Preface

Consider the system of equations

$$\begin{cases} i\frac{d\chi}{dt} + T\chi(t) = KJ\psi_-(t), \\ \chi(0) = x \in \mathcal{H}, \\ \psi_+ = \psi_- - 2iK^*\chi(t), \end{cases} \tag{0.1}$$

where T is a bounded linear operator from a Hilbert space $\mathcal{H}$ into itself, K is a bounded linear operator from a Hilbert space E ($\dim E < \infty$) into $\mathcal{H}$, $J = J^* = J^{-1}$ maps E into itself, and $\operatorname{Im} T = KJK^*$. If for a given continuous in E function $\psi_-(t) \in L^2_{[0,\tau_0]}(E)$ we have that $\chi(t) \in \mathcal{H}$ and $\psi_+(t) \in L^2_{[0,\tau_0]}(E)$ satisfy the system (0.1), then the following metric conservation law holds:

$$2\|\chi(\tau)\|^2 - 2\|\chi(0)\|^2 = \int_0^\tau (J\psi_-, \psi_-)_E\, dt - \int_0^\tau (J\psi_+, \psi_+)_E\, dt, \quad \tau \in [0, \tau_0]. \tag{0.2}$$

Given an input vector $\psi_- = \varphi_- e^{izt} \in E$, we seek solutions to the system (0.1) as an output vector $\psi_+ = \varphi_+ e^{izt} \in E$ and a state-space vector $\chi(t) = xe^{izt}$ in $\mathcal{H}$, ($z \in \mathbb{C}$). Substituting the expressions for $\psi_\pm(t)$ and $\chi(t)$ in (0.1) allows us to cancel exponential terms and convert the system to stationary algebraic format

$$\begin{cases} (T - zI)x = KJ\varphi_-, \\ \varphi_+ = \varphi_- - 2iK^*x; \end{cases} \qquad \operatorname{Im} T = KJK^*, \quad z \in \rho(T), \tag{0.3}$$

where $\rho(T)$ is the set of regular points of the operator T. The type of an open system in (0.3) was introduced and studied by Livšic [191]. A brief form of an open system (0.3) can be written as a rectangular array known in operator theory as an operator colligation [89]

$$\Theta = \begin{pmatrix} T & K & J \\ \mathcal{H} & & E \end{pmatrix}, \quad \operatorname{Im} T = KJK^*. \tag{0.4}$$

The transfer function of the system Θ of the form (0.3)-(0.4) is given by

$$W_\Theta(z) = I - 2iK^*(T - zI)^{-1}KJ, \tag{0.5}$$

and satisfies, for $z \in \rho(T)$,

$$\begin{aligned} W_\Theta^*(z) J W_\Theta(z) &\geq J, \quad (\operatorname{Im} z > 0),\\ W_\Theta^*(z) J W_\Theta(z) &= J, \quad (\operatorname{Im} z = 0),\\ W_\Theta^*(z) J W_\Theta(z) &\leq J, \quad (\operatorname{Im} z < 0). \end{aligned} \tag{0.6}$$

We call the function

$$V_\Theta(z) = K^*(\operatorname{Re} T - zI)^{-1} K = i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I]J, \tag{0.7}$$

the impedance function of the system Θ. This function $V_\Theta(z)$ is a Herglotz-Nevanlinna function in E. The condition $\operatorname{Im} T = KJK^*$ plays a crucial role in determining the analytical properties of the functions $W_\Theta(z)$ and $V_\Theta(z)$. The open system Θ in (0.3)-(0.4) has the property that its transfer function $W_\Theta(z)$ becomes a J-unitary operator for real $z \in \rho(T)$, i.e.,

$$[\varphi_+, \varphi_+] = [W_\Theta(z)\varphi_-, W_\Theta(z)\varphi_-],$$

where $[\cdot,\cdot] = (J\cdot,\cdot)_E$, and $(\cdot,\cdot)_E$ is an inner product in E.

