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★**Wiener chaos: moments, cumulants and diagrams.**

A survey with computer implementation.
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The original idea of Wiener chaos as an expansion of square-integrable functionals of the Brownian motion in multiple stochastic integrals goes back to the work of N. Wiener and K. Itô in the 1940s. Around the same time, R. H. Cameron and W. T. Martin introduced an alternative approach, based on a Fourier series-type expansion. While both approaches suggest a close connection with combinatorics, it was only in the 1990s that G.-C. Rota and C. Wallstrom developed a combinatorial approach to stochastic integration.

The objective of this book is to provide a detailed account of the combinatorial structures arising from the study of multiple stochastic integrals. One of the advantages of this approach is that many constructions can be generalized in various directions well beyond the original Wiener-Itô setting.

Chapter 1, the introduction, besides the usual summary of what is coming up, presents a detailed account, with references, of many topics that have been left out.

Chapter 2 introduces the set partitions and related combinatorial objects (Bell and Touchard polynomials, Stirling numbers, and Möbius functions). There are proofs of many key results, such as the general Möbius inversion.

Chapter 3 discusses, in the combinatorial context, various connections between moments and cumulants of a random vector. Also presented, with proofs, are the Stein and Chen-Stein lemmas, characterizing, respectively, Gaussian and Poisson distributions.

Chapter 4, despite having only 12 pages, is one of the central chapters in the book. It is about graphical representation of two partitions by a diagram. One of the main applications is graphical computation of cumulants from moments and, more generally, the method of moments and cumulants in the derivation of central limit theorems. The authors provide a unified approach to these graphical computations using set partitions.

Chapter 5, the longest in the book, constructs the stochastic integral as a stochastic product measure. The starting point is a centered, completely random measure φ having a non-atomic control measure ν on a measurable space $(Z,\mathcal{Z})$. In other words, for every set A with $\nu(A)<\infty$, $\varphi(A)$ is a zero-mean random variable such that $E[\varphi(A)\varphi(B)]=\nu(A\cap B)$ and if the sets $A_1,\dots,A_n$ are disjoint, then the random variables $\varphi(A_1),\dots,\varphi(A_n)$ are independent. These properties of φ are enough to construct a single integral $\varphi(f)=\int_Z f(z)\varphi(dz)$ for deterministic functions $f\in L_2(Z,\nu)$, but construction of multiple integrals requires additional technical conditions on φ to induce a suitable random product measure on Z^n. A centered completely random measure satisfying these additional conditions is called good; of course, the centered Gaussian and compensated

Poisson measures are good.

Some of the results of the chapter are: (a) a proof that, for every completely random measure φ with a non-atomic control measure ν and every deterministic function $f \in L_2(Z, \nu)$, the random variable $\varphi(f)$ is infinitely divisible; (b) construction of a multiple integral of order n for a good completely random measure over a given partition of the set $\{1, 2, \dots, n\}$—the classical multiple Wiener-Itô integral corresponds to the particular partition $\{\{1\}, \{2\}, \dots, \{n\}\}$; (c) specific computations in the case of Gaussian and Poisson measures.

Chapter 6 presents two proofs of the general multiplication formula for multiple Wiener-Itô integrals and the specifications in the Gaussian and Poisson cases. Also discussed is the operation of contraction of two symmetric functions. Chapter 7 continues with the formulas for the expectation and cumulants of the products of multiple Wiener-Itô integrals (diagram formulas). Derivation of those formulas requires additional restrictions on the measure φ to ensure that a certain multi-dimensional analogue of ν stays non-atomic. This leads to the notion of multiplicative good centered, completely random measure (Definition 7.1.1). Gaussian and Poisson measures continue to work.

Chapters 8, 9, and 11 consider the Gaussian case. In Chapter 8, the construction based on the completely chaotic measure is further generalized to the isonormal Gaussian process, that is, a collection of zero-mean Gaussian random variables indexed by the elements of a separable Hilbert space H. In the particular case emphasized by the authors in Chapter 9, H is a collection of linear combinations with real coefficients of complex-valued Hermitian functions that are square integrable with respect to a real non-atomic symmetric measure on the real line. This leads to the family of Hermite processes $X_{n,\gamma}$, which are elements of the nth chaos space and are self-similar with parameter $\gamma \in (1/2, 1)$; $n = 1$ corresponds to the fractional Brownian motion with Hurst parameter γ. Chapter 11 discusses chaotic limit theorems. As an example, the central limit theorem is stated for the random variables in a fixed chaos space and the rate of convergence to the normal distribution in the total variation metric is established.

Chapters 10 and 12 discuss some of the results of Chapters 8, 9, and 11 in the Poisson case. Of special interest could be Chapter 10, where the authors present a detailed account of the Charlier polynomials, the polynomials orthogonal with respect to the Poisson distribution.

The main part of the appendix describes Mathematica functions implementing various ideas from the book, such as generating set partitions; connecting moments and cumulants; computing symmetrization and contraction of functions, multiple Gaussian and Poisson integrals over a given set partition; and products of multiple integrals. There are detailed instructions on using the corresponding Mathematica notebook, as well as on getting the notebook itself. The second part of the appendix gives explicit connections between cumulants and moments, both central and non-central, up to the order 11–14.

Despite its relatively short size (about 210 pages without the appendices), the book is not a quick read. Non-experts in combinatorics might spend extra time on the first four chapters. The good news is that the presentation is very clear, with all the necessary proofs and examples.

The authors clearly accomplish the three goals they list in the introduction (to provide a unified approach to the diagram method using set partition, to give a combinatorial analysis of multiple stochastic integrals in the most general setting, and to discuss chaotic limit theorems). The result could be increased and improved collaboration between the “discrete” and “continuous” probabilists, a big accomplishment indeed.

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