trD[solFin3,t]//FullSimplify}
```

□

2.6.3 Generalized Separation of Variables

Problem 2.31

Maple:

```
with(PDEtools): declare((u,W)(x,t),(psi1,psi2)(t),phi2(x));
tr1:=psi1(t)+phi2(x)*psi2(t);
PDE1:=u->diff(u(x,t),t)=a*u(x,t)*diff(u(x,t),x$2)
        +b*(diff(u(x,t),x))^2+c;
Eq2:=expand(PDE1(W)); Eq3:=expand(subs(W(x,t)=tr1,Eq2));
Eq4:=expand(Eq3/psi2(t)^2); Eq5:=diff(diff(Eq4,t),x);
Eq51:=collect(lhs(Eq5),diff(phi2(x),x))=
      collect(rhs(Eq5),diff(phi2(x),x$3));
TermX:=select(has,rhs(Eq51),a);
Eq6:=evala(Eq51/TermX/diff(phi2(x),x));
Eq71:=rhs(Eq6)=C; Eq72:=factor(lhs(Eq6)=C);
SolPhi2:=dsolve(subs(C=0,Eq71),phi2(x));
SolPhi21:=sort(subs(_C2=A2,_C3=A3,algsubs(coeff(
 rhs(SolPhi2),x^2)=A1,SolPhi2)));
SolPsi2:=dsolve(expand(subs(C=0,Eq72)),psi2(t));
SolPsi21:=lhs(SolPsi2)=subs(_C1=1,_C2=C1,rhs(SolPsi2)*(-B));
Eq8:=algsubs(SolPhi21,algsubs(SolPsi21,Eq3));
SolPsi1:=subs(_C1=C2,dsolve(Eq8,psi1(t)));
SolFin1:=rhs(SolPsi1); SolFin2:=rhs(SolPhi21)*rhs(SolPsi21);
tr2:=A1*x^2+A2*x+A3=(x+C3)^2; tr3:={A1=1,A2=2*C3,A3=C3^2};
tr4:=B=-1/(2*(a+2*b)); SolFin21:=factor(subs(tr2,tr4,SolFin2));
SolFin12:=sort(collect(expand(subs(t+C1=T,simplify(subs(
  tr3,tr4,SolFin1)))),t)); SolFin13:=map(factor,collect(
  SolFin12,[C1,C2])); SolFin14:=factor(select(has,SolFin13,c))
  +select(has,SolFin13,T);
SolFin15:=u(x,t)=subs(T=t+C1,SolFin14+SolFin21);
Test1:=pdetest(SolFin15,PDE1(u));
```

Mathematica:

```
tr1=w[x,t]->psi1[t]+phi2[x]*psi2[t]; trD[u_,var_]:=Table[D[u,
 {var,i}],{i,1,2}]//Flatten; pde1[u_]:=D[u[x,t],t]==a*u[x,t]*D[
 u[x,t],{x,2}]+b*(D[u[x,t],x])^2+c; {eq2=Expand[pde1[w]],
 eq3=Expand[eq2/.tr1/.trD[tr1,x]/.trD[tr1,t]], eq4=Expand[Thread[
 eq3/psi2[t]^2,Equal]], eq5=Thread[D[Thread[D[eq4,t],Equal],x],
 Equal], eq51=Collect[eq5[[1]],D[phi2[x],x]]==Collect[eq5[[2]],
 D[phi2[x],{x,3}]], termX=Cases[eq51[[2]],_Plus]//First}
eq6=Expand[Thread[eq51/termX/D[phi2[x],x],Equal]]
```

```
{eq71=eq6[[2]]==c, eq72=Factor[eq6[[1]]]==c}
{solPhi2=DSolve[eq71/.c->0,phi2[x],x]//First, solPhi21=solPhi2/.
 Coefficient[solPhi2[[1,2]],x^2]->a1/.{C[2]->a2,C[1]->a3},
 solPsi2=DSolve[Expand[eq72/.c->0],psi2[t],t]//First,
 solPsi21=solPsi2[[1,1]]->solPsi2[[1,2]]*bN/.C[2]->1/.C[1]->c1}
eq8=eq3/.solPsi21/.solPhi21/.trD[solPsi21,t]/.trD[solPhi21,x]
solPsi1=DSolve[eq8,psi1[t],t]/.C[1]->c2//First
{tr2=a1*x^2+a2*x+a3->(x+c3)^2, tr3={a1->1,a2->2*c3,a3->c3^2},
 tr4=bN->-1/(2*(a+2*b)), tr5=c1+t->tN}
{solFin11=tr1[[2]]/.solPsi1/.solPhi21/.solPsi21/.tr2/.tr3/.
 tr4/.tr5, solFin12=u[x,t]->Map[FullSimplify,Collect[solFin11,
 {tN,c}]]/.tN->tr5[[1]], test1=pde1[u]/.solFin12/.
 trD[solFin12,x]/.trD[solFin12,t]//FullSimplify}
```

□

Problem 2.32

Maple:

```
with(PDEtools): declare((u,U,W)(x,t),phi(x),psi(t));
tr1:=4*arctan(U(x,t)); tr2:=phi(x)/psi(t);
PDE1:=u->diff(u(x,t),x$2)-diff(u(x,t),t$2)=sin(u(x,t));
Eq2:=expand(PDE1(W)); Eq3:=expand(algsubs(W(x,t)=tr1,Eq2));
Eq31:=collect(normal(subs(U(x,t)=tr2,Eq3)),diff);
Eq32:=normal(map(expand,Eq31)); Eq33:=map(expand,Eq32/4*
 denom(rhs(Eq32))/phi(x)/psi(t)); Eq34:=collect(Eq33,diff);
Eq35:=map(normal,lhs(Eq34))=rhs(Eq34); Eq4:=diff(Eq35,x,t);
Eq41:=expand(Eq4/2/psi(t)/diff(psi(t),t)/phi(x)/diff(phi(x),x));
ODEs:=[selectremove(has,lhs(Eq41),phi(x))]; ODE1:=-ODEs[1]=C;
ODE2:=ODEs[2]=C; Eq51:=expand(ODE1*phi(x)*diff(phi(x),x));
Eq52:=int(lhs(Eq51),x)=int(rhs(Eq51),x)+A1;
Eq53:=expand(Eq52*phi(x)*diff(phi(x),x));
Eq54:=int(lhs(Eq53),x)=int(rhs(Eq53),x)+A2;
Eq55:=algsubs(-A1=A1,algsubs(-2*A2=A2,Eq54*(-2)));
Eq61:=expand(ODE2*psi(t)*diff(psi(t),t));
Eq62:=int(lhs(Eq61),t)=int(rhs(Eq61),t)+B1;
Eq63:=expand(Eq62*psi(t)*diff(psi(t),t));
Eq64:=int(lhs(Eq63),t)=int(rhs(Eq63),t)+B2;
Eq65:=subs(2*B2=B2,Eq64*2); Eq7:=collect(subs(Eq55,Eq65,Eq35),
 [A1,B1,A2,B2,C]); Eq71:=select(has,lhs(Eq7),[B1,A1])=select(
 has,rhs(Eq7),[phi,psi]); Eq72:=collect(lhs(Eq71)-rhs(Eq71),
 [phi,psi]); Eq73:=(coeff(Eq72,phi(x),2)-coeff(Eq72,
 psi(t),2))/2; Eq74:=select(has,lhs(Eq7)/2,[B2,A2])=0;
```

```
Consts:=[isolate(Eq73,B1),isolate(Eq74,B2)];
Eqs:=collect(subs(Consts,[Eq55,Eq65]),psi(t)^2);
```

Mathematica:

```
tr1=w[x,t]->4*ArcTan[uN[x,t]]; tr2=uN[x,t]->phi[x]/psi[t];
trD[u_,var_]:=Table[D[u,{var,i}],{i,1,2}]//Flatten;
trS[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
pde1[u_]:=D[u[x,t],{x,2}]-D[u[x,t],{t,2}]==Sin[u[x,t]];
{eq2=Expand[pde1[w]], eq3=FunctionExpand[eq2/.tr1/.trD[tr1,x]/.
 trD[tr1,t]]//Expand, eq31=Map[Simplify,eq3/.tr2 /.trD[tr2,x]/.
 trD[tr2,t]], eq33=Map[Expand,Thread[eq31/4*Denominator[
 eq31[[2]]]/phi[x]/psi[t],Equal]], eq34=Collect[eq33,
 {psi''[t],phi''[x]}], eq35=Map[Factor,eq34[[1]]==eq34[[2]]}
{eq4=Thread[D[eq35,x,t],Equal], eq41=Expand[Thread[eq4/2/
 psi[t]/psi'[t]/phi[x]/phi'[x],Equal]]}
{odes=Level[eq41,{2}], ode1=-(Plus@@trS[odes,phi[x]])==c,
 ode2=Plus@@trS[odes,psi[t]]==c}
eq51=Expand[Thread[ode1*phi[x]*phi'[x],Equal]]
eq52=Integrate[eq51[[1]],x]==Integrate[eq51[[2]],x]+a1
eq53=Expand[Thread[eq52*phi[x]*phi'[x],Equal]]
eq54=Integrate[eq53[[1]],x]==Integrate[eq53[[2]],x]+a2
eq55=(Thread[eq54*(-2),Equal]//Expand)/.-a1->a1/.-2*a2->a2
eq61=Expand[Thread[ode2*psi[t]*psi'[t],Equal]]
eq62=Integrate[eq61[[1]],t]==Integrate[eq61[[2]],t]+b1
eq63=Expand[Thread[eq62*psi[t]*psi'[t],Equal]]
eq64=Integrate[eq63[[1]],t]==Integrate[eq63[[2]],t]+b2
eq65=(Thread[eq64*2,Equal]//Expand)/.-2*b2->b2
{tr55=ToRules[eq55]//First, tr65=ToRules[eq65]//First,
 eq701=eq35/.tr55, eq702=eq701/.tr65, eq7=Collect[eq702,
 {a1,b1,a2,b2,c}], eq71=Plus@@{trS[eq7[[1]],b1],
 trS[eq7[[1]],a1]}==Plus@@{trS[eq7[[2]],phi[x]],
 trS[eq7[[2]],psi[t]]}, eq72=Collect[eq71[[1]]-eq71[[2]],
 {phi[x],psi[t]}], eq73=(Coefficient[eq72,phi[x],2]-
 Coefficient[eq72,psi[t],2])/2//Expand}
eq74=Plus @@ {trS[eq7[[1]],b2],trS[eq7[[1]],a2]}/4//Expand
consts={Solve[eq73==0,b1],Solve[eq74==0,b2]}//Flatten
eqs=Collect[{eq55,eq65}/.consts,psi[t]^2]
```

Maple:

```
with(plots): AG:=array(1..2): BG:=array(1..2): CG:=array(1..4):
setoptions(animate,thickness=3,scaling=constrained):
Eqs1:=subs(C=0,A2=0,Eqs); Sol11:=expand([dsolve(Eqs1[1],
 phi(x))]); Sol12:=expand([dsolve(Eqs1[2],psi(t))]);
tr3:=[select(has,rhs(Sol11[1]),_C1)=D1,select(has,
 rhs(Sol11[2]),_C1)=D2,select(has,rhs(Sol12[1]),_C1)=E1,
 select(has,rhs(Sol12[2]),_C1)=E2];
for i from 1 to 2 do Solphi1||i:=subs(tr3,Sol11[i]);
                     Solpsi1||i:=subs(tr3,Sol12[i]); od;
SolFin10:=subs(U(x,t)=tr2,tr1);
SolFin11:=u(x,t)=subs(Solphi11,Solpsi11,SolFin10);
SolFin12:=algsubs(D1/E1=A,combine(SolFin11,exp));
SolFin21:=u(x,t)=subs(Solphi11,Solpsi12,SolFin10);
SolFin22:=algsubs(D1/E2=A,combine(SolFin21,exp));
SolFin31:=u(x,t)=subs(Solphi12,Solpsi11,SolFin10);
SolFin32:=algsubs(D2/E1=A,combine(SolFin31,exp));
SolFin41:=u(x,t)=subs(Solphi12,Solpsi12,SolFin10);
SolFin42:=algsubs(D2/E2=A,combine(SolFin41,exp));
Params:=[A1=2,A=1]; xR:=-10..10; tR:=-10..10; N:=200; P:=200:
Op1:=frames=N,numpoints=P;
for i from 1 to 4 do
 plot(subs(Params,t=0,rhs(SolFin||i||2)),x=xR);
 pdetest(SolFin||i||1,PDE1(u)); od;
for i from 1 to 4 do  CG[i]:=animate(subs(Params,
                      rhs(SolFin||i||2)),x=xR,t=tR,Op1): od:
for i from 1 to 2 do
 SolFin||i||2||Dx:=diff(rhs(SolFin||i||2),x);
 SolFin||i||2||Dt:=diff(rhs(SolFin||i||2),t); od;
for i from 1 to 2 do
 AG[i]:=animate(subs(Params,SolFin||i||2||Dx),x=xR,t=tR,Op1):
 BG[i]:=animate(subs(Params,SolFin||i||2||Dt),x=xR,t=tR,Op1):
od: display(AG); display(BG); display(CG);
```

Mathematica:

```
p=10; SetOptions[Plot,ImageSize->300,PlotStyle->{Hue[0.9],
 Thickness[0.01]},PlotRange->All]; SetOptions[Animate,
AnimationRate->0.9]; {eqs1=eqs/.c->0/.a2->0,
sol11=Expand[DSolve[eqs1[[1]],phi[x],x]], sol12=Expand[
 DSolve[eqs1[[2]],psi[t],t]],solphi12=phi[x]->sol11[[1,1,2]]/.
 C[1]->d2, solphi11=phi[x]->sol11[[2,1,2]]/.C[1]->d1,
 solpsi12=psi[t]->sol12[[1,1,2]]/.C[1]->e2,
 solpsi11=psi[t]->sol12[[2,1,2]]/.C[1]->e1, solFin10=tr1/.tr2}
{solFin11=u[x,t]->solFin10[[2]]/.solphi11/.solpsi11,
 solFin[1]=solFin11/.d1->e1*a, solFin21=u[x,t]->solFin10[[2]]/.
 solphi11/.solpsi12,solFin[2]=solFin21/.d1->e2*a}
{solFin31=u[x,t]->solFin10[[2]]/.solphi12/.solpsi11,
 solFin[3]=solFin31/.d2->e1*a, solFin41=u[x,t]->solFin10[[2]]/.
 solphi12/.solpsi12, solFin[4]=solFin41/.d2->e2*a,
 pars={a1->2,a->1}}
GraphicsRow[Table[Plot[Evaluate[solFin[i][[2]]/.pars/.t->0],
 {x,-p,p}],{i,1,4}]]
{Table[pde1[u]/.solFin[i]/.trD[solFin[i],x]/.trD[solFin[i],t]//
 FullSimplify,{i,1,4}], Table[solFinDx[i]=D[solFin[i][[2]],x],
 {i,1,2}],Table[solFinDt[i]=D[solFin[i][[2]],t],{i,1,2}]}
Do[g11[i_,xN_,tN_]:=solFin[i][[2]]/.pars/.{x->xN,t->tN},{i,1,4}];
Do[g12[i_,xN_,tN_]:=solFinDx[i]/.pars/.{x->xN,t->tN},{i,1,2}];
Do[g13[i_,xN_,tN_]:=solFinDt[i]/.pars/.{x->xN,t->tN},{i,1,2}];
Manipulate[Plot[Evaluate[g11[1,x,t]],{x,-p,p},PlotRange->All],
 {t,-p,p}]
Animate[Plot[g11[3,x,t],{x,-p,p},PlotRange->{0,2*Pi}],{t,-p,p}]
Animate[Plot[g11[4,x,t],{x,-p,p},PlotRange->{0,2*Pi}],{t,-p,p}]
Animate[Plot[g12[1,x,t],{x,-p,p},PlotRange->{0,Pi}],{t,-p,p}]
Animate[Plot[g13[1,x,t],{x,-p,p},PlotRange->{-Pi,0}],{t,-p,p}]
```

Maple:

```
DG:=array(1..2): EG:=array(1..2): FG:=array(1..2):
Eqs2:=subs(C=0,Eqs); Sol21:=expand([dsolve(Eqs2[1],phi(x))]);
Params:=[A1=2,A2=1,_C1=0]; xR:=-10..10; tR:=-10..10;
Sol22:=expand([dsolve(Eqs2[2],psi(t))]);
SolFin20:=subs(U(x,t)=tr2,tr1);
SolFin21:=u(x,t)=simplify(subs(Sol21[3],Sol22[3],SolFin20));
SolFin22:=u(x,t)=simplify(subs(Sol21[4],Sol22[4],SolFin20));
pdetest(SolFin21,PDE1(u));  pdetest(SolFin22,PDE1(u));
```

```
SolFin21Dx:=simplify(diff(SolFin21,x)); SolFin22Dx:=simplify(
 diff(SolFin22,x)); SolFin21Dt:=simplify(diff(SolFin21,t));
SolFin22Dt:=simplify(diff(SolFin22,t));
for i from 1 to 2 do
 DG[i]:=animate(expand(subs(Params,rhs(SolFin2||i))),x=xR,t=tR,
  color=blue,Op1): EG[i]:=animate(expand(subs(Params,
  rhs(SolFin2||i||Dx))),x=xR,t=tR,color=green,Op1):
 FG[i]:=animate(expand(subs(Params,rhs(SolFin2||i||Dt))),x=xR,
  t=tR,color=red,Op1): od: display(DG); display(EG); display(FG);
```

Mathematica:

```
{eqs2=eqs/.c->0/.a1->2, sol21=Flatten[Expand[DSolve[eqs2[[1]],
 phi[x],x]]], sol22=Expand[DSolve[eqs2[[2]],psi[t],t]]//Flatten}
{solFin20=tr1/.tr2, solFin2[1]=u[x,t]->Simplify[solFin20[[2]]/.
 sol21[[1]]/.sol22[[1]]], solFin2[2]=u[x,t]->Simplify[
 solFin20[[2]]/.sol21[[2]]/.sol22[[2]]]}
test12={pde1[u]/.solFin2[1]/.trD[solFin2[1],x]/.trD[solFin2[1],
 t]//FullSimplify, pde1[u]/.solFin2[2]/.trD[solFin2[2],x]/.
 trD[solFin2[2],t]//FullSimplify}
{solFinDx2[1]=Simplify[D[solFin2[1],x]], solFinDx2[2]=Simplify[
 D[solFin2[2],x]], solFinDt2[1]=Simplify[D[solFin2[1],t]],
 solFinDt2[2]=Simplify[D[solFin2[2],t]]}
params={a1->2,a2->1,C[1]->0}
Do[g21[i_,xN_,tN_]:=solFin2[i][[2]]/.params/.{x->xN,t->tN},
 {i,1,2}]; Do[g22[i_,xN_,tN_]:=solFinDx2[i][[2]]/.params/.{x->xN,
 t->tN},{i,1,2}]; Do[g23[i_,xN_,tN_]:=solFinDt2[i][[2]]/.params/.
 {x->xN,t->tN},{i,1,2}]; p=2*Pi;
Animate[Plot[g21[1,x,t],{x,-10,10},PlotRange->{-p,p}],{t,-10,10}]
Animate[Plot[g22[1,x,t],{x,-10,10},PlotRange->{-p,p}],{t,-10,10}]
Animate[Plot[g23[1,x,t],{x,-10,10},PlotRange->{-p,p}],{t,-10,10}]
```

Mathematica:

```
consts={C[1]->0,C[2]->0}; params1={a2->0,c->2}; params2=a1->4;
{eqs3=eqs/.params1, sols1=(DSolve[eqs3[[1]],phi[x],x]/.consts//
 Flatten)/.params2, sols2=(DSolve[eqs3[[2]],psi[t],t]/.consts//
 Flatten)/.params2, sol3[1]=sols1[[1]], sol3[2]=sols1[[2]],
 sol3[3]=sols2[[1]], sol3[4]=sols2[[2]], solFin30=tr1[[2]]/.tr2}
solFin3[1]=u[x,t]->solFin30/.sol3[1]/.sol3[3]//FullSimplify
solFin3[2]=u[x,t]->solFin30/.sol3[2]/.sol3[4]//FullSimplify
p3=2*Pi; xR=10*Pi; tR=10;
```

```
Map[FullSimplify,{pde1[u]/.solFin3[1]/.trD[solFin3[1],x]/.
 trD[solFin3[1],t], pde1[u]/.solFin3[2]/.
 trD[solFin3[2],x]/.trD[solFin3[2],t]}]
g31[xN_,tN_]:=solFin3[1][[2]]/.{x->xN,t->tN};
g32[xN_,tN_]:=solFin3[2][[2]]/.{x->xN,t->tN};
Animate[Plot[g31[x,t],{x,-xR,xR},PlotRange->{-p3,p3}],{t,-tR,tR}]
Animate[Plot[g32[x,t],{x,-xR,xR},PlotRange->{-p3,p3}],{t,-tR,tR}]
```

Maple:

```
interface(showassumed=0):assume(omega,'real',omega>0,t,'real');
omega1:=sqrt(1-0.9); HG:=array(1..2): xR:=-20..20; tR:=-150..150;
Params:=[C=4,A2=0]; Consts:=[_C1=0,_C2=0];
Eqs4:=subs(A1=1-omega^2,Params,Eqs); Sols:=[dsolve(Eqs4)];
N1:=nops(Sols);for i from 1 to N1 do Sols[i]; od;
Sol41:=subs(Consts,[op(op(1,Sols[N1]))]); Sol42:=subs(Consts,
 [op(op(2,Sols[N1]))]); SolFin40:=subs(U(x,t)=tr2,tr1);
for i from 1 to 2 do for j from 1 to 2 do
 SolFin4||i:=u(x,t)=subs(Sol41[2],Sol42[j],SolFin40); od; od;
for i from 1 to 2 do SolFin4||i; od;
for i from 1 to 2 do Z||i:=subs(omega=omega1,omega^2=omega1^2,
 subs(omega*t=OmegaT,rhs(SolFin4||i))); od;
for i  to 2 do HG[i]:=animate(evalf(Z||i),x=xR,OmegaT=tR,
 scaling=unconstrained,numpoints=70): od:
display(HG); for i to 2 do convert(SolFin4||i,abs); od;
```
□

Problem 2.33

Maple:

```
with(PDEtools): declare((u,W)(x,t),(phi1,phi2)(x));
interface(showassumed=0): assume(n,'integer',n>0);
assume(lambda,'integer',lambda>0);
tr1:=phi1(x)*exp(lambda*t)+phi2(x);
PDE1:=u->diff(u(x,t),t)*diff(u(x,t),x,t)-diff(u(x,t),x)
 *diff(u(x,t),t$2)-F(x)*diff(u(x,t),t$n)=0;
Eq2:=expand(PDE1(W)); Eq3:=expand(subs(W(x,t)=tr1,Eq2));
Eq4:=factor(Eq3); ODE1:=select(has,lhs(Eq4),F(x))=0;
Solphi2:=simplify(dsolve(ODE1,phi2(x)));
SolFin1:=u(x,t)=subs(Solphi2,tr1); pdetest(SolFin1,PDE1(u));
```

Mathematica:

```
tr1=w[x,t]->phi1[x]*Exp[lambda*t]+phi2[x];
trD[u_,var_]:=Table[D[u,{var,i}],{i,1,10}]//Flatten;
trDM[u_,var1_,var2_]:=D[u,var1,var2]//Flatten;
trS[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
pde1[u_]:=D[u[x,t],t]*D[u[x,t],x,t]-D[u[x,t],x]*D[u[x,t],
 {t,2}]-f[x]*D[u[x,t],{t,10}]==0;
eq2=Expand[pde1[w]]
{eq3=Expand[eq2/.tr1/.trD[tr1,t]/.trD[tr1,x]/.
 trDM[tr1,x,t]/.{10->n}], eq4=Factor[eq3]}
{ode1=trS[eq4[[1]],f[x]]==0, solphi2=Simplify[DSolve[ode1,
 phi2[x],x]]//First, solFin1=u[x,t]->tr1[[2]]/.solphi2}
pde1[u]/.solFin1/.trD[solFin1,t]/.trD[solFin1,x]/.
 trDM[solFin1,x,t]/.{10->n}//FullSimplify
```
□

Problem 2.34

Maple:

```
with(PDEtools): declare((u,W)(x,t),(psi1,psi2)(t));
interface(showassumed=0): assume(n,'integer',n>0);
assume(lambda,'integer',lambda>0);
tr1:=psi1(t)*exp(lambda*x)+psi2(t);
PDE1:=u->diff(u(x,t),x,t)+(diff(u(x,t),x))^2-u(x,t)
        *diff(u(x,t),x$2)-F(t)*diff(u(x,t),x$n)=0;
Eq2:=expand(PDE1(W)); Eq3:=expand(subs(W(x,t)=tr1,Eq2));
Eq4:=factor(Eq3); Solpsi2:=expand(isolate(Eq4,psi2(t)));
SolFin1:=u(x,t)=subs(Solpsi2,tr1); pdetest(SolFin1,PDE1(u));
```

Mathematica:

```
tr1=w[x,t]->psi1[t]*Exp[lambda*x]+psi2[t];
trD[u_,var_]:=Table[D[u,{var,i}],{i,1,10}]//Flatten;
trDM[u_,var1_,var2_]:=D[u,var1,var2]//Flatten;
pde1[u_]:=D[u[x,t],x,t]+(D[u[x,t],x])^2-u[x,t]*D[u[x,t],
 {x,2}]-f[t]*D[u[x,t],{x,10}]==0; {eq2=Expand[pde1[w]],
 eq3=Expand[eq2/.tr1/.trD[tr1,t]/.trD[tr1,x]/.
 trDM[tr1,x,t]/.{10->n}], eq4=Factor[eq3]}
{solpsi2=Expand[Solve[eq4,psi2[t]]]//First, solFin1=u[x,t]->
 tr1[[2]]/.solpsi2, pde1[u]/.solFin1/.trD[solFin1,t]/.
 trD[solFin1,x]/.trDM[solFin1,x,t]/.{10->n}//FullSimplify}
```
□

2.6.4 Functional Separation of Variables

Problem 2.35

Maple:

```
with(PDEtools): declare(psi1(t),psi2(t),W(z));
interface(showassumed=0); assume(z>0);
for i from 1 to 4 do assume(A||i,constant); od:
 tr1:=psi1(t)*x+psi2(t)=z; tr2:=expand(isolate(tr1,x));
PDE1:=u->diff(u,t)=a(t)*diff(u,x$2)+b(t)*diff(u,x)+c(t)*F(u);
Eq2:=expand(PDE1(W(lhs(tr1)))); Eq3:=convert(
     algsubs(tr1,Eq2),diff); Eq4:=expand(Eq3/diff(W(z),z));
Eq5:=expand(subs(tr2,Eq4)); Eq51:=-Eq5+rhs(Eq5);
```

Mathematica:

```
{tr1=psi1[t]*x+psi2[t]==z, tr2=Expand[Solve[tr1,x]]//First}
pde1[u_]:=D[u,t]==a[t]*D[u,{x,2}]+b[t]*D[u,x]+c[t]*f[u];
{eq2=pde1[w[tr1[[1]]]]//Expand, eq3=eq2/.ToRules[tr1]}
{eq4=Expand[Thread[eq3/w'[z],Equal]], eq5=Expand[eq4/.tr2]}
eq51=Thread[Thread[(-1)*eq5,Equal]+eq5[[2]],Equal]
```

Maple:

```
FunDiffEq1:=add(Phi[i]*Psi[i],i=1..4)=0; SolFunDiffEq1:=
 [Psi[1]=A1*Psi[3]+A2*Psi[4], Psi[2]=A3*Psi[3]+A4*Psi[4],
 Phi[3]=-A1*Phi[1]-A3*Phi[2], Phi[4]=-A2*Phi[1]-A4*Phi[2]];
L11:=[selectremove(has,op(2,lhs(Eq51)),[psi1,psi2])];
L12:=[selectremove(has,op(3,lhs(Eq51)),[-1,psi2])];
L13:=[selectremove(has,op(5,lhs(Eq51)),[1])];
L2:=[selectremove(has,op(1,lhs(Eq51)),[-1,psi1])];
L3:=[selectremove(has,op(4,lhs(Eq51)),[a(t),psi1])];
L4:=[selectremove(has,op(6,lhs(Eq51)),[c(t)])];
tr3:=[Phi[1]=L11[2]*L12[2]*L13[1],Psi[1]=L11[1]+L12[1]+L13[2],
 Phi[2]=L2[2],Psi[2]=L2[1],Phi[3]=L3[2],Psi[3]=L3[1],
 Phi[4]=L4[2],Psi[4]=L4[1]]; sys1:=subs(tr3,SolFunDiffEq1);
S2:=subs(_C1=B1,[dsolve(sys1[2],psi1(t))]); S2[1];
S1[1]:=simplify(subs(S2[1],subs(_C1=B2,map(expand,dsolve(
       sys1[1],psi2(t)))))); S1[2]:=simplify(subs(S2[2],
       subs(_C1=B2,map(expand,dsolve(sys1[1],psi2(t))))));
Eq3sys1:=int(lhs(sys1[3]),z)=int(rhs(sys1[3]),z);
Eq3sys11:=isolate(Eq3sys1,diff(W(z),z));
S3:=int(lhs(Eq3sys11),z)=B3*Int(rhs(Eq3sys11),z)+B4;
```

```
Eq4sys1:=expand(subs(S3,(subs(F(W(z))=F1,sys1[4]))));
S4:=F(W(z))=rhs(simplify(isolate(Eq4sys1,F1)));
Sol1:=[S1[1],S2[1],S3,S4]; Sol2:=[S1[2],S2[2],S3,S4];
```

Mathematica:

```
funDiffEq1=Sum[phi[i]*psi[i],{i,1,4}]==0;
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
trS2[eq_,var1_,var2_]:=Select[eq,MemberQ[#,var1,Infinity] &&
 MemberQ[#,var2,Infinity]&]; trS3[eq_,var_]:=Select[eq,
 FreeQ[#,var]&]; solFunDiffEq1={psi[1]==a1*psi[3]+a2*psi[4],
 psi[2]==a3*psi[3]+a4*psi[4],phi[3]==-a1*phi[1]-a3*phi[2],
 phi[4]==-a2*phi[1]-a4*phi[2]}
{l110=trS2[eq51[[1]],psi1[t],psi2[t]],l11={l110,trS3[l110,t]}}
{l120=trS1[eq51[[1]],psi2'[t]], l12={l120,-trS3[l120,t]}}
{l130=trS1[eq51[[1]],b[t]], l13={trS3[l130,t],l130}}
{l20=trS2[eq51[[1]],psi1'[t],z],l2={trS3[l20,z],-trS3[l20,t]}}
{l30=trS2[eq51[[1]],a[t],z], l3={trS3[l30,z],trS3[l30,t]}}
{l40=trS1[eq51[[1]],c[t]], l4={trS3[l40,z],trS3[l40,t]}}
tr3={phi[1]->l11[[2]]*l12[[2]]*l13[[1]], psi[1]->l11[[1]]+
 l12[[1]]+l13[[2]],phi[2]->l2[[2]], psi[2]->l2[[1]], phi[3]->
 l3[[2]],psi[3]->l3[[1]], phi[4]->l4[[2]],psi[4]->l4[[1]]}
{sys1=solFunDiffEq1/.tr3, s2=DSolve[sys1[[2]],psi1[t],t]/.
 C[1]->b1, s1[1]=DSolve[sys1[[1]],psi2[t],t]/.C[1]->b2/.
 s2[[1]]//Simplify//First, s1[2]=DSolve[sys1[[1]],psi2[t],t]/.
 C[1]->b2/.s2[[2]]//Simplify//First}
eq3sys1=Integrate[sys1[[3,1]],z]==Integrate[sys1[[3,2]],z]
eq3sys11=Solve[eq3sys1,w'[z]]//First
{s3=Integrate[eq3sys11[[1,1]],z]->b3*Hold[Integrate[
 eq3sys11[[1,2]],z]]+b4, eq4sys1=Expand[sys1[[4]]/.f[w[z]]->
 f1/.s3/.D[s3,z]]//ReleaseHold}
s4=f[w[z]]->(Solve[eq4sys1,f1]//First)[[1,2]]//Simplify
Print["sol1="]; sol1={s1[1],s3,s4}//Flatten
Print["sol2="]; sol2={s1[2],s3,s4}//Flatten
```

Maple:

```
Consts1:=[A1=-1,A3=0,B3=1,B4=0,a(t)=1,b(t)=0,c(t)=1];
Sol11:=factor(value(subs(Consts1,Sol1)));
Sol12:=factor(value(subs(Consts1,Sol2)));
tr41:=expand(ln(rhs(Sol11[3])))=combine(ln(lhs(Sol11[3])));
F1:=F(W(z))=factor(expand(subs(tr41,rhs(Sol11[4]))));
Sol11:=subsop(4=F1,Sol11); Sol12:=subsop(4=F1,Sol12);
```

```
Consts2:=[A1=-1,A3=0,B3=1,B4=0,a(t)=1,b(t)=1,c(t)=1];
Sol21:=simplify(value(subs(Consts2,Sol1)));
Sol22:=simplify(value(subs(Consts2,Sol2)));
tr42:=isolate(subs(W(z)=W,Sol21[3]),z);
F2:=F(W)=factor(expand(subs(tr42,rhs(Sol21[4]))));
Sol21:=simplify(subsop(4=F2,Sol21),symbolic);
Sol22:=simplify(subsop(4=F2,Sol22),symbolic);
```

Mathematica:

```
{consts1={a1->-1,a3->0,b3->1,b4->0,a[K[1]]->1,b[K[1]]->0,
 c[K[1]]->1}, solpsi11=s2[[1]]/.consts1, solpsi12=s2[[2]]/.
 consts1, sol11=Factor[sol1/.consts1/.psi1[K[1]]->
 solpsi11[[1,2]]]//FullSimplify, sol12=Factor[sol2/.consts1/.
 psi1[K[1]]->solpsi11[[1,2]]]//FullSimplify}
{tr41=Log[sol11[[2,2]]]==Simplify[Log[sol11[[2,1]]]]//
 PowerExpand, f1=sol11[[3,2]]/.ToRules[tr41]}
sol111={sol11/.{sol11[[3,2]]->f1},solpsi11}//Flatten
sol112={sol12/.{sol12[[3,2]]->f1},solpsi12}//Flatten
{consts2={a1->-1,a3->0,b3->1,b4->0,a[K[1]]->1,b[K[1]]->1,
 c[K[1]]->1}, sol21=Factor[sol1/.consts2/.psi1[K[1]]->
 solpsi11[[1,2]]]//FullSimplify, sol22=Factor[sol2/.consts2/.
 psi1[K[1]]->solpsi11[[1,2]]]//FullSimplify}
{tr42=Log[sol21[[2,2]]]==Simplify[Log[sol21[[2,1]]]]//
 PowerExpand, f2=sol21[[3,2]]/.ToRules[tr42]}
sol211={sol21/.{sol21[[3,2]]->f2},solpsi11}//Flatten
sol212={sol22/.{sol22[[3,2]]->f2},solpsi12}//Flatten
sol21[[2]]=w[z]->Integrate[eq3sys11[[1,2]]/.consts1,z]; sol21
sol22[[2]]=w[z]->Integrate[eq3sys11[[1,2]]/.consts2,z]; sol22
```
□

Problem 2.36

Maple:

```
with(PDEtools): declare(psi1(t),psi2(t),W(z));
interface(showassumed=0); assume(z>0);
for i from 1 to 4 do assume(A||i,constant); od:
tr1:=psi1(t)*x+psi2(t)=z; tr2:=expand(isolate(tr1,x));
PDE1:=u->diff(u,t)=diff(G(u)*diff(u,x),x)+F(u);
Eq2:=expand(PDE1(W(lhs(tr1)))); Eq3:=algsubs(tr1,Eq2);
Eq4:=expand(Eq3/diff(W(z),z)); Eq5:=expand(subs(tr2,Eq4));
Eq51:=map(convert,Eq5,diff); Eq52:=-Eq51+rhs(Eq51);
```

Mathematica:

```
{tr1=psi1[t]*x+psi2[t]==z, tr2=Expand[Solve[tr1,x]]//First}
pde1[u_]:=D[u,t]==D[g[u]*D[u,x],x]+f[u];
{eq2=pde1[w[tr1[[1]]]]//Expand, eq3=eq2/.ToRules[tr1]}
{eq4=Expand[Thread[eq3/w'[z],Equal]], eq5=Expand[eq4/.tr2]}
eq51=Thread[Thread[(-1)*eq5,Equal]+eq5[[2]],Equal]
```

Maple:

```
FunDiffEq1:=add(Phi[i]*Psi[i],i=1..4)=0; SolFunDiffEq1:=
 [Psi[1]=A1*Psi[3]+A2*Psi[4], Psi[2]=A3*Psi[3]+A4*Psi[4],
  Phi[3]=-A1*Phi[1]-A3*Phi[2], Phi[4]=-A2*Phi[1]-A4*Phi[2]];
L11:=[selectremove(has,op(2,lhs(Eq52)),[psi1,psi2])];
L12:=[selectremove(has,op(3,lhs(Eq52)),[-1,psi2])];
L2:=[selectremove(has,op(1,lhs(Eq52)),[-1,psi1])];
L31:=[selectremove(has,op(4,lhs(Eq52)),[psi1])];
L32:=[selectremove(has,op(5,lhs(Eq52)),[psi1])];
L4:=[selectremove(has,op(6,lhs(Eq52)),[1])];
tr3:=[Phi[1]=L11[2]*L12[2],Psi[1]=L11[1]+L12[1],
 Phi[2]=L2[2],Psi[2]=L2[1],Phi[3]=L31[2]+L32[2],
 Psi[3]=L31[1],Phi[4]=L4[2],Psi[4]=L4[1]];
tr31:=subs(rhs(tr3[5])=expand(diff(G(W(z))*diff(W(z),z),z)
      /diff(W(z),z)),tr3); sys0:=subs(tr31,SolFunDiffEq1);
```

Mathematica:

```
funDiffEq1=Sum[phi[i]*psi[i],{i,1,4}]==0; trS1[eq_,var_]:=
 Select[eq,MemberQ[#,var,Infinity]&]; trS2[eq_,var1_,var2_]:=
 Select[eq,MemberQ[#,var1,Infinity]&&MemberQ[#,var2,Infinity]&];
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&]; solFunDiffEq1=
{psi[1]==a1*psi[3]+a2*psi[4],psi[2]==a3*psi[3]+a4*psi[4],
 phi[3]==-a1*phi[1]-a3*phi[2],phi[4]==-a2*phi[1]-a4*phi[2]}
{l110=trS2[eq51[[1]],psi1[t],psi2[t]], l11={l110,trS3[l110,t]},
 l120=trS1[eq51[[1]],psi2'[t]], l12={l120,-trS3[l120,t]},
 l20=trS2[eq51[[1]],psi1'[t],z], l2={trS3[l20,z],-trS3[l20,t]},
 l310=trS2[eq51[[1]],psi1[t],g'[w[z]]], l31={trS3[l310,z],
 trS3[l310,t]}, l320=trS2[eq51[[1]],psi1[t],w''[z]]}
{l32={trS3[l320,z],trS3[l320,t]}, l40=trS1[eq51[[1]],f[w[z]]],
 l4={trS3[l40,z],trS3[l40,t]}}
tr3={phi[1]->l11[[2]]*l12[[2]], psi[1]->l11[[1]]+l12[[1]],
 phi[2]->l2[[2]], psi[2]->l2[[1]], phi[3]->l31[[2]]+l32[[2]],
 psi[3]->l31[[1]], phi[4]->l4[[2]], psi[4]->l4[[1]]}
sys0=solFunDiffEq1/.tr3
```

Maple:

```
Consts1:=[A1=-1,A3=0,B3=1,B4=0]; sys1:=subs(Consts1,sys0);
tr4:=G(W(z))=1; Eq3sys1:=subs(tr4,(D(G))(W(z))=0,sys1[3]);
S3:=subs(_C1=B1,_C2=B2,dsolve(Eq3sys1,W(z)));
Eq4sys1:=algsubs(S3,subs(F(W(z))=F(W),sys1[4]));
S4:=isolate(Eq4sys1,F(W)); tr5:=expand(isolate(
    subs(W(z)=W,S3),z)); F1:=combine(subs(tr5,S4));
S2:=subs(_C1=B1,dsolve(sys1[2],psi1(t))); S2;
S1:=value(subs(S2,subs(_C1=B3,map(expand,
  dsolve(sys1[1],psi2(t)))))); Sol1:=[S1,S2,S3,F1,tr4];
```

Mathematica:

```
Off[Solve::ifun]; consts1={a1->-1,a3->0,b3->1,b4->0}
{sys1=sys0/.consts1, tr4=g[w[z]]->1,
 eq3sys1=sys1[[3]]/.tr4/.g'[w[z]]->0,
 s3=DSolve[eq3sys1,w[z],z]/.C[1]->b2/.C[2]->b1//First}
eq4sys1=sys1[[4]]/.f[w[z]]->f[w]/.s3/.D[s3,z]
{s4=Solve[eq4sys1,f[w]], tr5=Expand[Solve[s3/.w[z]->w1/.
 Rule->Equal,z]]//First, f1=s4/.tr5//First}
s2=DSolve[sys1[[2]],psi1[t],t]/.C[1]->b1//First
{s1=DSolve[sys1[[1]],psi2[t],t]/.C[1]->b3/.psi1[K[1]]->
 s2[[1,2]]/.s2//FullSimplify, sol1={s1,s2,s3,f1,tr4}//Flatten}
```
□

Problem 2.37

Maple:

```
with(PDEtools):declare((psi1,psi2)(x));interface(showassumed=0);
assume(n,'integer',n>0,m,'integer',m>0,C,'integer',C>0):
tr1:=psi1(x)*y+psi2(x)=z; tr2:=expand(isolate(tr1,y));
PDE1:=u->diff(u,y)*diff(u,x,y)-diff(u,x)*diff(u,y$2)=f(x)*
 diff(u,y$2)^(n-1)*diff(u,y$m); Eq2:=expand(PDE1(W(lhs(tr1))));
Eq21:=lhs(Eq2)=rhs(Eq2)*psi1(x)^m; Eq3:=subs(y=z,subs(tr1,Eq21));
Eq4:=expand(Eq3/psi1(x)); Eq51:=combine(map(convert,Eq4,diff));
Eq61:=[selectremove(has,lhs(Eq51),[psi1])];
Eq62:=[selectremove(has,rhs(Eq51),[psi1,f])];
Eq7:=expand(Eq51/Eq61[2]/Eq62[1]);
Eq8:=subs(_C1=B1,factor(combine(dsolve(lhs(Eq7)=C,psi1(x)))));
Eq9:=combine(rhs(Eq7)=C); Eq10:=simplify(Eq9*denom(lhs(Eq9))/C);
EqC:=[selectremove(has,op(1,op(1,op(1,rhs(Eq8)))),[C,n,m])];
trC:=isolate(EqC[1]=EqC[2]/f(x),C); Eqpsi1:=subs(trC,Eq8);
EqW:=subs(trC,Eq10);
```

Mathematica:

```
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
{tr1=psi1[x]*y+psi2[x]==z, tr11=psi1[x]*z+psi2[x]->z}
tr2=Expand[Solve[tr1,y]]//First
pde1[u_]:=D[u,y]*D[u,x,y]-D[u,x]*D[u,{y,2}]==f[x]*D[u,
 {y,2}]^(n-1)*D[u,{y,m}]; {eq2=pde1[w[tr1[[1]]]]//Expand,
 eq21=eq2[[1]]==eq2[[2]]*psi1[x]^m, eq3=eq21/.y->z/.tr11}
{eq4=Expand[Thread[eq3/psi1[x],Equal]], eq51=eq4//PowerExpand//
 Together, eq61={trS1[eq51[[1]],x],trS1[eq51[[1]],z]},
 eq62={trS1[eq51[[2]],x],trS1[eq51[[2]],z]}}
eq7=Expand[Thread[eq51/eq61[[2]]/eq62[[1]],Equal]]
eq8=DSolve[eq7[[1]]==c,psi1[x],x]/.C[1]->b1//First
{eq9=eq7[[2]]==c//Together, eq10=Thread[eq9*Denominator[
 eq9[[1]]]/c,Equal], eq80=Level[eq8,{3}], eqC={c*eq80[[2,1]],
 -f[x]}, trC=Solve[eqC[[1]]==eqC[[2]]/f[x],c], eqpsi1=eq8/.
 trC//First, eqw=Thread[(eq10/.trC//First)/w''[z],Equal]}
```

□

Problem 2.38

Maple:

```
with(PDEtools): declare(psi1(t),psi2(t),psi3(t));
interface(showassumed=0);  assume(z(x,t)>0):
PDE1:=u->diff(u,t)-a*diff(u,x$2)-F(t)*u*ln(u)-G(t)*u=0;
Eq2:=combine(PDE1(exp(z(x,t)))); Eq3:=expand(Eq2/exp(z(x,t)));
tr1:=x^2*psi1(t)+x*psi2(t)+psi3(t); PDE2:=unapply(Eq3,z);
Eq4:=PDE2(z); Eq5:=expand(subs(z(x,t)=tr1,Eq4));
Eq6:=collect(Eq5,[x,a]);
EDOs:=[select(has,op(1,lhs(Eq6)),t)=0, select(has,op(2,
       lhs(Eq6)),t)=0,convert([op(3..6,lhs(Eq6))],`+`)=0];
dsolve(EDOs,{psi1(t),psi2(t),psi3(t)});
```

Mathematica:

```
pde1[u_]:=D[u,t]-a*D[u,{x,2}]-f[t]*u*Log[u]-g[t]*u==0;
trD[u_,var_]:=Table[D[u,{var,i}],{i,1,2}]//Flatten;
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
{eq2=pde1[Exp[z[x,t]]]//Together, eq3=Expand[Thread[eq2/Exp[
 z[x,t]],Equal]]//PowerExpand}
tr1=z[x,t]->x^2*psi1[t]+x*psi2[t]+psi3[t]
{eq4=eq3/.tr1/.trD[tr1,t]/.trD[tr1,x], eq5=Collect[eq4,{x,a}]}
```

```
odes={Coefficient[eq5[[1]],x^2]==0,Coefficient[eq5[[1]],x]==0,
 Coefficient[eq5[[1]],x,0]==0}
{DSolve[odes[[1]],psi1[t],t], DSolve[odes[[2]],psi2[t],t],
 DSolve[odes[[3]],psi3[t],t]}//Flatten
```

□

Problem 2.39

Maple:

```
with(PDEtools): declare(phi(x),psi(t)); tr1:=phi(x)+psi(t)=z;
PDE1:=u->diff(u,t)=diff(F(u)*diff(u,x),x);
phi1:=diff(phi(x),x)^2; Eq2:=expand(PDE1(W(lhs(tr1))));
Eq3:=subs(tr1,Eq2); Eq4:=expand(Eq3/diff(W(z),z));
trH:=H(z)=expand(convert([op(1..2,rhs(Eq4))],`+`)/phi1);
trH1:=H(z)=map(convert,op(1,rhs(trH)),diff)
          +map(convert,op(2,rhs(trH)),diff);
trH2:=H(z)=subs(F(W(z))=F1,subsop(1=diff(F1(z),z),
      rhs(trH1))); trH3:=algsubs(F1=F1(z),trH2);
Eq5:=lhs(Eq4)=op(3,rhs(Eq4))+lhs(trH1)*phi1; Eq6:=convert(
     simplify(Eq5),diff); Eq7:=subs(z=z(x),Eq6);
Eq8:=diff(Eq7,x); Eq9:=subs(diff(z(x),x)=diff(lhs(tr1),x),Eq8);
Eq10:=convert(subs(W(z(x))=W,z(x)=z,Eq9),diff);
termPhi:=select(has,op(1,rhs(Eq10)),phi);
Eq11:=subsop(1=diff(F1(z),z)*termPhi,rhs(Eq10))=0;
Eq12:=subs(F(W)=F1(W),Eq11);
FunDiffEq1:=add(Phi[i]*Psi[i],i=1..3)=0;
SolFun1:=[Psi[1]=A2*Psi[2],Psi[3]=A1*Psi[2],Phi[2]=-A1*Phi[3]
 -A2*Phi[1]]; SolFun2:=[Phi[1]=A1*Phi[3],Phi[2]=A2*Phi[3],
  Psi[3]=-A1*Psi[1]-A2*Psi[2]];
L11:=[selectremove(has,op(1,lhs(Eq12)),[phi])];
L12:=[selectremove(has,op(4,lhs(Eq12)),[phi])];
L2:=[selectremove(has,op(3,lhs(Eq12)),[phi])];
L3:=[selectremove(has,op(2,lhs(Eq12)),[phi])];
tr3:=[Phi[1]=L11[2]+L12[2],Phi[2]=L2[2],Phi[3]=L3[2],
 Psi[1]=L11[1], Psi[2]=L2[1],Psi[3]=L3[1]];
sys01:=subs(tr3,SolFun1); sys02:=subs(tr3,SolFun2);
```

Mathematica:

```
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&];
pde1[u_]:=D[u,t]==D[(f[u]*D[u,x]),x]; {tr1=phi[x]+psi[t]==z,
 phi1=(phi'[x])^2, eq2=pde1[w[tr1[[1]]]]//Expand,
 eq3=eq2/.ToRules[tr1], eq4=Expand[Thread[eq3/w'[z],Equal]]}
trH=h[z]->Expand[(eq4[[2,1]]+eq4[[2,3]])/phi1]
trH2=trH/.f[w[z]]->f1[z]/.trH[[2,1]]->f1'[z]
{eq5=eq4[[1]]==eq4[[2,2]]+trH2[[1]]*phi1, eq6=eq5/.z->z[x],
 eq7=Thread[D[eq6,x],Equal], eq8=eq7/.z'[x]->D[tr1[[1]],x]}
{eq9=eq8/.w[z[x]]->w/.z[x]->z, termphi=trS1[trS1[eq9[[2]],
 h[z]],x], termf=trS1[eq9[[2]],f'[w]]}
{eq10=eq9/.termf->f1'[z]*termphi, eq12=eq10/.f[w]->f1[w]}
funDiffEq1=Sum[phi[i]*psi[i],{i,1,3}]==0
{solFun1={psi[1]==a2*psi[2], psi[3]==a1*psi[2], phi[2]==
 -a1*phi[3]-a2*phi[1]}, solFun2={phi[1]==a1*phi[3], phi[2]==
 a2*phi[3], psi[3]==-a1*psi[1]-a2*psi[2]}}
{l110=trS1[eq12[[2]],f1'[z]], l11={trS3[l110,z],trS3[l110,x]},
 l120=trS1[eq12[[2]],h[z]], l12={trS3[l120,z]/2,trS3[l120,x]},
 l20=trS1[eq12[[2]],h'[z]], l2={trS3[l20,z],trS3[l20,x]},
 l30=trS1[eq12[[2]],f1[w]], l3={trS3[l30,w],trS3[l30,x]}}
tr3={phi[1]->l11[[2]]+l12[[2]],phi[2]->l2[[2]],phi[3]->l3[[2]],
 psi[1]->l11[[1]],psi[2]->l2[[1]],psi[3]->l3[[1]]}
{sys01=solFun1/.tr3, sys02=solFun2/.tr3}
```

Maple:

```
Consts1:={A1=0,A2=0}; sys11:=subs(Consts1,{sys01[1],sys01[2]});
Solphi1:=subs(_C1=B1,_C2=B2,op(op(dsolve(sys11,phi(x))[2])));
trF:=F1(z)=exp(z); Eq3sys01:=algsubs(trF,subs(F1(W)=F1(z),
     sys01[3])); SolH:=subs(_C1=B1,dsolve(Eq3sys01,H(z)));
SolH1:=eval(SolH,Consts1); EqW:=subs(SolH1,algsubs(trF,trH3));
SolW:=simplify(value(subs(Consts1,_C1=B2,_C2=B3,dsolve(EqW,
      W(z))))); sysFW:=subs(W(z)=W1,F1(z)=F2,{trF,SolW});
SolF1:=eliminate(sysFW,z); SolF2:=F(W)=subs(W1=W,solve(
       op(SolF1[2]),F2));
Eqpsi:=eval(subs(Solphi1,algsubs(z=lhs(tr1),subs(SolH,algsubs(
       Solphi1,Eq6)))),Consts1); Solpsi1:=subs(_C1=B4,
       dsolve(Eqpsi,psi(t)));
z=subs(B2=0,B4=0,B1^3=B2,rhs(Solphi1+Solpsi1));
```

Mathematica:

```
Off[Solve::ifun]; trD[u_,var_]:=Table[D[u,{var,i}],{i,1,2}]//
 Flatten; {consts1={a1->0,a2->0}, sys11={sys01[[1]],sys01[[2]]}/.
 consts1, solphi1=(DSolve[sys11[[1]],phi[x],x]/.C[2]->b1/.
 C[1]->b2)[[2]], trF=f1[z]->Exp[z], eq3sys01=sys01[[3]]/.trF/.
 f1[w]->f1[z], solH=DSolve[eq3sys01,h[z],z]/.C[1]->b1//First,
 solH1=solH/.consts1, eqW=trH2/.solH1/.trF/.D[trF,z],
 solW=DSolve[eqW/.Rule->Equal,w[z],z]/.C[2]->b2/.C[1]->b3/.
 consts1//First, sysFW={trF,solW}/.w[z]->w1/.f1[z]->f2//Flatten}
{solF1=Reduce[sysFW/.Rule->Equal,z], termw1=trS1[solF1,b3]}
solF2=f[w]->Solve[termw1,f2][[1,1,2]]/.w1->w
eqpsi=eq5/.solH/.z->tr1[[1]]/.solphi1/.trD[solphi1,x]/.consts1
solpsi1=DSolve[eqpsi,psi[t],t]/.C[1]->b4//First
solFin=tr1/.solphi1/.solpsi1/.b2->0/.b4->0/.b1^3->b2
```

□

2.7 Transformation Groups

2.7.1 One-Parameter Groups of Transformations

Problem 2.40

Maple:

