Chapter 1

KINEMATICS OF CONTINUA

1.1. Material and Spatial Descriptions of Continuum Motion

1.1.1. Lagrangian and Eulerian Coordinates. The Motion Law. Let us consider a continuum $\mathcal{B}$. Due to Axiom 2, at time $t = 0$ there is a one-to-one correspondence between every material point $\mathcal{M} \in \mathcal{B}$ and its radius-vector $\overset{\circ}{\mathbf{x}} = \overrightarrow{O\mathcal{M}}$ in a Cartesian coordinate system $O\bar{\mathbf{e}}_i$. Denote Cartesian coordinates of the radius-vector by $\overset{\circ}{x}{}^i$ ($\overset{\circ}{\mathbf{x}} = \overset{\circ}{x}{}^i \bar{\mathbf{e}}_i$) and introduce curvilinear coordinates X^i of the same material point $\mathcal{M}$ in the form of some differentiable one-to-one functions

$$\overset{\circ}{x}{}^i = \overset{\circ}{x}{}^i(X^k). \tag{1.1}$$

Since $\overset{\circ}{\mathbf{x}} = \overset{\circ}{x}{}^i \bar{\mathbf{e}}_i$, the relationship (1.1) takes the form

$$\overset{\circ}{\mathbf{x}} = \overset{\circ}{\mathbf{x}}(X^k). \tag{1.2}$$

Let us fix curvilinear coordinates of the point $\mathcal{M}$, and then material points of the continuum $\mathcal{B}$ are considered to be numbered by these coordinates X^i. For any motion of the continuum $\mathcal{B}$, coordinates X^i of material points are considered to remain unchanged; they are said to be 'frozen' into the medium and move together with the continuum. Coordinates X^i introduced in this way for a material point $\mathcal{M}$ are called *Lagrangian* (or *material*).

Due to Axiom 3, at every time t there is a one-to-one correspondence between every point $\mathcal{M} \in \mathcal{B}$ with Lagrangian coordinates X^i and its radius-vector $\mathbf{x} = \overrightarrow{O\mathcal{M}}$ with Cartesian coordinates x^i, where $\mathbf{x}$ and x^i depend on t. This means that there is a connection between Lagrangian X^i, and the Cartesian x^i coordinates of point $\mathcal{M}$ and time, i.e. there exist functions in the form (0.3)

$$x^i = x^i(X^k, t) \quad \forall X^k \in V_X. \tag{1.3}$$

These functions determine a motion of the material point $\mathcal{M}$ in the Cartesian coordinate system $O\bar{\mathbf{e}}_i$ of space $\mathcal{E}_3^a$. The relationships (1.3) are said to be the *law of the motion of the continuum $\mathcal{B}$*.

Coordinates x^i in (1.3) are called *Eulerian* (or *spatial*) coordinates of the material point $\mathcal{M}$.

Since $\mathbf{x} = x^i \bar{\mathbf{e}}_i$ and the coordinate system $O\bar{\mathbf{e}}_i$ is the same for all times t, the equivalent form of the motion law follows from (1.2):

$$\mathbf{x} = \mathbf{x}(X^k,\ t). \tag{1.4}$$

Since the consistency conditions (0.4) must be satisfied, from (1.2) and (1.4) we get the relationships

$$\mathbf{x}(X^k,\ 0) = \overset{\circ}{\mathbf{x}}(X^k), \qquad x^i(X^k,\ 0) = \overset{\circ}{x}{}^i(X^k). \tag{1.5}$$

Here the initial time $t = 0$ is considered as the time t_1 in (0.4), because just at time $t = 0$ we introduced Lagrangian coordinates X^i of point $\mathcal{M}$.

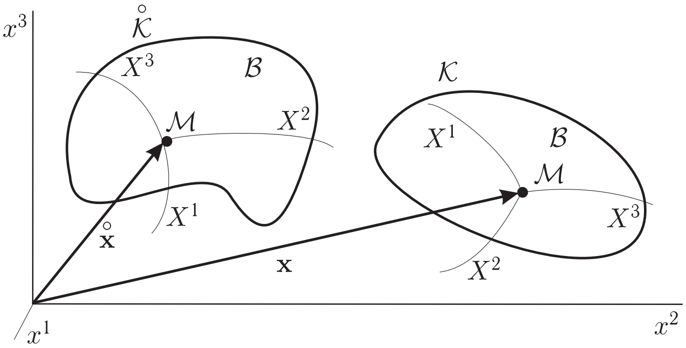

Figure 1.1. The motion of a continuum: positions of continuum $\mathcal{B}$ and material point $\mathcal{M}$ in reference and actual configurations

Unless otherwise stipulated, functions (1.3) are assumed to be regular in the domain $V_X \subset \mathbb{R}^3$ for all t, thus there exist the inverse functions

$$X^k = X^k(x^i,\ t) \quad \forall x^i \in V_x \subset \mathbb{R}^3.$$

The closed domain $\overset{\circ}{V} = \mathcal{W}(\mathcal{B},\ 0)$ in a fixed coordinate system $O\bar{\mathbf{e}}_i$, which is occupied by continuum $\mathcal{B}$ at the initial time $t = 0$, is called the *reference configuration* $\overset{\circ}{\mathcal{K}}$, and the domain $V = \mathcal{W}(\mathcal{B}, t)$ occupied by the same continuum $\mathcal{B}$ at the time $t > 0$ is called the *actual configuration* $\mathcal{K}$.

Figure 1.1 shows a geometric picture of the motion of a continuum from the reference configuration $\overset{\circ}{\mathcal{K}}$ to the actual one $\mathcal{K}$ at time t in space E_3^a.

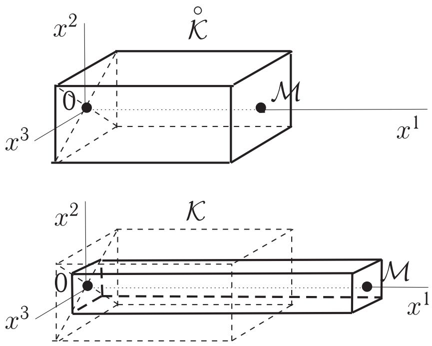

Figure 1.2. Extension of a beam

It should be noticed that if the continuum motion law (1.3) (or (1.4)) is known, then one of the main problems of continuum mechanics (to determine coordinates of all material points of the continuum at any time) will be resolved. However, in actual problems of continuum mechanics this law, as a rule, is unknown and must be found by solving some mathematical problems, whose statements are to be formulated. One of our objectives is to derive these statements.

Example 1.1. Let us consider a continuum $\mathcal{B}$, which at time $t = 0$ in the reference configuration $\overset{\circ}{\mathcal{K}}$ is a rectangular parallelepiped (a beam) with edge lengths $\overset{\circ}{h}_1$, $\overset{\circ}{h}_2$ and $\overset{\circ}{h}_3$, and in an actual configuration $\mathcal{K}$ at $t > 0$ the continuum is also a rectangular parallelepiped but with different edge lengths: h_1, h_2 and h_3. We assume that corresponding sides of both the parallelepipeds lie on parallel planes, and for one of the sides, which for example is situated on the plane (x^2, x^3), points of diagonals' intersection in $\mathcal{K}$ and in $\overset{\circ}{\mathcal{K}}$ are coincident (Figure 1.2). Then the motion law (1.3) for this continuum takes the form

$$x^\alpha = k_\alpha(t)\, X^\alpha, \qquad \alpha = 1,\ 2,\ 3, \tag{1.6}$$

i.e. coordinates $x^i, \overset{\circ}{x}{}^i = X^i$ of any material point $\mathcal{M}$ in $\overset{\circ}{\mathcal{K}}$ and $\mathcal{K}$ are proportional, and $k_\alpha(t) = h_\alpha(t)/\overset{\circ}{h}_\alpha$ is the proportion function. The motion law (1.6) is called the *beam extension law*. □

Example 1.2. In $\overset{\circ}{\mathcal{K}}$, let a continuum $\mathcal{B}$ be a rectangular parallelepiped oriented as shown in Figure 1.3; its motion law (1.3) has the form

$$\begin{cases} x^1 = X^1 + a(t) X^2, \\ x^2 = X^2, \\ x^3 = X^3, \end{cases} \tag{1.7}$$

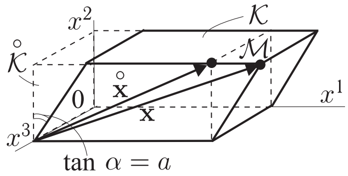

Figure 1.3. Simple shear of a beam

where $\overset{\circ}{x}{}^i = X^i$, $a(t)$ is a given function. In $\mathcal{K}$ this continuum $\mathcal{B}$ has become a parallelepiped, all cross-sections of which are planes orthogonal to the Ox^3 axis and are the same parallelograms.

This motion law is called *simple shear*; the tangent of the shear angle α is equal to a. □

Example 1.3. Consider a continuum $\mathcal{B}$, which in $\overset{\circ}{\mathcal{K}}$ is a rectangular parallelepiped (a beam) shown in Figure 1.4; under the transformation from $\overset{\circ}{\mathcal{K}}$ to $\mathcal{K}$ this parallelepiped changes its dimensions without a change in its angles (as in Example 1.1) and rotates by an angle $\varphi(t)$ in the plane Ox^1x^2 around the point O (Figure 1.4). The motion law for the continuum is called the *rotation of a beam with extension*. In this case equations (1.1) have the form

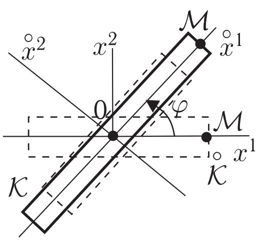

Figure 1.4. Rotation of a beam with extension

$$x^i = F_0{}^i{}_j \overset{\circ}{x}{}^j, \qquad \overset{\circ}{x}{}^j = X^j, \tag{1.8}$$

where the matrix $F_0{}^i{}_j$ is the product of two matrices, the rotation matrix O_0 and the stretch matrix U_0:

$$F_0{}^i{}_j = O_0{}^i{}_k U_0{}^k{}_j, \qquad O_0{}^i{}_j = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$U_0{}^i{}_j = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}, \qquad F_0{}^i{}_j = \begin{pmatrix} k_1\cos\varphi & -k_2\sin\varphi & 0 \\ k_1\sin\varphi & k_2\cos\varphi & 0 \\ 0 & 0 & k_3 \end{pmatrix},$$

and $k_\alpha(t) = h_\alpha(t)/h^0_\alpha$ are the proportion functions characterizing the ratio of lengths of the beam edges in $\mathcal{K}$ and $\overset{\circ}{\mathcal{K}}$ (as in Example 1.1). $\square$

1.1.2. Material and Spatial Descriptions. In continuum mechanics, physical processes occurring in bodies are characterized by a certain set of *varying scalar fields* $\phi = \phi(\mathcal{M}, t)$, *vector fields* $\mathbf{a} = \mathbf{a}(\mathcal{M}, t)$, and *tensor fields* of the nth order ${}^n\mathbf{\Omega}(\mathcal{M}, t)$. We will consider tensors and tensor fields in detail in paragraph 1.1.4 (see also [12]).

Since in the Cartesian coordinate system $O\bar{\mathbf{e}}_i$ a material point $\mathcal{M}$ corresponds to both Lagrangian coordinates X^i and Eulerian coordinates x^i, varying scalar and vector fields can be written as follows:

$$\phi(X^i,\ t) = \phi(X^i(x^j,\ t),\ t) = \widetilde{\phi}(x^j,\ t) = \widetilde{\phi}(\mathbf{x},\ t),$$
$$\mathbf{a}(X^i,\ t) = \widetilde{\mathbf{a}}(x^j,\ t) = \widetilde{\mathbf{a}}(\mathbf{x},\ t), \tag{1.9}$$

With the help of the motion law (1.3) (or (1.4)), we can pass from functions of Lagrangian coordinates to functions of Eulerian coordinates in formulae (1.9). In continuum mechanics the tilde $\widetilde{\ }$ is usually omitted (we will do this below).

For a fixed time t in (1.9), we obtain stationary scalar and vector fields. If in (1.9) a material point $\mathcal{M}$ is fixed, and time t changes within the interval $0 \leqslant t \leqslant t'$, then we get an ordinary scalar function $\phi = \phi(\mathcal{M},\ t)$ and vector function $\mathbf{a} = \mathbf{a}(\mathcal{M},\ t)$ depending on time.

