

Chapter 2

Symmetry

The concept of symmetry is equally important for understanding properties of individual molecules, crystals and liquid crystals [1]. The symmetry is of special importance in physics of liquid crystal because it allows us to distinguish numerous liquid crystalline phases from each other. In fact, all properties of mesophases are determined by their symmetry [2]. In the first section we consider the so-called *point group* symmetry very often used for discussion of the most important liquid crystalline phases. A brief discussion of the *space group* symmetry will be presented in Section 2.2.

2.1 Point Group Symmetry

2.1.1 Symmetry Elements and Operations

There are only few symmetry elements, which generates a number of symmetry operations [3, 4]. We may illustrate them by their applications to simple geometrical objects.

(a) *Proper rotation axis of nth-order, C_n*

Consider first *rotational symmetry*. Let us take an equilateral triangle and rotate it clockwise about its center by $360^\circ/3 = 120^\circ$, Fig. 2.1a. The new triangle would be undistinguishable from the original one (but not identical). The symmetry *element* we used is the proper rotation *axis* of order 3 (C_3 -axis). The same triangle can be rotated by a half of the full turn about one of the three other axes (medians going out of each vertex), Fig. 2.1b. The corresponding symmetry element is a C_2 axis (rotation angle $360^\circ/2$, $n = 2$). For a square, we can find one C_4 axis and four C_2 axes, for a hexagonal benzene molecule one C_6 axis and six C_2 axes.

The symmetry element C_3 may generate two other symmetry operations. For instance, applying C_3 rotation twice we again obtain an indistinguishable triangle. Symbolically, C_3^2 means rotation by $2 \times 2\pi/3 = 240^\circ$. The same C_3 rotation applied three times result in the exactly the same triangle. Therefore C_3^3 is one of

Fig. 2.1 Rotational symmetry. The illustration of operations made by proper rotation axes of third (a) and second (b) order

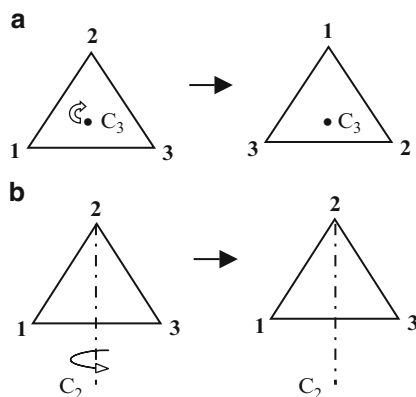
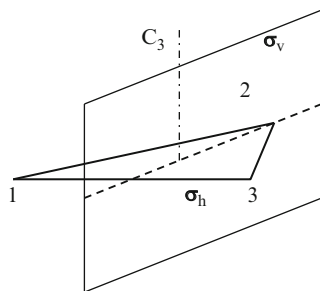


Fig. 2.2 Bilateral symmetry. Plane σ_v (vertical) is plane of reflection that contains the axis of the highest order C_3 for the equilateral triangle. After applying this element points 1 and 3 exchange their positions. Plane of the triangle is reflection plane σ_h (horizontal) perpendicular to C_3



the *identity operations*, $C_3^3 = E$. Generally, the identity operation corresponds to doing nothing with any figure and can also be obtained with C_2 or C_4 axes or with any axis of order n : $C_2^2 = C_4^4 = C_n^n = E$.

(b) A plane of symmetry, σ

This element generates only one operation, a reflection in the plane as in a mirror, Fig.2.2. Repeated twice this operation results in the initial structure that is $\sigma^2 = E$. Taking again our triangle we can see that plane σ interchanges points 1 and 3 leaving point 2 at the same place. Such symmetry is called *bilateral symmetry*. There may be several symmetry planes and they designated either as σ_h (the plane perpendicular to the axis C_n with highest number n) or σ_v (plane containing the C_n axis). By convention, the C_n axis with highest number n is taken as a vertical, therefore, indices h and v mean “horizontal” and “vertical”. In our figure we see the σ_v plane, and the plane of the triangle is σ_h . Note that a chiral object, for instance a hand, has no mirror plane (however, two hands in praying position have a mirror plane between them [4]).