```
with(PDEtools): declare((u,U)(x,t)); interface(showassumed=0):
assume(alpha>0,beta>0,zeta>0,a>0,b>0,c>0,k>0);
T:=(a)->[a^alpha,a^beta,a^zeta]; T(a); T(b);
for i from 1 to 3 do
 expand(T(a)[i]*T(b)[i]*x=T(a*b)[i]*x);
 expand(T(a)[i]*(T(b)[i]*T(c)[i])*x=T(a*b*c)[i]*x);
 expand(T(1)[i]*T(a)[i]*x=T(1*a)[i]*x);
 simplify(T(a)[i]*T(a^(-1))[i]*x=T(a*a^(-1))[i]*x)=T(1)[i]*x; od;
```

Mathematica:

```
tT[a_]:={a^alpha,a^beta,a^zeta}; tT[a]; tT[b];
Map[PowerExpand, {tT[a]*tT[b]*x==tT[a*b]*x,
  tT[a]*(tT[b]*tT[c])*x==tT[a*b*c]*x, tT[1]*tT[a]*x==tT[1*a]*x,
  tT[a]*tT[a^(-1)]*x==tT[a*a^(-1)]*x==tT[1]*x}]
```

Maple:

```
PDEL1:=diff(u(x,t),t)-k*diff(u(x,t),x$2)=0; tr1:={x=X/a^alpha,
 t=T/a^beta,u(x,t)=U(X,T)/a^zeta}; PDEL2:=combine(dchange(tr1,
 PDEL1,[X,T,U])); Eq1:=map(lhs,PDEL1=PDEL2);
termL1:=select(has,select(has,select(has,rhs(Eq1),a),beta),a);
termL2:=expand(rhs(Eq1)/termL1); Eq2:=select(has,select(has,
 termL2,a),a)=a^0; tr2:=isolate(Eq2,beta); tr3:=subs(tr2,tr1);
PDEL3:=dchange(tr3,PDEL1,[X,T,U]); termL3:=select(has,op(1,
 lhs(PDEL3)),a); PDEL4:=expand(PDEL3/termL3); PDEL1=PDEL4;
PDENL1:=diff(u(x,t),t)-k*diff(u(x,t)*diff(u(x,t),x),x)=0;
PDENL2:=combine(dchange(tr1,PDENL1,[X,T,U]));
Eq1:=map(lhs,PDENL1=PDENL2);
termNL1:=select(has,select(has,select(has,rhs(Eq1),a),beta),a);
termNL2:=expand(rhs(Eq1)/termNL1);
Eq2:=select(has,select(has,select(has,op(1,termNL2),a),a),a)=a^0;
tr4:=simplify(isolate(Eq2,zeta)); tr5:=subs(tr4,tr1);
PDENL3:=expand(dchange(tr5,PDENL1,[X,T,U]));
termNL3:=select(has,op(1,lhs(PDENL3)),a);
PDENL4:=collect(expand(PDENL3/termNL3),k); PDENL1=PDENL4;
```

Mathematica:

```
Off[Solve::ifun]; pdeL1[x_,t_]:=D[u[x,t],t]-k*D[u[x,t],{x,2}]==0;
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&]; pdeL1[x,t]
{tr11={x->xN/a^alpha,t->tN/a^beta},tr12=u[x,t]->uN[xN,tN]/a^zeta}
pdeL2[v_]:=First[((pdeL1[x,t]/.u->Function[{x,t},
 u[a^alpha*x,a^beta*t]/a^zeta])/.tr11)/.{u->v}]==0; pdeL2[uN]
{eq1=Map[First,pdeL1[x,t]==pdeL2[uN]], termL1=trS3[trS1[eq1[[2]],
 beta],xN], termL2=Thread[eq1[[2]]/termL1,Equal]//Simplify}
{eq2=trS1[trS1[termL2,a],a]==a^0, tr2=Solve[eq2,beta]//First,
 tr31=tr11/.tr2, Solve[tr31/.Rule->Equal,{xN,tN}]}
pdeL3[v_]:=First[((pdeL1[x,t]/.u->Function[{x,t},u[a^alpha*x,
 a^(2*alpha)*t]/a^zeta])/.tr31)/.{u->v}]==0; pdeL3[uN]
{termL3=trS1[trS3[pdeL3[uN][[1]],k],a], pdeL4=Thread[
 pdeL3[uN]/termL3,Equal]//Expand, pdeL1[x,t]==pdeL4}
pdeNL1[x_,t_]:=D[u[x,t],t]-k*D[u[x,t]*D[u[x,t],x],x]==0;
pdeNL2[v_]:=First[((pdeNL1[x,t]/.u->Function[{x,t},
 u[a^alpha*x,a^beta*t]/a^zeta])/.tr11)/.{u->v}]==0; pdeNL2[uN]
{eq1NL=Map[First,pdeNL1[x,t]==pdeNL2[uN]], termNL1=trS3[trS1[
 eq1NL[[2]],beta],xN], termNL2=Thread[eq1NL[[2]]/termNL1,Equal]//
 Expand, eq2=-trS3[trS3[termNL2[[2]],k],xN]==a^0}
```

```
{tr4=Solve[eq2,zeta]//First, tr5=tr11/.tr4, tr51=tr12/.tr4}
{Solve[tr5/.Rule->Equal,{xN,tN}], Solve[tr51/.Rule->Equal,
 uN[xN,tN]]}
pdeNL3[v_]:=First[((pdeNL1[x,t]/.u->Function[{x,t},u[a^alpha*x,
 a^beta*t]/a^(2*alpha-beta)])/.tr11)/.{u->v}]==0; pdeNL3[uN]
{termNL3=trS3[trS3[pdeNL3[uN][[1]],k],xN], pdeNL4=Collect[
 Expand[Thread[pdeNL3[uN]/termNL3,Equal]],k]}
pdeNL1[x,t]==pdeNL4
```
□

Problem 2.41

Maple:

```
with(PDEtools): declare(u(x,t),U(X,T));interface(showassumed=0):
assume(a>0,p>0); W:=diff_table(u(x,t));
KdV:=W[t]+6*W[]*W[x]+W[x,x,x]=0; KdVT:=W[t]+W[]*W[x]+W[x,x,x]=0;
mKdV:=W[t]+6*W[]^2*W[x]+W[x,x,x]=0;
mKdVT:=W[t]+W[]^p*W[x]+W[x,x,x]=0;
InvKdV:=proc(PDE)
 local Eq1,Eq2,Eq3,Eq4,tr1,tr2,tr3,tr4,term1,term2,term3,
 term4,sys1; tr1:={x=X/a^alpha,t=T/a^beta,u(x,t)=U(X,T)/a^zeta};
 tr2:=eval(tr1,zeta=1); Eq1:=combine(dchange(tr2,PDE,[X,T,U]));
 Eq2:=map(lhs,PDE=Eq1);
 term1:=select(has,select(has,select(has,rhs(Eq2),a),beta),a);
 term2:=expand(rhs(Eq2)/term1);
 term3:=select(has,select(has,term2,a),a);
 sys1:={select(has,op(1,term3),a)=a^0,
        select(has,op(2,term3),a)=a^0};
 tr3:=solve(sys1,{alpha,beta}); tr4:=subs(tr3,tr2);
 print(tr3,tr4); Eq3:=dchange(tr4,PDE,[X,T,U]);
 term4:=select(has,op(1,lhs(Eq3)),a);
 Eq4:=expand(Eq3/term4); PDE=simplify(Eq4); end proc:
InvKdV(KdV);  InvKdV(KdVT); InvKdV(mKdV); InvKdV(mKdVT);
```

Mathematica:

```
kdV=D[u[x,t],t]+6*u[x,t]*D[u[x,t],x]+D[u[x,t],{x,3}]==0
kdVT=D[u[x,t],t]+u[x,t]*D[u[x,t],x]+D[u[x,t],{x,3}]==0
mKdV=D[u[x,t],t]+6*u[x,t]^2*D[u[x,t],x]+D[u[x,t],{x,3}]==0
mKdVT=D[u[x,t],t]+u[x,t]^p*D[u[x,t],x]+D[u[x,t],{x,3}]==0
Off[Solve::ifun];
```

```
invKdV[pde_]:=Module[{eq1,eq2,eq3,eq4,tr1,tr2,tr3,tr4,term1,
 term2,term3,term4,sys1},
 trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
 trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&]; tr11={x->xN/a^alpha,
  t->tN/a^beta}; tr12=u[x,t]->uN[xN,tN]/a^zeta; tr2=tr12/.
  zeta->1; eq1[v_]:=First[((pde/.u->Function[{x,t},u[a^alpha*x,
  a^beta*t]/a])/.tr11)/.{u->v}]==0; eq2=Map[First,
  pde==eq1[uN]]; Print[eq2]; term1=trS1[trS1[eq2[[2]],beta],a];
 term2=Thread[eq2[[2]]/term1,Equal]//Expand; term3=trS1[term2,
  a]//PowerExpand; sys1={trS1[term3[[1]],a]==a^0,trS1[
  term3[[2]],a]==a^0}; tr3=Solve[sys1,{alpha,beta}]//First;
 tr41=tr11/.tr3; tr42=tr2/.tr3; Print[{tr3,tr41,tr42}//
  Flatten]; alpha1=tr3[[1,2]]; beta1=tr3[[2,2]];
 eq3[v_]:=First[((pde/.u->Function[{x,t},u[a^alpha1*x,
  a^beta1*t]/a])/.tr41)/.{u->v}]==0; term4=trS1[eq3[uN][[1,1]],
  a]; eq4=Thread[eq3[uN]/term4,Equal]//Expand;
 pde==PowerExpand[eq4]];
{invKdV[kdV], invKdV[kdVT], invKdV[mKdV], invKdV[mKdVT]}
```

□

Problem 2.42

Maple:

```
with(PDEtools): declare(u(x,t),U(X,T));interface(showassumed=0):
assume(a>0,zeta<0); W:=diff_table(u(x,t));
NLS:=I*W[t]+W[x,x]+gamma*W[]*abs(W[])^2=0;
tr1:={x=X/a^alpha,t=T/a^beta,u(x,t)=U(X,T)/a^zeta};
tr3:={x=X-a^alpha,t=T/a^beta,u(x,t)=U(X,T)/a^zeta};
tr2:=eval(tr1,alpha=1); tr4:=eval(tr3,alpha=1);
InvNLS:=proc(PDE,tr)
 local Eq1,Eq2,Eq3,Eq4,tr1,tr2,tr3,term1,term2,term3,term4,sys1;
 Eq1:=combine(dchange(tr,PDE,[X,T,U])); Eq2:=map(lhs,PDE=Eq1);
 term1:=select(has,select(has,select(has,rhs(Eq2),a),beta),a);
 term2:=expand(rhs(Eq2)/term1); term3:=select(has,select(has,
  term2,a),a); sys1:={select(has,op(1,term3),a)=a^0, select(has,
 op(2,term3),a)=a^0}; tr1:=solve(sys1,{beta,zeta}) assuming z<0;
 print(tr1); tr2:=subs(tr1,tr); print(tr2); Eq3:=dchange(tr2,
  PDE,[X,T,U]); term4:=select(has,op(1,lhs(Eq3)),a);
 Eq4:=expand(Eq3/term4); PDE=Eq4; end proc:
InvNLS(NLS,tr2); InvNLS(NLS,tr4);
```

Mathematica:

```
nLS=I*D[u[x,t],t]+D[u[x,t],{x,2}]+gamma*u[x,t]*Abs[u[x,t]]^2==0
Off[Solve::ifun]; invNLS[pde_]:=Module[{eq1,eq11,eq2,eq3,eq31,
 eq4,tr1,tr3,term1,term2,term3,term4,sys1,tr11,tr2,tr31,t1,t2},
 trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
 trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&]; tr11={x->xN/a^alpha,
  t->tN/a^beta}/.alpha->1; tr2=u[x,t]->uN[xN,tN]/a^zeta;
 eq1[v_]:=First[((pde/.u->Function[{x,t},
 u[a*x,a^beta*t]/a^zeta])/.tr11)/.{u->v}]==0;
 eq11=Assuming[{a\[Element]Reals,a<0,zeta\[Element]Reals},
  ExpandAll[PowerExpand[FullSimplify[PowerExpand[eq1[uN]]]]]];
 eq2=Map[First,pde==eq11]; term1=trS1[trS1[eq2[[2]],beta],a];
 term2=Thread[eq2[[2]]/term1,Equal]//Expand; term3=trS1[term2,
  a]//PowerExpand; sys1={trS1[term3[[1]],a]==a^0,trS1[term3[[2]],
  a]==a^0}; tr3=Solve[sys1,{beta,zeta}]//First; tr41=tr11/.tr3;
 tr42=tr2/.tr3; Print[{tr3,tr41,tr42}//Flatten];
 beta1=tr3[[1,2]]; zeta1=tr3[[2,2]]; eq3[v_]:=First[((pde/.u->
 Function[{x,t},u[a*x,a^beta1*t]/a^zeta1])/.tr41)/.{u->v}]==0;
 eq31=Assuming[{a\[Element]Reals,a< 0,zeta\[Element]Reals},
  ExpandAll[PowerExpand[FullSimplify[PowerExpand[eq3[uN]]]]]];
 pde==PowerExpand[eq31]]; invNLS[nLS]
```

```
tr11={x->xN-a^alpha,t->tN/a^beta}/.alpha->1;
eq1[v_]:=First[((pde/.u->Function[{x,t},u[a+x,a^beta*t]/
 a^zeta])/.tr11)/.{u->v}]==0; eq3[v_]:=First[((pde/.u->
 Function[{x,t},u[a+x,a^beta1*t]/a^zeta1])/.tr41)/.{u->v}]==0;
```
□

2.7.2 Group Analysis

Problem 2.43

Maple:

```
with(PDEtools): declare(u(x,y),(xi1,xi2,eta,zeta1,zeta2,
 H,Q)(x,y,u)); U:=diff_table(u(x,y)); DepVars:=u(x,y);
PDE1:=U[x,x]+U[y,y]-G(u)=0; F:=U[x,x]+U[y,y]-G(u);
UF,Uxi1,Uxi2,Ueta:=diff_table(F),diff_table(xi1(x,y,u)),
 diff_table(xi2(x,y,u)),diff_table(eta(x,y,u));
DT:=(Q,Z)->Q[Z]+U[Z]*Q[u];
DTM:=(Q,Z)->diff(Q,Z)+diff(u(x,y),Z)*diff(Q,u);
zeta1:=expand(DT(Ueta,x)-U[x]*DT(Uxi1,x)-U[y]*DT(Uxi2,x));
zeta2:=expand(DT(Ueta,y)-U[x]*DT(Uxi1,y)-U[y]*DT(Uxi2,y));
```

```
zeta11:=expand(map(DTM,zeta1,x)-U[x,x]*DT(Uxi1,x)-U[x,y]*
 DT(Uxi2,x)); zeta12:=expand(map(DTM,zeta1,y)-U[x,x]*DT(Uxi1,y)
 -U[x,y]*DT(Uxi2,y)); zeta22:=expand(map(DTM,zeta2,y)-U[x,y]*
 DT(Uxi1,y)-U[y,y]*DT(Uxi2,y));
tr1:={F_x=0,F_y=0,F_ux=0,F_uy=0,F_uxx=1,F_uxy=0,F_uyy=1};
tr2:={Zeta1=zeta1,Zeta2=zeta2,Zeta11=zeta11,Zeta12=zeta12,
 Zeta22=zeta22}; InvCond:=xi1*F_x+xi2*F_y+eta*UF[u]+Zeta1*F_ux+
 Zeta2*F_uy+Zeta11*F_uxx+Zeta12*F_uxy+Zeta22*F_uyy;
Eq1:=eval(InvCond,tr1); Eq2:=eval(Eq1,tr2);
Eq3:=expand(subs(U[y,y]=G(u)-U[x,x],Eq2));
```

Maple:

```
LD:=[U[x],U[y],U[x,y],U[x,x]]; LE1:=[[U[x,x],U[x]],[U[x,x],
 U[y]],[U[x],U[y]]]; LE2:=[U[x,x],U[x,y],U[x]^2,U[y]^2,U[x],
 U[y]]; E00:=sort(remove(has,Eq3,LD)=0); E0:=subsop(
 1=-eta(x,y,u)*diff(G(u),u),lhs(E00))=0; k1:=nops(LE1);
k2:=nops(LE2); S1:=NULL: for i from 1 to k1 do
 T||i:=expand(select(has,Eq3,LE1[i])/LE1[i][1]/LE1[i][2]):
 E||i:=remove(has,T||i,LD)=0; print(cat(E,i),LE1[i],E||i); od:
for i from 1 to k2 do
 T||(i+k1):=expand(select(has,Eq3,LE2[i])/LE2[i]):
 E||(i+k1):=remove(has,T||(i+k1),LD)=0;
 print(cat(E,i+k1),LE2[i],E||(i+k1));  od:
LE1D:=[[U[x,y],U[x]],[U[x,y],U[x]],[U[x]^2,U[y]],[U[y]^2,
 U[x]]]; LE2D:=[U[x]^3,U[y]^3]; k3:=nops(LE1D); k4:=nops(LE2D);
for i from 1 to k3 do
 T||(i+k1+k2):=expand(select(has,Eq3,LE1D[i])/LE1D[i][1]
   /LE1D[i][2]): E||(i+k1+k2):=remove(has,T||(i+k1+k2),LD)=0;
 print(cat(E,i+k1+k2),LE1D[i], E||(i+k1+k2)); od:
for i from 1 to k4 do
 T||(i+k1+k2+k3):=expand(select(has,Eq3,LE2D[i])/LE2D[i]):
 E||(i+k1+k2+k3):=remove(has,T||(i+k1+k2+k3),LD)=0;
 print(cat(E,i+k1+k2+k3),LE2D[i], E||(i+k1+k2+k3)); od:
for i from 1 to 1+k1+k2+k3+k4 do E||(i-1); S1:=S1,E||(i-1) od:
Sys1:=[S1];
```

Maple:

```
Sol0:=pdsolve(Sys1,{eta(x,y,u),xi1(x,y,u),xi2(x,y,u)});
L10:=subs(_C1=0,_C2=0,_C3=1,Sol0); L20:=subs(_C3=0,_C2=1,_C1=0,
 Sol0); L30:=subs(_C1=-1,_C2=0,_C3=0,Sol0);
```

```
X0:=f->xi1(x,y,u)*diff(f(x,y,u),x)+xi2(x,y,u)*diff(f(x,y,u),y)
 +eta(x,y,u)*diff(f(x,y,u),u);
X10:=eval(X0(f),L10); X20:=eval(X0(f),L20); X30:=eval(X0(f),L30);
A1:=pdsolve({Sys1[2],Sys1[3],Sys1[7]},{eta(x,y,u),xi1(x,y,u),
 xi2(x,y,u)}); A2:=convert(subs(_F1(x,y)=xi1(x,y),
 _F2(x,y)=xi2(x,y),_F3(x,y)=a(x,y),_F4(x,y)=b(x,y),A1),list);
A560:=[expand((diff(Sys1[5],x)+diff(Sys1[6],y))/(2)),
       expand((diff(Sys1[5],y)-diff(Sys1[6],x))/2)];
A56:=subs(xi1(x,y,u)=xi1(x,y),xi2(x,y,u)=xi2(x,y),A560);
A890:={Sys1[8],Sys1[9]};A891:=algsubs(A2[3],algsubs(A2[2],A890));
A892:=simplify(algsubs(A2[1],A891),{A56[1]});
A89:=subs(_C1=A,pdsolve(A892 union {diff((a(x,y),y))=0},a(x,y)));
Sys20:=convert(simplify(algsubs(A89[1],algsubs(A2[3],
 algsubs(A2[2],algsubs(A2[1],Sys1)))),convert(A56,set)),set);
Sys2:=convert(Sys20 minus {0=0},list); Sys21:=algsubs(diff(
 eta(x,y),y)=0,algsubs(b(x,y)=0,subs(A=0,Sys2)));
Sol1:=pdsolve(Sys21,{xi1(x,y),xi2(x,y)}) union {eta(x,y)=0};
L1:=subs(_C1=0,_C2=0,_C3=1,Sol1); L2:=subs(_C3=0,_C2=1,_C1=0,
 Sol1); L3:=subs(_C1=-1,_C2=0,_C3=0,Sol1);
X:=f->xi1(x,y)*diff(f(x,y,u),x)+xi2(x,y)*diff(f(x,y,u),y)+
 eta(x,y)*diff(f(x,y,u),u); X1:=eval(X(f),L1); X2:=eval(X(f),L2);
X3:=eval(X(f),L3);
```

Maple:

```
Equat1:=algsubs(b(x,y)=B,Sys2[3]); Equat2:=simplify(dsolve(
 Equat1,G(u))); Equat3:=algsubs(diff(xi2(x,y),y)=M,Equat2);
Equat31:=subs({_C1=1,B=0,A=1},Equat3); trG:=subs(-2*M+1=k,
 Equat31); trA0:=A=solve(algsubs(trG,subs(B=0,algsubs(
 diff(xi2(x,y),y)=M,Equat1))),A); trM:=M=1;
trA:=simplify(subs(trM,trA0)); Solxi1:=subs(_F1(y)=0,
 pdsolve(subs(trM,algsubs(diff(xi2(x,y),y)=M,Sys2[1])),
 xi1(x,y))); Sys2[2]; Solxi2:=subs(_F1(y)=y,
 pdsolve(algsubs(Solxi1,Sys2[2]),xi2(x,y)));
test1:=algsubs(xi2(x,y)=y,algsubs(xi1(x,y)=x,{Sys2[1]/2,
 Sys2[2]/(-2)})); L40:=subs(a(x,y)=rhs(trA),b(x,y)=0,xi1(x,y)=
 rhs(Solxi1),xi2(x,y)=rhs(Solxi2),A2);
L41:=subs(eta(x,y,u)=eta(x,y),xi1(x,y,u)=xi1(x,y),xi2(x,y,u)=
 xi2(x,y),L40); X41:=eval(X(f),L41);
Equat11:=subs(A=0,algsubs(b(x,y)=B,Sys2[3]));
Equat21:=simplify(dsolve(Equat11,G(u))); Equat311:=algsubs(
 diff(xi2(x,y),y)=M,subs(_C1=1,Equat21));
```

```
trG:=subs(-2*u*M/B=u,Equat31); trB0:=isolate(-2*u*M/B=u,B);
trB:=subs(M=1,trB0); L402:=subs(a(x,y)=0,b(x,y)=rhs(trB),
 xi1(x,y)=rhs(Solxi1),xi2(x,y)=rhs(Solxi2),A2);
L42:=subs(eta(x,y,u)=eta(x,y),xi1(x,y,u)=xi1(x,y),
 xi2(x,y,u)=xi2(x,y),L402); X42:=eval(X(f),L42);
```
□

Problem 2.44

Maple:

```
with(PDEtools): declare(u(x,y)); U:=diff_table(u(x,y));
interface(showassumed=0): assume(k>0,u>0);
PDE1:=U[x,x]+U[y,y]-G(u)=0; show; DepVars:=u(x,y);
Infs:=Infinitesimals(PDE1,DepVars,split=false,
      displayfunction=false);
Infs1:=eval(Infs,{_C1=1,_C2=0,_C3=0}); Infs2:=eval(Infs,
 {_C1=0,_C2=1,_C3=0}); Infs3:=eval(Infs,{_C1=0,_C2=0,_C3=1});
G1:=InfinitesimalGenerator(Infs1,DepVars,expanded);
G2:=InfinitesimalGenerator(Infs2,DepVars,expanded);
G3:=InfinitesimalGenerator(Infs3,DepVars,expanded);
PDE2:=subs(G(u)=u^k,PDE1);
Infs2:=simplify(Infinitesimals(PDE2,DepVars,split=false,
       displayfunction=false));
Infs41:=eval(Infs2,{_C1=0,_C2=0,_C3=1,_C4=0});
G41:=InfinitesimalGenerator(Infs41,DepVars,expanded);
PDE3:=subs(G(u)=exp(u),PDE1);
Infs3:=simplify(Infinitesimals(PDE3,DepVars,split=false,
       displayfunction=false));
Infs42:=eval(Infs3,{_C1=0,_F2=0}); G42:=simplify(
        InfinitesimalGenerator(Infs42,DepVars,expanded));
```
□

Problem 2.45

Maple:

```
with(PDEtools): declare((u,G)(x,t)); U,GU:=diff_table(u(x,t)),
 diff_table(G(u(x,t))); PDE1:=U[t]-G(u)*U[x,x]-GU[x]*U[x]=0;
show; DepVars:=u(x,t); Op1:=split=false,displayfunction=false;
Infs:=Infinitesimals(expand(PDE1),DepVars,Op1); Infs1:=eval(Infs,
 {_C1=1,_C2=0,_C3=0}); Infs2:=eval(Infs,{_C1=0,_C2=1,_C3=0});
Infs3:=eval(Infs,{_C1=0,_C2=0,_C3=1});
G1:=InfinitesimalGenerator(Infs1,DepVars,expanded);
```

```
G2:=InfinitesimalGenerator(Infs2,DepVars,expanded);
G3:=InfinitesimalGenerator(Infs3,DepVars,expanded);
PDE2:=diff(u(x,t),t)-u(x,t)^2*diff(u(x,t),x$2)-diff(u(x,t)^2,x)*
 diff(u(x,t),x)=0; Infs2:=Infinitesimals(PDE2,DepVars,Op1);
Infs41:=eval(Infs2,{_C1=0,_C2=0,_C3=1,_C4=0});
G41:=InfinitesimalGenerator(Infs41,DepVars,expanded);
PDE3:=diff(u(x,t),t)-exp(u(x,t))*diff(u(x,t),x$2)-
 diff(exp(u(x,t)),x)*diff(u(x,t),x)=0;
Infs3:=simplify(Infinitesimals(PDE3,DepVars,Op1));
Infs42:=eval(Infs3,{_C1=0,_C2=0,_C3=1,_C4=0});
G42:=InfinitesimalGenerator(Infs42,DepVars,expanded);
```
□

Problem 2.46

Maple:

```
with(PDEtools): declare(u(x,t),G(u(x,t))); DepVars:=u(x,t);
U,GU:=diff_table(u(x,t)),diff_table(G(u(x,t)));
PDE1:=U[t,t]-G(u)*U[x,x]-GU[x]*U[x]=0; show;
Op1:=split=false,displayfunction=false;
Infs:=Infinitesimals(expand(PDE1),DepVars,Op1);
Infs1:=eval(Infs,{_C1=1,_C2=0,_C3=0});
Infs2:=eval(Infs,{_C1=0,_C2=1,_C3=0});
Infs3:=eval(Infs,{_C1=0,_C2=0,_C3=1});
G1:=InfinitesimalGenerator(Infs1,DepVars,expanded);
G2:=InfinitesimalGenerator(Infs2,DepVars,expanded);
G3:=InfinitesimalGenerator(Infs3,DepVars,expanded);
PDE2:=diff(u(x,t),t$2)-u(x,t)^2*diff(u(x,t),x$2)
 -diff(u(x,t)^2,x)*diff(u(x,t),x)=0;
Infs2:=Infinitesimals(PDE2,DepVars,Op1);
Infs41:=eval(Infs2,{_C1=1,_C2=0,_C3=0,_C4=0});
G41:=InfinitesimalGenerator(Infs41,DepVars,expanded);
PDE3:=diff(u(x,t),t$2)-exp(u(x,t))*diff(u(x,t),x$2)
 -diff(exp(u(x,t)),x)*diff(u(x,t),x)=0;
Infs3:=simplify(Infinitesimals(PDE3,DepVars,Op1));
Infs42:=eval(Infs3,{_C1=1,_C2=0,_C3=0,_C4=0});
G42:=InfinitesimalGenerator(Infs42,DepVars,expanded);
```
□

2.7.3 Invariant Solutions

Problem 2.47

Maple:

```
with(PDEtools): interface(showassumed=0): assume(k>0,x>0);
declare(u(x,y),In(x,y,u),Phi(z)); U:=diff_table(u(x,y));
PDE1:=U[x,x]+U[y,y]-F(u)=0; show; DepVars:=u(x,y);
term0:=2/(1-k); X41:=f->x*diff(f,x)+y*diff(f,y)+term0*u*
 diff(f,u); X41(In(x,y,u))=0; ODEs:=dx/x=(dy/y=1/term0*du/u);
Eq1:=lhs(ODEs)=lhs(rhs(ODEs)); Eq2:=(lhs(ODEs)=rhs(rhs(ODEs)))*
 term0; FI10:=int(lhs(Eq1)/dx,x)=int(rhs(Eq1)/dy,y)+C1;
FI11:=subs(1/exp(C1)=C1,simplify(map(exp,FI10-lhs(FI10)))/
 exp(C1)); FI20:=int(lhs(Eq2)/dx,x)=int(rhs(Eq2)/du,u)+C2;
FI21:=subs(exp(C2)=C2,simplify(map(exp,FI20)));
FI22:=simplify(subs(1/C2=C2,(FI21/lhs(FI21))/C2));
I1:=rhs(FI11); I2:=rhs(FI22);
InvSol:=u(x,y)=simplify(solve(I2=Phi(I1),u));
PDE2:=u->diff(u,x$2)+diff(u,y$2)-u^k=0; OdePhi:=PDE2(rhs(
 InvSol)); OdePhi1:=expand(algsubs(y/x=z,OdePhi));
term1:=select(has,op(2,lhs(OdePhi1)),x);
OdePhi2:=map(simplify,expand(lhs(OdePhi1)/term1));
term2:=select(has,op(2,OdePhi2),k); OdePhi3:=map(simplify,
 OdePhi2/term2); OdePhi4:=convert(collect(OdePhi3,[Phi,k,z]),
 diff); SolPhi:=[dsolve(OdePhi4,Phi(z))];
SolPhi1:=unapply(SolPhi[2],z);
InvSolFin:=simplify(subs(SolPhi1(y/x),InvSol),exp);
expand(pdetest(subs(k=3,InvSolFin),subs(k=3,PDE2(u(x,y)))));
```

Mathematica:

```
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&]; {term0=2/(1-k),
 pde1=D[u[x,y],{x,2}]+D[u[x,y],{y,2}]-f[u]==0}
x41[f_]:=x*D[f,x]+y*D[f,y]+term0*u*D[f,u]; x41[iN[x,y,u]]==0
{odes=dx/x==(dy/y==1/term0*du/u), eq1=odes[[1]]==odes[[2, 1]]}
eq2=Thread[(odes[[1]]==odes[[2,2]])*term0,Equal]//Expand
fI10=Integrate[eq1[[1]]/dx,x]==Integrate[eq1[[2]]/dy,y]+c1
{fI11=Solve[Map[Exp,Thread[fI10-fI10[[2]],Equal]]/.Exp[c1_]->
 -c1,c1]//First, termeq21=trS1[Thread[eq2[[1]]/dx,Equal],x],
 termeq22=trS3[Thread[eq2[[1]]/dx,Equal],x]}
```

```
{fI20=termeq22*Integrate[Thread[termeq21,Equal],x]==Integrate[
 Thread[eq2[[2]]/du,Equal],u]+c2, fI21=Map[Exp,fI20]/.
 Exp[c2]->c2, fI22=Thread[fI21/fI21[[1]]/c2,Equal]/.
 {1/c2->c2}//Simplify, i1=fI11[[1,2]], i2=fI22[[2]]}
invSol=Solve[i2==phi[i1],u]//First
pde2[u_]:=D[u,{x,2}]+D[u,{y,2}]-u^k==0;
{odePhi=pde2[invSol[[1,2]]], odePhi1=odePhi/.{y->x*z}//Expand}
{term1=trS1[odePhi1[[1,1]],x], odePhi2=Thread[odePhi1[[1]]/
 term1,Equal]//Expand, term2=trS1[odePhi2[[1]],k]}
{odePhi3=Thread[odePhi2/term2,Equal]//FullSimplify//Together//
 PowerExpand, odePhi4=Collect[odePhi3,{phi[z],k,z}]//
 FullSimplify, solPhi=DSolve[odePhi4==0,phi[z],z]}
```

Maple:

```
Infs:=Infinitesimals(PDE2(u(x,y)),DepVars,split=false,
      displayfunction=false);
Infs1:=simplify(Infs,{_C1=0,_C2=0,_C3=1,_C4=0}) assuming u>0;
X41M:=InfinitesimalGenerator(Infs1,DepVars,expanded);
DetSys:=DeterminingPDE(PDE2(u(x,y)));
SymmetryTest(Infs1,PDE2(u(x,y)));
SymmetryTransformation(Infs1,DepVars,V(X,T));
SimilarityTransformation(Infs1,DepVars,V(X,T));
InvariantTransformation(Infs1,u(x,t),V(X,T));
S:=[x,y,2/(1-k)*u];
CC:=CanonicalCoordinates(S,DepVars,V(X,T));
Invs:=[Invariants(S,u(x,t),jetnotation=false)];
```

□

Problem 2.48

Maple:

```
with(PDEtools): declare(u(x,t),G(u(x,t)),In(x,t,u),Phi(z));
DepVars:=u(x,t); U,GU:=diff_table(u(x,t)),diff_table(G(u(x,t)));
PDE1:=U[t,t]-G(u)*U[x,x]-GU[x]*U[x]=0; show;
Infs:=Infinitesimals(expand(PDE1),DepVars,split=false,
      displayfunction=false);
Infs1:=eval(Infs,{_C1=1,_C2=0,_C3=0});
X3:=InfinitesimalGenerator(Infs1,DepVars,expanded); S:=[x,t,0];
Invs:=[Invariants(S,u(x,t),jetnotation=false)];
expand(map(X3,Invs)); X3(f(x,t)); X3(In(x,t,u))=0;
ODEs:=dx/x=(dt/t=du/zero); Eq1:=lhs(ODEs)=lhs(rhs(ODEs));
```

```
Eq2:=numer(rhs(rhs(ODEs)))=0;
FI10:=int(lhs(Eq1)/dx,x)=int(rhs(Eq1)/dt,t)+C1;
FI11:=subs(1/exp(C1)=C1,simplify(map(exp,
      FI10-lhs(FI10)))/exp(C1));
FI20:=int(lhs(Eq2)/du,u)=int(rhs(Eq2)/du,u)+C2; I1:=rhs(FI11);
I2:=lhs(FI20); InvSol:=u(x,t)=simplify(solve(I2=Phi(I1),u));
PDE2:=u->diff(u,t$2)-diff(G(u)*diff(u,x),x)=0;
OdePhi:=PDE2(rhs(InvSol));
OdePhi1:=expand(algsubs(t/x=z,OdePhi));
term1:=select(has,op(1,lhs(OdePhi1)),x);
OdePhi2:=map(simplify,expand(lhs(OdePhi1)/term1));
```

Mathematica:

```
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity] &];
pde2[u_]:=D[u,{t,2}]-D[g[u]*D[u,x],x]==0;
{odes=dx/x==(dt/t==du/zero), eq1=odes[[1]]==odes[[2,1]]}
eq2=Numerator[odes[[2,2]]]==0
fI10=Integrate[eq1[[1]]/dx,x]==Integrate[eq1[[2]]/dt,t]+c1
fI11=Solve[Map[Exp,Thread[fI10-fI10[[2]],Equal]]/.
 Exp[c1_]->-c1,c1]//First
fI20=Integrate[eq2[[1]]/du,u]==Integrate[eq2[[2]]/du,u]+c2
{i1=fI11[[1,2]], i2=fI20[[1]], invSol=Solve[i2==phi[i1],u]//
 First, odePhi=pde2[invSol[[1,2]]], odePhi1=odePhi/.t->x*z//
 Expand, term1=trS1[odePhi1[[1,1]],x]}
odePhi2=Collect[Thread[odePhi1[[1]]/term1,Equal]//Expand,z]
```
□

Problem 2.49

Maple:

```
with(PDEtools): declare(u(x,t),G(u(x,t)),In(x,t,u),Phi(z));
DepVars:=u(x,t); U,GU:=diff_table(u(x,t)),diff_table(G(u(x,t)));
PDE1:=U[t]-G(u)*U[x,x]-GU[x]*U[x]=0; show;
Infs:=Infinitesimals(expand(PDE1),DepVars,split=false,
 displayfunction=false); Infs1:=eval(Infs,{_C1=2,_C2=0,_C3=0});
S:=[x,2*t,0]; X3:=InfinitesimalGenerator(Infs1,DepVars,expanded);
Invs:=[Invariants(S,u(x,t),jetnotation=false)];
X3(f(x,t)); X3(In(x,t,u))=0; ODEs:=dx/x=(dt/(2*t)=du/zero);
Eq1:=lhs(ODEs)=lhs(rhs(ODEs)); Eq2:=numer(rhs(rhs(ODEs)))=0;
FI10:=(int(lhs(Eq1)/dx,x)=int(rhs(Eq1)/dt,t)+C1)*2;
```

```
FI11:=simplify(map(exp,FI10-lhs(FI10)));
termC1:=select(has,rhs(FI11),C1); FI12:=subs(1/termC1=C1,
 FI11/termC1); FI20:=int(lhs(Eq2)/du,u)=int(rhs(Eq2)/du,u)+C2;
I1:=rhs(FI12); I2:=lhs(FI20); InvSol:=u(x,y)=simplify(solve(
 I2=Phi(I1),u)); PDE2:=u->diff(u,t)-diff(G(u)*diff(u,x),x)=0;
OdePhi:=PDE2(rhs(InvSol)); OdePhi1:=expand(algsubs(t/x^2=z,
 OdePhi)); term1:=select(has,op(1,lhs(OdePhi1)),x);
OdePhi2:=map(simplify,expand(lhs(OdePhi1)/term1));
```

Mathematica:

```
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
pde2[u_]:=D[u,t]-D[g[u]*D[u,x],x]==0;
{odes=dx/x==(dt/(2*t)==du/zero),eq1=odes[[1]]==odes[[2,1]]}
eq2=Numerator[odes[[2,2]]]==0
fI10=Integrate[eq1[[1]]/dx,x]==Integrate[eq1[[2]]/dt,t]+c1
fI11=Solve[Map[Exp,Thread[fI10-fI10[[2]],Equal]]/.
 Exp[c1_]->-c1,c1]//First
fI20=Integrate[eq2[[1]]/du,u]==Integrate[eq2[[2]]/du,u]+c2
{i1=fI11[[1,2]]^2, i2=fI20[[1]]}
invSol=Solve[i2==phi[i1],u]//First
{odePhi=pde2[invSol[[1,2]]], odePhi1=odePhi/.t->x^2*z//Expand,
 term1=trS1[odePhi1[[1,1]],x]}
odePhi2=Collect[Thread[odePhi1[[1]]/term1,Equal]//Expand,z]
```

□

Problem 2.50

Maple:

```
with(PDEtools): interface(showassumed=0): assume(k>1,x>0,u>0);
declare(u(x,t),In(x,t,u),Phi(z)); DepVars:=u(x,t);
PDE1:=u->diff(u,t)-diff(u^k*diff(u,x),x)=0;
Infs:=Infinitesimals(PDE1(u(x,t)),DepVars,split=false,
 displayfunction=false); Infs1:=eval(Infs,{_C1=0,_C2=0,_C3=1,
 _C4=0}); S:=[k*x/2,0,u];
X4:=InfinitesimalGenerator(Infs1,DepVars,expanded);
Invs:=[Invariants(S,u(x,t),jetnotation=false)];
X4(f(x,t)); X4(In(x,t,u))=0; ODEs:=dx/(k*x/2)=(dt/zero=du/u);
Eq1:=lhs(ODEs)=rhs(rhs(ODEs)); Eq2:=numer(lhs(rhs(ODEs)))=0;
FI10:=(int(lhs(Eq1)/dx,x)=int(rhs(Eq1)/du,u)+C1);
FI11:=expand(map(exp,FI10-lhs(FI10))); termC1:=select(has,
 rhs(FI11),C1); FI12:=subs(1/termC1=C1,FI11/termC1);
FI20:=int(lhs(Eq2)/dt,t)=int(rhs(Eq2)/du,u)+C2;
```

```
I2:=rhs(FI12); I1:=lhs(FI20);
InvSol:=u(x,t)=simplify(solve(I2=Phi(I1),u)) assuming u>0;
OdePhi:=expand(PDE1(rhs(InvSol))); term1:=select(has,op(2,
 lhs(OdePhi)),x); OdePhi1:=map(simplify,expand(
 lhs(OdePhi)/term1)); SolPhi:=[dsolve(OdePhi1,Phi(t))];
SolPhi1:=lhs(SolPhi[1])=subs(_C1=A,combine(rhs(SolPhi[1])));
SolPhi2:=unapply(SolPhi1,t); SolPhi2(t);
InvSolFin:=factor(subs(SolPhi2(t),InvSol));
simplify(pdetest(subs(k=3,InvSolFin),subs(k=3,PDE1(u(x,t)))));
```

Mathematica:

```
Off[Solve::ifun]; trS1[eq_,var_]:=Select[eq,MemberQ[#,var,
 Infinity]&]; pde1[u_]:=D[u,t]-D[u^k*D[u,x],x]==0;
{odes=dx/(k*x/2)==(dt/zero==du/u),
 eq1=odes[[1]]==odes[[2,2]],eq2=Numerator[odes[[2,1]]]==0}
{fI10=Integrate[eq1[[1]]/dx,x]==Integrate[eq1[[2]]/du,u]+
 c1//Simplify, fI11=Solve[(Map[Exp,Thread[fI10-fI10[[1]],
 Equal]]//ExpandAll)/.Exp[-c1]->c1//Expand,c1]//First}
fI20=Integrate[eq2[[1]]/dt,t]==Integrate[eq2[[2]]/du,u]+c2
{i2=fI11[[1,2]], i1=fI20[[1]]}
invSol=Solve[i2==phi[i1],u]//First
odePhi=pde1[invSol[[1,2]]]//PowerExpand//ExpandAll
term1=trS1[odePhi[[1,1]],x]
odePhi1=Collect[Thread[odePhi[[1]]/term1,Equal]//Expand,z]
solPhi=DSolve[odePhi1==0,phi[t],t]//ExpandAll//First
{solPhi1=solPhi/.C[1]->a, invSolFin=invSol/.solPhi1}
test1=pde1[invSolFin[[1,2]]]//PowerExpand//FullSimplify
```

□

Problem 2.51

Maple:

```
with(PDEtools): declare(u(x,t),G(u(x,t)),In(x,t,u),Phi(z));
DepVars:=u(x,t); U,GU:=diff_table(u(x,t)),diff_table(G(u(x,t)));
PDE1:=U[t]-G(u)*U[x,x]-GU[x]*U[x]=0; Infs:=Infinitesimals(PDE1,
 DepVars,split=false,displayfunction=false); Infs1:=eval(Infs,
 {_C1=0,_C2=1,_C3=0}); Infs2:=eval(Infs,{_C1=0,_C2=0,_C3=1});
Infs3:=eval(Infs,{_C1=1,_C2=0,_C3=0});
for i from 1 to 3 do
 X||i:=InfinitesimalGenerator(Infs||i,DepVars,expanded); od;
collect(X1(In(x,t,u))+a*X2(In(x,t,u)),diff)=0;
ODEs:=dx/a=(dt/1=du/zero); Eq1:=lhs(ODEs)=lhs(rhs(ODEs));
```

```
Eq2:=numer(rhs(rhs(ODEs)))=0;
FI10:=expand((int(lhs(Eq1)/dx,x)=int(rhs(Eq1)/dt,t)+C1)*a);
termC1:=select(has,rhs(FI10),C1); FI11:=isolate(subs(termC1=C1,
 FI10),C1); FI20:=int(rhs(Eq2)/du,u)+C2=int(lhs(Eq2)/du,u);
I1:=rhs(FI11); I2:=rhs(FI20);
InvSol:=u(x,t)=simplify(solve(I2=Phi(I1),u));
```

Mathematica:

```
Off[Solve::ifun]; pde1[u_]:=D[u,t]-g[u]*D[u,{x,2}]-
 D[g[u],x]*D[u,x]==0; {odes=dx/a==(dt/1==du/zero),
 eq1=odes[[1]]==odes[[2,1]], eq2=Numerator[odes[[2,2]]]==0}
{fI10=Integrate[eq1[[1]]/dx,x]==Integrate[eq1[[2]]/dt,t]+c1//
 Simplify, fI11=Solve[fI10,c1]//Expand//First}
fI20=Integrate[eq2[[2]]/du,u]+c2==Integrate[eq2[[1]]/du,u]
{i1=fI11[[1,2]]*a//Expand, i2=fI20[[2]]}
invSol=Solve[i2==phi[i1],u]//First
```

□

Problem 2.52

Maple:

```
with(PDEtools): declare(u(x,t),In(x,t,u),Phi(z));
DepVars:=u(x,t); PDE1:=u->diff(u,t)-diff(exp(u)*diff(u,x),x)=0;
Infs:=Infinitesimals(PDE1(u(x,t)),DepVars,split=false,
 displayfunction=false);
Infs3:=eval(Infs,{_C1=1,_C2=0,_C3=0,_C4=0});
Infs4:=eval(Infs,{_C1=0,_C2=0,_C3=1,_C4=0});
for i from 3 to 4 do
 X||i:=InfinitesimalGenerator(Infs||i,DepVars,expanded); od;
collect(X3(In(x,t,u))+a*X4(In(x,t,u)),diff)=0;
ODEs:=dx/(1/2*x*(a+1))=(dt/t=du/a);
Eq1:=lhs(ODEs)=lhs(rhs(ODEs)); Eq2:=rhs(ODEs);
FI10:=expand((int(lhs(Eq1)/dx,x)=int(rhs(Eq1)/dt,t)+C1)*(a+1));
FI11:=expand(map(exp,FI10-lhs(FI10))); termC1:=select(has,
 rhs(FI11),C1); FI12:=combine(subs(1/termC1=1/C1,FI11/termC1));
FI13:=isolate(FI12,C1);
FI20:=simplify((int(lhs(Eq2)/dt,t)=int(rhs(Eq2)/du,u)+C2)*a);
termC2:=select(has,rhs(FI20),C2);
FI21:=subs(termC2=C2,FI20); FI22:=isolate(FI21,C2);
I1:=combine(rhs(FI13)); I2:=rhs(FI22);
InvSol:=u(x,t)=simplify(solve(I2=Phi(I1),u));
OdePhi:=expand(PDE1(rhs(InvSol)));
```

```
OdePhi1:=simplify(algsubs(I1=z,OdePhi));
term1:=denom(lhs(OdePhi1)); OdePhi2:=expand(OdePhi1*term1);
OdePhi3:=numer(lhs(factor(OdePhi2)))=0;
OdePhi4:=convert(collect(OdePhi3,[exp,a,z]),diff);
```

Mathematica:

```
Off[Solve::ifun]; trS1[eq_,var_]:=Select[eq,MemberQ[#,var,
 Infinity]&]; pde1[u_]:=D[u,t]-D[Exp[u]*D[u,x],x]==0;
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&];
{odes=dx/(1/2*x*(a+1))==(dt/t==du/a), eq1=odes[[1]]==
 odes[[2,1]],eq2=odes[[2]], fI10=Thread[(Integrate[
 eq1[[1]]/dx,x]==Integrate[eq1[[2]]/dt,t]+c1)*(a+1),
 Equal]//FullSimplify, fI11=Map[Exp,fI10]//ExpandAll}
{termC1=trS1[fI11[[1]],a]/.t->1, fI12=Thread[fI11/termC1,
 Equal]/.Exp[-c1-a*c1]->1/c1, fI13=Solve[fI12,c1]//First}
{fI20=Thread[(Integrate[eq2[[1]]/dt,t]==Integrate[
 eq2[[2]]/du,u]+c2)*a,Equal]//Expand, termC2=trS1[
 fI20[[2]], c2], fI21=fI20/.termC2->c2, fI22=Solve[
 fI21,c2]//First, i1=fI13[[1,2]]/(a+1)^2, i2=fI22[[1,2]]}
invSol=Solve[i2==phi[i1],u]//First
odePhi=pde1[invSol[[1,2]]]//PowerExpand//ExpandAll
{odePhi1=odePhi/.x^2->z/t^(-1-a), term1=trS1[odePhi1[[1,1]],t],
 odePhi2=Thread[odePhi1/term1,Equal]//Expand,
 odePhi3=Numerator[Factor[odePhi2[[1]]]]==0}
Collect[Thread[odePhi3,Equal],{Exp[phi[z]],a,z}]
```
□

2.8 Nonlinear Systems

2.8.1 Traveling Wave Reductions

Problem 2.53

Maple:

```
with(PDEtools): interface(showassumed=0):
assume(k>0,lambda>0,x>0,t>0,xi>0); tr1:=k*x-lambda*t=xi;
declare((W1,W2)(xi),(w1,w2,u,v)(x,t),(F,G)(u,v));
U,V:=diff_table(u(x,t)),diff_table(v(x,t));
F:=(u,v)->a1*u*ln(v); G:=(u,v)->a2*v*ln(u);
Sys1:=(w1,w2)->[diff(w1,x)-F(w1,w2)=0,diff(w2,t)-G(w1,w2)=0];
tr1:=k*x-lambda*t=xi;
Sys2:=expand(Sys1(W1(lhs(tr1)),W2(lhs(tr1))));
```

```
Sys3:=algsubs(tr1,Sys2); Sys4:=map(convert,Sys3,diff);
Sol1:=combine(dsolve(Sys4,{W1(xi),W2(xi)}));
Sol11:=simplify(subs(_C1=0,_C2=1,Sol1));
trW1:=expand(subs(Sol11[1],Sol11[2])); trW2:=Sol11[1];
test1:=simplify(expand(subs(trW1,trW2,Sys4)),symbolic);
```

Mathematica:

```
Off[InverseFunction::"ifun"]; f[u_, v_]:=a1*u*Log[v];
g[u_,v_]:=a2*v*Log[u]; sys1[w1_,w2_]:={D[w1,x]-f[w1, w2]==0,
 D[w2,t]-g[w1,w2]==0}; {tr1=k*x-lambda*t->xi,
 sys2=sys1[w1[tr1[[1]]],w2[tr1[[1]]]]//Expand, sys3=sys2/.tr1}
sol1=DSolve[sys3,{w1[xi],w2[xi]},xi];
{sol2=sol1/.{C[1]->1,C[2]->0}//PowerExpand, n=Length[sol2]}
sols=Table[sol2[[i]]//FullSimplify,{i,1,n}]
{trW1=sols[[1,2]], trW2=sols[[1,1]], test1=sys3/.trW1/.
 D[trW1,xi]/.trW2/.D[trW2,xi]//PowerExpand//FullSimplify}
```

□

Problem 2.54

Maple:

```
with(PDEtools): interface(showassumed=0):
assume(k>0,lambda>0,x>0,t>0,xi>0,a1>0,a2>0);
declare((W1,W2)(xi),(w1,w2,u,v)(x,t),(F,G)(u,v));
U,V:=diff_table(u(x,t)),diff_table(v(x,t));
F:=(u,v)->u*ln(v); G:=(u,v)->u*v^n; tr1:=k*x-lambda*t=xi;
Sys1:=(w1,w2)->[diff(w1,t)-a1*diff(w1,x$2)-F(w1,w2)=0,
                diff(w2,t)-a2*diff(w2,x$2)-G(w1,w2)=0];
Sys2:=expand(Sys1(W1(lhs(tr1)),W2(lhs(tr1))));
Sys3:=algsubs(tr1,Sys2); Sys4:=map(convert,Sys3,diff);
```

Mathematica:

```
f[u_,v_]:=u*Log[v]; g[u_,v_]:=u*v^n; tr1=k*x-lambda*t->xi
sys1[w1_,w2_]:={D[w1,t]-a1*D[w1,{x,2}]-f[w1,w2]==0,
                D[w2,t]-a2*D[w2,{x,2}]-g[w1,w2]==0};
sys2=sys1[w1[tr1[[1]]],w2[tr1[[1]]]]//Expand
sys3=sys2/.tr1
```

□

2.8.2 Special Reductions

Problem 2.55

Maple:

```
with(PDEtools): interface(showassumed=0): assume(k>0,x>0,t>0,
 U(x,t)>0,W(x,t)>0); declare((u,v)(x,t),(xi,phi)(x),psi(t));
Sys1:=[diff(u(x,t),x)=u(x,t)*F(v(x,t)),
       diff(v(x,t),t)/G(v(x,t))=u(x,t)^k];
tr1:={u(x,t)=U(x,t)^(1/k),
      G(v(x,t))=diff(v(x,t),t)/diff(W(x,t),t)};
Sys2:=dchange(tr1,Sys1,[U(x,t),W(x,t)]);
Sys3:=expand(simplify(Sys2));
Sys31:=[Sys3[1]*k/U(x,t)^(1/k)*U(x,t),Sys3[2]];
tr2:={rhs(Sys31[2])=lhs(Sys31[2])}; Eq1:=sort(subs(tr2,
 Sys31[1])); tr3:=k*F(v(x,t))=Phi(W);
Eq2:=algsubs(k*F(v(x,t))=Phi(W),Eq1);
Eq3:=int(lhs(Eq2),t)=int(Phi(W),W)+theta(x);
tr4:={W(x,t)=Int(1/G(v),v)}; Eq4:=expand(subs(rhs(tr3)=lhs(tr3),
   Eq3*G(v(x,t)))); Eq5:=subs(tr4,subs(W=W(x,t),subs(tr4,Eq4)));
Eq6:=subs(lhs(Eq5)=diff(v(x,t),x),Eq5); Eq7:=combine(Eq6);
F:=v->2*a1*v; G:=v->a2*v; Eq8:=expand(value(subs(k=1,Eq7)));
term1:=select(has,rhs(Eq8),int); tr5:=term1=a1*v^2;
Eq9:=subs(tr5,Eq8); Eq10:=lhs(Eq9)=algsubs(v=v(x,t),rhs(Eq9));
Solv:=pdsolve(Eq10); term2:=select(has,numer(rhs(Solv)),a2);
tr6:={term2=xi(x)}; Solv1:=expand(subs(_F1(t)=a2*psi(t),
   subs(tr6,Solv))); term3:=select(has,denom(rhs(Solv1)),xi(x));
tr7:={-term3=phi(x)*a1}; Solv2:=expand(subs(tr7,Solv1));
tr8:={diff(term3/a1,x)=-diff(phi(x),x)};
Solv3:=expand(subs(tr8,Solv2));
```

Mathematica:

```
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&]; fF[v_]:=2*a1*v;
gG[v_]:=a2*v; sys1={D[u[x,t],x]==u[x,t]*f[v[x,t]],
 D[v[x,t],t]/g[v[x,t]]==u[x,t]^k}
tr1={u[x,t]->uN[x,t]^(1/k),g[v[x,t]]->D[v[x,t],t]/D[w[x,t],t]}
{sys2=sys1/.tr1/.D[tr1[[1]],x], sys3=sys2//PowerExpand}
sys31={Thread[sys3[[1]]*k/uN[x,t]^(1/k)*uN[x,t],Equal],sys3[[2]]}
{tr2=sys31[[2,2]]->sys31[[2,1]], eq1=sys31[[1]]/.tr2/.D[tr2,x]}
{tr3=k*f[v[x,t]]->phi[w], eq2=eq1/.k*f[v[x,t]]->phi[w]}
eq3=Integrate[eq2[[1]],t]==Integrate[phi[w],w]+theta[x]
```

```
{tr4=w[x,t]->Hold[Integrate[1/g[v],v]], eq4=Thread[eq3*g[v[x,t]],
 Equal]/.tr3[[2]]->tr3[[1]]//Expand, eq5=(eq4/.w->w[x,t])/.tr4}
{eq6=eq5/.eq5[[1]]->D[v[x,t],x], eq70=eq6//PowerExpand}
{termk=trS1[eq70[[2]],k], eq71=eq70/.termk->trS3[termk,g]}
{trk=k*f[v]->Integrate[k*f[v],v], eq7=eq71[[1]]==eq71[[2]]/.
 v[x,t]->v/.trk, eq8=eq7/.k->1/.f[v]->fF[v]/.g[v]->gG[v]}
{eq9=eq8[[1]]==(eq8[[2]]/.v->v[x,t]), solv=DSolve[eq9,v[x,t],
 {x,t}]//First, term2=-Numerator[solv[[1,2]]], tr6=term2->xi[x],
 tr61=tr6/.x->K[2], solv1=solv/.tr6/.tr61/.C[1][t]->-a2*psi[t]}
term3=term3=trS1[Denominator[solv1[[1,2]]],K[2]]
{tr7=term3->phi[x]*a1, solv2=solv1/.tr7,
 tr8=D[term3/a1,x]->phi'[x], solv3=solv2/.tr8}
```

Maple:

```
Eq11:=subs(Solv3,Sys1[1]); Solu:=pdsolve(Eq11,u(x,t));
tr9:=diff(isolate(Solv3,psi(t)),t); simplify(expand(subs(Solv3,
 tr9))); EqF1:=simplify(expand(subs(Solu,Solv3,k=1,Sys1)));
SolF1:=factor(solve(EqF1,_F1(t))) assuming a>0,b>0;
SolF11:=subs(tr9,SolF1); Solu1:=simplify(expand(subs(Solv3,subs(
 SolF11,Solu)))); simplify(expand(subs(Solu1,Solv3,k=1,Sys1)));
```

Mathematica:

```
trF=f[v[x,t]]->fF[v[x,t]];  trG=g[v[x,t]]->gG[v[x,t]];
{eq11=(sys1[[1]]/.trF)/.solv3, solu=DSolve[eq11,u[x,t],
 {x,t}]//First}
tr9=D[Solve[solv3/.Rule->Equal,psi[t]],t]//First//Simplify
eqf1=sys1/.k->1/.solu/.D[solu,x]/.trF/.trG/.solv3/.D[solv3,t]
{solf1=Solve[eqf1,C[1][t]]//First, solf11=solf1/.tr9}
solu1=solu/.solf11/.solv3/.D[solv3,t]
sys1/.k->1/.solu1/.D[solu1,x]/.trF/.trG/.solv3/.D[solv3,t]
```

□

2.8.3 Separation of Variables

Problem 2.56

Maple:

```
with(PDEtools): declare((u,v,W1,W2)(x,t),(phi1,phi2,psi1,psi2,
 phi)(t),theta(x,t)); tr1:=phi1(t)*theta(x,t)+psi1(t);
tr2:=phi2(t)*theta(x,t)+psi2(t); Sys1:=(u,v)->[diff(u(x,t),t)-
 diff(u(x,t),x$2)-u(x,t)*a1*ln(u(x,t)-v(x,t))-a2*ln(u(x,t)-
 v(x,t)),diff(v(x,t),t)-diff(v(x,t),x$2)-a1*v(x,t)*ln(u(x,t)-
 v(x,t))-a2*ln(u(x,t)-v(x,t))]; Sys2:=expand(Sys1(W1,W2));
Sys3:=expand(subs(W1(x,t)=tr1, W2(x,t)=tr2,Sys2));
Cond1:=diff(u(x,t)-v(x,t),x); Cond11:=collect(expand(subs(
 u(x,t)=tr1,v(x,t)=tr2,Cond1)),diff)=0; Sol1:=isolate(lhs(
 Cond11)=0,phi2(t)); tr31:={phi1(t)=phi(t)}; tr32:={subs(tr31,
 Sol1)}; Sys4:=collect(expand(subs(tr31,tr32,Sys3)),diff);
Sys5:=expand([Sys4[1]/phi(t),Sys4[2]/phi(t)]);
Sys6:=collect(expand(factor(Sys5)),theta);
Eq1:=remove(has,expand(select(has,Sys6[1],ln)/theta(x,t)),
 theta); Eq2:=remove(has,Sys6[1],[theta]); Eq3:=remove(has,
 Sys6[2],theta); Eq21:=simplify(isolate(Eq2,diff(psi1(t),t)));
Eq31:=simplify(isolate(Eq3,diff(psi2(t),t)));
Eq40:=[selectremove(has,Sys6[1],[a1,a2,phi])];
Eq4:=op(Eq40[2]); Solphi:=subs(_C1=1,dsolve(Eq1,phi(t)));
Solu:=u(x,t)=subs(Solphi,subs(tr31,tr1)); Solv:=v(x,t)=subs(
 Solphi,subs(tr32,tr2)); Consts:={_C1=0,_C2=1,_C3=1,_c[1]=1};
solpsi:=dsolve({Eq21,Eq31},{psi1(t),psi2(t)}); soltheta:=pdsolve(
 diff(theta(x,t),t)-diff(theta(x,t),x$2)=0,build);
solpsi1:=[solpsi[1,1],subs(solpsi[1,1],op(solpsi[2]))];
SoluF:=rhs(simplify(subs(Consts,subs(solpsi1,soltheta,Solu))));
SolvF:=rhs(simplify(subs(Consts,subs(solpsi1,soltheta,Solv))));
uv:=SoluF-SolvF; Test1:=simplify(diff(SolvF,t)-diff(SolvF,x$2)-
 a1*SolvF*ln(uv)-a2*ln(uv))=0; Test2:=simplify(diff(SoluF,t)-
 diff(SoluF,x$2)-a1*SoluF*ln(uv)-a2*ln(uv))=0;
```

Mathematica:

```
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&]; trD[u_,var_]:=Table[
 D[u,{var,i}],{i,1,2}]//Flatten; sys1[u_,v_]:={D[u,t]-D[u,{x,2}]-
 a1*u*Log[u-v]-a2*Log[u-v], D[v,t]-D[v,{x,2}]-a1*v*Log[u-v]-
 a2*Log[u-v]}; {tr1=phi1[t]*theta[x,t]+psi1[t], tr2=phi2[t]*
 theta[x,t]+psi2[t], sys2=sys1[w1,w2]//Expand, sys3=sys1[tr1,
 tr2]//Expand, cond1=D[u[x,t]-v[x,t],x], cond11=(cond1/.
 {D[u[x,t],x]->D[tr1,x],D[v[x,t],x]->D[tr2,x]}//Expand)==0}
{sol1=Solve[cond11,phi2[t]], tr31=phi1[t]->phi[t],
 tr32=sol1/.tr31, sys4=(sys3/.tr31/.tr32/.D[tr31,t]/.D[tr32,t]//
 Simplify)[[1,1]], sys5={sys4[[1]]/phi[t],sys4[[2]]/phi[t]}//
 Expand, sys6=Collect[Expand[Factor[sys5]],theta[x,t]]}
eq1=Coefficient[trS1[sys6[[1]],a1],theta[x,t]]==0
{eq2=trS3[sys6[[1]],theta], eq3=trS3[sys6[[2]],theta]}
{Solve[eq2==0,psi1'[t]], Solve[eq3==0,psi2'[t]]}
{eq4=trS3[trS3[trS3[sys6[[1]],a1],a2],phi], solphi=DSolve[eq1,
 phi[t],t]/.C[1]->1//First, solu=u[x,t]->tr1/.tr31/.solphi,
 solv=v[x,t]->tr2/.tr32/.solphi}
consts={C[1]->0,C[2]->0,C[3]->1,c1->1}; solpsi=DSolve[{eq2==0,
 eq3==0},{psi1[t],psi2[t]},t]//FullSimplify//First
tr4={psi1[K[1]]->solpsi[[2,2]],psi2[K[1]]->solpsi[[1,2]]};
DSolve[eq4==0,theta,{x,t}]
soltheta=theta[x,t]->Exp[Sqrt[c1]*x]*C[3]*Exp[c1*t]*C[1]+C[3]*
 Exp[c1*t]*C[2]/Exp[Sqrt[c1]*x];
{soluF=solu/.solpsi/.soltheta/.tr4/.consts//Factor, solvF=solv/.
 solpsi/.soltheta/.tr4/.consts//Factor//First}
Assuming[{t>0,a1>0},sys1[u[x,t],v[x,t]]/.soluF/.solvF/.trD[soluF,
 x]/.trD[soluF,t]/.trD[solvF,x]/.trD[solvF,t]//FullSimplify]
```
□

Chapter 3
Geometric-Qualitative Approach

3.1 Method of Characteristics

3.1.1 Characteristic Directions. General Solution

Problem 3.1

Maple:

```
with(plots): R1:=-Pi..Pi: N1:=10: V1:=[-24,72]; V2:=[-59,-29];
A1:=`THICK`;setoptions3d(fieldplot3d,grid=[N1,N1,N1],axes=boxed);
r1:=(x,t,u)->[u,1,-u]; r2:=(x,t,u)->[x,t,-(x^2+t^2)];
fieldplot3d(r1(x,t,u),x=R1,t=R1,u=R1,arrows=SLIM,orientation=V1);
fieldplot3d(r2(x,t,u),x=R1,t=R1,u=R1,arrows=A1,orientation=V2);
```

Mathematica:

```
r1[x_,t_,u_]:={u,1,-u}; r2[x_,t_,u_]:={x,t,-(x^2+t^2)};
{n1=10, p=Pi, v1={1,-3,1}, v2={1,2,3}}
SetOptions[VectorPlot3D,VectorColorFunction->Hue,VectorPoints->
 {n1,n1,n1},VectorStyle->Arrowheads[0.02],PlotRange->All];
VectorPlot3D[r1[x,t,u],{x,-p,p},{t,-p,p},{u,-p,p},ViewPoint->v1]
VectorPlot3D[r2[x,t,u],{x,-p,p},{t,-p,p},{u,-p,p},
 ViewPoint->v2]
```

□

Problem 3.2

Maple:

```
with(PDEtools): declare(u(x,y)); U:=diff_table(u(x,y));
F:=U[]*(x+y); G:=U[]*(x-y); H:=x^2+y^2;
PDE:=F*U[x]+G*U[y]=H; CharEqs:=[dx/F,dy/G,du/H];
Eq1:=expand([U[]*CharEqs[1],U[]*CharEqs[2],-U[]*CharEqs[3]]);
```

```
Eq21:=numer(Eq1[1])*y+numer(Eq1[2])*x+numer(Eq1[3]);
Eq22:=expand(denom(Eq1[1])*y+denom(Eq1[2])*x-denom(Eq1[3]));
Eq31:=numer(Eq1[1])*x-numer(Eq1[2])*y+numer(Eq1[3]);
Eq3:=expand(denom(Eq1[1])*x-denom(Eq1[2])*y-denom(Eq1[3]));
I11:=(x*y-int(u,u))=C1; I1:=subs(2*C1=C1,I11*2);
I21:=int(op(1,Eq31)/dx,x)+int(op(2,Eq31)/dy,y)-int(u,u)=C2;
I2:=subs(2*C2=C2,I21*2);  f(lhs(I1),lhs(I2));
```

Mathematica:

```
{fU=u[x,y], fF=u[x,y]*(x+y), fG=u[x,y]*(x-y), fH=x^2+y^2}
{pde=fF*D[fU,x]+fG*D[fU,y]==fH,charEqs={dx/fF,dy/fG,du/fH}}
{eq1={fU*charEqs[[1]],fU*charEqs[[2]],-fU*charEqs[[3]]}//
 Expand, eq21=Numerator[eq1[[1]]]*y+Numerator[eq1[[2]]]*x+
 Numerator[eq1[[3]]], eq22=Denominator[eq1[[1]]]*y+
 Denominator[eq1[[2]]]*x-Denominator[eq1[[3]]]//Expand}
{eq31=Numerator[eq1[[1]]]*x-Numerator[eq1[[2]]]*y+
 Numerator[eq1[[3]]], eq3=Denominator[eq1[[1]]]*x-
 Denominator[eq1[[2]]]*y-Denominator[eq1[[3]]]//Expand}
{i11=(x*y-Integrate[u,u])==c1, i1=(Thread[i11*2, Equal]//
 Expand)/.{2*c1->c1}, i21=Integrate[eq31[[1]]/dx,x]+
 Integrate[eq31[[2]]/dy,y]-Integrate[u,u]==c2}
i2=Thread[i21*2,Equal]//Expand/.{2*c2->c2}
f[i1[[1]],i2[[1]]]
```

□

3.1.2 Integral Surfaces. Cauchy Problem

Problem 3.3

Maple:

```
with(plots); tR:=0..5; xR:=-20..20; ODE:=diff(U(t),t)=0;
Sol_Ch:=dsolve({ODE,U(0)=X[0]}); Eq_Ch:=diff(x(t),t)=U(t);
Eq_Ch:=subs(Sol_Ch,Eq_Ch); Cur_Ch:=dsolve({Eq_Ch, x(0)=X[0]});
display([seq(plot([subs(X[0]=x,eval(x(t),Cur_Ch)),t,t=tR],
         color=blue,thickness=2),x=xR)],view=[xR,tR]);
u:=unapply(subs(X[0]=solve(subs(x(t)=x+1,Cur_Ch),X[0]),
   eval(U(t),Sol_Ch)),x,t);
```

Mathematica:

```
SetOptions[Plot,ImageSize->500,PlotStyle->{Hue[0.7],
 Thickness[0.001]}]; {ode=D[uN[t],t]==0, solCh=DSolve[{ode,
 uN[0]==xN[0]},uN[t],t], eqCh=D[x[t],t]==uN[t]/.solCh[[1]]}
curCh=DSolve[{eqCh,x[0]==xN[0]},x[t],t]//Simplify
g=Table[ParametricPlot[{(curCh[[1,1,2]]/.xN[0]->x),t},
 {t,0,5}],{x,-20,20}]; Show[g,PlotRange->{{-20,20},{0,5}},
 AspectRatio->1]
u1=solCh[[1]]/.Solve[curCh[[1,1,2]]==x+1,xN[0]]
u[xN_,tN_]:=u1[[1,1,2]]/.{x->xN,t->tN}; u[x,t]
```

□

Problem 3.4

Maple:

```
PDE:=u->diff(u(x,t),t)+u(x,t)*diff(u(x,t),x)=x;
ODEs:=dt/1=(dx/u=du/x); f1:=x->1; f2:=x->x;
Eq2:=lhs(rhs(ODEs))+rhs(rhs(ODEs))=d(x+u)/(x+u);
Eq3:=rhs(Eq2)=lhs(ODEs); Eq4:=log(x+u)-log(C1)=t;
tr1:=isolate(Eq4,C1); Eq5:=lhs(rhs(ODEs))=rhs(rhs(ODEs));
Eq6:=Eq5*x*u; Eq61:=op(1,lhs(Eq6))=op(1,rhs(Eq6));
Eq7:=int(lhs(Eq61),x)+C2/2=int(rhs(Eq61),u);
Eq71:=simplify(Eq7*2); tr2:=isolate(Eq7,C2);
sys1:=simplify(subs(t=0,u=f1(x),[tr1,tr2]));
tr3:=isolate(sys1[1],x); tr21:=simplify(subs(tr3,sys1[2]));
Sol1:=subs(tr1,subs(tr21,tr2)); Sol11:=lhs(Sol1)=factor(
 rhs(Sol1)); Sol12:=normal(Sol11/(x+u));
Sol13:=collect(lhs(Sol12),[exp(t),u])=rhs(Sol12);
Sol14:=combine(convert(combine(Sol13),trigh));
Sol15:=expand(Sol14); Sol16:=u=solve(Sol15,u);
Sol17:=collect(Sol16,x); SolFin1:=expand(simplify(Sol17,trig));
sys2:=simplify(subs(t=0,u=f2(x),[tr1,tr2]));
tr3:=isolate(sys2[1],x); tr21:=sys2[2];
Sol1:=subs(tr1,subs(tr21,tr2)); SolFin2:=u=[solve(Sol1,u)];
u1:=unapply(rhs(SolFin1),x,t);expand(PDE(u1));
u21:=unapply(rhs(SolFin2)[1],x,t); expand(PDE(u21));
u22:=unapply(rhs(SolFin2)[2],x,t); expand(PDE(u22));
```

Mathematica:

```
pde[u_]:=D[u,t]+u*D[u,x]==x; f1[x_]:=1; f2[x_]:=x;
{odes=dt/1==(dx/u==du/x), eq2=odes[[2,1]]+odes[[2,2]]==
 HoldForm[d(x+u)/(x+u)], eq3=eq2[[2]]==odes[[1]]}
{eq4=Log[x+u]-Log[c1]==t, tr1=Solve[eq4,c1]//First}
{eq5=odes[[2,1]]==odes[[2,2]], eq6=Thread[eq5*x*u,Equal]}
eq7=Integrate[eq6[[1]]/dx,x]+c2/2==Integrate[eq6[[2]]/du,u]
{eq71=Thread[eq7*2,Equal]//Expand,tr2=Solve[eq7,c2]//First}
{sys1={tr1,tr2}/.t->0/.u->f1[x]//Flatten, tr3=Solve[
 sys1[[1]]/.Rule->Equal,x], tr21=sys1[[2]]/.tr3}
sol1=tr2/.tr21/.tr1/.Rule->Equal//Expand//First
{sol11=sol1[[1]]==Factor[sol1[[2]]], sol12=Thread[
 sol11/(x+u),Equal]//Factor, sol13=Collect[sol12[[1]],
 {Exp[t],u}]==sol12[[2]], sol14=sol13//ExpToTrig}
{sol15=sol14//FullSimplify//ExpToTrig, sol16=Solve[
 sol15,u]//First, solFin1=Collect[sol16,x]}
sys2={tr1,tr2}/.t->0/.u->f2[x]//Flatten
tr3=Solve[sys2[[1]]/.Rule->Equal,x]//First
{tr21=sys2[[2]], sol1=tr2/.tr21/.tr1/.Rule->Equal}
solFin2=Solve[sol1,u]//Flatten
test1=Map[FullSimplify,{pde[solFin1[[1,2]]],
 pde[solFin2[[1,2]]],pde[solFin2[[2,2]]]}]
```

□

Problem 3.5

Maple:

```
with(PDEtools): with(Student[Precalculus]): with(plots):
setoptions(implicitplot,numpoints=100): declare(u(x,t));
alias(u=u(x,t)); Chars:=NULL: GrU1:=NULL: GrU2:=NULL:
PDE1:=u->diff(u(x,t),t)+u(x,t)*diff(u(x,t),x)=1; PDE1(u);
IniCurve1:=u(r,2*r)=r; IniCurve2:=[T(0,r)=2*r,X(0,r)=r,U(0,r)=r];
CharEqs:=[diff(T(s),s)=1,diff(X(s),s)=U(s),diff(U(s),s)=1];
sys1:={subs(_C1=rhs(IniCurve2[3]),dsolve(CharEqs[3],U(s))),
       subs(_C1=rhs(IniCurve2[1]),dsolve(CharEqs[1],T(s)))};
sys11:=subs(_C1=rhs(IniCurve2[2]),dsolve(subs(U(s)=rhs(sys1[2]),
       CharEqs[2]),X(s))); tr2:={X(s)=X,T(s)=T,U(s)=U};
CharsEq:=eliminate(subs(tr2,{sys1[1],sys11}),s);
CharsEq1:=subs(r=R,CompleteSquare(op(CharsEq[2])));
for R from -10 to 10 by 0.5 do Chars:=Chars,CharsEq1; od:
Chars; implicitplot([Chars],X=-10..10,T=0..2,color=blue);
```

```
sys3:=subs(tr2,sys1 union {sys11}); Ch1:=op(op(2,CharsEq));
r1:=isolate(Ch1,r); s1:=simplify(isolate(subs(r1,sys3[1]),s));
SolFin:=simplify(subs(r1,s1,sys3[2])); U1:=rhs(SolFin);
for i from 0 to 0.999 by 0.1 do
 GU1||round(i*10):=plot(subs(T=i,U1),X=-10..10,-10..10,
   color=[blue,blue]): GrU1:=GrU1,GU1||round(i*10): od:
for j from 1.001 to 2 by 0.1 do
 GU2||round(j*10):=plot(subs(T=j,U1),X=-10..10,-10..10,
   color=[blue,blue]): GrU2:=GrU2,GU2||round(j*10) od:
 display({GrU1,GrU2});
```

Mathematica:

```
cS[a_.x_^2+b_.x_+c_.]:=a*((x+b/(2*a))^2-(b^2-4*a*c)/(4*a^2));
completeSquare[x_]:=If[TrueQ[x==Expand[x]],x,cS[Expand[x]]];
p=10; SetOptions[ContourPlot,Frame->True,ImageSize->500,
ContourShading->False,ContourStyle->Blue]; guN1={}; guN2={};
SetOptions[Plot,Frame->True,ImageSize->500,PlotRange->
 {{-p,p},{-p,p}},PlotStyle->Blue]; chars={}; gC={};
pde1[u_]:=D[u[x,t],t]+u[x,t]*D[u[x,t],x]==1; pde1[u]
{iniCurve1=u[r,2*r]->r, iniCurve2={tN[0,r]->2*r,xN[0,r]->r,
 uN[0,r]->r}, charEqs={tN'[s]==1,xN'[s]==uN[s],uN'[s]==1}}
sys1={DSolve[charEqs[[3]],uN[s],s]/.C[1]->iniCurve2[[3,2]],
 DSolve[charEqs[[1]],tN[s],s]/.C[1]->iniCurve2[[1,2]]}//Flatten
sys11=DSolve[(charEqs[[2]]/.uN[s]->sys1[[1,2]]),xN[s],s]/.
 C[1]->iniCurve2[[2,2]]//Flatten
tr2={xN[s]->xN,tN[s]->tN,uN[s]->uN}
{charsEq=Eliminate[Flatten[{sys1[[2]],sys11}/.tr2/.Rule->Equal],
 s]//Together, charsEq1=Map[Factor,completeSquare[charsEq[[1]]-
 charsEq[[2]]]]/.r->rN}
Do[chars=Append[chars,charsEq1],{rN,-p,p,0.5}]; chars
Do[gC=Append[gC,ContourPlot[chars[[i]]==0,{xN,-p,p},{tN,0,2}]],
 {i,1,Length[chars]}]; Show[gC]
{sys3={sys1,sys11}/.tr2//Flatten, ch1=charsEq[[1]]-
 charsEq[[2]]==0, r1=Solve[ch1,r]//First, s1=Solve[sys3[[2]]/.
 Rule->Equal/.r1,s]//First, solFin=sys3[[1]]/.Rule->Equal/.
 r1/.s1//Simplify, uN1=solFin[[2]]}
Do[guN1=Append[guN1,Plot[uN1/.tN->i,{xN,-p,p}]],{i,0,0.999,0.1}];
Do[guN2=Append[guN2,Plot[uN1/.tN->i,{xN,-p,p}]],{i,1.001,2,0.1}];
Show[{guN1,guN2}]
```

□

Problem 3.6

Maple:

```
with(PDEtools): declare(u(x,t)); alias(u=u(x,t));
PDE:=u->diff(u(x,t),t)+c(u(x,t))*diff(u(x,t),x)=0; PDE(u);
IniCurve:=u(x,0)=f(x); IniData:=u(xi,0)=f(xi);
tr1:={X(t)=X,U(t)=U}; CharEqs:=[diff(X(t),t)=c(U),
 diff(U(t),t)=0]; Eq1:=dsolve(CharEqs[1],X(t));
Eq2:=isolate(Eq1,_C1); Eq3:=subs(tr1,Eq2); Eq4:=isolate(Eq3,X);
Eq5:=subs(_C1=xi,c(U)=c(rhs(IniData)),Eq4);
Eq6:=u(x,t)=(u(rhs(Eq5),t)=IniData);
Eq7:={u(x,t)=rhs(rhs(rhs(Eq6))),-Eq5+xi+X};
PDE1:=u->diff(u(x,t),t)+u(x,t)*diff(u(x,t),x)=0; PDE1(u);
IniCurve1:=(u(x,0)=x^2)=f(x);
u1:=unapply(subs(f(xi)=rhs(lhs(IniCurve1)),rhs(Eq7[2])),x,t);
Eq8:=u1=u1(xi,t);
Eq9:=subs(c(f(xi))=rhs(lhs(IniCurve1)),Eq7[1]);
Eq91:=subs(x=xi,Eq9); Eq10:=[solve(Eq91,xi)];
Eq11:=xi^2=expand(Eq10[1]^2); Eq12:=xi^2=expand(Eq10[2]^2);
SolFin1:=subs(Eq11,Eq8); SolFin2:=subs(Eq12,Eq8);
limit(rhs(SolFin1),t=0); limit(rhs(SolFin2),t=0);
CharsEq:=subs(xi=eta,Eq91); Chars1:=NULL:
for eta from -5 to 5 by 0.5 do Chars1:=Chars1,CharsEq; od:
Chars1; with(plots): setoptions(plot,numpoints=100):
implicitplot([Chars1],X=-5..5,t=0..2,color=blue);
U1:=rhs(SolFin1); GU0:=plot(rhs(op(1,IniCurve1)),x=-2..2):
GU3:=plot(subs(t=0.1,U1),X=-2..2,color=blue):
GU5:=plot(subs(t=0.2,U1),X=-2..2,color=blue):
display({GU0,GU3,GU5});
```

Mathematica:

```
p1=5; p2=2; chars1={}; gC={}; SetOptions[ContourPlot,Frame->True,
 ImageSize->500,ContourShading->False,ContourStyle->Blue];
SetOptions[Plot,Frame->True,ImageSize->500,PlotRange->{{-p2,p2},
 {0,7}}]; pde[u_]:=D[u[x,t],t]+c[u[x,t]]*D[u[x,t],x]==0;
{pde[u], iniCurve=u[x,0]->f[x], iniData=u[xi,0]->f[xi],
 tr1={xN[t]->xN,uN[t]->uN}, charEqs={xN'[t]==c[uN],uN'[t]==0}}
```

```
{eq1=DSolve[charEqs[[1]],xN[t],t]//First,
 eq2=Solve[eq1/.Rule->Equal,C[1]]//First, eq3=eq2/.tr1,
 eq4=Solve[eq3/.Rule->Equal,xN]//First,
 eq5=eq4/.C[1]->xi/.c[uN]->c[iniData[[2]]],
 eq6=u[x,t]->u[eq5[[1,2]],t]==iniData}
eq51=eq5/.Rule->Equal//First
eq52=Thread[Thread[eq51*(-1),Equal]+xi+xN,Equal]//Expand//ToRules
eq7={u[x,t]->eq6[[2,2,2]],eq52}//Flatten
pde1[u_]:=D[u[x,t],t]+u[x,t]*D[u[x,t],x]==0; pde1[u]
u1[xN_,tN_]:=eq7[[1,2]]/.f[xi]->iniCurve1[[1,2]]/.{x->xN,t->tN}
{iniCurve1=(u[x,0]==x^2)==f[x], u1[xi,t],
 eq9=eq7[[2]]/.c[f[xi]]->iniCurve1[[1,2]], eq91=eq9/.x->xi,
 eq10=Solve[eq91/.Rule->Equal,xi]//Flatten,
 eq11=xi^2->eq10[[1,2]]^2//Expand}
{eq12=xi^2->eq10[[2,2]]^2//Expand, solFin1=u1[xi,t]/.eq11}
{solFin2=u1[xi,t]/.eq12,Limit[solFin1,t->0],Limit[solFin2,t->0]}
{charsEq=eq91/.Rule->Equal/.xi->eta,
 charsEq1=charsEq[[1]]-charsEq[[2]], uN1=solFin2}
Do[chars1=Append[chars1,charsEq1],{eta,-p1,p1,0.5}]; chars1
Do[gC=Append[gC,ContourPlot[chars1[[i]]==0,{xN,-p1,p1},{t,0,2}]],
 {i,1,Length[chars1]}]; Show[gC]
gU0=Plot[iniCurve1[[1,2]],{x,-2,2},PlotStyle->Red]; gU3=Plot[
 uN1/.t->0.1,{xN,-2,2},PlotStyle->Blue]; gU5=Plot[uN1/.t->0.2,
 {xN,-2,2},PlotStyle->Blue]; Show[{gU0,gU3,gU5}]
```
□

Problem 3.7

Maple:

```
with(PDEtools): with(plots): setoptions(plot,numpoints=100):
declare(u(x,t)); alias(u=u(x,t)); BB:=color=blue: tR:=0..1:
PDE:=u->diff(u(x,t),t)+c(u(x,t))*diff(u(x,t),x)=0; PDE(u);
IniCurve:=u(x,0)=f(x); IniData:=u(xi,0)=f(xi);
tr1:={X(t)=X,U(t)=U}; Chs1:=NULL: Chs2:=NULL: Chs3:=NULL:
CharEqs:=[diff(X(t),t)=c(U),diff(U(t),t)=0];
Eq1:=dsolve(CharEqs[1],X(t)); Eq2:=isolate(Eq1,_C1);
Eq3:=subs(tr1,Eq2); Eq4:=isolate(Eq3,X); Eq5:=subs(_C1=xi,
 c(U)=c(rhs(IniData)),Eq4); Eq6:=u(x,t)=(u(rhs(Eq5),t)=IniData);
Eq7:={u(x,t)=rhs(rhs(rhs(Eq6))),-Eq5+xi+X};
PDE1:=u->diff(u(x,t),t)+u(x,t)*diff(u(x,t),x)=0; PDE1(u);
InCur1:=(u(x,0)=piecewise(abs(x)<=1,1^2-x^2,abs(x)>1,0))=f(x);
u1:=unapply(subs(f(xi)=rhs(lhs(InCur1)),rhs(Eq7[2])),x,t);
```

```
Eq8:=u1=u1(xi,t); Eq9:=subs(c(f(xi))=op(2,rhs(lhs(InCur1))),
 Eq7[1]); Eq91:=subs(x=xi,Eq9); Eq10:=[solve(Eq91,xi)];
Eq11:=xi^2=expand(Eq10[1]^2); Eq12:=xi^2=expand(Eq10[2]^2);
SolFin1:=subs(Eq11,Eq8); SolFin2:=subs(Eq12,Eq8);
ChsEq:=subs(xi=eta,Eq91);
for eta from -1 to 1 by 0.1 do Chs1:=Chs1,ChsEq;
od: for eta from -3 to -1 by 0.1 do Chs2:=Chs2,subs(t=0,ChsEq);
od: for eta from 1 to 3 by 0.1 do Chs3:=Chs3,subs(t=0,ChsEq);
od: Chs1; Chs2; Chs3;
G1:=implicitplot([Chs1],X=-1..1,t=tR,color=blue):
G2:=implicitplot([Chs2],X=-3..-1,t=tR):
G3:=implicitplot([Chs3],X=1..3,t=tR): display({G1,G2,G3});
U1:=[op(2,rhs(SolFin1)),op(2,rhs(SolFin2))];
GU0:=plot(rhs(op(1,InCur1)),x=-10..40): GU3:=plot(subs(t=3,U1),
 X=-10..40,0..3,BB): GU5:=plot(subs(t=30,U1),X=-10..40,0..3,BB):
display({GU0,GU3,GU5});
```

Mathematica:

```
p=10; tR=1; chs1={}; chs2={}; chs3={}; gC1={}; gC2={}; gC3={};
SetOptions[ContourPlot,Frame->True,ImageSize->500,
 ContourShading->False,ContourStyle->Red]; SetOptions[Plot,
 Frame->True,ImageSize->500,PlotRange->{{-p,p*4},{0,3}}];
pde[u_]:=D[u[x,t],t]+c[u[x,t]]*D[u[x,t],x]==0; pde[u]
{iniCurve=u[x,0]->f[x], iniData=u[xi,0]->f[xi],
 tr1={xN[t]->xN,uN[t]->uN}, charEqs={xN'[t]==c[uN],uN'[t]==0}}
{eq1=DSolve[charEqs[[1]],xN[t],t]//First, eq2=Solve[eq1/.
 Rule->Equal,C[1]]//First, eq3=eq2/.tr1, eq4=Solve[eq3/.
 Rule->Equal,xN]//First, eq5=eq4/.C[1]->xi/.c[uN]->
 c[iniData[[2]]], eq6=u[x,t]->u[eq5[[1,2]],t]==iniData}
{eq51=eq5/.Rule->Equal//First, eq52=Thread[Thread[eq51*(-1),
 Equal]+xi+xN,Equal]//Expand//ToRules}
eq7={u[x,t]->eq6[[2,2,2]],eq52}//Flatten
pde1[u_]:=D[u[x,t],t]+u[x,t]*D[u[x,t],x]==0; pde1[u]
u1[xN_,tN_]:=eq7[[1,2]]/.f[xi]->inCur1[[1]]/.{x->xN,t->tN}
{inCur1=(u[x,0]=Piecewise[{{1-x^2,Abs[x]<=1},
 {0,Abs[x]>1}}])==f[x], u1[xi,t], eq9=eq7[[2]]/.c[f[xi]]->
 inCur1[[1,1,1,1]], eq91=eq9/.x->xi, eq10=Solve[eq91/.
 Rule->Equal,xi]//Flatten, eq11=xi^2->eq10[[1,2]]^2//Expand}
{eq12=xi^2->eq10[[2,2]]^2//Expand, solFin1=u1[xi,t]/.eq11}
{solFin2=u1[xi,t]/.eq12, chsEq=eq91/.Rule->Equal/.xi->eta}
chsEq1=chsEq[[1]]-chsEq[[2]]
uN1={solFin1[[1,1,1]],solFin2[[1,1,1]]}
```

```
Do[chs1=Append[chs1,chsEq1],{eta,-1,1,0.1}]; chs1
Do[chs2=Append[chs2,chsEq1/.t->0],{eta,-3,-1,0.1}]; chs2
Do[chs3=Append[chs3,chsEq1/.t->0],{eta,1,3,0.1}]; chs3
Do[gC1=Append[gC1,ContourPlot[chs1[[i]]==0,{xN,-1,1},{t,0,tR},
 ContourStyle->Blue]],{i,1,Length[chs1]}]; Do[gC2=Append[gC2,
 ContourPlot[chs2[[i]]==0,{xN,-3,-1},{t,0,tR}]],{i,1,Length[
 chs2]}]; Do[gC3=Append[gC3,ContourPlot[chs3[[i]]==0,{xN,1,3},
 {t,0,tR}]],{i,1,Length[chs3]}];
Show[{gC1,gC2,gC3},PlotRange->{{-3,3},{0,1}}]
gU0=Plot[inCur1[[1]],{x,-10,40},PlotStyle->Red];
gU3=Plot[uN1/.t->3,{xN,-10,40},PlotStyle->Blue]; gU5=Plot[uN1/.
 t->30,{xN,-10,40},PlotStyle->Blue]; Show[{gU0,gU3,gU5}]
```
□

3.1.3 Solution Profile at Infinity

Problem 3.8

Maple:

```
with(PDETools): declare(u(x,t),w(x,t)); U:=diff_table(u(x,t));
interface(showassumed=0);assume(f(x)>=0,lambda1>0,lambda2>0,t>0);
tr1:={u(x,t)=1/w(x,t)}; Eq1:=U[t]+U[x]-U[]*(lambda1-lambda2*U[]);
Eq2:=simplify(dchange(tr1,Eq1,[w(x,t)])*w(x,t)^2);
Solw1:=pdsolve({Eq2,w(x,0)=1/f(x)},w(x,t)); Solu1:=1/rhs(Solw1);
ProfInf:=limit(Solu1,t=infinity);
```

Mathematica:

```
eq1[u_]:=D[u,t]+D[u,x]-u*(lambda1-lambda2*u); inf=Infinity
{eq2=eq1[1/w[x,t]]//FullSimplify, eq21=eq2*w[x,t]^2//Expand}
solw1=DSolve[{eq2==0,w[x,0]==1/f[x]},w[x,t],{x,t}]//First
solu1=1/solw1[[1,2]]/.{f[-t+x]->f[z]}//FullSimplify
profInf=Limit[solu1,t->inf,Assumptions->{lambda1>0,lambda2>0}]
```
□

Problem 3.9

Maple:

```
with(PDETools): declare(u(x,t)); U:=diff_table(u(x,t));
interface(showassumed=0); assume(t>0);
Eq1:=U[t]+U[]*U[x]+U[]=0;
Solu:=pdsolve({Eq1,u(x,0)=x+1},u(x,t),HINT=f(x)*g(t),explicit);
Solu1:=rhs(Solu); SN:=numer(Solu1)*exp(-t);
```

```
SD:=combine(expand(denom(Solu1)*exp(-t))); SoluFin:=SN/SD;
Sol1:=expand(SoluFin); Sol11:=sort(op(1,Sol1));
Sol12:=sort(op(2,Sol1)); ProfInf1:=op(2,Sol11)*op(3,Sol11)
 *limit(op(1,Sol11),t=infinity);
ProfInf2:=op(2,Sol12)*limit(op(1,Sol12),t=infinity);
ProfInf:=simplify(ProfInf1+ProfInf2);
plot3d(ProfInf,x=-5..5,t=0..5,axes=boxed);
plots[contourplot](ProfInf,x=-5..5,t=0..5,filledregions=true,
      coloring=[magenta,blue]);
```

Mathematica:

```
eq1=D[u[x,t],t]+u[x,t]*D[u[x,t],x]+u[x,t]==0
solu=DSolve[{eq1,u[x,0]==x+1},u[x,t],{x,t}]//First
{solu1=solu[[1,2]], sN=Numerator[solu1]*Exp[-t]//Simplify}
{sD=Denominator[solu1]*Exp[-t]//Simplify, soluFin=sN/sD}
profInf=sN/Limit[sD,t->Infinity]
Plot3D[profInf,{x,-5,5},{t,0,5},PlotRange->All,Mesh->False]
ContourPlot[profInf,{x,-5,5},{t,0,5},PlotRange->All,
 ContourStyle->Hue[0.9],ContourShading->Automatic]
```

□

3.2 Generalized Method of Characteristics

3.2.1 Complete Integrals. General Solution

Problem 3.10

Maple:

```
with(PDEtools): declare(u(x,t)); alias(u=u(x,t));
IntSurf:=(t-a)^2+(x-b)^2+u(x,t)^2=c^2;
PDE:=u->u(x,t)^2*(diff(u(x,t),t)^2+diff(u(x,t),x)^2+1)=c^2;
expand(PDE(u)); Eq1:=diff(IntSurf,t)/2;
Eq2:=diff(IntSurf,x)/2; Eq11:=isolate(Eq1,-a)+t;
Eq21:=isolate(Eq2,-b)+x;
Eq3:=lhs(Eq11)^2+lhs(Eq21)^2=rhs(Eq11)^2+rhs(Eq21)^2;
Eq4:=lhs(Eq3)=factor(rhs(Eq3)); Eq5:=Eq4+u^2;
Eq6:=expand(subs(lhs(Eq5)=rhs(IntSurf),Eq5));
expand(PDE(u)=Eq6); eliminate({IntSurf,Eq1,Eq2},{a,b});
```

Mathematica:

```
intSurf=(t-a)^2+(x-b)^2+u[x,t]^2==c^2
pde[u_]:=u[x,t]^2*(D[u[x,t],t]^2+D[u[x,t],x]^2+1)==c^2;
{pde[u]//Expand, eq1=Thread[D[intSurf,t]/2,Equal]//Expand}
{eq2=Thread[D[intSurf,x]/2,Equal]//Expand, eq11=Thread[
 eq1-eq1[[1,3]],Equal], eq21=Thread[eq2-eq2[[1,3]],Equal],
 eq3=Thread[Map[#^2&,eq11]+Map[#^2&,eq21],Equal]}
{eq4=Map[FullSimplify,eq3], eq5=Thread[eq4+u[x,t]^2,Equal]}
{eq6=eq5/.eq5[[1]]->intSurf[[2]], pde[u]==eq6//Expand}
Eliminate[{intSurf,eq1,eq2},{a,b}]//FullSimplify
```

□

Problem 3.11

Maple:

```
with(Student[Precalculus]): F:=p^2+q*y-u;
CharEqs:=[diff(x(t),t)=diff(F,p),diff(y(t),t)=diff(F,q),
 diff(u(t),t)=p*diff(F,p)+q*diff(F,q), diff(p(t),t)=-(diff(F,x)+
 p*diff(F,u)),diff(q(t),t)=-(diff(F,y)+q*diff(F,u))];
tr1:=q(t)=q; tr2:=du=p*dx+q*dy; tr3:=u-a*y=v;
Eq1:=map(int,CharEqs[5],t); Eq2:=lhs(Eq1)=rhs(Eq1)+a;
Eq3:=[solve(subs(subs(tr1,Eq2),F),p)]; Eq4:=p=Eq3[1];
Eq5:=(dx/rhs(CharEqs[1])=dy/rhs(CharEqs[2]))=du/rhs(CharEqs[3]);
Eq6:=subs(tr2,Eq5); Eq7:=subs(p=rhs(Eq4),q=rhs(Eq2),tr2);
Eq8:=expand(isolate(Eq7,dx));
Eq81:=rhs(Eq8)=d(lhs(tr3))/denom(rhs(Eq8)); Eq82:=subs(tr3,Eq81);
Eq83:=int(1/denom(rhs(Eq82)),v)=int(lhs(Eq8)/dx,x)+b;
Eq84:=subs(v=lhs(tr3),lhs(Eq83))=rhs(Eq83);
Sol:=solve(Eq84,u); CompleteInt:=u(x,y)=CompleteSquare(Sol);
```

Mathematica:

```
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
cS[a_.x_^2+b_.x_+c_.]:=a*((x+b/(2*a))^2-(b^2-4*a*c)/(4*a^2));
completeSquare[x_]:=If[TrueQ[x==Expand[x]],x,cS[Expand[x]]];
{fF=p^2+q*y-u, charEqs={x'[t]==D[fF,p],y'[t]==D[fF,q],
 u'[t]==p*D[fF,p]+q*D[fF,q], p'[t]==-(D[fF,x]+p*D[fF,u]),
 q'[t]==-(D[fF,y]+q*D[fF,u])}}
{tr1=q[t]->q, tr2=du->p*dx+q*dy, tr3=u-a*y->v}
eq1=Thread[Integrate[charEqs[[5]],t],Equal]
```

```
{eq2=eq1[[1]]->eq1[[2]]+a, eq3=Solve[fF==0/.{eq2/.tr1},p],
 eq4=p==eq3[[1,1,2]], eq5=(dx/charEqs[[1,2]]==
 dy/charEqs[[2,2]])==du/charEqs[[3,2]], eq6=eq5/.tr2}
eq7=du==(tr2/.{p->eq4[[2]],q->eq2[[2]]})[[2]]
{eq8=Solve[eq7,dx]//Expand//First, eq81=eq8[[1,2]]==
 d[tr3[[1]]]*trS1[eq8[[1,2,1]],a],eq82=eq81/.tr3,
 eq83=Integrate[eq82[[2,1]],v]==Integrate[eq8[[1,1]]/dx,x]+b}
{eq84=eq83[[1]]==eq83[[2]]/.v->tr3[[1]], sol=Solve[eq84,u]}
completeInt=u[x,y]==completeSquare[sol[[1,1,2]]]
```
□

Problem 3.12

Maple:

```
F:=p^2+q^2-1; CharEqs:=[diff(x(t),t)=diff(F,p),
 diff(y(t),t)=diff(F,q), diff(u(t),t)=p*diff(F,p)+q*diff(F,q),
 diff(p(t),t)=-(diff(F,x)+p*diff(F,u)),
 diff(q(t),t)=-(diff(F,y)+q*diff(F,u))];
tr1:={q(t)=q,p(t)=p}; tr2:=du=p*dx+q*dy;
Eq1:=dsolve(CharEqs[4],p(t)); Eq2:=subs(tr1,_C1=a,Eq1);
Eq3:=dsolve(CharEqs[5],q(t)); Eq4:=subs(tr1,_C1=c,Eq3);
Eq5:=subs(Eq2,Eq4,F); Eq51:=[solve(Eq5,c)]; Eq52:=c=Eq51[1];
Eq6:=subs(p=rhs(Eq2),q=rhs(Eq4),Eq52,tr2);
CompleteInt:=int(lhs(Eq6)/du,u)=int(op(1,rhs(Eq6))/dx,x)
             +int(op(2,rhs(Eq6))/dy,y)+b;
```

Mathematica:

```
{fF=p^2+q^2-1, charEqs={x'[t]==D[fF,p], y'[t]==D[fF,q],
 u'[t]==p*D[fF,p]+q*D[fF,q], p'[t]==-(D[fF,x]+p*D[fF,u]),
 q'[t]==-(D[fF,y]+q*D[fF,u])}}
{tr1={q[t]->q, p[t]->p}, tr2=du->p*dx+q*dy}
{eq1=DSolve[charEqs[[4]],p[t],t], eq2=eq1/.tr1/.C[1]->a//First}
{eq3=DSolve[charEqs[[5]],q[t],t], eq4=eq3/.tr1/.C[1]->c//First}
{eq5=fF/.eq2/.eq4, eq51=Solve[eq5==0,c]//Flatten, eq52=c->
 eq51[[2,2]], eq6=tr2/.p->eq2[[1,2]]/.q->eq4[[1,2]]/.eq52}
completeInt=Integrate[eq6[[1]]/du,u]==
 Integrate[eq6[[2,1]]/dx,x]+Integrate[eq6[[2,2]]/dy,y]+b
```
□

Problem 3.13

Maple:

```
CInt:=u=a*x+sqrt(-a^2+1)*y+b; CI1:=subs(b=1,a=0.1,CInt);
CI2:=subs(b=-1,a=0.5,CInt);
plot3d({rhs(CI1),rhs(CI2)},x=-10..10,y=-10..10,axes=boxed);
CInt1:=simplify(subs(a=cos(alpha),CInt)) assuming sin(alpha)>0;
Eq1:=subs(b=G(alpha),CInt1); Eq2:=diff(Eq1,alpha);
Eq3:=eliminate([Eq1,Eq2],alpha); tralpha:=op(Eq3[1]);
GenSol:=subs(tralpha,u=v,Eq1);
sys1:=eval([Eq1,Eq2], G(alpha)=0);
Eq31:=eliminate(sys1,alpha); tralpha1:=op(Eq31[1]);
Sol1:=simplify(subs(tralpha1,u=v,sys1[1])) assuming x>0, y>0;
Sol2:=simplify(subs(tralpha1,u=v,sys1[1])) assuming x<0, y<0;
```
□

3.2.2 The Monge Cone. Characteristic Directions

Problem 3.14

Maple:

```
with(plots): TPlane:=u-u0=p*(x-x0)+q*(y-y0); PDE1:=p^2+q^2-1=0;
Solsq:=[solve(PDE1,q)];  uR:=0..1; vR:=0..2*Pi; Op1:=color=blue;
Eq11:=subs(q=Solsq[1],TPlane); Eq12:=subs(q=Solsq[2],TPlane);
Eq21:=diff(Eq11,p); Eq22:=diff(Eq12,p);
tr1:={x-x0=X,y-y0=Y,u-u0=U,-y+y0=-Y};
tr2:={X=x-x0,Y=y-y0,U=u-u0};
Eq3:=subs(tr1,Eq11); Eq4:=subs(tr1,collect(Eq21*sqrt(1-p^2),p));
Eq31:=map(`^`, Eq3, 2); Eq41:=map(`^`, Eq4, 2);
EqMC1:=subs(tr2,simplify(expand(Eq31+Eq41))); sys1:=[Eq11,Eq21];
sys2:=[Eq12,Eq22]; params:={x0=0,y0=0,u0=0};
TP1:=subs(p=0,params,sys1[1]); TP2:=subs(p=1,params,sys1[1]);
TP3:=subs(p=-1,params,sys1[1]);
GMC:=plot3d([u*cos(v),u*sin(v),u],u=uR,v=vR,axes=boxed,Op1):
for i from 1 to 3 do G||i:=plot3d(rhs(TP||i),x=-1..1,y=-1..1,
 color=x+y): od: display({GMC,G1,G2,G3});
```

Mathematica:

```
{tPlane=u-u0==p*(x-x0)+q*(y-y0), pde1=p^2+q^2-1==0,
 solsq=Solve[pde1,q]//Flatten, eq11=tPlane/.solsq[[2]],
 eq12=tPlane/.solsq[[1]], eq21=Thread[D[eq11,p],Equal],
 eq22=Thread[D[eq12,p],Equal]}
```

```
{tr1={x-x0->xN,y-y0->yN,u-u0->uN,-y+y0->-yN}, tr2={xN->x-x0,
 yN->y-y0,uN->u-u0}, eq3=eq11/.tr1}
eq4=FullSimplify[Thread[eq21*Sqrt[1-p^2],Equal]]/.tr1
{eq31=Map[Power[#,2]&,eq3], eq41=Map[Power[#,2]&,eq4]}
{eq42=0==eq41[[1]]-eq41[[2]], eqMC1=FullSimplify[Expand[Thread[
 eq31+eq42,Equal]]/.tr2], sys1={eq11,eq21}, sys2={eq12,eq22}}
{params={x0->0,y0->0,u0->0}, tP[1]=sys1[[1]]/.p->0/.params,
 tP[2]=sys1[[1]]/.p->1/.params, tP[3]=sys1[[1]]/.p->-1/.params}
gMC=ParametricPlot3D[{u*Cos[v],u*Sin[v],u},{u,0,1},{v,0,2*Pi},
 PlotStyle->Blue]; Do[g[i]=Plot3D[tP[i][[2]],{x,-1,1},{y,-1,1},
 ColorFunction->Function[t,Hue[.95*(0.01+t)]]],{i,1,3}];
Show[{gMC,g[1],g[2],g[3]}]
```
□

Problem 3.15

Maple:

```
with(plots): G:=NULL: F:=p^2-q^2-sin(u);
CharEqs:=[dx/dr=subs(p=p(r),diff(F,p)), dt/dr=subs(q=q(r),
 diff(F,q)),diff(u(r),r)=p(r)*subs(p=p(r),diff(F,p))+
 q(r)*subs(q=q(r),diff(F,q)),diff(p(r),r)=-(diff(F,x)+
 p(r)*diff(F,u)),diff(q(r),r)=-(diff(F,t)+q(r)*diff(F,u))];
Sol45:=subs(u=u(r),dsolve({CharEqs[4],CharEqs[5]},
 {p(r),q(r)}));
Eq21:=combine(lhs(CharEqs[2])/lhs(CharEqs[1])=
 rhs(CharEqs[2])/rhs(CharEqs[1])); Eq45:=subs(Sol45,Eq21);
Eq451:=simplify(2*map(`^`,Sol45[1],2)-2*map(`^`,Sol45[2],2));
Eq3:=subs(Eq451,CharEqs[3]); dsolve(Eq3); tr1:={_C1=1,_C2=1};
tr2:={_C1=-1,_C2=1}; Eq31:=subs(tr1,Eq3); Eq31:=subs(tr2,Eq3);
Chars:=[subs(tr1,Eq45),subs(tr2,Eq45)];
for x from -10 to 10 by 2 do
 G:=G,plot({-xi-x,xi+x},xi=-10..10,color=[blue,magenta]); od:
display([G], view=[-10..10,0..10]);
```

Mathematica:

```
{fF=p^2-q^2-Sin[u], charEqs={dx/dr==D[fF,p]/.p->p[r],
 dt/dr==D[fF,q]/.q->q[r], u'[r]==p[r]*(D[fF,p]/.p->p[r])+
 q[r]*(D[fF,q]/.q->q[r]), p'[r]==-(D[fF,x]+p[r]*D[fF,u]),
 q'[r]==-(D[fF,t]+q[r]*D[fF,u])}}
{sol45=DSolve[{charEqs[[4]],charEqs[[5]]},{p[r],q[r]},r]/.
 u->u[r], eq21=charEqs[[2,1]]/charEqs[[1,1]]==
 charEqs[[2,2]]/charEqs[[1,2]], eq45=eq21/.sol45[[1]]}
```

```
{eq451=Apply[Subtract,Map[2*#^2&,{sol45[[1,1,1]],
 sol45[[1,2,1]]}]]->Apply[Subtract,Map[2*#^2&,{sol45[[1,1,2]],
 sol45[[1,2,2]]}]]//Simplify, eq3=charEqs[[3]]/.eq451}
{tr1={C[1]->1,C[2]->1}, tr2={C[1]->-1,C[2]->1}}
{eq31=eq3/.tr1, eq31=eq3/.tr2, chars={eq45/.tr1,eq45/.tr2}}
g=Table[Plot[{-xi1-x1,xi1+x1},{xi1,-10,10}],{x1,-10,10,2}];
Show[g,PlotRange->{{-10,10},{0,10}}]
```
□

3.2.3 Integral Surfaces. Cauchy Problem

Problem 3.16

Maple:

```
F:=p^2*q-1; IniData:={u=x,y=0}; CharEqs:=[diff(x(t),t)=
 diff(F,p),diff(y(t),t)=diff(F,q), diff(u(t),t)=p*diff(F,p)+
 q*diff(F,q), diff(p(t),t)=-(diff(F,x)+p*diff(F,u)),
 diff(q(t),t)=-(diff(F,y)+q*diff(F,u))]; tr1:={q(t)=q,p(t)=p};
tr2:=du=p*dx+q*dy; Eq1:=dsolve(CharEqs[4],p(t));
Eq2:=subs(tr1,_C1=a,Eq1); Eq3:=dsolve(CharEqs[5],q(t));
Eq4:=subs(tr1,_C1=c,Eq3); Eq5:=subs(Eq2,Eq4,F);
Eq51:=isolate(Eq5,c); Eq6:=subs(Eq51,tr2); Eq7:=subs(p=rhs(Eq2),
 q=rhs(Eq4),Eq51,tr2); Eq8:=expand(isolate(Eq7,du));
Sol:=int(lhs(Eq8)/du,u)=int(op(1,rhs(Eq8))/dx,x)
 +int(op(2,rhs(Eq8))/dy,y)+b; Sol1:=subs(IniData,Sol);
Consts:={a=coeff(lhs(Sol1),x),b=coeff(lhs(Sol1),x,0)};
SolCauchy:=subs(Consts,Sol);
```

Mathematica:

```
{fF=p^2*q-1, iniData={u->x,y->0},charEqs={x'[t]==D[fF,p],
 y'[t]==D[fF,q], u'[t]==p*D[fF,p]+q*D[fF,q], p'[t]==
 -(D[fF,x]+p*D[fF,u]), q'[t]==-(D[fF,y]+q*D[fF,u])}}
{tr1={q[t]->q, p[t]->p}, tr2=du->p*dx+q*dy}
{eq1=DSolve[charEqs[[4]],p[t],t], eq2=eq1/.tr1/.C[1]->a}
{eq3=DSolve[charEqs[[5]],q[t],t], eq4=eq3/.tr1/.C[1]->c}
{eq5=Flatten[fF==0/.eq2/.eq4], eq51=Solve[eq5,c], eq6=tr2/.eq51}
eq7=tr2/.p->eq2[[1,1,2]]/.q->eq4[[1,1,2]]/.eq51/.Rule->Equal
{sol=Integrate[eq7[[1,1]]/du,u]==Integrate[eq7[[1,2,1]]/dx,x]+
 Integrate[eq7[[1,2,2]]/dy,y]+b, sol1=sol/.iniData}
{consts={a->Coefficient[sol1[[1]],x],b->Coefficient[
 sol1[[1]],x,0]}, solCauchy=sol/.consts}
```
□

Problem 3.17

Maple:

```
interface(showassumed=0); assume(t>0); F:=p^2+q+u;
F1:=p(r,t)^2+q(r,t)+u(r,t); IniData:=[x(r,0)=r,y(r,0)=0,
 u(r,0)=r]; IniData1:=[x(0)=r,y(0)=0,u(0)=r];
Eq1:=p(r,0)=rhs(diff(IniData[1],r));
Eq2:=subs(Eq1,IniData[3],subs(t=0,F1)); Eq3:=isolate(Eq2,q(r,0));
CharEqs:=[diff(x(t),t)=diff(F,p),diff(y(t),t)=diff(F,q),
          diff(u(t),t)=p*diff(F,p)+q*diff(F,q),
          diff(p(t),t)=-(diff(F,x)+p*diff(F,u )),
          diff(q(t),t)=-(diff(F,y)+q*diff(F,u))];
tr1:={p(t)=p(r,t),q(t)=q(r,t)}; tr2:={p=p(r,t),q=q(r,t)};
Eq4:=dsolve({subs(tr1,lhs(CharEqs[4]))=
             subs(tr2,rhs(CharEqs[4])),Eq1},p(r,t));
Eq5:=dsolve({subs(tr1,lhs(CharEqs[5]))=
             subs(tr2,rhs(CharEqs[5])),Eq3},q(r,t));
tr4:={p=rhs(Eq4),q=rhs(Eq5)}; tr5:={x(t)=x,y(t)=y,u(t)=u};
Eq6:=combine(subs(tr4,[CharEqs[i] $ i=1..3]));
Eq7:=dsolve({op(Eq6),op(IniData1)},{x(t),y(t),u(t)});
Eq8:=eliminate(subs(tr5,Eq7),{t,r}); SolFin:=collect(combine(
 expand(isolate(op(Eq8[2]),u))),exp); pdetest(u(x,y)=rhs(SolFin),
 diff(u(x,y),x)^2+diff(u(x,y),y)+u(x,y)=0);
```

Mathematica:

```
{fF=p^2+q+u, f1=p[r,t]^2+q[r,t]+u[r,t], iniData={x[r,0]->r,
 y[r,0]->0,u[r,0]->r}, iniData1={x[0]==r,y[0]==0,u[0]==r}}
{eq1=p[r,0]->D[iniData[[1]],r][[2]], eq2=(f1/.t->0)/.eq1/.
 iniData[[3]], eq3=Solve[eq2==0,q[r,0]]}
{charEqs={x'[t]==D[fF,p], y'[t]==D[fF,q], u'[t]==p*D[fF,p]+
 q*D[fF,q], p'[t]==-(D[fF,x]+p*D[fF,u]),
 q'[t]==-(D[fF,y]+q*D[fF,u])}, tr1={p->p[r,t],q->q[r,t]}}
tr2={p'[t]->D[p[r,t],t], q'[t]->D[q[r,t],t]}
eq4=DSolve[{(charEqs[[4,1]]/.tr2)==(charEqs[[4,2]]/.tr1),
 (eq1/.Rule->Equal)},p[r,t],{r,t}]
eq5=DSolve[{(charEqs[[5,1]]/.tr2)==(charEqs[[5,2]]/.tr1),
 (eq3/.Rule->Equal)},q[r,t],{r,t}]
{tr4={p->eq4[[1,1,2]],q->eq5[[1,1,2]]},tr5={x[t]->x,y[t]->y,
 u[t]->u}, eq6=Table[charEqs[[i]],{i,1,3}]/.tr4}
eq7=DSolve[{eq6,iniData1},{x[t],y[t],u[t]},t]
```

```
{sys7=(eq7/.tr5/.Rule->Equal)[[1]]//Expand, sys8=Eliminate[
 sys7,r], eq8=sys8[[2]]/.(sys8[[1]]/.Equal->Rule),
 solFin=(Solve[eq8,u]//Simplify)/.Rule->Equal}
```

□

Problem 3.18

Maple:

```
interface(showassumed=0); assume(t>0); F:=p^2+q+u;
F1:=p(r,t)^2+q(r,t)+u(r,t); IniData:=[x(r,0)=r,y(r,0)=r,
 u(r,0)=2*r-1]; IniData1:=[x(0)=r,y(0)=r,u(0)=2*r-1];
CharEqs:=[diff(x(t),t)=diff(F,p), diff(y(t),t)=diff(F,q),
 diff(u(t),t)=p*diff(F,p)+q*diff(F,q),
 diff(p(t),t)=-(diff(F,x)+p*diff(F,u)),
 diff(q(t),t)=-(diff(F,y)+q*diff(F,u))];
tr1:={p(t)=p(r,t),q(t)=q(r,t)}; tr2:={p=p(r,t),q=q(r,t)};
Eq1:=subs(IniData,subs(t=0,F1))=0; StripCond:=diff(u(r,0),r)=
 p(r,0)*diff(x(r,0),r)+q(r,0)*diff(y(r,0),r);
IniData2:=diff(IniData,r); Eq2:=subs(IniData2,StripCond);
sys1:={Eq1,Eq2}; vars:=indets(sys1) minus {r};
sols:=[allvalues(solve(sys1,vars))];
for k from 4 to 5 do  Eq||k||1:=subs(tr1,lhs(CharEqs[k]))=
 subs(tr2,rhs(CharEqs[k])); od;
Solp:=dsolve({Eq41} union {op(1,sols[1])},p(r,t));
Solq:=dsolve({Eq51} union {op(2,sols[1])},q(r,t));
tr4:={p=rhs(Solp),q=rhs(Solq)}; tr5:={x(t)=x,y(t)=y,u(t)=u};
Eq6:=combine(subs(tr4,[CharEqs[i] $ i=1..3]));
Eq7:=dsolve({op(Eq6),op(IniData1)},{x(t),y(t),u(t)});
Eq71:= Eq7 union {Solp} union {Solq};
SolFin:=subs({u(t)=u(r,t),x(t)=x(r,t),y(t)=y(r,t)},Eq71);
test1:=simplify(subs(SolFin,F1));
```

Mathematica:

```
{fF=p^2+q+u, f1=p[r,t]^2+q[r,t]+u[r,t], iniData={x[r,0]->r,
 y[r,0]->r,u[r,0]->2*r-1}, iniData1={x[0]==r,y[0]==r,
 u[0]==2*r-1}, charEqs={x'[t]==D[fF,p], y'[t]==D[fF,q],
 u'[t]==p*D[fF,p]+q*D[fF,q], p'[t]==-(D[fF,x]+p*D[fF,u]),
 q'[t]==-(D[fF,y]+q*D[fF,u])}}
{tr1={p->p[r,t], q->q[r,t]}, tr12={p'[t]->D[p[r,t],t],
 q'[t]->D[q[r,t],t]}, tr2={p->p[r,t],q->q[r,t]}}
```

```
{eq1=(f1/.t->0/.iniData)==0, stripCond=D[u[r,0],r]==p[r,0]*
 D[x[r,0],r]+q[r,0]*D[y[r,0],r], iniData2=D[iniData,r],
 eq2=stripCond/.iniData2, sys1={eq1,eq2}}
{vars=Complement[Variables[{eq1[[1]],eq2[[1]]}],{r}],
 sols=Solve[sys1,vars], sols1=sols//First//Sort}
{Table[eq[k]=(charEqs[[k]][[1]]/.tr12)==(charEqs[[k]][[2]]/.tr2),
 {k,4,5}], sols2={vars[[1]]==sols1[[1,2]],
 vars[[2]]==sols1[[2,2]]}, solp=DSolve[{eq[4],sols2[[1]]},p[r,t],
 {r,t}], solq=DSolve[{eq[5],sols2[[2]]},q[r,t],{r,t}]}
{tr4={p->solp[[1,1,2]], q->solq[[1,1,2]]}, tr5={x[t]->x,y[t]->y,
 u[t]->u}, tr6={u[t]->u[r,t], x[t]->x[r,t], y[t]->y[r,t]}}
eq6=Table[charEqs[[i]],{i,1,3}]/.tr4//Together
{eq7=DSolve[{eq6,iniData1},{x[t],y[t],u[t]},{t}],
 eq71={eq7,solp,solq}, solFin=eq71/.tr6//Flatten,
 test1=f1/.solFin//Simplify}
```

Maple:

```
Sol1:=[op(SolFin)];
tr6:={p(r,t)=p,q(r,t)=q,u(r,t)=u,x(r,t)=x,y(r,t)=y};
Sol2:=eliminate(subs(tr6,{Sol1[3],Sol1[4],Sol1[5]}),{r,t});
Sol3:=combine(op(Sol2[2]))=0;
SolFin1:=map(combine,collect(expand(isolate(Sol3,u)),exp(y)));
```
□

Problem 3.19

Maple:

```
with(Student[Precalculus]): interface(showassumed=0);
assume(t>0); assume(x>x0); assume(y>y0); assume(u>u0);
F:=p^2+q^2-n^2; F1:=p(r,t)^2+q(r,t)^2-n^2;
IniData:=[x(0,t)=x0(t),y(0,t)=y0(t),u(0,t)=u0(t)];
IniData1:=[x(0)=x0(t),y(0)=y0(t),u(0)=u0(t)];
tr1:={p(t)=p(r,t),q(t)=q(r,t)}; tr2:={p=p(r,t),q=q(r,t)};
CharEqs:=[diff(x(r),r)=diff(F,p),diff(y(r),r)=diff(F,q),
 diff(u(r),r)=p*diff(F,p)+q*diff(F,q),
 diff(p(r),r)=-(diff(F,x)+p*diff(F,u )), diff(q(r),r)=
 -(diff(F,y)+q*diff(F,u))];
Eq1:=subs(_C1=p0(t),p(r,t)=rhs(dsolve(CharEqs[4],p(r))));
Eq2:=subs(_C1=q0(t),q(r,t)=rhs(dsolve(CharEqs[5],q(r))));
tr4:={p=rhs(Eq1),q=rhs(Eq2)};
```

```
tr5:={x(r)=x,y(r)=y,u(r)=u,p0(t)=p0,q0(t)=q0,x0(t)=x0,
 y0(t)=y0,u0(t)=u0}; tr6:=p0(t)^2+q0(t)^2=n^2;
Eq3:=combine(subs(tr4,[CharEqs[i] $ i=1..3]));
Eq4:=dsolve({op(Eq3),op(IniData1)},{x(r),y(r),u(r)});
Eq41:=op(1,Eq4); Eq42:=map(factor,collect(rhs(Eq41),r));
Eq43:=lhs(Eq41)=subs(tr6,Eq42); Eq5:=eliminate(subs(tr5,
 {Eq43,Eq4[2],Eq4[3],tr6}),{p0,q0,r});
Eq6:=CompleteSquare(op(Eq5[2])); Eq7:=collect(isolate(Eq6,
 (u-u0)^2),n^2); SolFin:=factor([solve(Eq7,u) assuming u>u0]);
```

Mathematica:

```
{fF=p^2+q^2-n^2, f1=p[r,t]^2+q[r,t]^2-n^2}
iniData={x[0,t]->x0[t],y[0,t]->y0[t],u[0,t]->u0[t]}
iniData1={x[0]->x0[t],y[0]->y0[t],u[0]->u0[t]}
{tr1={p[t]->p[r,t],q[t]->q[r,t]}, tr2={p->p[r,t],q->q[r,t]}}
charEqs={x'[r]==D[fF,p], y'[r]==D[fF,q], u'[r]==p*D[fF,p]+
 q*D[fF,q],p'[r]==-(D[fF,x]+p*D[fF,u]),
 q'[r]==-(D[fF,y]+q*D[fF,u])}
{eq1=p[r,t]==(First[DSolve[charEqs[[4]],p[r],r]/.
 C[1]->p0[t]])[[1,2]], eq2=q[r,t]==(First[DSolve[charEqs[[5]],
 q[r],r]/.C[1]->q0[t]])[[1,2]]}
{tr4={p->eq1[[2]],q->eq2[[2]]}, tr5={x[r]->x,y[r]->y,u[r]->u,
 p0[t]->p0,q0[t]->q0,x0[t]->x0,y0[t]->y0,u0[t]->u0},
 tr6=p0[t]^2+q0[t]^2->n^2}
{eq3=Table[charEqs[[i]],{i,1,3}]/.tr4, eq4=DSolve[{eq3,
 iniData1/.Rule->Equal}//Flatten,{x[r],y[r],u[r]},r]//Flatten}
{eq41=eq4[[3]], eq42=eq41[[2]]//FullSimplify,
 eq43=eq41[[1]]==eq42/.tr6, eq5=Eliminate[{eq43,eq4[[1]],
 eq4[[2]],tr6}/.Rule->Equal/.tr5,{p0,q0,r}]}
{eq51=eq5[[1]]-eq5[[2]], eq6=FullSimplify[eq51]==0,
 eq7=Thread[eq6+(u-u0)^2,Equal]}
solFin=Solve[eq7,u]//FullSimplify//Flatten
```

Maple:

```
SolFin:=u=u0+n*sqrt((x-x0)^2+(y-y0)^2); trn:={n=1};
U:=unapply(rhs(SolFin),x,y); tr0:={p0(t)=n*cos(alpha),
 q0(t)=n*sin(alpha)}; combine(subs(tr0,p0(t)^2+q0(t)^2=n^2));
tr1:={x0=0,y0=0,u0=0}; tr2:={x0(t)=t,y0(t)=t,u0(t)=t^2};
plot3d(subs(tr1,trn,U(x,y)),x=-10..10,y=-10..10);
EikonalEq:=diff(U(x,y),x)^2+diff(U(x,y),y)^2-n^2;
```

```
sys1:={u=2*n^2*r+u0(t),x=2*p0(t)*r+x0(t),
       y=2*q0(t)*r+y0(t)}; sys2:=subs(tr2,tr0,sys1);
Sol2:=eliminate(sys2,{r,t}); Sol21:=isolate(op(Sol2[2]),u);
U2:=unapply(collect(rhs(Sol21),[n,x,y]),x,y);
plot3d(subs(trn,alpha=Pi,U2(x,y)),x=-10..10,y=-10..10);
```

Mathematica:

```
SetOptions[Plot3D,PlotRange->All,Mesh->False,BoxRatios->{1,1,1}];
{solFin=u==u0+n*Sqrt[(x-x0)^2+(y-y0)^2], trn=n->1}
uN[xN_,yN_]:=solFin[[2]]/.x->xN/.y->yN; uN[x,y]
{tr0={p0[t]->n*Cos[alpha],q0[t]->n*Sin[alpha]},
 p0[t]^2+q0[t]^2==n^2/.tr0, tr1={x0->0,y0->0,u0->0},
 tr2={x0[t]->t,y0[t]->t,u0[t]->t^2}}
Plot3D[Evaluate[uN[x,y]/.tr1/.trn],{x,-10,10},{y,-10,10}]
eikonalEq=D[uN[x,y],x]^2+D[uN[x,y],y]^2-n^2
sys1={u==2*n^2*r+u0[t], x==2*p0[t]*r+x0[t], y==2*q0[t]*r+y0[t]}
{sys2=sys1/.tr0/.tr2, sol2=Eliminate[sys2,{r,t}],
 sol21=Solve[sol2,u]//First}
uN2[xN_,yN_]:=sol21[[1,2]]//FullSimplify/.x->Xn/.y->yN;
Plot3D[Evaluate[uN2[x,y]/.alpha->Pi/.trn],{x,-10,10},
 {y,-10,10}]
```

□

3.3 Qualitative Analysis

3.3.1 Nonlinear PDEs

Problem 3.20

Maple:

```
with(PDEtools): with(plots): with(DEtools): with(LinearAlgebra):
Ops:=arrows=medium,dirgrid=[20,20],stepsize=0.1,thickness=2,
 linecolour=blue,color=green;
tr1:=x-c*t=z; tr2:={V(z)=V,W(z)=W}; vars:=[W(z),V(z)];
Eq1:=u->diff(u,t)-diff(u,x$2)-u*(1-u);
Eq2:=expand(Eq1(W(lhs(tr1))));  Eq3:=algsubs(tr1,Eq2)=0;
Eq4:=map(convert,Eq3,diff); Eq5:=diff(W(z),z)=V(z);
Eq6:=isolate(subs(Eq5,Eq4),diff(V(z),z)); Eq7:=subs(tr2,
 Diff(V,W)=rhs(Eq6)/rhs(Eq5)); Eq81:=denom(rhs(Eq7))=0;
Eq82:=[solve(subs(Eq81,numer(rhs(Eq7))),W)];
SingularPoints:=[[Eq82[1],rhs(Eq81)],[Eq82[2],rhs(Eq81)]];
P:=subs(tr2,rhs(Eq5)); Q:=subs(tr2,rhs(Eq6));
```

```
A:=<diff(P,W),diff(Q,W)|diff(P,V),diff(Q,V)>;
A1:=subs({W=0,V=0},A); A2:=subs({W=1,V=0},A);
map(Eigenvalues,{A1,A2}); c>=2; Eqs:=subs(c=3,[Eq5,Eq6]);
IC:=[[W(0)=-0.5,V(0)=-1.],[W(0)=0.1,V(0)=1.],
 [W(0)=0.96968212,V(0)=0.1],[W(0)=0.35,V(0)=1.],
 [W(0)=0.5,V(0)=1.],[W(0)=0.8,V(0)=-1.],[W(0)=0.4,V(0)=-1.],
 [W(0)=0.1,V(0)=-1.],[W(0)=0.999,V(0)=-1.],[W(0)=1.1,V(0)=-1.],
 [W(0)=1.29889,V(0)=-1.],[W(0)=0.65999,V(0)=1.1]];
phaseportrait(Eqs,vars,z=0..60,IC,Ops,view=[-0.7..1,-1..1]);
```

Mathematica:

```
eq1[u_]:=D[u,t]-D[u,{x,2}]-u*(1-u);
{tr1=x-c*t->z, tr2={vN[z]->vN,uN[z]->uN}}
tr3={uN'[z]->vN[z],uN''[z]->vN'[z]}
{eq2=eq1[uN[tr1[[1]]]]//Expand, eq3=(eq2/.tr1)==0,
 eq4=eq3//TraditionalForm, eq5=eq3/.tr3}
{eq6=Solve[eq5,vN'[z]], eq7=Hold[D[vN,uN]]==eq6[[1,1,2]]/vN[z]}
{eq81=eq7[[2,1,1]]->0, eq82=Solve[eq7[[2,2]]==0/.eq81,uN[z]]}
singPoints={{eq82[[1,1,2]],eq81[[2]]},{eq82[[2,1,2]],eq81[[2]]}}
{p1=tr3[[1,2]]/.tr2, q1=eq6[[1,1,2]]/.tr2}
ma={{D[p1,uN],D[p1,vN]},{D[q1,uN],D[q1,vN]}}
{a1=ma/.{uN->0,vN->0}, a2=ma/.{uN->1,vN->0}}
{Eigenvalues[a1], Eigenvalues[a2], c>=2}
{eqs=({tr3[[1]],eq6[[1,1]]}/.c->3)/.Rule->Equal, vars={uN,vN}}
ic={{-0.5,-1.},{0.1,1.},{0.96968212,0.1},{0.35,1.},{0.5,1.},
    {0.8,-1.},{0.4,-1.},{0.1,-1.},{0.999,-1.},{1.1,-1.},
    {1.29889,-1.},{0.65999,1.1}}; n=Length[ic]; zF=60;
Do[{sys[i]={eqs[[1]],eqs[[2]],uN[0]==ic[[i,1]],vN[0]==ic[[i,2]]};
  sols=NDSolve[sys[i],vars,{z,0,zF}]; cu=uN/.sols[[1]];
  cv=vN/.sols[[1]]; c[i]=ParametricPlot[Evaluate[{cu[z],cv[z]}],
  {z,0,zF},PlotStyle->{Hue[0.1*i+0.2],Thickness[.01]}];},{i,1,n}]
fu=eqs[[1,2]]/.tr2; fv=eqs[[2,2]]/.tr2;
fd=VectorPlot[{fu,fv},{uN,-0.7,1.},{vN,-1.,1.},
   VectorColorFunction->Hue]; Show[fd,Table[c[i],{i,1,n}]]
```

□

Problem 3.21

Maple:

```
with(PDEtools): with(plots): with(DEtools): declare(u(x,t));
Ops:=arrows=medium,dirgrid=[20,20],stepsize=0.1,thickness=2,
     linecolour=blue,color=green;
```

```
tr1:=x-c*t=z; tr2:={V(z)=V,W(z)=W};
Eq1:=u->diff(u,x$2)-diff(u,t$2)-sin(u);
Eq2:=expand(Eq1(W(lhs(tr1)))); Eq3:=algsubs(tr1,Eq2)=0;
Eq4:=map(convert,Eq3,diff); Eq5:=diff(W(z),z)=V(z);
Eq6:=isolate(collect(subs(Eq5,Eq4),diff),V(z));
Eq7:=subs(tr2,Diff(V,W)=rhs(Eq6)/rhs(Eq5));
Eqs:=subs(c=0.9,[Eq5,Eq6]); vars:=[W(z),V(z)];
IC:=[[W(0)=0.01,V(0)=0.],[W(0)=-0.01,V(0)=0.],[W(0)=2,V(0)=0.],
 [W(0)=-2,V(0)=0.],[W(0)=4.1,V(0)=4.1],[W(0)=-4.1,V(0)=-4.1],
 [W(0)=6.28,V(0)=0.],[W(0)=-6.28,V(0)=0.]];
P:=(X,Y)->phaseportrait(Eqs,vars,z=0..2*Pi,IC,scene=[X,Y],
          Ops,view=[-7..7,-7..7]); P(W,V); P(z,W);
```

Mathematica:

```
n1=20; p=15; zF=2*Pi//N;
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&];
SetOptions[ParametricPlot,AspectRatio->1,
 PlotStyle->{Blue,Thickness[.01]}];
SetOptions[VectorPlot,VectorColorFunction->Hue,
 VectorPoints->{n1,n1},VectorStyle->Arrowheads[0.02],
 PlotRange->All]; {tr1=x-c*t->z, tr2={v[z]->v,w[z]->w}}
eq1[u_]:=D[u,{x,2}]-D[u,{t,2}]-Sin[u];
{eq2=eq1[w[tr1[[1]]]]//Expand, eq3=(eq2/.tr1)==0,
 eq4=eq3//TraditionalForm, eq5={w'[z]->v[z],w''[z]->v'[z]}}
{eq6=eq3/.eq5//FullSimplify, termc=trS3[trS1[eq6[[1]],c],z]}
eq61=Thread[eq6/termc,Equal]//FullSimplify
eq62=Thread[Thread[(-1)*eq61,Equal]+v'[z],Equal]
eq7=HoldForm[D[v,w]]==(eq62[[1]]/eq5[[1,2]])/.tr2
eqs={eq5[[1]]/.Rule->Equal, eq62[[2]]==eq62[[1]]}/.c->0.9
vars={w,v};  ic={{0.01,0.},{-0.01,0.},{2,0.},{-2,0.},
 {4.1,4.1},{-4.1,-4.1},{6.28,0.},{-6.28,0.}}
n=Length[ic];
Do[{sys[i]={eqs[[1]],eqs[[2]], w[0]==ic[[i,1]],
 v[0]==ic[[i,2]]}; sols=NDSolve[sys[i],vars,{z,0,zF}];
 cw=w/.sols[[1]]; cv=v/.sols[[1]];
 wv[i]=ParametricPlot[Evaluate[{cw[z],cv[z]}],{z,0,zF},
  PlotRange->{{-p,p},{-zF,zF}}];
 zw[i]=ParametricPlot[Evaluate[{z,cw[z]}],{z,0,zF},
  PlotRange->{{0,zF},{-p,p}}];},{i,1,n}];
fw=eqs[[1,2]]/.tr2; fv=eqs[[2,2]]/.tr2;
fd=VectorPlot[{fw,fv},{w,-p,p},{v,-zF,zF}];
```

```
Show[fd,Table[wv[i],{i,1,n}]]
Show[Table[zw[i],{i,1,n}]]
```

□

3.3.2 Nonlinear Systems

Problem 3.22

Maple:

```
with(plots): with(DEtools):
Ops:=arrows=medium,dirgrid=[20,20],stepsize=0.1,thickness=2,
 linecolour=blue,color=magenta;
phi_1:=1; phi_2:=1; epsilon:=0.1; delta:=-1/2;
Eq1:=D(v)(t)=epsilon*u(t)*(-delta+1/4+phi_1/2*(u(t)^2+v(t)^2)
 -phi_2/4*(u(t)^2+v(t)^2)^2);
Eq2:=D(u)(t)=epsilon*v(t)*(delta+1/4-phi_1/2*(u(t)^2+v(t)^2)
 +phi_2/4*(u(t)^2+v(t)^2)^2);
Eqs:=[Eq1,Eq2]; vars:=[v(t),u(t)];
IC:=[[u(0)=0,v(0)=1.1033],[u(0)=0,v(0)=-1.1033],
 [u(0)=1.1055,v(0)=0],[u(0)=-1.1055,v(0)=0],
 [u(0)=0,v(0)=1.613],[u(0)=0,v(0)=-1.613],
 [u(0)=0.2,v(0)=0],[u(0)=0.4,v(0)=0],
 [u(0)=1.3,v(0)=1.2],[u(0)=-1.3,v(0)=-1.2],
 [u(0)=0,v(0)=1.5]];
phaseportrait(Eqs,vars,t=-48..400,IC,Ops);
```

Mathematica:

```
n1=20; tI=-48; tF=400;
{delta=-1/2,phi1=1,phi2=1,epsilon=0.1}
{eq1=epsilon*u[t]*(-delta+1/4+phi1/2*(u[t]^2+v[t]^2)-
 phi2/4*(u[t]^2+v[t]^2)^2),
 eq2=epsilon*v[t]*(delta+1/4-phi1/2*(u[t]^2+v[t]^2)+
 phi2/4*(u[t]^2+v[t]^2)^2)}
ic={{0,1.1033},{0,-1.1033},{1.1055,0},{-1.1055,0},{0,1.613},
 {0,-1.613},{0.2,0},{0.4,0},{1.3,1.2},{-1.3,-1.2},{0,1.5}}
n=Length[ic];
Do[{sys[i]={v'[t]==eq1,u'[t]==eq2,v[0]==ic[[i,2]],
 u[0]==ic[[i,1]]}; sols=NDSolve[sys[i],{v,u},{t,tI,tF}];
 cv=v/.sols[[1]]; cu=u/.sols[[1]];
 c[i]=ParametricPlot[Evaluate[{cv[t],cu[t]}],{t,tI,tF},
  AspectRatio->1,PlotStyle->{Blue,Thickness[.01]}];},
{i,1,n}];
```

```
fv=eq1/.{v[t]->v,u[t]->u}; fu=eq2/.{v[t]->v,u[t]->u};
fd=VectorPlot[{fv,fu},{v,-2,2},{u,-2,2},Frame->True,
 VectorColorFunction->Function[{x},Hue[0.8-Log[x]/30]],
 VectorPoints->{n1,n1},VectorStyle->Arrowheads[0.02]];
Show[fd,Table[c[i],{i,1,n}]]
```
□

Chapter 4
General Analytical Approach. Integrability

4.1 Painlevé Test and Integrability

4.1.1 Painlevé Property and Test

Problem 4.1

Maple:

```
interface(showassumed=0): assume(j>0): with(PDETools):
declare(u(x,t),u[0](t),psi(t)); U:=diff_table(u(x,t));
tr11:=N->1/phi^(rho)*sum(u[j](t)*phi^j,j=0..N);
tr21:=phi=x-psi(t); tr22:=rhs(tr21)=phi;
tr23:=-rhs(tr21)=-phi; Eq1:=U[t]+a*U[]*U[x]+b*U[x,x]=0;
Eq2:=algsubs(u(x,t)=subs(tr21,tr11(0)),Eq1)*(rhs(tr21))^(rho+2);
Eq3:=collect(subs(tr22,simplify(Eq2)),[u[0](t),diff]);
Eq4:=collect(map(factor,lhs(Eq3)),diff); Eq5:=combine(subs(
 tr23,Eq4)); Eq51:=remove(has,Eq5,diff);
termH:=select(has,select(has,indets(Eq51),phi),rho);
tr3:=isolate(op(2,op(termH)),rho);
tr4:=u[0](t)=[solve(remove(has,subs(tr3,Eq5),phi),u[0](t))][2];
tr5:=subs(tr21,u[0](t)*phi^(-rho)+u[j](t)*phi^(j-rho));
tr51:=u(x,t)=subs(tr3,tr4,tr5);
Eq6:=expand(algsubs(tr51,a*U[]*U[x]+b*U[x,x]));
Eq7:=subs(tr23,subs(tr22,combine(Eq6))); Eq9:=collect(Eq7,
 [u[j](t)]); Eq10:=factor(combine(coeff(Eq9,u[j](t),1)));
FI:=[solve(remove(has,Eq10,phi),j) assuming j<>0]; FI[2];
tr8:=u(x,t)=subs(tr21,tr3,expand(tr11(FI[2])));
Eq11:=simplify(subs(tr23,expand(algsubs(tr8,Eq1)))*phi^3);
Eq12:=collect(Eq11,[diff,u[1](t),u[2](t)]);
```

```
Eq13:=map(factor,lhs(Eq12)); Eq14:=subs(tr23,Eq13);
Eq15:=collect(Eq14,[phi,u[1](t),u[2](t)]);
Eq16:=map(factor,Eq15); Eq17:=collect(Eq16,[u[2](t)]);
Eq18:=map(factor,Eq17); Eq19:=subs(tr23,Eq18);
sys1:=factor([coeff(Eq19,phi,2)=0,coeff(Eq19,phi,1)=0,
 coeff(Eq19,phi,0)=0]); tru1:=isolate(sys1[2],u[1](t));
SolFin:=u(x,t)=expand(subs(tr4,subs(tr3,tr21,tru1,tr11(FI[2]))),
 rhs(tr21)); Sol2:=subs(a=1,b=-nu,SolFin);
```

Mathematica:

```
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&]; tr11[n_]:=1/phi^(rho)*
 Sum[u[t][j]*phi^j,{j,0,n}]; trD[u_,var_]:=Table[D[u,{var,i}],
 {i,1,2}]//Flatten; {tr21=phi->x-psi[t], tr22=tr21[[2]]->phi,
 tr23=-tr21[[2]]->-phi}
eq1=D[u[x,t],t]+a*u[x,t]*D[u[x,t],x]+b*D[u[x,t],{x,2}]==0
{tr24={u[x,t]->tr11[0]/.tr21}, eq11=eq1/.tr24/.trD[tr24,t]/.
 trD[tr24,x], eq2=Thread[eq11*(tr21[[2]])^(rho+2),Equal]//
 Simplify, eq3=Collect[eq2/.tr22,{u[t][0],u'[t][0]}]}
{eq4=Map[Factor,eq3[[1]]], eq51=trS1[eq4,a],
 termH=trS1[eq51,phi], tr3=Solve[termH[[2]]==0,rho]//First}
tr4=Solve[trS3[Expand[eq4/.tr3],phi]==0,u[t][0]]//Flatten
{tr5=u[t][0]*phi^(-rho)+u[t][j]*phi^(j-rho)/.tr21, tr51=u[x,t]->
 tr5/.tr4[[2]]/.tr3, eq6=a*u[x,t]*D[u[x,t],x]+b*D[u[x,t],
 {x,2}]/.tr51/.trD[tr51,x]//Expand//FullSimplify}
{eq7=eq6/.tr22/.tr23, eq9=Collect[eq7//Expand,u[t][j]]}
eq10=Coefficient[eq9,u[t][j],1]//FullSimplify
{fI=Solve[trS3[eq10,phi]==0,j]//Flatten, fI[[2]]}
tr8=u[x,t]->(tr11[fI[[2,2]]]//Expand)/.tr3/.tr21
eq11=(((eq1/.tr8/.trD[tr8,x]/.trD[tr8,t])//Expand)/.tr22)
eq111=Thread[eq11*phi^3,Equal]//Expand
{eq12=(eq111/.x->phi+psi[t])//Factor, eq13=eq12[[1]]}
{sys1={Coefficient[eq13,phi,2]==0,Coefficient[eq13,phi,1]==0,
 Coefficient[eq13,phi,0]==0}//Factor, tru1=Solve[sys1[[2]],
 u[t][1]]//First, sol1=Collect[tr11[fI[[2,2]]]/.tru1/.tr21/.
 tr3/.tr4[[2]]//Expand,{b,u[t][2],a}]}
{solFin=u[x,t]->Map[Factor,sol1], sol2=solFin/.a->1/.b->-nu}
```
□

Problem 4.2

Maple:

```
interface(showassumed=0): assume(j>0,rho>0): with(PDETools):
declare(u(x,t),u[0](t),psi(t)); U:=diff_table(u(x,t));
tr11:=N->1/phi^(rho)*sum(u[j](t)*phi^j,j=0..N);
tr21:=phi=x-psi(t); tr22:=rhs(tr21)=phi; tr23:=-rhs(tr21)=-phi;
Eq1:=U[t]+U[x]+U[]^2*U[x]+a*U[x,x,x]+b*U[x,x,x,x,x]=0;
Eq2:=algsubs(u(x,t)=subs(tr21,tr11(0)),Eq1)*(rhs(tr21))^(rho+5);
Eq3:=collect(subs(tr22,simplify(Eq2)),[u[0](t),diff]);
Eq4:=map(factor,lhs(Eq3)); Eq5:=combine(subs(tr23,Eq4));
Eq51:=remove(has,Eq5,diff);
termH:=select(has,select(has,indets(Eq51),phi),rho);
tr3:=isolate(op(2,op(termH)),rho);
tr4:=u[0](t)=[solve(remove(has,subs(tr3,Eq5),phi),u[0](t))][2];
tr5:=subs(tr21,u[0](t)*phi^(-rho)+u[j](t)*phi^(j-rho));
tr51:=u(x,t)=subs(tr3,tr4,tr5);
Eq6:=expand(algsubs(tr51,U[]^2*U[x]+b*U[x,x,x,x,x])):
Eq7:=subs(tr23,subs(tr22,combine(Eq6))):
Eq8:=remove(has,Eq7,diff); Eq9:=collect(Eq8,[u[j](t)]);
Eq10:=factor(combine(coeff(Eq9,u[j](t),1)));
Eq11:=remove(has,subs(tr23,Eq10),phi);
Eq12:=map(factor,collect(Eq11,[x,psi,a,b]));
Eq13:=select(has,Eq12,[b]); FI:=[solve(Eq13,j) assuming j<>0];
```

Mathematica:

```
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&]; tr11[n_]:=1/phi^(rho)*
 Sum[u[t][j]*phi^j,{j,0,n}]; trD[u_,var_]:=Table[D[u,{var,i}],
 {i,1,5}]//Flatten; {tr21=phi->x-psi[t], tr22=tr21[[2]]->phi,
 tr23=-tr21[[2]]->-phi}
{eq1=D[u[x,t],t]+D[u[x,t],x]+u[x,t]^2*D[u[x,t],x]+a*D[u[x,t],
 {x,3}]+b*D[u[x,t],{x,5}]==0, tr24={u[x,t]->tr11[0]/.tr21}}
eq11=eq1/.tr24/.trD[tr24,t]/.trD[tr24,x]
eq2=Thread[eq11*(tr21[[2]])^(rho+5),Equal]//Simplify
eq3=Collect[eq2/.tr22,{u[t][0],u'[t][0]}]
{eq4=Map[Factor,eq3[[1]]], eq51=trS1[eq4,u[t][0]^3],
 termH=trS1[eq51,phi], tr3=Solve[termH[[2]]==0,rho]//First}
tr4=Solve[trS3[Expand[eq4/.tr3],phi^5]==0,u[t][0]]//Flatten
tr5=u[t][0]*phi^(-rho)+u[t][j]*phi^(j-rho)/.tr21
```

```
{tr51=u[x,t]->tr5/.tr4[[3]]/.tr3, eq6=u[x,t]^2*D[u[x,t],x]+
 b*D[u[x,t],{x,5}]/.tr51/.trD[tr51,x]//Expand//FullSimplify}
{eq7=eq6/.tr22/.tr23, eq9=Collect[eq7//Expand,u[t][j]]}
eq10=Coefficient[eq9,u[t][j],1]//FullSimplify
termphi=trS1[trS3[eq10//Factor,psi],phi]
{eq11=trS3[trS3[Thread[eq10/termphi,Equal]//Expand,psi],phi]//
 Factor, fI=Solve[eq11==0,j]//Flatten}
```
□

4.1.2 Truncated expansions

Problem 4.3

Maple:

```
interface(showassumed=0): assume(j>0): with(PDETools):
declare(u(x,t),u[0](x,t),u[1](x,t),phi(x,t));
U:=diff_table(u(x,t)); Eq1:=U[t]+a*U[]*U[x]+b*U[x,x]=0;
tr1:=N->u(x,t)=sum(u[j](x,t)/phi(x,t)^(N-j),j=0..N); tr1(1);
Eq2:=algsubs(tr1(1),Eq1); Eq3:=collect(lhs(Eq2),[phi(x,t)]);
for i from 1 to 3 do
 E||i:=select(has,select(has,Eq3,phi(x,t)^(-i)),diff)=0; od;
Eq0:=remove(has,Eq3,phi(x,t))=0;
tru0:=u[0](x,t)=[solve(factor(E3),u[0](x,t))][2];
EE1:=factor(algsubs(tru0,E1)); EE2:=factor(algsubs(tru0,E2));
tr11:=subs(tru0,tr1(1)); tr2:=u[1](x,t)=0;
trHC:=subs(tr2,tr11); EE21:=subs(tr2,EE2);
  subs(a=1,b=-nu,trHC); subs(a=1,b=-nu,EE21);
```

Mathematica:

```
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&];
trD[u_,var_]:=Table[D[u,{var,i}],{i,1,5}]//Flatten;
tr1[n_]:=u[x,t]->Sum[u[x,t][j]/phi[x,t]^(n-j),{j,0,n}];
{tr1[1], eq1=D[u[x,t],t]+a*u[x,t]*D[u[x,t],x]+b*D[u[x,t],
 {x,2}]==0, eq2=eq1/.tr1[1]/.trD[tr1[1],t]/.trD[tr1[1],x]}
{eq3=Collect[eq2,phi[x,t]], Table[e[i]=trS3[trS1[
 eq3[[1]],phi[x,t]^(-i)],phi[x,t]]==0,{i,1,3}],
 eq0=trS3[eq3[[1]],phi[x,t]]==0}
tru0=Solve[Factor[e[3]],u[x,t][0]]//Flatten
{tru01=tru0[[2]], ee1=e[1]/.tru01/.trD[tru01,x]/.trD[tru01,t],
 ee2=e[2]/.tru01/.trD[tru01,x]/.trD[tru01,t]}
{tr11=tr1[1]/.tru01, tr2=u[x,t][1]->0, tr3={a->1,b->-nu}}
{trHC=tr11/.tr2, ee21=ee2/.tr2, trHC/.tr3, ee21/.tr3//Factor}
```
□

Problem 4.4

Maple:

```
with(PDETools): declare(u(x,t),u[0](x,t),u[1](x,t),u[2](x,t),
 phi(x,t)); U:=diff_table(u(x,t));
Eq1:=U[t]+U[x]+U[x]*U[]^2+a*U[x,x,x]+b*U[x,x,x,x,x]=0;
tr1:=N->u(x,t)=sum(u[j](x,t)/phi(x,t)^(N-j),j=0..N); tr1(2);
Eq2:=algsubs(tr1(2),Eq1); Eq3:=collect(lhs(Eq2),[phi(x,t)]);
for i from 1 to 7 do
 E||i:=select(has,select(has,Eq3,phi(x,t)^(-i)),diff)=0; od;
Eq0:=remove(has,Eq3,phi(x,t))=0;
tru01:=u[0](x,t)=[solve(E7,u[0](x,t))][2];
tru02:=u[0](x,t)=[solve(E7,u[0](x,t))][3];
for i from 1 to 2 do
 EE6||i:=algsubs(tru0||i,E6);
 tru1||i:=u[1](x,t)=solve(EE6||i,u[1](x,t));
 EE5||i:=algsubs(tru1||i,algsubs(tru0||i,E5));
 tru2||i:=u[2](x,t)=solve(EE5||i,u[2](x,t));
 tr1||i:=subs(tru0||i,tru1||i,tru2||i,tr1(2)); od;
```

Mathematica:

```
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&]; trD[u_,var_]:=Table[
 D[u,{var,i}],{i,1,5}]//Flatten;
tr1[n_]:=u[x,t]->Sum[u[x,t][j]/phi[x,t]^(n-j),{j,0,n}]; tr1[2]
{eq1=D[u[x,t],t]+D[u[x,t],x]+u[x,t]^2*D[u[x,t],x]+a*D[u[x,t],
 {x,3}]+b*D[u[x,t],{x,5}]==0, eq2=eq1/.tr1[2]/.trD[tr1[2],t]/.
 trD[tr1[2],x], eq3=Collect[eq2,phi[x,t]]}
{Table[e[i]=trS3[trS1[eq3[[1]],phi[x,t]^(-i)],phi[x,t]]==0,
 {i,1,7}], Do[Print["e[",i,"]=",e[i]],{i,1,7}];}
{eq0=trS3[eq3[[1]],phi[x,t]]==0, tru0=Solve[Factor[e[7]],
 u[x,t][0]]//Flatten, tru0N[1]=tru0[[3]], tru0N[2]=tru0[[2]]}
Do[ee6[i]=e[6]/.tru0N[i]/.trD[tru0N[i],x]/.trD[tru0N[i],t];
 tru1[i]=Solve[ee6[i],u[x,t][1]]//Flatten; ee5[i]=e[5]/.
 tru0N[i]/.trD[tru0N[i],x]/.trD[tru0N[i],t]/.tru1[i]/.
 trD[tru1[i],x]/.trD[tru1[i],t]; tru2[i]=Solve[ee5[i],
 u[x,t][2]]//Flatten; tr1N[i]=tr1[2]/.tru0N[i]/.tru1[i]/.tru2[i];
Map[Print,{ee6[i],tru1[i],ee5[i],tru2[i],tr1N[i]}],{i,1,2}];
```

Maple:

```
S1:=algsubs(tr11,Eq1); S2:=collect(simplify(S1),phi(x,t));
S3:=op(1,S2); Solphi1:=convert(pdsolve(expand(op(5,S3)),
 phi(x,t)),tanh); Solphi2:=subs(_F1(t)=1,_F4(t)=0,Solphi1);
for i from 1 to 4 do
Q||i:=simplify(subs(Solphi2,algsubs(Solphi2,op(i,S3)))); od;
SolF3:=[dsolve(Q1,_F3(t))][2]; SolF2:=dsolve(algsubs(
 SolF3,Q4),_F2(t)); Solphi3:=subs(SolF3,SolF2,Solphi2);
SolFin1:=simplify(subs(Solphi3,algsubs(Solphi3,tr11)));
params:={a=1,b=-1,_C1=0}; SolFin2:=subs(params,SolFin1);
Test1:=pdetest(SolFin2,subs(params,Eq1));
```

□

4.2 Complete Integrability. Evolution Equations

4.2.1 Conservation Laws

Problem 4.5

Maple:

```
with(PDEtools): declare(T(x,t),X(x,t),u(x,t));
alias(u=u(x,t),T=T(x,t),X=X(x,t)); U:=diff_table(u(x,t));
EqC:=(T,X)->diff(T,t)+diff(X,x)=0; ab:=-infinity..infinity;
CM:=T->int(T,x=ab)=C; PDE1:=U[t]-6*U[]*U[x]+U[x,x,x]=0;
T1:=U[]; X1:=U[x,x]-3*U[]^2; Eq11:=expand(EqC(T1,X1));
CM1:=CM(T1); Eq20:=expand(PDE1*u); T2:=U[]^2/2;
X2:=U[]*U[x,x]-U[x]^2/2-2*U[]^3; Eq21:=expand(EqC(T2,X2));
CM2:=subs(2*C=C,expand(CM(T2)*2)); Eq301:=PDE1*3*U[]^2;
Eq302:=diff(PDE1,x)*U[x]; Eq303:=expand(Eq301+Eq302);
T3:=U[]^3+U[x]^2/2; X3:=-9/2*U[]^4+3*U[]^2*U[x,x]
    -6*U[]*U[x]^2+U[x]*U[x,x,x]-U[x,x]^2/2;
Eq31:=expand(EqC(T3,X3)); CM3:=simplify(CM(T3));
Test1:=Eq11-PDE1; Test2:=Eq21-Eq20; Test3:=Eq303-Eq31;
```

Mathematica:

```
var=Sequence[x,t]; eqC[tN_,xN_]:=D[tN,t]+D[xN,x]==0;
inf=Infinity; cM[tN_]:=Integrate[tN,{x,-inf,inf}]==c;
pde1=D[u[var],t]-6*u[var]*D[u[var],x]+D[u[var],{x,3}]==0
{tN1=u[var], xN1=D[u[var],{x,2}]-3*u[var]^2}
```

```
{eq11=eqC[tN1,xN1]//Expand, cM1=cM[tN1]}
{eq20=Thread[pde1*u[var],Equal]//Expand}
{tN2=1/2*u[var]^2, xN2=u[var]*D[u[var],{x,2}]-
 D[u[var],x]^2/2-2*u[var]^3}
{eq21=eqC[tN2, xN2]//Expand, cM2=(Thread[cM[tN2]*2,
 Equal]//Expand)/.{2*c->c}//Expand, eq301=Thread[
 pde1*3*u[var]^2,Equal], eq302=Thread[D[pde1,x]*D[u[var],x],
 Equal], eq303=Thread[eq301+eq302,Equal]//Expand}
{tN3=u[var]^3+D[u[var],x]^2/2, xN3=-9/2*u[var]^4+3*u[var]^2*
 D[u[var],{x,2}]-6*u[var]*D[u[var],x]^2+D[u[var],x]*D[u[var],
 {x,3}]-D[u[var],{x,2}]^2/2}
{eq31=eqC[tN3,xN3]//Expand, cM3=cM[tN3]//Simplify}
{eq11==pde1,eq21==eq20,eq303==eq31}
```

□

Problem 4.6

Maple:

```
with(PDEtools): declare((u,v,w,T,X)(x,t));
alias(u=u(x,t),v=v(x,t),w=w(x,t),T=T(x,t),X=X(x,t));
W:=diff_table(w(x,t));  trMi:=u=v^2+diff(v,x);
KdV:=diff(u,t)-6*u*diff(u,x)+diff(u,x$3)=0; Eq1:=algsubs(trMi,
 KdV); mKdV:=diff(v,t)-6*v^2*diff(v,x)+diff(v,x$3)=Mv;
subs(mKdV,Eq1); MvL:=lhs(mKdV); Eq2:=2*v*Mv+Diff(Mv,x)=0;
Eq3:=expand(2*v*MvL+diff(MvL,x))=0; evalb(Eq1=Eq3);
trv:=v=1/(2*epsilon)+epsilon*w; Eq4:=expand(algsubs(trv,trMi));
trGar:=subs(1/epsilon^2=0,Eq4);
Eq5:=collect(expand(subs(trGar,KdV)),diff);
Eq51:=map(factor,lhs(Eq5)); tr1:=1+2*epsilon^2*w=G;
Eq52:=collect(subs(tr1,Eq51),G);
Eq53:=collect(expand((Eq52-op(1,Eq52))/epsilon),diff);
term1:=op(1,op(2,Eq53))=G1;
EqGar:=subs(G1=lhs(term1),map(int,subs(term1,Eq53),x))=0;
Eq54:=expand(subs(G=lhs(tr1),G*EqGar+epsilon*diff(EqGar,x)));
Test1:=expand(Eq54-Eq5);
Test2:=evalb(KdV=algsubs(w=u,subs(epsilon=0,EqGar)));
```

Mathematica:

```
trD[u_,var_]:=Table[D[u,{var,i}],{i,1,6}]//Flatten;
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&]; var=Sequence[x,t];
{trMi=u[var]->v[var]^2+D[v[var],x], kdV=D[u[var],t]-6*u[var]*
 D[u[var],x]+D[u[var],{x,3}]==0}
eq1=kdV/.trMi/.trD[trMi,x]/.trD[trMi,t]
mKdV=D[v[var],t]-6*v[var]^2*D[v[var],x]+D[v[var],{x,3}]->mv
{eq1/.mKdV, mvL=mKdV[[1]], eq2=2*v[var]*mv+Hold[D[mv,x]]==0}
{eq3=Expand[2*v[var]*mvL+D[mvL,x]]==0, Map[Expand,eq1==eq3]}
{trv=v[var]->1/(2*epsilon)+epsilon*w[var], eq4=trMi/.trv/.
 trD[trv,x]//Expand, trGar=eq4/.{1/epsilon^2->0}}
l1={trD[w[x,t],x],trD[w[x,t],t]}//Flatten
eq5=Collect[kdV/.trGar/.trD[trGar,x]/.trD[trGar,t]//Expand,l1]
{eq51=Map[Factor,eq5[[1]]], tr1=1+2*epsilon^2*w[var]->g}
{eq52=Collect[eq51/.tr1,g], termg=trS1[eq52,g]}
eq53=Collect[Thread[(eq52-termg)/epsilon,Equal]//Expand,l1]
termepsilon=trS3[trS1[eq53,epsilon],D[w[var],{x,2}]]
{term1=termepsilon->g1, eqGar=(Map[Integrate[#,x]&,eq53/.term1]/.
 g1->term1[[1]])==0, eq541=Thread[g*eqGar,Equal]//Expand}
eq542=Thread[epsilon*D[eqGar,x],Equal]//Expand
eq54=Thread[eq541+eq542,Equal]/.g->tr1[[1]]//Expand
test1=Thread[eq54-Expand[eq5],Equal]//Expand
test2=kdV==(eqGar/.epsilon->0/.w->u)
```

Maple:

```
EqC:=(T,X)->diff(T,t)+diff(X,x)=0; ab:=-infinity..infinity;
CM:=T->int(T,x=ab)=C; T1:=w;X1:=W[x,x]-3*W[]^2-2*epsilon^2*W[]^3;
EqC1:=expand(EqC(T1,X1)); CM1:=CM(T1); test1:=expand(EqC1-EqGar);
SolSer:=(j,K)->w=Sum(epsilon^j*V[j](x,t),j=0..K);
PEq0:=value(algsubs(SolSer(i1,0),trGar)); S0:=coeff(lhs(PEq0),
 epsilon,0)=coeff(rhs(PEq0),epsilon,0);
S01:=isolate(S0,V[0](x,t)); SS:={S01}: for k from 1 to 6 do
 PEq||k:=expand(value(algsubs(SolSer(i1,k),trGar)));
 S||k:=coeff(lhs(PEq||k),epsilon,k)=coeff(rhs(PEq||k),epsilon,k);
 S||k||1:=isolate(S||k,V[k](x,t)); SS:=SS union
  {seq(S||j||1,j=1..k)}; S||k||2:=expand(subs(SS,subs(SS,subs(SS,
  subs(SS,subs(SS,subs(SS,subs(SS,S||k||1))))))));
  print(V[k]=S||k||2); od: CM0:=int(rhs(S01),x);
for i from 1 to 6 do CM||i:=map(int,rhs(S||i||2),x); od;
```

Mathematica:

```
eqC[tN_,xN_]:=D[tN,t]+D[xN,x]==0; inf=Infinity;
cM[tN_]:=Integrate[tN,{x,-inf,inf}]==c; {tN1=w[var],
 xN1=D[w[var],{x,2}]-3*w[var]^2-2*epsilon^2*w[var]^3}
{eqC1=eqC[tN1,xN1]//Expand, cM1=cM[tN1]}
test1=Thread[eqC1-Expand[eqGar],Equal]//Simplify
solSer[j_,k_]:=w[var]->Sum[epsilon^j*v[var][j],{j,0,k}];
pEq0=trGar/.solSer[i1,0]/.trD[solSer[i1,0],x]
{s0=Coefficient[pEq0[[1]],epsilon,0]==Coefficient[pEq0[[2]],
 epsilon,0], s01=Solve[s0,v[var][0]]//First, sS={s01}}
Do[pEq[k]=trGar/.solSer[i1,k]/.trD[solSer[i1,k],x]//Expand;
 s[k]=Coefficient[pEq[k][[1]],epsilon,k]==Coefficient[
  pEq[k][[2]],epsilon,k]; s1[k]=(Solve[s[k],v[var][k]]//First);
 sS=Union[sS,Table[s1[j],{j,1,k}]]//Flatten;
 s2[k]=s1[k]//.sS//.trD[sS,x]//.sS//.trD[sS,x]//.sS//Expand;
 Print[v[var][k],"=",s2[k]],{k,1,6}];
cM0=Integrate[s01[[1,2]],x]
Do[cM[k]=Map[Integrate[#,x]&,s2[k][[1,2]]]//Simplify;
 Print["cM[",k,"]=",cM[k]],{k,1,6}];
```
□

4.2.2 Nonlinear Superposition Formulas

Problem 4.7

Maple:

