

Chapter 2

Geometry of Rigged Hilbert Spaces

In this chapter we study extensions of symmetric non-densely defined operators in the triplets $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ of rigged Hilbert spaces. The Krasnoselskii formulas discussed in Section 1.7 are based upon the indirect decomposition (1.33), where deficiency subspaces and the domain of symmetric operator may be linearly dependent. Introduction of the rigged Hilbert spaces allows us to obtain the direct decomposition and parameterization for the domain of the adjoint operator. This direct decomposition is written in terms of the semi-deficiency subspaces and is an analogue of the von Neumann formulas (1.7) and (1.13) for the case of the symmetric operator A whose domain is not dense in \mathcal{H} .

2.1 The Riesz-Berezansky operator

In this section we are going to equip our Hilbert space \mathcal{H} with spaces \mathcal{H}_+ and \mathcal{H}_- , called spaces with positive and negative norms, respectively.

We start with a Hilbert space \mathcal{H} with inner product (x, y) and norm $\|\cdot\|$. Let \mathcal{H}_+ be a dense in \mathcal{H} linear set that is a Hilbert space itself with respect to another inner product $(x, y)_+$ generating the norm $\|\cdot\|_+$. We assume that $\|x\| \leq \|x\|_+$, ($x \in \mathcal{H}_+$), i.e., the norm $\|\cdot\|_+$ generates a stronger than $\|\cdot\|$ topology in \mathcal{H}_+ . The space \mathcal{H}_+ is called the **space with positive norm**.

Now let \mathcal{H}_- be a space dual to \mathcal{H}_+ . It means that \mathcal{H}_- is a space of linear functionals defined on \mathcal{H}_+ and continuous with respect to $\|\cdot\|_+$. By the $\|\cdot\|_-$ we denote the norm in \mathcal{H}_- that has a form

$$\|h\|_- = \sup_{u \in \mathcal{H}_+} \frac{|(h, u)|}{\|u\|_+}, \quad h \in \mathcal{H}.$$

The value of a functional $f \in \mathcal{H}_-$ on a vector $u \in \mathcal{H}_+$ is denoted by (u, f) . The space \mathcal{H}_- is called a **space with negative norm**.

Further on in this chapter we will need to consider an embedding operator $\sigma : \mathcal{H}_+ \mapsto \mathcal{H}$ that embeds \mathcal{H}_+ into \mathcal{H} . Since $\|\sigma f\| \leq \|f\|_+$ for all $f \in \mathcal{H}_+$, then $\sigma \in [\mathcal{H}_+, \mathcal{H}]$. The adjoint operator σ^* maps \mathcal{H} into \mathcal{H}_- and satisfies the condition $\|\sigma^* f\|_- \leq \|f\|$ for all $f \in \mathcal{H}$. Since σ is a monomorphism with a (\cdot) -dense range, then σ^* is a monomorphism with $(-)$ -dense range. By identifying $\sigma^* f$ with f ($f \in \mathcal{H}$) we can consider \mathcal{H} embedded in \mathcal{H}_- as a $(-)$ -dense set and $\|f\|_- \leq \|f\|$. Also, the relation

$$(\sigma f, h) = (f, \sigma^* h), \quad f \in \mathcal{H}_+, h \in \mathcal{H},$$

implies that the value of the functional $\sigma^* h \in \mathcal{H}$ calculated at a vector $f \in \mathcal{H}_+$ as $(f, \sigma^* h)$ corresponds to the value (f, h) in the space \mathcal{H} .

It follows from the Riesz representation theorem that there exists an isometric operator \mathcal{R} which maps \mathcal{H}_- onto \mathcal{H}_+ such that $(f, g) = (f, \mathcal{R}g)_+$ ($\forall f \in \mathcal{H}_+$, $g \in \mathcal{H}_-$) and $\|\mathcal{R}g\|_+ = \|g\|_-$. Now we can turn \mathcal{H}_- into a Hilbert space by introducing $(f, g)_- = (\mathcal{R}f, \mathcal{R}g)_+$. Thus,

$$\begin{aligned} (f, g)_- &= (f, \mathcal{R}g) = (\mathcal{R}f, g) = (\mathcal{R}f, \mathcal{R}g)_+, & (f, g \in \mathcal{H}_-), \\ (u, v)_+ &= (u, \mathcal{R}^{-1}v) = (\mathcal{R}^{-1}u, v) = (\mathcal{R}^{-1}u, \mathcal{R}^{-1}v)_-, & (u, v \in \mathcal{H}_+). \end{aligned} \quad (2.1)$$

The operator \mathcal{R} (or \mathcal{R}^{-1}) will be called the **Riesz-Berezansky operator**. Applying the above reasoning, we define a triplet $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ to be called the **rigged Hilbert space**.

In what follows we use symbols $(+)$, (\cdot) , and $(-)$ to indicate the norms $\|\cdot\|_+$, $\|\cdot\|$, and $\|\cdot\|_-$ by which geometrical and topological concepts are defined in \mathcal{H}_+ , \mathcal{H} , and \mathcal{H}_- . When considering continuity or closeness of an operator, we first indicate the topology of its domain and then the topology of the range. For instance, an operator B is called $(-, \cdot)$ -continuous, if

$$\text{Dom}(B) \subset \mathcal{H}_-, \quad \text{Ran}(B) \subset \mathcal{H}, \quad \sup_{h \in \text{Dom}(B)} \frac{\|Bh\|}{\|h\|_-} < \infty.$$

Similarly, the closure of a set \mathcal{L} using the norms $\|\cdot\|_+$, $\|\cdot\|$, and $\|\cdot\|_-$ is denoted by $\overline{\mathcal{L}}^{(+)}$, $\overline{\mathcal{L}}$, and $\overline{\mathcal{L}}^{(-)}$, respectively. If \mathcal{L} is a subset from \mathcal{H}_+ , then its orthogonal complement \mathcal{L}^\perp will be a set of those functionals from \mathcal{H}_- that annihilate \mathcal{L} . Thus, $\mathcal{L}^\perp \subset \mathcal{H}_-$. Likewise, if $\mathcal{L} \subset \mathcal{H}_-$ then its orthogonal complement $\mathcal{L}^\perp \subset \mathcal{H}_+$ is a set of those elements $x \in \mathcal{H}_+$ that annihilate all the functionals from \mathcal{L} . If $\mathcal{L} \subset \mathcal{H}$, then $\mathcal{L}^\perp \subset \mathcal{H}$ and is defined as a set of elements from \mathcal{H} that are (\cdot) -orthogonal to \mathcal{L} .

The following theorem establishes the relationship between the orthogonal complements of the set \mathcal{L} .

Theorem 2.1.1. 1. If \mathcal{L} is a subspace in \mathcal{H} , then

$$\left(\overline{\mathcal{L}}^{(-)}\right)^\perp = \mathcal{L}^\perp \cap \mathcal{H}_+, \quad (\mathcal{L} \cap \mathcal{H}_+)^\perp = \overline{\mathcal{L}}^\perp. \quad (2.2)$$

2. If \mathcal{L} is a subspace in \mathcal{H}_+ , then

$$(\overline{\mathcal{L}})^\perp = \mathcal{L}^\perp \cap \mathcal{H}. \quad (2.3)$$

3. If \mathcal{L} is a subspace in \mathcal{H}_- , then

$$(\mathcal{L} \cap \mathcal{H})^\perp = \overline{\mathcal{L}}^\perp. \quad (2.4)$$

Proof. Let $f \in \overline{\mathcal{L}}^{(-)}$. Then there is a sequence of elements $\{f_n\} \subset \mathcal{L}$ such that $f_n \xrightarrow{(-)} f$. Therefore $(h, f_n) \rightarrow (h, f)$ for any $h \in \mathcal{H}_+$. In particular, if $h \in \mathcal{L}^\perp \cap \mathcal{H}_+$, then $(h, f_n) = 0$, and consequently, $(h, f) = 0$. Thus, $\mathcal{L}^\perp \cap \mathcal{H}_+ \subset (\overline{\mathcal{L}}^{(-)})^\perp$.

Conversely, let $h \in (\overline{\mathcal{L}}^{(-)})^\perp$. Then $h \in \mathcal{H}_+$ and $(h, f) = 0$ for any $f \in \overline{\mathcal{L}}^{(-)}$, and, in particular, for any $f \in \mathcal{L}$. Hence $h \in \mathcal{L}^\perp$ and $h \in \mathcal{L}^\perp \cap \mathcal{H}_+$. This proves the first part of (2.2). The second part is being proved similarly by substituting \mathcal{L}^\perp for \mathcal{L} .

Statements (2.3) and (2.4) can be proved in a similar way. We just note that one can obtain (2.4) from (2.3) by the respective substitution of $\mathcal{L}(\subset \mathcal{H}_+)$ for $\mathcal{L}^\perp(\subset \mathcal{H}_-)$. \square

Theorem 2.1.1 can be interpreted as asserting the commutativity of the following diagram:

$$\begin{array}{ccc} \mathfrak{R} & \xleftrightarrow{\quad} & \mathfrak{R} \\ \downarrow \uparrow & & \downarrow \uparrow \\ \mathfrak{R}_+ & \xleftrightarrow{\quad} & \mathfrak{R}_- \end{array}$$

Here \mathfrak{R}_+ , \mathfrak{R} , and \mathfrak{R}_- are the classes of all $(+)$ -closed, (\cdot) -closed, and $(-)$ -closed linear manifolds in \mathcal{H}_+ , \mathcal{H} , and \mathcal{H}_- , respectively. The horizontal arrows denote the passage to the orthogonal complement. The short down arrow \downarrow denotes intersection with \mathcal{H}_+ , and the long down arrow stands for the $(-)$ -closure. The long up arrow on the left represents (\cdot) -closure whenever the short up arrow \uparrow on the right denotes intersection with \mathcal{H} .