Let us look at a simple but motivating example leading to a system of the form (0.3)-(0.4). Consider a four-terminal electrical circuit in Figure 1. Let C denote the capacity of the capacitor and let L represent the inductance of an induction coil. Given a harmonic input

$$\psi_- = \varphi_- e^{i\omega t}, \quad \varphi_- = \begin{pmatrix} \sqrt{2}\, I^- \\ \sqrt{2}\, U^- \end{pmatrix},$$

where I^- is the current and U^- is the voltage, we are trying to find the harmonic output

$$\psi_+ = \varphi_+ e^{i\omega t}, \quad \varphi_+ = \begin{pmatrix} I^+ \\ U^+ \end{pmatrix},$$

and also describe the state of the capacitor and the induction coil

$$\chi = x e^{i\omega t}, \quad x = \begin{pmatrix} \sqrt{L}\, I \\ \sqrt{C}\, U \end{pmatrix}.$$

Here I is the current on the induction coil, U is the voltage on the capacitor, I^+ and U^+ are the output current and voltage, respectively. Using electrical circuit equations

$$L\frac{dI}{dt} = U^-(t), \qquad C\frac{dU}{dt} = -I(t) + I^-(t),$$

we can obtain system (0.1), separate variables, and arrive at the system

$$\begin{cases} (T - \omega I)x = KJ\varphi_-, \\ \varphi_+ = \varphi_- - 2iK^* x, \end{cases} \quad , \quad \omega \in \rho(T),$$

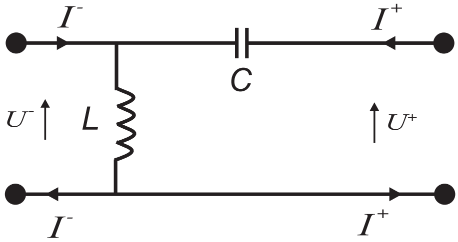

Figure 1: Four-terminal circuit

where

$$T = \begin{pmatrix} 0 & 0 \\ \frac{i}{\sqrt{LC}} & 0 \end{pmatrix}, \quad K = \begin{pmatrix} \frac{i}{\sqrt{2L}} & 0 \\ 0 & \frac{i}{\sqrt{2C}} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{0.8}$$

and x, $\varphi_\pm$ are defined above. By a routine argument one obtains $\operatorname{Im} T = KJK^*$. This open system can be re-written in the form (0.4)

$$\Theta = \begin{pmatrix} T & K & J \\ \mathbb{C}^2 & & \mathbb{C}^2 \end{pmatrix}, \quad \operatorname{Im} T = KJK^*,$$

whose transfer function is of the form (0.5) and actually reads

$$W_\Theta(\omega) = I - 2iK^*(T - \omega I)^{-1}KJ = \begin{pmatrix} 1 & \frac{i}{\omega L} \\ \frac{i}{\omega C} & 1 - \frac{1}{\omega LC} \end{pmatrix},$$

where T, K, and J are defined in (0.8). It is easy to see that $W_\Theta(\omega)$ satisfies the conditions (0.6) with $z = \omega \in \rho(T)$.

When a physical system (for instance a lengthy line) has distributed parameters, the state-space operator T of the system becomes unbounded. As a result, the above mentioned system Θ does not (as an algebraic structure) have any meaning, since the imaginary part of an unbounded operator T may not be defined properly because the domains of T and T^* may not coincide. However, some examples [191] of systems with unbounded operators show that the ranges of the channel operators K belong to some triplets of Hilbert spaces $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ while not being a part of $\mathcal{H}$. In the 1960s Livšic formulated a problem [191] of developing a theory of open systems and their transfer functions that would involve unbounded operators and at the same time preserve the algebraic structure existing in the case when the state-space operator of the system is bounded. The importance of a problem of using generalized functions in system theory (especially in systems with distributed parameters) was pointed out in the 1970s independently by Helton [155]. The solution to this difficult problem is the main subject of the current book. The monograph also covers the research in this area for the last three decades. Different approaches to realization problems of various types of systems with continuous time and conservativity condition (or without) have been considered by Arov-Dym [56], [57], [58], [119], Ball-Staffans [69], Staffans [235], [237], Bart-Gohberg-Kaashoek-Ran [70], [71], and others.

Below we provide a brief description of the results considered in the current text.

- In Chapter 1 we consider some basic facts related to the theory of extensions of linear symmetric operators. In particular, we study the parameterizations of the domains of all self-adjoint extensions in dense and non-dense domain cases. This includes the von Neumann and Krasnoselkiĭ decomposition and parametrization formulas.