```
with(PDEtools): declare(u(x,t)); U:=diff_table(u(x,t));
PDE1:=U[x]^2+U[t]^2-U[]^4=0; u1:=-1/x; u2:=1/(1-x);
uN:=u1+u2; pdetest(u(x,t)=u1,PDE1);
pdetest(u(x,t)=u2,PDE1); pdetest(u(x,t)=u1+u2,PDE1);
```

Mathematica:

```
trD[u_,var_]:=Table[D[u,{var,i}],{i,1,2}]//Flatten;
pde1=D[u[x,t],x]^2+D[u[x,t],t]^2-u[x,t]^4==0
{u1=u[x,t]->-1/x, u2=u[x,t]->1/(1-x), uN=u[x,t]->
 u1[[2]]+u2[[2]]}
{pde1/.u1/.trD[u1,x]/.trD[u1,t], pde1/.u2/.trD[u2,x]/.
 trD[u2,t], pde1/.uN/.trD[uN,x]/.trD[uN,t]//Factor}
```
□

Problem 4.8

Maple:

```
alias(u[n]=u[n](x,t),uT[n]=uT[n](x,t),u[n-1]=u[n-1](x,t),
      u[n+1]=u[n+1](x,t)); L1:=[lambda[n],lambda[n+1]];
phi1:=(u,v,lambda)->diff((u+v),x)-2*lambda*sin((u-v)/2)=0;
Eq1:=phi1(u[n],u[n-1],L1[1]); Eq2:=phi1(uT[n],u[n-1],L1[2]);
Eq3:=phi1(u[n+1],u[n],L1[2]); Eq4:=phi1(u[n+1],uT[n],L1[1]);
Eq34:=Eq3-Eq4; Eq12:=Eq1-Eq2; Eq1234:=factor((Eq34-Eq12)/2);
Eq10:=collect(Eq1234,L1); k1:=u[n+1]; k2:=u[n-1];
A:=(u[n+1]-u[n-1])/4; B:=(uT[n]-u[n])/4;
tr1:=sin(X)+sin(Y)=2*sin((X+Y)/2)*cos((X-Y)/2);
Eq11:=select(has,lhs(Eq10),L1[1])/L1[1];
Eq21:=select(has,lhs(Eq10),L1[2])/L1[2];
A1:=op(1,select(has,Eq11,k1)); A2:=op(1,select(has,Eq11,k2));
A3:=op(1,select(has,Eq21,k1)); A4:=op(1,select(has,Eq21,k2));
Eq13:=subs(X=A1,Y=A2,tr1); Eq23:=subs(X=A3,Y=A4,tr1);
Eq24:=rhs(L1[2]*Eq23+L1[1]*Eq13)/2; tr2:=(L1[2]-L1[1])*sin(A)*
 cos(B)+(L1[2]+L1[1])*cos(A)*sin(B); T1:=factor(Eq24/op(3,
 op(1,Eq24))); T2:=combine(tr2); test1:=factor(T1+T2);
Eq25:=map(`/`,tr2,cos(B)); Eq26:=map(`/`,Eq25,cos(A));
Eq27:=convert(Eq26,tan); tlam:=select(has,op(1,Eq27),
 lambda[n+1]); Eq28:=map(`/`,Eq27,tlam);
Eq29:=(-op(1,op(2,op(1,Eq28)))+arctan(op(2,Eq28)))*4=0;
```

Mathematica:

```
trD[u_,var_]:=Table[D[u,{var,i}],{i,1,6}]//Flatten;
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&]; var=Sequence[x,t];
var=Sequence[x,t]; l1={lambda[n],lambda[n+1]}
phi1[u_,v_,lambda_]:=D[(u+v),x]-2*lambda*Sin[(u-v)/2]==0;
{eq1=phi1[u[var][n],u[var][n-1],l1[[1]]], eq2=phi1[uT[var][n],
 u[var][n-1],l1[[2]]], eq3=phi1[u[var][n+1],u[var][n],l1[[2]]],
 eq4=phi1[u[var][n+1],uT[var][n],l1[[1]]]}
eq34=Thread[eq3+Thread[(-1)*eq4,Equal],Equal]
eq12=Thread[eq1+Thread[(-1)*eq2,Equal],Equal]
{eq1234=Thread[Thread[eq34+Thread[(-1)*eq12,Equal],Equal]/2,
 Equal]//Expand, eq10=Collect[eq1234,l1]}
```

```
{k1=u[var][n+1], k2=u[var][n-1], a=(u[var][n+1]-u[var][n-1])/4,
 b=(uT[var][n]-u[var][n])/4, tr1=Sin[xN]+Sin[yN]->2*Sin[
 (xN+yN)/2]*Cos[(xN-yN)/2],eq11=trS1[eq10[[1]],l1[[1]]]/l1[[1]],
 eq21=trS1[eq10[[1]],l1[[2]]]/l1[[2]]}
aF[eq_,var_]:=(trS1[eq,var]//Simplify)[[1]]//Expand;
{a1=aF[eq11,k1],a2=aF[eq11,k2],a3=aF[eq21,k1],a4=aF[eq21,k2]}
eq13=tr1/.xN->a1/.yN->a2/.Rule->Equal//ExpandAll
eq23=tr1/.xN->a3/.yN->a4/.Rule->Equal//ExpandAll
{eq24=(Thread[Thread[l1[[2]]*eq23,Equal]+Thread[l1[[1]]*eq13,
 Equal],Equal])[[2]]/2//Expand, tr2=(l1[[2]]-l1[[1]])*Sin[a]*
 Cos[b]+(l1[[2]]+l1[[1]])*Cos[a]*Sin[b]}
termCos=trS3[trS3[trS1[eq24,l1[[2]]],l1[[2]]],Sin[_]]
{t1=eq24/termCos//Factor, t2=Together[tr2//ExpandAll,
 Trig->True], test1=(t1+t2)//FullSimplify}
{eq25=Map[Divide[#,Cos[b]]&,tr2], eq26=Map[Divide[#,Cos[a]]&,
 eq25], tlam=trS3[trS1[eq26,l1[[2]]][[1]],Tan[_]]}
{eq28=Map[Divide[#,tlam]&,eq26], eq29=Simplify[(eq29=Map[
 ArcTan[#]&,eq28]*4)//PowerExpand//ExpandAll]}
```

□

Problem 4.9

Maple:

```
with(PDEtools): with(plots): declare(w1(x,t),w2(x,t));
alias(w1=w1(x,t),w2=w2(x,t),u[n]=u[n](x,t),uT[n]=uT[n](x,t),
 u[n-1]=u[n-1](x,t),u[n+1]=u[n+1](x,t));
L1:=[lambda[n],lambda[n+1]];
phi1:=(u,v,lambda)->diff((u+v),x)-2*lambda-(u-v)^2/2=0;
Eq1:=phi1(u[n],u[n-1],L1[1]); Eq2:=phi1(uT[n],u[n-1],L1[2]);
Eq3:=phi1(u[n+1],u[n],L1[2]); Eq4:=phi1(u[n+1],uT[n],L1[1]);
Eq34:=Eq3-Eq4; Eq12:=Eq1-Eq2; Eq1234:=simplify(Eq34-Eq12);
Eq5:=collect(Eq1234,[u[n],uT[n]]);
```

Mathematica:

```
var=Sequence[x,t]; l1={lambda[n],lambda[n+1]}
phi1[u_,v_,lambda_]:=D[(u+v),x]-2*lambda-(u-v)^2/2==0;
eq1=phi1[u[var][n],u[var][n-1],l1[[1]]]
eq2=phi1[uT[var][n],u[var][n-1],l1[[2]]]
eq3=phi1[u[var][n+1],u[var][n],l1[[2]]]
eq4=phi1[u[var][n+1],uT[var][n],l1[[1]]]
eq34=Thread[eq3+Thread[(-1)*eq4,Equal],Equal]
```

```
eq12=Thread[eq1+Thread[(-1)*eq2,Equal],Equal]
eq1234=Thread[eq34+Thread[(-1)*eq12,Equal],Equal]//Expand
eq5=Collect[eq1234,{u[var][n],uT[var][n]}]
```

Maple:

```
Eq6:=collect(subs(n=1,u[0](x,t)=0,Eq5),[u[2](x,t)]);
Eq7:=subs(u[2](x,t)=u[12],u[1](x,t)=u[1],uT[1](x,t)=u[2],Eq6);
Eq8:=isolate(Eq7,u[12]); params:={lambda[1]=-1,lambda[2]=-4};
BTx:=diff(w1+w2,x)-2*lambda-1/2*(w1-w2)^2;
BTt:=diff(w1-w2,t)-3*(diff(w1,x)^2-diff(w2,x)^2)+diff(w1-w2,x$3);
Eq11:=subs(w2=0,BTx); Eq12:=subs(w2=0,BTt);
sol1:=pdsolve({Eq11,Eq12},w1,HINT='TWS(tanh)');
Eq21:=subs(w1=0,BTx); Eq22:=subs(w1=0,BTt);
sol2:=pdsolve({Eq21,Eq22},w2,HINT='TWS(coth)');
sols:=simplify({u[1]=subs(lambda=-1,_C1=0,
 rhs(op(sol1[4]))),u[2]=subs(lambda=-4,_C1=0,rhs(op(sol2[4])))});
Eq9:=factor(subs(params,sols,Eq8));
U12:=unapply(rhs(diff(Eq9,x)),x,t);
EqKdV:=diff(u(x,t),t)-6*u(x,t)*diff(u(x,t),x)+diff(u(x,t),x$3)=0;
test1:=simplify(algsubs(-u(x,t)=-U12(x,t),EqKdV));
animate(-U12(x,t),x=-50..50,t=-10..10,numpoints=400,frames=200);
```

□

4.2.3 Hirota Method

Problem 4.10

Maple:

```
with(PDEtools): with(plots):  declare((u,w,F,F1,F2,f,g)(x,t));
alias(u=u(x,t),w=w(x,t),F=F(x,t),F1=F1(x,t),F2=F2(x,t),f=f(x,t),
 g=g(x,t)); tr1:=u=diff(w,x$2);
PDE1:=u->diff(u,t)+6*u*diff(u,x)+diff(u,x$3)=0;
Eq1:=expand(PDE1(rhs(tr1))); Eq2:=map(int,Eq1,x);
tr2:=w=2*log(F); Eq4:=simplify(algsubs(tr2,Eq2)/2*F^2);
Dxn:=(f,g,n)->sum((-1)^k*binomial(n,k)*diff(f,x$(n-k))
     *diff(g,x$k),k=0..n); Dxn(f,g,1); Dxn(f,g,2); Dxn(f,g,4);
DxDt:=(f,g)->f*diff(g,x,t)-diff(f,x)*diff(g,t)
      -diff(f,t)*diff(g,x)+diff(f,x,t)*g;
Eq5:=(Dxn(F,F,4)+DxDt(F,F)=0)/2; test1:=Eq4-Eq5;
```

Mathematica:

```
trD[u_,var_]:=Table[D[u,{var,i}],{i,1,6}]//Flatten;
var=Sequence[x,t]; tr1=u[var]->D[w[var],{x,2}]
pde1[u_]:=D[u,t]+6*u*D[u,x]+D[u,{x,3}]==0
{eq1=pde1[tr1[[2]]]//Expand, eq2=Map[Integrate[#,x]&,eq1]}
tr2=w[var]->2*Log[f[var]]
eq41=(eq2/.tr2/.trD[tr2,x]/.D[tr2,x,t])//Expand
eq4=Thread[eq41/2*f[var]^2,Equal]//Expand
dxn[f_,g_,n_]:=Sum[(-1)^k*Binomial[n,k]*D[f[var],{x,n-k}]*
 D[g[var],{x,k}],{k,0,n}]; {dxn[f,g,1], dxn[f,g,2], dxn[f,g,4]}
dxdt[f_,g_]:=f[var]*D[g[var],x,t]-D[f[var],x]*D[g[var],t]-
 D[f[var],t]*D[g[var],x]+D[f[var],x,t]*g[var];
{eq50=dxn[f,f,4]+dxdt[f,f]==0, eq5=Thread[eq50/2,Equal]//Expand}
test1=Thread[eq4-eq5,Equal]//Expand
```

Maple:

```
Eq41:=lhs(collect(Eq4,F)); Eq6:=subs(tr2,tr1);
LD:=op(1,Eq41)/F; ND:=Eq41-op(1,Eq41);
M:=1; Eq7:=simplify(exp(lhs(Eq6))=exp(rhs(Eq6)));
Eq71:=op(1,lhs(Eq7))=op(1,rhs(Eq7)); Eq72:=subs(u=0,Eq71);
Eq73:=subs(F=V[0],pdsolve(Eq72,F,explicit));
arbFun:={_F1(t)=0,_F2(t)=1}; SolVac:=eval(Eq73,arbFun);
SolSer:=N->F=V[0]+Sum(epsilon^n*V[n],n=1..N);
tr3:=i->theta[i]=k[i]*x-c[i]*t;
tr4:=(j,K)->subs(tr3(j),V[1]='Sum(exp(theta[j]),j=1..K)');
tr5:=subs(tr3(i),u=exp(theta[i])); Eq8:=PDE1(rhs(tr5));
Eq81:=op(1,lhs(Eq8))+op(3,lhs(Eq8))=0;
tr6:=unapply(isolate(Eq81,c[i]),i); tr31:=subs(tr6(i),tr3(i));
Corr11:=value(subs(tr6(i),tr4(i,M)));
SS1:=subs(epsilon=1,Corr11,value(subs(SolVac,SolSer(M))));
SS1u:=simplify(subs(SS1,Eq6)); SS1uG:=subs(k[1]=1,rhs(SS1u));
animate(SS1uG,x=-50..50,t=-10..10,numpoints=200,frames=50);
```

Mathematica:

```
SetOptions[Plot,PlotPoints->300,ImageSize->500,PlotStyle->
 {Hue[0.9],Thickness[0.01]}]; eq41=Collect[eq4,f[var]][[1]]
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&]; var=Sequence[x,t];
```

```
{eq6=tr1/.tr2/.trD[tr2,x]/.Rule->Equal,
 termf=trS1[eq41,f[var]], lD=termf/f[var]//Simplify,
 nD=eq41-termf//Expand, m=1, eq7=Map[Exp[#]&,eq6],
 eq71=eq7[[1,2]]==eq7[[2,2]], eq72=eq71/.u[var]->0}
eq73=DSolve[eq72,f[var],{x,t}]/.f[var]->v[var][0]//First
{arbFun={C[1][t]->0,C[2][t]->1}, solVac=eq73/.arbFun}
solSer[nN_]:=f[var]->v[var][0]+Sum[epsilon^n*v[var][n],{n,1,nN}];
tr3[i_]:=theta[i]->k[i]*x-c[i]*t; tr4[j_,k_]:=v[var][1]->
 Sum[Exp[theta[j]],{j,1,k}]/.tr3[j]; {tr5=u[var]->Exp[theta[i]]/.
 tr3[i], eq8=pde1[tr5[[2]]], eq81=trS3[eq8[[1]],6]==0}
tr6[i1_]:=((Solve[eq81,c[i]][[1,1]]))/.i->i1; tr31[i1_]:=tr3[i]/.
 tr6[i]/.i->i1; corr11=tr4[i,m]/.tr5/.tr31[1]
sS1=solSer[m]/.solVac/.corr11/.epsilon->1
sS1u=eq6/.sS1/.trD[sS1,x]/.trD[sS1,t]//Simplify
sS1uG[xN_,tN_]:=(sS1u[[2]]/.k[1]->1)/.{x->xN,t->tN}; sS1uG[x,t]
Animate[Plot[Evaluate[sS1uG[x,t]],{x,-50,50},
 PlotRange->{{-50,50},{0,0.6}}],{t,-10,10},AnimationRate->0.9]
```

Maple:

```
M:=2;  tr32:=seq(tr3(i),i=1..M); tr2V1:=value(tr4(i,M));
tr2V2:=subs(tr32,V[2]=a[1,2]*exp(theta[1]+theta[2]));
Eq9L:=algsubs(F=rhs(tr2V2),LD); Eq9R:=-algsubs(F=rhs(tr2V1),ND);
Eq91:=collect(simplify(Eq9L-Eq9R),exp);
a12:=factor(isolate(subs(seq(tr6(i),i=1..M),Eq91),a[1,2]));
Corr21:=value(subs(tr6(i),tr4(i,M))); trk:={k[1]=1,k[2]=2};
Corr22:=subs(a12,seq(tr6(i),i=1..M),tr2V2);
SS2:=subs(epsilon=1,Corr21,Corr22,value(subs(SolVac,SolSer(M))));
SS2u:=simplify(subs(SS2,Eq6)); SS2uG:=subs(trk,rhs(SS2u));
animate(SS2uG,x=-50..50,t=-10..10,numpoints=200,frames=50);
```

Mathematica:

```
{m=2, tr32=Table[tr3[i],{i,1,m}], tr2v1=tr4[i,m]/.tr5/.tr31[1]/.
 tr31[2], tr2v2=v[var][2]->a[1,2]*Exp[theta[1]+theta[2]]/.tr32,
 trf2=f[var]->tr2v2[[2]],trf1=f[var]->tr2v1[[2]]}
eq9L=lD/.trf2/.trD[trf2,x]/.D[trf2,x,t]
eq9R=-nD/.trf1/.trD[trf1,x]/.trD[trf1,t]/.D[trf1,x,t]
{eq91=eq9L-eq9R//FullSimplify, tr6seq=Table[tr6[i],{i,1,m}]}
a12=Solve[(eq91/.tr6seq)==0,a[1,2]]//First
corr21=tr4[i,m]/.tr6[i]/.tr31[1]/.tr31[2]
{trk={k[1]->1,k[2]->2}, corr22=tr2v2/.a12/.tr6seq}
```

```
sS2=solSer[m]/.solVac/.corr22/.corr21/.epsilon->1
sS2u=eq6/.sS2/.trD[sS2,x]/.trD[sS2,t]//Simplify
sS2uG[xN_,tN_]:=(sS2u[[2]]/.trk)/.{x->xN,t->tN}; sS2uG[x,t]
Animate[Plot[Evaluate[sS2uG[x,t]],{x,-50,50},
 PlotRange->{{-50,50},{0,2}}],{t,-10,10},AnimationRate->0.9]
```

Maple:

```
M:=3; tr33:=seq(tr3(i),i=1..M); tr3V1:=value(tr4(j,M));
LP0:=combinat[permute](3,2); LP1:=NULL: NLP0:=nops(LP0);
for i from 1 to NLP0 do
 if op(1,LP0[i])<op(2,LP0[i]) then LP1:=LP1,LP0[i]; fi; od;
LP2:=[LP1]; NLP2:=nops(LP2);
tr3V2:=subs(tr33,V[2]=add(a[op(LP2[i])]*exp(theta[op(1,LP2[i])]
 +theta[op(2,LP2[i])]),i=1..NLP2));
tr3V3:=subs(tr33,V[3]=b[1,2,3]*exp(theta[1]+theta[2]+theta[3]));
Eq10L:=algsubs(F=rhs(tr3V3),LD);
Eq51:=collect(Dxn(F1,F2,4)+DxDt(F1,F2),[F1,F2]);
Eq511:=Eq51-op(1,Eq51)-op(2,Eq51);
ND1:=algsubs(F2=rhs(tr3V2),algsubs(F1=rhs(tr3V1),Eq511)):
Eq10R:=-rhs(tr3V1)*algsubs(F=rhs(tr3V2),LD)-rhs(tr3V2)
 *algsubs(F=rhs(tr3V1),LD)-ND1:
Eq101:=simplify(Eq10L-Eq10R):
Eq102:=subs(seq(tr6(i),i=1..M),collect(Eq101,exp)); a12;
a13:=subs(a[1,2]=a[1,3],k[2]=k[3],a12);
a23:=subs(a[1,3]=a[2,3],k[1]=k[2],a13);
Eq103:=subs(a12,a13,a23,Eq102);
for i from 1 to nops(Eq103) do A||i:=simplify(op(i,Eq103)); od;
b123:=factor(isolate(subs(seq(tr6(i),i=1..3),A1),b[1,2,3]));
Corr31:=value(subs(tr6(j),tr4(j,M)));
Corr32:=subs(a12,a13,a23,subs(seq(tr6(j),j=1..M),tr3V2));
Corr33:=subs(b123,seq(tr6(i),i=1..M),tr3V3);
SS3:=subs(epsilon=1,Corr31,Corr32,Corr33,
 value(subs(SolVac,SolSer(M)))); SS3u:=subs(SS3,Eq6);
SS3uG:=subs(k[1]=1,k[2]=2,k[3]=3,rhs(SS3u));
animate(SS3uG,x=-50..50,t=-10..10,numpoints=200,frames=50);
```

Mathematica:

```
{m=3, tr33=Table[tr3[i],{i,1,m}], tr3v1=tr4[j,m]/.tr31[1]/.
 tr31[2]/.tr31[3], lP0=Permutations[Range[3],{2}],
 lP2=Select[lP0,#[[1]]<#[[2]]&], nLP2=Length[lP2]}
```

```
tr3v2=v[var][2]->Sum[a[lP2[[i,1]],lP2[[i,2]]]*Exp[
 theta[lP2[[i,1]]]+theta[lP2[[i,2]]]],{i,1,nLP2}]/.tr33
tr3v3=v[var][3]->b[1,2,3]*Exp[theta[1]+theta[2]+theta[3]]/.tr33
{trff33=f[var]->tr3v3[[2]], trff23=f[var]->tr3v2[[2]],
 trff13=f[var]->tr3v1[[2]], trf23=f2[var]->tr3v2[[2]],
 trf13=f1[var]->tr3v1[[2]]}
eq10L=lD/.trff33/.trD[trff33,x]/.D[trff33,x,t]
eq51=Collect[dxn[f1,f2,4]+dxdt[f1,f2],{f1[var],f2[var]}]
{termf1=trS1[eq51,f1[var]], termf2=trS1[eq51,f2[var]]}
eq511=eq51-termf1-termf2
nD1=eq511/.trf23/.trD[trf23,x]/.trD[trf23,t]/.D[trf23,x,t]/.
 trf13/.trD[trf13,x]/.trD[trf13,t]/.D[trf13,x,t]
eq10R=-tr3v1[[2]]*(lD/.trff23/.trD[trff23,x]/.D[trff23,x,t])-
 tr3v2[[2]]*(lD/.trff13/.trD[trff13,x]/.D[trff13,x,t])-nD1
{eq101=eq10L-eq10R//Simplify, tr63seq=Table[tr6[i],{i,1,m}]}
eq102=Collect[eq101/.tr63seq,Exp[_]]
eq1021=Map[Factor,Collect[eq102,Exp[_]]]//Simplify
{a12, a13=a12/.a[1,2]->a[1,3]/.k[2]->k[3], a23=a13/.
 a[1,3]->a[2,3]/.k[1]->k[2], eq103=eq1021/.a12/.a13/.a23}
Do[aN[i]=eq103[[i]]//Simplify; Print[aN[i]],{i,1,Length[eq103]}];
b123=Solve[(aN[3]/.tr63seq)==0,b[1,2,3]]//First//FullSimplify
corr31=tr4[j,m]/.tr6[i]/.tr31[1]/.tr31[2]/.tr31[3]
{corr32=tr3v2/.a12/.a13/.a23/.tr63seq, corr33=tr3v3/.b123/.
 tr63seq, sS3=solSer[m]/.solVac/.corr33/.corr32/.corr31/.
 epsilon->1, sS3u=eq6/.sS3/.trD[sS3,x]/.trD[sS3,t],
 trk3={k[1]->1,k[2]->2,k[3]->3}}
sS3uG[xN_,tN_]:=(sS3u[[2]]/.trk3)/.{x->xN,t->tN}; sS3uG[x,t]
Animate[Plot[Evaluate[sS3uG[x,t]],{x,-50,50},
 PlotRange->{{-50,50},{0,5}}],{t,-10,10},AnimationRate->0.9]
```
□

4.2.4 Lax Pairs

Problem 4.11

Maple:

```
with(PDEtools): with(Ore_algebra); declare(u(x,t));
A:=diff_algebra([Dx,x],[Dt,t],[comm,c],func={u});
L:=-Dx^2+u(x,t); M1:=c*Dx; CommLM1:=skew_product(L,M1,A)
 -skew_product(M1,L,A); LaxEq1:=diff(L,t)+CommLM1=0;
M2:=-4*Dx^3+3*(u(x,t)*Dx+skew_product(Dx,u(x,t),A));
CommLM2:=skew_product(L,M2,A)-skew_product(M2,L,A);
LaxEq2:=diff(L,t)+CommLM2=0;
```
□

Problem 4.12

Maple:

```
with(PDEtools): with(LinearAlgebra); declare(u(x,t),r(x,t),
 q(x,t)); A0:=<<1,0>|<0,-1>>; A1:=<<0,r(x,t)>|<q(x,t),0>>;
B0:=<<0,0>|<0,0>>; A:=I*lambda*A0+I*A1; B1:=2*I*lambda^2*A0
 +2*I*lambda*A1+<<0,-diff(r(x,t),x)>|<diff(q(x,t),x),0>>
             -I*<<q(x,t)*r(x,t),0>|<0,-q(x,t)*r(x,t)>>;
CompCond:=(L,M)->simplify(map(diff,L,t)-map(diff,M,x)+(L.M-M.L));
Eqs1:=[CompCond(A,B1)[1,2]=B0[1,2],CompCond(A,B1)[2,1]=B0[2,1]];
NSEq1:=simplify(subs(q(x,t)=conjugate(r(x,t)),Eqs1))[2];
NSEq2:=simplify(subs(q(x,t)=-conjugate(r(x,t)),Eqs1))[2];
B2:=1/(4*I*lambda)*<<cos(u(x,t)),I*sin(u(x,t))>|<-I*sin(u(x,t)),
 -cos(u(x,t))>>; Eqs2:=[seq(seq(CompCond(A,B2)[i,j]=B0[i,j],
 j=1..2),i=1..2)]; SineG:=expand(subs(q(x,t)=1/2*diff(u(x,t),x),
 r(x,t)=1/2*diff(u(x,t),x),Eqs2[2]*2/I));
B3:=1/(4*I*lambda)*<<cosh(u(x,t)),-I*sinh(u(x,t))>|
 <-I*sinh(u(x,t)),-cosh(u(x,t))>>;
Eqs3:=[seq(seq(CompCond(A,B3)[i,j]=B0[i,j],j=1..2),i=1..2)];
SinhG:=expand(subs(q(x,t)=1/2*diff(u(x,t),x),r(x,t)=
 -1/2*diff(u(x,t),x),Eqs3[2]*2/I));
```

Mathematica:

```
trD[u_,var_]:=Table[D[u,{var,i}],{i,1,2}]//Flatten;
var=Sequence[x,t]; {a0={{1,0},{0,-1}},
 a1={{0,q[var]},{r[var],0}}, b0={{0,0},{0,0}}}
{a=I*lambda*a0+I*a1, b1=2*I*lambda^2*a0+2*I*lambda*a1+
 {{0,D[q[var],x]},{-D[r[var],x],0}}-I*{{q[var]*r[var],0},
 {0,-q[var]*r[var]}}}
Map[MatrixForm,{a0,a1,b0,a,b1}]
compCond[l_,m_]:=(D[l,t]-D[m,x]+(l.m-m.l))//Simplify;
eqs1={compCond[a,b1][[1,2]]==b0[[1,2]],
 compCond[a,b1][[2,1]]==b0[[2,1]]}//ComplexExpand//Expand
trq[s_]:=q[var]->s*Conjugate[r[var]];
{trr1=r[x,t]^2->r[x,t]*r1[x,t], trr2=r1[x,t]->r[x,t]}
nSEq1=(eqs1[[2]]/.trq[1]/.trr1//FullSimplify)/.trr2
nSEq2=(eqs1[[2]]/.trq[-1]/.trr1//FullSimplify)/.trr2
b2=1/(4*I*lambda)*{{Cos[u[var]],-I*Sin[u[var]]},{I*Sin[u[var]],
 -Cos[u[var]]}}
eqs2=Table[compCond[a,b2][[i,j]]==b0[[i,j]],{j,1,2},{i,1,2}]//
 Flatten//Expand
```

```
trrq={q[var]->1/2*D[u[var],x],r[var]->1/2*D[u[var],x]}
sineG=Thread[eqs2[[2]]*2/I,Equal]/.trrq/.D[trrq[[2]],t]//Expand
{b3=1/(4*I*lambda)*{{Cosh[u[var]],-I*Sinh[u[var]]},
 {-I*Sinh[u[var]],-Cosh[u[var]]}}, eqs3=Table[compCond[
 a,b3][[i,j]]==b0[[i,j]],{j,1,2},{i,1,2}]//Flatten//Expand}
sinhG=Thread[eqs3[[3]]*2/I,Equal]/.trrq/.trD[trrq,t]/.
 trD[trrq,x]//Expand
```

□

Problem 4.13

Maple:

```
with(PDEtools): with(LinearAlgebra); declare((u,r,q)(x,t));
A0:=<<1,0>|<0,-1>>; A11:=<<0,u(x,t)>|<1,0>>; B0:=<<0,0>|<0,0>>;
A1:=I*lambda*A0+A11; B1:=4*I*lambda^3*A0+4*lambda^2*A11
 +2*I*lambda*<<u(x,t),diff(u(x,t),x)>|<0,-u(x,t)>>
 +<<-diff(u(x,t),x),2*u(x,t)^2-diff(u(x,t),x$2)>|<2*u(x,t),
 diff(u(x,t),x)>>; CompCond:=(L,M)->simplify(map(diff,L,t)
 -map(diff,M,x)+(L.M-M.L));
Eqs1:=[seq(seq(CompCond(A1,B1)[m,n]=B0[m,n],n=1..2),m=1..2)];
KdV:=Eqs1[3]; A2:=<<-I*lambda,u(x,t)>|<u(x,t),I*lambda>>;
B2:=<<-4*I*lambda^3-2*I*lambda*u(x,t)^2,4*lambda^2*u(x,t)
 -2*I*lambda*diff(u(x,t),x)-diff(u(x,t),x$2)+2*u(x,t)^3>|
 <4*lambda^2*u(x,t)+2*I*lambda*diff(u(x,t),x)-diff(u(x,t),x$2)
 +2*u(x,t)^3,4*I*lambda^3+2*I*lambda*u(x,t)^2>>;
Eqs2:=[seq(seq(CompCond(A2,B2)[m,n]=B0[m,n],n=1..2),m=1..2)];
mKdV:=Eqs2[2];
```

Mathematica:

```
trD[u_,var_]:=Table[D[u,{var,i}],{i,1,2}]//Flatten;
{var=Sequence[x,t], a0={{1,0},{0,-1}}, a11={{0,1},{u[var],0}},
 b0={{0,0},{0,0}}, a1=I*lambda*a0+a11, b1=4*I*lambda^3*a0+
 4*lambda^2*a11+2*I*lambda*{{u[var],0},{D[u[var],x],-u[var]}}+
 {{-D[u[var],x],2*u[var]},
 {2*u[var]^2-D[u[var],{x,2}],D[u[var],x]}}}
 Map[MatrixForm,{a0,a11,b0,a1,b1}]
compCond[l_,m_]:=(D[l,t]-D[m,x]+(l.m-m.l))//Simplify;
{eqs1=Table[compCond[a1,b1][[m,n]]==b0[[m,n]],{m,1,2},
 {n,1,2}]//Flatten, kdV=eqs1[[3]]}
```

```
{a2={{-I*lambda,u[var]},{u[var],I*lambda}}, b2={{-4*I*lambda^3-
 2*I*lambda*u[var]^2,4*lambda^2*u[var]+2*I*lambda*D[u[var],x]-
 D[u[var],{x,2}]+2*u[var]^3},{4*lambda^2*u[var]-
 2*I*lambda*D[u[var],x]-D[u[var],{x,2}]+2*u[var]^3,
 4*I*lambda^3+2*I*lambda*u[var]^2}}}
Map[MatrixForm,{a2,b2}]
{eqs2=Table[compCond[a2,b2][[m,n]]==b0[[m,n]],{m,1,2},
 {n,1,2}]//Flatten, mKdV=eqs2[[2]]}
```
□

4.2.5 Variational Principle

Problem 4.14

Maple:

```
with(PDEtools): declare(u(x,t)); U:=diff_table(u(x,t));
PDE1:=U[t,t]-U[x,x]+diff(F(u),u); tr1:={U[x]=p,U[t]=q};
tr2:={p=U[x],q=U[t]}; Lag1:=1/2*(U[t]^2-U[x]^2)-F(U[]);
L:=subs(tr1,Lag1); EulerLagEq:=Diff(L,U[])-diff(subs(tr2,
   diff(L,p)),x)-diff(subs(tr2,diff(L,q)),t)=0;
value(subs(u(x,t)=u,EulerLagEq));
```

Mathematica:

```
pde1=D[u[x,t],{t,2}]-D[u[x,t],{x,2}]+D[f[u[x,t]],u]
tr1={D[u[x,t],x]->p,D[u[x,t],t]->q}
tr2={p->D[u[x,t],x],q->D[u[x,t],t]}
lag1=1/2*(D[u[x,t],t]^2-D[u[x,t],x]^2)-f[u[x,t]]
{l=lag1/.tr1, eulerLagEq=D[l,u[x,t]]-D[(D[l,p]/.tr2),x]-
 D[(D[l,q]/.tr2),t]==0}
```
□

4.3 Nonlinear Systems. Integrability Conditions

Problem 4.15

Maple:

```
with(PDEtools): declare(z(x,y),(F,G)(x,y,z)); Uz,UF,UG:=
 diff_table(z(x,y)),diff_table(F(x,y,z)),diff_table(G(x,y,z));
F:=(x,y,z)->y*z; G:=(x,y,z)->z^2+a*x*z;
Sys1:=[Uz[x]=y*z(x,y),Uz[y]=z(x,y)^2+a*x*z(x,y)];
ConsCond:=factor(UF[y]+G(x,y,z)*UF[z]-UG[x]-F(x,y,z)*UG[z]=0);
Solz:=[solve(ConsCond,z)];
for i from 1 to 2 do simplify(algsubs(z(x,y)=Solz[i],Sys1)); od;
```

Mathematica:

```
trD[u_,var_]:=Table[D[u,{var,i}],{i,1,2}]//Flatten;
var=Sequence[x,y,z]; fF[x_,y_,z_]:=y*z; fG[x_,y_,z_]:=z^2+a*x*z;
sys1={D[z[x,y],x]==y*z[x,y],D[z[x,y],y]==z[x,y]^2+a*x*z[x,y]}
{consCond=D[fF[var],y]+fG[var]*D[fF[var],z]-D[fG[var],x]-fF[var]*
 D[fG[var],z]==0//Factor, solz=Solve[consCond,z]//Flatten}
n=Length[solz]; Do[tr[i]=z[x,y]->solz[[i,2]]; test[i]=(sys1/.
 tr[i]/.trD[tr[i],x]/.trD[tr[i],y])//Factor; Print[tr[i]];
 Print[test[i]],{i,1,2}];
```
□

Problem 4.16

Maple:

```
with(PDEtools): declare(z(x,y),(F,G)(x,y,z)); Uz,UF,UG:=
 diff_table(z(x,y)),diff_table(F(x,y,z)),diff_table(G(x,y,z));
F:=(x,y,z)->a*exp(y-z); G:=(x,y,z)->b*exp(y-z)+1;
Sys1:=[Uz[x]=a*exp(y-z(x,y)), Uz[y]=b*exp(y-z(x,y))+1];
ConsCond:=factor(UF[y]+G(x,y,z)*UF[z]-UG[x]-F(x,y,z)*UG[z]=0);
Sol1:=simplify(pdsolve(Sys1[1]));
Eq1:=expand(algsubs(Sol1,Sys1[2])); SolF1:=dsolve(Eq1,_F1(y));
SolFin:=simplify(subs(SolF1,Sol1)); pdetest(SolFin,Sys1);
```

Mathematica:

```
Off[Solve::ifun]; trD[u_,var_]:=Table[D[u,{var,i}],{i,1,2}]//
 Flatten; var=Sequence[x,y,z]; fF[x_,y_,z_]:=a*Exp[y-z];
fG[x_,y_,z_]:=b*Exp[y-z]+1; {sys1={D[z[x,y],x]==a*Exp[y-z[x,y]],
 D[z[x,y],y]==b*Exp[y-z[x,y]]+1}, consCond=D[fF[var],y]+fG[var]*
 D[fF[var],z]-D[fG[var],x]-fF[var]*D[fG[var],z]==0//Factor}
{sol1=DSolve[sys1[[1]],z[x,y],{x,y}]//First, eq1=sys1[[2]]/.
 sol1/.trD[sol1,y]//Expand, solC1=DSolve[eq1,C[1][y],y]//First}
solFin=sol1/.solC1//FullSimplify
sys1/.solFin/.trD[solFin,x]/.trD[solFin,y]//FullSimplify
```
□

Problem 4.17

Maple:

```
with(PDEtools): declare(z(x,y),phi(x),(P,Q,R,F,G)(x,y,z));
UP,UQ,UR,Uz,UF:=diff_table(P(x,y,z)),diff_table(Q(x,y,z)),
 diff_table(R(x,y,z)),diff_table(z(x,y)),diff_table(F(x,y,z));
UG:=diff_table(G(x,y,z)); P:=(x,y,z)->y*(x*z+a);
Q:=(x,y,z)->x*(y+b); R:=(x,y,z)->x^2*y; Eq1:=P(x,y,z)*dx
 +Q(x,y,z)*dy+R(x,y,z)*dz=0; IntCond:=factor(R(x,y,z)*(UP[y]
 -UQ[x])+P(x,y,z)*(UQ[z]-UR[y])+Q(x,y,z)*(UR[x]-UP[z])=0);
F:=(x,y,z)->-P(x,y,z)/R(x,y,z); G:=(x,y,z)->-Q(x,y,z)/R(x,y,z);
Sys1:=[Uz[x]=F(x,y,z),Uz[y]=G(x,y,z)];
ConsCond:=factor(UF[y]+G(x,y,z)*UF[z]-UG[x]-F(x,y,z)*UG[z]=0);
sol1:=simplify(pdsolve(Sys1[1])); eq1:=expand(algsubs(sol1,
 Sys1[2])); solF1:=dsolve(eq1,_F1(y)); solFin:=simplify(subs(
 solF1,sol1)); ODE1:=expand(subs(dx=0,Eq1)/x/dy);
ODE11:=eval(ODE1,{dz=dy*diff(z(x,y),y)});
Sol1:=dsolve(ODE11,z(x,y)); Sol11:=subs(z(x,y)=z,Sol1);
Sol12:=combine(algsubs(Sol11,Eq1)); tr1:={dz=diff(rhs(Sol11),x)
 *dx+diff(rhs(Sol11),y)*dy}; EqF1:=simplify(subs(tr1,Sol12));
SolF1:=dsolve(select(has,lhs(EqF1),_F1),_F1(x));
SolFin:=factor(eval(Sol1,SolF1)); trC1:=isolate(SolFin,_C1);
simplify(subs(trC1,simplify(algsubs(solFin,Sys1))));
```

Mathematica:

```
var=Sequence[x,y,z]; trS1[eq_,var_]:=Select[eq,MemberQ[#,var,
 Infinity]&]; trD[u_,var_]:=Table[D[u,{var,i}],{i,1,2}]//Flatten;
fP[x_,y_,z_]:=y*(x*z[x,y]+a); fQ[x_,y_,z_]:=x*(y+b);
fR[x_,y_,z_]:=x^2*y; eqP=fP[var]*dx+fQ[var]*dy+fR[var]*dz==0
fF[x_,y_,z_]:=-fP[var]/fR[var]; fG[x_,y_,z_]:=-fQ[var]/fR[var];
sys1={D[z[x,y],x]==fF[var], D[z[x,y],y]==fG[var]}
intCond=fR[var]*(D[fP[var],y]-D[fQ[var],x])+fP[var]*
 (D[fQ[var],z]-D[fR[var],y])+fQ[var]*(D[fR[var],x]-
 D[fP[var],z])==0//Factor
consCond=D[fF[var],y]+fG[var]*D[fF[var],z]-D[fG[var],x]-
 fF[var]*D[fG[var],z]==0//Factor
{sol1=DSolve[sys1[[1]],z[x,y],{x,y}]//First, eq1=sys1[[2]]/.
 sol1/.trD[sol1,y], solC1=DSolve[eq1,C[1][y],y]//First}
solFin=sol1/.solC1/.C[2]->C[1]//Factor
{ode1=Thread[(eqP/.dx->0)/x/dy,Equal]//Expand, ode11=ode1/.
 {dz->dy*D[z[x,y],y]}, sol10=DSolve[ode11,z[x,y],{x,y}]//First}
```

```
{trz=z[x,y]->z, sol11=sol10/.trz, sol12=eqP/.trz/.sol11//
 Simplify, tr1={dz->D[sol11[[1,2]],x]*dx+D[sol11[[1,2]],y]*dy}}
{eqC1=sol12/.tr1//Simplify, solC11=DSolve[trS1[eqC1[[1]],
 C[1][x]]==0,C[1][x],x]//First, solFin=sol11/.solC11/.C[2]->
 C[1]//Simplify, solFin1=solFin/.z->z[x, y]}
trC1=Solve[solFin/.Rule->Equal,C[1]]
sys1/.solFin1/.trD[solFin1,x]/.trD[solFin1,y]/.trC1//Simplify
```

□

Chapter 5
Approximate Analytical Approach

5.1 Adomian Decomposition Method

5.1.1 Adomian Polynomials

Problem 5.1

Maple:

```
with(PDETools): declare(U(x)): N:=9; ADM1:=i->convert(subs(
 lambda=0,value(1/i!*Diff(F(Sum(lambda^k*u[k],k=0..i)),
 lambda$i))),diff);
A0:=F(u[0]);  for i from 1 to N do A||i:=ADM1(i); od;
for i from 0 to N do expand(subs(F(u[0])=u[0]^2,A||i)); od;
for i from 0 to N do expand(subs(F(u[0])=u[0]^3,A||i)); od;
for i from 0 to N do subs({seq(u[k]=u[k][x],k=0..N)},
                     expand(subs(F(u[0])=u[0]^2,A||i))); od;
for i from 0 to N do subs({seq(u[k]=u[k][x],k=0..N)},
                     expand(subs(F(u[0])=u[0]^3,A||i))); od;
for i from 0 to N do
 convert(1/2*diff(subs({seq(u[k]=U[k](x),k=0..N)},
 expand(subs(F(u[0])=u[0]^2,A||i))),x),diff); od;
for i from 0 to N do expand(subs(F(u[0])=sin(u[0]),A||i)); od;
for i from 0 to N do expand(subs(F(u[0])=cos(u[0]),A||i)); od;
for i from 0 to N do expand(subs(F(u[0])=sinh(u[0]),A||i)); od;
for i from 0 to N do expand(subs(F(u[0])=cosh(u[0]),A||i)); od;
for i from 0 to N do expand(subs(F(u[0])=exp(u[0]),A||i)); od;
for i from 0 to N do expand(subs(F(u[0])=ln(u[0]),A||i)); od;
```

Mathematica:

```
n=9; trD[u_,var_]:=Table[D[u,{var,i}],{i,1,n}]//Flatten;
fADM1[i_]:=1/i!*D[f[Sum[lambda^k*u[k],{k,0,i}]],{lambda,i}]/.
 lambda->0//Expand; a[0]=f[u[0]]
Do[a[i]=fADM1[i]; Print["a[",i,"]=",a[i]],{i,1,n}];
{trp1=f[u[0]]->u[0]^2, trp2=f[u[0]]->u[0]^3, trp3=Table[u[k]->
 D[u[x][k],x],{k,0,n}],trp5=Table[u[k]->fU[x][k],{k,0,n}]}
trp[g_]:=f[u[0]]->g[u[0]];
Do[p1[i]=a[i]/.trp1/.trD[trp1,u[0]];Print["p1[",i,"]=",p1[i]],
 {i,0,n}]; Do[p2[i]=a[i]/.trp2/.trD[trp2,u[0]];
 Print["p2[",i,"]=",p2[i]],{i,0,n}];
Do[p3[i]=(a[i]/.trp1/.trD[trp1,u[0]])/.trp3;
 Print["p3[",i,"]=",p3[i]],{i,0,n}]; Do[p4[i]=(a[i]/.trp2/.
 trD[trp2,u[0]])/.trp3;Print["p4[",i,"]=",p4[i]],{i,0,n}];
Do[p5[i]=1/2*D[(a[i]/.trp1/.trD[trp1,u[0]])/.trp5,x]//Expand;
 Print["p5[",i,"]=",p5[i]],{i,0,n}];
pFun[p_,fun_]:=Do[p[i]=a[i]/.trp[fun]/.trD[trp[fun],u[0]];
 Print[ToString[p],"[",i,"]=",p[i]],{i,0,n}];
pFun[p6,Sin]; pFun[p7,Cos]; pFun[p8,Sinh]; pFun[p9,Cosh];
pFun[p10,Exp]; pFun[p11,Log];
```

□

5.1.2 Nonlinear PDEs

Problem 5.2

Maple:

```
with(PDEtools): declare((u,W)(x,t)); KN:=9;
ADM1:=n->convert(subs(lambda=0,value(1/n!*Diff(F(Sum(
 lambda^i*U[i],i=0..n)),lambda$n))),diff); A0[0]:=F(U[0]);
for n from 1 to KN do A0[n]:=ADM1(n); od:
for n from 0 to KN do
 A[n]:=convert(1/2*diff(subs({seq(U[i]=W[i](x,t),i=0..KN)},
       expand(subs(F(U[0])=U[0]^2,A0[n]))),x),diff); od;
L:=w->diff(w(x,t),t); NL:=w->w*diff(w(x,t),x);  L(u); NL(u);
PDE1:=w->L(w)=-NL(w); PDE1(u); IC1:=u(x,0)=x+1;
LI:=w->Int(w(x,t),t=0..t); LI(u); tr1:=u-rhs(IC1);
Eq1:=LI(lhs(PDE1(u)))=LI(rhs(PDE1(u)));
Eq2:=simplify(subs(lhs(Eq1)=tr1,Eq1));
trL:=u=add(u[j](x,t),j=0..KN);
trN:=LI(NL(u))=Int(Sum(A[i],i=0..KN),t=0..t);
Eq3:=subs(trL,lhs(Eq2))=subs(trN,rhs(Eq2));
```

```
Apr[0]:=u[0](x,t)=rhs(IC1);
AprK:=u[k+1](x,t)=-Int(AD[k],t=0..t);
for i from 0 to KN do
 Apr[i+1]:=value(subs({seq(Apr[m],m=0..i)},subs(
 {seq(W[m]=u[m],m=0..i)},subs(k=i,AD[i]=A[i],AprK))));  od;
trSol:={seq(Apr[i],i=0..KN)}; Sol:=value(subs(trSol,trL));
Sol1:=collect(combine(Sol),t); factor(Sol1);
SolF:=(x+1)*sum((-1)^j*t^j,j=0..infinity);
Test1:=subs(u=SolF,(algsubs(u(x,t)=SolF,PDE1(u))));
```

Mathematica:

```
kN=9; var=Sequence[x,t]; trD[u_,var_]:=Table[D[u,{var,i}],
 {i,1,n}]//Flatten; fADM1[n_]:=1/n!*D[f[Sum[lambda^i*fU[i],
 {i,0,n}]],{lambda,n}]/.lambda->0//Expand; a0[0]=f[fU[0]]
{trp1=f[fU[0]]->fU[0]^2, trp2=Table[fU[i]->fW[var][i],{i,0,kN}]}
Do[a0[n]=fADM1[n];Print["a0[",n,"]=",a0[n]],{n,1,kN}];
Do[a[n]=1/2*D[(a0[n]/.trp1/.trD[trp1,fU[0]])/.trp2,x]//Expand;
 Print["a[",n,"]=",a[n]],{n,0,kN}]; fL[w_]:=D[w[var],t];
fNL[w_]:=w[var]*D[w[var],x]; {fL[u], fNL[u]}
pdeT[w_]:=fL[w]==fNL[w]; pde1[w_]:=fL[w]==-fNL[w];
{pde1[u], ic1=u[x,0]->x+1}
fLI[w_]:=Integrate[w,{t,0,t}]; {fLI[u[var]], tr1=u[var]-ic1[[2]]}
{eq1=fLI[pdeT[u][[1]]]==-fLI[pdeT[u][[2]]],
 eq2=eq1/.eq1[[1]]->tr1, trL=u[var]->Sum[u[var][j],{j,0,kN}],
 trN=fLI[fNL[u]]->Hold[Integrate[Sum[a[i],{i,0,kN}],{t,0,t}]]}
{eq3=(eq2[[1]]/.trL)==(eq2[[2]]/.trN),apr[0]=u[var][0]->ic1[[2]]}
aprK=u[var][k+1]->-Hold[Integrate[ad[k],{t,0,t}]]
Do[apr[i+1]=ReleaseHold[aprK/.k->i/.ad[i]->a[i]/.
 Table[fW[var][m]->u[x,t][m],{m,0,i}]/.Table[D[fW[var][m],x]->
 D[u[x,t][m],x],{m,0,i}]/.Table[apr[m],{m,0,i}]/.
 Table[D[apr[m],x],{m,0,i}]]; Print["apr[",i+1,"]=",apr[i+1]],
{i,0,kN}]; {trSol=Table[apr[i],{i,0,kN}], sol=trL/.trSol,
 sol1=Collect[sol//Together,t], sol1//Factor}
{solF=(x+1)*Sum[(-1)^j*t^j,{j,0,Infinity}], trp3=u[x,t]->solF}
pde1[u]/.trp3/.D[trp3,x]/.D[trp3,t]
```
□

Problem 5.3

Maple:

```
with(PDEtools): declare((u,W)(x,t)); KN:=9;
ADM1:=n->convert(subs(lambda=0,value(1/n!*Diff(F(Sum(
 lambda^i*U[i],i=0..n)),lambda$n))),diff); A0[0]:=F(U[0]);
for n from 1 to KN do A0[n]:=ADM1(n); od:
for n from 0 to KN do
 A[n]:=convert(1/2*diff(subs({seq(U[i]=W[i](x,t),i=0..KN)},
       expand(subs(F(U[0])=U[0]^2,A0[n]))),x),diff); od;
L:=w->diff(w(x,t),t); NL:=w->w*diff(w(x,t),x);  L(u); NL(u);
PDE1:=w->L(w)=-NL(w); PDE1(u); IC1:=u(x,0)=cos(x);
LI:=w->Int(w(x,t),t=0..t); LI(u); tr1:=u-rhs(IC1);
Eq1:=LI(lhs(PDE1(u)))=LI(rhs(PDE1(u)));
Eq2:=simplify(subs(lhs(Eq1)=tr1,Eq1));
trL:=u=add(u[j](x,t),j=0..KN);
trN:=LI(NL(u))=Int(Sum(A[i],i=0..KN),t=0..t);
Eq3:=subs(trL,lhs(Eq2))=subs(trN,rhs(Eq2)); Apr[0]:=u[0](x,t)
 =rhs(IC1); AprK:=u[k+1](x,t)=-Int(AD[k],t=0..t);
for i from 0 to KN do
 Apr[i+1]:=value(subs({seq(Apr[m],m=0..i)},subs({seq(W[m]=u[m],
  m=0..i)},subs(k=i,AD[i]=A[i],AprK))));  od;
trSol:={seq(Apr[i],i=0..KN)}; Sol:=value(subs(trSol,trL));
SolF:=collect(combine(Sol),t);
```

Mathematica:

```
kN=9; var=Sequence[x,t];
trD[u_,var_]:=Table[D[u,{var,i}],{i,1,n}]//Flatten;
fADM1[n_]:=1/n!*D[f[Sum[lambda^i*fU[i],{i,0,n}]],{lambda,n}]/.
 lambda->0//Expand; a0[0]=f[fU[0]]
{trp1=f[fU[0]]->fU[0]^2, trp2=Table[fU[i]->fW[var][i],{i,0,kN}]}
Do[a0[n]=fADM1[n];Print["a0[",n,"]=",a0[n]],{n,1,kN}];
Do[a[n]=1/2*D[(a0[n]/.trp1/.trD[trp1,fU[0]])/.trp2,x]//Expand;
 Print["a[",n,"]=",a[n]],{n,0,kN}]; fL[w_]:=D[w[var],t];
fNL[w_]:=w[var]*D[w[var],x]; {fL[u], fNL[u]}
pdeT[w_]:=fL[w]==fNL[w]; pde1[w_]:=fL[w]==-fNL[w];
{pde1[u], ic1=u[x,0]->Cos[x]}
fLI[w_]:=Integrate[w,{t,0,t}]; {fLI[u[var]], tr1=u[var]-ic1[[2]]}
{eq1=fLI[pdeT[u][[1]]]==-fLI[pdeT[u][[2]]],
 eq2=eq1/.eq1[[1]]->tr1, trL=u[var]->Sum[u[var][j],{j,0,kN}],
 trN=fLI[fNL[u]]->Hold[Integrate[Sum[a[i],{i,0,kN}],{t,0,t}]]}
```

```
{eq3=(eq2[[1]]/.trL)==(eq2[[2]]/.trN),apr[0]=u[var][0]->ic1[[2]],
 aprK=u[var][k+1]->-Hold[Integrate[ad[k],{t,0,t}]]}
Do[apr[i+1]=ReleaseHold[aprK/.k->i/.ad[i]->a[i]/.
 Table[fW[var][m]->u[x,t][m],{m,0,i}]/.Table[D[fW[var][m],x]->
 D[u[x,t][m],x],{m,0,i}]/.Table[apr[m],{m,0,i}]/.
 Table[D[apr[m],x],{m,0,i}]]; Print["apr[",i+1,"]=",apr[i+1]],
{i,0,kN}]; {trSol=Table[apr[i],{i,0,kN}], sol=trL/.trSol}
solF=u[var]->Collect[Map[Together[#,Trig->True]&,sol[[2]]],t]
```
□

Problem 5.4

Maple:

```
with(PDEtools): declare((u,W,U)(x,t)); KN:=9; ADM1:=n->
 convert(subs(lambda=0,value(1/n!*Diff(F(Sum(lambda^i*U[i],
 i=0..n)),lambda$n))),diff); A0[0]:=F(U[0]);
for n from 1 to KN do A0[n]:=ADM1(n); od:
for n from 0 to KN do A[n]:=subs({seq(U[i]=W[i](x,t),
 i=0..KN)},expand(subs(F(U[0])=U[0]^2,A0[n]))); od;
Dt:=w->diff(w(x,t),t$2); Dx:=w->-diff(w(x,t),x$2);
L:=w->w; NL:=w->w^2; f:=(x,t)->x*t+x^2*t^2; g1:=x->0; g2:=x->x;
PDE1:=w->Dt(w)+Dx(w)+L(w)+NL(w)=f(x,t); PDE1(u);
IC1:=[u(x,0)=g1(x),D[2](u)(x,0)=g2(x)];
LI:=w->Int(Int(w(x,t),t=0..t),t=0..t); LI(u);
tr1:=u-g1(x)-t*g2(x); KN:=1; Eq1:=LI(lhs(PDE1(u)))=LI(f);
Eq2:=simplify(subs(lhs(Eq1)=tr1,Eq1)); trL:=u=add(u[j](x,t),
 j=0..KN); trN:=LI(NL(u))=Int(Int(Sum(A[i],i=0..KN),t=0..t),
 t=0..t); Eq3:=value(subs(trL,lhs(Eq2))=subs(trN,rhs(Eq2)));
Eq31:=Eq3+g1(x)+t*g2(x); Apr[0]:=u[0](x,t)=rhs(Eq31);
AprK:=u[k+1](x,t)=Int(Int(diff(u[k](x,t),x$2),t=0..t),t=0..t)
 -Int(Int(AD[k],t=0..t),t=0..t); for i from 0 to KN do
  Apr[i+1]:=value(subs({seq(Apr[m],m=0..i)},subs(
  {seq(W[m]=u[m],m=0..i)},subs(k=i,AD[i]=A[i],AprK)))); od;
termN:=select(has,rhs(Apr[0]),[t^3,t^4]); Apr[0];
AApr[0]:=rhs(Apr[0])-termN; AApr[1]:=rhs(Apr[1])+termN;
U:=unapply(AApr[0],x,t); U(x,t); PDE1(U); SolF:=AApr[0];
Test1:=subs(u=SolF,(algsubs(u(x,t)=SolF,PDE1(u))));
```

Mathematica:

```
trS2[eq_,var1_,var2_]:=Select[eq,MemberQ[#,var1,Infinity]||
 MemberQ[#,var2,Infinity]&]; kN=9; var=Sequence[x,t];
trD[u_,var_]:=Table[D[u,{var,i}],{i,1,9}]//Flatten;
```

```
fADM1[n_]:=1/n!*D[f[Sum[lambda^i*fU[i],{i,0,n}]],{lambda,n}]/.
 lambda->0//Expand;  a0[0]=f[fU[0]]
{trp1=f[fU[0]]->fU[0]^2, trp2=Table[fU[i]->fW[var][i],{i,0,kN}]}
Do[a0[n]=fADM1[n]; Print["a0[",n,"]=",a0[n]],{n,1,kN}];
Do[a[n]=(a0[n]/.trp1/.trD[trp1,fU[0]])/.trp2//Expand;
  Print["a[",n,"]=",a[n]],{n,0,kN}]; fDt[w_]:=D[w[var],{t,2}];
fDx[w_]:=D[w[var],{x,2}]; fL[w_]:=w[var]; fNL[w_]:=w[var]^2;
f[x_, t_]:=x*t+x^2*t^2; g1[x_]:=0; g2[x_]:=x;
pde1[w_]:=fDt[w]+fDx[w]+fL[w]+fNL[w]==f[x,t];
{pde1[u], ic1={u[x,0]->g1[x],(D[u[x,t],t]/.t->0)->g2[x]}}
fLI[w_]:=Hold[Integrate[w,t,t]]; fLI[u[var]]
{tr1= u[var]-g1[x]-t*g2[x], kN=1,
 eq1=fLI[pde1[u][[1]]]==fLI[f[var]], eq2=eq1/.eq1[[1]]->tr1,
 trL=u[var]->Sum[u[var][j],{j,0,kN}], trN=fLI[fNL[u]]->
 Hold[Integrate[Sum[a[i],{i,0,1}],t,t]]}
eq3=(eq2[[1]]/.trL)==(ReleaseHold[(eq2[[2]]/.trN)])//Expand
{eq31=Thread[eq3+g1[x]+t*g2[x],Equal],
 apr[0]=u[var][0]->eq31[[2]], aprK=u[var][k+1]->Hold[Integrate[
 D[u[var][k],{x,2}],t,t]]-Hold[Integrate[ad[k],t,t]]}
Do[apr[i+1]=ReleaseHold[aprK/.k->i/.ad[i]->a[i]]/.
 Table[fW[var][m]->u[x,t][m],{m,0,i}]/.Table[D[fW[var][m],x]->
 D[u[x,t][m],x],{m,0,i}]/.Table[apr[m],{m,0,i}]/.Table[D[apr[m],
 {x,2}],{m,0,i}]; Print["apr[",i+1,"]=",apr[i+1]],{i,0,kN}];
{termN=trS2[apr[0][[2]],t^3,t^4], apr[0],
 aApr[0]=apr[0][[2]]-termN, aApr[1]=apr[1][[2]]+termN}
uN[xN_,tN_]:=aApr[0]/.x->xN/.t->tN; uN[x,t]
{pde1[uN], solF=aApr[0], trsolF=u[var]->solF,
 test1=pde1[u]/.trsolF/.trD[trsolF,x]/.trD[trsolF,t]}
```
□

Problem 5.5

Maple:

```
with(PDEtools): declare((u,W)(x,t)); KN:=9;
ADM1:=n->convert(subs(lambda=0,value(1/n!*Diff(F(Sum(
 lambda^i*U[i],i=0..n)),lambda$n))),diff); A0[0]:=F(U[0]);
for n from 1 to KN do A0[n]:=ADM1(n); od:
for n from 0 to KN do
 A[n]:=convert(1/2*diff(subs({seq(U[i]=W[i](x,t),i=0..KN)},
       expand(subs(F(U[0])=U[0]^2,A0[n]))),x),diff); od;
Dt:=w->diff(w(x,t),t); NL:=w->w*diff(w(x,t),x);
Dx:=w->diff(w(x,t),x$2); PDE1:=w->Dx(w)=Dt(w)+NL(w); PDE1(u);
```

```
BC1:=[u(0,t)=g1(t),D[1](u)(0,t)=g2(t)]; g1:=t->-2/(q*t);
g2:=t->1/t+2/(q^2*t^2); LI:=w->Int(Int(w(x,t),x=0..x),x=0..x);
LI(u); tr1:=u+(g1(t)+x*g2(t)); KN:=4; Eq1:=LI(lhs(PDE1(u)))=
 map(LI,rhs(PDE1(u))); Eq2:=expand(subs(lhs(Eq1)=tr1,Eq1));
trL:=diff(u(x,t),t)=add(diff(u[j](x,t),t),j=0..KN);
trN:=LI(NL(u))=Int(Int(Sum(A[i],i=0..KN),x=0..x),x=0..x);
Eq3:=lhs(Eq2)=value(subs(trL,trN,rhs(Eq2)));
```

Mathematica:

```
kN=9; trD[u_,var_]:=Table[D[u,{var,i}],{i,1,9}]//Flatten;
fADM1[n_]:=1/n!*D[f[Sum[lambda^i*fU[i],{i,0,n}]],
 {lambda,n}]/.lambda->0//Expand; a0[0]=f[fU[0]]
{trp1=f[fU[0]]->fU[0]^2,trp2=Table[fU[i]->fW[x,t][i],{i,0,kN}]}
Do[a0[n]=fADM1[n]; Print["a0[",n,"]=",a0[n]],{n,1,kN}];
Do[a[n]=1/2*D[(a0[n]/.trp1/.trD[trp1,fU[0]])/.trp2,x]//Expand;
  Print["a[",n,"]=",a[n]],{n,0,kN}];
fDt[w_]:=D[w[x,t],t]; fDx[w_]:=D[w[x,t],{x,2}];
fNL[w_]:=w[x,t]*D[w[x,t],x]; pde1[w_]:=fDx[w]==fDt[w]+fNL[w];
{pde1[u], bc1={u[0,t]->g1[t],(D[u[x,t],x]/.x->0)->g2[t]}}
g1[t_]:=-2/(q*t); g2[t_]:=1/t+2/(q^2*t^2); fLI[w_]:=Hold[
 Integrate[w,x,x]]; {fLI[u[x,t]], tr1=u[x,t]+(g1[t]+x*g2[t])}
{kN=4, eq1=fLI[pde1[u][[1]]]==Map[fLI, pde1[u][[2]]]}
eq2=eq1/.eq1[[1]]->tr1
trL=D[u[x,t],t]->Sum[D[u[x,t][j],t],{j,0,kN}]
trN=fLI[fNL[u]]->Hold[Integrate[Sum[a[i],{i,0,2}],x,x]]
eq3=eq2[[1]]==(eq2[[2]]/.trL/.trN)//Expand
```

Maple:

```
Apr[0]:=u[0](x,t)=select(has, lhs(Eq3),t);
AprK:=u[k+1](x,t)=Int(Int(diff(u[k](x,t),t),x=0..x),x=0..x)
 +Int(Int(AD[k],x=0..x),x=0..x);
for i from 0 to KN do
 Apr[i+1]:=expand(value(subs({seq(Apr[m],m=0..i)},
 subs({seq(W[m]=u[m],m=0..i)},subs(k=i,AD[i]=A[i],AprK)))));
od; trSol:={seq(Apr[i],i=0..KN)}; Sol:=value(subs(trSol,
 u=add(u[j](x,t),j=0..KN)));
Sol1:=expand((rhs(Sol)-x/t)/(-2/(q*t)));
SolF:=expand(x/t-(2/(q*t))*sum((-1)^j*(1/q^j)*x^j/t^j,
 j=0..infinity)); Test1:=pdetest(u(x,t),PDE1(u));
Test2:=[subs(x=0,SolF)=g1(t),subs(x=0,diff(SolF,x))=g2(t)];
```

Mathematica:

```
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&];
apr[0]=u[x,t][0]->trS3[eq3[[1]],u[x,t]]
aprK=u[x,t][k+1]->Hold[Integrate[D[u[x,t][k],t],x,x]]+
 Hold[Integrate[ad[k],x,x]]
Do[apr[i+1]=ReleaseHold[aprK/.k->i/.ad[i]->a[i]/.
 Table[fW[x,t][m]->u[x,t][m],{m,0,i}]/.Table[D[fW[x,t][m],x]->
 D[u[x,t][m],x],{m,0,i}]/.Table[apr[m],{m,0,i}]/.
 Table[D[apr[m],t],{m,0,i}]/.Table[D[apr[m],x],{m,0,i}]]//Expand;
 Print["apr[",i+1,"]=",apr[i+1]],{i,0,kN}];
{trSol=Table[apr[i],{i,0,kN}], sol=u[x,t]->Sum[u[x,t][j],
 {j,0,kN}]/.trSol, sol1=(sol[[2]]-x/t)/(-2/(q*t))//Expand}
{solF=x/t-(2/(q*t))*Sum[(-1)^j*(1/q^j)*x^j/t^j,
 {j,0,Infinity}]//Expand, trsolF=u[x,t]->solF}
pde1[u]/.trsolF/.trD[trsolF,x]/.trD[trsolF,t]//Simplify
test2={(solF/.x->0)==g1[t],(D[solF,x]/.x->0)==g2[t]}//Simplify
```
□

5.1.3 Nonlinear Systems

Problem 5.6

Maple:

```
with(PDEtools): declare((u,v)(x,t)); KN:=9; ADM1:=n->convert(
 subs(lambda=0,value(1/n!*Diff(F(Sum(lambda^i*U[i],i=0..n)),
 lambda$n))),diff); A0[0]:=F(U[0]);
for n from 1 to KN do A0[n]:=ADM1(n); od:
for n from 0 to KN do
 A[n]:=sort(convert(1/2*diff(subs({seq(U[i]=u[i](x,t),
  i=0..KN)},expand(subs(F(U[0])=U[0]^2,A0[n]))),x),diff)); od;
A1[0]:=A[0]/u[0](x,t)*v[0](x,t); A2[0]:=A[0]/u[0,x](x,t)
 *v[0,x](x,t); for j from 1 to KN do k1:=nops(A[j]):
 A1[j]:=add(op(i,A[j])/u[k1-i](x,t)*v[k1-i](x,t),i=1..k1);
 A2[j]:=add(op(i,A[j])/u[i-1,x](x,t)*v[i-1,x](x,t),i=1..k1); od;
```

Mathematica:

```
kN=9; trD[u_,var_]:=Table[D[u,{var,i}],{i,1,9}]//Flatten;
fADM1[n_]:=1/n!*D[f[Sum[lambda^i*fU[i],{i,0,n}]],
 {lambda,n}]/.lambda->0//Expand; a0[0]=f[fU[0]]
{trp1=f[fU[0]]->fU[0]^2,trp2=Table[fU[i]->u[x,t][i],{i,0,kN}]}
Do[a0[n]=fADM1[n]; Print["a0[",n,"]=",a0[n]],{n,1,kN}];
```

```
Do[a[n]=1/2*D[(a0[n]/.trp1/.trD[trp1,fU[0]])/.trp2,x]//Expand;
 Print["a[",n,"]=",a[n]],{n,0,kN}];
{a1[0]=a[0]/u[x,t][0]*v[x,t][0],
 a2[0]=a[0]/D[u[x,t][0],x]*D[v[x,t][0],x]}
Do[k1=Length[a[j]]; a1[j]=Sum[a[j][[i]]/u[x,t][k1-i]*
 v[x,t][k1-i],{i,1,k1}]; a2[j]=Sum[a[j][[i]]/D[u[x,t][i-1],x]*
 D[v[x,t][i-1],x],{i,1,k1}]; Print["a1[",j,"]=",a1[j]];
 Print["a2[",j,"]=",a2[j]],{j,1,kN}];
```

Maple:

```
Dt:=w->diff(w(x,t),t); L:=w->w; NL:=(w1,w2)->w1*diff(w2(x,t),x);
f:=(x,t)->1; g1:=x->exp(-x); g2:=x->exp(x);
PDE1:=(w1,w2)->Dt(w1)=NL(w2,w1)+L(w1)+f(x,t); PDE1(u,v);
PDE2:=(w1,w2)->Dt(w2)=-NL(w1,w2)-L(w2)+f(x,t); PDE2(u,v);
IC1:=u(x,0)=g1(x); IC2:=v(x,0)=g2(x); LI:=w->Int(w(x,t),
 t=0..t); LI(u); tr1:=u-rhs(IC1); tr2:=v-rhs(IC2);
Eq11:=LI(lhs(PDE1(u,v)))=map(LI,rhs(PDE1(u,v)));
Eq12:=LI(lhs(PDE2(u,v)))=map(LI,rhs(PDE2(u,v)));
Eq21:=simplify(subs(lhs(Eq11)=tr1,Eq11)); Eq22:=simplify(subs(
 lhs(Eq12)=tr2,Eq12)); trL1:=u=add(u[j](x,t),j=0..KN);
trL2:=v=add(v[j](x,t),j=0..KN); trN1:=LI(NL(v,u))=Int(Sum(A[i],
 i=0..KN),t=0..t); trN2:=LI(NL(u,v))=Int(Sum(A[i],i=0..KN),
 t=0..t); Eq31:=subs(trL1,lhs(Eq21))=subs(trN1,rhs(Eq21));
Eq32:=subs(trL2,lhs(Eq22))=subs(trN2,rhs(Eq22));
```

Mathematica:

```
fDt[w_]:=D[w[x,t],t]; fL[w_]:=w[x,t]; fNL[w1_,w2_]:=w1[x,t]*
 D[w2[x,t],x]; f[x_,t_]:=1; g1[x_]:=Exp[-x]; g2[x_]:=Exp[x];
pde1[w1_,w2_]:=fDt[w1]==fNL[w2,w1]+fL[w1]+f[x,t];
pde2[w1_,w2_]:=fDt[w2]==-fNL[w1,w2]-fL[w2]+f[x,t];
{pde1[u,v],pde2[u,v], ic1=u[x,0]->g1[x], ic2=v[x,0]->g2[x]}
fLI[w_]:=Hold[Integrate[w,{t,0,t}]]; fLI[u[x,t]]
{tr1=u[x,t]-ic1[[2]], tr2=v[x,t]-ic2[[2]]}
eq11=fLI[pde1[u,v][[1]]]==Map[fLI,pde1[u,v][[2]]]
eq12=fLI[pde2[u,v][[1]]]==Map[fLI,pde2[u,v][[2]]]
{eq21=eq11/.eq11[[1]]->tr1, eq22=eq12/.eq12[[1]]->tr2}
trL1=u[x,t]->Sum[u[x,t][j],{j,0,kN}]
trL2=v[x,t]->Sum[v[x,t][j],{j,0,kN}]
trN1=fLI[fNL[v,u]]->Hold[Integrate[Sum[a[i],{i,0,kN}],{t,0,t}]]
trN2=fLI[fNL[u,v]]->Hold[Integrate[Sum[a[i],{i,0,kN}],{t,0,t}]]
```

```
eq31=(eq21[[1]]/.trL1)==(eq21[[2]]/.trN1)
eq32=(eq22[[1]]/.trL2)==(eq22[[2]]/.trN2)
```

Maple:

```
Ap1[0]:=u[0](x,t)=rhs(IC1); Ap2[0]:=v[0](x,t)=rhs(IC2);
Apr1[1]:=u[1](x,t)=t+Int(A1[0],t=0..t)+Int(u[0](x,t),t=0..t);
Apr2[1]:=v[1](x,t)=t-Int(A2[0],t=0..t)-Int(v[0](x,t),t=0..t);
Apr1K:=u[k+1](x,t)=Int(A1[k],t=0..t)+Int(u[k](x,t),t=0..t);
Apr2K:=v[k+1](x,t)=-Int(A2[k],t=0..t)-Int(v[k](x,t),t=0..t);
trD[1]:={u[0,x](x,t)=-exp(-x),v[0,x](x,t)=exp(x)};
Ap1[1]:=simplify(value(subs(Ap1[0],Ap2[0],Apr1[1])));
Ap2[1]:=simplify(value(subs(trD[1],subs(Ap1[0],Ap2[0],
 Apr2[1]))));
for i from 2 to KN do
 trD[i]:={u[i-1,x](x,t)=diff(rhs(Ap1[i-1]),x),v[i-1,x](x,t)
  =diff(rhs(Ap2[i-1]),x)};
 Ap1[i]:=simplify(expand(subs({seq(Ap1[m],m=0..i-1),
  seq(Ap2[m],m=0..i-1)},value(subs(k=i-1,Apr1K)))));
 Ap2[i]:=simplify(expand(subs(seq(trD[m],m=1..i),seq(Ap1[m],
  m=0..i-1),seq(Ap2[m],m=0..i-1),value(subs(k=i-1,Apr2K)))));
od; trSol:={seq(Ap1[i],i=0..KN),seq(Ap2[i],i=0..KN)};
Solu:=value(subs(trSol,trL1)); Solv:=value(subs(trSol,trL2));
Solu1:=collect(Solu,exp); Solv1:=collect(Solv,exp);
SoluF:=combine(exp(-x)*sum((1/l!)*t^l,l=0..infinity));
SolvF:=combine(exp(x)*sum((1/l!)*(-t)^l,l=0..infinity));
Test1:=pdetest({u(x,t)=SoluF,v(x,t)=SolvF},diff(u(x,t),t)
 =v(x,t)*diff(u(x,t),x)+u(x,t)+1);
Test2:=pdetest({u(x,t)=SoluF,v(x,t)=SolvF},diff(v(x,t),t)
 =-u(x,t)*diff(v(x,t),x)-v(x,t)+1);
Test3:=subs(t=0,SoluF)=g1(x); Test4:=subs(t=0,SolvF)=g2(x);
```

Mathematica:

```
{ap1[0]=u[x,t][0]->ic1[[2]], ap2[0]=v[x,t][0]->ic2[[2]]}
{apr1[1]=u[x,t][1]->t+Hold[Integrate[a1[0],{t,0,t}]+
 Integrate[u[x,t][0],{t,0,t}]], apr2[1]=v[x,t][1]->t+
 Hold[-Integrate[a2[0],{t,0,t}]-Integrate[v[x,t][0],{t,0,t}]]}
{apr1K=u[x,t][k+1]->Hold[Integrate[a1[k],{t,0,t}]+
 Integrate[u[x,t][k],{t,0,t}]], apr2K=v[x,t][k+1]->
 Hold[-Integrate[a2[k],{t,0,t}]-Integrate[v[x,t][k],{t,0,t}]]}
ap1[1]=ReleaseHold[apr1[1]]/.ap1[0]/.ap2[0]/.D[ap1[0],x]
```

```
ap2[1]=ReleaseHold[apr2[1]]/.ap1[0]/.ap2[0]/.D[ap1[0],x]/.
 D[ap2[0],x]
Do[ap1[i]=ReleaseHold[apr1K/.k->i-1/.Flatten[Table[{D[ap2[m],x],
 D[ap1[m],x],ap2[m],ap1[m]},{m,0,i-1}]]]/.Flatten[
 Table[{D[ap2[m],x],D[ap1[m],x],ap2[m],ap1[m]},{m,0,i-1}]];
 ap2[i]=ReleaseHold[apr2K/.k->i-1/.Flatten[Table[{D[ap2[m],x],
 D[ap1[m],x],ap2[m],ap1[m]},{m,0,i-1}]]]/.Flatten[Table[
 {D[ap2[m],x],D[ap1[m],x],ap2[m],ap1[m]},{m,0,i-1}]];
 Print["ap1[",i,"]=",ap1[i]]; Print["ap2[",i,"]=",ap2[i]],
{i,2,kN}];
trSol={Table[ap1[i],{i,0,kN}],Table[ap2[i],{i,0,kN}]}//Flatten
{solu=trL1/.trSol, solv=trL2/.trSol, solu1=Collect[solu,Exp[_]],
 solv1=Collect[solv,Exp[_]], soluF=Exp[-x]*Sum[(1/l!)*t^l,
 {l,0,Infinity}], solvF=Exp[x]*Sum[(1/l!)*(-t)^l,{l,0,Infinity}]}
{trsoluF=u[x,t]->soluF, trsolvF=v[x,t]->solvF}
test1=pde1[u,v]/.trsoluF/.trsolvF/.D[trsoluF,x]/.D[trsoluF,t]
test2=pde2[u,v]/.trsoluF/.trsolvF/.D[trsolvF,x]/.D[trsolvF,t]
{test3=(soluF/.t->0)==g1[x], test4=(solvF/.t->0)==g2[x]}
```
□

5.2 Asymptotic Expansions. Perturbation Methods

5.2.1 Nonlinear PDEs

Problem 5.7

Maple:

```
with(PDEtools): declare(u(xi)); tr0:=x-c*t=xi; PDE1:=v->diff(v,t)
 -diff(v,x$2)-v*(1-v)=0; Eq0:=expand(PDE1(u(lhs(tr0))));
Eq1:=collect(convert(algsubs(tr0,Eq0),diff),diff);
BC1:=(u,A)->u(-infinity)=A; BC2:=(u,B)->u(infinity)=B;
IC:=(u,C)->u(0)=C; tr1:=[xi=z*c,xi=z*epsilon^(-1/2),
 c=epsilon^(-1/2)]; Eq2:=expand(eval(dchange(tr1[2],Eq1,[z,u]),
 tr1[3])); tr2:=u(z,epsilon)=add(u||i(z)*epsilon^i,i=0..2);
Eq3:=collect(algsubs(tr2,Eq2),epsilon); Pr0:=remove(has,
 lhs(Eq3),[epsilon])=0; Sol0:=dsolve({Pr0} union {IC(u0,1/2)},
 u0(z)); limit(Sol0,z=-infinity)=BC1(u0,1);
limit(Sol0,z=infinity)=BC2(u0,0); Pr1:=remove(has,expand(
 lhs(Eq3)/epsilon),[epsilon])=0; Pr11:=collect(algsubs(Sol0,Pr1),
 [diff,u1]); Pr12:=map(factor,lhs(Pr11));
Sol1:=combine(dsolve({Pr12} union {IC(u1,0)},u1(z)));
combine(Sol1) assuming z>0; limit(Sol1,z=-infinity)=BC1(u1,0);
limit(Sol1,z=infinity)=BC2(u1,0);
```

```
Pr2:=select(has,lhs(Eq3),epsilon^2)/epsilon^2=0;
Pr21:=collect(algsubs(Sol1,algsubs(Sol0,Pr2)),[diff,u2]);
Pr22:=map(factor,lhs(Pr21)); Pr23:=op(1,Pr22)+op(2,Pr22)+
 combine(collect(factor(op(3,Pr22)),[exp,ln]));
Sol2:=combine(dsolve({Pr23} union {IC(u2,0)},u2(z)));
```

Mathematica:

```
Off[Solve::ifun]; trD[u_,var_]:=Table[D[u,{var,i}],
 {i,1,2}]//Flatten; pde1[v_]:=D[v,t]-D[v,{x,2}]-v*(1-v)==0;
{tr0=x-c*t->xi, eq0=pde1[u[tr0[[1]]]]//Expand, eq1=eq0/.tr0}
bc1[u_,a_]:=u[-Infinity]==a; bc2[u_,b_]:=u[Infinity]==b;
ic[u_,c_]:=u[z]==c; {tr1={xi->z*c,xi->z*epsilon^(-1/2),
 c->epsilon^(-1/2)}, tr11=xi->z/Sqrt[epsilon]}
eq1F[xi_]:=-u[xi]+u[xi]^2-c*u'[xi]-u''[xi]==0; eq1F[xi]
eq1FT[v_]:=First[((eq1F[xi]/.u->Function[xi,u[xi*Sqrt[
 epsilon]]])/.tr11)/.{u->v}]; eq2=(eq1FT[u]//Expand)/.tr1[[3]]
tr2=u[z]->Sum[u[i][z]*epsilon^i,{i,0,2}]
eq3=Collect[Expand[eq2/.tr2/.trD[tr2,z]],epsilon]
{pr0=Coefficient[eq3,epsilon,0]==0, sol00=DSolve[pr0,u[0][z],
 z]//First, trC10=Solve[(sol00/.z->0)[[1,2]]==ic[u,1/2][[2]],
 C[1]]//First, sol0=sol00/.trC10}
(TraditionalForm[Limit[sol0[[1,1]],z->-Infinity]]==
 Limit[sol0[[1,2]],z->-Infinity])==bc1[u[0],1]
(TraditionalForm[Limit[sol0[[1,1]],z->Infinity]]==
 Limit[sol0[[1,2]],z->Infinity])==bc2[u[0],0]
{pr1=Coefficient[eq3,epsilon,1]==0, pr11=Collect[pr1/.sol0/.
 trD[sol0,z],u[1][z]], sol10=DSolve[pr11,u[1][z],z]//First}
trC11=Solve[(sol10/.z->0)[[1,2]]==ic[u,0][[2]],C[1]]//First
sol1=sol10/.trC11//Simplify
(TraditionalForm[Limit[sol1[[1,1]],z->-Infinity]]==
 Limit[sol1[[1,2]],z->-Infinity])==bc1[u[1],0]
(TraditionalForm[Limit[sol1[[1,1]],z->Infinity]]==
 Limit[sol1[[1,2]],z->Infinity])==bc2[u[1],0]
{pr2=Coefficient[eq3,epsilon,2]==0, pr21=Collect[pr2/.sol0/.
 sol1/.trD[sol0,z]/.trD[sol1,z],u[2][z]], sol20=DSolve[
 pr21,u[2][z],z]//First, trC12=Solve[(sol20/.z->0)[[1,2]]==
 ic[u,0][[2]],C[1]]//First//FullSimplify}
sol2=sol20/.trC12//FullSimplify
```

□

Problem 5.8

Maple:

```
with(PDEtools): declare(u(x,t));
Eq1:=diff(u(x,t),t$2)-diff(u(x,t),x$2)-epsilon*diff(u(x,t)^2+
 diff(u(x,t),x$2),x$2)=0;
tr1:=u(x,t)=add(u||i(x,t)*epsilon^i,i=0..1);
Eq2:=collect(algsubs(tr1,Eq1),epsilon);
Pr0:=remove(has,lhs(Eq2),epsilon)=0;
Pr1:=remove(has,expand(lhs(Eq2)/epsilon),epsilon)=0;
ICs:=[u0(x,0)=g(x),D[2](u0)(x,0)=-diff(g(x),x)];
Sol0D:=subs(_F1(t+x)=f(t+x),_F2(t-x)=g(x-t),subs(g(t-x)=g(x-t),
 pdsolve(Pr0,u0(x,t)))); sysIP:={eval(rhs(Sol0D),t=0)=
 rhs(ICs[1]), eval(diff(rhs(Sol0D),t),t=0)=rhs(ICs[2])};
SolsysIP:=dsolve(sysIP,{f(x),g(x)}); Sol0:=subs(f(t+x)=0,Sol0D);
Pr11:=algsubs(Sol0,Pr1); tr2:={x=(xi+eta)/2,t=(eta-xi)/2};
Pr12:=dchange(tr2,Pr11,[xi,eta])/(-4); Pr13:=[selectremove(has,
 lhs(Pr12),eta)]; Pr14:=Pr13[1]=-Pr13[2];
Sol11:=map(int,Pr14,eta); Sol12:=map(int,Sol11,xi);
Sol13:=lhs(Sol12)=rhs(Sol12)+A(xi)+B(eta);
ICs1:=[eval(u1(x,t)=0,t=0),eval(D[2](u1)(x,t),t=0)=0];
sysIP1:={eval(rhs(Sol13),eta=xi)=rhs(ICs1[1]),
         eval(diff(rhs(Sol13),eta),eta=xi)=rhs(ICs1[2])};
SolsysIP:=dsolve(sysIP1,{A(xi),B(xi)});
SolsysIP1:=sort(convert(eval(SolsysIP,_C1=0),list));
A_xi:=SolsysIP1[1]; B_eta:=eval(SolsysIP1[2],xi=eta);
Sol14:=eval(Sol13,{A_xi,B_eta}); Sol15:=collect(Sol14*4,
 [xi,eta]); tr31:={g(eta)^2+diff(g(eta),eta$2)=G(eta)};
tr32:={-g(xi)^2-diff(g(xi),xi$2)=-G(xi)};
Sol16:=collect(simplify(Sol15,tr31),[xi,eta]);
Sol17:=collect(collect(simplify(Sol16,tr32),[xi,eta])/4,diff);
Sol18:=subs(diff(G(xi),xi)=Diff(G(xi),xi),Sol17);
tr4:={xi=x-t,eta=x+t}; Sol19:=u1(x,t)=subs(tr4,rhs(Sol18));
SolFin:= subs(Sol0,Sol19,tr1);
```

Mathematica:

```
trD[u_,var_]:=Table[D[u,{var,i}],{i,1,4}]//Flatten;
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&]; {eq1=D[u[x,t],{t,2}]-
 D[u[x,t],{x,2}]-epsilon*D[u[x,t]^2+D[u[x,t],{x,2}],{x,2}]==0,
 tr1=u[x,t]->Sum[u[i][x,t]*epsilon^i,{i,0,1}]}
```

```
eq2=Collect[eq1/.tr1/.trD[tr1,x]/.trD[tr1,t],epsilon]
{pr0=Coefficient[eq2[[1]],epsilon,0]==0, pr1=Coefficient[
 eq2[[1]],epsilon,1]==0, ics={u[0][x,0]->g[x],(D[u[0][x,t],t]/.
 {t->0})->-g'[x]}, sol0D0=DSolve[pr0,u[0][x,t],{x,t}]//First}
{sol0D=sol0D0/.C[2][t+x]->f[t+x]/.C[1][t-x]->g[x-t]/.
 g[t-x]->g[x-t], sysIP={(sol0D[[1,2]]/.t->0)==ics[[1,2]],
 (D[sol0D[[1,2]],t]/.t->0)==ics[[2,2]]}, solsysIP=DSolve[sysIP,
 {f[x],g[x]},x], solfSysC=DSolve[sysIP[[2]],{f[x]},x]//First,
 trC1=Solve[sysIP[[1]]/.solfSysC,C[1]]//First,
 solfSys=solfSysC/.trC1, solgSys=ToRules[sysIP[[1]]]/.solfSys/.
 trD[solfSys,x], sol0=sol0D/.f[t+x]->0}
{pr11=pr1/.sol0/.trD[sol0,x]/.trD[sol0,t], tr2={x->(xi+eta)/2,
 t->(eta-xi)/2}, tr4=Solve[tr2/.Rule->Equal,{xi,eta}]}
pr11F[xN_,tN_]:=pr11/.x->xN/.t->tN; pr11FT[v_]:=First[((
 pr11F[x,t]/.u[1]->Function[{x,t},u[1][x-t,x+t]])/.tr2)/.
 {u->v}]//Simplify; {pr11F[x,t], pr12=Thread[(pr11FT[u]//
 Expand)/(-4),Equal]//Expand, pr14=trS1[pr12,eta]==-trS3[pr12,
 eta], sol11=Map[Integrate[#,eta]&,pr14], sol12=Map[Integrate[
 #,xi]&,sol11], sol13=sol12[[1]]==sol12[[2]]+a[xi]+b[eta]}
{ics1={u[1][x,0]->0,(D[u[1][x,t],t]/.t->0)->0},
sysIP1={(sol13[[2]]/.eta->xi)==ics1[[1,2]],(D[sol13[[2]],eta]/.
 eta->xi)==ics1[[2,2]]}, solsysIP=DSolve[sysIP1,{a[xi],b[xi]},
 xi], solsysIP1=solsysIP/.C[1]->0//First, axi=solsysIP1[[1]],
 beta=solsysIP1[[2]]/.xi->eta, sol14=sol13/.axi/.beta}
sol15=Collect[Thread[sol14*4,Equal]//Expand,{xi,eta}]
{tr31={g[eta]^2+g''[eta]->gN[eta]},tr32={-g[xi]^2-g''[xi]->
 -gN[xi]}, tr33=Map[Times[#,-1]&,tr32//First]}
{sol16=Collect[sol15/.tr31,{xi,eta}], sol17=Thread[Collect[
 sol16/.tr32/.D[tr32,xi]/.D[tr33,xi],{xi,eta}]/4,Equal]//Factor,
 sol19=Collect[u[1][x,t]->sol17[[2]]/.tr4,gN_[_]]}
solFin=tr1/.sol0/.sol19
```
□

Problem 5.9

Maple:

```
with(PDEtools): declare(u(x,t),U(xi,tau));
FU0:=diff_table(U0(xi,tau)); Eq1:=u->diff(u,t$2)-diff(u,x$2)-
 epsilon*diff(u^2+diff(u,x$2),x$2)=0;
tr1:={x=xi+tau/epsilon,t=tau/epsilon};
Eq2:=expand(dchange(tr1,Eq1(U(x,t)),[xi,tau]));
tr2:=U(xi,tau,epsilon)=add(U||i(xi,tau)*epsilon^i,i=0..1);
```

```
Eq3:=collect(algsubs(tr2,Eq2),epsilon);
Pr0:=remove(has,expand(lhs(Eq3)/epsilon),epsilon)*(-1)=0;
Eq4:=op(1,lhs(Pr0))+Diff(U0(xi,tau)^2+FU0[xi,xi],xi,xi)=0;
Pr0-value(Eq4); Eq5:=map(int,lhs(Eq4),xi); Eq6:=value(Eq5);
Pr1:=remove(has,expand(lhs(Eq3)/epsilon^2),epsilon)=0;
```

Mathematica:

```
trD[u_,var_]:=Table[D[u,{var,i}],{i,1,4}]//Flatten;
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
eq1[x_,t_]:=D[u[x,t],{t,2}]-D[u[x,t],{x,2}]-epsilon*D[
 u[x,t]^2+D[u[x,t],{x,2}],{x,2}]==0; {tr1={x->xi+tau/epsilon,
 t->tau/epsilon}, Solve[tr1/.Rule->Equal,{xi,tau}]}
eq1T[v_]:=First[((eq1[x,t]/.u->Function[{x,t},
 u[x-t,t*epsilon]])/.tr1)/.{u->v}]//Simplify; eq1[x,t]
eq2=eq1T[uN]//Expand
tr2=uN[xi,tau]->Sum[v[i][xi,tau]*epsilon^i,{i,0,1}]
eq3=Collect[eq2/.tr2/.trD[tr2,xi]/.trD[tr2,tau]/.
 D[tr2,xi,tau],epsilon]
pr0=Coefficient[eq3,epsilon,1]*(-1)==0
eq4=trS1[pr0[[1]],D[v[0][xi,tau],xi,tau]]+
 Hold[D[v[0][xi,tau]^2+D[v[0][xi,tau],{xi,2}],{xi,2}]]==0
Thread[pr0-ReleaseHold[eq4],Equal]
eq5=Map[Integrate[#,xi]&,eq4[[1]]]
{eq6=ReleaseHold[eq5], pr1=Coefficient[eq3,epsilon,2]==0}
```
□

5.2.2 Nonlinear Systems

Problem 5.10

Maple:

```
Th:=proc(Expr) local AA,Z1,Z2,j; global Z3;
 AA:=sqrt((1+th)/(1-th)); Z1:=convert(Expr, exp);
 Z2:=simplify(subs({'exp(j*chi)=AA^j' $ 'j'=-10..10},Z1));
 Z3:=combine(Z2); RETURN(Z3) end:
NA:=2: NP:=NA-1: vars1:=alpha,z,C,theta,psi: vars2:=alpha,z,C(t),
 theta(t),psi(t); Fxi:=(x->xi(alpha,z,C(x),theta(x),psi(x))):
Feta:=(x->eta(alpha,z,C(x),theta(x),psi(x))):
dxi:=diff(Fxi(t),t): deta:=diff(Feta(t),t):
FA1:=(x->A1(C(x),theta(x))): FB1:=(x->B1(C(x),theta(x))):
CT:=convert(['epsilon^i*F||A||i(t)' $ 'i'=1..NP],`+`):
psiT:=omega+convert(['epsilon^i*F||B||i(t)' $ 'i'=1..NP],`+`):
```

```
thetaT:=epsilon^4*D11+convert(['epsilon^i*F||B||i(t)'
 $ 'i'=1..NP],`+`): setsubt:={C(t)=C,psi(t)=psi,theta(t)=theta}:
setsub:={diff(C(t),t)=CT,diff(psi(t),t)=psiT,diff(theta(t),t)
 =thetaT}: sub1xi:=subs(setsub,dxi): sub2xi:=subs(setsub,
 diff(sub1xi,t)): sub1eta:=subs(setsub,deta):
sub2eta:=subs(setsub,diff(sub1eta,t)):
Xi||NP:=-sin(alpha)*cosh(I*z)/Th(sinh(chi))*C*cos(psi):
Eta||NP:=cos(alpha)*sinh(I*z)/Th(sinh(chi))*C*cos(psi):
Sigma||NP:=cos(alpha)*cosh(I*z)/Th(sinh(chi))*C*cos(psi):
xi:=(x1,x2,x3,x4,x5)->-sin(x1)*cosh(I*x2)/Th(sinh(chi))*
 x3*x4/x4*cos(x5): eta:=(x1,x2,x3,x4,x5)->cos(x1)*sinh(I*x2)
 /Th(sinh(chi))*x3*x4/x4*cos(x5): sigma:=(x1,x2,x3,x4,x5)->
 cos(x1)*cosh(I*x2)/Th(sinh(chi))*x3*x4/x4*cos(x5): xi(vars1):
eta(vars1): sigma(vars1): xi2T:=collect(sub2xi,epsilon):
xi2T:=convert(['coeff(xi2T,epsilon,i)*epsilon^(i)'$'i'=NP-1..NP],
 `+`); eta2T:=collect(sub2eta,epsilon): eta2T:=convert(
 ['coeff(eta2T,epsilon,i)*epsilon^(i)'$'i'=NP-1..NP],`+`);
```

Maple:

```
F1:=-omega^2*diff(sigma(vars1),alpha)-epsilon*(xi2T*diff(
 xi(vars1),alpha)+eta2T*diff(eta(vars1),alpha))-xi2T:
F2:=-omega^2*(-I)*diff(sigma(vars1),z)-epsilon*(xi2T*(-I)
 *diff(xi(vars1),z)+eta2T*(-I)*diff(eta(vars1),z))-eta2T:
F3:=-diff(xi(vars1),alpha)-(-I)*diff(eta(vars1),z)+epsilon
 *((-I)*diff(xi(vars1),z)*diff(eta(vars1),alpha)-(-I)
 *diff(eta(vars1),z)*diff(xi(vars1),alpha)):
F1:=subs({D[2](B1)(C,theta)=0,D[2](A1)(C,theta)=0},subs(
 setsubt,F1)): F1:=coeff(F1,epsilon,NP)*epsilon^NP;
F2:=subs({D[2](B1)(C,theta)=0,D[2](A1)(C,theta)=0},subs(
 setsubt,F2)): F2:=coeff(F2,epsilon,NP)*epsilon^NP;
F3:=subs(setsubt,F3): F3:=coeff(F3,epsilon,NP)*epsilon^NP;
setAB:={A1(C,theta)=A1,B1(C,theta)=B1}: F1S:=expand(subs(
 setAB,F1)); F2S:=expand(subs(setAB,F2)); F3S:=expand(F3);
Bc11:=eval(coeff(xi(vars2),epsilon,NP-1),alpha=0);
Bc12:=simplify(eval(coeff(xi(vars2),epsilon,NP-1),alpha=Pi*n))
 assuming n::natural; Bc2:=eval(coeff(eta(vars2),epsilon,NP-1),
 z=0); bc3:=(x,y)->lambda*x-y-epsilon^4*y*cos(2*psi-2*theta):
Bc3:=bc3(sigma(vars2),eta(vars2)): lambda:=Th(tanh(chi));
Bc3:=subs(setsubt,eval(Bc3,z=-I*chi)):
Bc3:=coeff(Bc3,epsilon,NP)*epsilon^NP;
```

Maple:

```
Eq1:=(N,M,K)->-K^2*Xi||N||M||K-N*Sigma||N||M||K=EqC(F1S,N,M,K)
 /epsilon^NP; Eq2:=(N,M,K)->-K^2*Eta||N||M||K+M*Sigma||N||M||K
 =EqC(F2S,N,M,K)/epsilon^NP; Eq3:=(N,M,K)->N*Xi||N||M||K+
 M*Eta||N||M||K=EqC(F3S,N,M,K)/epsilon^NP; Eq4:=(N,M,K)->lambda
 *Sigma||N||M||K*Th(cosh(M*chi))-Eta||N||M||K*Th(sinh(M*chi));
Eq1s:=(N,M,K)->-K^2*Xi||N||M||K-N*Sigma||N||M||K=EqCA(F1S,N,M,K)
 /epsilon^NP;
Eq2s:=(N,M,K)->-K^2*Eta||N||M||K+M*Sigma||N||M||K
 =EqCA(F2S,N,M,K)/epsilon^NP; Eq3s:=(N,M,K)->N*Xi||N||M||K+
 M*Eta||N||M||K=EqCA(F3S,N,M,K)/epsilon^NP; Eq4s:=(N,M,K)->lambda
 *Sigma||N||M||K*Th(cosh(M*chi))-Eta||N||M||K*Th(sinh(M*chi));
```

Maple:

```
S1_NMK:=proc(x,N,M,K) local I_N,I_M,I_K; global I_T;
 I_N:=1/Pi*int(x*sin(N*alpha),alpha=0..2*Pi);
 if M=0 then I_M:=1/(2*Pi)*int(I_N*cos(M*z),z=0..2*Pi);
 else I_M:=1/Pi*int(I_N*cos(M*z),z=0..2*Pi); fi:
 if K=0 then I_K:=1/(2*Pi)*int(I_M*cos(K*psi),psi=0..2*Pi);
 else I_K:=1/Pi*int(I_M*cos(K*psi),psi=0..2*Pi); fi:
 I_T:=I_K/omega^2; RETURN(I_T); end:
S1A_NMK:=proc(x,N,M,K) local I_N,I_M,I_K; global I_T;
 I_N:=1/Pi*int(x*sin(N*alpha),alpha=0..2*Pi);
 if M=0 then I_M:=1/(2*Pi)*int(I_N*cos(M*z),z=0..2*Pi);
 else I_M:=1/Pi*int(I_N*cos(M*z),z=0..2*Pi); fi:
 I_K:=1/Pi*int(I_M*sin(K*psi),psi=0..2*Pi);
 I_T:=I_K/omega^2; RETURN(I_T); end:
S2_NMK:=proc(x,N,M,K) local I_N,I_M,I_K; global I_T;
 if N=0 then I_N:=1/(2*Pi)*int(x*cos(N*alpha),alpha=0..2*Pi);
 else I_N:=1/Pi*int(x*cos(N*alpha),alpha=0..2*Pi); fi:
 I_M:=1/(Pi*I)*int(I_N*sin(M*z),z=0..2*Pi);
 if K=0 then I_K:=1/(2*Pi)*int(I_M*cos(K*psi),psi=0..2*Pi);
 else I_K:=1/Pi*int(I_M*cos(K*psi),psi=0..2*Pi); fi:
 I_T:=I_K/omega^2; RETURN(I_T); end:
S2A_NMK:=proc(x,N,M,K) local I_N,I_M,I_K; global I_T;
 if N=0 then I_N:=1/(2*Pi)*int(x*cos(N*alpha),alpha=0..2*Pi);
 else I_N:=1/Pi*int(x*cos(N*alpha),alpha=0..2*Pi); fi:
 I_M:=1/(Pi*I)*int(I_N*sin(M*z),z=0..2*Pi);
 I_K:=1/Pi*int(I_M*sin(K*psi),psi=0..2*Pi);
 I_T:=I_K/omega^2; RETURN(I_T); end:
```

Maple:

```
S3_NMK:=proc(x,N,M,K) local I_N,I_M,I_K; global I_T;
 if N=0 then I_N:=1/(2*Pi)*int(x*cos(N*alpha),alpha=0..2*Pi);
 else I_N:=1/Pi*int(x*cos(N*alpha),alpha=0..2*Pi); fi:
 if M=0 then I_M:=1/(2*Pi)*int(I_N*cos(M*z),z=0..2*Pi);
 else I_M:=1/(Pi)*int(I_N*cos(M*z),z=0..2*Pi); fi:
 if K=0 then I_K:=1/(2*Pi)*int(I_M*cos(K*psi),psi=0..2*Pi);
 else I_K:=1/Pi*int(I_M*cos(K*psi),psi=0..2*Pi); fi:
 I_T:=I_K; RETURN(I_T); end:
S3A_NMK:=proc(x,N,M,K) local I_N,I_M,I_K; global I_T;
 if N=0 then I_N:=1/(2*Pi)*int(x*cos(N*alpha),alpha=0..2*Pi);
 else I_N:=1/Pi*int(x*cos(N*alpha),alpha=0..2*Pi); fi:
 if M=0 then I_M:=1/(2*Pi)*int(I_N*cos(M*z),z=0..2*Pi);
 else I_M:=1/(Pi)*int(I_N*cos(M*z),z=0..2*Pi); fi:
 I_K:=1/Pi*int(I_M*sin(K*psi),psi=0..2*Pi);
 I_T:=I_K; RETURN(I_T); end:
```

Maple:

```
EqC:=proc(Eq,N,M,K) local NEq,i; global SS; NEq:=nops(Eq); SS:=0:
 for i from 1 to NEq do
  if Eq=F1S then SS:=SS+S1_NMK(factor(op(i,Eq)),N,M,K):
  elif Eq=F2S then SS:=SS+S2_NMK(factor(op(i,Eq)),N,M,K):
  elif Eq=F3S then SS:=SS+S3_NMK(factor(op(i, Eq)),N,M,K): fi:
 od: RETURN(SS) end:
EqCA:=proc(Eq,N,M,K) local NEq,i; global SS; NEq:=nops(Eq);SS:=0:
  for i from 1 to NEq do
   if Eq=F1S then SS:=SS+S1A_NMK(op(i,Eq),N,M,K):
   elif Eq=F2S then SS:=SS+S2A_NMK(op(i,Eq),N,M,K):
   elif Eq=F3S then SS:=SS+S3A_NMK(op(i,Eq),N,M,K): fi:
  od: RETURN(SS) end:
Bc3_NK:=proc(x,N,K) local I_N,I_K; global I_T;
 if N=0 then I_N:=1/(2*Pi)*int(x*cos(N*alpha),alpha=0..2*Pi);
 else I_N:=1/Pi*int(x*cos(N*alpha),alpha=0..2*Pi); fi:
 if K=0 then I_K:=1/(2*Pi)*int(I_N*cos(K*psi),psi=0..2*Pi);
 else I_K:=1/Pi*int(I_N*cos(K*psi),psi=0..2*Pi); fi:
 I_T:=I_K; RETURN(I_T); end:
```

```
Bc3A_NK:=proc(x,N,K) local I_N,I_K; global I_T;
 if N=0 then I_N:=1/(2*Pi)*int(x*cos(N*alpha),alpha=0..2*Pi);
 else I_N:=1/Pi*int(x*cos(N*alpha),alpha=0..2*Pi); fi:
 I_K:=1/Pi*int(I_N*sin(K*psi),psi=0..2*Pi);
 I_T:=I_K; RETURN(I_T); end:
FORS:=proc(Syst) local i,nm; global SYST;
 nm:=nops(Syst): SYST:={}:
 for i from 1 to nm do SYST:=SYST union
  {cat(lhs(op(i,Syst)),9)=rhs(op(i,Syst))} od: RETURN(SYST) end:
```

Maple:

```
SetSol:={}:
for N from 0 to NA do for M from 0 to NA do for K from 0 to NA do
 SetSol:=SetSol union {Xi||N||M||K=0} union {Eta||N||M||K=0}
         union {Sigma||N||M||K=0}: od: od: od:
sys1:=[Eq1(2,0,0),Eq2(2,0,0),Eq3(2,0,0),Eq1(2,2,0),Eq2(2,2,0),
 Eq3(2,2,0),Eq4(2,0,0)+Eq4(2,2,0)=-Bc3_NK(Bc3,2,0)/epsilon^NP];
sys1:=convert(sys1,set) union {Eta200=0}: var1:={Xi200,Eta200,
 Sigma200,Xi220,Eta220,Sigma220}: DF1:=solve(sys1,var1);
sys2:=[Eq1(2,0,2),Eq2(2,0,2),Eq3(2,0,2),Eq1(2,2,2),Eq2(2,2,2),
 Eq3(2,2,2),Eq4(2,0,2)+Eq4(2,2,2)=-Bc3_NK(Bc3,2,2)/epsilon^NP];
sys2:=convert(sys2,set) union {Eta202=0}: var2:={Xi202,Eta202,
 Sigma202,Xi222,Eta222,Sigma222}: DF2:=solve(sys2,var2);
sys3:=[Eq1(0,2,0),Eq2(0,2,0),Eq3(0,2,0),Eq1(0,0,0),Eq2(0,0,0),
 Eq3(0,0,0),Eq4(0,2,0)+Eq4(0,0,0)=-Bc3_NK(Bc3,0,0)/epsilon^NP];
sys3:=convert(sys3,set) union {Eta000=0,Xi020=0,Xi000=0}:
var3:={Xi020,Eta020,Sigma020,Xi000,Eta000,Sigma000}: DF3:=solve(
 sys3,var3); sys4:=[Eq1(0,2,2),Eq2(0,2,2),Eq3(0,2,2),Eq1(0,0,2),
 Eq2(0,0,2),Eq3(0,0,2),Eq4(0,2,2)+Eq4(0,0,2)=-Bc3_NK(Bc3,0,2)/
 epsilon^NP]; sys4:=convert(sys4,set) union {Eta002=0}:
var4:={Xi022,Eta022,Sigma022,Xi002,Eta002,Sigma002}:
DF4:=solve(sys4,var4); C_2:=0: S_2:=0:
sys5:=[Eq1(1,1,1),Eq2(1,1,1),Eq3(1,1,1),Eq4(1,1,1)=
 -Bc3_NK(Bc3,1,1)/epsilon^N]; sys5:=convert(sys5,set) union
 {Sigma111=C_2}: var5:={Xi111,Eta111,Sigma111,B||NP}: DF5:=
 solve(sys5,var5); sys6:=[Eq1s(1,1,1),Eq2s(1,1,1),Eq3s(1,1,1),
 Eq4s(1,1,1)=-Bc3A_NK(Bc3,1,1)/epsilon^N]; sys6:=convert(sys6,
 set) union {Sigma111=S_2}: var6:={Xi111,Eta111,Sigma111,A||NP}:
```

```
DF6S:=solve(sys6,var6); for i from 1 to nops(DF6S) do
 if has(op(i,DF6S),A||NP) then ZZ:=i fi: od: DF6s:=subsop(
 ZZ=NULL,DF6S): DF6:=FORS(DF6s); sys7:=[Eq1s(2,0,0),Eq2s(2,0,0),
 Eq3s(2,0,0),Eq1s(2,2,0),Eq2s(2,2,0),Eq3s(2,2,0),Eq4s(2,0,0)+
 Eq4s(2,2,0)=-Bc3A_NK(Bc3,2,0)/epsilon^NP]; sys7:=convert(sys7,
 set): var7:={Xi200,Eta200,Sigma200,Xi220,Eta220,Sigma220}:
DF7S:=solve(sys7,var7); DF7:=FORS(DF7S);
sys8:=[Eq1s(2,0,2),Eq2s(2,0,2),Eq3s(2,0,2),Eq1s(2,2,2),
 Eq2s(2,2,2),Eq3s(2,2,2),Eq4s(2,0,2)+Eq4s(2,2,2)=
 -Bc3A_NK(Bc3,2,2)/epsilon^NP]; sys8:=convert(sys8,set)
 union {Eta202=0}: var8:={Xi202,Eta202,Sigma202,Xi222,Eta222,
 Sigma222}: DF8S:=solve(sys8,var8); DF8:=FORS(DF8S);
```

Maple:

```
FF:=proc(x::list) global Xi_S,Eta_S,Sigma_S,NA; local i,Nx,a,b,
 c,s,N,M,K; Nx:=nops(x);  Xi_S:={}: Eta_S:={}: Sigma_S:={}:
 for i from 1 to Nx do a:=lhs(x[i]); for N from 0 to NA do
 for M from 0 to NA do for K from 0 to NA do
 if a=Xi||N||M||K then XiS||N||M||K:=rhs(x[i])*sin(N*alpha)
 *cosh(M*(beta+chi))*cos(K*psi): Xi_S:=Xi_S union {XiS||N||M||K};
 elif a=Eta||N||M||K then EtaS||N||M||K:=rhs(x[i])*cos(N*alpha)
 *sinh(M*(beta+chi))*cos(K*psi): Eta_S:=Eta_S union
 {EtaS||N||M||K};
 elif a=Sigma||N||M||K then SigmaS||N||M||K:=rhs(x[i])
 *cos(N*alpha)*cosh(M*(beta+chi))*cos(K*psi):
 Sigma_S:=Sigma_S union {SigmaS||N||M||K}; fi:
 if a=Xi||N||M||K||9 then XiS||N||M||K:=rhs(x[i])*sin(N*alpha)
 *cosh(M*(beta+chi))*sin(K*psi): Xi_S:=Xi_S union {XiS||N||M||K};
 elif a=Eta||N||M||K||9 then EtaS||N||M||K:=rhs(x[i])
 *cos(N*alpha)*sinh(M*(beta+chi))*sin(K*psi):
 Eta_S:=Eta_S union {EtaS||N||M||K};
 elif a=Sigma||N||M||K||9 then SigmaS||N||M||K:=rhs(x[i])
 *cos(N*alpha)*cosh(M*(beta+chi))*sin(K*psi):
 Sigma_S:=Sigma_S union {SigmaS||N||M||K}; fi: od: od: od: od:
RETURN(Xi_S, Eta_S, Sigma_S); end:
SetSol:=convert({op(SetSol)} union {'op(DF||i)'$'i'=1..8},list):
Xi||NA:=convert(FF(SetSol)[1],`+`); Eta||NA:=convert(
 FF(SetSol)[2],`+`); Sigma||NA:=convert(FF(SetSol)[3],`+`);
 Xi||NP:=-sin(alpha)*cosh(beta+chi)/Th(sinh(chi))*C*cos(psi);
Eta||NP:=cos(alpha)*sinh(beta+chi)/Th(sinh(chi))*C*cos(psi);
Sigma||NP:=cos(alpha)*cosh(beta+chi)/Th(sinh(chi))*C*cos(psi);
```

```
XiSer:=Xi||NP+epsilon^NP*(Xi||NA); EtaSer:=Eta||NP+epsilon^NP
 *(Eta||NA); SigmaSer:=Sigma||NP+epsilon^NP*(Sigma||NA);
AN:=nops(DF6S): for j from 1 to AN do if lhs(op(j,DF6S))=A||NP
 then AA||NP:=rhs(op(j,DF6S)): fi: od: BN:=nops(DF5):
for j from 1 to BN do if lhs(op(j,DF5))=B||NP then BB||NP:=
 rhs(op(j,DF5)): fi: od: combine(AA||NP), combine(BB||NP);
```

Maple:

```
with(plots): Digits:=30: tr1:=th=tanh(chi);
xiS:=subs(tr1,op(1,XiSer)+epsilon^(1/4)*op(2,op(2,XiSer)));
etaS:=subs(tr1,op(1,EtaSer)+epsilon^(1/4)*op(2,op(2,EtaSer)));
beta:=0: alpha:=a*kappa: rho:=1: psi:=OmegaT/2+Pi: g:=981.7:
n:=2: L:=10.5: w:=1.7: h:=15: s:=0.11: Omega:=44.647;
kappa:=evalf((Pi*n)/L): chi:=evalf(h*kappa):
lambda:=evalf(tanh(chi)): omega:=evalf(sqrt(lambda*g*kappa)):
Lambda:=evalf((2*Pi)/kappa): epsilon:=evalf(s*Omega^2/g):
phi2:=subs(z=lambda,(2*z^6+3*z^4+12*z^2-9)/(64*z^4));
BB2:=-omega*phi2*C^2; Delta:=evalf(omega-Omega/2);
EqC:=Delta+BB2*epsilon^(1/2)+(1/4)*epsilon*omega;
C:=fsolve(EqC,C=0..100); C2:=epsilon^(1/4)*C;
y2:=epsilon^(1/4)*etaS/kappa; x2:=a+epsilon^(1/4)*xiS/kappa;
X2:=unapply(x2,[a,OmegaT]); Y2:=unapply(y2,[a,OmegaT]);
animate([X2(a,t),Y2(a,t),a=0..L],t=0..40,color=blue,
 thickness=4,scaling=constrained,frames=50);
Y2Max:=Y2(L/2,0); Y2Min:=Y2(L/2,2*Pi); DeltaY:=(Y2Max-Y2Min)/2;
HL:=(Y2Max-Y2Min)/Lambda; lprint(`Omega=`,evalf(Omega/(2*Pi)),
 `2Pi Omega=`,Omega,`h=`,h,`s=`,s); lprint(`C2=`,C2,`HL=`,HL,
 `Y2Max=`,Y2Max); lprint(`DeltaY=`,DeltaY);
X2mx:=subs(OmegaT=0,x2): Y2mx:=subs(OmegaT=0,y2):
Mmx:=plot([X2mx,Y2mx,a=0..L],color=blue,thickness=2):
display(Mmx);
```

□

Chapter 6
Numerical Approach

6.1 Embedded Numerical Methods

6.1.1 Nonlinear PDEs

Problem 6.1

Maple:

```
with(plots): with(PDEtools): declare(u(x,t)); nu:=0.009; A:=0.4:
S:=1/100; tR:=0..40; xR:=0..1; NF:=30; NP:=100; N:=3; L:=1;
L1:=[red,blue,green]; L2:=[0,1/2,1]; Ops:=spacestep=S,timestep=S;
Op1:=frames=NF,numpoints=NP; f:=x->A*sin(Pi*x/L);
PDE1:=diff(u(x,t),t)-nu*diff(u(x,t),x$2)=0;
PDE2:=diff(u(x,t),t)-nu*diff(u(x,t),x$2)+u(x,t)*diff(u(x,t),x)=0;
IC:={u(x,0)=f(x)}; BC:={u(0,t)=0,u(L,t)=0};
for i from 1 to 2 do
 Sol||i:=pdsolve(PDE||i,IC union BC,numeric,u(x,t),Ops); od;
for i from 1 to N do for j from 1 to 2 do
 G||j||i:=Sol||j:-plot(t=L2[i],color=L1[i],numpoints=200): od:od:
display({seq(G1||i,i=1..N)}); display({seq(G2||i,i=1..N)});
for i from 1 to 2 do
 Num_vals||i:=Sol||i:-value(); Num_vals||i(1/2,Pi);
 Sol||i:-plot3d(u(x,t),t=tR,shading=zhue,axes=boxed);
 Sol||i:-animate(u(x,t),x=xR,t=tR,Op1,thickness=3); od;
```

Mathematica:

```
f[x_]:=a*Sin[Pi*x/l]; {nu=0.009,a=0.4,s=0.01,nP=100,nN=3,l=1,
 l1={Red,Blue,Green},l2={0,1/2,1}}
SetOptions[Plot,ImageSize->500,PlotRange->All,PlotPoints->nP*2,
 PlotStyle->{Blue,Thickness[0.01]}];
{pde[1]=D[u[x,t],t]-nu*D[u[x,t],{x,2}]==0,
 pde[2]=D[u[x,t],t]-nu*D[u[x,t],{x,2}]+u[x,t]*D[u[x,t],x]==0,
 ic={u[x,0]==f[x]},bc={u[0,t]==0,u[l,t]==0}}
Do[sol[i]=NDSolve[{pde[i],ic,bc},u,{x,0,1},{t,0,40},
 MaxStepSize->s],{i,1,2}]; Do[g[j,i]=Plot[Evaluate[u[x,l2[[i]]]/.
 sol[j]],{x,0,1},PlotStyle->{l1[[i]],Thickness[0.01]}],
 {j,1,2},{i,1,nN}]; GraphicsRow[{Show[Table[g[1,i],{i,1,nN}]],
 Show[Table[g[2,i],{i,1,nN}]]}]
Do[numVals[i]=Evaluate[u[1/2,Pi]/.sol[i]]; Print[numVals[i]];
 g3D[i]=Plot3D[Evaluate[u[x,t]/.sol[i]],{x,0,1},{t,0,40},
 ColorFunction->Function[{x,y},Hue[x]],BoxRatios->1,ViewPoint->
 {-1,2,1},ImageSize->900]; gCP[i]=ContourPlot[Evaluate[
  u[x,t]/.sol[i]],{x,0,1},{t,0,40},ColorFunction->Hue,
  ImageSize->300],{i,1,2}]; GraphicsRow[{g3D[1],g3D[2]}]
GraphicsRow[{gCP[1],gCP[2]}]
Animate[Row[{Plot[Evaluate[u[x,t]/.sol[1],{x,0,1}],PlotRange->
 {0,0.4}], Plot[Evaluate[u[x,t]/.sol[2],{x,0,1}],PlotRange->
 {0,0.4}]}],{t,0,40},AnimationRate->0.5]
```

Maple:

```
N1:=1; N2:=50; tk:=1: X:=(A*L)/(2*Pi*nu); q:=Pi*x/L;
p:=nu*Pi^2*t/L^2; GrF:=(x,xi,t,N)->2/L*add(sin(n*q)
 *sin(n*Pi*xi/L)*exp(-p*n^2),n=1..N1);
SolAn1:=unapply(int(expand(f(xi)*GrF(x,xi,t,N)),xi=0..L),x,t);
A0:=evalf(1/L*int(exp(-X*(1-cos(q))),x=0..L));
An:=2/L*int(exp(-X*(1-cos(q)))*cos(n*q),x=0..L);
phi:=A0+add(evalf(An)*exp(-n^2*p)*cos(n*q),n=1..N2);
SolAn2:=unapply(-2*nu*diff(phi,x)/phi,x,t);
for i from 1 to 2 do
 G1||i:=Sol||i:-plot(t=tk,color=red,numpoints=2*NP):
 G2||i:=plot(SolAn||i(x,tk),x=xR,color=blue): od:
display({G11,G21}); display({G12,G22});
```

Mathematica:

```
{n1=1,n2=50,tk=1,xN=(a*l)/(2*Pi*nu),q=Pi*x/l,p=nu*Pi^2*t/l^2}
grF[x_,xi_,t_,nN]:=2/l*Sum[Sin[n*q]*Sin[n*Pi*xi/l]*
 Exp[-p*n^2],{n,1,nN}];
solAn[1][x1_,t1_]:=Integrate[Expand[f[xi]*grF[x,xi,t,nN]],
 {xi,0,l}]/.{x->x1,t->t1}; solAn[1][x,t]
a0=1/l*NIntegrate[Exp[-xN*(1-Cos[q])],{x,0,l}]
an=2/l*Hold[NIntegrate[Exp[-xN*(1-Cos[q])]*Cos[n*q],{x,0,l},
 AccuracyGoal->6]]
phi=a0+Sum[Re[ReleaseHold[an]]*Exp[-n^2*p]*Cos[n*q],{n,1,n2}]
solAn[2][x1_,t1_]:=-2*nu*D[phi,x]/phi/.x->x1/.t->t1;
solAn[2][x,t]
Do[g1[i]=Plot[Evaluate[u[x,tk]/.sol[i]],{x,0,l},PlotStyle->
 {Red,Thickness[0.01]},PlotPoints->nP*2];
   g2[i]=Plot[Evaluate[solAn[i][x,tk]],{x,0,l},PlotStyle->
 {Blue,Thickness[0.005]}],{i,1,2}]; Show[{g1[1],g2[1]}]
Show[{g1[2],g2[2]}]
```

Maple:

```
ErFun2:=rhs(Sol2:-value(abs(u(x,t)-SolAn2(x,t)),t=tk,output=
 listprocedure)[3]); seq(ErFun2(n/10),n=0..10);
utk:=Sol2:-value(t=tk,output=listprocedure);
uVal:=rhs(op(3,utk)); plot(uVal(x),x=xR);
for x from 0 to L by 0.1 do print(uVal(x),evalf(SolAn2(x,tk)),
      abs(uVal(x)-evalf(SolAn2(x,tk)))); od;
```

Mathematica:

```
uVal[x1_]:=Evaluate[u[x,tk]/.sol[2]]/.{x->x1}; Plot[Evaluate[
 u[x,tk]/.sol[2]],{x,0,l},PlotStyle->{Red,Thickness[0.01]},
 PlotPoints->nP*2]
PaddedForm[Table[{First[uVal[x1]],solAn[2][x1,tk]//N,Abs[First[
 uVal[x1]]-solAn[2][x1,tk]//N]},{x1,0,l,0.1}]//TableForm,
 {12,5}]
```

□

Problem 6.2

Maple:

```
with(VectorCalculus): with(plots): with(PDEtools):
declare(u(x,t)); c:=evalf(1/(4*Pi)): lambda:=1: L:=0.5; T:=1.5;
S:=1/100; tR:=0..T; xR:=0..L; NF:=30; NP:=100; Ops:=spacestep=S,
 timestep=S; Op1:=frames=NF,numpoints=NP,thickness=3; N:=3;
L1:=[red,blue,green];L2:=[0.3,0.7,1.5]; F:=u->exp(lambda*u(x,t));
PDE1:=diff(u(x,t),t$2)-c^2*Laplacian(u(x,t),'cartesian'[x])=0;
PDE2:=diff(u(x,t),t$2)=c^2*diff(u(x,t),x$2)+F(u);
Ics:={u(x,0)=0,D[2](u)(x,0)=sin(4*Pi*x)};
Bcs:={u(0,t)=0,u(L,t)=0};
for i from 1 to 2 do
 Sol||i:=pdsolve(PDE||i,Ics union Bcs,numeric,u(x,t),Ops); od;
for i from 1 to N do for j from 1 to 2 do
 G||j||i:=Sol||j:-plot(t=L2[i],color=L1[i],numpoints=NP*2): od:
od: display({seq(G1||i,i=1..N)}); display({seq(G2||i,i=1..N)});
for i from 1 to 2 do
 Num_vals||i:=Sol||i:-value(); Num_vals||i(0.1,1/2);
 Sol||i:-plot3d(u(x,t),t=tR,shading=zhue,axes=boxed);
 Sol||i:-animate(u(x,t),x=xR,t=tR,Op1); od;
```

Mathematica:

```
<<DifferentialEquations`InterpolatingFunctionAnatomy`
SetOptions[Plot,ImageSize->500,PlotRange->{All,{-5,5}},
 PlotPoints->nP*2,PlotStyle->{Blue,Thickness[0.01]}];
SetOptions[Plot3D,ImageSize->500,PlotRange->All]; c=N[1/(4*Pi)];
lambda=1; f[u_]:=Exp[lambda*u]; {l=0.5,tF=1.5,s=1/100,
 nP=100,nN=3,l1={Red,Blue,Green},l2={0.3,0.7,1.5}}
pde[1]=D[u[x,t],{t,2}]-c^2*D[u[x,t],{x,2}]==0
pde[2]=D[u[x,t],{t,2}]-c^2*D[u[x,t],{x,2}]-f[u[x,t]]==0
ic={u[x,0]==0,(D[u[x,t],t]/.t->0)==Sin[4*Pi*x]}
bc={u[0,t]==0,u[l,t]==0}
Do[sol[j]=NDSolve[Flatten[{pde[j],ic,bc}],u,{x,0,l},{t,0,tF},
 MaxStepSize->s,PrecisionGoal->2];
 f[j]=u/.First[sol[j]],{j,1,2}];
Map[Length,InterpolatingFunctionCoordinates[f[2]]]
Do[g[j,i]=Plot[Evaluate[u[x,l2[[i]]]/.sol[j]],{x,0,l},
 PlotStyle->{l1[[i]],Thickness[0.01]}],{j,1,2},{i,1,nN}];
GraphicsRow[{Show[Table[g[1,i],{i,1,nN}]],
 Show[Table[g[2,i],{i,1,nN}]]}]
```

```
Do[numVals[i]=Evaluate[u[0.1,1/2]/.sol[i]]; Print[numVals[i]];
 g3D[i]=Plot3D[Evaluate[u[x,t]/.sol[i]],{x,0,l},{t,0,tF},
 ColorFunction->Function[{x,y},Hue[x]],BoxRatios->1,
 ViewPoint->{-1,2,1},ImageSize->500]; gCP[i]=ContourPlot[
 Evaluate[u[x,t]/.sol[i]],{x,0,l},{t,0,tF},ColorFunction->Hue,
 ImageSize->300],{i,1,2}]; GraphicsRow[{g3D[1],g3D[2]}]
GraphicsRow[{gCP[1],gCP[2]}]
Animate[Row[{Plot[Evaluate[u[x,t]/.sol[1],{x,0,l}],PlotRange->
 {-1,5}],Plot[Evaluate[u[x,t]/.sol[2],{x,0,l}],PlotRange->
 {-1,5}]}],{t,0,tF,0.001},AnimationRate->0.1]
```

□

Problem 6.3

Maple:

```
with(plots): with(PDEtools): declare(u(x,t));
S:=1/100; NF:=30; NP:=100; N:=3; L:=1; L1:=[red,blue,green];
L2:=[0,1/2,1]; Ops:=spacestep=S,timestep=S; Ops1:=frames=NF,
 numpoints=NP; Ops2:=grid=[20,20],shading=zhue,axes=boxed;
F:=u->b*u(x,t)-c*u(x,t)^3; F(u);
KG:=diff(u(x,t),t$2)-a^2*diff(u(x,t),x$2)+F(u)=0;
IC:=(f1,f2)->{u(x,0)=f1,D[2](u)(x,0)=f2};
BC:=(c,f1,d,f2)->{D[1](u)(c,t)=f1,D[1](u)(d,t)=f2};
```

Mathematica:

```
SetOptions[Plot,ImageSize->500,PlotRange->All,PlotPoints->
 nP*2,PlotStyle->{Blue,Thickness[0.01]}]; {s=1/100,nP=100}
{nN=3,l=1,l1={Red,Blue,Green},l2={0,1/2,1}}
f[u_]:=b*u[x,t]-c*u[x,t]^3; f[u]
eKG=D[u[x,t],{t,2}]-a^2*D[u[x,t],{x,2}]+f[u]==0
fIC1[f1_]:=u[x,0]==f1; fIC2[f2_]:=(D[u[x,t],t]/.t->0)==f2;
fBC1[c_,f1_]:=(D[u[x,t],x]/.x->c)==f1;
fBC2[d_,f2_]:=(D[u[x,t],x]/.x->d)==f2;
{fIC1[f1],fIC2[f2],fBC1[c,f1],fBC2[d,f2]}
```

Maple:

```
params2:={a=0.1,b=0.1,c=1,lambda=0.3}; c2:=-5; d2:=5; tR2:=0..4;
xR2:=c2..d2; k:=sqrt(a/(2*(lambda^2-a^2))); SolEx2:=sqrt(a/c)*
 tanh(k*(x-lambda*t)); f12:=eval(SolEx2,t=0); f22:=eval(diff(
 SolEx2,t),t=0); f32:=eval(diff(SolEx2,x),x=c2); f42:=eval(
 diff(SolEx2,x),x=d2); KG2:=evalf(subs(params2,KG));
IC2:=evalf(subs(params2,IC(f12,f22)));
BC2:=evalf(subs(params2,BC(c2,f32,d2,f42)));
Sol2:=pdsolve(KG2,IC2 union BC2,numeric,u(x,t),Ops);
for i from 1 to N do G||i:=Sol2:-plot(t=L2[i],color=L1[i],
 numpoints=NP*2): od: display({seq(G||i,i=1..N)});
Sol2:-plot3d(u(x,t),t=tR2,Ops2,orientation=[95,55]);
Sol2:-animate(u(x,t),x=xR2,t=tR2,Ops1,thickness=3);
```

Mathematica:

```
{params2={a->0.1,b->0.1,c->1,lambda->0.3},c2=-5,d2=5,tF2=4,
 xI2=c2,xF2=d2,k=Sqrt[a/(2*(lambda^2-a^2))]}
{solEx2=Sqrt[a/c]*Tanh[k*(x-lambda*t)], f12=solEx2/.t->0,
 f22=D[solEx2,t]/.t->0, f32=D[solEx2,x]/.x->c2,
 f42=D[solEx2,x]/.x->d2}
{eKG2=N[eKG/.params2], ic2=N[{fIC1[f12],fIC2[f22]}/.
 params2], bc2=N[{fBC1[c2,f32],fBC2[d2,f42]}/.params2]}
sol2=NDSolve[Flatten[{eKG2,ic2,bc2}],u,{x,xI2,xF2},{t,0,tF2},
 MaxStepSize->s,PrecisionGoal->2]
Do[g[i]=Plot[Evaluate[u[x,l2[[i]]]/.sol2],{x,xI2,xF2},
 PlotStyle->{l1[[i]],Thickness[0.01]}],{i,1,nN}];
Show[Table[g[i],{i,1,nN}]]
Plot3D[Evaluate[u[x,t]/.sol2],{x,xI2,xF2},{t,0,tF2},
 ColorFunction->Function[{x,y},Hue[x]],BoxRatios->1,ViewPoint->
 {-1,2,1},ImageSize->500]
Animate[Plot[Evaluate[u[x,t]/.sol2,{x,xI2,xF2}],PlotRange->
 {-0.5,0.5}],{t,0,tF2},AnimationRate->0.5]
```

Maple:

```
params3:={a=0.3,b=0.3,c=1,lambda=0.25}; c3:=-5; d3:=5;
tR3:=0..4; xR3:=c3..d3; k:=sqrt(a/(a^2-lambda^2));
SolEx3:=sqrt(2*a/c)*sech(k*(x-lambda*t));
f13:=eval(SolEx3,t=0); f23:=eval(diff(SolEx3,t),t=0);
f33:=eval(diff(SolEx3,x),x=c3); f43:=eval(diff(SolEx3,x),x=d3);
KG3:=evalf(subs(params3,KG));
```

```
IC3:=evalf(subs(params3,IC(f13,f23)));
BC3:=evalf(subs(params3,BC(c3,f33,d3,f43)));
Sol3:=pdsolve(KG3,IC3 union BC3,numeric,u(x,t),Ops);
for i from 1 to N do
 G||i:=Sol3:-plot(t=L2[i],color=L1[i],numpoints=NP*2): od:
display({seq(G||i,i=1..N)}); Sol3:-plot3d(u(x,t),t=tR3,Ops2);
Sol3:-animate(u(x,t),x=xR3,t=tR3,Ops1,thickness=3);
```

Mathematica:

```
{params3={a->0.3,b->0.3,c->1,lambda->0.25},c3=-5,d3=5,tF3=4,
 xI3=c3,xF3=d3,k=Sqrt[a/(a^2-lambda^2)]}
{solEx3=Sqrt[2*a/c]*Sech[k*(x-lambda*t)],f13=solEx3/.t->0,
 f23=D[solEx3,t]/.t->0,f33=D[solEx3,x]/.x->c3, f43=D[solEx3,x]/.
 x->d3, eKG3=N[eKG/.params3], ic3=N[{fIC1[f13],fIC2[f23]}/.
 params3], bc3=N[{fBC1[c3,f33],fBC2[d3,f43]}/.params3]}
sol3=NDSolve[Flatten[{eKG3,ic3,bc3}],u,{x,xI3,xF3},{t,0,tF3},
 MaxStepSize->s,PrecisionGoal->2]
Do[g[i]=Plot[Evaluate[u[x,l2[[i]]]/.sol3],{x,xI3,xF3},PlotStyle->
 {l1[[i]],Thickness[0.01]}],{i,1,nN}]; Show[Table[g[i],{i,1,nN}]]
Plot3D[Evaluate[u[x,t]/.sol3],{x,xI3,xF3},{t,0,tF3},
 ColorFunction->Function[{x,y},Hue[x]],BoxRatios->1,ViewPoint->
 {1,2,1},PlotRange->All,PlotPoints->{20,20},ImageSize->500]
Animate[Plot[Evaluate[u[x,t]/.sol3,{x,xI3,xF3}],PlotRange->
 {0,1}],{t,0,tF3},AnimationRate->0.5]
```

Maple:

```
params4:={a=0.3,b=0.3,c=1,lambda[1]=0.25,lambda[2]=-0.25,x0[1]=
 -2,x0[2]=2}; for i to 2 do k[i]:=sqrt(a/(a^2-lambda[i]^2)); od:
c4:=-5; d4:=5; tR4:=0..7; xR4:=c4..d4;
SolEx4:=sqrt(2*a/c)*add(sech(k[i]*(x-lambda[i]*t-x0[i])),i=1..2);
f14:=eval(SolEx4,t=0); f24:=eval(diff(SolEx4,t),t=0);
f34:=eval(diff(SolEx4,x),x=c4); f44:=eval(diff(SolEx4,x),x=d4);
KG4:=evalf(subs(params4,KG)); IC4:=evalf(subs(params4,
 IC(f14,f24))); BC4:=evalf(subs(params4,BC(c4,f34,d4,f44)));
Sol4:=pdsolve(KG4,IC4 union BC4,numeric,u(x,t),Ops);
for i from 1 to N do
 G||i:=Sol4:-plot(t=L2[i],color=L1[i],numpoints=NP*2): od:
display({seq(G||i,i=1..N)}); Sol4:-plot3d(u(x,t),t=tR4,Ops2);
Sol4:-animate(u(x,t),x=xR4,t=tR4,Ops1,thickness=3);
```

Mathematica:

```
params4={a->0.3,b->0.3,c->1,lambdaN[1]->0.25,lambdaN[2]->-0.25,
 x0[1]->-2,x0[2]->2}; Do[kN[i]=Sqrt[a/(a^2-lambdaN[i]^2)],
 {i,1,2}]; {c4=-5,d4=5,tF4=7,xI4=c4,xF4=d4}
{solEx4=Sqrt[2*a/c]*Sum[Sech[kN[i]*(x-lambdaN[i]*t-x0[i])],
 {i,1,2}], f14=solEx4/.t->0,f24=D[solEx4,t]/.t->0,
 f34=D[solEx4,x]/.x->c4, f44=D[solEx4,x]/.x->d4,
 eKG4=N[eKG/.params4],ic4=N[{fIC1[f14], fIC2[f24]}/.params4],
 bc4=N[{fBC1[c4,f34],fBC2[d4,f44]}/.params4]}
sol4=NDSolve[Flatten[{eKG4,ic4,bc4}],u,{x,xI4,xF4},{t,0,tF4},
 MaxStepSize->s,PrecisionGoal->2]
Do[g[i]=Plot[Evaluate[u[x,l2[[i]]]/.sol4],{x,xI4,xF4},
 PlotStyle->{l1[[i]],Thickness[0.01]}],{i,1,nN}];
Show[Table[g[i],{i,1,nN}]]
Plot3D[Evaluate[u[x,t]/.sol4],{x,xI4,xF4},{t,0,tF4},
 ColorFunction->Function[{x,y},Hue[x]],BoxRatios->1,ViewPoint->
 {1,2,1},PlotRange->All,PlotPoints->{20,20},ImageSize->500]
Animate[Plot[Evaluate[u[x,t]/.sol4,{x,xI4,xF4}],PlotRange->
 {0,1.8}],{t,0,tF4},AnimationRate->0.5]
```

Maple:

```
params5:={a=1,b=1,c=-1,A=1.5}; c5:=-5; d5:=5; tR5:=0..4;
xR5:=c5..d5; f15:=A*(1+cos(2*Pi*x/d5)); f25:=0; f35:=0;
f45:=0; KG5:=evalf(subs(params5,KG)):
IC5:=evalf(subs(params5,IC(f15,f25))):
BC5:=evalf(subs(params5,BC(c5,f35,d5,f45)));
Sol5:=pdsolve(KG5,IC5 union BC5,numeric,u(x,t),Ops);
for i from 1 to N do
 G||i:=Sol5:-plot(t=L2[i],color=L1[i],numpoints=NP): od:
 display({seq(G||i,i=1..N)}); Sol5:-plot3d(u(x,t),t=tR5,Ops2);
Sol5:-animate(u(x,t),x=xR5,t=tR5,Ops1,thickness=3);
```

Mathematica:

```
params5={a->1,b->1,c->-1,aN->1.5}; {c5=-5,d5=5,tF5=4,
 xI5=c5,xF5=d5,f15=aN*(1+Cos[2*Pi*x/d5]),f25=0,f35=0,f45=0,
 eKG5=N[eKG/.params5], ic5=N[{fIC1[f15],fIC2[f25]}/.params5],
 bc5=N[{fBC1[c5,f35],fBC2[d5,f45]}/.params5]}
```

```
sol5=NDSolve[Flatten[{eKG5,ic5,bc5}],u,{x,xI5,xF5},{t,0,tF5},
 MaxStepSize->s,PrecisionGoal->2]
Do[g[i]=Plot[Evaluate[u[x,l2[[i]]]/.sol5],{x,xI5,xF5},PlotStyle->
 {l1[[i]],Thickness[0.01]}],{i,1,nN}]; Show[Table[g[i],{i,1,nN}]]
Plot3D[Evaluate[u[x,t]/.sol5],{x,xI5,xF5},{t,0,tF5},
 ColorFunction->Function[{x,y},Hue[x]],BoxRatios->1,ViewPoint->
 {1,2,1},PlotRange->All,PlotPoints->{20,20},ImageSize->500]
Animate[Plot[Evaluate[u[x,t]/.sol5,{x,xI5,xF5}],PlotRange->
 {-3,3}],{t,0,tF5},AnimationRate->0.5]
```
□

6.1.2 Specifying Classical Numerical Methods

Problem 6.4

Maple:

```
with(PDEtools): with(plots): declare(u(x,t)); NF:=30; NP:=100;
xR:=0..1; tR:=0..1; S:=1/100; Ops:=timestep=S,spacestep=S;
N:=3; L1:=[red,blue,magenta]; L2:=[0,0.15,0.3];
PDE1:=diff(u(x,t),t)+diff(u(x,t),x)*u(x,t)=0;
f:=x->exp(-10*(4*x-2)^2)/4; IBC:={u(x,0)=f(x),u(0,t)=exp(-10)/4};
M1:=ForwardTime1Space[backward];
Sol1:=pdsolve(PDE1,IBC,numeric,time=t,range=xR,method=M1,Ops);
Num_vals1:=Sol1:-value(); Num_vals1(0,0.5);
for i from 1 to N do
 G||i:=Sol1:-plot(t=L2[i],color=L1[i],numpoints=NP*2): od:
display({seq(G||i,i=1..N)}); Ops1:=frames=NF,numpoints=NP;
Sol1:-animate(u(x,t),x=xR,t=tR,Ops1,thickness=3);
```
□

Problem 6.5

Maple:

```
with(PDEtools): with(plots): declare(u(x,t)); NF:=30; NP:=100;
xR:=-1..1; tR:=0..5; S:=1/100; Ops:=timestep=S,spacestep=S;
N:=2; L1:=[red,blue]; L2:=[0,1/2]; nu:=0.1; NBCs:=[Box,n];
PDE1:=diff(u(x,t),t)=nu*diff(u(x,t),x)*u(x,t)^3;
IBC:={u(x,0)=cos(Pi*x/2),u(-1,t)=u(1,t)}; M1:=CrankNicholson;
Sol1:=pdsolve(PDE1,IBC,numeric,numericalbcs=NBCs,method=M1,Ops);
for i from 1 to N do
 G||i:=Sol1:-plot(t=L2[i],color=L1[i],numpoints=NP): od:
display({seq(G||i,i=1..N)}); Ops1:=frames=NF,numpoints=NP;
Sol1:-animate(u(x,t),x=xR,t=tR,Ops1,thickness=3);
```
□

Problem 6.6

Maple:

```
with(PDEtools): with(plots): declare(u(x,t));
NF:=30; NP:=100; xR:=-10..10; tR:=0..5; S:=1/50;
Ops:=timestep=S,spacestep=S;
N:=3; L1:=[red,blue,magenta]; L2:=[0,0.4,0.5];
PDE1:=diff(u(x,t),t)+u(x,t)*diff(u(x,t),x)=0;
f:=x->piecewise(x<-1,0,x>=-1 and x<=0,x+1,1);
IBC:={u(x,0)=f(x),(D[1](u))(-10,t)=0}; M1:=DuFortFrankel;
Sol1:=pdsolve(PDE1,IBC,type=numeric,time=t,range=xR,
      numericalbcs=u[1,n]-u[1,n-1],method=M1,startup=Euler,Ops);
for i from 1 to N do
 G||i:=Sol1:-plot(t=L2[i],color=L1[i],numpoints=NP,thickness=2):
od: display({seq(G||i,i=1..N)}); Ops1:=frames=NF,numpoints=NP;
Sol1:-animate(u(x,t),x=xR,t=tR,Ops1,thickness=3);
```

□

6.1.3 Nonlinear Systems

Problem 6.7

Maple:

```
Digits:=30: with(PDEtools):  with(plots): Ops1:=numpoints=100:
Ops2:=color=magenta: Ops3:=color=blue: Ops4:=color="BlueViolet":
Ops5:=axes=boxed,shading=zhue,orientation=[40,50]; a:=0:
b:=1: Tf:=0.5; U,V:=diff_table(u(x,t)),diff_table(v(x,t)):
sys1:={U[t]=V[]*U[x]+U[]+1,V[t]=-U[]*V[x]-V[]+1};
IBC1:={u(x,0)=exp(-x),v(x,0)=exp(x),u(1,t)=exp(t-1),
 v(0,t)=exp(-t)}; S:=1/100; Ops:=spacestep=S,timestep=S;
Sol1:=pdsolve(sys1,IBC1,[u,v],numeric,time=t,range=a..b,Ops);
L1:=[0.1,0.2,0.5]; NL1:=nops(L1); for i from 1 to NL1 do
 G||i:=Sol1:-plot(t=L1[i],Ops1,Ops||(i+1)); od:
display({G1,G2,G3}); GU:=Sol1:-plot(u(x,t),t=Tf,Ops1,Ops2):
GV:=Sol1:-plot(v(x,t),t=Tf,Ops1,Ops3): display({GU,GV});
PU:=r->eval(u,(Sol1:-value(u,t=Tf))(r)):
PV:=r->eval(v,(Sol1:-value(v,t=Tf))(r)):
plot([PU,PV,a..b],Ops2,labels=[u,v],Ops1); Sol1:-animate(
 t=Tf,Ops1,Ops2); GU3D:=Sol1:-plot3d(u(x,t),t=0..Tf):
GV3D:=Sol1:-plot3d(v(x,t),t=0..Tf): display(GU3D,Ops5);
display(GV3D,Ops5); P1:=Sol1:-value(u,t=0); P1(0.1,0);
```

Mathematica:

```
{nP=100, iS=500, l1={0.1,0.2,0.5}, l2={Magenta,Blue,Hue[0.8]}}
nl1=Length[l1]; SetOptions[Plot,ImageSize->iS,PlotRange->All,
 PlotPoints->nP*2]; SetOptions[Plot3D,ImageSize->500,
 ColorFunction->Function[{x,y},Hue[x]],PlotRange->All,
 BoxRatios->1,ViewPoint->{1,2,1}]; {s=1/100,a=0,b=1,tF=0.5}
sys1={D[u[x,t],t]==v[x,t]*D[u[x,t],x]+u[x,t]+1,
      D[v[x,t],t]==-u[x,t]*D[v[x,t],x]-v[x,t]+1}
{ibc1={u[x,0]==Exp[-x],v[x,0]==Exp[x],u[1,t]==Exp[t-1],
 v[0,t]==Exp[-t]}, sol1=NDSolve[Flatten[{sys1,ibc1}],{u,v},
 {x,a,b},{t,0,tF},MaxStepSize->s]}
Do[g[i]=Plot[Evaluate[u[x,l1[[i]]]/.sol1],{x,a,b},PlotStyle->
 {l2[[i]],Thickness[0.01]}],{i,1,nl1}];
Show[Table[g[i],{i,1,nl1}]]
gu=Plot[Evaluate[u[x,tF]/.sol1],{x,a,b},PlotStyle->{l2[[1]],
 Thickness[0.01]}]; gv=Plot[Evaluate[v[x,tF]/.sol1],{x,a,b},
 PlotStyle->{l2[[2]],Thickness[0.01]}]; Show[{gu,gv}]
ParametricPlot[{Evaluate[{u[x,tF],v[x,tF]}/.sol1]},{x,a,b},
 PlotRange->All,PlotStyle->{l2[[1]],Thickness[0.01]}]
Animate[Plot[{Evaluate[{u[x,t]}/.sol1]},{x,a,b},PlotRange->
 {0.3,1.6},PlotStyle->{l2[[1]],Thickness[0.01]}],{t,0,tF},
 AnimationRate->0.1]
gu3D=Plot3D[Evaluate[u[x,t]/.sol1],{x,a,b},{t,0,tF}];
gv3D=Plot3D[Evaluate[v[x,t]/.sol1],{x,a,b},{t,0,tF}];
GraphicsRow[{gu3D,gv3D}]
p1=Evaluate[u[0.1,0]/.sol1]
```

□

Problem 6.8

Maple:

```
with(PDEtools):  with(plots): Ops1:=numpoints=100:
Ops2:=color=magenta: Ops3:=color=blue: Ops4:=color="BlueViolet":
alpha:=0.1: beta:=0.01: a:=-50: b:=50:
U,V:=diff_table(u(x,t)),diff_table(v(x,t)):
sys1:=[U[t]=U[x,x]+U[]*(alpha-U[])*(U[]-1)-V[],V[t]=beta*U[]];
IBC1:={u(x,0)=piecewise(x<0,1,0),v(x,0)=piecewise(x<0,0,1),
 D[1](u)(a,t)=0, D[1](u)(b,t)=0}; L1:=[0.5,5,7]; NL1:=nops(L1);
Sol1:=pdsolve(sys1,IBC1,[u,v],numeric);
for i from 1 to NL1 do
 G||i:=Sol1:-plot(t=L1[i],Ops1,Ops||(i+1)); od:
```

```
display({G1,G2,G3}); GU:=Sol1:-plot(u(x,t),t=5,Ops1,Ops2):
GV:=Sol1:-plot(v(x,t),t=5,Ops1,Ops3): display({GU,GV});
Sol1:-animate(t=0.5,Ops1,Ops2);
```

Mathematica:

```
{nP=100,iS=500,l1={0.5,5,7},l2={Magenta,Blue,Hue[0.8]}}
nl1=Length[l1]; SetOptions[Plot,ImageSize->iS,PlotRange->All,
 PlotPoints->nP*2]; SetOptions[Plot3D,ImageSize->iS,
 ColorFunction->Function[{x,y},Hue[x]], PlotRange->All,
 BoxRatios->1, ViewPoint->{1,2,1}]; {alpha=0.1,beta=0.01,
 a=-50,b=50,tF=7, sys1={D[u[x,t],t]==D[u[x,t],{x,2}]+
 u[x,t]*(alpha-u[x,t])*(u[x,t]-1)-v[x,t],D[v[x,t],t]==
 beta*u[x,t]}, ibc1={u[x,0]==Piecewise[{{1,x<0},{0,x>0}}],
 v[x,0]==Piecewise[{{0,x<0},{1,x>0}}],(D[u[x,t],x]/.x->a)==0,
 (D[u[x,t],x]/.x->b)==0}}
sol1=NDSolve[Flatten[{sys1,ibc1}],{u,v},{x,a,b},{t,0,7},
 AccuracyGoal->1,PrecisionGoal->1,MaxSteps->{15,Infinity},
 Method->{"MethodOfLines","SpatialDiscretization"->
 {"TensorProductGrid"}}]
Do[g[i]=Plot[Evaluate[u[x,l1[[i]]]/.sol1],{x,a,b},PlotStyle->
 {l2[[i]],Thickness[0.01]}],{i,1,nl1}];
Show[Table[g[i],{i,1,nl1}]]
gu=Plot[Evaluate[u[x,5]/.sol1],{x,a,b},PlotStyle->{l2[[1]],
 Thickness[0.01]}]; gv=Plot[Evaluate[v[x,5]/.sol1],{x,a,b},
 PlotStyle->{l2[[2]],Thickness[0.01]}]; Show[{gu,gv}]
Animate[Plot[{Evaluate[{u[x,t]}/.sol1]},{x,a,b},PlotRange->
 {-1,1.3},PlotStyle->{l2[[1]],Thickness[0.01]}],{t,0,tF},
 AnimationRate->0.5]
```

□

6.2 Finite Difference Methods

6.2.1 Evolution Equations

Problem 6.9

Maple:

```
with(plots): nu:=0.009: NX:=100: NT:=100: a:=0: b:=1: L:=1.:
T:=0.4; h:=evalf((b-a)/NX); k:=evalf(T/NT); r:=nu*k/h^2;
f:=x->evalf(sin(Pi*x/L)); for i from 0 to NX do X[i]:=a+i*h od:
Ops1:=thickness=3,labels=["X","U"]; IC:={seq(U(i,0)=f(X[i]),
 i=0..NX)}; BC:={seq(U(a,j)=0,j=0..NT),seq(U(NX,j)=0,j=0..NT)}:
```

```
IBC:=IC union BC: FD:=(i,j)->U(i,j)+r*(U(i+1,j)-2*U(i,j)
 +U(i-1,j))-k/h*U(i,j)*(U(i+1,j)-U(i,j));
for j from 0 to NT do for i from 1 to NX-1 do
 U(i,j+1):=subs(IBC,FD(i,j)); od: od:
G:=j->plot([seq([X[i],subs(IBC,U(i,j))],i=0..NX)],color=blue):
display([seq(G(j),j=0..NT)],insequence=true,Ops1);
```

Mathematica:

```
SetOptions[ListPlot,ImageSize->500,PlotRange->{{0,1},{0,1.05}},
 Joined->True]; {nu=0.009,nX=100,nT=100,a=0,b=1,l=1.,tF=0.4,
 h=(b-a)/nX//N, k=tF/nT//N, r=nu*k/h^2}
f[x_]:=Sin[Pi*x/l]//N; Table[xN[i]=a+i*h,{i,0,nX}];
ic=Table[uN[i,0]->f[xN[i]],{i,0,nX}];
bc={Table[uN[a,j]->0,{j,0,nT}], Table[uN[nX,j]->0,{j,0,nT}]};
ibc=Flatten[{ic,bc}]
fd[i_,j_]:=uN[i,j]+r*(uN[i+1,j]-2*uN[i,j]+uN[i-1,j])-k/h*uN[i,j]*
 (uN[i+1,j]-uN[i,j]);
Do[uN[i,j+1]=fd[i,j]/.ibc,{j,0,nT},{i,1,nX-1}];
g[j_]:=ListPlot[Table[{xN[i],uN[i,j]/.ibc},{i,0,nX}],PlotStyle->
 {Blue,Thickness[0.01]},AxesLabel->{"X","U"}];
grs=Evaluate[Table[g[j],{j,0,nT}]]; ListAnimate[grs]
```

□

Problem 6.10

Maple:

```
with(plots): NX:=50: NT:=9: a:=-4; b:=6; T:=0.4;
h:=evalf((b-a)/NX); k:=evalf(T/NT); f:=x->arctan(4*x)+2;
for i from 0 to NX do X[i]:=a+i*h od: L1:=[1,3,5,7,9];
IC:={seq(U[i,0]=f(X[i]),i=0..NX)}; BC:={seq(U[a,j]=0,j=0..NT)}:
IBC:=IC union BC:
FD:=(i,j)->1/2*(U[i+1,j]+U[i-1,j])-k/(2*h)*(U[i+1,j]^2/2-
 U[i-1,j]^2/2);
for j from 0 to NT do for i from 1 to NX-1 do
 U[i,j+1]:=subs(IBC,FD(i,j)); od: od: NL1:=nops(L1);
for i from 1 to NL1 do
 G||(L1[i]):=[seq([X[j],U[j,L1[i]]],j=1..NX+1)]; od:
plot([seq(G||(L1[i]),i=1..NL1)]);
```

Mathematica:

```
SetOptions[ListPlot,ImageSize->300,PlotRange->{All,All},Joined->
 True]; {nX=50,nT=9,a=-4,b=6,tF=0.4,h=(b-a)/nX//N,k=tF/nT//N}
f[x_]:=ArcTan[4*x]+2//N; Table[xN[i]=a+i*h,{i,0,nX}];
ic=Table[uN[i,0]->f[xN[i]],{i,0,nX}]; bc={Table[uN[a,j]->0,
 {j,0,nT}]}; ibc=Flatten[{ic,bc}]
fd[i_,j_]:=0.5*(uN[i+1,j]+uN[i-1,j])-k/(2*h)*(uN[i+1,j]^2/2-
 uN[i-1,j]^2/2); Do[uN[i,j+1]=fd[i,j]/.ibc,{j,0,nT},
 {i,1,nX-1}]; {l1={1,3,5,7,9},nl1=Length[l1]}
Do[g[j]=ListPlot[Table[{xN[i],uN[i,l1[[j]]]/.ibc},{i,0,nX}],
 PlotStyle->{Hue[0.7+i/10],Thickness[0.01]},AxesLabel->
 {"X","U"}],{j,1,nl1}];
Show[Table[g[i],{i,1,nl1}],Frame->True,Axes->False]
```
□

Problem 6.11

Maple:

```
with(plots): nu:=1: NX:=15: NT:=100: L:=1.: T:=0.2; h:=L/NX;
k:=T/NT; r:=nu*k/h^2; f:=x->evalf(sin(Pi*x));
for i from 0 to NX do X[i]:=i*h od:
IC:={seq(U(i,0)=f(X[i]),i=0..NX)}; BC:={seq(U(0,j)=0,j=0..NT),
 seq(U(NX,j)=0,j=0..NT)}: IBC:=IC union BC:
FD:=(i,j)->(1-2*r)*U(i,j)+r*(U(i+1,j)+U(i-1,j));
for j from 0 to NT do for i from 1 to NX-1 do
 U(i,j+1):=subs(IBC,FD(i,j)); od: od:
G:=j->plot([seq([X[i],subs(IBC,U(i,j))],i=0..NX)],color=blue):
Ops1:=thickness=3,labels=["X","U"];
display([seq(G(j),j=0..NT)],insequence=true,Ops1);
```

Mathematica:

```
SetOptions[ListPlot,ImageSize->300,PlotRange->{{0,1},{0,1}},
 Joined->True]; f[x_]:=N[Sin[Pi*x]]; {nu=1,nX=15,nT=100,l=1,
 tF=0.2,h=l/nX,k=tF/nT,r=nu*k/h^2}
Table[xN[i]=i*h,{i,0,nX}]; ic=Table[uN[i,0]->f[xN[i]],{i,0,nX}];
bc={Table[uN[0,j]->0,{j,0,nT}],Table[uN[nX,j]->0,{j,0,nT}]};
ibc=Flatten[{ic,bc}]
fd[i_,j_]:=(1-2*r)*uN[i,j]+r*(uN[i+1,j]+uN[i-1,j]);
Do[uN[i,j+1]=fd[i,j]/.ibc,{j,0,nT},{i,1,nX-1}];
g[j_]:=ListPlot[Table[{xN[i],uN[i,j]/.ibc},{i,0,nX}],
 PlotStyle->{Blue,Thickness[0.01]},AxesLabel->{"X","U"}];
grs=Evaluate[Table[g[j],{j,0,nT}]]; ListAnimate[grs]
```

Maple:

```
with(plots): with(LinearAlgebra): nu:=1: NX:=40: NT:=800:
L:=1.: T:=0.2; h:=L/NX; k:=T/NT; r:=nu*k/h^2; NG:=90;
interface(rtablesize=NX): M:=BandMatrix([r,1-2*r,r],1,NX-1);
f:=x->evalf(sin(Pi*x)); U0:=Vector([[seq(f(i*h),i=1..NX-1)]]);
for k from 1 to NG do  U||k:=M.U||(k-1);
 G||k:=plot([[0,0],seq([i/NX,U||k[i]],i=1..NX-1),[L,0]]); od:
display([seq(G||i,i=1..NG)],insequence=true,thickness=3);
```

Mathematica:

```
SetOptions[ListPlot,ImageSize->500,PlotStyle->{Blue,
 Thickness[0.01]},PlotRange->{{0,1},{0,1}},Joined->True];
{nu=1,nX=40,nT=800,l=1,tF=0.2,h=l/nX,k=tF/nT,r=nu*k/h^2,nG=90}
f[x_]:=N[Sin[Pi*x]]; mat=SparseArray[{Band[{2,1}]->r,
 Band[{1,1}]->1-2*r,Band[{1,2}]->r},{nX-1,nX-1}];
Print[MatrixForm[mat]]; uN[0]=Table[f[i*h],{i,1,nX-1}]
Do[uN[k]=mat.uN[k-1]; gr[k]={{0,0}};
 Do[gr[k]=Append[gr[k],{i/nX,uN[k][[i]]}],{i,1,nX-1}];
 gr[k]=Append[gr[k],{l,0}]; g[k]=ListPlot[gr[k]],{k,1,nG}];
grs=Evaluate[Table[g[j],{j,1,nG}]]; ListAnimate[grs]
```

Maple:

```
with(plots): nu:=1: NX:=50: NT:=50: L:=1.: T:=0.2; h:=L/NX;
 k:=T/NT; r:=nu*k/h^2; for i from 0 to NX do X[i]:=i*h; od:
f:=i->evalf(sin(Pi*X[i])): IBC:={seq(U[i,0]=f(i),i=0..NX),
 seq(U[0,j]=0,j=0..NT),seq(U[NX,j]=0,j=0..NT)}: Sol0:=IBC;
FD:=(i,j)->(1+2*r)*U[i,j]-r*(U[i+1,j]+U[i-1,j])-U[i,j-1];
for j from 1 to NT do
 Eqs||j:={seq(FD(i,j)=0,i=1..NX-1)}; Eqs1||j:=subs(
  Sol||(j-1),IBC,Eqs||j); vars||j:={seq(U[i,j],i=1..NX-1)};
  Sol||j:=fsolve(Eqs1||j,vars||j); od:
G:=j->plot([seq([X[i],subs(Sol||j,IBC,U[i,j])],i=0..NX)],
            color=blue,thickness=3):
display([seq(G(j),j=0..NT)],insequence=true,labels=["X","U"]);
```

Mathematica:

```
SetOptions[ListPlot,PlotRange->{{0,1},{0,1}},Joined->True];
{nu=1,nX=50,nT=50,l=1,tF=0.2,h=l/nX,k=tF/nT,r=nu*k/h^2}
Table[xN[i]=i*h,{i,0,nX}]; f[i_]:=N[Sin[Pi*xN[i]]];
```

```
ibc={Table[uN1[i,0]->f[i],{i,0,nX}],Table[uN1[0,j]->0,
 {j,0,nT}], Table[uN1[nX,j]->0,{j,0,nT}]}//Flatten
sol[0]=ibc; fd[i_,j_]:=(1+2*r)*uN1[i,j]-
 r*(uN1[i+1,j]+uN1[i-1,j])-uN1[i,j-1];
Do[eqs[j]=Table[Expand[fd[i,j]]==0,{i,1,nX-1}];
 eqs1[j]=eqs[j]/.sol[j-1]/.ibc; vars[j]=Table[uN1[i,j],
 {i,1,nX-1}]; sol[j]=NSolve[eqs1[j],vars[j]],{j,1,nT}];
g[j_]:=ListPlot[Table[{xN[i],uN1[i,j]/.Flatten[sol[j]]/.ibc},
 {i,0,nX}],PlotStyle->{Blue,Thickness[0.01]},AxesLabel->
 {"X","U"}]; grs=Evaluate[Table[g[j],{j,1,nT}]];
ListAnimate[grs]
```

Maple:

```
with(PDEtools): with(plots): declare(v(x1,t1)); L:=1; T:=0.2:
nu:=1: NX:=20: NT:=20: NX1:=NX-1: NX2:=NX-2: h:=L/NX; k:=T/NT;
r:=nu*k/(h^2); SX:=h; ST:=k; tR:=0..T; xR:=0..L; NF:=30:
NP:=100: tk:=0.2: Ops1:=spacestep=SX,timestep=ST:
F:=i->sin(Pi*i); IC:={v(x1,0)=F(x1)}; BC:={v(0,t1)=0,v(L,t1)=0};
U[NX-1]:=0: PDE1:=diff(v(x1,t1),t1)-nu*diff(v(x1,t1),x1$2)=0;
for i from 1 to NX1 do U[i-1]:=evalf(F(i*h)); od:
LM[0]:=1+r: UM[0]:=-r/(2*LM[0]):
for i from 2 to NX2 do LM[i-1]:=1+r+r*UM[i-2]/2;
 UM[i-1]:=-r/(2*LM[i-1]); od: LM[NX1-1]:=1+r+0.5*r*UM[NX2-1]:
for j from 1 to NT do t:=j*k; Z[0]:=((1-r)*U[0]+r*U[1]/2)/LM[0];
 for i from 2 to NX1 do Z[i-1]:=((1-r)*U[i-1]+0.5*r*(U[i]+U[i-2]+
  Z[i-2]))/LM[i-1]; od:
 U[NX1-1]:=Z[NX1-1]:
 for i1 to NX2 do i:=NX2-i1+1; U[i-1]:=Z[i-1]-UM[i-1]*U[i]; od:
od: ExSol:=(x,t)->exp(-Pi^2*t)*sin(Pi*x):
printf(`Crank-Nicolson Method\n`); for i from 1 to NX1 do X:=i*h;
 printf(`%3d %11.8f %13.8f %13.8f,%13.8f\n`,i,X,U[i-1],
 evalf(ExSol(X,tk)),U[i-1]-evalf(ExSol(X,tk))); od:
NSol1:=pdsolve(PDE1,IC union BC,numeric,v(x1,t1),Ops1,time=t1,
 range=0..L,method=CrankNicholson):
printf(`Crank-Nicolson Method\n`); vtk:=NSol1:-value(t1=tk,
 output=listprocedure): vVal:=rhs(op(3,vtk)):
for i from 1 to NX1 do X1:=i*h:
 printf(`%3d %11.8f %13.8f %13.8f,%13.8f\n`,i,evalf(X1),vVal(X1),
  evalf(ExSol(X1,tk)),abs(vVal(X1)-evalf(ExSol(X1,tk)))); od:
```

Mathematica:

```
fF[i_] := Sin[Pi*i]; {l=1,tF=0.2,nu=1,nX=20,nT=20,nX1=nX-1,
 nX2=nX-2,h=l/nX,k=tF/nT,r=nu*k/(h^2),tk=0.2}
{ic={v[x1,0]==fF[x1]}, bc={v[0,t1]==0,v[l,t1]==0}}
{lM=Table[0,{i,0,nX}],uN=Table[0,{i,0,nX}],uM=Table[0,{i,0,nX}],
 z=Table[0,{i,0,nX}], uN[[nX-1]]=0}
pde1=D[v[x1,t1],t1]-nu*D[v[x1,t1],{x1,2}]==0 Do[uN[[i-1]]=
 N[fF[i*h]],{i,1,nX1}]; {lM[[0]]=1+r, uM[[0]]=-r/(2*lM[[0]])}
Do[lM[[i-1]]=1+r+r*uM[[i-2]]/2; uM[[i-1]]=-r/(2*lM[[i-1]]),
 {i,2,nX2}]; lM[[nX1-1]]=1+r+0.5*r*uM[[nX2-1]]
Do[t=j*k; z[[0]]=((1-r)*uN[[0]]+r*uN[[1]]/2)/lM[[0]];
 Do[z[[i-1]]=((1-r)*uN[[i-1]]+0.5*r*(uN[[i]]+uN[[i-2]]+
  z[[i-2]]))/lM[[i-1]],{i,2,nX1}]; uN[[nX1-1]]=z[[nX1-1]];
 Do[i=nX2-i1+1; uN[[i-1]]=z[[i-1]]-uM[[i-1]]*uN[[i]],{i1,1,nX2}],
{j,1,nT}]; nD=10; extSol[x1_,t1_]:=Exp[-Pi^2*t]*Sin[Pi*x]/.
 {x->x1,t->t1}; Print["Crank-Nicolson Method"];
Do[xN=i*h; Print[i," ",PaddedForm[N[xN,nD],{12,10}]," ",
 PaddedForm[uN[[i-1]],{12,10}]," ", PaddedForm[N[extSol[xN,tk],
 nD],{12,10}]," ", PaddedForm[uN[[i-1]]-N[extSol[xN,tk],nD],
 {12,10}]],{i,1,nX1}];
```

Maple:

```
restart: with(plots): NX:=20: NT:=300: A:=0.4; L:=1; a:=0; b:=1;
T:=40.; h:=evalf((b-a)/NX); k:=evalf(T/NT); f:=x->evalf(
 A*sin(Pi*x/L)); for i from 0 to NX do X[i]:=a+i*h od:
IC:={seq(U(i,0)=f(X[i]),i=0..NX)};
BC:={seq(U(a,j)=0,j=0..NT),seq(U(NX,j)=0,j=0..NT)}:
IBC:=IC union BC: FD:=(i,j)->0.5*(U(i+1,j)+U(i-1,j))-
 k/(2*h)*(U(i+1,j)^2/2-U(i-1,j)^2/2);
for j from 0 to NT do for i from 1 to NX-1 do
 U(i,j+1):=subs(IBC,FD(i,j)); od: od:
G:=j->plot([seq([X[i],subs(IBC,U(i,j))],i=0..NX)],color=blue,
 thickness=3,numpoints=100):
display([seq(G(j),j=0..NT)],insequence=true,labels=["X","U"]);
```

Mathematica:

```
ClearAll["Global`*"]; SetOptions[ListPlot,PlotRange->{{0,1},
 {0,0.4}},Joined->True]; {nX=20,nT=300,am=0.4,l=1,a=0,b=1,
 tF=40,h=N[(b-a)/nX],k=N[tF/nT]}
f[x_]:=N[am*Sin[Pi*x/l]]; Table[xN[i]=a+i*h,{i,0,nX}];
```

```
ic=Table[uN[i,0]->f[xN[i]],{i,0,nX}]; bc={Table[uN[a,j]->0,
 {j,0,nT}],Table[uN[nX,j]->0,{j,0,nT}]}; ibc=Flatten[{ic,bc}]
fd[i_,j_]:=0.5*(uN[i+1,j]+uN[i-1,j])-k/(2*h)*(uN[i+1,j]^2/2-
 uN[i-1,j]^2/2); Do[uN[i,j+1]=fd[i,j]/.ibc,{j,0,nT},{i,1,nX-1}];
g[j_]:=ListPlot[Table[{xN[i],uN[i,j]/.ibc},{i,0,nX}],
 PlotStyle->{Blue,Thickness[0.01]},AxesLabel->{"X","U"}];
grs=Evaluate[Table[g[j],{j,0,nT}]]; ListAnimate[grs]
```

□

Problem 6.12

Maple:

```
c:=1/(4*Pi); L:=0.5; T:=1.5; NX:=40; NT:=40; NX1:=NX+1;
NX2:=NX-1; NT1:=NT+1; NT2:=NT-1; h:=L/NX; k:=T/NT;
r:=evalf(c*k/h); F:=i->0; G:=i->sin(4*Pi*i);
for j from 2 to NT1 do U[0,j-1]:=0; U[NX1-1,j-1]:=0; od:
U[0,0]:=evalf(F(0)); U[NX1-1,0]:=evalf(F(L));
for i from 2 to NX do
 U[i-1,0]:=F(h*(i-1)); U[i-1,1]:=(1-r^2)*F(h*(i-1))
  +r^2*(F(i*h)+F(h*(i-2)))/2+k*G(h*(i-1)); od:
for j from 2 to NT do for i from 2 to NX do
 U[i-1,j]:=evalf(2*(1-r^2)*U[i-1,j-1]+r^2*(U[i,j-1]
  +U[i-2,j-1])-U[i-1,j-2]); od; od;
printf(` i X(i) U(X(i),NT)\n`);
for i from 1 to NX1 do  X[i-1]:=(i-1)*h:
 printf(`%3d %11.8f %13.8f\n`,i,X[i-1],U[i-1,NT1-1]); od:
Points:=[seq([X[i-1],U[i-1,NT1-1]],i=1..NX1)];
plot(Points,style=point,color=blue,symbol=circle);
```

Mathematica:

```
SetOptions[ListPlot,PlotRange->All,Joined->False];  f[x_]:=0;
g[x_]:=Sin[4*Pi*x]; {c=N[1/(4*Pi)],l=0.5,tF=1.5,nX=40,nX1=nX+1,
 nT=40,nT1=nT+1,h=l/nX,k=tF/nT,r=N[c*k/h]}
fi[i_]:=f[h*(i-1)]; gi[i_]:=g[h*(i-1)]; uN=Table[0,{nT1},{nX1}];
For[i=1,i<=nT1,i++,uN[[i,1]]=fi[i]]; For[i=2,i<=nT,i++,
 uN[[i,2]]=(1-r^2)*fi[i]+r^2*(fi[i+1]+fi[i-1])/2+k*gi[i]];
For[j=3,j<=nX1,j++, For[i=2,i<=nT,i++, uN[[i,j]]=2*(1-r^2)*
 uN[[i,j-1]]+r^2*(uN[[i+1,j-1]]+uN[[i-1,j-1]])-uN[[i,j-2]]//N];];
Print["  i      xN[i]       uN[xN[i],nT1]", "\n"];
For[i=1,i<=nT1,i++,Print[PaddedForm[i,2],PaddedForm[h*(i-1),7],
   "   ",PaddedForm[uN[[i,nT1]],10]]];
points=Table[{h*(i-1),uN[[i,nT1]]},{i,1,nT1}]
```

```
ListPlot[points,PlotStyle->{Blue,PointSize[0.02]}]
Print[NumberForm[TableForm[Transpose[Chop[uN]]],3]];
ListPlot3D[uN,ViewPoint->{3,1,3},ColorFunction->Hue]
```

Maple:

```
with(plots): c:=evalf(1/(4*Pi)); L:=0.5: T:=1.5: NX:=40:
NT:=40: h:=L/NX; k:=T/NT; r:=(c*k/h)^2; f:=x->0:
g:=x->evalf(sin(4*Pi*x)): IC:={seq(U1(i,0)=f(i*h),i=1..NX-1),
 seq(U1(i,1)=f(i*h)+k*g(i*h),i=1..NX-1)}:
BC:={seq(U1(0,j)=0,j=0..NT),seq(U1(NX,j)=0,j=0..NT)}:
IBC:=IC union BC: FD:=(i,j)->2*(1-r)*U1(i,j)+r*(U1(i+1,j)
 +U1(i-1,j))-U1(i,j-1);
for j from 1 to NT-1 do for i from 1 to NX-1 do
 U1(i,j+1):=subs(IBC,FD(i,j)); od: od:
G:=j->plot([seq([i*h,subs(IBC,U1(i,j))],i=0..NX)],color=blue):
display([seq(G(j),j=0..NT)],insequence=true,thickness=3,
 labels=["X","U"]);
```

Mathematica:

```
SetOptions[ListPlot,ImageSize->500,PlotRange->{{0,0.5},{-1,1}},
 Joined->True]; f[x_]:=0; g[x_]:=N[Sin[4*Pi*x]]; {c=N[1/(4*Pi)],
 l=0.5,tF=1.5,nX=40,nT=40,h=l/nX,k=tF/nT,r=(c*k/h)^2}
ic={Table[uN1[i,0]->f[i*h],{i,1,nX-1}],Table[uN1[i,1]->f[i*h]+
 k*g[i*h],{i,1,nX-1}]}; bc={Table[uN1[0,j]->0,{j,0,nT}],
 Table[uN1[nX,j]->0,{j,0,nT}]}; ibc=Flatten[{ic,bc}]
fd[i_,j_]:=2*(1-r)*uN1[i,j]+r*(uN1[i+1,j]+uN1[i-1,j])-uN1[i,j-1];
Do[uN1[i,j+1]=fd[i,j]/.ibc,{j,1,nT-1},{i,1,nX-1}];
g[j_]:=ListPlot[Table[{i*h,uN1[i,j]/.ibc},{i,0,nX}],PlotStyle->
 {Blue,Thickness[0.01]},AxesLabel->{"X","U"}];
grs=N[Table[g[j],{j,0,nT}]]; ListAnimate[grs]
```

Maple:

```
restart: with(plots): L:=0.5: T:=1.5: NX:=25: NT:=20:
c:=evalf(1/(4*Pi)): h:=L/NX; lambda:=1: k:=T/NT; r:=(c*k/h)^2;
f:=x->0: g:=x->evalf(sin(4*Pi*x)): IC:={seq(U(i,0)=f(i*h),
 i=1..NX-1),seq(U(i,1)=f(i*h)+k*g(i*h),i=1..NX-1)}:
BC:={seq(U(0,j)=0,j=0..NT),seq(U(NX,j)=0,j=0..NT)}:
IBC:=IC union BC: FD:=(i,j)->evalf(2*(1-r)*U(i,j)+r*(U(i+1,j)
 +U(i-1,j))-U(i,j-1)+exp(lambda*U(i,j))*k^2);
```

```
for j from 1 to NT-1 do for i from 1 to NX-1 do
  U(i,j+1):=subs(IBC,FD(i,j)); od: od:
G:=j->plot([seq([i*h,subs(IBC,U(i,j))],i=0..NX)],color=blue):
display([seq(G(j),j=0..NT)],insequence=true,thickness=3,
  labels=["X","U"]);
```

Mathematica:

```
ClearAll["Global`*"]; SetOptions[ListPlot,ImageSize->500,
 PlotRange->{{0,0.5},{-1,5}},Joined->True]; {l=0.5,tF=1.5,
 nX=20,nT=20,c=N[1/(4*Pi)],h=l/nX,k=tF/nT,r=(c*k/h)^2}
f[x_]:=0; g[x_]:=Sin[4*Pi*x]//N; ic={Table[uN[i,0]->f[i*h],
 {i,1,nX-1}],Table[uN[i,1]->f[i*h]+k*g[i*h],{i,1,nX-1}]};
bc={Table[uN[0,j]->0,{j,0,nT}],Table[uN[nX,j]->0,{j,0,nT}]};
ibc=Flatten[{ic,bc}]
fd[i_,j_]:=2*(1-r)*uN[i,j]+r*(uN[i+1,j]+uN[i-1,j])-uN[i,j-1]+
 Exp[uN[i,j]]*k^2;
Do[uN[i,j+1]=fd[i,j]/.ibc,{j,1,nT-1},{i,1,nX-1}];
g[j_]:=ListPlot[Table[{i*h,uN[i,j]/.ibc},{i,0,nX}],
  PlotStyle->{Blue,Thickness[0.01]},AxesLabel->{"X","U"}];
grs=Evaluate[Table[g[j],{j,0,nT}]]; ListAnimate[grs]
```
□

6.2.2 Interaction of Solitons

Problem 6.13

Maple:

```
with(plots):  Digits:=50: L:=100: T:=70: NX:=100: NT:=360:
h:=evalf(L/NX); k:=evalf(T/NT); r:=evalf(k/h^3); a:=2: b:=1:
xR:=0..NX; tR:=0..NT; tk:=0; NP:=1000: L1:=[20,40];
L2:=[0.7,0.2]; K:=nops(L1); Ops1:=numpoints=NP,thickness=3:
S:=(xk,c)->3*c/a*(sech(sqrt(c*b)/(2*b)*(-(x-xk)+c*t))^2):
F1:=unapply(add(S(L1[i],L2[i]),i=1..K),x,t);
F2:=unapply(diff(F1(x,t),t),x,t);  F1(x,t); F2(x,t);
for i from 0 to NX do for j from 0 to NT do U(i,j):=0: od: od:
for i from 0 to NX do U(i,0):=evalf(F1(i,tk)): od:
for i from 0 to NX do U(i,1):=evalf(F1(i,tk)+k*F2(i,tk)): od:
FD:=(i,j)->U(i,j-1)-a*r*h^2*(U(i+1,j)+U(i,j)+U(i-1,j))*
 (U(i+1,j)-U(i-1,j))/3-r*b*(U(i+2,j)-2*U(i+1,j)+2*U(i-1,j)-
 U(i-2,j));
for j from 1 to NT do for i from 2 to NX-2 do
 U(i,j+1):=evalf(FD(i,j)): od: od:
```

```
G:=j->plot([seq([i,U(i,j)],i=2..NX-2)],Ops1,color=blue,
 view=[0..100,0..1.2]): LG:=[seq(G(4*j),j=0..NT/4)]:
display(LG,insequence=true);
```

Mathematica:

```
{l=100,tF=70,nX=100,nT=360,h=N[l/nX],k=N[tF/nT],r=N[k/h^3],
 a=2,b=1,tk=0,l1={20,40},l2={0.7,0.2},kl1=Length[l1]}
SetOptions[ListPlot,ImageSize->500,PlotRange->{{0,100},{0,1.2}},
 PlotStyle->{Blue,Thickness[0.01]},Joined->True];
fS[xk_,c_]:=3*c/a*(Sech[Sqrt[c*b]/(2.*b)*(-(x-xk)+c*t)]^2);
f1[x1_,t1_]:=Sum[fS[l1[[i]],l2[[i]]],{i,1,kl1}]/.x->x1/.t->t1;
f2[x1_,t1_]:=D[f1[x,t],t]/.x->x1/.t->t1; {f1[x,t],f2[x,t]}
Do[uN[i,j]=0,{j,0,nT},{i,0,nX}]; Do[uN[i,0]=f1[i,tk],{i,0,nX}];
Do[uN[i,1]=f1[i,tk]+k*f2[i,tk],{i,0,nX}];
fd[i_,j_]:=uN[i,j-1]-a*r*h^2*(uN[i+1,j]+uN[i,j]+uN[i-1,j])*
 (uN[i+1,j]-uN[i-1,j])/3-r*b*(uN[i+2,j]-2*uN[i+1,j]+2*uN[i-1,j]-
 uN[i-2,j]); Do[uN[i,j+1]=fd[i,j],{j,1,nT},{i,2,nX-2}];
g[j_]:=ListPlot[Table[{i,uN[i,j]},{i,2,nX-2}],PlotStyle->
 {Blue,Thickness[0.01]},AxesLabel->{"X","U"}];
grs=Evaluate[Table[g[4*j],{j,0,nT/4}]]; ListAnimate[grs]
```
□

Problem 6.14

Maple:

```
with(plots): NX:=100: NT:=100: xR:=0..NX; tR:=0..NT; NP:=100;
A2:=1.; A1:=2; L:=10; L1:=[-L,L]; K:=nops(L1); Ops1:=
 numpoints=NP,thickness=3,color=blue: S:=(x,t)->-4*arctan(
 (A2-exp(2*x*sqrt(A1)))*exp(-x*sqrt(A1)+t*sqrt(A1-1))
 *sqrt(A1-1)/(sqrt(A1)*(A2+exp(2*t*sqrt(A1-1)))));
h:=evalf((L1[2]-L1[1])/NX);
Sx:=unapply(diff(S(x,t),x),x,t):St:=unapply(diff(S(x,t),t),x,t):
for i from 0 to NX by 2 do
 X:=L1[1]+i*h; U[i,0]:=evalf(S(X,-10)); Up[i,0]:=evalf(
 Sx(X,-10)); Uq[i,0]:=evalf(St(X,-10)); od:
for j from 2 to NT by 2 do
 U[0,j]:=evalf(-2*Pi): Up[0,j]:=0: Uq[0,j]:=0;
 U[NX,j]:=evalf(2*Pi): Up[NX,j]:=0: Uq[NX,j]:=0; od:
```

```
for j from 0 to NT-1 do
 if modp(j,2)=0 then k:=0: else k:=1: fi;
 for i from k to NX-2 by 2 do
  Up[i+1,j+1]:=0.5*(Up[i+2,j]+Up[i,j]+Uq[i+2,j]-Uq[i,j]
   +(-sin(U[i+2,j])+sin(U[i,j]))*h);
  Uq[i+1,j+1]:=0.5*(Up[i+2,j]-Up[i,j]+Uq[i+2,j]+Uq[i,j]
   +(-sin(U[i+2,j])-sin(U[i,j]))*h);
  UL:=U[i,j]+0.5*h*(Up[i,j]+Up[i+1,j+1]+Uq[i,j]+Uq[i+1,j+1]);
  UR:=U[i+2,j]+0.5*h*(-Up[i+2,j]-Up[i+1,j+1]+Uq[i+2,j]
   +Uq[i+1,j+1]); U[i+1,j+1]:=0.5*(UL+UR); od: od:
for j from 0 to NT by 2 do
 G:=j->plot([seq([L1[1]+2*i*h,U[2*i,j]],i=0..NX/2)],Ops1): od:
LG:=[seq(G(2*j),j=0..NT/2)]: display(LG,insequence=true);
```

Mathematica:

```
{nX=100,nT=100,a2=1,a1=2,l=10,l1={-l,l},ll1=Length[l1]}
SetOptions[ListPlot,ImageSize->500,PlotRange->{{-l,l},{-7,7}},
 PlotStyle->{Blue,Thickness[0.01]},Joined->True];
fS[x_,t_]:=-4*ArcTan[(a2-Exp[2*x*Sqrt[a1]])*Exp[-x*Sqrt[a1]+
 t*Sqrt[a1-1]]*Sqrt[a1-1]/(Sqrt[a1]*(a2+Exp[2*t*Sqrt[a1-1]]))];
h=N[(l1[[2]]-l1[[1]])/nX];
sx[x1_,t1_]:=D[fS[x,t],x]/.x->x1/.t->t1;
st[x1_,t1_]:=D[fS[x,t],t]/.x->x1/.t->t1;
Do[xN=l1[[1]]+i*h; uN[i,0]=N[fS[xN,-10]];
   up[i,0]=N[sx[xN,-10]]; uq[i,0]=N[st[xN,-10]],{i,0,nX,2}];
Do[uN[0,j]=N[-2*Pi]; up[0,j]=0; uq[0,j]=0;
   uN[nX,j]=N[2*Pi]; up[nX,j]=0; uq[nX,j]=0,{j,2,nT,2}];
For[j=0,j<=nT-1,j++, If[Mod[j,2]==0,k=0,k=1];
 For[i=k,i<=nX-2,i=i+2, up[i+1,j+1]=0.5*(up[i+2,j]+up[i,j]+
 uq[i+2,j]-uq[i,j]+(-Sin[uN[i+2,j]]+Sin[uN[i,j]])*h);
  uq[i+1,j+1]=0.5*(up[i+2,j]-up[i,j]+uq[i+2,j]+uq[i,j]+
   (-Sin[uN[i+2,j]]-Sin[uN[i,j]])*h);
  uL=uN[i,j]+0.5*h*(up[i,j]+up[i+1,j+1]+uq[i,j]+uq[i+1,j+1]);
  uR=uN[i+2,j]+0.5*h*(-up[i+2,j]-up[i+1,j+1]+uq[i+2,j]+
   uq[i+1,j+1]); uN[i+1,j+1]=0.5*(uL+uR)]];
Do[g[j_]:=ListPlot[Table[{l1[[1]]+2*i*h,uN[2*i,j]},{i,0,nX/2}],
 PlotStyle->{Blue,Thickness[0.01]},AxesLabel->{"X","U"}],
 {j,0,nT,2}]; grs=Evaluate[Table[g[2*j],{j,0,nT/2}]];
ListAnimate[grs]
```

□

6.2.3 Elliptic Equations

Problem 6.15

Maple:

```
with(linalg): with(PDEtools): f:=(x,y)->sin(x)*cos(y);
PDE1:=laplacian(u(x,y),[x,y])-f(x,y)=0;
Sol1:=pdsolve(PDE1,build); Test1:=pdetest(Sol1,PDE1);
Sol11:=unapply(subsop(1=0,2=0,rhs(Sol1)),x,y);
Sol12:=expand(Sol11(x,y));
plot3d(Sol12,x=0..Pi,y=0..2*Pi,shading=zhue);
Sol11(x,0);  Sol11(x,2*Pi); Sol11(0,y); Sol11(Pi,y);
```

Maple:

```
with(plots): a:=0; b:=Pi; c:=0; d:=2*Pi; NX:=20; NY:=20;
h:=(b-a)/NX; k:=(d-c)/NY; r:=(h/k)^2; XY:=seq(x[i]=a+i*h,
 i=0..NX),seq(y[j]=c+j*k,j=0..NY); FD:=(i,j)->2*(1+r)*U[i,j]
 -U[i+1,j]-U[i-1,j]-r*U[i,j+1]-r*U[i,j-1]-cos(j*k)*sin(i*h)=0;
F1:=i->-1/2*sin(i*h); F2:=i->-1/2*sin(i*h); F3:=j->0; F4:=j->0;
BC:=seq(U[i,0]=F1(i),i=0..NX),seq(U[i,NY]=F2(i),i=0..NX),
    seq(U[0,j]=F3(j),j=0..NY),seq(U[NX,j]=F4(j),j=0..NY);
Eqs:={seq(seq(FD(i,j),i=1..NX-1),j=1..NY-1)}: Eqs1:=subs(BC,Eqs):
vars:={seq(seq(U[i,j],i=1..NX-1),j=1..NY-1)}: Sol:=evalf(
 fsolve(Eqs1,vars)); Points:=[seq(seq([x[i],y[j],U[i,j]],
 i=0..NX),j=0..NY)]: Points1:=subs({XY,BC,op(Sol)},Points):
pointplot3d(Points1,symbol=solidsphere,shading=z,
 orientation=[50,70],axes=frame);
```

Mathematica:

```
SetAttributes[{x,y},NHoldAll]; {nD=10,a=0,b=Pi,c=0,d=2*Pi,nX=20,
 nY=20,h=N[(b-a)/nX],k=N[(d-c)/nY],r=N[(h/k)^2]}
xN=Table[x[i]->a+i*h,{i,0,nX}]//N; yN=Table[y[j]->c+j*k,
 {j,0,nY}]//N; fd[i_,j_]:=2*(1+r)*uN[i,j]-uN[i+1,j]-uN[i-1,j]-
 r*uN[i,j+1]-r*uN[i,j-1]-Cos[j*k]*Sin[i*h]; f1[i_]:=
 -1/2*Sin[i*h]; f2[i_]:=-1/2*Sin[i*h]; f3[j_]:=0; f4[j_]:=0;
bc=Flatten[{Table[uN[i,0]->f1[i],{i,0,nX}],Table[uN[i,nY]->
 f2[i],{i,0,nX}], Table[uN[0,j]->f3[j],{j,0,nY}],
 Table[uN[nX,j]->f4[j],{j,0,nY}]}]
eqs=Flatten[Table[fd[i,j]==0,{i,1,nX-1},{j,1,nY-1}]];
eqs1=Flatten[eqs/.bc]; vars=Flatten[Table[uN[i,j],{i,1,nX-1},
 {j,1,nY-1}]]; sol=NSolve[eqs1,vars]
```

```
points=Table[{x[i],y[j],uN[i,j]},{i,0,nX},{j,0,nY}];
points1=Flatten[N[points/.xN/.yN/.bc/.sol[[1]],nD],1];
g3D=ListPointPlot3D[points1,BoxRatios->{1,1,1},PlotStyle->
 PointSize[0.01],PlotRange->All]; gCP=ListContourPlot[points1,
 PlotRange->All]; GraphicsRow[{g3D,gCP},ImageSize->800]
```

Maple:

```
with(plots): a:=0; b:=Pi; c:=0; d:=2*Pi; NX:=15; NY:=15;
h:=(b-a)/NX; k:=(d-c)/NY; r:=(h/k)^2;
XY:=seq(x[i]=a+i*h,i=0..NX),seq(y[j]=c+j*k,j=0..NY);
FD:=(i,j)->evalf(2*(1+r)*U[i,j]-U[i+1,j]-U[i-1,j]-r*U[i,j+1]
          -r*U[i,j-1]-h^2*sin(U[i,j]))=0;
F1:=i->-1/2*sin(i*h); F2:=i->-1/2*sin(i*h);
F3:=j->0; F4:=j->0;
BC:=seq(U[i,0]=F1(i),i=0..NX),seq(U[i,NY]=F2(i),i=0..NX),
    seq(U[0,j]=F3(j),j=0..NY),seq(U[NX,j]=F4(j),j=0..NY);
Eqs:={seq(seq(FD(i,j),i=1..NX-1),j=1..NY-1)}:
Eqs1:=subs(BC,Eqs);
vars:={seq(seq(U[i,j]=0.,i=1..NX-1),j=1..NY-1)};
Sol:=fsolve(Eqs1,vars,real);
Points:=[seq(seq([x[i],y[j],U[i,j]],i=0..NX),j=0..NY)];
Points1:=evalf(subs(XY,BC,op(Sol),Points));
pointplot3d(Points1,symbol=solidsphere,shading=z,
 orientation=[50,60],axes=frame);
Points2:=[seq([seq([x[i],y[j],U[i,j]],i=0..NX)],j=0..NY)]:
Points3:=evalf(subs(XY,BC,op(Sol),Points2)):
surfdata(Points3,axes=boxed,labels=[x,y,u], shading=zhue);
```

Mathematica:

```
{nD=30,a=0,b=Pi,c=0,d=2*Pi,nX=15,nY=30,h=(b-a)/nX,k=(d-c)/nY}
r=(h/k)^2; xN=Table[x[i]->a+i*h,{i,0,nX}]; yN=Table[y[j]->c+j*k,
 {j,0,nY}]; fd[i_,j_]:=2*(1+r)*uN[i,j]-uN[i+1,j]-uN[i-1,j]-
 r*uN[i,j+1]-r*uN[i,j-1]-h^2*Sin[uN[i,j]]; f1[i_]:=
 -1/2*Sin[i*h]; f2[i_]:=-1/2*Sin[i*h]; f3[j_]:=0; f4[j_]:=0;
bc=Flatten[{Table[uN[i,0]->f1[i],{i,0,nX}],Table[uN[i,nY]->
 f2[i],{i,0,nX}], Table[uN[0,j]->f3[j],{j,0,nY}],
 Table[uN[nX,j]->f4[j],{j,0,nY}]}]
eqs=Flatten[Table[fd[i,j]==0,{i,1,nX-1},
 {j,1,nY-1}]]; eqs1=Flatten[eqs/.bc];
vars=Flatten[Table[{uN[i,j],0},{i,1,nX-1},{j,1,nY-1}],1];
```

```
sol=FindRoot[eqs1,vars]
points=Table[{x[i],y[j],uN[i,j]},{i,0,nX},{j,0,nY}];
points1=Flatten[N[points/.xN/.yN/.bc/.sol,nD],1];
g3D=ListPointPlot3D[points1,BoxRatios->Automatic,PlotStyle->
 PointSize[0.01],PlotRange->All]; gCP=ListContourPlot[points1,
 PlotRange->All]; GraphicsRow[{g3D,gCP},ImageSize->800]
```

Mathematica:

```
f1[x_]:=-1/2*Sin[x]; f2[x_]:=-1/2*Sin[x]; f3[y_]:=0;
f4[y_]:=0; {a=0,b=Pi,c=0,d=2*Pi,tF=100}
{pde1=D[u[x,y,t],t]==D[u[x,y,t],{x,2}]+D[u[x,y,t],{y,2}]+
 Sin[u[x,y,t]], ic={u[x,y,0]==-1/2*Sin[x]},
 bc={u[x,c,t]==f1[x],u[x,d,t]==f2[x],u[a,y,t]==f3[y],
 u[b,y,t]==f4[y]}}
sol1=NDSolve[{pde1,ic,bc},u,{x,a,b},{y,c,d},{t,0,tF},
 Method->{"MethodOfLines","SpatialDiscretization"->
 {"TensorProductGrid","DifferenceOrder"->"Pseudospectral"}}]
Plot3D[Evaluate[u[x,y,tF]/.sol1],{x,a,b},{y,c,d},
 BoxRatios->Automatic,PlotRange->All]
```

□

Chapter 7
Analytical-Numerical Approach

7.1 Method of Lines

7.1.1 Nonlinear PDEs

Problem 7.1

Maple:

```
with(LinearAlgebra):  with(PDEtools): declare(u(x,t)):
with(plots): Digits:=15; nu:=0.009; L:=1; a:=0; b:=1; Tf:=0.4;
NX:=20; tr1:=h=evalf((b-a)/(NX+1)); tr2:=x=i*h; NT:=20;
EAbs:=0.1e-6; Guxt:=Matrix(NX+2,NT+1);
PDE1:=diff(u(x,t),t)=nu*diff(u(x,t),x$2)-u(x,t)*diff(u(x,t),x);
BC1:=u(x,t)=0; BC2:=u(x,t)=0; IC1:=u(x,0)=sin(Pi*x/L);
FWD1:=(-u[k+2](t)-3*u[k](t)+4*u[k+1](t))/(2*h);
BWD1:=(u[k-2](t)+3*u[k](t)-4*u[k-1](t))/(2*h);
CD1:=(u[k+1](t)-u[k-1](t))/(2*h);
CD2:=(u[k-1](t)-2*u[k](t)+u[k+1](t))/h^2;
BC1D:=subs(lhs(BC1)=u[0](t),x=a,BC1);
BC2D:=subs(lhs(BC2)=u[NX+1](t),x=b,BC2);
tr3:={diff(u(x,t),x$2)=subs(k=i,CD2),diff(u(x,t),x)=
 subs(k=i,CD1),u(x,t)=u[i](t)}; EqD[0]:=BC1D; EqD[NX+1]:=BC2D;
for i from 1 to NX do
 EqD[i]:=diff(u[i](t),t)=expand(subs(tr3,tr2,rhs(PDE1))); od;
for i from 1 to NX do EqD[i]:=eval(EqD[i],tr1); od;
Eqs:=seq(EqD[i],i=1..NX); Eqs1:=subs(EqD[0],EqD[NX+1],[Eqs]);
vars:=seq(u[i](t),i=1..NX);
ICs:=seq(u[i](0)=evalf(subs(tr1,subs(tr2,rhs(IC1)))),i=1..NX);
```

Maple:

```
SolN:=dsolve({op(Eqs1),ICs},{vars},type=numeric,output=
 listprocedure,abserr=EAbs);
for i from 1 to NX do U[i]:=subs(SolN,u[i](t)); od;
U[0]:=subs(u[1](t)=U[1],u[2](t)=U[2],u[0](t));
U[NX+1]:=subs(u[NX](t)=U[NX],u[NX-1](t)=U[NX-1],u[NX+1](t));
for i from 1 to NX do
 Gut[i]:=plot(U[i](t),t=0..Tf,thickness=2,color=blue): od:
display({seq(Gut[i],i=1..NX)},labels=[t,u],axes=boxed);
X1:=[seq(subs(tr1,(i-1)*h),i=1..NX+2)]; T1:=[seq(i*Tf/NT,
 i=0..NT)]; for j from 1 to NT+1 do
 Gux[j]:=plot([seq([subs(tr1,i*h),U[i](T1[j])],i=0..NX+1)],
 style=line,thickness=2,color=blue,axes=boxed,numpoints=100):
od: display({seq(Gux[j],j=1..NT+1)},labels=[x,u]);
for i from 1 to NX+2 do
 Guxt[i,1]:=evalf(subs(tr1,subs(x=(i-1)*h,rhs(IC1)))); od;
for i from 1 to NX+2 do for j from 2 to NT+1 do
  Guxt[i,j]:=eval(U[i-1](t),t=T1[j]): od: od:
Points:=[evalf(seq(seq([X1[i],T1[j],Guxt[i,j]],i=2..NX-1),
 j=1..NT-1))]; pointplot3d(Points,symbol=solidsphere,
 shading=z,labels=[x,t,u],orientation=[-50,60],axes=frame);
Points1:=[seq([seq([X1[i],T1[j],Guxt[i,j]],i=2..NX-1)],
 j=1..NT-1)]: surfdata(Points1,axes=boxed,labels=[x,t,u],
 shading=zhue,grid=[20,20]);
```

Mathematica:

```
{l=1,a=0,b=1,tF=0.4, pde1=D[u[x,t],t]==nu*D[u[x,t],{x,2}]-u[x,t]*
 D[u[x,t],x],bc1=u[a,t]==0,bc2=u[b,t]==0,ic1=u[x,0]==Sin[Pi*x/l]}
sol1=Block[{nu=0.009}, NDSolve[{pde1,ic1,bc1,bc2},u,{x,0,l},
 {t,0,tF},Method->{"MethodOfLines","SpatialDiscretization"->
 {"TensorProductGrid","MinPoints"->500,"DifferenceOrder"->
 "Pseudospectral"}}]][[1]]
Plot3D[u[x,t]/.sol1,{x,0,l},{t,0,tF},BoxRatios->{1,1,1},
 AxesLabel->{x,t,u},Mesh->{70,0},ImageSize->500,
 ViewPoint->{-2,-2,3}]
Plot[Table[u[x,i]/.sol1,{i,0,tF,0.01}],{x,0,l},ImageSize->500]
```
□

7.1.2 Nonlinear Systems

Problem 7.2

Maple:

```
with(LinearAlgebra):  with(PDEtools): declare(u(x,t)):
with(plots): Digits:=15; L:=1; a:=0; b:=1; Tf:=0.5;
N:=2; NX:=10; tr1:=h=evalf((b-a)/(NX+1)); tr2:=x=i*h; NT:=40;
EAbs:=0.1e-6; Guxt:=Matrix(NX+2,NT+1); Gvxt:=Matrix(NX+2,NT+1);
PDE1:=diff(u[1](x,t),t)=u[2](x,t)*diff(u[1](x,t),x)+u[1](x,t)+1;
PDE2:=diff(u[2](x,t),t)=-u[1](x,t)*diff(u[2](x,t),x)-u[2](x,t)+1;
IC1:=u[1](x,0)=exp(-x); IC2:=u[2](x,0)=exp(x); BC1:=u[1](x,t)
 =exp(t-1); BC2:=u[2](x,t)=exp(-t); BC3:=u[1](x,t)=exp(t);
BC4:=u[2](x,t)=exp(1-t); for i from 1 to N do
 FWD1||i:=1/2*(-u[k+2,i](t)-3*u[k,i](t)+4*u[k+1,i](t))/h;
 BWD1||i=1/2*(u[k-2,i](t)+3*u[k,i](t)-4*u[k-1,i](t))/h;
 CD1||i:=1/2*(u[k+1,i](t)-u[k-1,i](t))/h;
 CD2||i:=(u[k-1,i](t)-2*u[k,i](t)+u[k+1,i](t))/h^2; od;
BC1D:=subs(lhs(BC1)=u[NX+1,1](t),x=b,BC1); BC2D:=subs(lhs(BC2)=
 u[0,2](t),x=a,BC2); BC3D:=subs(lhs(BC3)=u[0,1](t),x=a,BC3);
BC4D:=subs(lhs(BC4)=u[NX+1,2](t),x=b,BC4);
tr3:={diff(u[1](x,t),x$2)=subs(k='i',CD21), diff(u[2](x,t),x$2)
 =subs(k='i',CD22), diff(u[1](x,t),x)=subs(k='i',CD11),
 diff(u[2](x,t),x)=subs(k='i',CD12), u[1](x,t)=u['i',1](t),
 u[2](x,t)=u['i',2](t)}; EqD[NX+1,1]:=BC1D; EqD[0,2]:=BC2D;
EqD[0,1]:=BC3D; EqD[NX+1,2]:=BC4D; for i from 1 to NX do
 EqD[i,1]:=diff(u[i,1](t),t)=expand(subs(tr3,tr2, rhs(PDE1)));
 EqD[i,2]:=diff(u[i,2](t),t)=expand(subs(tr3,tr2, rhs(PDE2)));od;
for i from 1 to NX do EqD[i,1]:=eval(EqD[i,1],tr1); EqD[i,2]:=
 eval(EqD[i,2],tr1); od; Eqs:=seq(seq(EqD[i,j],i=1..NX),j=1..N);
Eqs1:=subs(EqD[NX+1,1],EqD[0,2],EqD[0,1],EqD[NX+1,2],[Eqs]);
vars:=seq(seq(u[i,j](t),i=1..NX),j=1..N);
ICs:=seq(u[i,1](0)=evalf(subs(tr1,subs(tr2,rhs(IC1)))),i=1..NX),
 seq(u[i,2](0)=evalf(subs(tr1,subs(tr2,rhs(IC2)))),i=1..NX);
```

Maple:

```
SolN:=dsolve({op(Eqs1),ICs},{vars},type=numeric,
 output=listprocedure,abserr=EAbs);
for j from 1 to N do for i from 1 to NX do U[i,j]:=subs(SolN,
 u[i,j](t)); od; od; X1:=[seq(subs(tr1,(i-1)*h),i=1..NX+2)];
T1:=[seq(i*Tf/NT,i=0..NT)]; for i from 1 to NX+2 do
 Guxt[i,1]:=evalf(subs(tr1,subs(x=(i-1)*h,rhs(IC1)))); od;
for i from 1 to NX+2 do for j from 2 to NT+1 do
  Guxt[i,j]:=eval(U[i-1,1](t),t=T1[j]): od: od:
PsU:=[evalf(seq(seq([X1[i],T1[j],Guxt[i,j]],i=2..NX+1),
 j=1..NT+1))]; pointplot3d(PsU,symbol=solidsphere,shading=z,
 labels=[x,t,u],orientation=[40,60],axes=frame);
PsU1:=[seq([seq([X1[i],T1[j],Guxt[i,j]],i=2..NX)],j=1..NT+1)]:
surfdata(PsU1,axes=boxed,labels=[x,t,u],shading=zhue);
for i from 1 to NX+2 do
 Gvxt[i,1]:=evalf(subs(tr1,subs(x=(i-1)*h,rhs(IC2)))); od;
for i from 1 to NX+2 do for j from 2 to NT+1 do
  Gvxt[i,j]:=eval(U[i-1,2](t),t=T1[j]): od: od:
PsV:=[evalf(seq(seq([X1[i],T1[j],Gvxt[i,j]],i=2..NX+1),
 j=1..NT+1))]; pointplot3d(PsV,symbol=solidsphere,shading=z,
 labels=[x,t,v],orientation=[40,60],axes=frame);
PsV1:=[seq([seq([X1[i],T1[j],Gvxt[i,j]],i=2..NX)],j=1..NT+1)]:
surfdata(PsV1,axes=boxed,labels=[x,t,v], shading=zhue);
eval(U[1,1](t),t=0);
```

Mathematica:

```
SetOptions[Plot3D,BoxRatios->{1,1,1},AxesLabel->{x,t,u},
 Mesh->{70,0},ImageSize->500,ViewPoint->{-2,-2,3},
 PlotRange->All]; {l=1,a=0,b=1,tF=0.5, pde1=D[u[x,t],t]==
 v[x,t]*D[u[x,t],x]+u[x,t]+1, pde2=D[v[x,t],t]==-u[x,t]*
 D[v[x,t],x]-v[x,t]+1, ic1=u[x,0]==Exp[-x],ic2=v[x,0]==Exp[x],
 bc1=u[1,t]==Exp[t-1], bc2=v[0,t]==Exp[-t]}
sol1=NDSolve[{pde1,pde2,ic1,ic2,bc1,bc2},{u,v},
 {x,a,b},{t,0,tF},Method->{"MethodOfLines",
 "SpatialDiscretization"->{"TensorProductGrid"}}]
Plot3D[u[x,t]/.sol1,{x,a,b},{t,0,tF}]
Plot3D[v[x,t]/.sol1,{x,a,b},{t,0,tF}]
p1=Evaluate[u[0.1,0]/.sol1]
```

□

7.2 Spectral Collocation Method

7.2.1 Nonlinear Systems

Problem 7.3

Maple:

```
N:=4: S:=0; tr1:={y=C*eta}; tr2:={x=0,t=0}; tr3:={x=0,t=Pi};
for n from 1 to N-1 do for m from 1 to N do
 if type(n+m,odd) then S:=S+B||n||m*cos(n*x)*cos((m-1)*t) fi:
od: od: eta:=S; S:=0:
for n from 1 to N do for m from 1 to N-1 do
 if type(n+m,odd) then
  S:=S+A||n||m*cos((n-1)*x)*cosh((n-1)*(y+h))*sin(m*t)
 fi: od: od: phi:=A0*t+S;
Eq1:=eval(eta*(1+eps*omega^2*cos(2*t))+omega*diff(phi,t)
 +1/2*C*((diff(phi,x))^2+(diff(phi,y))^2),tr1);
Eq2:=eval(diff(phi,y)-omega*diff(eta,t)-C*diff(phi,x)
 *diff(eta,x),tr1);
Eq3:=2-eval(eta,tr2)+eval(eta,tr3); Eq4:=eval(Eq1,tr3);
```

Mathematica:

```
{nN=4, sN=0, tr1=y->cN*eta, tr2={x->0,t->0}, tr3={x->0,t->Pi}}
Do[If[OddQ[n+m],sN=sN+bN[n,m]*Cos[n*x]*Cos[(m-1)*t]],
 {m,1,nN},{n,1,nN-1}]; {eta=sN, sN=0}
Do[If[OddQ[n+m], sN=sN+aN[n,m]*Cos[(n-1)*x]*Cosh[(n-1)*(y+h)]*
 Sin[m*t]],{m,1,nN-1},{n,1,nN}]; phi=a0*t+sN
eq1=(eta*(1+eps*omega^2*Cos[2*t])+omega*D[phi,t]+1/2*cN*
 (D[phi,x]^2+D[phi,y]^2))/.tr1
eq2=(D[phi,y]-omega*D[eta,t]-cN*D[phi,x]*D[eta,x])/.tr1
{eq3=2-(eta/.tr2)+(eta/.tr3), eq4=eq1/.tr3}
```

Maple:

```
tr4:=(j,i)->[x=X||j,t=T||i];
for i from 1 to N/2 do T||i:=Pi/N*(i-1/2):
for j to N-1 do
 if N=4 then X||j:=Pi/3*(j-1/2) else X||j:=Pi/(N-2)*(j-1) fi;
 Eq1||i||j:=eval(subs(tr4(j,i),Eq1));
 Eq2||i||j:=eval(subs(tr4(j,i),Eq2));
 print("Eq1"||i||j,Eq1||i||j); print("Eq2"||i||j,Eq2||i||j);
od; od; L1:=NULL: L2:=NULL:
```

```
for n to N do for m to N-1 do if type(n+m,odd) then
 L1:=L1,A||n||m fi: od: od:
for n to N-1 do for m to N do if type(n+m,odd) then
 L2:=L2,B||n||m fi: od: od: LA:=[L1]; LB:=[L2]; k:=1:
for i from 1 to N/2 do for j from 1 to N-1 do
 F||k:=simplify(Eq1||i||j); F||(k+N/2*(N-1)):=
 simplify(Eq2||i||j); k:=k+1: od; od; LF:=NULL:
F||(N*(N-1)+1):=simplify(Eq3); F||(N*(N-1)+2):=simplify(Eq4);
for i from 1 to N*(N-1)+2 do LF:=LF,F||i; od: LFforNum:={LF}:
```

Mathematica:

```
tr4[j_,i_]:={x->xN[j],t->tN[i]};
Do[tN[i]=Pi/nN*(i-1/2); Print["T","[",i,"]=",tN[i]];
 Do[If[nN==4, xN[j]=Pi/3*(j-1/2), xN[j]=Pi/(nN-2)*(j-1)];
  eqD1[i,j]=eq1/.tr4[j,i]; eqD2[i,j]=eq2/.tr4[j,i];
  Print["Eq1","[",i,",",j,"]=",eqD1[i,j]];
  Print["Eq2","[",i,",",j,"]=",eqD2[i,j]],
{j,1,nN-1}],{i,1,nN/2}]; {l1={}, l2={},lF={}}
Do[If[OddQ[n+m],l1=Append[l1,aN[n,m]]],{m,1,nN-1},{n,1,nN}];
Do[If[OddQ[n+m],l2=Append[l2,bN[n,m]]],{m,1,nN},{n,1,nN-1}];
{lA=l1, lB=l2, k=1}
Do[fF[k]=eqD1[i,j]; fF[k+nN/2*(nN-1)]=eqD2[i,j]; k=k+1,
 {j,1,nN-1},{i,1,nN/2}];
{fF[nN*(nN-1)+1]=eq3,fF[nN*(nN-1)+2]=eq4}
Do[lF=Append[lF,fF[i]],{i,1,nN*(nN-1)+2}]; lFforNum=lF;
```

Maple:

```
SolOmega0:=proc(E,H,c,LEqs::set,A::list,B::list)
 local param,param1,LF1,IVA,IVB,IVA1,IVB1,IVA0,IVals,Sol0,
  OmegaNum0; param:=[C=c,h=H,eps=E];
 param1:=omega0=sqrt(tanh(H));
 LF1:=evalf(eval(LEqs,param)); IVA:={op(A)} minus {A21};
 IVB:={op(B)} minus {B12}; IVA:=[op({op(A)} minus {A21})];
 IVB:=[op({op(B)} minus {B12})];
 IVA1:={seq(IVA[i]=0.,i=1..nops(IVA))};
 IVB1:={seq(IVB[i]=0.,i=1..nops(IVB))}; IVA0:={A0=0.};
 IVals:=eval(eval({A21=-omega0/sinh(H),B12=1.,omega=omega0},
  param1),param) union IVA1 union IVB1 union IVA0;
 Sol0:=fsolve(LF1,IVals);
 OmegaNum0:=eval(rhs(op(select(has,Sol0,omega))),param1);
RETURN(OmegaNum0,Sol0); end proc;
```

Mathematica:

```
solOmega0[eN_,hN_,c_,lEqs_,lA_,lB_]:=Module[
 {param,param1,lF1,ivA,ivB,ivA1,ivB1,ivA0,iVals,sol0,
  omegaNum0},param={cN->c,h->hN,eps->eN}; param1=omega0->
  Sqrt[Tanh[hN]]; lF1=lEqs/.param;
 ivA=Complement[lA,{aN[2,1]}]; ivB=Complement[lB,{bN[1,2]}];
 ivA1=Table[{ivA[[i]],0.},{i,1,Length[ivA]}];
 ivB1=Table[{ivB[[i]],0.},{i,1,Length[ivB]}];
 ivA0={{a0,0.}}; iVals=Union[{{aN[2,1],-omega0/Sinh[hN]},
  {bN[1,2],1.},{omega,omega0}}/.param1/.param,ivA1,ivB1,ivA0];
 sol0=FindRoot[lF1,iVals,PrecisionGoal->Infinity];
 omegaNum0=Select[sol0,MemberQ[#,omega]&][[1,2]]/.param1/.param;
 Return[{omegaNum0,sol0}]];
```

Maple:

```
SolOmega:=proc(E,H,c,LEqs::set,IVals::set)
 local param,param1,LF1,Sol,OmegaNum;
 param:=[C=c,h=H,eps=E]; param1:=omega0=sqrt(tanh(H));
 LF1:=evalf(eval(LEqs,param));
 Sol:=fsolve(LF1,IVals);
 OmegaNum:=eval(rhs(op(select(has,Sol,omega))),param1);
 RETURN(OmegaNum,Sol); end proc;
```

Mathematica:

```
solOmega[eN_,hN_,c_,lEqs_,iVals_]:=Module[
 {param,param1,lF1,sol,omegaNum},
 param={cN->c,h->hN,eps->eN};
 param1=omega0->Sqrt[Tanh[hN]]; lF1=lEqs/.param;
 sol=FindRoot[lF1,iVals,PrecisionGoal->Infinity];
 omegaNum=Select[sol,MemberQ[#,omega]&][[1,2]]/.param1/.param;
 Return[{omegaNum,sol}]];
```

Maple:

```
Digits:=30: with(plots): Ac:=0.11; M:=2; L:=10.5;
kappa:=evalf(Pi*M/L); Hd:=3.*kappa; E:=evalf(-4*Ac*kappa);
H:=3.; g:=981.7;  Cn:=20.; C0:=0.03; Cf:=0.6; Ch:=(Cf-C0)/Cn;
Ampl:=C0; LAmpl:=[seq(C0+i*Ch,i=0..Cn)];
LNum:=NULL: for i to nops(LAmpl) do
 if i=1 then Sol||i:=SolOmega0(E,H,LAmpl[i],LFforNum,LA,LB);
    LNum:=LNum,Sol||i[1];
 else Sol||i:=SolOmega(E,H,LAmpl[i],LFforNum,Sol||(i-1)[2]);
    LNum:=LNum,Sol||i[1]; fi; od: LOmegaNum:=[LNum];
PNum:=evalf([seq([LOmegaNum[i]*sqrt(kappa*g)*2,LAmpl[i]/kappa],
 i=1..nops(LOmegaNum))]);
GNum:=plot(PNum,color=blue,thickness=3): display(GNum);
```

Mathematica:

```
SetOptions[ListPlot,ImageSize->500,Joined->True,PlotRange->
 {All,{0.,1.1}},TicksStyle->Directive[Blue,9],Ticks->
 {{44.5,45.,45.5},Automatic}]; {nD=30, ac=0.11, mN=2, lN=10.5,
 kappa=Pi*mN/lN, hd=3.*kappa, eN=-4*ac*kappa, hN=3., g=981.7,
 cn=20., c0=0.03, cf=0.6, ch=(cf-c0)/cn, ampl=c0}
{lAmpl=Table[c0+i*ch,{i,0,cn}], lNum={}, nlAmpl=Length[lAmpl]}
Do[If[i==1, {sol[i]=solOmega0[eN,hN,lAmpl[[i]],lFforNum,lA,lB],
 lNum=Append[lNum,sol[i][[1]]]},
 {solP[i]=Table[{sol[i-1][[2]][[k,1]],sol[i-1][[2]][[k,2]]},
 {k,1,Length[sol[1][[2]]]}];
 sol[i]=solOmega[eN,hN,lAmpl[[i]],lFforNum,solP[i]],
 lNum=Append[lNum,sol[i][[1]]]}],
{i,1,nlAmpl}]; lOmegaNum=lNum
pNum=Table[{lOmegaNum[[i]]*Sqrt[kappa*g]*2,lAmpl[[i]]/kappa},
 {i,1,Length[lOmegaNum]}]
gNum=ListPlot[pNum,PlotStyle->{Blue,Thickness[0.01]}]; Show[gNum]
```

Maple:

```
LT:=[0,Pi/2,Pi]; LC:=[blue,magenta,"BlueViolet"]; eta; Sol11;
C11:=LAmpl[11]; Prof11:=eval(eta,Sol11[2]); Sol11[1];
Prof12:=subs(x=xD*kappa,t=tD*Sol11[1]*sqrt(kappa*g),Prof11);
Prof13:=Prof12*kappa/C11;
```

```
animate(Prof13,xD=0..L,tD=0..1,color=blue,frames=50,
 numpoints=100,thickness=3,scaling=constrained);
for j from 1 to 3 do Gr||j:=plot(subs(tD=LT[j],Prof13),
 xD=0..L,scaling=constrained,color=LC[j],thickness=2); od:
display({Gr1, Gr2, Gr3});
plot3d(Prof13,xD=0..L,tD=0..1,shading=z,style=patchnogrid,
 transparency=0.9,axes=boxed);
contourplot(Prof13,xD=0..L,tD=0..1,grid=[50,50],contours=10,
 filled=true,coloring=["BlueViolet",magenta]);
```

Mathematica:

```
lTime={0,Pi/2,Pi}; lAspRat={1/2,1/2,0.1}; nP=100;
SetOptions[Plot,ImageSize->300,PlotRange->All,PlotStyle->
 {Blue,Thickness[0.01]},PlotPoints->nP]; {eta,sol[11]}
{cN11=lAmpl[[11]], sol[11][[1]], prof11=eta/.sol[11][[2]]}
prof12=prof11/.x->xD*kappa/.t->tD*sol[11][[1]]*Sqrt[kappa*g]
prof13[x1_,t1_]:=(prof12*kappa/cN11)/.xD->x1/.tD->t1;
Animate[Plot[prof13[xD,tD],{xD,0,lN},AspectRatio->Automatic,
 ImageSize->500,PlotRange->{All,{-2,3}}],{tD,0,1},
 AnimationRate->0.1]
Do[gr[j]=Plot[prof13[x,lTime[[j]]],{x,0,lN},AspectRatio->
 lAspRat[[j]]],{j,1,3}];
GraphicsRow[{gr[1],gr[2],gr[3]},ImageSize->1000]
gr3D=Plot3D[prof13[x,t],{x,0,lN},{t,0,1},ImageSize->500,
 PlotRange->All,PlotStyle->{Hue[0.47],Thickness[0.001]},
 Mesh->None]; grCP=ContourPlot[prof13[x,t],{x,0,lN},{t,0,1},
 ImageSize->500,PlotRange->All]; GraphicsRow[{gr3D,grCP}]
```

□

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