According to relationships (1.9), there are two ways to describe different physical processes in continua.

In the *material* (*Lagrangian) description* of a continuum, all tensor fields describing physical processes are considered as functions of X^i and t.

In the *spatial* (*Eulerian) description*, all tensor fields describing physical processes are functions of x^i and t.

Both the descriptions are equivalent. It should be noted that for solids we more often use the material description, where it is convenient to fix coordinates X^i of a material point $\mathcal{M}$ and to observe its motion at different times t. For gaseous and fluid continua, Eulerian description is more convenient; when an observer fixes a geometric point with coordinates x^i and monitor the material points $\mathcal{M}$ passing through this point x^i at different times t.

1.1.3. Local Bases in $\mathcal{K}$ and $\overset{\circ}{\mathcal{K}}$. Using the motion law (1.4) and relationship (1.1), at every material point $\mathcal{M}$ with coordinates X^i in the actual and reference configurations we can introduce its *local basis vectors*:

$$\mathbf{r}_k = \frac{\partial \mathbf{x}}{\partial X^k} = \frac{\partial x^i}{\partial X^k}\bar{\mathbf{e}}_i = Q^i{}_k\bar{\mathbf{e}}_i, \qquad \overset{\circ}{\mathbf{r}}_k = \frac{\partial \overset{\circ}{\mathbf{x}}}{\partial X^k} = \frac{\partial \overset{\circ}{x}{}^i}{\partial X^k}\bar{\mathbf{e}}_i = \overset{\circ}{Q}{}^i{}_k\bar{\mathbf{e}}_i, \tag{1.10}$$

where

$$Q^i{}_k = \partial x^i/\partial X^k, \qquad \overset{\circ}{Q}{}^i{}_k = \partial \overset{\circ}{x}{}^i/\partial X^k,$$
$$P^i{}_k = \partial X^i/\partial x^k, \qquad \overset{\circ}{P}{}^i{}_k = \partial X^i/\partial \overset{\circ}{x}{}^k \tag{1.11}$$

are *Jacobian matrices* and *inverse Jacobian matrices*.

Here and below all values referred to the configuration $\overset{\circ}{\mathcal{K}}$ will be denoted by superscript $^\circ$. As follows from the definition (1.11), local bases vectors $\mathbf{r}_k$ and $\overset{\circ}{\mathbf{r}}_k$ are directed tangentially to corresponding coordinate lines X^k (Figure 1.5)

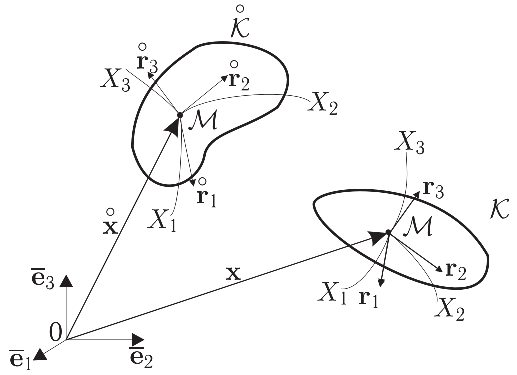

Figure 1.5. Local basis vectors in reference and actual configurations

In $\mathcal{K}$ and $\overset{\circ}{\mathcal{K}}$ introduce *metric matrices* g_{kl}, $\overset{\circ}{g}_{kl}$ and *inverse metric matrices* g^{kl}, $\overset{\circ}{g}{}^{kl}$ as follows:

$$
\begin{aligned}
g_{kl} &= \mathbf{r}_k \cdot \mathbf{r}_l = Q^i{}_k Q^j{}_l \delta_{ij} = \frac{\partial x^i}{\partial X^k}\frac{\partial x^j}{\partial X^l}\delta_{ij},\\
\overset{\circ}{g}_{kl} &= \overset{\circ}{\mathbf{r}}_k \cdot \overset{\circ}{\mathbf{r}}_l = \frac{\partial \overset{\circ}{x}{}^i}{\partial X^k}\frac{\partial \overset{\circ}{x}{}^j}{\partial X^l}\delta_{ij},\\
g^{kl} g_{lm} &= \delta^k_m, \qquad \overset{\circ}{g}{}^{kl}\overset{\circ}{g}_{lm} = \delta^k_m,
\end{aligned}
\tag{1.12}
$$

and also *vectors of reciprocal local bases*

$$
\mathbf{r}^i = g^{im}\mathbf{r}_m, \qquad \overset{\circ}{\mathbf{r}}{}^i = \overset{\circ}{g}{}^{im}\overset{\circ}{\mathbf{r}}_m, \tag{1.13}
$$

which satisfy the reciprocity relations

$$
\mathbf{r}_i \cdot \mathbf{r}^j = \delta_i{}^j, \qquad \overset{\circ}{\mathbf{r}}_i \cdot \overset{\circ}{\mathbf{r}}{}^j = \delta_i{}^j, \tag{1.14a}
$$

and also the following relations:

$$
\mathbf{r}_n \times \mathbf{r}_m = \sqrt{g}\ \epsilon_{nmk}\mathbf{r}^k, \qquad \overset{\circ}{\mathbf{r}}_n \times \overset{\circ}{\mathbf{r}}_m = \sqrt{\overset{\circ}{g}}\ \epsilon_{nmk}\ \overset{\circ}{\mathbf{r}}{}^k, \tag{1.14b}
$$

With the help of the mixed multiplication, i.e. sequentially applying scalar and vector products to three different local bases vectors, we can determine the volumes $|V|$ and $|\overset{\circ}{V}|$ constructed by these vectors:

$$
\begin{aligned}
|\overset{\circ}{V}| &= \overset{\circ}{\mathbf{r}}_1 \cdot (\overset{\circ}{\mathbf{r}}_2 \times \overset{\circ}{\mathbf{r}}_3) = \sqrt{\overset{\circ}{g}} = \sqrt{\det\,(\overset{\circ}{\mathrm{g}}_{\mathrm{ij}})} = \left|\partial \overset{\circ}{x}{}^k / \partial X^i\right|,\\
|V| &= \mathbf{r}_1 \cdot (\mathbf{r}_2 \times \mathbf{r}_3) = \sqrt{g} = |\partial x^k / \partial X^i|.
\end{aligned}
\tag{1.15}
$$

It should be noted that although local bases $\mathbf{r}_i$ and $\overset{\circ}{\mathbf{r}}_i$ have been introduced in different configurations $\mathcal{K}$ and $\overset{\circ}{\mathcal{K}}$, they correspond to the same coordinates X^i (if one consider the same point $\mathcal{M}$); therefore each of the bases can be carried as a rigid whole into the same point in $\mathcal{K}$ or in $\overset{\circ}{\mathcal{K}}$. Due to this, we can resolve any vector field $\mathbf{a}(\mathcal{M})$ for each of the bases $\mathbf{r}_i$, $\mathbf{r}^i$, $\overset{\circ}{\mathbf{r}}{}^i$ and $\overset{\circ}{\mathbf{r}}_i$:

$$\mathbf{a} = a^i \mathbf{r}_i = \overset{\circ}{a}{}^i \overset{\circ}{\mathbf{r}}_i = a_i \mathbf{r}^i = \overset{\circ}{a}_i \overset{\circ}{\mathbf{r}}{}^i. \tag{1.16}$$

If curvilinear coordinates X^i are orthogonal, then the vectors $\overset{\circ}{\mathbf{r}}_i$ are orthogonal as well: $(\overset{\circ}{\mathbf{r}}_i \cdot \overset{\circ}{\mathbf{r}}_j = \delta_{ij})$, and matrices $\overset{\circ}{g}_{ij}$ and $\overset{\circ}{g}{}^{ij}$ are diagonal; hence we can introduce *Lamé's coefficients* $\overset{\circ}{H}_\alpha = \sqrt{\overset{\circ}{g}_{\alpha\alpha}}$ $(\alpha = 1, 2, 3)$ and the *physical orthonormal basis*

$$\widehat{\overset{\circ}{\mathbf{r}}}_\alpha = \overset{\circ}{\mathbf{r}}_\alpha / \overset{\circ}{H}_\alpha = \overset{\circ}{\mathbf{r}}{}^\alpha \overset{\circ}{H}_\alpha. \tag{1.17}$$

Components of a vector $\mathbf{a}$ with respect to this basis are called *physical*:

$$\mathbf{a} = \widehat{\overset{\circ}{a}}{}^i \, \widehat{\overset{\circ}{\mathbf{r}}}_i. \tag{1.18}$$

The actual basis $\mathbf{r}_i$ is in general not orthogonal even if the basis $\overset{\circ}{\mathbf{r}}_i$ is orthogonal; therefore we cannot introduce the corresponding physical basis in $\mathcal{K}$. One can introduce a physical basis in $\mathcal{K}$ not with the help of $\mathbf{r}_i$, but with the help of another special basis (see paragraph 1.1.7).

1.1.4. Tensors and Tensor Fields in Continuum Mechanics. For different local bases $\mathbf{r}_i$, $\overset{\circ}{\mathbf{r}}_i$, $\mathbf{r}^i$, $\overset{\circ}{\mathbf{r}}{}^i$ or $\bar{\mathbf{e}}_i$ at every point $\mathcal{M}$, and with the help of formulae given in the work [12] we can introduce different *dyadic (tensor) bases*: $\mathbf{r}_i \otimes \mathbf{r}_j$, $\overset{\circ}{\mathbf{r}}_i \otimes \overset{\circ}{\mathbf{r}}_j$, $\overset{\circ}{\mathbf{r}}_i \otimes \mathbf{r}^j$, $\mathbf{r}_i \otimes \overset{\circ}{\mathbf{r}}_j$, $\overset{\circ}{\mathbf{r}}{}^i \otimes \mathbf{r}_j$ etc., which are equivalence classes of vector sets consisting of $2 \cdot 3 = 6$ vectors (for example, $\mathbf{r}_1 \otimes \mathbf{r}_j = [\mathbf{r}_1 \mathbf{r}_j \mathbf{r}_2 \mathbf{0} \mathbf{r}_3 \mathbf{0}]$, where [ ] is the notation of an equivalence class), and $\otimes$ is the sign of *tensor product*. A field of *second-order tensor* $\mathbf{T}(\mathcal{M})$ can be represented as a linear combination of dyadic basis elements:

$$\mathbf{T} = T^{ij} \mathbf{r}_i \otimes \mathbf{r}_j = \overset{\circ}{T}{}^{ij} \overset{\circ}{\mathbf{r}}_i \otimes \overset{\circ}{\mathbf{r}}_j = \bar{T}^{ij} \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j = \overset{\circ}{T}_{ij} \overset{\circ}{\mathbf{r}}{}^i \otimes \overset{\circ}{\mathbf{r}}{}^j. \tag{1.19}$$

During the passage from one basis to another, tensor components T^{ij} are transformed by the *tensor law*:

$$\bar{T}^{ij} = P^i{}_k P^j{}_l T^{kl} = \overset{\circ}{P}{}^i{}_k \overset{\circ}{P}{}^j{}_l \overset{\circ}{T}{}^{kl}. \tag{1.20}$$

Metric matrices g^{im}, g_{im}, $\overset{\circ}{g}{}^{im}$ and $\overset{\circ}{g}_{im}$ are components of the *unit (metric) tensor* $\mathbf{E}$ with respect to different bases:

$$\mathbf{E} = g_{im} \mathbf{r}^i \otimes \mathbf{r}^m = \overset{\circ}{g}_{im} \overset{\circ}{\mathbf{r}}{}^i \otimes \overset{\circ}{\mathbf{r}}{}^m = g^{im} \mathbf{r}_i \otimes \mathbf{r}_m = \overset{\circ}{g}{}^{im} \overset{\circ}{\mathbf{r}}_i \otimes \overset{\circ}{\mathbf{r}}_m. \tag{1.21}$$

For a second-order tensor $\mathbf{T}$, in continuum mechanics one often uses the transpose tensor $\mathbf{T}^{\mathrm{T}} = T^{ij} \mathbf{r}_j \otimes \mathbf{r}_i$ and the inverse tensor $\mathbf{T}^{-1}$, where $\mathbf{T}^{-1} \cdot \mathbf{T} = \mathbf{E}$. The inverse tensor exists only for a nonsingular tensor (when

$\det \mathbf{T} \neq 0$). The determinant of a tensor is defined by the determinant of its mixed components matrix: $\det \mathbf{T} = \det T^i{}_j$.