- In Chapter 2 we study the extensions of symmetric non-densely defined operators in triplets of rigged Hilbert spaces. The Krasnoselskiĭ formulas discussed in Chapter 1 are based on indirect decomposition, where linear manifolds such as the domain of the symmetric operator and its deficiency subspaces may be linearly dependent. Introduction of rigged Hilbert spaces allows us to obtain direct decomposition for the domain of the adjoint operator and parametrization of all self-adjoint extensions. This direct decomposition is written in terms of the semi-deficiency subspaces and is an analogue of the classical von Neumann formulas.

- Chapter 3 is dedicated to the development of a new extension theory of symmetric operators in triplets of Hilbert spaces, the so-called *bi-extension theory* that will be put to extensive use later in the text.

- Chapter 4 studies quasi-self-adjoint extensions of symmetric operators and contains the definition of the so-called $(*)$-extension. The $(*)$-extensions will be used later in the book in the definition of L-systems. We also present an analysis of these extensions together with their description and parametrization.

- Chapter 5 contains the main concepts and ideas of the classical theory of the Livšic canonical systems (operator colligations) with bounded operators. A comprehensive study of operator colligations of the form (0.4) in operator theory was developed by Brodskiĭ and Livšic [89], [91]. We provide a collection of known results for such a type of systems in terms of transfer functions and their linear-fractional transformations. These results include couplings of these systems and multiplication and factorization theorems for the transfer functions.

- In Chapter 6, we introduce rigged canonical systems with an unbounded operator T, $(\rho(T) \neq \emptyset)$, where $T \supset \dot{A}$, $T^* \supset \dot{A}$, $\dot{A}$ is a symmetric operator, $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ is a rigged Hilbert space generated by $\dot{A}$. If we consider system (0.1) and its stationary version (0.3) with an unbounded operator T such that $\mathrm{Ran}(K) \subset \mathcal{H}_-$, we will run into substantial difficulties even at the stage of defining the solution of the system. This happens because, for a given input $\varphi_- \in E$, the first equation of system (0.3) does not have regular solutions $x \in \operatorname{Dom}(T)$. In order to treat this case adequately, we need to perform a certain regularization of the system that is based on the bi-extension and $(*)$-extension theory developed in Chapters 3 and 4. This regularization also allows us to determine the imaginary part $\operatorname{Im} T$ of the unbounded operator T. Let $\mathbb{A}, \mathbb{A}^* \in [\mathcal{H}_+, \mathcal{H}_-]$ be $(*)$-extensions of T and consider the system

$$\begin{cases} i\frac{d\chi}{dt} + \mathbb{A}\chi(t) = KJ\psi_-(t), \\ \chi(0) = x \in \mathcal{H}_+, \\ \psi_+ = \psi_- - 2iK^*\chi(t), \end{cases} \tag{0.9}$$

where $K \in [E, \mathcal{H}_-]$, $K^* \in [\mathcal{H}_+, E]$, J is a self-adjoint and unitary operator in E, and $\operatorname{Im}\mathbb{A} = KJK^*$. If for a given continuous in E function $\psi_-(t) \in L^2_{[0,\tau_0]}(E)$ we have that a continuous in $\mathcal{H}_+$ and strongly differentiable in $\mathcal{H}$ function $\chi(t) \in \mathcal{H}_+$ and a function $\psi_+(t) \in L^2_{[0,\tau_0]}(E)$ satisfy the system (0.9), then the metric conservation law (0.2) also holds. In line with the approach of the bounded case, we look for stationary solutions and convert our system to the algebraic form

$$\begin{cases} (\mathbb{A} - zI)x = KJ\varphi_-, \\ \varphi_+ = \varphi_- - 2iK^*x, \end{cases} \qquad z \in \rho(T), \tag{0.10}$$

where φ_- and φ_+ are input and output vectors in E, respectively and vector $x \in \mathcal{H}_+$ is a vector of the state space. The system (0.10) above is called *the Livšic rigged, canonical system* or *L-system* and can be written as an array

$$\Theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & E \end{pmatrix}, \quad z \in \rho(T), \tag{0.11}$$

where $\mathbb{A}$ is a $(*)$-extension of T such that

$$\operatorname{Im} T = \operatorname{Im}\mathbb{A} = KJK^*, \quad \operatorname{Ran}(\operatorname{Im}\mathbb{A}) = \operatorname{Ran}(K). \tag{0.12}$$