If B is an operator in the class $[\mathcal{H}_+, \mathcal{H}_-]$, then its **adjoint operator** B^* is defined by the formula $(Bf, g) = (f, B^*g)$ ($\forall f, g \in \mathcal{H}_+$). This operator B^* acts from \mathcal{H}_+ into \mathcal{H}_- , is bounded, and therefore $B^* \in [\mathcal{H}_+, \mathcal{H}_-]$ as well. Thus, the class $[\mathcal{H}_+, \mathcal{H}_-]$ is invariant under taking adjoint. The class $[\mathcal{H}_-, \mathcal{H}_+]$ has a similar property. The concept of a bounded **self-adjoint operator** is, therefore, well defined in both of these classes. For instance, for the class $[\mathcal{H}_+, \mathcal{H}_-]$ such an operator is characterized by the quadratic functional (Bf, f) ($f \in \mathcal{H}_+$) taking real values

only. If $(Bf, f) \geq 0$ for all $f \in \mathcal{H}_+$, then B is called **non-negative**. For an operator $B \in [\mathcal{H}_+, \mathcal{H}_-]$ we introduce a new operator

$$\hat{B} = B \upharpoonright \text{Dom}(\hat{B}), \quad \text{Dom}(\hat{B}) = \{f \in \mathcal{H}_+ : Bf \in \mathcal{H}\}. \quad (2.5)$$

This operator \hat{B} is called a **quasi-kernel** of the operator B .

For the remainder of this text we will need the following theorem.

Theorem 2.1.2. *Let \mathcal{H}_1 , \mathcal{H}_2 , and \mathcal{H} be Hilbert spaces and let B and C be operators in $[\mathcal{H}_1, \mathcal{H}]$ and $[\mathcal{H}_2, \mathcal{H}]$, respectively. The following conditions are equivalent:*

- (i) $\text{Ran}(B) \subset \text{Ran}(C)$;
- (ii) $\ker(C^*) \subset \ker(B^*)$ and

$$\sup_{h \in \mathcal{H}, h \notin \ker(C^*)} \frac{\|B^*h\|}{\|C^*h\|} < \infty;$$

- (iii) *there exists an operator $W \in [\mathcal{H}_1, \mathcal{H}_2]$ such that $B = CW$.*

Proof. First we show that (i) \Rightarrow (ii). The first part of condition (ii) can be derived from (i) by passing to the orthogonal complements. Now let us assume that the second part of the condition (ii) is not true. Then there exists a sequence of $f_n \in \mathcal{H}$ such that $\|B^*f_n\| \rightarrow \infty$ and $\|C^*f_n\| \rightarrow 0$. By condition (i) for any $h_1 \in \mathcal{H}_1$ there is an element $h_2 \in \mathcal{H}_2$ such that $Bh_1 = Ch_2$. We have

$$(B^*f_n, h_1) = (f_n, Bh_1) = (f_n, Ch_2) = (C^*f_n, h_2) \rightarrow 0.$$

Therefore, B^*f_n converges weakly to zero which contradicts $\|B^*f_n\| \rightarrow \infty$.

(ii) \Rightarrow (iii). To every vector $\varphi = C^*f \in \text{Ran}(C^*) \subset \mathcal{H}_2$ we assign a vector $\psi \in B^*f \in \text{Ran}(B^*) \subset \mathcal{H}_1$. According to condition (ii), the operator $\psi = U'\varphi$ is well defined and bounded. We extend U' onto \mathcal{H}_2 to an operator $U \in [\mathcal{H}_2, \mathcal{H}_1]$ for which $B^* = UC^*$. Then, $B = CU^*$ and we can defined $W = U^*$.

It is very easy to see that (iii) \Rightarrow (i). □

Remark 2.1.3. Theorem 2.1.2 can be stated equivalently in the form: For every $A, B \in [\mathcal{H}, \mathcal{H}]$ the following statements are equivalent:

- (i) $\text{Ran}(A) \subset \text{Ran}(B)$;
- (ii) $A = BC$ for some $C \in [\mathcal{H}, \mathcal{H}]$;
- (iii) $AA^* \leq \lambda BB^*$ for some $\lambda \geq 0$.

In this case there is a unique C satisfying $\|C\|^2 = \inf\{\lambda : AA^* \leq \lambda BB^*\}$ and $\text{Ran}(C) \subset \overline{\text{Ran}(B^*)}$, in which case $\ker C = \ker A$.

The following three results are based upon Theorem 2.1.2.

Theorem 2.1.4. *Let $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ be a rigged Hilbert space and \mathcal{G} be a Hilbert space. If $B \in [\mathcal{G}, \mathcal{H}_-]$ and $\text{Ran}(B) \subset \mathcal{H}$, then $B \in [\mathcal{G}, \mathcal{H}]$. Moreover, if $B \in [\mathcal{G}, \mathcal{H}_-]$ or $B \in [\mathcal{G}, \mathcal{H}]$ and $\text{Ran}(B) \subset \mathcal{H}_+$, then $B \in [\mathcal{G}, \mathcal{H}_+]$.*

Proof. For the embedding operator σ defined in the beginning of this section we have that $\sigma^* \in [\mathcal{H}, \mathcal{H}_-]$ and $\text{Ran}(B) \subset \text{Ran}(\sigma)$. According to Theorem 2.1.2 there exists an operator $W \in [\mathcal{G}, \mathcal{H}]$ such that $B = \sigma^* W$. Since σ^* is an embedding of \mathcal{H} into \mathcal{H}_- , then $B \in [\mathcal{G}, \mathcal{H}]$. The proof of the second statement is similar. \square

Theorem 2.1.5. *Let $C \in [\mathcal{H}_+, \mathcal{H}_-]$. Then C is a monomorphism and C^{-1} is $(-, \cdot)$ -continuous if and only if $\text{Ran}(C^*) \supset \mathcal{H}$.*

Proof. The existence and $(-, \cdot)$ -continuity of operator C^{-1} are equivalent to

$$\inf_{h \in \mathcal{H}_+} \frac{\|Ch\|_-}{\|h\|} > 0,$$

i.e., $\ker(C) = \{0\}$ and $\sup \frac{\|\sigma h\|}{\|Ch\|_-} < \infty$. Applying Theorem 2.1.2 we see that the latter is equivalent to $\text{Ran}(\sigma^*) \subset \text{Ran}(C^*)$ that means $\mathcal{H} \subset \text{Ran}(C^*)$. \square

The following theorem can be proven similarly.

Theorem 2.1.6. *Let $C \in [\mathcal{H}_+, \mathcal{H}_-]$. Then C is a monomorphism and C^{-1} is $(-, -)$ -continuous if and only if $\text{Ran}(C^*) \supset \mathcal{H}_+$.*

2.2 Construction of the operator generated rigging

Let now \dot{A} be a closed symmetric operator whose domain $\text{Dom}(\dot{A})$ is not assumed to be dense in \mathcal{H} . Setting $\overline{\text{Dom}(\dot{A})} = \mathcal{H}_0$, we can consider \dot{A} as a densely defined operator from \mathcal{H}_0 into \mathcal{H} . Clearly, $\text{Dom}(\dot{A}^*)$ is dense in \mathcal{H} and $\text{Ran}(\dot{A}^*) \subset \mathcal{H}_0$.

We introduce a new Hilbert space $\mathcal{H}_+ = \text{Dom}(\dot{A}^*)$ with inner product

$$(f, g)_+ = (f, g) + (\dot{A}^* f, \dot{A}^* g), \quad (f, g \in \mathcal{H}_+), \quad (2.6)$$

and then construct the rigged Hilbert space $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$.

Theorem 2.2.1. *Let \dot{A} be a closed symmetric operator in \mathcal{H} . Then*

1. *The operator \dot{A} is $(\cdot, -)$ -continuous.*
2. *If $\overline{\dot{A}}$ is an extension of \dot{A} by $(\cdot, -)$ -continuity to \mathcal{H}_0 , then the Riesz-Berezansky operator is given by the formula*

$$\mathcal{R}^{-1} = I + \overline{\dot{A}} \dot{A}^*. \quad (2.7)$$

3. $\mathcal{R}\mathcal{H} = \text{Dom}(\dot{A} \dot{A}^*)$.

Proof. (1) Since $\|\dot{A}^*h\| \leq \|h\|_+$, ($\forall h \in \mathcal{H}_+$), then

$$\|\dot{A}g\|_- = \sup_{h \in \mathcal{H}_+} \frac{|(\dot{A}g, h)|}{\|h\|_+} = \sup_{h \in \mathcal{H}_+} \frac{|(g, \dot{A}^*h)|}{\|h\|_+} \leq \sup_{h \in \mathcal{H}_+} \frac{\|g\| \cdot \|\dot{A}^*h\|}{\|h\|_+} \leq \|g\|,$$

for all $g \in \text{Dom}(\dot{A})$. This yields the $(\cdot, -)$ -continuity of \dot{A} . Now let \overline{A} be an extension of \dot{A} onto $\text{Dom}(\overline{A}) = \mathcal{H}_0$ using $(\cdot, -)$ -continuity. We will show that

$$(\overline{A}g, f) = (g, \dot{A}^*f), \quad (g \in \mathcal{H}_0, f \in \mathcal{H}_+). \quad (2.8)$$

Let $g \in \mathcal{H}_0$. Then there is a sequence $\{g_n\} \subset \text{Dom}(\dot{A})$ such that $g_n \rightarrow g$ in (\cdot) -metric. Hence $\dot{A}g_n \rightarrow \overline{A}g$ in $(-)$ -metric. Letting $n \rightarrow \infty$ in

$$(\dot{A}g_n, f) = (g_n, \dot{A}^*f),$$

we get (2.8). We should note that (2.8) indicates that $\overline{A} \in [\mathcal{H}_0, \mathcal{H}_-]$ is the adjoint to $\dot{A}^* \in [\mathcal{H}_+, \mathcal{H}_0]$ operator.

The condition that $\overline{A}g \in \mathcal{H}$ for some $g \in \mathcal{H}_0$ implies $g \in \text{Dom}(\dot{A})$. Indeed, it follows from (2.8) that for an arbitrary $f \in \mathcal{H}_+$ we have $g \in \text{Dom}(\dot{A}^{**})$ and $\overline{A}g = \dot{A}^{**}g$. Since $\dot{A} = \dot{A}^{**}$ we have that $g \in \text{Dom}(\dot{A})$.