Besides second-order tensors, in continuum mechanics one sometimes uses tensors of higher orders [12]. To introduce the tensors, we define *polyadic bases* by induction: $\mathbf{r}_{i_1} \otimes \ldots \otimes \mathbf{r}_{i_n}$; the bases are equivalence classes of vector sets consisting of $n \cdot 3 = 3n$ vectors. A field of n-th order tensor ${}^n\mathbf{\Omega}(\mathcal{M})$ can be represented by a linear combination of polyadic basis elements:

$$ {}^n\mathbf{\Omega} = \Omega^{i_1 \ldots i_n} \mathbf{r}_{i_1} \otimes \ldots \otimes \mathbf{r}_{i_n} = \overset{\circ}{\Omega}{}^{i_1 \ldots i_n} \overset{\circ}{\mathbf{r}}_{i_1} \otimes \ldots \otimes \overset{\circ}{\mathbf{r}}_{i_n}, $$

where $\Omega^{i_1 \ldots i_n}$ and $\overset{\circ}{\Omega}{}^{i_1 \ldots i_n}$ are components of the n-th order tensor with respect to the corresponding polyadic basis.

For fourth-order tensors, analogs of the tensor $\mathbf{E}$ are the first, second and third unit tensors defined as follows:

$$ \begin{gathered} \mathbf{\Delta}_{\mathrm{I}} = \mathbf{e}_i \otimes \mathbf{e}^i \otimes \mathbf{e}_k \otimes \mathbf{e}^k = \mathbf{E} \otimes \mathbf{E}, \\ \mathbf{\Delta}_{\mathrm{II}} = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}^i \otimes \mathbf{e}^k, \quad \mathbf{\Delta}_{\mathrm{III}} = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}^k \otimes \mathbf{e}^i, \end{gathered} \tag{1.22} $$

and also the symmetric fourth-order unit tensor

$$ \mathbf{\Delta} = \frac{1}{2}(\mathbf{\Delta}_{\mathrm{II}} + \mathbf{\Delta}_{\mathrm{III}}). \tag{1.22a} $$

$$ \mathbf{\Delta} = \Delta^{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \qquad \Delta^{ijkl} = \frac{1}{2}(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}). $$

We can transpose fourth-order tensors as follows:

$$ {}^4\mathbf{\Omega}^{(m_1 m_2 m_3 m_4)} = \Omega^{i_1 i_2 i_3 i_4} \mathbf{r}_{i_{m_1}} \otimes \mathbf{r}_{i_{m_2}} \otimes \mathbf{r}_{i_{m_3}} \otimes \mathbf{r}_{i_{m_4}}, $$

where $(m_1 m_2 m_3 m_4)$ is some substitution, for example, ${}^4\mathbf{\Omega}^{(4321)} = \Omega^{i_1 i_2 i_3 i_4} \mathbf{r}_{i_4} \otimes \mathbf{r}_{i_3} \otimes \mathbf{r}_{i_2} \otimes \mathbf{r}_{i_1}$.

1.1.5. Covariant Derivatives in $\mathcal{K}$ and $\overset{\circ}{\mathcal{K}}$. Introduce the following *nabla-operators* in configurations $\mathcal{K}$ and $\overset{\circ}{\mathcal{K}}$, respectively:

$$ \mathbf{\nabla} = \mathbf{r}^k \otimes \frac{\partial}{\partial X^k}, \qquad \overset{\circ}{\mathbf{\nabla}} = \overset{\circ}{\mathbf{r}}{}^k \otimes \frac{\partial}{\partial X^k}. \tag{1.23} $$

Applying the nabla-operators to a vector field, we get the *gradients of a vector* in $\mathcal{K}$ and $\overset{\circ}{\mathcal{K}}$:

$$ \mathbf{\nabla} \otimes \mathbf{a} = \mathbf{r}^k \otimes \frac{\partial \mathbf{a}}{\partial X^k} = \nabla_k a_i \mathbf{r}^k \otimes \mathbf{r}^i, \tag{1.24} $$

$$ \overset{\circ}{\mathbf{\nabla}} \otimes \mathbf{a} = \overset{\circ}{\mathbf{r}}{}^k \otimes \frac{\partial \mathbf{a}}{\partial X^k} = \overset{\circ}{\nabla}_k \overset{\circ}{a}_i \overset{\circ}{\mathbf{r}}{}^k \otimes \overset{\circ}{\mathbf{r}}{}^i = \overset{\circ}{\nabla}{}^k \overset{\circ}{a}_i \overset{\circ}{\mathbf{r}}_k \otimes \overset{\circ}{\mathbf{r}}{}^i = \overset{\circ}{\nabla}{}^k \overset{\circ}{a}{}^i \overset{\circ}{\mathbf{r}}_k \otimes \overset{\circ}{\mathbf{r}}_i, $$

where we have denoted the following *covariant derivatives* in different *tensor bases* in configurations $\overset{\circ}{\mathcal{K}}$ and $\mathcal{K}$:

$$ \overset{\circ}{\nabla}_k \overset{\circ}{a}_i = \frac{\partial \overset{\circ}{a}_i}{\partial X^k} - \overset{\circ}{\Gamma}{}^m_{ik} \overset{\circ}{a}_m, \qquad \overset{\circ}{\nabla}_k \overset{\circ}{a}{}^i = \frac{\partial \overset{\circ}{a}{}^i}{\partial X^k} + \overset{\circ}{\Gamma}{}^i_{km} \overset{\circ}{a}{}^m \tag{1.25} $$

$$\nabla_k a_i = \frac{\partial a_i}{\partial X^k} - \Gamma^m_{ik} a_m, \qquad \nabla_k a^i = \frac{\partial a^i}{\partial X^k} + \Gamma^i_{km} a^m.$$

Here Γ^m_{ij} and $\overset{\circ}{\Gamma}{}^m_{ij}$ are the *Christoffel symbols* in configurations $\mathcal{K}$ and $\overset{\circ}{\mathcal{K}}$. For the Christoffel symbols the following relations (see [12]) hold:

$$\Gamma^m_{ij} = \frac{1}{2} g^{km} \left(\frac{\partial g_{kj}}{\partial X^i} + \frac{\partial g_{ki}}{\partial X^j} - \frac{\partial g_{ij}}{\partial X^k} \right),$$

$$\overset{\circ}{\Gamma}{}^m_{ij} = \frac{1}{2} \overset{\circ}{g}{}^{km} \left(\frac{\partial \overset{\circ}{g}_{kj}}{\partial X^i} + \frac{\partial \overset{\circ}{g}_{ki}}{\partial X^j} - \frac{\partial \overset{\circ}{g}_{ij}}{\partial X^k} \right). \tag{1.26}$$

Contravariant derivatives in $\mathcal{K}$ and $\overset{\circ}{\mathcal{K}}$ are introduced as follows:

$$\overset{\circ}{\nabla}{}^k \overset{\circ}{a}_i = \overset{\circ}{g}{}^{km} \overset{\circ}{\nabla}_m \overset{\circ}{a}_i, \qquad \nabla^k a^i = g^{km} \nabla_m a^i. \tag{1.27}$$

The covariant derivatives (1.25) are components of the second-order tensors $\overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{a}$ and $\boldsymbol{\nabla} \otimes \mathbf{a}$, therefore during the passage from the local basis $\mathbf{r}_i$ to another one they are transformed by the tensor law (1.20).

The nabla-operators $\overset{\circ}{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$ in $\overset{\circ}{\mathcal{K}}$ and $\mathcal{K}$ can be applied to a tensor field ${}^n\boldsymbol{\Omega}(X^i)$ of n-th order:

$$\overset{\circ}{\boldsymbol{\nabla}} \otimes {}^n\boldsymbol{\Omega} = \overset{\circ}{\mathbf{r}}{}^k \otimes \frac{\partial\, {}^n\boldsymbol{\Omega}}{\partial X^k} = \overset{\circ}{\nabla}_k \overset{\circ}{\Omega}{}^{i_1 \ldots i_n} \overset{\circ}{\mathbf{r}}{}^k \otimes \overset{\circ}{\mathbf{r}}_{i_1} \otimes \ldots \otimes \overset{\circ}{\mathbf{r}}_{i_n}, \tag{1.28}$$

$$\boldsymbol{\nabla} \otimes {}^n\boldsymbol{\Omega} = \mathbf{r}^k \otimes \frac{\partial\, {}^n\boldsymbol{\Omega}}{\partial X^k} = \nabla_k \Omega^{i_1 \ldots i_n} \mathbf{r}^k \otimes \mathbf{r}_{i_1} \otimes \ldots \otimes \mathbf{r}_{i_n},$$

where $\nabla_k \Omega^{i_1 \ldots i_n}$ and $\overset{\circ}{\nabla}_k \overset{\circ}{\Omega}{}^{i_1 \ldots i_n}$ are the covariant derivatives in $\mathcal{K}$ and $\overset{\circ}{\mathcal{K}}$, respectively:

$$\overset{\circ}{\nabla}_k \overset{\circ}{\Omega}{}^{i_1 \ldots i_n} = \frac{\partial}{\partial X^k} \overset{\circ}{\Omega}{}^{i_1 \ldots i_n} + \sum_{s=1}^{n} \overset{\circ}{\Gamma}{}^{i_s}_{mk} \overset{\circ}{\Omega}{}^{i_1 \ldots i_s = m \ldots i_n}. \tag{1.29}$$

In the same way we can define operations of scalar product of the nabla-operator in $\overset{\circ}{\mathcal{K}}$ (*the divergence of a tensor*):

$$\overset{\circ}{\boldsymbol{\nabla}} \cdot {}^n\boldsymbol{\Omega} = \overset{\circ}{\mathbf{r}}_k \cdot \frac{\partial\, {}^n\boldsymbol{\Omega}}{\partial X^k} = \overset{\circ}{\nabla}_k \overset{\circ}{\Omega}{}^{k i_2 \ldots i_n} \overset{\circ}{\mathbf{r}}_{i_2} \otimes \ldots \otimes \overset{\circ}{\mathbf{r}}_{i_n}, \tag{1.30}$$

and vector product of the nabla-operator in $\overset{\circ}{\mathcal{K}}$ (*the curl of a tensor*):

$$\overset{\circ}{\boldsymbol{\nabla}} \times {}^n\boldsymbol{\Omega} = \overset{\circ}{\mathbf{r}}{}^k \times \frac{\partial\, {}^n\boldsymbol{\Omega}}{\partial X^k} = \frac{1}{\sqrt{\overset{\circ}{g}}} \epsilon^{ijk} \overset{\circ}{\nabla}_i \overset{\circ}{\Omega}_{j i_2 \ldots i_n} \overset{\circ}{\mathbf{r}}_k \otimes \overset{\circ}{\mathbf{r}}{}^{i_2} \otimes \ldots \otimes \overset{\circ}{\mathbf{r}}{}^{i_n}. \tag{1.31}$$

1.1.6. The Deformation Gradient. Consider how a local *neighborhood* of a point $\mathcal{M}$ is transformed during the passage from configuration $\overset{\circ}{\mathcal{K}}$ to $\mathcal{K}$. Take an arbitrary elementary radius-vector $d\overset{\circ}{\mathbf{x}}$ connecting in $\overset{\circ}{\mathcal{K}}$ two infinitesimally close points $\mathcal{M}$ and $\mathcal{M}'$ (Figure 1.6). In configuration $\mathcal{K}$, these material points $\mathcal{M}$ and $\mathcal{M}'$ are connected by the elementary radius-vector $d\mathbf{x}$.