The transfer function of the system Θ has the form

$$W_\Theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}KJ, \quad z \in \rho(T),$$

and satisfies analytical conditions (0.6). The impedance function of the system Θ is

$$V_\Theta(z) = K^*(\operatorname{Re}\mathbb{A} - zI)^{-1}K = i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I]J.$$

Clearly, systems of the form (0.11)-(0.12) have the same algebraic structure as systems (0.3)-(0.4) with bounded operators. Therefore we can refer to the canonical systems in Chapter 5 as L-systems as well. We recall that, for a holomorphic function that maps the open upper half-plane into itself, one can find the names Herglotz [137], Nevanlinna [42], and R-functions [159] (sometimes depending on the geographical origin of authors). In this text we adopt the term Herglotz-Nevanlinna function and extend it to both scalar and operator-valued cases. An important criteria is obtained for the class of Herglotz-Nevanlinna functions in Hilbert space E ($\dim E < \infty$) of the form

$$V(z) = Q + Lz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\Sigma(t)$$

where $Q = Q^*$, $L \geq 0$, and $\int_{\mathbb{R}} \frac{(d\Sigma(t)x,x)_E}{1+t^2} < \infty$ for all $x \in E$ that can be realized as impedance functions of some scattering ($J = I$) L-system. The proof of this criteria relies on the constructed minimal L-system involving an operator of multiplication by the independent variable in the model functional Hilbert space. This model L-system is extensively used in the following chapters.

- In Chapter 7 we introduce three distinct subclasses of the class of all realizable Herglotz-Nevanlinna operator-valued functions. Complete proofs of direct and inverse realization theorems are given for each subclass. We also provide several multiplication theorems related to this class partition.

- In Chapter 8 we give the definition and describe the properties of normalized canonical L-systems. We also prove an important theorem about the constant J-unitary factor. This theorem states that if an operator-valued function $W(z)$ is realizable as a transfer function of an L-system Θ, then for an arbitrary constant J-unitary operator B the functions $W(z)B$ and $BW(z)$ can be realized as transfer functions of the same type of L-system that contains the same unbounded operator T as Θ but a different channel operator.

- In Chapter 9 we present the solution to the restricted Phillips-Kato extension problem on the existence and description of all proper accretive and sectorial maximal extensions of a given densely defined non-negative symmetric operator. This description (parametrization) is presented in terms of contractive extensions of a given symmetric contraction that are linear fractional transformations of the corresponding accretive operators. The established parametrization strengthens the classical Krein theorem on self-adjoint contractive extensions of symmetric contractions. On the basis of this new parametrization we establish a criterion in terms of the impedance function $V_\Theta(z)$ of an L-system Θ when the state-space operator T of this system is a contraction or so-called an α-co-sectorial contraction. Also in this chapter we establish the criteria for a given Stieltjes or inverse Stieltjes function to be realized as an impedance function of some L-system Θ whose state-space operator T is maximal accretive or α-sectorial.

