Andrea Pascucci

PDE and Martingale Methods in Option Pricing

Bocconi University Press

Springer

Andrea Pascucci
Dipartimento di Matematica
Università di Bologna
andrea.pascucci@unibo.it

B&SS – Bocconi & Springer Series
ISSN print edition: 2039-1471 ISSN electronic edition: 2039-148X

ISBN 978-88-470-1780-1 e-ISBN 978-88-470-1781-8

DOI 10.1007/978-88-470-1781-8

Library of Congress Control Number: 2010929483

Springer Milan Dordrecht Heidelberg London New York

9 8 7 6 5 4 3 2 1

Cover-Design: K Design, Heidelberg

Typesetting with LaTeX: PTP-Berlin, Protago TeX-Production GmbH, Germany (www.ptp-berlin.eu)
Printing and Binding: Grafiche Porpora, Segrate (Mi)

Printed in Italy

Springer-Verlag Italia srl – Via Decembrio 28 – 20137 Milano
Springer is a part of Springer Science+Business Media (www.springer.com)

Preface

This book gives an introduction to the mathematical, probabilistic and numerical methods used in the modern theory of option pricing. It is intended as a textbook for graduate and advanced undergraduate students, but I hope it will be useful also for researchers and professionals in the financial industry.

Stochastic calculus and its applications to the arbitrage pricing of financial derivatives form the main theme. In presenting these, by now classic, topics, the emphasis is put on the more quantitative rather than economic aspects. Being aware that the literature in this field is huge, I mention the following incomplete list of monographs whose contents overlap with those of this text: in alphabetic order, Avellaneda and Laurence [14], Benth [43], Björk [47], Dana and Jeanblanc [84], Dewynne, Howison and Wilmott [340], Dothan [100], Duffie [102], Elliott and Kopp [120], Epps [121], Follmer and Schied [134], Glasserman [158], Huang and Litzenberger [171], Ingersoll [178], Karatzas [200; 202], Lamberton and Lapeyre [226], Lipton [239], Merton [252], Musiela and Rutkowski [261], Neftci [264], Shreve [310; 311], Steele [315], Zhu, Wu and Chern [349].

What distinguishes this book from others is the attempt to present the matter by giving equal weight to the probabilistic point of view, based on the martingale theory, and the analytical one, based on partial differential equations. The present book does not claim to describe the latest developments in mathematical finance: that target would indeed be very ambitious, given the speed of progress of research in the field. Instead, I have chosen to develop some of the essential ideas of the classical pricing theory to devote space to the fundamental mathematical and numerical tools when they arise. Thus I hope to provide a sound background of basic knowledge which may facilitate the independent study of newer problems and more advanced models.

The theory of stochastic calculus, for continuous and discontinuous processes, constitutes the bulk of the book: Chapters 3 on stochastic processes, 4 on Brownian integration and 9 on stochastic differential equations may form the material for an introductory course on stochastic calculus. In these chapters, I have constantly sought to combine the theoretical concepts to the insight on the financial meaning, in order to make the presentation less abstract and more motivated: in fact many theoretical concepts naturally lend themselves to an intuitive and meaningful economic interpretation.

The origin of this book can be traced to courses on option pricing which I taught at the master program in Quantitative Finance of the University of Bologna, which I have directed with Sergio Polidoro since its beginning, in 2004. I wrote the first version as lecture notes for my courses. During these years, I substantially improved and extended the text with the inclusion of sections on numerical methods and the addition of completely new chapters on stochastic calculus for jump processes and Fourier methods. Nevertheless, during these years the original structure of the book remained essentially unchanged.

I am grateful to many people for the suggestions and helpful comments with which supported and encouraged the writing of the book: in particular I would like to thank several colleagues and PhD students for many valuable suggestions on the manuscript, including David Applebaum, Francesco Caravenna, Alessandra Cretarola, Marco Di Francesco, Piero Foscari, Paolo Foschi, Ermanno Lanconelli, Antonio Mura, Cornelis Oosterlee, Sergio Polidoro, Valentina Prezioso, Enrico Priola, Wolfgang Runggaldier, Tiziano Vargiolu, Valeria Volpe. I also express my thanks to Rossella Agliardi, co-author of Chapter 13, and to Matteo Camaggi for helping me in the translation of the book.