(c) *Inversion center, I*

This symmetry element I generates an operation of inversion through a point called the inversion center, Fig. 2.3. Therefore now we deal with *inversion symmetry*. We can take any point of an object and connect it by a straight line with the center O . Then, along the same line behind the center and at the equal distance from it we must find the point equivalent to the first one. A good example is a parallelogram. Note, that two inversions result in the identical object, $I^2 = E$.

(d) *Improper rotation, S_n*

This element is also called a rotation–reflection axis or mirror–rotation axis. It consists of two steps, a rotation through $1/n$ of the full turn followed by reflection in a plane perpendicular to the rotation axis, Fig. 2.4. A molecule of ethane in the staggered configuration is a good illustration of S_6 rotation–reflection axis, see the figure. Note, that this object has neither C_6 axis nor σ plane on their own. But after combined operations C_6 (60° clockwise) and σ we obtain an indistinguishable object with interchanged positions of all hydrogen atoms. Therefore S_6 , and, more generally, S_n is independent symmetry operation. Like element C_n , element S_6 may generate several operations, for instance, $S_6^2 = C_3$ because this operation consists of rotation by $2 \times 2\pi/6 = 120^\circ = 2\pi/3$ and identity operation $\sigma^2 = E$. Totally, $S_6^2 = C_3E$. Other examples are: $S_3^2 = C_3^2$; $S_3^3 = \sigma$; $S_3^4 = C_3$; $S_3^6 = E$.

Finally, we have five independent symmetry elements: identity E , proper rotation axis C_n , symmetry plane σ , inversion center I and rotation–reflection axis S_n , generating single (elements E , σ , I) or multiple (elements C_n , S_n) symmetry operations.

Fig. 2.3 Inversion symmetry. Point O is inversion center and the inversion operation exchanges positions of points 1–1', 2–2', 3–3' etc.

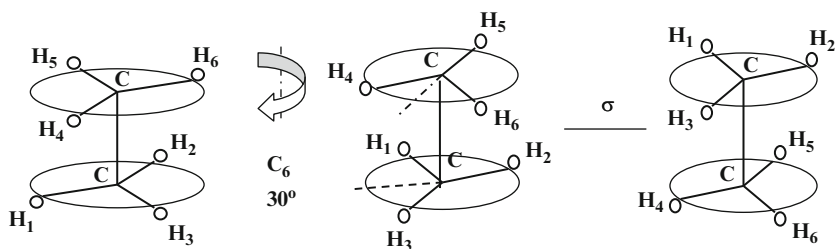
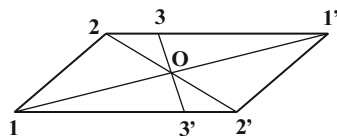


Fig. 2.4 Improper rotation. Axis connecting points C–C is a rotation–reflection axis S_6 . An ethane molecule has a symmetry element including two subsequent operations, the rotation of the whole structure through an angle of 30° with a subsequent reflection by plane σ . After this the left and right sketch become identical

2.1.2 Groups

Now, for each geometrical object or a molecule we can write a *set of symmetry operations*, which transform the object into its equivalent formal representation. Let consider three examples.

- (i) Water molecule, Fig. 2.5a. It has the C_2 axis, two symmetry planes and together with identity element we have a full set of symmetry operations E , C_2 , σ , σ' . As we shall see soon this set corresponds to symmetry group C_{2v} .
- (ii) The next is an ion $[\text{Co}(\text{NH}_3)_4\text{ClBr}]^{+3}$ shown in Fig. 2.5b. Its set of symmetry operations is E , C_4 , C_4^2 , C_4^3 , $4\sigma_v$ (group C_{4v}).
- (iii) Finally we take a flat borate molecule BCl_3 having the symmetry of an equilateral triangle, therefore allowing the following operations:
 E , C_3 , C_3^2 , C_2 , C_2' , C_2'' , σ_h , S_3 , S_3^2 , σ_v , σ_v' , σ_v'' (group D_{3h}).

Some operations belong to the same classes (see below), therefore we may write the set in a more compact way: E , $2C_3$, $3C_2$, σ_h , $3S_3$, $3\sigma_v$.

It is essential that any element of each set of operations can be obtained by a combination of other elements from the same set. Application of the subsequent symmetry operations is called *multiplication*. For a water molecule we can write the corresponding *multiplication table*, Table 2.1. For instance, the multiplication of operations C_2 from the first row and the first column corresponds to the identity operation, $C_2 \cdot C_2 = E$, shown in the table. Further, $C_2\sigma = \sigma'$ and $\sigma C_2 = \sigma'$, so these operations commute in our particular case. Generally they may not commute and the order of operations is important (by convention, multiplication operation $C_2\sigma$ means that first we apply operation σ and then C_2).