- Chapter 10 is dedicated to an important application: it provides the descriptions of all accretive and sectorial boundary value problems T_h, where T_h is a Schrödinger operator in $L_2[a, +\infty)$ with a complex boundary parameter h. We also provide a complete description of all L-systems with Schrödinger operator T_h. Moreover, we describe the class of scalar Stieltjes (inverse Stieltjes) like functions that can be realized as impedance functions of L-systems with Schrödinger operator T_h. It is shown that the Schrödinger operator T_h of an L-system is accretive if and only if the impedance function of this L-system is either a Stieltjes or inverse Stieltjes function. We derive the formulas that restore an L-system uniquely from a given Stieltjes (inverse Stieltjes) like function to become the impedance function of this L-system. These formulas allow us to solve the inverse problem and find the exact value of the parameter h in the definition of T_h, as well as a real parameter μ that appears in the construction of the elements of the realizing L-system. An elaborate investigation of these formulas shows the dynamics of the restored parameters h and μ in terms of the changing constant term from the integral representation of the realizable function. We also point out an important connection between the impedance functions of L-systems with Schrödinger operator T_h and the Krein-von Neumann and Friedrichs extensions of a minimal non-negative Schrödinger operator.
- In Chapter 11 we consider a new type of solutions of Nevanlinna-Pick interpolation problems for the class of scalar Herglotz-Nevanlinna functions. These are explicit system solutions that are impedance functions of some L-systems with bounded operators. The conditions for the existence and uniqueness of solutions are presented in terms of interpolation data. We also find new properties of the classical Pick matrices. The exact formula for the angle of sectoriality of the corresponding state-space operator in the explicit system solution is derived. The criterion for this operator to be accretive, but not α-sectorial for any angle $\alpha \in (0, \pi/2)$, is obtained in terms of interpolation data and classic Pick matrices. We find conditions on interpolation data under which the explicit system solution of a scalar Nevanlinna-Pick interpolation problem is generated by the dissipative L-system whose state-space operator is a non-self-adjoint, prime dissipative Jacobi matrix with a rank-one imaginary part. In order to obtain these results, we establish a new model for prime, bounded, dissipative operators with rank-one imaginary part and show that a semi-infinite (finite) bounded Jacobi matrix is a new model. In addition, an inverse spectral problem for finite non-self-adjoint Jacobi matrices with rank-one imaginary part is solved. It is shown that any finite sequence of non-real numbers in the open upper half-plane is the set of eigenvalues (counting multiplicities) of some dissipative non-self-adjoint Jacobi matrix with rank-one imaginary part. The algorithm of reconstruction of the unique Jacobi matrix from its non-real eigenvalues is presented.
- In Chapter 12 we consider non-canonical rigged systems and show that the metric conservation law holds for them as well. Moreover, it is easily seen

that, in the special case when a non-canonical system becomes canonical, the metric conservation law for the non-canonical system matches its canonical version. Later on in this chapter we utilize non-canonical systems to present the solution of the general realization problem for an arbitrary Herglotz-Nevanlinna function. In particular it is shown that an arbitrary Herglotz-Nevanlinna operator-valued function can be realized as the transfer function of the corresponding non-canonical impedance system or NCI-system. The conditions on an arbitrary Herglotz-Nevanlinna function to be an impedance function of a non-canonical L-system (or NCL-system) are also provided.

Over the last several decades many books and papers have been dedicated to the analysis of infinite-dimensional systems and realization problems for different function classes. The literature on this subject is too extensive to be discussed exhaustively but we refer in this matter to [1]–[274] and the literature therein. In this text we propose a comprehensive analysis of the above mentioned L-systems with, generally speaking, unbounded operators that satisfy the *metric conservation law*. We also treat realization problems for Herglotz-Nevanlinna functions and their various subclasses when members of these subclass are realized as impedance functions of L-systems. This type of realizations is called *conservative*. The detailed study provided relies on a new method involving extension theory of linear operators with the exit into rigged triplets of Hilbert spaces. In particular, it is possible to set a one-to-one correspondence between the impedance of L-systems and related $(*)$-extensions of unbounded operators. The theory of *singular systems* developed in the current monograph leads to several useful and important applications including systems with non-self-adjoint Schrödinger operator, non-self-adjoint Jacobi matrices, and system interpolation. In summary, we hope that this book contains new developments and will be of value and interest to researchers in the field of operator theory, spectral analysis of differential operators, and system theory. We also think that this text may be used to teach a graduate level special topics course on this subject.

Acknowledgements

We are very grateful to Harm Bart and Olof Staffans for their reviews and valuable comments. We are also indebted to Vladimir Peller and Peter Kuchment for their helpful suggestions. Special thanks goes to our co-authors and colleagues Vladimir Derkach, Seppo Hassi, Fritz Gesztesy, Konstantin Makarov, Mark Malamud, and Henk de Snoo for their valuable input and fruitful discussions.

Lugansk, Troy, Niagara Falls,

October 2010

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Springer

http://www.springer.com/978-3-7643-9995-5

Conservative Realizations of Herglotz-Nevanlinna Functions

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2011, XVIII, 530 p., Hardcover
ISBN: 978-3-7643-9995-5
A product of Birkhäuser Basel