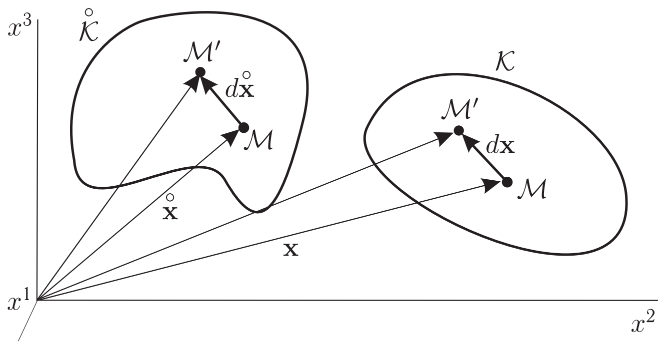

Figure 1.6. Transformation of an elementary radius-vector during the passage from the reference configuration to the actual one

The vectors $d\overset{\circ}{\mathbf{x}}$ and $d\mathbf{x}$ can always be resolved for local bases:

$$d\mathbf{x}(X^k) = \frac{\partial \mathbf{x}}{\partial X^k}\, dX^k = \mathbf{r}_k\, dX^k, \quad d\overset{\circ}{\mathbf{x}}(X^k) = \frac{\partial \overset{\circ}{\mathbf{x}}}{\partial X^k}\, dX^k = \overset{\circ}{\mathbf{r}}_k\, dX^k. \tag{1.32}$$

On multiplying the first equation by $\mathbf{r}^m$ and the second — by $\overset{\circ}{\mathbf{r}}{}^m$, we get

$$\mathbf{r}^m \cdot d\mathbf{x} = \mathbf{r}^m \cdot \mathbf{r}_k\, dX^k = dX^m, \quad \overset{\circ}{\mathbf{r}}{}^m \cdot d\overset{\circ}{\mathbf{x}} = \overset{\circ}{\mathbf{r}}{}^m \cdot \overset{\circ}{\mathbf{r}}_k\, dX^k = dX^m. \tag{1.33}$$

Substitution of dX^m (1.33) into the first equation of (1.32) yields $d\mathbf{x} = \mathbf{r}_k \otimes \overset{\circ}{\mathbf{r}}{}^k \cdot d\overset{\circ}{\mathbf{x}}$. Changing the order of the tensor and scalar products (that is permissible by the tensor analysis rules), we get the relation between $d\overset{\circ}{\mathbf{x}}$ and $d\mathbf{x}$:

$$d\mathbf{x} = \mathbf{F} \cdot d\overset{\circ}{\mathbf{x}}. \tag{1.34}$$

Here we have denoted the linear transformation tensor

$$\mathbf{F} = \mathbf{r}_k \otimes \overset{\circ}{\mathbf{r}}{}^k, \tag{1.35}$$

called the *deformation gradient*. As follows from (1.34), the deformation gradient connects elementary radius-vectors $d\overset{\circ}{\mathbf{x}}$ and $d\mathbf{x}$ of the same material point $\mathcal{M}$ in configurations $\mathcal{K}$ and $\overset{\circ}{\mathcal{K}}$.

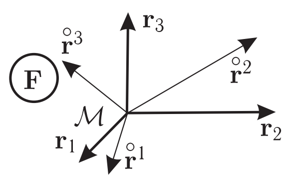

Figure 1.7. Geometric representation of the deformation gradient

Definition (1.19) allows us to give a geometric representation of the deformation gradient: if $\mathbf{r}_i$ are considered as the left vectors and $\overset{\circ}{\mathbf{r}}{}^i$ — as the right vectors, then, by formulae of paragraph 1.1.4 (see [12]), the tensor $\mathbf{F}$ takes the form

$$\mathbf{F} = \mathbf{r}_i \otimes \overset{\circ}{\mathbf{r}}{}^i = [\mathbf{r}_1 \overset{\circ}{\mathbf{r}}{}^1 \mathbf{r}_2 \overset{\circ}{\mathbf{r}}{}^2 \mathbf{r}_3 \overset{\circ}{\mathbf{r}}{}^3].$$

According to the geometric definition of a tensor (see paragraph 1.1.4), the tensor $\mathbf{F}$ can be represented as equivalence class of the ordered set of six vectors $\mathbf{r}_i, \overset{\circ}{\mathbf{r}}{}^i$ (Figure 1.7).

Besides $\mathbf{F}$, in continuum mechanics one often uses the transpose tensor $\mathbf{F}^{\mathrm{T}}$, the inverse tensor $\mathbf{F}^{-1}$ and the inverse to the transpose tensor $\mathbf{F}^{-1\mathrm{T}}$:

$$\mathbf{F}^{\mathrm{T}} = \overset{\circ}{\mathbf{r}}{}^k \otimes \mathbf{r}_k = \overset{\circ}{\mathbf{r}}{}^k \otimes \frac{\partial \mathbf{x}}{\partial X^k} = \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{x}, \qquad \mathbf{F}^{-1} = \overset{\circ}{\mathbf{r}}_k \otimes \mathbf{r}^k, \tag{1.35a}$$

$$\mathbf{F}^{-1\mathrm{T}} = (\overset{\circ}{\mathbf{r}}_k \otimes \mathbf{r}^k)^{\mathrm{T}} = \mathbf{r}^k \otimes \overset{\circ}{\mathbf{r}}_k = \mathbf{r}^k \otimes \frac{\partial \overset{\circ}{\mathbf{x}}}{\partial X^k} = \boldsymbol{\nabla} \otimes \overset{\circ}{\mathbf{x}}.$$

It follows from (1.35) that

$$\mathbf{F} \cdot \overset{\circ}{\mathbf{r}}_i = \mathbf{r}_k \otimes \overset{\circ}{\mathbf{r}}{}^k \cdot \mathbf{r}_i = \mathbf{r}_k \delta_i^k = \mathbf{r}_i. \tag{1.36}$$

i.e. the deformation gradient transforms local bases vectors of the same material point $\mathcal{M}$ from $\overset{\circ}{\mathcal{K}}$ to $\mathcal{K}$.

Theorem 1.1. *The transpose deformation gradient* $\mathbf{F}^{\mathrm{T}}$ *connects gradients of an arbitrary vector* $\mathbf{a}$ *in* $\overset{\circ}{\mathcal{K}}$ *and* $\mathcal{K}$*:*

$$\overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{a} = \mathbf{F}^{\mathrm{T}} \cdot \boldsymbol{\nabla} \otimes \mathbf{a}, \qquad \boldsymbol{\nabla} \otimes \mathbf{a} = \mathbf{F}^{-1\mathrm{T}} \cdot \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{a}. \tag{1.37}$$

▼ To derive formulae (1.37), we apply the definitions (1.24) and (1.35):

$$\boldsymbol{\nabla} \otimes \mathbf{a} = \mathbf{r}^i \otimes \frac{\partial \mathbf{a}}{\partial X^i} = \mathbf{r}^j \delta_j^i \otimes \frac{\partial \mathbf{a}}{\partial X^i} = \mathbf{r}^j \otimes \overset{\circ}{\mathbf{r}}_j \cdot \overset{\circ}{\mathbf{r}}{}^i \otimes \frac{\partial \mathbf{a}}{\partial X^i} = \mathbf{F}^{-1\mathrm{T}} \cdot \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{a}. \ \blacktriangle \tag{1.38}$$

1.1.7. Curvilinear Spatial Coordinates. Notice that the choice of Cartesian basis $O\bar{\mathbf{e}}_i$ as a fixed (immovable) system in the spatial (Eulerian) description of the continuum motion is not a necessary condition. For some problems of continuum mechanics it is convenient to consider a *moving system* $O'\bar{\mathbf{e}}'_i$ with the origin at a moving point O' ($\mathbf{x}_0 = \overrightarrow{OO'}$) and a moving orthonormal basis $\bar{\mathbf{e}}'_i$ (Figure 1.8), which is connected to $\bar{\mathbf{e}}_i$ by the orthogonal tensor $\mathbf{Q}$:

$$\bar{\mathbf{e}}'_i = \mathbf{Q} \cdot \bar{\mathbf{e}}_i. \tag{1.39}$$

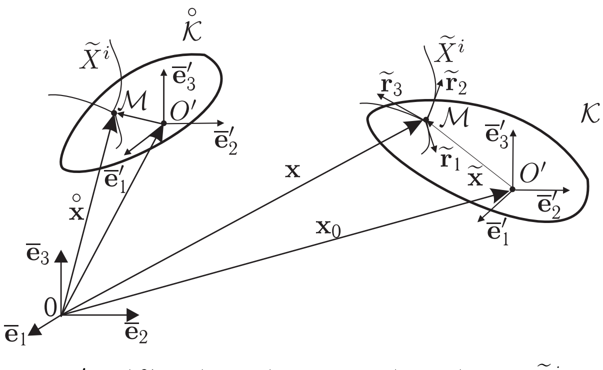

Figure 1.8. Moving bases $\bar{\mathbf{e}}'_i$ and $\widetilde{\mathbf{r}}_i$ and curvilinear spatial coordinates $\widetilde{X}^i$ in moving system $O'\bar{\mathbf{e}}'_i$

In this case, instead of Cartesian coordinates x^i of a point $\mathcal{M}$ in the basis $\bar{\mathbf{e}}_i$ one consider its Cartesian coordinates $\widetilde{x}^i$ in basis $\bar{\mathbf{e}}'_i$:

$$\widetilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0 = \widetilde{x}^i \bar{\mathbf{e}}'_i. \tag{1.40}$$

Let $Q^i{}_j$ be components of the tensor $\mathbf{Q}$ with respect to the basis $\bar{\mathbf{e}}_i$:

$$\mathbf{Q} = Q^i{}_j \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}^j, \tag{1.41}$$

then relation (1.39) takes the form

$$\bar{\mathbf{e}}'_i = Q^j{}_i \bar{\mathbf{e}}_j, \tag{1.42}$$

and coordinates $\widetilde{x}^i$ and x^i are connected as follows:

$$\begin{gathered} \widetilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0 = (x^i - x_0^i)\bar{\mathbf{e}}_i = \widetilde{x}^i \bar{\mathbf{e}}'_i = \widetilde{x}^i Q^j{}_i \bar{\mathbf{e}}_j, \\ x^i - x_0^i = Q^i{}_j \widetilde{x}^j, \\ \partial x^i / \partial \widetilde{x}^j = Q^i{}_j, \qquad \partial \widetilde{x}^j / \partial x^i = P^j{}_i\, . \end{gathered} \tag{1.43}$$

Instead of Cartesian coordinates $\widetilde{x}^i$, we can consider special curvilinear coordinates $\widetilde{X}^k$ with the origin at point O':

$$\widetilde{x}^i = \widetilde{x}^i(\widetilde{X}^k), \tag{1.44}$$

which, due to (1.27), are connected to x^i by the relations

$$x^i = x_0^i(t) + Q^i{}_j(t)\widetilde{x}^j(\widetilde{X}^k) \equiv x^i(\widetilde{X}^k, t) \quad \text{or} \quad \widetilde{X}^j = \widetilde{X}^j(x^i, t). \tag{1.45}$$

The dependence on t in the relations is defined by functions $x_0^i(t)$ and $Q^i{}_j(t)$ (i.e. only by the motion of system $O'\bar{\mathbf{e}}'_i$), which are assumed to be known in continuum mechanics.

Coordinates $\widetilde{X}^i$ are no longer Lagrangian (material): at different times they correspond to different material points. However, it is often convenient to choose coordinates $\widetilde{X}^i$ coincident with X^i in the reference configuration $\overset{\circ}{\mathcal{K}}$. In this case we have the relations

$$\overset{\circ}{x}{}^i(X^i) = x^i(X^j, 0) = x^i(\widetilde{X}^j, 0). \tag{1.46}$$

With the help of transformation (1.45) we can use the spatial description in coordinates $\widetilde{X}^i$ as well when consider the functions

$$\mathbf{a} = \mathbf{a}(x^i, t) = \widetilde{\mathbf{a}}(\widetilde{X}^i, t); \tag{1.47}$$

therefore coordinates $\widetilde{X}^i$ are called *curvilinear spatial coordinates*.

Introduce local vectors

$$\widetilde{\mathbf{r}}_i = \frac{\partial \mathbf{x}}{\partial \widetilde{X}^i} = \frac{\partial x^j}{\partial \widetilde{X}^i}\bar{\mathbf{e}}_j. \tag{1.48}$$

In particular, the basis $\bar{\mathbf{e}}'_i$ may be fixed (Figure 1.9), then $\bar{\mathbf{e}}'_i = \bar{\mathbf{e}}_i$, $\widetilde{\mathbf{x}} = \mathbf{x}$, and curvilinear spatial coordinates $\widetilde{X}^j = \widetilde{X}^j(x^i)$ are independent of t; the basis $\widetilde{\mathbf{r}}_i$ is independent of t as well, and from (1.46) and (1.48) it follows that the basis coincides with $\overset{\circ}{\mathbf{r}}_i$:

$$\widetilde{\mathbf{r}}_i = \frac{\partial x^j}{\partial \widetilde{X}^i}\bar{\mathbf{e}}_j = \frac{\partial \overset{\circ}{x}{}^j}{\partial X^i}\bar{\mathbf{e}}_j = \overset{\circ}{\mathbf{r}}_i. \tag{1.49}$$

When the basis $\bar{\mathbf{e}}'_i$ is moving, bases $\widetilde{\mathbf{r}}_i$ and $\overset{\circ}{\mathbf{r}}_i$ are no longer coincident.