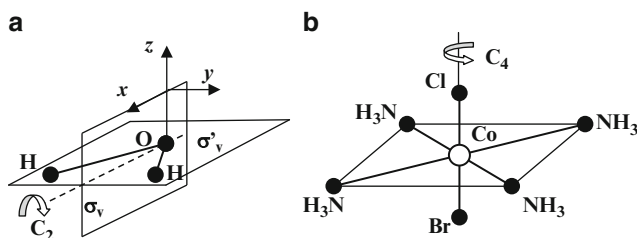


Fig. 2.5 Set of symmetry operations and the structure for water molecule belonging to group C_{2v} (a) and the structure of ion $[\text{Co}(\text{NH}_3)_4\text{ClBr}]^{+3}$ belonging to group C_{4v} (b)

Table 2.1 Multiplication table for symmetry elements of water molecule

	E	C_2	σ	σ'
E	E	C_2	σ	σ'
C_2	C_2	E	σ'	σ
σ	σ	σ'	E	C_2
σ'	σ'	σ	C_2	E

In the mathematical sense, each set of the operations shown above and, more generally, characterizing the symmetry of objects of different shape forms a *group*. So, what is a group? A set of elements A, B, C , etc. form a group if there is a rule for combining any two elements to form their “product” AB (or $A \cdot B$) such that:

- Every “product” of two elements is an element of the same set.
- There is a unit element E such that $EA = AE = A$.
- The associative law is valid, that is $A(BC) = (AB)C$, etc.
- Each element A has its inverse element A^{-1} belonging to the same set such that $AA^{-1} = A^{-1}A = E$.

The group “product” might be different (multiplication, addition, permutation, space rotation, etc).

For example, a set of integers including negative, zero and positive numbers $-\infty, \dots, -n, \dots, -3, -2, -1, 0, 1, 2, 3, \dots, n, \dots, \infty$ form a group under *addition* as a group operation. Indeed, all the group rules are fulfilled:

$A + B = C$ (e.g. $2 + 18 = 20$) belongs to the same set,

evidently $A + (B + C) = (A + B) + C$

$E = 0$ (e.g., $0 + A = A$, that is addition of zero changes nothing)

$A + (-A) = 0 = E$ (under *addition* as group operation ($-A$) symbolically means A^{-1})

This group contains infinite number of elements, therefore it is an *infinite group* or group of infinite order.

Another example is a set of four 2×2 matrices that form a *finite* group of order 4 (because we have four elements in the group) under *multiplication* operation (it may directly be checked using the rule for matrix multiplication):

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (2.1)$$

Here the identity element is matrix A , the associative law for matrices is always valid, the inverse matrices are B for D and vice versa ($BD = DB = A$), elements A and C are inverses on their own ($AA = A, CC = A$). It is interesting that there is a one-to-one correspondence between the elements of the group of our four matrices and the elements of the group of complex numbers $1, i, -1, -i$. The two groups have the same multiplication table. For example, the product of the elements of the first group (matrices) $B_1C_1 = D_1$ corresponds to the product $B_2C_2 = D_2$ of the second group (numbers). Therefore, we say that these two groups are *isomorphic*.

The concept of isomorphism is very important. Here, we are interested in the groups of elements whose products are symmetry operations. Each set of symmetry operations is a group, which may be *represented* by a group of matrices isomorphic to our group. To demonstrate this, let us go back to group C_{2v} in Fig. 2.5a and pay attention to the coordinate system: this is the *basis* for representation of the selected group in the matrix form. Since the water molecule is two-dimensional the group will be represented by 4×4 matrices. We shall consider the hydrogen–oxygen

bonds as vectors x , y and the symmetry operation will transform them into vectors x' and y' . The identity operation E does not change anything, and this operation is represented by the unit matrix E :

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \sigma_v = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \sigma'_v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (2.2)$$

The C_2 operation exchanges only y -projections of the two vectors, and this is given by matrix C_2 . In our particular case, the result of σ_v operation is the same and the corresponding matrices C_2 and σ_v are equal. Operation σ'_v does not change positions of the two vectors and its matrices E and σ'_v are identical. The sums of the diagonal elements of a matrix is a trace or a character of the matrix. Our matrices have the following characters 2, 0, 0, 2 corresponding to one of possible representations of group C_{2v} .