It is greatly appreciated if readers could forward any errors, misprints or suggested improvements to: `andrea.pascucci@unibo.it`
Corrections received after publication will be posted on the website:
`http://www.dm.unibo.it/~pascucci/`

Bologna, November 2010 *Andrea Pascucci*

Contents

Preface V

General notations XV

1 Derivatives and arbitrage pricing 1
- 1.1 Options 1
  - 1.1.1 Main purposes 3
  - 1.1.2 Main problems 4
  - 1.1.3 Rules of compounding 4
  - 1.1.4 Arbitrage opportunities and Put-Call parity formula . . 5
- 1.2 Risk-neutral price and arbitrage pricing 7
  - 1.2.1 Risk-neutral price 7
  - 1.2.2 Risk-neutral probability 8
  - 1.2.3 Arbitrage price 8
  - 1.2.4 A generalization of the Put-Call parity 10
  - 1.2.5 Incomplete markets 11

2 Discrete market models 15
- 2.1 Discrete markets and arbitrage strategies 15
  - 2.1.1 Self-financing and predictable strategies 16
  - 2.1.2 Normalized market 19
  - 2.1.3 Arbitrage opportunities and admissible strategies 20
  - 2.1.4 Equivalent martingale measure 21
  - 2.1.5 Change of numeraire 24
- 2.2 European derivatives 26
  - 2.2.1 Pricing in an arbitrage-free market 27
  - 2.2.2 Completeness 30
  - 2.2.3 Fundamental theorems of asset pricing 31
  - 2.2.4 Markov property 34
- 2.3 Binomial model 35
  - 2.3.1 Martingale measure and arbitrage price 38
  - 2.3.2 Hedging strategies 40

2.3.3 Binomial algorithm . . . 45
2.3.4 Calibration . . . 50
2.3.5 Binomial model and Black-Scholes formula . . . 53
2.3.6 Black-Scholes differential equation . . . 60
2.4 Trinomial model . . . 62
2.4.1 Pricing and hedging in an incomplete market . . . 66
2.5 American derivatives . . . 72
2.5.1 Arbitrage price . . . 74
2.5.2 Optimal exercise strategies . . . 80
2.5.3 Pricing and hedging algorithms . . . 83
2.5.4 Relations with European options . . . 88
2.5.5 Free-boundary problem for American options . . . 90
2.5.6 American and European options in the binomial model 93

3 Continuous-time stochastic processes . . . 97
3.1 Stochastic processes and real Brownian motion . . . 97
3.1.1 Markov property . . . 100
3.1.2 Brownian motion and the heat equation . . . 102
3.2 Uniqueness . . . 103
3.2.1 Law of a continuous process . . . 103
3.2.2 Equivalence of processes . . . 105
3.2.3 Modifications and indistinguishable processes . . . 107
3.2.4 Adapted and progressively measurable processes . . . 110
3.3 Martingales . . . 111
3.3.1 Doob's inequality . . . 113
3.3.2 Martingale spaces: $\mathscr{M}^2$ and $\mathscr{M}_c^2$ . . . 114
3.3.3 The usual hypotheses . . . 117
3.3.4 Stopping times and martingales . . . 120
3.4 Riemann-Stieltjes integral . . . 125
3.4.1 Bounded-variation functions . . . 127
3.4.2 Riemann-Stieltjes integral and Itô formula . . . 131
3.4.3 Regularity of the paths of a Brownian motion . . . 134

4 Brownian integration . . . 139
4.1 Stochastic integral of deterministic functions . . . 140
4.2 Stochastic integral of simple processes . . . 141
4.3 Integral of $\mathbb{L}^2$-processes . . . 145
4.3.1 Itô and Riemann-Stieltjes integral . . . 149
4.3.2 Itô integral and stopping times . . . 151
4.3.3 Quadratic variation process . . . 153
4.3.4 Martingales with bounded variation . . . 156
4.3.5 Co-variation process . . . 157
4.4 Integral of $\mathbb{L}^2_{\mathrm{loc}}$-processes . . . 159
4.4.1 Local martingales . . . 161
4.4.2 Localization and quadratic variation . . . 163

5 Itô calculus 167
5.1 Itô processes 168
5.1.1 Itô formula for Brownian motion 169
5.1.2 General formulation 174
5.1.3 Martingales+and parabolic equations 176
5.1.4 Geometric Brownian motion 176
5.2 Multi-dimensional Itô processes 179
5.2.1 Multi-dimensional Itô formula 183
5.2.2 Correlated Brownian motion+and martingales 188
5.3 Generalized Itô formulas 191
5.3.1 Itô formula and+weak derivatives 191
5.3.2 Tanaka formula+and local times 194
5.3.3 Tanaka+formula for Itô processes 197
5.3.4 Local+time and Black-Scholes formula 198

