Preface

One may say that the history of hypergeometric functions started practically with a paper by Gauss (cf. [Gau]). There, he presented most of the properties of hypergeometric functions that we see today, such as power series, a differential equation, contiguous relations, continued fractional expansion, special values and so on. The discovery of a hypergeometric function has since provided an intrinsic stimulation in the world of mathematics. It has also motivated the development of several domains such as complex functions, Riemann surfaces, differential equations, difference equations, arithmetic theory and so forth. The global structure of the Gauss hypergeometric function as a complex function, i.e., the properties of its monodromy and the analytic continuation, has been extensively studied by Riemann. His method is based on complex integrals. Moreover, when the parameters are rational numbers, its relation to the period integral of algebraic curves became clear, and a fascinating problem on the uniformization of a Riemann surface was proposed by Riemann and Schwarz. On the other hand, Kummer has contributed a lot to the research of arithmetic properties of hypergeometric functions. But there, the main object was the Gauss hypergeometric function of one variable.

In contrast, for more general hypergeometric functions, including the case of several variables, the question arises: *What in fact are hypergeometric functions ?* Since Gauss and Riemann, many researchers tried generalizing the Gauss hypergeometric function. Those which are known under the names of Goursat, Pochhammer, Barnes, Mellin, and Appell are such hypergeometric functions. Although these functions interested some researchers as special objects, they didn't attract many researchers and no significant result came about. If anything, those expressed with the aid of some properties of hypergeometric functions appeared interestingly in several situations, either in partial or another form. The orthogonal polynomials studied in Szegö's book, several formulas that we can find everywhere in Ramanujan's enormous notebooks, spherical functions on Lie groups, and applications to mathematical physics containing quantum mechanics, are such examples. Simply, they were not considered from a general viewpoint of hypergeometric functions.

In this book, hypergeometric functions of several variables will be treated. Our point of view is that the hypergeometric functions are complex integrals of complex powers of polynomials. Most of the properties of hypergeometric functions which have appeared in the literature up to now can be reconsidered from this point of view. In addition, it turns out that these functions establish interesting connections among several domains in mathematics.

One of the prominent properties of hypergeometric functions is the so-called contiguity relations. We understand them based on the classical paper by G. D. Birkhoff [Bir1] about difference equations and their generalization. This is an approach treating hypergeometric functions as solutions of difference equations with respect to shifts of parameters, and characterizing by analysis of asymptotic behaviors when the parameters tend to infinity. One sees a relation between the Padé approximation and the continued fractional expansion. For this purpose, we use either analytic or algebraic de Rham cohomology (twisted de Rham cohomology) as a natural form of complex integrals. In Chapter 2, several relations satisfied by hypergeometric functions will be derived and explained in terms of twisted de Rham cohomology. There, the reader may notice that the excellent idea due to J. Hadamard about a "*finite part of a divergent integral*" developed in his book [Had] will be naturally integrated into the theory. In Chapter 4, we will construct cycles via the saddle point method and apply the Morse theory on affine varieties to describe the global structure of an asymptotic behavior of solutions to difference equations.

Another prominent feature is a holonomic system of partial differential equations satisfied by hypergeometric functions, in particular, an infinitesimal concept called integrable connection (the Gauss−Manin connection) that has a form of partial differential equations of the first order, and a topological concept called monodromy that is its global realization. The latter means to provide a linear representation of a fundamental group, in other words, a local system on the underlying topological space. But here, what is important is not only the topological concept but the mathematical substance that provides it. Hypergeometric functions provide such typical examples. As a consequence, they also help us understand the fundamental group itself.

We will treat complex integrals of complex powers of polynomials, but the main point is not only to state general theorems in an abstract form but also to provide a concrete form of the statements. In Chapters 3 and 4, for linear polynomials, concrete formulas of differential equations, difference equations, integral representations, etc. will be derived, applying the idea from the invariant theory of general linear groups.

In the world surrounding hypergeometric functions, there are several subjects studying power series, orthogonal functions, spherical functions, differential equations, difference equations, etc. in a broad scope such as real (complex) analysis, arithmetic analysis, geometry, algebraic topololgy and combinatorics, which are mutually related and attract researchers. This book explains one such idea. In particular, micro-local analysis and the theory of holonomic $\mathcal{D}$-modules developed in Japan provided considerable impacts. In Chapter 3 of this book, we will treat a holonomic system of Fuchsian partial differential equations over Grassmannians satisfied by the hypergeometric functions, introduced by Gelfand et al., defined as integrals of complex powers of functions as described above. But there, we will explain them only by concrete computations. For a general theory of $\mathcal{D}$-modules, we propose that the reader consult the book written by Hotta and Tanisaki[1] in this series. Here, we will not treat either arithmetic aspects or the problem of the uniformization of complex manifolds. There are also several applications to mathematical physics such as conformal field theory, and solvable models in statistical mechanics. For these topics, the reader may consult Appendix D and the references in this book.

If this book serves as the first step to understanding hypergeometric functions and motivate the reader's interest towards further topics, we should say that our aim has been accomplished.

We asked Toshitake Kohno to write Appendix D including his recent result. We express our gratitude to him.

Lastly, our friends Takeshi Sasaki, Keiji Matsumoto and Masaaki Yoshida gave us precious remarks and criticisms on this manuscript. We also express our gratitude to them.

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[1] The translation is published as [H-T-T].

Preface to English Edtion

After the publication of the original Japanese edition, hypergeometric functions attracted researchers both domestic and abroad, and some aspects are now fairly developed, for example in relation to arrangement of hyperplanes, conformal field theory and random matrix theory. Some related books have also been published: those by M. Yoshida [Yos3], which treats the uniformaization via period matrix, by M. Saito, B. Sturmfels and N. Takayama [S-S-T], which treats algebraic $\mathcal{D}$-modules satisfied by hypergeometric functions, and by P. Orlik and H. Terao [Or-Te3], which sheds light on hypergeometric functions from viewpoint of arrangements of hyperplanes, are particularly related to the contents of this book.

In this English edition, the contents are almost the same as the original except for a minor revision. In particular, in spite of its importance, hypergeometric functions of confluent type are not treated in this book (they can be treated in the framework of twisted de Rham theory but the situation becomes much more complicated). As for the references, we just added several that are directly related to the contents of this book. For more detailed and up-to-date references, the reader may consult the book cited above, etc.

The co-author of this book, who had been going to produce outstanding results unfortunately passed away in 1995. May his soul rest in peace.

Finally, I am indebted to Dr. Kenji Iohara, who has taken the trouble to translate the original version into English.

August, 2010. *Kazuhiko Aomoto*

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