Therefore, we can operate with the matrices using a powerful apparatus of modern mathematics and obtain important results not at all evident from the beginning. Such a *theory of group representations* will not be considered here although that theory is very powerful and widely used not only in crystallography [2] but in many areas of physics [1, 5, 6]. In this section we shall only list the sets of symmetry operations corresponding to the most important *point symmetry groups*. The term “point” reflects the fact, that under any operation of the groups listed, at least, one point of the object is not changed. For comparison, when an object is translated in space we should discuss its translational symmetry and corresponding *space symmetry groups* [7].

2.1.3 Point Groups

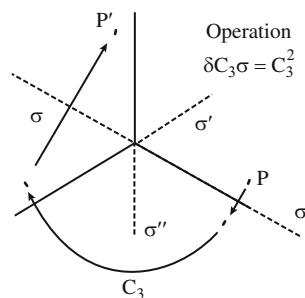
Generally, there is infinite number of point groups, but not all of them correspond to real physical objects such as molecules or crystals. For example, only 32 point groups are compatible with crystal lattices. Each of them is labeled by a certain symbol according to Schönflies or according to the International classifications. The Schönflies symbols are vivid and more often used in scientific literature. Here we present only those point groups we may encounter in the literature on liquid crystals.

In the Table 2.2 the Schönflies symbols are given on the left side, symmetry operations (not symmetry elements) are given on the right side. As we see a number of symmetry operations increases from top to bottom. Therefore, we say that D_{6h} is more symmetric phase than, say, C_{2h} . The capital letter D (with index $n = 2, 3, 6, \dots$) is used when n number of C_2 axes appear, which are perpendicular to the principal C_n axis. All symmetry operations except σ_d have been discussed above. The operation σ_d appears in D_{4h} , D_{6h} etc. to distinguish between vertical reflection planes σ_v containing the C_2 axes and additional planes (also vertical) passing along bisectors between already available pairs of the C_2 axes.

Table 2.2 Some point groups with Schönflies and international symbols

Schönflies symbol	International	Symmetry operations
C_1	1	E
C_s or C_{1h}	m	E, σ_h
C_i or S_2	$\bar{1}$	E, I
C_2	2	E, C_2
C_{2h}	$2/m$	E, C_2 , I, σ_h
C_{2v}	mm	E, C_2 , σ_v , σ_v'
C_{3v}	$3m$	E, $2C_3$, $3\sigma_v$
D_{2h}	mmm	E, $3C_2$, I, σ_h , σ_v , σ_v'
D_{3h}	$\bar{6}m2$	E, $2C_3$, $3C_2$, σ_h , $2S_6$, $3\sigma_v$
D_{6h}	$6/mmm$	E, $2C_6$, $2C_3$, C_2 , $3C_2'$, $3C_2''$, I, $2S_6$, $2S_6$, σ_h , $3\sigma_d$, $3\sigma_v$

Fig. 2.6 The procedure illustrating that, in NH_3 molecule (group C_{3v}), operations C_3 and C_3^2 , belong to the same class and operations σ , σ' , σ'' belong to another class



For the sake of brevity, some operations form *classes* consisted of conjugate symmetry operations. For example, consider the C_{3v} group: operation $2C_3$ includes two conjugate operations C_3 and C_3^2 (not two C_3 axes!). The definition of the conjugate elements of a group is as follows: we say operations A and B of a group are conjugate and belong to the same class if $XAX^{-1} = B$ where X is any of the same group operations and X^{-1} is its inverse operation. Therefore, B is similarity transform of A (note, that $XX^{-1} = E$). Note that single operation E forms the class on its own because, for any X , $XEX^{-1} = EXX^{-1} = E$.