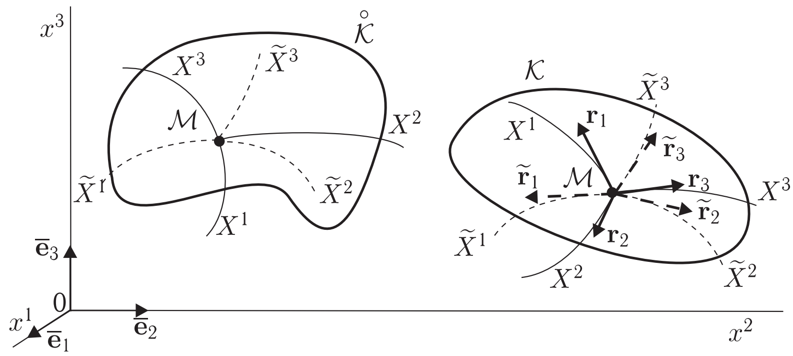

Figure 1.9. Curvilinear spatial coordinates $\widetilde{X}^i$ and Lagrangian coordinates X^i for the fixed basis $\bar{\mathbf{e}}'_i = \bar{\mathbf{e}}_i$

The vectors $\widetilde{\mathbf{r}}_i$ are directed tangentially to the coordinate lines $\widetilde{X}^i$ and defined simultaneously with $\mathbf{r}_i$ at every point $\mathcal{M}$ at any time $t \geqslant 0$.

A change of vectors $\widetilde{\mathbf{r}}_i$ in time is defined only by the motion of basis $\bar{\mathbf{e}}'_i$, because from (1.42), (1.43) and (1.48) it follows that

$$\bar{\mathbf{e}}'_i = Q^j{}_i \frac{\partial \widetilde{X}^k}{\partial x^j} \widetilde{\mathbf{r}}_k = \frac{\partial x^j}{\partial \widetilde{x}^i} \frac{\partial \widetilde{X}^k}{\partial x^j} \widetilde{\mathbf{r}}_k = \Big(\frac{\partial \widetilde{X}^k}{\partial \widetilde{x}^i}\Big) \widetilde{\mathbf{r}}_k, \tag{1.50}$$

and the matrix $\widetilde{P}^k{}_i \equiv \partial \widetilde{X}^k / \partial \widetilde{x}^i$ is independent of t according to (1.44).

The bases vectors $\mathbf{r}_i$ and $\widetilde{\mathbf{r}}_i$ are connected as follows:

$$\mathbf{r}_i = \frac{\partial \mathbf{x}}{\partial X^i} = \frac{\partial \mathbf{x}}{\partial \widetilde{X}^k} \frac{\partial \widetilde{X}^k}{\partial X^i} = \frac{\partial \widetilde{X}^k}{\partial X^i} \widetilde{\mathbf{r}}_k. \tag{1.51}$$

Just as in paragraph 1.1.2, we define the metric matrix $\widetilde{g}_{ij}$ and the inverse metric matrix $\widetilde{g}^{ij}$:

$$\widetilde{g}_{ij} = \widetilde{\mathbf{r}}_i \cdot \widetilde{\mathbf{r}}_j, \qquad \widetilde{g}^{ij} \widetilde{g}_{jk} = \delta^i_k, \tag{1.52}$$

and the reciprocal basis vectors

$$\widetilde{\mathbf{r}}^i = \widetilde{g}^{ik} \widetilde{\mathbf{r}}_k = \frac{\partial \widetilde{X}^i}{\partial x^j} \bar{\mathbf{e}}^j = \frac{\partial \widetilde{X}^i}{\partial \widetilde{x}^k} \bar{\mathbf{e}}'^k. \tag{1.53}$$

According to formulae (1.51) and (1.52), we find the relation between matrices g_{ij} and $\widetilde{g}_{kl}$:

$$g_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j = \frac{\partial \widetilde{X}^k}{\partial X^i} \frac{\partial \widetilde{X}^l}{\partial X^j} \widetilde{\mathbf{r}}_k \cdot \widetilde{\mathbf{r}}_l = \frac{\partial \widetilde{X}^k}{\partial X^i} \frac{\partial \widetilde{X}^l}{\partial X^j} \widetilde{g}_{kl}. \tag{1.54}$$

The inverse matrix g^{ij} is found from (1.54) by the rule of matrix product inversion (see Exercise 1.1.13):

$$g^{ij} = \frac{\partial X^i}{\partial \widetilde{X}^k} \frac{\partial X^j}{\partial \widetilde{X}^l} \widetilde{g}^{kl}. \tag{1.55}$$

From (1.51), (1.53) and (1.55) we can find the relation between vectors of reciprocal bases $\mathbf{r}^i$ and $\widetilde{\mathbf{r}}^i$:

$$\mathbf{r}^i = g^{ij}\mathbf{r}_j = \frac{\partial X^i}{\partial \widetilde{X}^k}\frac{\partial X^j}{\partial \widetilde{X}^l}\widetilde{g}^{kl}\frac{\partial \widetilde{X}^m}{\partial X^j}\widetilde{\mathbf{r}}_m = \frac{\partial X^i}{\partial \widetilde{X}^k}\widetilde{\mathbf{r}}^k. \tag{1.56}$$

Let there be a tensor ${}^n\mathbf{\Omega}$, then it can be resolved for the basis $\mathbf{r}_i$ and for the basis $\widetilde{\mathbf{r}}_i$:

$${}^n\mathbf{\Omega} = \Omega^{i_1\ldots i_n}\mathbf{r}_{i_1}\otimes\ldots\otimes\mathbf{r}_{i_n} = \widetilde{\Omega}^{i_1\ldots i_n}\widetilde{\mathbf{r}}_{i_1}\otimes\ldots\otimes\widetilde{\mathbf{r}}_{i_n}. \tag{1.57}$$

On substituting (1.51) into (1.57), we derive transformation formulae for tensor components during the passage from coordinates X^i to $\widetilde{X}^i$:

$$\widetilde{\Omega}^{i_1\ldots i_n} = \Omega^{j_1\ldots j_n}\frac{\partial \widetilde{X}^{i_1}}{\partial X^{j_1}}\ldots\frac{\partial \widetilde{X}^{i_n}}{\partial X^{j_n}}. \tag{1.58}$$

Introduce the nabla-operator $\widetilde{\mathbf{\nabla}}$ of covariant differentiation in coordinates $\widetilde{X}^i$:

$$\widetilde{\mathbf{\nabla}} = \widetilde{\mathbf{r}}^i\frac{\partial}{\partial \widetilde{X}^i} \tag{1.59}$$

and contravariant derivatives of components $\widetilde{a}_i$ of a vector $\mathbf{a} = \widetilde{a}_i\widetilde{\mathbf{r}}^i$ in coordinates $\widetilde{X}^i$:

$$\widetilde{\nabla}_k\widetilde{a}_i = \frac{\partial \widetilde{a}_i}{\partial \widetilde{X}^k} - \widetilde{\Gamma}^m_{ik}\widetilde{a}_m, \qquad \widetilde{\nabla}_k\widetilde{a}^i = \frac{\partial \widetilde{a}^i}{\partial \widetilde{X}^k} + \widetilde{\Gamma}^i_{km}\widetilde{a}^m. \tag{1.60}$$

The Christoffel symbols $\widetilde{\Gamma}^m_{ij}$ in coordinates $\widetilde{X}^i$ are connected to $\widetilde{g}_{ij}$ by the relations which are similar to (1.26).

Theorem 1.2. *The results of covariant differentiation in coordinates $\widetilde{X}^i$ and X^i (in the configuration $\mathcal{K}$) are coincident:*

$$\mathbf{\nabla}\otimes{}^n\mathbf{\Omega} = \widetilde{\mathbf{\nabla}}\otimes{}^n\mathbf{\Omega}, \qquad \mathbf{\nabla}\cdot{}^n\mathbf{\Omega} = \widetilde{\mathbf{\nabla}}\cdot{}^n\mathbf{\Omega}, \qquad \mathbf{\nabla}\times{}^n\mathbf{\Omega} = \widetilde{\mathbf{\nabla}}\times{}^n\mathbf{\Omega}. \tag{1.61}$$

▼ Prove the first formula in (1.61). Due to (1.23), we have

$$\mathbf{\nabla}\otimes{}^n\mathbf{\Omega} = \mathbf{r}^i\otimes\frac{\partial\,{}^n\mathbf{\Omega}}{\partial X^i} = \frac{\partial X^i}{\partial \widetilde{X}^k}\widetilde{\mathbf{r}}^k\otimes\frac{\partial\,{}^n\mathbf{\Omega}}{\partial \widetilde{X}^l}\frac{\partial \widetilde{X}^l}{\partial X^i} = \delta^l_k\mathbf{r}^k\otimes\frac{\partial\,{}^n\mathbf{\Omega}}{\partial \widetilde{X}^l} = \widetilde{\mathbf{\nabla}}\otimes{}^n\mathbf{\Omega}. \tag{1.62}$$

The remaining two formulae in (1.61) can be derived in the same way (see Exercise 1.1.8). ▲

Going to components of a tensor ${}^n\mathbf{\Omega}$ with respect to bases $\mathbf{r}_i$ and $\widetilde{\mathbf{r}}_i$, from (1.58) we get the relation between the covariant derivatives:

$$\nabla_i\Omega^{j_1\ldots j_n} = \widetilde{\nabla}_i\widetilde{\Omega}^{j_1\ldots j_n}. \tag{1.63}$$

Determine the tensor $\mathbf{H}$ transforming coordinates $\widetilde{X}^i$ into X^j:

$$\mathbf{H} = \overset{\circ}{\mathbf{r}}_j\otimes\widetilde{\mathbf{r}}^j = \widetilde{H}^i{}_j\widetilde{\mathbf{r}}_i\otimes\widetilde{\mathbf{r}}^j = \bar{H}^i{}_j\bar{\mathbf{e}}_i\otimes\bar{\mathbf{e}}^j. \tag{1.64}$$

Then we get the relations (see Exercises 1.1.10 and 1.1.11):

$$\overset{\circ}{\mathbf{r}}_i = \mathbf{H}\cdot\widetilde{\mathbf{r}}_i = \widetilde{H}^j{}_i\widetilde{\mathbf{r}}_j, \qquad \overset{\circ}{\mathbf{r}}^i = \mathbf{H}^{-1\mathrm{T}}\cdot\widetilde{\mathbf{r}}^i = (\widetilde{H}^i{}_j)^{-1}\widetilde{\mathbf{r}}^j. \tag{1.65}$$

The coordinates $\widetilde{X}^i$ are often chosen orthogonal, then the bases $\widetilde{\mathbf{r}}_i$ and $\widetilde{\mathbf{r}}^i$ are orthogonal as well, and the matrices $\widetilde{g}_{ij}$ and $\widetilde{g}^{ij}$ are diagonal; and we can introduce the physical (orthonormal) basis:

$$\widehat{\widetilde{\mathbf{r}}}_\alpha = \widetilde{\mathbf{r}}_\alpha / \widetilde{H}_\alpha, \tag{1.66}$$

where $\widetilde{H}_\alpha = \sqrt{\widetilde{g}_{\alpha\alpha}}$ are Lamé's coefficients, which are in general not coincident with the coefficients $\overset{\circ}{H}_\alpha = \sqrt{\overset{\circ}{g}_{\alpha\alpha}}$. Tensor components with respect to the basis $\widehat{\widetilde{\mathbf{r}}}_\alpha$ are called *physical*:

$$\mathbf{T} = \widehat{\widetilde{T}}^{ij} \widehat{\widetilde{\mathbf{r}}}_i \otimes \widehat{\widetilde{\mathbf{r}}}_j. \tag{1.67}$$

Relations between physical and covariant components of a tensor are determined by the known formulae (see [12]).

Exercises for 1.1.