(2) For any $g, f \in \mathcal{H}_+$,

$$\begin{aligned} (\mathcal{R}^{-1}g, f) &= (g, f)_+ = (g, f) + (\dot{A}^*g, \dot{A}^*f) \\ &= (g, f) + (\overline{A}\dot{A}^*g, f) = ((I + \overline{A}\dot{A}^*)g, f), \end{aligned}$$

which implies (2.7).

(3) Obviously, for $g \in \text{Dom}(\dot{A}\dot{A}^*)$ $\mathcal{R}^{-1}g = (I + \overline{A}\dot{A}^*)g \in \mathcal{H}$. Conversely, if $g \in \mathcal{H}_+$, $\mathcal{R}^{-1}g \in \mathcal{H}$. Then $\overline{A}\dot{A}^*g = \mathcal{R}^{-1}g - g \in \mathcal{H}$. As we have shown above $\dot{A}^*g \in \text{Dom}(\dot{A})$ and thus $g \in \text{Dom}(\dot{A}\dot{A}^*)$. So, the conditions $g \in \text{Dom}(\dot{A}\dot{A}^*)$ and $\mathcal{R}^{-1}g \in \mathcal{H}$ are equivalent. \square

Theorem 2.2.2. Let $f \in \mathcal{H}_0$ and \overline{A} be an extension of \dot{A} by $(\cdot, -)$ -continuity to \mathcal{H}_0 . Then $\overline{A}f$ belongs to \mathcal{H} if and only if $f \in \text{Dom}(\dot{A})$.

Proof. The sufficiency part is obvious because $\overline{A}f = \dot{A}f \in \mathcal{H}$ for $f \in \text{Dom}(\dot{A})$. Assume that for a $g \in \mathcal{H}_+ = \text{Dom}(\dot{A}^*)$, $(\dot{A}^*g, f) = (g, \overline{A}f)$. Since \dot{A} is closed and $\dot{A}^{**} = \dot{A}$, we have that $f \in \text{Dom}(\dot{A})$ and $\dot{A}f = \overline{A}f$. \square

2.3 Direct decomposition and an analogue of the first von Neumann's formula

We call an operator \dot{A} **regular**, if $P\dot{A}$ is a closed operator in \mathcal{H}_0 . Here P is an orthogonal projection in \mathcal{H} onto $\mathcal{H}_0 = \overline{\text{Dom}(\dot{A})}$. Obviously, any densely defined

closed symmetric operator is regular. For a regular operator \dot{A} we construct a rigged Hilbert space $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ using the technique from the previous section. If \dot{A} is densely defined, then by the first von Neumann formula (1.7) we have

$$\mathcal{H}_+ = \text{Dom}(\dot{A}) \dot{+} \mathfrak{N}_\lambda \dot{+} \mathfrak{N}_{\bar{\lambda}}. \quad (2.9)$$

This decomposition is $(+)$ -orthogonal for $\lambda = \pm i$. When the domain of \dot{A} is not dense in \mathcal{H} , Theorem 1.7.1 implies an indirect decomposition

$$\mathcal{H}_+ = \text{Dom}(\dot{A}) + \mathfrak{N}_\lambda + \mathfrak{N}_{\bar{\lambda}}, \quad (2.10)$$

where $\text{Dom}(\dot{A})$, \mathfrak{N}_λ , and $\mathfrak{N}_{\bar{\lambda}}$ may be linearly dependent (see Theorem 1.7.4). Now we are going to derive an analogue of the first von Neumann's formula that has a direct decomposition of the involved linear manifolds. Define two subspaces of \mathcal{H}_+ :

$$\mathcal{D}^* := \mathcal{H}_+ \cap \mathcal{H}_0, \quad \mathcal{D} := \overline{\text{Dom}(\dot{A})}^{(+)}. \quad (2.11)$$

Clearly, \mathcal{D} is the domain of the closure of a densely defined in \mathcal{H}_0 symmetric operator $P\dot{A}$ and $\mathcal{D}^* = \text{Dom}((P\dot{A})^*)$. Hence

$$\mathcal{D}^* = \mathcal{D} \dot{+} \mathfrak{N}'_\lambda \dot{+} \mathfrak{N}'_{\bar{\lambda}}, \quad \text{Im } \lambda \neq 0,$$

where \mathfrak{N}'_λ and $\mathfrak{N}'_{\bar{\lambda}}$ are defined by (1.23). If \dot{A} is regular, then $\mathcal{D} = \text{Dom}(\dot{A})$. According to Theorem 2.1.1 the orthogonal complement of the subspace \mathcal{D}^* is $\overline{\mathfrak{L}}^{(-)}$ where

$$\mathfrak{L} = \mathcal{H} \ominus \mathcal{H}_0. \quad (2.12)$$

This makes

$$\mathfrak{N} = \mathcal{R}\overline{\mathfrak{L}}^{(-)} \quad (2.13)$$

a $(+)$ -orthogonal complement of \mathcal{D}^* . Thus we have

$$\mathcal{H}_+ = \mathcal{D} \dot{+} \mathfrak{N}'_\lambda \dot{+} \mathfrak{N}'_{\bar{\lambda}} \dot{+} \mathfrak{N}, \quad (\text{Im } \lambda \neq 0). \quad (2.14)$$

This is a generalization of the first von Neumann's formula. For $\lambda = \pm i$ we obtain the $(+)$ -orthogonal decomposition

$$\mathcal{H}_+ = \mathcal{D} \oplus \mathfrak{N}'_i \oplus \mathfrak{N}'_{-i} \oplus \mathfrak{N}. \quad (2.15)$$

Let

$$\mathfrak{M} = \mathfrak{N}'_i \oplus \mathfrak{N}'_{-i} \oplus \mathfrak{N}, \quad (2.16)$$

and let $\mathcal{F} (\subset \mathcal{H}_-)$ be the (\cdot) -orthogonal complement of $\text{Dom}(\dot{A}) (\subset \mathcal{H}_+)$, i.e.,

$$\mathcal{F} = \left\{ \varphi \in \mathcal{H}_- : (\varphi, f) = 0 \quad \text{for all } f \in \text{Dom}(\dot{A}) \right\}. \quad (2.17)$$

It is clear that

$$\mathcal{F} = \mathcal{R}^{-1}\mathfrak{M} = \mathcal{R}^{-1}\mathfrak{N}'_i \oplus \mathcal{R}^{-1}\mathfrak{N}'_{-i} \oplus \overline{\mathfrak{L}}^{(-)}, \quad (2.18)$$

and the last decomposition is $(-)$ -orthogonal.

Here and below by $P_{\mathcal{G}}^+$ we denote the orthogonal projection in \mathcal{H}_+ onto a subspace \mathcal{G} of \mathcal{H}_+ . Respectively, $P_{\mathcal{G}}$ would represent the orthogonal projection in \mathcal{H} onto a subspace \mathcal{G} of \mathcal{H} .

Theorem 2.3.1. 1. *The operator $\dot{A}^* \upharpoonright \mathfrak{N}$ is symmetric in \mathcal{H} and $\dot{A}^* \mathfrak{N} \subset \mathcal{D}$.*

2. *The operator $\dot{A}^* \pm iI$ maps $\mathfrak{N} (+, \cdot)$ -isometrically on $\overline{\mathfrak{B}}_{\pm i}$, where \mathfrak{B}_{λ} is defined in (1.22).*

3. *The operator \dot{B} given by the relations*

$$\text{Dom}(\dot{B}) = \text{Dom}(\dot{A}) \oplus \mathfrak{N}, \quad \dot{B} = \dot{A}P_{\text{Dom}(\dot{A})}^+ + \dot{A}^*P_{\mathfrak{N}}^+, \quad (2.19)$$

is closed, densely defined and symmetric in \mathcal{H} , and

$$\text{Dom}(\dot{B}^*) = \text{Dom}(\dot{B}) \oplus \mathfrak{N}'_i \oplus \mathfrak{N}'_{-i}, \quad \dot{B}^* = AP_{\text{Dom}(\dot{A})}^+ + \dot{A}^*P_{\mathfrak{M}}^+. \quad (2.20)$$

Proof. (1) Let $g \in \text{Dom}(\dot{A})$ and $f \in \mathfrak{M}$. Then

$$0 = (g, f)_+ = (g, f) + (\dot{A}^*g, \dot{A}^*f) = (g, f) + (P\dot{A}g, \dot{A}^*f) = (g, f) + (\dot{A}g, \dot{A}^*f).$$

Hence,

$$(\dot{A}g, \dot{A}^*f) = (g, -Pf), \quad \forall g \in \text{Dom}(\dot{A}).$$

Consequently, $\dot{A}^*f \in \text{Dom}(\dot{A}^*) = \mathcal{H}_+$ and $(\dot{A}^*)^2f = -Pf$. On the other hand, since $\text{Ran}(\dot{A}^*) \subset \mathcal{H}_0$, we get $\dot{A}^*f \in \mathcal{D}^*$, i.e., $\dot{A}^*\mathfrak{M} \subset \mathcal{D}^*$.

Let $\psi, f \in \mathfrak{M}$. Then

$$\begin{aligned} (\psi, \dot{A}^*f)_+ &= (\psi, \dot{A}^*f) + (\dot{A}^*\psi, (\dot{A}^*)^2f) = (\psi, \dot{A}^*f) + (\dot{A}^*\psi, -Pf) \\ &= (\psi, \dot{A}^*f) - (\dot{A}^*\psi, f). \end{aligned}$$

In particular, if $\psi \in \mathfrak{N}$, then ψ is $(+)$ -orthogonal to $\dot{A}^*f \in \mathcal{D}^*$. This implies that

$$(\psi, \dot{A}^*f) = (\dot{A}^*\psi, f), \quad \forall \psi \in \mathfrak{N}, \forall f \in \mathfrak{M}.$$

Therefore, the operator $\dot{A}^* \upharpoonright \mathfrak{N}$ is symmetric.

Now let $\psi \in \mathfrak{N}$, $f \in \mathfrak{N}'_{\pm i}$. Then

$$(\dot{A}^*\psi, f)_+ = (\dot{A}^*\psi, f) + (-P\psi, \dot{A}^*f) = (\dot{A}^*\psi, f) - (\psi, \dot{A}^*f) = 0.$$

Thus, $\dot{A}^*\mathfrak{N}$ is $(+)$ -orthogonal to \mathfrak{N}'_i and \mathfrak{N}'_{-i} . It follows from (2.15) that $\dot{A}^*\mathfrak{N} \subset \mathcal{D}$.