6 Parabolic PDEs with variable coefficients: uniqueness 203
6.1 Maximum principle and Cauchy-Dirichlet problem 206
6.2 Maximum principle and Cauchy problem 208
6.3 Non-negative solutions of the Cauchy problem 213

7 Black-Scholes model 219
7.1 Self-financing strategies 220
7.2 Markovian strategies and Black-Scholes equation 222
7.3 Pricing 225
7.3.1 Dividends and time-dependent parameters 228
7.3.2 Admissibility and absence of arbitrage 229
7.3.3 Black-Scholes analysis: heuristic approaches 231
7.3.4 Market price of risk 233
7.4 Hedging 236
7.4.1 The Greeks 236
7.4.2 Robustness of the model 245
7.4.3 Gamma and Vega-hedging 246
7.5 Implied volatility 248
7.6 Asian options 252
7.6.1 Arithmetic average 253
7.6.2 Geometric average 255

8 Parabolic PDEs with variable coefficients: existence 257
8.1 Cauchy problem and fundamental solution 258
8.1.1 Levi's parametrix method 260
8.1.2 Gaussian estimates and adjoint operator 261
8.2 Obstacle problem 263
8.2.1 Strong solutions 265
8.2.2 Penalization method 268

9 Stochastic differential equations . . . 275
9.1 Strong solutions . . . 276
9.1.1 Uniqueness . . . 278
9.1.2 Existence . . . 280
9.1.3 Properties of solutions . . . 283
9.2 Weak solutions . . . 286
9.2.1 Tanaka's example . . . 286
9.2.2 Existence: the martingale problem . . . 287
9.2.3 Uniqueness . . . 290
9.3 Maximal estimates . . . 292
9.3.1 Maximal estimates for martingales . . . 293
9.3.2 Maximal estimates for diffusions . . . 296
9.4 Feynman-Kač representation formulas . . . 298
9.4.1 Exit time from a bounded domain . . . 300
9.4.2 Elliptic-parabolic equations and Dirichlet problem . . . 302
9.4.3 Evolution equations and Cauchy-Dirichlet problem . . . 307
9.4.4 Fundamental solution and transition density . . . 308
9.4.5 Obstacle problem and optimal stopping . . . 310
9.5 Linear equations . . . 314
9.5.1 Kalman condition . . . 318
9.5.2 Kolmogorov equations and Hörmander condition . . . 323
9.5.3 Examples . . . 326

10 Continuous market models . . . 329
10.1 Change of measure . . . 329
10.1.1 Exponential martingales . . . 329
10.1.2 Girsanov's theorem . . . 332
10.1.3 Representation of Brownian martingales . . . 334
10.1.4 Change of drift . . . 339
10.2 Arbitrage theory . . . 340
10.2.1 Change of drift with correlation . . . 343
10.2.2 Martingale measures and market prices of risk . . . 345
10.2.3 Examples . . . 348
10.2.4 Admissible strategies and arbitrage opportunities . . . 352
10.2.5 Arbitrage pricing . . . 355
10.2.6 Complete markets . . . 357
10.2.7 Parity formulas . . . 358
10.3 Markovian models: the PDE approach . . . 359
10.3.1 Martingale models for the short rate . . . 361
10.3.2 Pricing and hedging in a complete model . . . 364
10.4 Change of numeraire . . . 366
10.4.1 LIBOR market model . . . 370
10.4.2 Change of numeraire for Itô processes . . . 372
10.4.3 Pricing with stochastic interest rate . . . 374
10.5 Diffusion-based volatility models . . . 376

10.5.1 Local and path-dependent volatility 377
10.5.2 CEV model 379
10.5.3 Stochastic volatility and the SABR model 386

11 American options 389
11.1 Pricing and hedging in the Black-Scholes model 389
11.2 American Call and Put options in the Black-Scholes model . . 395
11.3 Pricing and hedging in a complete market 398

12 Numerical methods 403
12.1 Euler method for ordinary equations 403
12.1.1 Higher order schemes 407
12.2 Euler method for stochastic differential equations 408
12.2.1 Milstein scheme 411
12.3 Finite-difference methods for parabolic equations 412
12.3.1 Localization 413
12.3.2 θ-schemes for the Cauchy-Dirichlet problem 414
12.3.3 Free-boundary problem 419
12.4 Monte Carlo methods 420
12.4.1 Simulation 423
12.4.2 Computation of the Greeks 425
12.4.3 Error analysis 427