Exercise 1.1.1. With the help of formulae (1.10), (1.12), (1.13) and (1.17) show that if the motion law of a continuum describes extension of a beam (1.6) (see Example 1.1), then the local basis vectors $\overset{\circ}{\mathbf{r}}_i$ and the metric matrices have the forms

$$\overset{\circ}{\mathbf{r}}_i = \mathbf{e}_i, \qquad \overset{\circ}{\mathbf{r}}{}^i = \mathbf{e}^i,$$
$$\mathbf{r}_\alpha = k_\alpha \bar{\mathbf{e}}_\alpha, \qquad \mathbf{r}^\alpha = (1/k_\alpha)\bar{\mathbf{e}}^\alpha, \qquad \alpha = 1, 2, 3,$$
$$\overset{\circ}{g}_{ij} = \delta_{ij}, \qquad \overset{\circ}{g}{}^{ij} = \delta^{ij},$$
$$(g_{ij}) = \begin{pmatrix} k_1^2 & 0 & 0 \\ 0 & k_2^2 & 0 \\ 0 & 0 & k_3^2 \end{pmatrix}, \qquad (g^{ij}) = \begin{pmatrix} k_1^{-2} & 0 & 0 \\ 0 & k_2^{-2} & 0 \\ 0 & 0 & k_3^{-2} \end{pmatrix},$$

i.e.

$$g_{\alpha\beta} = k_\alpha^2 \delta_{\alpha\beta}, \quad g^{\alpha\beta} = k_\alpha^{-2}\delta_{\alpha\beta}, \quad H_\alpha = \sqrt{g_{\alpha\alpha}} = k_\alpha, \quad \widehat{\mathbf{r}}_\alpha = \mathbf{e}_\alpha.$$

Exercise 1.1.2. Show that if the motion law of a continuum describes a simple shear (see Example 1.2), then the local basis vectors and the metric matrices have the forms

$$\overset{\circ}{\mathbf{r}}_i = \mathbf{e}_i, \qquad \overset{\circ}{\mathbf{r}}{}^i = \mathbf{e}^i, \qquad \overset{\circ}{g}_{ij} = \delta_{ij}, \qquad \overset{\circ}{g}{}^{ij} = \delta^{ij},$$
$$\mathbf{r}_1 = \bar{\mathbf{e}}_1, \quad \mathbf{r}_2 = a\bar{\mathbf{e}}_1 + \bar{\mathbf{e}}_2, \qquad \mathbf{r}_3 = \bar{\mathbf{e}}_3,$$
$$\mathbf{r}^1 = \bar{\mathbf{e}}^1 - a\bar{\mathbf{e}}^2, \qquad \mathbf{r}^2 = \bar{\mathbf{e}}^2, \qquad \mathbf{r}^3 = \bar{\mathbf{e}}^3,$$
$$g_{ij} = \begin{pmatrix} 1 & a & 0 \\ a & 1+a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad g^{ij} = \begin{pmatrix} 1+a^2 & -a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 1.1.3. Show that if the motion law describes rotation of a beam with extension (see Example 1.3), then with introducing the rotation tensor $\mathbf{O}_0$ and the stretch tensor $\mathbf{U}_0$:

$$\mathbf{O}_0 = O_0{}^i{}_j \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}^j, \qquad \mathbf{U}_0 = \sum_{\alpha=1}^{3} k_\alpha \bar{\mathbf{e}}_\alpha \otimes \bar{\mathbf{e}}_\alpha,$$

we can rewrite the beam motion law in the tensor form

$$\mathbf{x} = \mathbf{F}_0 \cdot \overset{\circ}{\mathbf{x}}, \qquad \mathbf{F}_0 = \mathbf{O}_0 \cdot \mathbf{U}_0.$$

Show that the local basis vectors and metric matrices for this problem have the forms

$$\mathbf{r}_i = F_0{}^k_{\ i}\,\bar{\mathbf{e}}_k, \qquad \overset{\circ}{\mathbf{r}}_i = \bar{\mathbf{e}}_i,$$

$$g_{ij} = F_0{}^k_{\ i}\,F_0{}^l_{\ j}\delta_{kl} = \begin{pmatrix} k_1^2\cos^2\varphi + k_2^2\sin^2\varphi & (k_1^2-k_2^2)\cos\varphi\sin\varphi & 0 \\ (k_1^2-k_2^2)\cos\varphi\sin\varphi & k_1^2\sin^2\varphi + k_2^2\cos^2\varphi & 0 \\ 0 & 0 & k_3^2 \end{pmatrix},$$

$$g = k_1k_2k_3,$$

$$g^{ij} = \begin{pmatrix} k_2^{-2}\sin^2\varphi + k_1^{-2}\cos^2\varphi & (k_1^{-2}-k_2^{-2})\cos\varphi\sin\varphi & 0 \\ (k_1^{-2}-k_2^{-2})\cos\varphi\sin\varphi & k_2^{-2}\cos^2\varphi + k_1^{-2}\sin^2\varphi & 0 \\ 0 & 0 & k_3^{-2} \end{pmatrix}.$$

Exercise 1.1.4. Using the property (1.14) of reciprocal basis vectors, show that the following relations hold:

$$\overset{\circ}{\mathbf{r}}{}^i = \frac{\partial X^i}{\partial \overset{\circ}{x}{}^k}\bar{\mathbf{e}}^k, \qquad \mathbf{r}^i = P^i_{\ k}\bar{\mathbf{e}}^k = \frac{\partial X^i}{\partial x^k}\bar{\mathbf{e}}^k.$$

Exercise 1.1.5. Show that $\mathbf{F}$, $\mathbf{F}^{\mathrm{T}}$, $\mathbf{F}^{-1}$ and $\mathbf{F}^{-1\mathrm{T}}$ in the Cartesian coordinate system take the forms

$$\mathbf{F} = \frac{\partial x^m}{\partial \overset{\circ}{x}{}^i}\bar{\mathbf{e}}_m \otimes \bar{\mathbf{e}}^i, \qquad \mathbf{F}^{\mathrm{T}} = \frac{\partial X^k}{\partial \overset{\circ}{x}{}^i}\bar{\mathbf{e}}^i \otimes \frac{\partial x^m}{\partial X^k}\bar{\mathbf{e}}_m = \frac{\partial x^m}{\partial \overset{\circ}{x}{}^i}\bar{\mathbf{e}}^i \otimes \bar{\mathbf{e}}_m,$$

$$\mathbf{F}^{-1} = \frac{\partial \overset{\circ}{x}{}^m}{\partial X^k}\bar{\mathbf{e}}_m \otimes \frac{\partial X^k}{\partial x^i}\bar{\mathbf{e}}^i = \frac{\partial \overset{\circ}{x}{}^m}{\partial x^i}\bar{\mathbf{e}}_m \otimes \mathbf{e}^i,$$

$$\mathbf{F}^{-1\mathrm{T}} = \frac{\partial X^k}{\partial x^i}\bar{\mathbf{e}}^i \otimes \frac{\partial \overset{\circ}{x}{}^m}{\partial X^k}\bar{\mathbf{e}}_m = \frac{\partial \overset{\circ}{x}{}^m}{\partial x^i}\bar{\mathbf{e}}^i \otimes \mathbf{e}_m.$$

Exercise 1.1.6. Substituting (1.54), (1.52) and (1.55) into (1.12), derive formula (1.55).

Exercise 1.1.7. Prove that

$$\mathbf{r}^i = \mathbf{F}^{-1\mathrm{T}} \cdot \overset{\circ}{\mathbf{r}}{}^i.$$

Exercise 1.1.8. Derive the third formula of (1.61).

Exercise 1.1.9. Prove that for any scalar function $\varphi(X^i)$ its gradients in $\overset{\circ}{\mathcal{K}}$ and $\mathcal{K}$ are connected by the relationship

$$\boldsymbol{\nabla}\varphi = \mathbf{F}^{-1\mathrm{T}} \cdot \overset{\circ}{\boldsymbol{\nabla}}\varphi.$$

Exercise 1.1.10. Show that formulae (1.65) follow from (1.64).

Exercise 1.1.11. Using (1.47), show that in formulae (1.64) the tensor $\mathbf{H}$ has the following components with respect to bases $\bar{\mathbf{e}}_i$ and $\widetilde{\mathbf{r}}_i$:

$$\bar{H}^i_{\ j} = \frac{\partial \overset{\circ}{x}{}^i}{\partial X^k}\frac{\partial \widetilde{X}^k}{\partial x^j}, \quad (\bar{H}^i_{\ j})^{-1} = \frac{\partial x^i}{\partial \widetilde{X}^k}\frac{\partial X^k}{\partial \overset{\circ}{x}{}^j}, \quad \widetilde{H}^i_{\ j} = \frac{\partial \overset{\circ}{x}{}^k}{\partial X^j}\frac{\partial \widetilde{X}^i}{\partial x^k}, \quad (\widetilde{H}^i_{\ j})^{-1} = \frac{\partial x^k}{\partial \widetilde{X}^j}\frac{\partial X^i}{\partial \overset{\circ}{x}{}^k}.$$

Exercise 1.1.12. Introducing the notation $\overset{\circ}{F}{}^{ij}$ for components of the deformation gradient $\mathbf{F}$ with respect to basis $\overset{\circ}{\mathbf{r}}_i$: $\mathbf{F} = \overset{\circ}{F}{}^{ij}\overset{\circ}{\mathbf{r}}_i \otimes \overset{\circ}{\mathbf{r}}_j = \overset{\circ}{F}{}^i_{\ j}\overset{\circ}{\mathbf{r}}_i \otimes \overset{\circ}{\mathbf{r}}{}^j$, show that formula (1.36) yields

$$\mathbf{r}_j = \overset{\circ}{F}{}^i_{\ j}\overset{\circ}{\mathbf{r}}_i.$$

Exercise 1.1.13. Show that the Levi-Civita symbols are connected by the relations

$$\epsilon_{ijk}\epsilon^{ijk} = 6, \quad \epsilon_{ijk}\epsilon^{ilm} = \delta^l_j\delta^m_k - \delta^l_k\delta^m_j, \quad \epsilon_{ijk}\epsilon^{ijl} = 2\delta^l_i,$$

$$\sqrt{g}\ \epsilon_{ijk} = (1/\sqrt{g})\ \epsilon^{mnl} g_{mi} g_{nj} g_{lk}. \quad \epsilon_{ijk}T^{jk} = 0$$

where T^{jk} are components of an arbitrary symmetric tensor: $T^{jk} = T^{kj}$.

Exercise 1.1.14. Using relations (1.14a), show that the local bases vectors are connected by the relations

$$\mathbf{r}_\alpha \times \mathbf{r}_\beta = \sqrt{g}\ \mathbf{r}^\gamma, \quad \mathring{\mathbf{r}}_\alpha \times \mathring{\mathbf{r}}_\beta = \sqrt{\mathring{g}}\ \mathring{\mathbf{r}}^\gamma, \quad \alpha \neq \beta \neq \gamma \neq \alpha.$$

Exercise 1.1.15. Show that the unit fourth-order tensors $\mathbf{\Delta}_{\mathrm{I}}$, $\mathbf{\Delta}_{\mathrm{II}}$ and $\mathbf{\Delta}_{\mathrm{III}}$ defined by formulae (1.22) have the following properties:

$$\mathbf{\Delta}_{\mathrm{I}} \cdot\cdot\, \mathbf{T} = I_1(\mathbf{T})\mathbf{E}, \quad I_1(\mathbf{T}) = \mathbf{T} \cdot\cdot\, \mathbf{E}, \quad \mathbf{\Delta}_{\mathrm{II}} \cdot\cdot\, \mathbf{T} = \mathbf{T}^{\mathrm{T}},$$

$$\mathbf{\Delta}_{\mathrm{III}} \cdot\cdot\, \mathbf{T} = \mathbf{T}, \quad \mathbf{\Delta} \cdot\cdot\, \mathbf{T} = \frac{1}{2}(\mathbf{T} + \mathbf{T}^{\mathrm{T}}),$$

and

$$\mathbf{\Delta}_{\mathrm{I}} \cdot\cdot\, {}^4\mathbf{\Omega} = \mathbf{E} \otimes \mathbf{E} \cdot\cdot\, {}^4\mathbf{\Omega}, \quad \mathbf{\Delta}_{\mathrm{II}} \cdot\cdot\, {}^4\mathbf{\Omega} = \mathbf{\Omega}^{(2134)},$$

$$\mathbf{\Delta}_{\mathrm{III}} \cdot\cdot\, {}^4\mathbf{\Omega} = {}^6\mathbf{\Omega}, \quad \mathbf{\Delta} \cdot\cdot\, {}^4\mathbf{\Omega} = \frac{1}{2}(\mathbf{\Omega}^{(2134)} + \mathbf{\Omega}),$$

for arbitrary second-order tensor $\mathbf{T}$ and fourth-order tensor ${}^4\mathbf{\Omega}$. As follows from these formulae, the tensor $\mathbf{\Delta}_{\mathrm{III}}$ is the 'true' unit fourth-order tensor.