(2) Using the symmetric property of $\dot{A}^* \upharpoonright \mathfrak{N}$ for $\psi \in \mathfrak{N}$ we get

$$\|(\dot{A}^* \pm iI)\psi\|^2 = \|\dot{A}^*\psi\|^2 + \|\psi\|^2 = \|\psi\|_+^2.$$

This implies that $(\dot{A}^* \pm iI)\upharpoonright \mathfrak{N}$ is an $(+, \cdot)$ -isometry. Letting $f \in \mathcal{H}_+$ and $g \in \text{Dom}(\dot{A})$ we have

$$\begin{aligned} \left((\dot{A}^* + iI)f, (\dot{A} + iI)g \right) &= (\dot{A}^* f, \dot{A} g) + i(f, \dot{A} g) - i(\dot{A}^* f, g) + (f, g) \\ &= (\dot{A}^* f, P \dot{A} g) + (f, g) = (f, g)_+. \end{aligned}$$

In particular, if $f \in \mathfrak{M}$, then $\left((\dot{A}^* + iI)f, (\dot{A} + iI)g \right) = 0$ and hence

$$(\dot{A}^* + iI)\mathfrak{M} \subset \mathfrak{N}_i.$$

Now we will show that for any $\phi \in \mathfrak{L}$,

$$P_i \phi = -i(\dot{A}^* + iI)\mathcal{R}\phi, \quad (2.21)$$

where \mathcal{R} is the Riesz-Berezansky operator and

$$\|P_i \phi\| = \|\phi\|_-. \quad (2.22)$$

Here and below

$$P_\lambda = P_{\mathfrak{N}_\lambda}.$$

Since $\mathcal{R}\phi \in \mathcal{R}\mathfrak{L} \subset \mathfrak{M}$, $(\dot{A}^* + iI)\mathcal{R}\phi \in \mathfrak{N}_i$. On the other hand, by Theorem 2.2.1, we have $\mathcal{R}\phi \in \text{Dom}(\dot{A}\dot{A}^*)$ and

$$\begin{aligned} \phi &= (I + \dot{A}\dot{A}^*)\mathcal{R}\phi = \left[(\dot{A} + iI)\dot{A}^* - i(\dot{A}^* + iI) \right] \mathcal{R}\phi \\ &= (\dot{A} + iI)\dot{A}^*\mathcal{R}\phi - i(\dot{A}^* + iI)\mathcal{R}\phi, \end{aligned}$$

so $\phi + i(\dot{A}^* + iI)\mathcal{R}\phi \in \mathfrak{M}_{-i}$. This proves (2.21). The latter also implies via (2.1) that

$$\|P_i \phi\| = \|(\dot{A} + iI)\mathcal{R}\phi\| = \|\mathcal{R}\phi\|_+ = \|\phi\|_-, \quad (\phi \in \mathfrak{L}).$$

Thus, the operator $(\dot{A}^* + iI)$ maps a linear $(+)$ -dense in \mathfrak{N} set $\mathcal{R}\mathfrak{L}$ $(+, \cdot)$ -isometrically onto $P_i\mathfrak{L} = \mathfrak{B}_i$. That is why

$$(\dot{A}^* + iI)\mathfrak{N} = \overline{\mathfrak{B}_i}.$$

Similarly we can show the same for $(\dot{A}^* - iI)$. Also

$$P_{-i}\phi = i(\dot{A}^* - iI)\mathcal{R}\phi, \quad \|P_{-i}\phi\| = \|\phi\|_-, \quad (\phi \in \mathfrak{L}), \quad (2.23)$$

so that $\|P_i \phi\| = \|P_{-i} \phi\|$.

(3) Let the operator \dot{B} be given by (2.19). If a vector $h \in \mathcal{H}$ is orthogonal to $\text{Dom}(\dot{B})$, then $h \in \mathfrak{L}$ and $(h, f) = 0$ for all $f \in \mathfrak{N}$. From definition (2.13) of \mathfrak{N} we get

$$0 = (h, \mathcal{R}h) = \|h\|_-^2.$$

Therefore, $h = 0$ and $\text{Dom}(\dot{B})$ is dense in \mathcal{H} .

Since $\dot{A}^* \upharpoonright \mathfrak{N}$ is symmetric in \mathcal{H} , the operator \dot{B} is also symmetric in \mathcal{H} . The relations $(\dot{A}^* \pm iI)\mathfrak{N} = \overline{\mathfrak{B}}_{\pm i}$ yield

$$(\dot{B} \pm iI)\text{Dom}(\dot{B}) = (\dot{A} \pm iI)\text{Dom}(\dot{A}) \oplus \overline{\mathfrak{B}}_{\pm i}.$$

Hence, the linear manifolds $(\dot{B} \pm iI)\text{Dom}(\dot{B})$ are closed in \mathcal{H} . It follows that the operator \dot{B} is closed, and

$$\mathcal{H} \ominus \left((\dot{B} \pm iI)\text{Dom}(\dot{B}) \right) = \mathfrak{N}'_{\pm i}.$$

Thus, semi-deficiency subspace $\mathfrak{N}'_{\pm i}$ of the operator \dot{A} coincides with the deficiency subspace of \dot{B} corresponding to the number $\pm i$. In accordance with the first von Neumann formula for \dot{B}^* we get (2.20). \square

Notice that the operator \dot{B} is a symmetric extension of the operator \dot{A} and \dot{B} is self-adjoint if and only if the semi-deficiency indices of \dot{A} are zero.

Corollary 2.3.2. *The operator $\dot{A}^* \pm iI$ maps $\mathfrak{N}'_{\pm i} \oplus \mathfrak{N}$ with $(+)$ -metric homeomorphically onto the subspace $\mathfrak{N}_{\pm i}$ with either (\cdot) - or $(+)$ -metric. Moreover,*

$$(\dot{A}^* \pm iI)\mathfrak{M} = \mathfrak{N}_{\pm i}.$$

Proof. Indeed, using (1.23) we get

$$(\dot{A}^* \pm iI)(\mathfrak{N}'_{\pm i} \oplus \mathfrak{N}) = \mathfrak{N}'_{\pm i} \dot{+} (\dot{A}^* \pm iI)\mathfrak{N} = \mathfrak{N}'_{\pm i} \dot{+} \overline{\mathfrak{B}}_{\pm i} = \mathfrak{N}_{\pm i}.$$

$(+, \cdot)$ -continuity of $(\dot{A}^* \pm iI)$ follows from

$$\|\dot{A}^* h \pm ih\| \leq \|\dot{A}^* h\| + \|h\| \leq 2\|h\|_+, \quad h \in \mathcal{H}_+.$$

Let $\phi \in \mathfrak{N}'_{\pm i} \oplus \mathfrak{N}$, i.e., $\phi = \varphi + \psi$, where $\varphi \in \mathfrak{N}'_{\pm i}$ and $\psi \in \mathfrak{N}$. Then $\|\phi\|_+^2 = \|\varphi\|_+^2 + \|\psi\|_+^2$. Further,

$$(\dot{A}^* \pm iI)\phi = 2i\varphi + (\dot{A}^* \pm iI)\psi,$$

and the terms $\pm 2i\varphi \in \mathfrak{N}'_{\pm i}$ and $(\dot{A}^* \pm iI)\psi \in \overline{\mathfrak{B}}_{\pm i}$ are (\cdot) -orthogonal. Consequently,

$$\|(\dot{A}^* \pm iI)\phi\|^2 = 4\|\varphi\|^2 + \|(\dot{A}^* \pm iI)\psi\|^2 = 2\|\varphi\|_+^2 + \|\psi\|_+^2 \geq \|\phi\|_+^2,$$

which implies the continuity of the inverse mapping. \square

We should note that the operator $\dot{A}^* \pm iI$ maps $\mathfrak{N}_{\mp i}$ to zero and acts like $\pm 2iI$ on $\mathfrak{N}'_{\pm i}$. Therefore, it maps \mathfrak{M} onto $\mathfrak{N}_{\pm i}$, the mapping is one-to-one, and mutually $(+, +)$ -continuous on $\mathfrak{N}'_{\pm i} \oplus \mathfrak{N}$.

Following Section 1.6 we use (1.26) to introduce an isometric exclusion operator $V = V_i : \mathfrak{B}_i \rightarrow \mathfrak{B}_{-i}$ defined by the formula

$$VP_i f = P_{-i} f, \quad f \in \mathfrak{L}, \quad P_{\pm i} = P_{\mathfrak{N}_{\pm i}}. \quad (2.24)$$

Its closure \overline{V} maps $\overline{\mathfrak{B}_i}$ isometrically onto $\overline{\mathfrak{B}_{-i}}$. It follows from (2.21) and (2.23) that

$$V(\dot{A}^* + iI)\mathcal{R}f = -(\dot{A}^* - iI)\mathcal{R}f, \quad (f \in \mathfrak{L}),$$

and hence

$$\overline{V}(\dot{A}^* + iI)\mathcal{R}f = -(\dot{A}^* - iI)\mathcal{R}f,$$

for all $f \in \overline{\mathfrak{L}}^{(-)}$. Thus

$$\overline{V}(\dot{A}^* + iI)\psi = -(\dot{A}^* - iI)\psi, \quad \psi \in \mathfrak{N}.$$

Theorem 2.3.3. *The operator $P_{\mathfrak{M}}^+$ is a bijection and a homeomorphism of $\mathfrak{N}_{\pm i}$ with the (\cdot) -metric onto $(\mathfrak{N}'_{\pm i} \oplus \mathfrak{N})$ with the $(+)$ -metric.*

Proof. Let $\phi \in \mathfrak{N}_i$. Then $\phi = \varphi + \psi$, where $\varphi \in \mathfrak{N}'_i$ and $\psi \in \overline{\mathfrak{B}_i}$. According to Theorem 2.3.1, there exists such an element $h \in \mathfrak{N}$ that $\dot{A}^*h + ih = \psi$ and $\|\psi\| = \|h\|_+$. Since $\dot{A}^*h \in \text{Dom}(\dot{A})$, then $P_{\mathfrak{M}}^+\psi = ih$, and hence, $P_{\mathfrak{M}}^+\phi = \varphi + ih$. Thus $P_{\mathfrak{M}}^+(\mathfrak{N}_i) = \mathfrak{N}'_{\pm i} \oplus \mathfrak{N}$.