Since operation σ is equal to its own inverse, it is convenient to use it as X and analyze whether operations C_n form a class or not. For example, consider NH_3 molecule (group C_{3v}) whose projection along the C_3 axis is shown in Fig. 2.6. Let us take point P and, at first, make twice C_3 operation to arrive at point P' . Then, we start again from point P and, guided by arrows in the figure, make operation $\sigma C_3 \sigma^{-1}$. We again arrive at point P' . Therefore $\sigma C_3 \sigma^{-1} = C_3^2$, and operations C_3 and C_3^2 belong to the same class. Now, since from the C_3 symmetry is evident that C_3 is inverse of C_3^2 , we may find a conjugate of σ : $C_3 \sigma C_3^2 = \sigma''$. Finally, all symmetry operations for the C_{3v} group, i.e. E , C_3 , C_3^2 , σ , σ' , σ'' can be combined in three classes E , $2C_3$, 3σ , as shown in Table 2.2.

2.1.4 Continuous Point Groups

There are also so-called *continuous point groups*, which include rotations of objects (or coordinate systems) by infinitesimal angles. Therefore the number of their elements tends to a limit that is infinity and the groups themselves are infinite. The continuous point groups were introduced by P. Curie and can be represented by physical objects [3]. Totally there are seven continuous groups. They are important for description of very symmetric liquid crystalline phases such as nematic, smectic A and polar nematic phase the existence of which is still under discussion. There are also continuous space groups describing helical (chiral) phases, see below. For the beginning consider the groups of cones. The symmetry of an immobile cone, see Fig. 2.7a, is $C_{\infty v}$ (or ∞m according to the International classification). It includes an infinite order axis C_{∞} , an infinite number of symmetry planes σ_v like the ABC plane but has no σ_h plane. Therefore, the C_{∞} -axis has properties of a genuine vector. We say this axis is a *polar axis* and the phase is also polar, in particular, it may possess spontaneous polarization. In the liquid crystal physics this group would describe the *polar nematic phase*, the very existence of which is still questionable. The rotating cone, see Fig. 2.7b, has polar symmetry reduced to C_{∞} (or ∞) because the only symmetry element is a rotation axis C_n with $n = \infty$. Due to rotation there is no symmetry plane. The cone may rotate either clockwise or anti-clockwise and we can say that it has two enantiomorphic modifications.

The next is a series of cylinders. The immobile cylinder has symmetry elements shown in Fig. 2.8a: a rotation axis of infinite order C_{∞} , an infinite number of C_2 axes perpendicular to C_{∞} , a horizontal symmetry plane σ_h and infinite number of vertical symmetry planes σ_v . Its point group $D_{\infty h}$ (or ∞/m) corresponds to symmetry of the conventional (non-chiral) nematic or smectic A phase. A rotating cylinder, Fig. 2.8b, has no C_2 axis perpendicular to the C_{∞} rotation axis but has a horizontal symmetry plane σ_h (no chirality). Its point group symmetry is $C_{\infty h}$ (or ∞/m). A twisted cylinder, Fig. 2.8c, is chiral therefore has lost all symmetry planes but still has the C_{∞} axis and infinite number of C_2 axes perpendicular to C_{∞} . Therefore, according to Schönflies, it keeps the D letter and its symmetry group is D_{∞} (or $\infty 2$) corresponding to chiral cholesteric or chiral smectic A* phase. Both the twisted and rotating cylinders may be encountered in

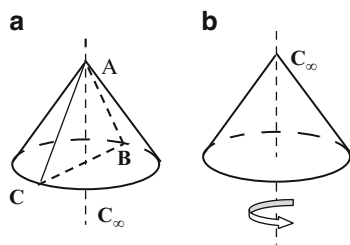


Fig. 2.7 Continuous groups of cones: symmetry elements of an immobile cone $C_{\infty v}$ (a) and group of a rotating cone C_{∞} (b)

Fig. 2.8 Continuous groups of cylinders: symmetry elements of an immobile cylinder, group $D_{\infty h}$ (a), the group of a rotating cylinder $C_{\infty h}$ (b) and chiral group of a twisted cylinder D_{∞} (c)

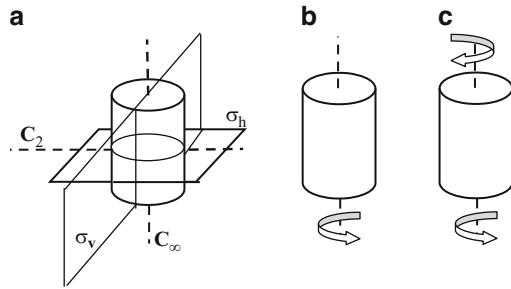
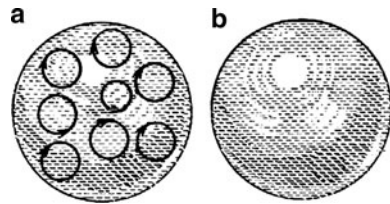


Fig. 2.9 Continuous groups of spheres: group of a chiral sphere K or R (a) and group of achiral (having mirror symmetry) sphere K_h or O (b)



two enantiomorphic modifications. Note that all cylinders and cones are optically uniaxial.