Exercise 1.1.16. Show that components of the symmetric unit fourth-order tensor $\mathbf{\Delta}$ with respect to a tetradic basis have the form

$$\mathbf{\Delta} = \frac{1}{2}(\mathbf{e}_i \otimes \mathbf{e}^l \otimes \mathbf{e}^i \otimes \mathbf{e}_l + \mathbf{e}_i \otimes \mathbf{e}^l \otimes \mathbf{e}_l \otimes \mathbf{e}^i) = \Delta^{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l.$$

$$\Delta^{ijkl} = \frac{1}{2}(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}).$$

Exercise 1.1.17. Show that for any second-order tensor $\mathbf{T}$ and for any vector $\mathbf{a}$ the following formula of covariant differentiation hold:

$$\nabla \cdot (\mathbf{T} \cdot \mathbf{a}) = \mathbf{T} \cdot\cdot\, (\nabla \otimes \mathbf{a})^{\mathrm{T}} + \mathbf{a} \cdot \nabla \cdot \mathbf{T}.$$

1.2. Deformation Tensors and Measures

1.2.1. Deformation Tensors. Besides $\mathbf{F}$, important characteristics of the motion of a continuum are *deformation tensors*, which are introduced as follows:

$$\begin{aligned} \mathbf{C} &= \frac{1}{2}(g_{ij} - \mathring{g}_{ij})\mathring{\mathbf{r}}^i \otimes \mathring{\mathbf{r}}^j = \varepsilon_{ij}\mathring{\mathbf{r}}^i \otimes \mathring{\mathbf{r}}^j, \\ \mathbf{A} &= \frac{1}{2}(g_{ij} - \mathring{g}_{ij})\mathbf{r}^i \otimes \mathbf{r}^j = \varepsilon_{ij}\mathbf{r}^i \otimes \mathbf{r}^j, \\ \mathbf{\Lambda} &= \frac{1}{2}(\mathring{g}^{ij} - g^{ij})\mathring{\mathbf{r}}_i \otimes \mathring{\mathbf{r}}_j = \varepsilon^{ij}\mathring{\mathbf{r}}_i \otimes \mathring{\mathbf{r}}_j, \end{aligned} \tag{2.1}$$

$$\mathbf{J} = \frac{1}{2}(\overset{\circ}{g}{}^{ij} - g^{ij})\mathbf{r}_i \otimes \mathbf{r}_j = \varepsilon^{ij}\mathbf{r}_i \otimes \mathbf{r}_j.$$

Here $\mathbf{C}$ is called the *right Cauchy–Green deformation tensor*, $\mathbf{A}$ — the *left Almansi deformation tensor*, $\mathbf{\Lambda}$ — the *right Almansi deformation tensor*, and $\mathbf{J}$ — the *left Cauchy–Green tensor*.

As follows from the definition of the tensors, covariant components of $\mathbf{C}$ and $\mathbf{A}$ are coincident, but they are defined with respect to different tensor bases. Components ε_{ij} are called *covariant components of the deformation tensor*.

Contravariant components of the tensors $\mathbf{\Lambda}$ and $\mathbf{J}$ are also coincident and called *contravariant components* ε^{ij} of the deformation tensor, but they are defined with respect to different tensor bases of the tensors $\mathbf{\Lambda}$ and $\mathbf{J}$.

Notice that the deformation tensor components

$$\varepsilon_{ij} = \frac{1}{2}(g_{ij} - \overset{\circ}{g}_{ij}), \qquad \varepsilon^{ij} = \frac{1}{2}(\overset{\circ}{g}{}^{ij} - g^{ij}), \tag{2.2}$$

have been defined independently of each other, therefore the formal rearrangement of indices is not permissible for these components, i.e.

$$\breve{\varepsilon}^{kl} = \varepsilon_{ij}g^{ik}g^{jl} \neq \varepsilon^{kl}, \qquad \breve{\varepsilon}_{kl} = \varepsilon^{ij}g_{ik}g_{jl} \neq \varepsilon_{kl}. \tag{2.3}$$

Thus, when there is a need to obtain contravariant components from ε_{ij} and covariant components from ε^{ij}, one should use the notation $\breve{\varepsilon}^{kl}$ and $\breve{\varepsilon}_{kl}$. We will also use the notation

$$\overset{\circ}{\varepsilon}_{kl} = \varepsilon^{ij}\overset{\circ}{g}_{ik}\overset{\circ}{g}_{jl}, \qquad \overset{\circ}{\varepsilon}{}^{kl} = \varepsilon_{ij}\overset{\circ}{g}{}^{ik}\overset{\circ}{g}{}^{jl}. \tag{2.4}$$

Theorem 1.3. *The deformation tensors* $\mathbf{C}$, $\mathbf{A}$, $\mathbf{\Lambda}$ *and* $\mathbf{J}$ *are connected to the deformation gradient* $\mathbf{F}$ *as follows:*

$$\mathbf{C} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} - \mathbf{E}), \qquad \mathbf{A} = \frac{1}{2}(\mathbf{E} - \mathbf{F}^{-1\mathrm{T}} \cdot \mathbf{F}^{-1}),$$
$$\mathbf{\Lambda} = \frac{1}{2}(\mathbf{E} - \mathbf{F}^{-1} \cdot \mathbf{F}^{-1\mathrm{T}}), \qquad \mathbf{J} = \frac{1}{2}(\mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} - \mathbf{E}). \tag{2.5}$$

▼ Let us derive a relation between $\mathbf{C}$ and $\mathbf{F}$. Having used the definitions of g_{ij}, $\overset{\circ}{g}_{ij}$ and $\mathbf{F}$, we get

$$\mathbf{C} = \frac{1}{2}\left((\mathbf{r}_i \cdot \mathbf{r}_j)\overset{\circ}{\mathbf{r}}{}^i \otimes \overset{\circ}{\mathbf{r}}{}^j - \mathbf{E}\right) = \frac{1}{2}\left(\overset{\circ}{\mathbf{r}}{}^i \otimes \mathbf{r}_i \cdot \ \mathbf{r}_j \otimes \overset{\circ}{\mathbf{r}}{}^j - \mathbf{E}\right) = \frac{1}{2}(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} - \mathbf{E}). \tag{2.6}$$

The remaining relations of (2.5) can be proved in the same way:

$$\mathbf{A} = \frac{1}{2}(\mathbf{E} - \mathbf{r}^i \otimes \overset{\circ}{\mathbf{r}}_i \cdot \overset{\circ}{\mathbf{r}}_j \otimes \mathbf{r}^j) = \frac{1}{2}(\mathbf{E} - \mathbf{F}^{-1\mathrm{T}} \cdot \mathbf{F}^{-1}),$$
$$\mathbf{\Lambda} = \frac{1}{2}(\mathbf{E} - \overset{\circ}{\mathbf{r}}_i \otimes \mathbf{r}^i \cdot \mathbf{r}^j \otimes \overset{\circ}{\mathbf{r}}_j) = \frac{1}{2}(\mathbf{E} - \mathbf{F}^{-1} \cdot \mathbf{F}^{-1\mathrm{T}}), \tag{2.6a}$$
$$\mathbf{J} = \frac{1}{2}(\mathbf{r}_i \otimes \overset{\circ}{\mathbf{r}}{}^i \cdot \overset{\circ}{\mathbf{r}}{}^j \otimes \mathbf{r}_j - \mathbf{E}) = \frac{1}{2}(\mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} - \mathbf{E}). \ ▲$$

1.2.2. Deformation Measures. Besides the deformation tensors, we define *deformation measures*: the *right Cauchy–Green measure* $\mathbf{G}$ and the *left Almansi measure* $\mathbf{g}$:

$$\mathbf{G} = g_{ij}\overset{\circ}{\mathbf{r}}{}^i \otimes \overset{\circ}{\mathbf{r}}{}^j = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} = \mathbf{E} + 2\mathbf{C},$$

$$\mathbf{g} = \overset{\circ}{g}_{ij}\mathbf{r}^i \otimes \mathbf{r}^j = \mathbf{F}^{-1\mathrm{T}} \cdot \mathbf{F}^{-1} = \mathbf{E} - 2\mathbf{A}, \tag{2.7}$$

and also the *left Cauchy–Green measure* $\mathbf{g}^{-1}$ and the *right Almansi measure* $\mathbf{G}^{-1}$:

$$\mathbf{g}^{-1} = \overset{\circ}{g}{}^{ij}\mathbf{r}_i \otimes \mathbf{r}_j = \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} = \mathbf{E} + 2\mathbf{J},$$

$$\mathbf{G}^{-1} = g^{ij}\overset{\circ}{\mathbf{r}}_i \otimes \overset{\circ}{\mathbf{r}}_j = \mathbf{F}^{-1} \cdot \mathbf{F}^{-1\mathrm{T}} = \mathbf{E} - 2\boldsymbol{\Lambda}. \tag{2.8}$$

1.2.3. Displacement Vector. Introduce a *displacement vector* $\mathbf{u}$ of a point $\mathcal{M}$ from the reference configuration to the actual one as follows (Figure 1.10):

$$\mathbf{u} = \mathbf{x} - \overset{\circ}{\mathbf{x}}. \tag{2.9}$$

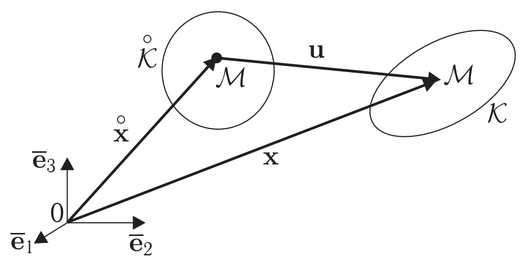

Figure 1.10. The displacement vector of a point $\mathcal{M}$ from the reference configuration to the actual one

Theorem 1.4. *The deformation tensors and the deformation gradient are connected to the displacement vector* $\mathbf{u}$ *by the relations*

$$\mathbf{F} = \mathbf{E} + (\overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u})^{\mathrm{T}}, \qquad \mathbf{F}^{-1} = \mathbf{E} - (\boldsymbol{\nabla} \otimes \mathbf{u})^{\mathrm{T}},$$

$$\mathbf{F}^{\mathrm{T}} = \mathbf{E} + \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u}, \qquad \mathbf{F}^{-1\mathrm{T}} = \mathbf{E} - \boldsymbol{\nabla} \otimes \mathbf{u}, \tag{2.10}$$

and also

$$\mathbf{C} = \frac{1}{2}\left(\overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u} + \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u}^{\mathrm{T}} + \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u} \cdot \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u}^{\mathrm{T}}\right),$$

$$\mathbf{A} = \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \mathbf{u} + \boldsymbol{\nabla} \otimes \mathbf{u}^{\mathrm{T}} - \boldsymbol{\nabla} \otimes \mathbf{u} \cdot \boldsymbol{\nabla} \otimes \mathbf{u}^{\mathrm{T}}\right), \tag{2.11}$$

$$\boldsymbol{\Lambda} = \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \mathbf{u} + (\boldsymbol{\nabla} \otimes \mathbf{u})^{\mathrm{T}} - \boldsymbol{\nabla} \otimes \mathbf{u}^{\mathrm{T}} \cdot \boldsymbol{\nabla} \otimes \mathbf{u}\right),$$

$$\mathbf{J} = \frac{1}{2}\left(\overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u} + \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u}^{\mathrm{T}} + \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u}^{\mathrm{T}} \cdot \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u}\right).$$

▼ The definition (2.9) of the displacement vector and the properties (1.35) of the deformation gradient yield

$$\mathbf{F}^{\mathrm{T}} = \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{x} = \overset{\circ}{\boldsymbol{\nabla}} \otimes (\overset{\circ}{\mathbf{x}} + \mathbf{u}) = \overset{\circ}{\mathbf{r}}{}^i \otimes \frac{\partial \overset{\circ}{\mathbf{x}}}{\partial X^i} + \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u} =$$
$$= \overset{\circ}{\mathbf{r}}{}^i \otimes \overset{\circ}{\mathbf{r}}_i + \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u} = \mathbf{E} + \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u}. \quad (2.12)$$