Furthermore,

$$\|\phi\|_+^2 \geq \|P_{\mathfrak{M}}^+\phi\|_+^2 = \|\varphi\|_+^2 + \|h\|_+^2 = 2\|\varphi\|^2 + \|\psi\|^2 \geq \|\varphi\|^2 + \|\psi\|^2 = \|\phi\|^2,$$

implies the conclusion of the theorem for \mathfrak{N}_i . The proof of the theorem for \mathfrak{N}_{-i} is similar. \square

Let us now denote by $P_{\mathfrak{N}}^+$, the orthogonal projection operator from \mathcal{H}_+ onto \mathfrak{N} . We introduce a new inner product $(\cdot, \cdot)_1$ defined by

$$(f, g)_1 = (f, g)_+ + (P_{\mathfrak{N}}^+f, P_{\mathfrak{N}}^+g)_+ \quad (2.25)$$

for all $f, g \in \mathcal{H}_+$. The obvious inequality

$$\|f\|_+^2 \leq \|f\|_1^2 \leq 2\|f\|_+^2$$

shows that the norms $\|\cdot\|_+$ and $\|\cdot\|_1$ are topologically equivalent. It is easy to see that the spaces $\text{Dom}(\dot{A})$, \mathfrak{N}'_i , \mathfrak{N}'_{-i} , \mathfrak{N} are (1)-orthogonal. We write \mathfrak{M}_1 for the Hilbert space $\mathfrak{M} = \mathfrak{N}'_i \oplus \mathfrak{N}'_{-i} \oplus \mathfrak{N}$ with inner product $(f, g)_1$. We denote by \mathcal{H}_{+1} the space \mathcal{H}_+ with norm $\|\cdot\|_1$, and by \mathcal{R} the corresponding Riesz-Berezansky operator related to the triplet $\mathcal{H}_{+1} \subset \mathcal{H} \subset \mathcal{H}_{-1}$. Both operators \mathcal{R} and \mathcal{R} act according to Fig. 2.1 below.

One can also see that

$$\mathcal{R}^{-1} = \mathcal{R}^{-1}(I + P_{\mathfrak{N}}^+), \quad \mathcal{R} = (I - \frac{1}{2}P_{\mathfrak{N}}^+)\mathcal{R}. \quad (2.26)$$

Note also that the operators $2^{-\frac{1}{2}}P_{\mathfrak{M}}^+ \upharpoonright \mathfrak{N}_i$ and $2^{-\frac{1}{2}}P_{\mathfrak{M}}^+ \upharpoonright \mathfrak{N}_{-i}$ mentioned in Theorem 2.3.3 are $(\cdot, 1)$ -isometries from \mathfrak{N}_i and \mathfrak{N}_{-i} onto $\mathfrak{N}'_i \oplus \mathfrak{N}$ and $\mathfrak{N}'_{-i} \oplus \mathfrak{N}$, respectively. It follows from the proof of Theorem 2.3.1 that the explicit expression for $P_{\mathfrak{M}}^+ \upharpoonright \mathfrak{N}_{\pm i}$ is of the form

$$P_{\mathfrak{M}}^+\phi = \pm i(2P_{\mathfrak{N}'_{\pm i}}^+ + P_{\mathfrak{N}}^+)(\dot{A}^* \pm iI)^{-1}\phi, \quad \phi \in \mathfrak{N}_{\pm i}. \quad (2.27)$$

Figure 2.1: Operators \mathcal{R} and \mathcal{R}

2.4 Regular and singular symmetric operators

At the beginning of Section 2.3 we introduced the definition of a regular operator \dot{A} . In this section we will provide a criteria for an operator \dot{A} to be regular. It was shown in Corollary 1.5.5 that for all λ ($\text{Im } \lambda \neq 0$) the manifolds \mathfrak{B}_λ are either all closed or all non-closed. We will show that operator \dot{A} is regular in the first case and is called **singular** in the second.

Theorem 2.4.1. *The following statements are equivalent for a closed symmetric operator \dot{A} :*

- (1) *The manifolds \mathfrak{B}_λ are (\cdot) -closed for all λ ($\text{Im } \lambda \neq 0$).*
- (2) *$\text{Dom}(\dot{A})$ is $(+)$ -closed.*
- (3) *\dot{A} is regular ($P\dot{A}$ is closed).*
- (4) *\dot{A} is $(+, \cdot)$ -bounded.*
- (5) *\mathfrak{L} is $(-)$ -closed.*
- (6) *\mathfrak{N} is (\cdot) -closed.*
- (7) *$\text{Dom}(\dot{B}^*) = \text{Dom}(\dot{A}^*)$, where \dot{B} is defined by (2.19).*

Proof. The equivalence (1) \iff (3) is already proved (see Theorem 1.5.4 and Corollary 1.5.5). The equivalence (2) \iff (3) follows from the definition of $(+)$ -norm.

(2) \Rightarrow (4) follows from the Closed Graph Theorem and the inequality $\|f\| \leq \|f\|_+$, $f \in \mathcal{H}_+$. Since \dot{A} is closed we get (4) \Rightarrow (2).

(1) \iff (5). Because of (2.22) the operator $P_i = P_{\mathfrak{N}_i}$ maps the set \mathfrak{L} isometrically onto the set \mathfrak{B}_i . Thus $(-)$ -closure of \mathfrak{L} is equivalent to the (\cdot) -closure of \mathfrak{B}_i .

(5) \iff (6). Since $\|f\|_-^2 = (\mathcal{R}f, f) = \|\mathcal{R}f\|_+^2$ for $f \in \mathcal{H}_-$, and hence $\|f\|_-^2 \leq \|\mathcal{R}f\| \|f\|$ for $f \in \mathcal{H}$, we get

$$\begin{aligned} \gamma \|f\| &\leq \|f\|_- \leq \|f\| \quad \text{for all } f \in \mathcal{L} \quad \text{and for some } \gamma \in (0, 1) \\ &\iff \|\mathcal{R}f\| \geq \gamma \|\mathcal{R}f\|_+. \end{aligned}$$

Comparing (2.15) with (2.19) and (2.20) we get that the equivalence (2) and (7) holds true. \square

This theorem immediately implies the following independently sufficient conditions for a closed symmetric operator \dot{A} to be regular:

- $\text{Dom}(\dot{A})$ has a finite codimension ($\dim \mathfrak{L} < \infty$);
- $\mathfrak{L} \subset \mathcal{H}_+$, where \mathfrak{L} is defined in (2.12).

Proposition 2.4.2. *If \dot{A} is a regular symmetric operator, then the direct decomposition*

$$\mathcal{H} = \mathcal{H}_0 \dot{+} \mathfrak{N}$$

holds.

Proof. Since \dot{A} is regular, by Theorem 2.4.1 and Theorem 1.5.4 the linear manifold \mathfrak{B}_i is (\cdot) -closed and $\Theta(\mathfrak{L}, \mathfrak{B}_i) < 1$. Now from Lemma 1.5.1 we get the equality

$$P_{\mathfrak{L}}\mathfrak{B}_i = \mathfrak{L}.$$

On the other hand, from Theorem 2.3.1 we have that $\mathfrak{B}_i = (\dot{A}^* + iI)\mathfrak{N}$. Hence, $P_{\mathfrak{L}}(\dot{A}^* + iI)\mathfrak{N} = \mathfrak{L}$. But $\dot{A}^*\mathfrak{N} \subset \text{Dom}(\dot{A}) \subset \mathcal{H}_0$. Therefore,

$$P_{\mathfrak{L}}\mathfrak{N} = \mathfrak{L}.$$

Taking into account that $\mathfrak{N} \cap \mathcal{H}_0 = \{0\}$ and $\mathcal{H} = \mathcal{H}_0 \oplus \mathfrak{L}$, we get the equality $\mathcal{H} = \mathcal{H}_0 \dot{+} \mathfrak{N}$. \square

A closed symmetric operator \dot{A} is said to be an **O-operator** if both its semi-deficiency indices equal zero. For such an operator $\mathfrak{M} = \mathfrak{N}$.

Theorem 2.4.3. *A closed symmetric operator \dot{A} is a regular O-operator if and only if $\mathcal{F} \subset \mathcal{H}$, where \mathcal{F} is of the form (2.17).*

The proof easily follows from the definition of a regular O-operator.

2.5 Closed symmetric extensions

Let A be a closed symmetric extension of a symmetric operator \dot{A} . Then

$$(Af, g) = (f, Ag), \quad (\forall f, g \in \text{Dom}(A)),$$

and, in particular

$$(\dot{A}f, g) = (f, Ag) = (f, PAg), \quad (\forall f \in \text{Dom}(\dot{A}), g \in \text{Dom}(A)).$$

It follows then that $g \in \text{Dom}(\dot{A}^*)$ and $PAg = \dot{A}^*g$, and thus $\text{Dom}(A) \subset \mathcal{H}_+$ and

$$PAf = \dot{A}^*f, \quad f \in \text{Dom}(A), \quad (2.28)$$

where P is an orthogonal projection operator in \mathcal{H} onto $\mathcal{H}_0 = \overline{\text{Dom}(\dot{A})}$. The next theorem is an immediate consequence of (2.28) and the Closed Graph Theorem.

Theorem 2.5.1. *For a closed symmetric extension A of an operator \dot{A} the following conditions are equivalent:*

- (1) *The set $\text{Dom}(A)$ is $(+)$ -closed.*
- (2) *The operator A is $(+, \cdot)$ -bounded.*
- (3) *The operator $\dot{A}^* \upharpoonright \text{Dom}(A)$ is (\cdot, \cdot) -closed.*

Moreover, under the conditions (1)–(3) the operator \dot{A} is regular.

A closed symmetric extension A of a closed symmetric operator \dot{A} satisfying the conditions (1)–(3) of Theorem 2.5.1 is called a **regular symmetric extension**.