The two objects shown in Fig. 2.9 are spheres made of different materials. The sphere (b) is made of non-chiral material. Such a sphere has full orthogonal symmetry group $O(3)$ or K_h (i.e. $\infty\infty m$): infinite number of C_{∞} axes, infinite number of reflection planes passing through the center of the sphere, an inversion center. Any isotropic achiral liquid has this point symmetry group. However, liquids consisting of chiral molecules, which rotate the light polarization plane (like some sugar solutions), have lower symmetry, Fig. 2.9a; they belong to the full rotational group $R(3)$ or K (or $\infty\infty$) because they have lost all symmetry planes and the inversion center. To conclude, the full list of seven continuous point groups includes (in the order of reducing symmetry): spheres (K_h , K), cylinders ($D_{\infty h}$, $C_{\infty h}$, D_{∞}), and cones ($C_{\infty v}$ and C_{∞}).

2.2 Translational Symmetry

(a) Crystals made of atoms with spherical symmetry

Such crystals have only *translational* (i.e., *positional*) order and no orientational order. Their structure is characterized by (i) the point group symmetry of an elementary cell which includes rotations, reflections and inversion as group operation and (ii) the group of translations which includes vectors with their addition as a group operation. The translation vector is $\mathbf{T} = n_1\mathbf{a} + n_2\mathbf{b} + n_3\mathbf{c}$ where \mathbf{a} , \mathbf{b} , \mathbf{c} are unit

Fig. 2.10 The translation vector $\mathbf{T} = n_1\mathbf{a} + n_2\mathbf{b} + n_3\mathbf{c}$ with $n_1 = 2, n_2 = 1, n_3 = 1$; (a), (b) and (c) are unit basis vectors

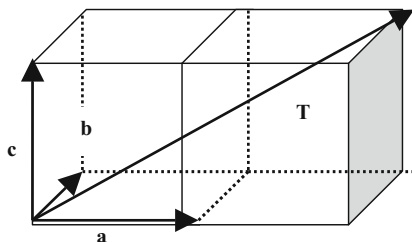
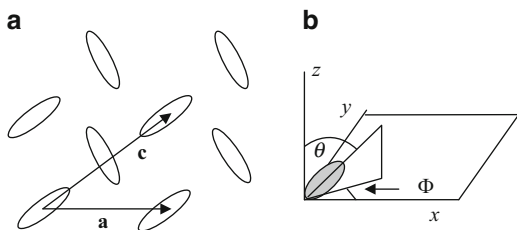


Fig. 2.11 Molecular crystal with rigidly fixed molecules (a) and Euler angles ϑ and Φ (b) for a particular molecule



vectors and n_i are integers. For example, $n_1 = 2, n_2 = 1, n_3 = 1$, see Fig. 2.10. The overall symmetry (the crystal group) is determined by combination of all these elements.

(b) *Molecular crystals*

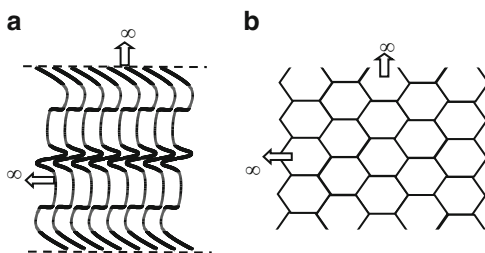
Due to anisometric (particularly elongated) shape of molecules, these crystals possess both the *translational* and *orientational* order. The latter is determined by Euler angles ϑ, Φ such molecules form with respect to selected coordinate frame as shown in the right part of Fig. 2.11. The third Euler angle Ψ describing rotation of a molecule about its longest axis is not shown for simplicity. The point group symmetry includes this orientational order.