13 Introduction to Lévy processes 429
13.1 Beyond Brownian motion 429
13.2 Poisson process 432
13.3 Lévy processes 437
13.3.1 Infinite divisibility and characteristic function 439
13.3.2 Jump measures of compound Poisson processes 444
13.3.3 Lévy-Itô decomposition 450
13.3.4 Lévy-Khintchine representation 457
13.3.5 Cumulants and Lévy martingales 460
13.4 Examples of Lévy processes 463
13.4.1 Jump-diffusion processes 464
13.4.2 Stable processes 466
13.4.3 Tempered stable processes 469
13.4.4 Subordination 471
13.4.5 Hyperbolic processes 478
13.5 Option pricing under exponential Lévy processes 480
13.5.1 Martingale modeling in Lévy markets 480
13.5.2 Incompleteness and choice of an EMM 485
13.5.3 Esscher transform 486
13.5.4 Exotic option pricing 491
13.5.5 Beyond Lévy processes 494

14 Stochastic calculus for jump processes 497
14.1 Stochastic integrals 497
14.1.1 Predictable processes 500
14.1.2 Semimartingales 504
14.1.3 Integrals with respect to jump measures 507
14.1.4 Lévy-type stochastic integrals 511
14.2 Stochastic differentials 514
14.2.1 Itô formula for discontinuous functions 514
14.2.2 Quadratic variation 515
14.2.3 Itô formula for semimartingales 518
14.2.4 Itô formula for Lévy processes 520
14.2.5 SDEs with jumps and Itô formula 525
14.2.6 PIDEs and Feynman-Kač representation 529
14.2.7 Linear SDEs with jumps 532
14.3 Lévy models with stochastic volatility 534
14.3.1 Lévy-driven models and pricing PIDEs 534
14.3.2 Bates model 537
14.3.3 Barndorff-Nielsen and Shephard model 539

15 Fourier methods 541
15.1 Characteristic functions and branch cut 542
15.2 Integral pricing formulas 545
15.2.1 Damping method 546
15.2.2 Pricing formulas 547
15.2.3 Implementation 551
15.2.4 Choice of the damping parameter 553
15.3 Fourier-cosine series expansions 562
15.3.1 Implementation 567

16 Elements of Malliavin calculus 577
16.1 Stochastic derivative 578
16.1.1 Examples 580
16.1.2 Chain rule 582
16.2 Duality 586
16.2.1 Clark-Ocone formula 588
16.2.2 Integration by parts and computation of the Greeks .. 590
16.2.3 Examples 594

Appendix: a primer in probability and parabolic PDEs 599
A.1 Probability spaces 599
A.1.1 Dynkin's theorems 601
A.1.2 Distributions 605
A.1.3 Random variables 608
A.1.4 Integration 610
A.1.5 Mean and variance 612

A.1.6 σ-algebras and information ... 618
A.1.7 Independence ... 619
A.1.8 Product measure and joint distribution ... 622
A.1.9 Markov inequality ... 625
A.2 Fourier transform ... 626
A.3 Parabolic equations with constant coefficients ... 630
A.3.1 A special case ... 631
A.3.2 General case ... 636
A.3.3 Locally integrable initial datum ... 637
A.3.4 Non-homogeneous Cauchy problem ... 638
A.3.5 Adjoint operator ... 639
A.4 Characteristic function and normal distribution ... 641
A.4.1 Multi-normal distribution ... 643
A.5 Conditional expectation ... 646
A.5.1 Radon-Nikodym theorem ... 646
A.5.2 Conditional expectation ... 648
A.5.3 Conditional expectation and discrete random variables 650
A.5.4 Properties of the conditional expectation ... 652
A.5.5 Conditional expectation in L^2 ... 655
A.5.6 Change of measure ... 656
A.6 Stochastic processes in discrete time ... 657
A.6.1 Doob's decomposition ... 659
A.6.2 Stopping times ... 661
A.6.3 Doob's maximal inequality ... 665
A.7 Convergence of random variables ... 669
A.7.1 Characteristic function and convergence of variables .. 670
A.7.2 Uniform integrability ... 674
A.8 Topologies and σ-algebras ... 676
A.9 Generalized derivatives ... 678
A.9.1 Weak derivatives in $\mathbb{R}$ ... 678
A.9.2 Sobolev spaces and embedding theorems ... 681
A.9.3 Distributions ... 682
A.9.4 Mollifiers ... 687
A.10 Separation of convex sets ... 690

References ... 691

Index ... 713

Springer

http://www.springer.com/978-88-470-1780-1

PDE and Martingale Methods in Option Pricing

Pascucci, A.

2011, XVII, 721 p., Hardcover
ISBN: 978-88-470-1780-1