Let us recall some aspects of the theory of extensions of closed symmetric operators \dot{A} with non-dense domain developed in Chapter 1. Once again we denote the exclusion operator (2.24) by V . Following Section 1.6 we call an isometric operator U ($\text{Dom}(U) = \overline{\text{Dom}(\dot{U})} \subset \mathfrak{N}_i$, $\text{Ran}(U) \subset \mathfrak{N}_{-i}$) an **admissible operator** if $Ux = Vx$ only for $x = 0$. The general form of a closed symmetric extension A of an operator \dot{A} follows from (1.33) and is given by formulas

$$\begin{aligned} \text{Dom}(A) &= \text{Dom}(\dot{A}) \dot{+} \text{Ran}(I - U), \\ A(g + \varphi - U\varphi) &= \dot{A}g + i(\varphi + U\varphi), \\ g &\in \text{Dom}(\dot{A}), \varphi \in \text{Dom}(U), \end{aligned} \quad (2.29)$$

where U is an admissible operator.

The operator A is self-adjoint if and only if $\text{Dom}(U) = \mathfrak{N}_i$ and $\text{Ran}(U) = \mathfrak{N}_{-i}$. Also, if $\varphi \in \mathfrak{B}_i$ then $\varphi - V\varphi \in \text{Dom}(\dot{A})$. Conversely, if $\varphi - \psi \in \text{Dom}(\dot{A})$, where $\varphi \in \mathfrak{N}_i$, $\psi \in \mathfrak{N}_{-i}$, then $\varphi \in \mathfrak{B}_i$, and $\psi = V\varphi \in \mathfrak{B}_{-i}$. Hence, in particular, for admissible U the equality $x = Ux$ holds only for $x = 0$. Thus, the operator $I - U$ is injective and $(\cdot, +)$ -continuous.

Now we are going to prove an auxiliary result of a general geometrical nature. First let us recall the notion of the **minimal angle** $\alpha(\mathcal{L}, \mathcal{M})$ **between subspaces** \mathcal{L} and \mathcal{M} of a Hilbert space \mathcal{H} :

$$\cos \alpha(\mathcal{L}, \mathcal{M}) := \sup_{\substack{x \in \mathcal{L}, x_2 \in \mathcal{M}, \\ \|x\| = \|y\| = 1}} |(x, y)| \quad (2.30)$$

Definition (2.30) yields the equalities

$$\cos \alpha(\mathcal{L}, \mathcal{M}) = \|P_{\mathcal{L}} \upharpoonright \mathcal{M}\| = \|P_{\mathcal{M}} \upharpoonright \mathcal{L}\|, \quad (2.31)$$

where $P_{\mathcal{L}}$ and $P_{\mathcal{M}}$ are orthogonal projections onto \mathcal{L} and \mathcal{M} , respectively.

Lemma 2.5.2. *Let \mathcal{L}_1 be a subspace of a Hilbert space \mathcal{H} and \mathcal{L}_2 be a linear manifold in \mathcal{H} that is a range of a bounded operator T mapping a Banach space into \mathcal{H} . Let also $P_{\mathcal{L}_1}$ be an orthogonal projection from \mathcal{H} onto \mathcal{L}_1 and $P_{\mathcal{L}_1^\perp} = I - P_{\mathcal{L}_1}$. Then the following statements are equivalent:*

- (1) $\mathcal{L}_1 \cap \mathcal{L}_2 = \{0\}$ and the linear set $\mathcal{L}_1 \dot{+} \mathcal{L}_2$ is closed.

(2) $\mathcal{L}_1 \cap \mathcal{L}_2 = \{0\}$ and the linear set $P_{\mathcal{L}_1^\perp} \mathcal{L}_2$ is closed.

(3) \mathcal{L}_2 is closed and the minimal angle between spaces \mathcal{L}_1 and \mathcal{L}_2 is positive.

(4) \mathcal{L}_2 is closed and $P_{\mathcal{L}_1^\perp} \mathcal{L}_2$ is a homeomorphism.

Proof. (1) \Leftrightarrow (2). It is clear that

$$\mathcal{L}_1 \dot{+} \mathcal{L}_2 = \mathcal{L}_1 \oplus P_{\mathcal{L}_1^\perp} \mathcal{L}_2,$$

and hence, the linear manifolds $\mathcal{L}_1 \dot{+} \mathcal{L}_2$ and $P_{\mathcal{L}_1^\perp} \mathcal{L}_2$ are closed simultaneously.

(1) \Leftrightarrow (3). Assume that $\mathcal{L}_2 = T\mathcal{L}_0$, where \mathcal{L}_0 is a Banach space, $T : \mathcal{L}_0 \rightarrow \mathcal{H}$ is bounded and injective. Consider a Banach space $\mathcal{M} = \mathcal{L}_1 \times \mathcal{L}_0$ with the norm

$$\| \langle f, g \rangle \|_{\mathcal{M}} := \|f\|_{\mathcal{L}_1} + \|g\|_{\mathcal{L}_0}, \quad f \in \mathcal{L}_1, \quad g \in \mathcal{L}_0.$$

Then the mapping $\mathcal{Z} : \mathcal{M} \rightarrow \mathcal{L}_1 \dot{+} \mathcal{L}_2$ given by

$$\mathcal{Z} \langle f, g \rangle = f + Tg$$

is continuous and bijective. Applying the Banach inverse mapping theorem, we get that $\mathcal{L}_2 = \mathcal{Z} \langle 0 \times \mathcal{L}_0 \rangle$ is closed in \mathcal{H} . Let us show that the minimal angle between \mathcal{L}_1 and \mathcal{L}_2 is positive, i.e., $\cos \alpha(\mathcal{L}_1, \mathcal{L}_2) < 1$. Since \mathcal{L}_2 is closed, $\mathcal{L}_1 \cap \mathcal{L}_2 = \{0\}$, and $P_{\mathcal{L}_1^\perp} \mathcal{L}_2$ is closed, from Banach's inverse mapping theorem we get that there exists a constant $\gamma \in (0, 1)$ such that

$$\|P_{\mathcal{L}_1^\perp} h\| \geq \gamma \|h\| \quad \text{for all } h \in \mathcal{L}_2. \quad (2.32)$$

It follows that $\|P_{\mathcal{L}_1} \mathcal{L}_2\| < 1$. This means $\alpha(\mathcal{L}_1, \mathcal{L}_2) > 0$. Thus (1) \Rightarrow (3).

Now suppose that (3) holds. Due to (2.31) we have

$$\cos \alpha(\mathcal{L}_1, \mathcal{L}_2) < 1 \iff (2.32).$$

Therefore, the operator $P_{\mathcal{L}_1^\perp} \mathcal{L}_2$ is a homeomorphism. Taking into account the relation

$$\|f + h\|^2 = \|f + P_{\mathcal{L}_1} h\|^2 + \|P_{\mathcal{L}_1^\perp} h\|^2, \quad f \in \mathcal{L}_1, \quad h \in \mathcal{L}_2,$$

we get that $\mathcal{L}_1 \dot{+} \mathcal{L}_2$ is closed. So, (3) \iff (4) and (3) \Rightarrow (1). \square

Now let \dot{A} be a closed symmetric operator and A be its closed symmetric extension with the corresponding admissible isometric operator U . Applying Lemma 2.5.2 to $\mathcal{H} = \mathcal{H}_+$, $\mathcal{L}_1 = \text{Dom}(\dot{A})$, and $\mathcal{L}_2 = \text{Dom}(U)$ and taking into account that $\text{Dom}(U)$ is the range of a $(\cdot, +)$ -continuous operator $I - U$, we obtain the following theorem:

Theorem 2.5.3. *The following statements are equivalent:*

- (1) A is a regular symmetric extension of \dot{A} (that is, $\text{Dom}(A)$ is a $(+)$ -closed set).

- (2) $P_{\mathfrak{M}}^+(\text{Ran}(I - U))$ is a (+)-closed set.
- (3) $\text{Ran}(I - U)$ is (+)-closed, and the minimal angle between $\text{Dom}(\dot{A})$ and $\text{Ran}(I - U)$ is positive (with respect to the (+)-metric).
- (4) $\text{Ran}(I - U)$ is (+)-closed, and $P_{\mathfrak{M}}^+ \upharpoonright \text{Ran}(I - U)$ is a homeomorphism.

We note that under the conditions of Theorem 2.5.3 there is a constant $c > 0$ such that

$$\|g + \varphi - U\varphi\|_+ \geq c\|\varphi\|, \quad (\forall g \in \text{Dom}(\dot{A}), \forall \varphi \in \text{Dom}(U)). \quad (2.33)$$

Indeed, the positivity of the minimal angle between $\text{Dom}(\dot{A})$ and $\text{Dom}(U)$ implies that orthoprojection on $\text{Dom}(U)$ parallel to $\text{Dom}(\dot{A})$ is (+)-continuous, i.e.,

$$c_1\|\varphi - U\varphi\|_+ \leq \|g + \varphi - U\varphi\|_+,$$

for some $c_1 > 0$. On the other hand, it follows from the Closed Graph Theorem that

$$\|\varphi - U\varphi\|_+ \geq c_2\|\varphi\|, \quad (\varphi \in \text{Dom}(U)),$$

for some $c_2 > 0$. This proves (2.33).

Theorem 2.5.4. *If A is a regular symmetric extension of a regular closed symmetric operator \dot{A} , then there is a constant $c > 0$ such that*

$$\|V\varphi - U\varphi\|_+ \geq c\|\varphi\|, \quad \forall \varphi \in \mathfrak{B}_i \cap \text{Dom}(U), \quad (2.34)$$

where V is of the form (2.24). The converse is valid if $\text{Dom}(U) \supset \mathfrak{B}_i$, in particular, for self-adjoint A .

Proof. For $\varphi \in \mathfrak{B}_i \cap \text{Dom}(U)$ we have

$$V\varphi - U\varphi = -(\varphi - V\varphi) + (\varphi - U\varphi).$$

Since $-(\varphi - V\varphi) \in \text{Dom}(\dot{A})$, then Theorem 2.5.3 yields (2.34).