(c) *Plastic crystals and liquid crystals*

A loss of the orientational order of a molecular crystal due to free rotation of molecules around x, y and z -axes with the positional order remained results in *plastic crystals*. The point group symmetry increases to that characteristic of crystals with spherical atoms. However, such crystals are much softer. An example is solid methane CH_4 at low temperature.

A loss of the translational order (at least, partially) results in *liquid crystals* of different rotational and translational symmetry. On heating, one can observe step-by-step melting and separate phase transitions to less ordered phases of enhanced symmetry. On cooling, correspondingly one observes step-by-step “crystallization”. An isotropic liquid is the most symmetric phase, it has full translational and orientational freedom, and this can be written as a product of group multiplication, $\text{O}(3) \times \text{T}(3)$, where $\text{O}(3)$ is the full orthogonal symmetry (infinite and

Fig. 2.12 One-dimensional (a) and two-dimensional (b) periodicity in the three-dimensional space



continuous) group and T is the full infinite continuous group of translations [8]. Upon cooling, a series of phase transitions occurs, and each time the symmetry is reduced. Each new point symmetry group is a subgroup of $O(3)$ and a new group of translations is a subgroup of $T(3)$. For example, on transition to the nematic phase the translational freedom is not confined but the rotational symmetry is lowered and the full symmetry is reduced to $D_{\infty h} \times T(3)$.

In Fig. 2.12a one-dimensional periodic structure is shown in the three-dimensional space. The curve line having point group C_{1h} (or C_s) is repeated with a certain period along the horizontal axis. Such translation may be symbolically written as T_1^3 . It is the same space group the smectic A phase has, see Fig. Int.1 in Chapter 1. In Fig. 2.12b the two-dimensional hexagonal lattice in the three-dimensional space (T_2^3) is presented. We shall discuss later the smectic B phase having this type of the hexagonal order of molecules.

It should be noted that cholesteric liquid crystals (chiral nematics) having point group symmetry D_∞ are also periodic with the pitch considerably exceeding a molecular size. The preferable direction of the local molecular orientation, i.e. the director oriented along the C_∞ axis, rotates additionally through subsequent infinitesimal angles in the direction perpendicular to that axis. Hence a helical structure forms with a screw axis and continuous translation group.

(d) *Classification of liquid crystals*

Liquid crystals can be classified according to

- (i) Their mean of formation: thermotropic (change of temperature and pressure), lyotropic (change of the molecular concentration in water and some other solvents), carbonized (change of polymerization degree), some rare special mechanisms (e.g., formation of chain structures in some inorganic substances).
- (ii) Molecular shape, as discussed in Chapter 2, like rod-like or calamitic (from Greek *καλαμίζ* that means “cane”), discotic, banana- or bent-like, dendrites, etc.
- (iii) Optical properties (uniaxial, biaxial, helical).
- (iv) Chemical classes (biphenyls, Schiff bases, pyrimidines, tolanes, etc).
- (v) The symmetry of a liquid crystalline phase which determines physical properties of the phase. This classification is a generalization of the earlier one suggested by G. Friedel. In Chapter 3 we consider symmetry and structure of the most important liquid crystalline phases.

References

1. Landau, L.D., Lifshits, E.M.: Quantum Mechanics, 3rd edn. Nauka, Moscow (1974). Ch.12 (in Russian)
2. Pikin, S.A.: Structural Transformations in Liquid Crystals. Gordon & Breach, New York (1991)
3. Sonin, A.S.: Besedy o kristalofizike (Talks on Crystal Physics). Atomizdat, Moscow (1976) (in Russian)
4. Hargittai I., Hargittai, M.: Symmetry Through the Eyes of a Chemist. VCH Verlagsgesellschaft, New York (1986)
5. Bhagavantam, S., Venkatarayudu, T.: Theory of Groups and Its Applications to Physical Problems, 2nd edn. Andhra University, Waltair (1951)
6. Weyl H.: Symmetry. Princeton University Press, Princeton (1952)
7. Kittel, Ch: Introduction to Solid State Physics, 4th edn. Wiley, New York (1971)
8. Kats, E.I.: New types of ordering in liquid crystals. Uspekhi Fis. Nauk **142**, 99–129 (1984)



<http://www.springer.com/978-90-481-8828-4>

Structure and Properties of Liquid Crystals

Blinov, L.M.

2011, XVIII, 439 p., Hardcover

ISBN: 978-90-481-8828-4