Conversely, let $\text{Dom}(U) \supset \mathfrak{B}_i$ and for all $\varphi \in \mathfrak{B}_i$

$$\|V\varphi - U\varphi\|_+ \geq c\|\varphi\|,$$

for some $c > 0$. We will show that $P_{\mathfrak{M}}^+ \text{Dom}(U)$ is (+)-closed. Let

$$P_{\mathfrak{M}}^+(g_n - U g_n) \xrightarrow{(+)} 0,$$

for $g_n = g'_n + g''_n$, ($g'_n \in \mathfrak{N}'_i$, $g''_n \in \mathfrak{B}_i$). Then

$$g'_n + P_{\mathfrak{M}}^+(g''_n - U g_n) \xrightarrow{(+)} 0.$$

It was shown in Theorem 2.3.3 that

$$P_{\mathfrak{M}}^+ \mathfrak{B}_i = \mathfrak{N}, \quad P_{\mathfrak{M}}^+ \mathfrak{N}_{-i} = \mathfrak{N} \oplus \mathfrak{N}'_i.$$

Therefore, $P_{\mathfrak{M}}^+(g_n - Ug_n) \subset \mathfrak{N} \oplus \mathfrak{N}'_i$, i.e., the elements g'_n and $P_{\mathfrak{M}}^+(g''_n - Ug_n)$ are $(+)$ -orthogonal. Hence, $g'_n \xrightarrow{(+)} 0$, and thus $P_{\mathfrak{M}}^+(g''_n - Ug_n) \xrightarrow{(+)} 0$.

Furthermore,

$$g''_n - Ug''_n = g''_n - Vg''_n + Vg''_n - Ug''_n.$$

Since the vector $g''_n - Vg''_n \in \text{Dom}(\dot{A})$ is $(+)$ -orthogonal to \mathfrak{M} , then

$$P_{\mathfrak{M}}^+(Vg''_n - Ug''_n) \xrightarrow{(+)} 0.$$

According to Theorem 2.3.3 we have $Vg''_n - Ug''_n \xrightarrow{(+)} 0$. It follows from (2.34) that $g''_n \xrightarrow{(+)} 0$. Therefore, $g_n \xrightarrow{(+)} 0$ which implies that $P_{\mathfrak{M}}^+(\text{Dom}(U))$ is $(+)$ -closed. Then, by Theorem 2.5.3, \dot{A} is a regular closed symmetric extension of the operator A . \square

In the case of a regular closed symmetric operator \dot{A} we can describe its symmetric (in particular, self-adjoint) extensions in terms other than those in Chapter 1. The following theorem gives a characterization of the regular extensions for a regular closed symmetric operator \dot{A} .

Theorem 2.5.5.

- I. For each closed symmetric extension A of a regular operator \dot{A} there exists a (1)-isometric (see formula (2.25) for the definition of (1)-metric) operator $\mathcal{U} = \mathcal{U}(A)$ on \mathfrak{M}_1 with the properties:

- (a) $\text{Dom}(\mathcal{U})$ is $(+)$ -closed and belongs to $\mathfrak{N} \oplus \mathfrak{N}'_i$, $\text{Ran}(\mathcal{U}) \subset \mathfrak{N} \oplus \mathfrak{N}'_{-i}$;
- (b) $\mathcal{U}\psi = \psi$ only for $\psi = 0$, and

$$\begin{aligned} \text{Dom}(A) &= \text{Dom}(\dot{A}) \oplus (I - \mathcal{U})\text{Dom}(\mathcal{U}), \\ A(g + (I - \mathcal{U})\psi) &= \dot{A}g + \dot{A}^*(I - \mathcal{U})\psi + i(\dot{A}\dot{A}^* + I)P_{\mathfrak{M}}^+(I + \mathcal{U})\psi, \end{aligned} \tag{2.35}$$

where $g \in \text{Dom}(\dot{A})$, $\psi \in \text{Dom}(\mathcal{U})$.

Conversely, for each (1)-isometric operator \mathcal{U} with the properties (a) and (b) the operator A defined by (2.35) is a closed symmetric extension of \dot{A} .

- II. The extension A is regular if and only if the manifold $\text{Ran}(I - \mathcal{U})$ is (1)-closed.
- III. The operator A is self-adjoint if and only if $\text{Dom}(\mathcal{U}) = \mathfrak{N} \oplus \mathfrak{N}'_i$, $\text{Ran}(\mathcal{U}) = \mathfrak{N} \oplus \mathfrak{N}'_{-i}$.

Proof. Let \dot{A} be a regular closed symmetric operator and A be its closed symmetric extension whose domain is defined by (2.29) via the corresponding operator U . Besides, $\text{Dom}(A)$ admits $(+)$ -orthogonal decomposition

$$\text{Dom}(A) = \text{Dom}(\dot{A}) \oplus P_{\mathfrak{M}}^+(\text{Ran}(I - U)).$$

We introduce a new operator $\mathcal{U} = \mathcal{U}(A)$ by the formula

$$\mathcal{U}P_{\mathfrak{M}}^+\varphi = P_{\mathfrak{M}}^+U\varphi, \quad \varphi \in \text{Dom}(U). \quad (2.36)$$

Therefore,

$$\begin{aligned} \text{Dom}(\mathcal{U}) &= P_{\mathfrak{M}}^+\text{Dom}(U) \subset \mathfrak{N} \oplus \mathfrak{N}'_i, \\ \text{Ran}(\mathcal{U}) &= P_{\mathfrak{M}}^+(\text{Ran}(U)) \subset \mathfrak{N} \oplus \mathfrak{N}'_{-i}. \end{aligned}$$

It follows from Theorem 2.3.3 that $\text{Dom}(\mathcal{U})$ is $(+)$ -closed and thus the definition (2.36) makes sense. Since the operator $2^{-\frac{1}{2}}P_{\mathfrak{M}}^+\upharpoonright \mathfrak{N}_{\pm i}$ is a $(\cdot, 1)$ -isometry and U is a (\cdot, \cdot) -isometry, the operator \mathcal{U} is isometric with respect to $\|\cdot\|_1$ metric, i.e., $(1, 1)$ -isometry. Let us assume that $\mathcal{U}\psi = \psi$ for some $\psi \in \text{Dom}(\mathcal{U})$. Then $P_{\mathfrak{M}}^+\varphi - P_{\mathfrak{M}}^+U\varphi = 0$ for some $\varphi \in \text{Dom}(U)$. This implies that $\varphi - U\varphi \in \text{Dom}(\dot{A})$. Consequently, $\varphi = 0$ and $\psi = 0$.

From (2.27) we get the parametric expression for \mathcal{U}

$$\begin{cases} \psi = i(2P_{\mathfrak{N}'_i}^+ + P_{\mathfrak{N}}^+)(\dot{A}^* + iI)^{-1}\varphi \\ \mathcal{U}\psi = -i(2P_{\mathfrak{N}'_{-i}}^+ + P_{\mathfrak{N}}^+)(\dot{A}^* - iI)^{-1}Ug \end{cases}, \quad \varphi \in \text{Dom}(U). \quad (2.37)$$

It follows that

$$\begin{cases} \varphi = -i(\dot{A}^* + iI)(\frac{1}{2}P_{\mathfrak{N}'_i}^+ + P_{\mathfrak{N}}^+)\psi \\ U\varphi = i(\dot{A}^* - iI)(\frac{1}{2}P_{\mathfrak{N}'_{-i}}^+ + P_{\mathfrak{N}}^+)\mathcal{U}\psi \end{cases}, \quad \psi \in \text{Dom}(\mathcal{U}). \quad (2.38)$$

Hence,

$$\begin{aligned} \varphi - U\varphi &= \psi - \mathcal{U}\psi - i\dot{A}^*P_{\mathfrak{N}}^+(I + \mathcal{U})\psi, \\ i(\varphi + U\varphi) &= i(I + \mathcal{U})\psi + \dot{A}^*P_{\mathfrak{N}}^+(I - \mathcal{U})\psi. \end{aligned}$$

Let $g \in \text{Dom}(\dot{A})$, $\varphi \in \text{Dom}(U)$. Then

$$A(g + (I - U)\varphi) = \dot{A}g + i(I + U)\varphi.$$

Therefore

$$\begin{aligned} A(g + (I - U)\psi) &= A(g + i\dot{A}^*P_{\mathfrak{N}}^+(I + \mathcal{U})\psi + \varphi - U\varphi) \\ &= \dot{A}g + i\dot{A}\dot{A}^*P_{\mathfrak{N}}^+(I + \mathcal{U})\psi + i(I + U)\varphi \\ &= \dot{A}g + i\dot{A}\dot{A}^*P_{\mathfrak{N}}^+(I + \mathcal{U})\psi + i(I + \mathcal{U})\psi + \dot{A}^*P_{\mathfrak{N}}^+(I - \mathcal{U})\psi \\ &= \dot{A}g + \dot{A}^*(I - \mathcal{U})\psi + i(\dot{A}\dot{A}^* + I)P_{\mathfrak{N}}^+(I + \mathcal{U})\psi. \end{aligned}$$

Conversely, let \mathcal{U} be a $(1, 1)$ -isometry in the subspace \mathfrak{M} with (1) -closed domain $\text{Dom}(\mathcal{U}) \subset \mathfrak{N} \oplus \mathfrak{N}'_i$, and $\text{Ran}(\mathcal{U}) \subset \mathfrak{N} \oplus \mathfrak{N}'_{-i}$ such that $\mathcal{U}\psi = \psi$ only for $\psi = 0$. Let the operator A be defined by (2.35). Then A is an extension of \dot{A} and by direct calculations one can check that A is symmetric. Relations (2.38) define

(\cdot, \cdot) -isometry U with $\text{Dom}(U) = \overline{\text{Dom}(U)} \subset \mathfrak{N}_i$ and $\text{Ran}(U) \subset \mathfrak{N}_{-i}$ such that (2.36) and consequently (2.29) hold.

It remains to show that U is an admissible operator. Suppose $U\varphi = V\varphi$ for some $\varphi \in \mathfrak{B}_i \cap \text{Dom}(U)$. Since the element $\varphi - V\varphi \in \text{Dom}(\dot{A})$ is $(+)$ -orthogonal to \mathfrak{M} , then $P_{\mathfrak{M}}^+\varphi = P_{\mathfrak{M}}^+V\varphi$ and thus $P_{\mathfrak{M}}^+U\varphi = P_{\mathfrak{M}}^+\varphi$, and $\mathcal{U}(P_{\mathfrak{M}}^+\varphi) = P_{\mathfrak{M}}^+\varphi$. Therefore, $P_{\mathfrak{M}}^+\varphi = 0$ and $\varphi = 0$. \square

Theorem 2.5.6. *A regular closed symmetric operator \dot{A} has a regular self-adjoint extension if and only if its semi-deficiency indices are equal.*

Proof. Let A be a regular self-adjoint extension of \dot{A} . Then by Theorem 2.5.5 the corresponding operator $\mathcal{U} = \mathcal{U}(A)$ maps $\mathfrak{N} \oplus \mathfrak{N}'_i$ (1)-isometrically onto $\mathfrak{N} \oplus \mathfrak{N}'_{-i}$. Hence it is clear that the dimensions $\dim \mathfrak{N}'_{-i}$ and $\dim \mathfrak{N}'_i$ are the same and the semi-deficiency indices of \dot{A} are equal.

Conversely, suppose $\dim \mathfrak{N}'_{-i} = \dim \mathfrak{N}'_i$. It is not hard to construct a (1)-isometry \mathcal{U} in \mathfrak{M}_1 that maps $\mathfrak{N} \oplus \mathfrak{N}'_i$ onto $\mathfrak{N} \oplus \mathfrak{N}'_{-i}$ with number 1 as a regular point. For example, we can take $\mathcal{U}|_{\mathfrak{N}'_i}$ to be arbitrary (1)-isometric operator with the range \mathfrak{N}'_{-i} and set $\mathcal{U}h = \varepsilon h$, $h \in \mathfrak{N}$, where $|\varepsilon| = 1$, $\varepsilon \neq 1$. Then the corresponding operator A will become a regular self-adjoint extension of the operator \dot{A} . \square

Theorem 2.5.7. *Let A be a regular self-adjoint extension of a regular symmetric operator \dot{A} . The following statements are valid:*

1. *the operator $PA|_{(\text{Dom}(A) \cap \mathcal{H}_0)}$ is self-adjoint in \mathcal{H}_0 ;*
2. *if $U \in [\mathfrak{N}_i, \mathfrak{N}_{-i}]$ is an admissible operator that determines A by formulas (2.29) and if*

$$\tilde{\mathfrak{N}}_i := \{\varphi \in \mathfrak{N}_i, (U - I)\varphi \in \mathcal{H}_0\}, \quad (2.39)$$

then

$$\mathcal{H}_+ = \text{Dom}(A) \dot{+} (U + I)\tilde{\mathfrak{N}}_i. \quad (2.40)$$

Proof. Let us set

$$\mathfrak{N}_i^0 := (A - iI)^{-1}\mathfrak{L}.$$

Then $\mathfrak{N}_i^0 \subset \mathfrak{N}_i$. Indeed, if $f \in \mathfrak{L}$ and $g \in \mathfrak{M}_{-i} = (\dot{A} + iI)\text{Dom}(\dot{A})$, then

$$(g, (A - iI)^{-1}f) = ((A + iI)^{-1}g, f) = ((\dot{A} + iI)^{-1}g, f) = 0,$$

Since A is a regular self-adjoint extension of \dot{A} , then the operator A is $(+, \cdot)$ -bounded and therefore the resolvent $(A - iI)^{-1}$ has the estimate

$$c_1\|f\| \leq \|(A - iI)^{-1}f\|_+ \leq c_2\|f\| \quad \text{for all } f \in \mathcal{H},$$

with $c > 0$. Therefore \mathfrak{N}_i^0 is a subspace in \mathcal{H}_+ (and simultaneously in \mathcal{H}).

Let $\tilde{\mathfrak{N}}_i$ be defined by (2.39). We are going to prove that

$$\mathfrak{N}_i^0 \oplus \tilde{\mathfrak{N}}_i = \mathfrak{N}_i, \quad (2.41)$$

where the sum is (\cdot) -orthogonal. By (2.29) we have

$$(A + iI)(U - I)\varphi = -i\varphi - iU\varphi + iU\varphi - i\varphi = -2i\varphi, \quad \varphi \in \mathfrak{N}_i,$$

and hence

$$(A + iI)^{-1}\varphi = -\frac{1}{2i}(U - I)\varphi.$$

This yields the equivalence relations

$$(U - I)\varphi \in \mathcal{H}_0 \iff (A + iI)^{-1}\varphi \perp \mathfrak{L} \iff \varphi \perp \mathfrak{N}_i^0,$$

and thus the decomposition (2.41) holds true. Thus, it is proved that

$$\text{Dom}(A) \cap \mathcal{H}_0 = \text{Dom}(\dot{A}) \dot{+} (U - I)\tilde{\mathfrak{N}}_i.$$

Suppose the vector $\psi \in \mathcal{H}_0$ is (\cdot) -orthogonal to $(PA + iI)(\text{Dom}(A) \cap \mathcal{H}_0)$. Then $\psi \in \mathfrak{N}'_i \cap \mathfrak{N}_i^0$. Since $\mathfrak{N}_i^0 \subset \text{Dom}(A)$, we get that $\psi \in \text{Dom}(A)$ and (see (2.28)) $PA\psi = \dot{A}^*\psi = i\psi$. Because $PA \upharpoonright (\text{Dom}(A) \cap \mathcal{H}_0)$ is symmetric, we get that $\psi = 0$. This means $(PA + iI)(\text{Dom}(A) \cap \mathcal{H}_0)$ is dense in \mathcal{H}_0 . Similarly it can be proved that $(PA - iI)(\text{Dom}(A) \cap \mathcal{H}_0)$ is dense in \mathcal{H}_0 . Thus the operator $(PA + iI)(\text{Dom}(A) \cap \mathcal{H}_0)$ is essentially self-adjoint in \mathcal{H}_0 . On the other hand, since A is a regular extension, Theorem 2.5.1 yields that the set $\text{Dom}(A) \cap \mathcal{H}_0$ is $(+)$ -closed. Consequently, the operator $(PA + iI)(\text{Dom}(A) \cap \mathcal{H}_0)$ is (\cdot) -closed and hence is self-adjoint in \mathcal{H}_0 . The first statement is proved.

Since for all $\phi \in \tilde{\mathfrak{N}}_i$, $(U + I)\phi = (U - I)\phi + 2\phi$, then $(U + I)\phi \in \text{Dom}(A)$ implies $\phi \in \text{Dom}(A)$. But then $PA\phi = \dot{A}^*\phi = iP\phi$ and thus $P(A - iI)\phi = 0$ or equivalently $(A - iI)\phi \in \mathfrak{L}$. Therefore, $\phi = 0$ and hence $\text{Dom}(A) \cap (U + I)\tilde{\mathfrak{N}}_i = 0$. It is easy to see that $(U + I)\mathfrak{N}_i^0 \subset \text{Dom}(A)$.

According to (2.10) $\text{Dom}(\dot{A}^*) = \text{Dom}(A) + \mathfrak{N}_i + \mathfrak{N}_{-i}$. Let

$$f = g + \varphi + \psi,$$

where $g \in \text{Dom}(\dot{A})$, $\varphi \in \mathfrak{N}_i$, and $\psi \in \mathfrak{N}_{-i}$. Then

$$f = g + \frac{1}{2}(U - I)(U^{-1}\psi - \varphi) + \frac{1}{2}(U + I)(U^{-1}\varphi + \varphi) = x + (U + I)y,$$

where $x = g + \frac{1}{2}(U - I)(U^{-1}\psi - \varphi)$ and $y = \frac{1}{2}(U + I)(U^{-1}\varphi + \varphi)$. This implies that $\text{Dom}(\dot{A}^*) \subseteq \text{Dom}(A) + (U + I)\mathfrak{N}_i$. Since the inverse inclusion is obvious, we conclude that

$$\text{Dom}(\dot{A}^*) = \text{Dom}(A) + (U + I)\mathfrak{N}_i. \quad (2.42)$$

Combining (2.42), $\text{Dom}(A) \cap (U + I)\tilde{\mathfrak{N}}_i = 0$, and $(U + I)\mathfrak{N}_i^0 \subset \text{Dom}(A)$ we obtain

$$\text{Dom}(\dot{A}^*) = \text{Dom}(A) \dot{+} (U + I)\tilde{\mathfrak{N}}_i.$$

This completes the proof of statement 2. □

Theorem 2.5.8. *Let \dot{A} be a regular O -operator. Then all its regular self-adjoint extensions are of the form*

$$\text{Dom}(A) = \mathcal{H}_+, \quad A = \dot{A}P_{\text{Dom}(\dot{A})}^+ + (\dot{A}^* + (\dot{A}\dot{A}^* + I)^{-1}S)P_{\mathfrak{N}}^+,$$

where S is an arbitrary $(+)$ -self-adjoint and $(+)$ -bounded operator in \mathfrak{N} .

Proof. We apply Theorem 2.5.5 for the case $\mathfrak{N}'_i = \mathfrak{N}'_{-i} = \{0\}$. Then there is a one-to-one correspondence between all (1) -unitary operators \mathcal{U} in \mathfrak{N} having the number 1 as a regular point and all regular self-adjoint extensions A of \dot{A} , which take the form (2.35). Let

$$S = i(I + \mathcal{U})(I - \mathcal{U})^{-1}.$$

Then $S \in [\mathfrak{N}, \mathfrak{N}]$, $S = S^*$, and (2.35) can be re-written as

$$\text{Dom}(A) = \text{Dom}(\dot{A}) \oplus \mathfrak{N} = \mathcal{H}_+, \quad A = \dot{A}P_{\text{Dom}(\dot{A})}^+ + (\dot{A}^* + (\dot{A}\dot{A}^* + I)^{-1}S)P_{\mathfrak{N}}^+. \quad \square$$

Conservative Realizations of Herglotz-Nevanlinna
Functions

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2011, XVIII, 530 p., Hardcover

ISBN: 978-3-7643-9995-5

A product of Birkhäuser Basel