Then the tensor $\mathbf{C}$ takes the form

$$\mathbf{C} = \frac{1}{2}\left((\mathbf{E} + \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u}) \cdot (\mathbf{E} + \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u}^{\mathrm{T}}) - \mathbf{E}\right) =$$
$$= \frac{1}{2}\left(\overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u} + \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u}^{\mathrm{T}} + \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u} \cdot \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u}^{\mathrm{T}}\right). \quad (2.13)$$

In a similar way, we can prove the remaining relations of the theorem. ▲

1.2.4. Relations between Components of Deformation Tensors and Displacement Vector. The displacement vector $\mathbf{u}$ can be resolved for both bases $\overset{\circ}{\mathbf{r}}_i$ and $\mathbf{r}_i$:

$$\mathbf{u} = \overset{\circ}{u}{}^i \overset{\circ}{\mathbf{r}}_i = u^i \mathbf{r}_i. \quad (2.14)$$

The derivative with respect to X^i can be determined in both the bases as well:

$$\frac{\partial \mathbf{u}}{\partial X^i} = \overset{\circ}{\nabla}_i \overset{\circ}{u}{}^k \overset{\circ}{\mathbf{r}}_k = \nabla_i u^k \mathbf{r}_k. \quad (2.15)$$

Then the displacement vector gradients take the forms

$$\overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{u} = \overset{\circ}{\mathbf{r}}{}^i \otimes \frac{\partial \mathbf{u}}{\partial X^i} = \overset{\circ}{\nabla}_i \overset{\circ}{u}{}^k \overset{\circ}{\mathbf{r}}{}^i \otimes \overset{\circ}{\mathbf{r}}_k = \overset{\circ}{\nabla}{}^i \overset{\circ}{u}{}^k \overset{\circ}{\mathbf{r}}_i \otimes \overset{\circ}{\mathbf{r}}_k, \quad (2.16)$$

$$\boldsymbol{\nabla} \otimes \mathbf{u} = \mathbf{r}^i \otimes \frac{\partial \mathbf{u}}{\partial X^i} = \nabla_i u^k \mathbf{r}^i \otimes \mathbf{r}_k = \nabla^i u^k \mathbf{r}_i \otimes \mathbf{r}_k. \quad (2.17)$$

Substitution of these expressions into (2.10) gives

$$\mathbf{F} = (\delta_i^k + \overset{\circ}{\nabla}_i \overset{\circ}{u}{}^k) \overset{\circ}{\mathbf{r}}_k \otimes \overset{\circ}{\mathbf{r}}{}^i = \overset{\circ}{F}{}^k{}_i \, \overset{\circ}{\mathbf{r}}_k \otimes \overset{\circ}{\mathbf{r}}{}^i. \quad (2.18)$$

Here we have introduced components of the deformation gradient in the reference configuration:

$$\overset{\circ}{F}{}^k{}_i = \delta_i^k + \overset{\circ}{\nabla}_i \overset{\circ}{u}{}^k. \quad (2.19)$$

The transpose gradient $\mathbf{F}^{\mathrm{T}}$ has the components

$$\mathbf{F}^{\mathrm{T}} = \overset{\circ}{F}{}^k{}_i \overset{\circ}{\mathbf{r}}{}^i \otimes \overset{\circ}{\mathbf{r}}_k = \overset{\circ}{F}{}_i{}^k \overset{\circ}{\mathbf{r}}_k \otimes \overset{\circ}{\mathbf{r}}{}^i, \quad (2.20)$$

$$\overset{\circ}{F}{}_i{}^k = \delta_i^k + \overset{\circ}{\nabla}{}^k \overset{\circ}{u}_i, \quad (2.21)$$

where $(\overset{\circ}{F}{}_i{}^k)^{\mathrm{T}} = \overset{\circ}{F}{}^k{}_i$.

In a similar way, one can find the expression for the inverse gradient

$$\mathbf{F}^{-1} = (\delta_i^k - \nabla_i u^k) \mathbf{r}_k \otimes \mathbf{r}^i = (F^{-1})^k{}_i \, \mathbf{r}_k \otimes \mathbf{r}^i \quad (2.22)$$

and for the inverse–transpose gradient

$$\mathbf{F}^{-1\mathrm{T}} = (F^{-1})^{k}_{\ i}\,\mathbf{r}^{i} \otimes \mathbf{r}_{k} = (F^{-1})_{i}^{\ k}\mathbf{r}_{k} \otimes \mathbf{r}^{i}, \tag{2.23}$$

where their components with respect to the actual configuration are expressed as follows:

$$(F^{-1})_{i}^{\ k} = \delta_{i}^{k} - \nabla^{k} u_{i}, \tag{2.24}$$

$$(F^{-1})^{k}_{\ i} = \delta_{i}^{k} - \nabla_{i} u^{k}. \tag{2.25}$$

Thus, we have proved the following theorem.

Theorem 1.5. *Components of the deformation gradients* $\mathbf{F}$, $\mathbf{F}^{\mathrm{T}}$, $\mathbf{F}^{-1}$ *and* $\mathbf{F}^{-1\mathrm{T}}$ *in local bases of configurations* $\overset{\circ}{\mathcal{K}}$ *and* $\mathcal{K}$ *are connected to components of the displacement vector* $\mathbf{u}$ *by relations (2.19), (2.20), (2.24) and (2.25).*

On substituting formulae (2.16) and (2.17) into (2.11) for $\mathbf{C}$ and $\mathbf{A}$ and comparing them with (2.1), we get

$$\varepsilon_{ij} = \frac{1}{2}(\overset{\circ}{\nabla}_{i}\overset{\circ}{u}_{j} + \overset{\circ}{\nabla}_{j}\overset{\circ}{u}_{i} + \overset{\circ}{\nabla}_{i}\overset{\circ}{u}^{k}\overset{\circ}{\nabla}_{j}\overset{\circ}{u}_{k}),$$

$$\varepsilon_{ij} = \frac{1}{2}(\nabla_{i}u_{j} + \nabla_{j}u_{i} - \nabla_{i}u^{k}\nabla_{j}u_{k}), \tag{2.26}$$

— the expressions for covariant components of the deformation tensor in terms of components of the displacement vector with respect to $\overset{\circ}{\mathcal{K}}$ and $\mathcal{K}$.

In a similar way, substituting (2.16) and (2.17) into (2.11) for $\mathbf{\Lambda}$ and $\mathbf{J}$, we obtain

$$\varepsilon^{ij} = \frac{1}{2}(\overset{\circ}{\nabla}{}^{i}\overset{\circ}{u}{}^{j} + \overset{\circ}{\nabla}{}^{j}\overset{\circ}{u}{}^{i} + \overset{\circ}{\nabla}{}^{k}\overset{\circ}{u}{}^{i}\overset{\circ}{\nabla}_{k}\overset{\circ}{u}_{j}),$$

$$\varepsilon^{ij} = \frac{1}{2}(\nabla^{i}u^{j} + \nabla^{j}u^{i} - \nabla^{k}u^{i}\nabla_{k}u^{j}) \tag{2.27}$$

— the relations between contravariant components of the deformation tensor and components of the displacement vector in $\overset{\circ}{\mathcal{K}}$ and $\mathcal{K}$.

Then with using relations (2.2), (2.26) and (2.27), we can find the connection between the metric matrices and displacement components:

$$g_{ij} = \overset{\circ}{g}_{ij} + \overset{\circ}{\nabla}_{i}\overset{\circ}{u}_{j} + \overset{\circ}{\nabla}_{j}\overset{\circ}{u}_{i} + \overset{\circ}{\nabla}_{i}\overset{\circ}{u}^{k}\nabla_{j}\overset{\circ}{u}_{k} = \overset{\circ}{g}_{ij} + \nabla_{i}u_{j} + \nabla_{j}u_{i} - \nabla_{i}u^{k}\nabla_{j}u_{k}, \tag{2.28}$$

$$g^{ij} = \overset{\circ}{g}{}^{ij} + \overset{\circ}{\nabla}{}^{i}\overset{\circ}{u}{}^{j} + \overset{\circ}{\nabla}{}^{j}\overset{\circ}{u}{}^{i} + \overset{\circ}{\nabla}{}^{k}\overset{\circ}{u}{}^{i}\overset{\circ}{\nabla}_{k}\overset{\circ}{u}{}^{j} = \overset{\circ}{g}{}^{ij} + \nabla^{i}u^{j} + \nabla^{j}u^{i} - \nabla^{k}u^{i}\nabla_{k}u^{j}. \tag{2.29}$$

Thus, we have proved the following theorem.

Theorem 1.6. *Components of the deformation tensor* ε_{ij}, ε^{ij} *and metric matrices* g_{ij}, g^{ij} *are connected to components of the displacement vector* $\mathbf{u}$ *by relations (2.26)–(2.29).*

1.2.5. Physical Meaning of Components of the Deformation Tensor. Let us clarify now a physical meaning of components of the deformation tensor:

$$\varepsilon_{ij} = \frac{1}{2}(g_{ij} - \overset{\circ}{g}_{ij}) = \frac{1}{2}(\mathbf{r}_{i}\cdot\mathbf{r}_{j} - \overset{\circ}{\mathbf{r}}_{i}\cdot\overset{\circ}{\mathbf{r}}_{j}). \tag{2.30}$$

By the definition of the scalar product (see [12]), we have

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(|\mathbf{r}_\alpha||\mathbf{r}_\beta|\cos\psi_{\alpha\beta} - |\overset{\circ}{\mathbf{r}}_\alpha||\overset{\circ}{\mathbf{r}}_\beta|\cos\overset{\circ}{\psi}_{\alpha\beta}\right), \tag{2.31}$$

where $\psi_{\alpha\beta}$ and $\overset{\circ}{\psi}_{\alpha\beta}$ are the angles between basis vectors $\mathbf{r}_\alpha$, $\mathbf{r}_\beta$ and $\overset{\circ}{\mathbf{r}}_\alpha$, $\overset{\circ}{\mathbf{r}}_\beta$ in $\mathcal{K}$ and $\overset{\circ}{\mathcal{K}}$, respectively.

Consider elementary radius-vectors $d\mathbf{x}$ and $d\overset{\circ}{\mathbf{x}}$ in configurations $\mathcal{K}$ and $\overset{\circ}{\mathcal{K}}$, and introduce their lengths ds and $d\overset{\circ}{s}$, respectively:

$$ds^2 = d\mathbf{x}\cdot d\mathbf{x}, \qquad d\overset{\circ}{s}{}^2 = d\overset{\circ}{\mathbf{x}}\cdot d\overset{\circ}{\mathbf{x}}. \tag{2.32}$$

Since $d\overset{\circ}{\mathbf{x}}$ is arbitrary, we can choose it to be oriented along one of the basis vectors $\overset{\circ}{\mathbf{r}}_\alpha$. Then $d\mathbf{x}$ will be directed along the corresponding vector $\mathbf{r}_\alpha$ as well, because under this transformation $\overset{\circ}{\mathbf{r}}_\alpha$ becomes $\mathbf{r}_\alpha$ for the same material point $\mathcal{M}$ with Lagrangian coordinates X^k. In this case we have

$$\begin{aligned} |d\overset{\circ}{\mathbf{x}}| &= d\overset{\circ}{s}_\alpha = \left|\frac{\partial\overset{\circ}{\mathbf{x}}}{\partial X^\alpha}\,dX^\alpha\right| = |\overset{\circ}{\mathbf{r}}_\alpha|\,dX^\alpha, \\ |d\mathbf{x}| &= ds_\alpha = \left|\frac{\partial x}{\partial X^\alpha}\,dX^\alpha\right| = |\mathbf{r}_\alpha|\,dX^\alpha. \end{aligned} \tag{2.33}$$

Hence

$$ds_\alpha/d\overset{\circ}{s}_\alpha = |\mathbf{r}_\alpha|/|\overset{\circ}{\mathbf{r}}_\alpha| = \delta_\alpha + 1, \tag{2.34}$$

where δ_α is called the *relative elongation*. Formula (2.34) yields

$$|\mathbf{r}_\alpha| = |\overset{\circ}{\mathbf{r}}_\alpha|(1+\delta_\alpha). \tag{2.35}$$

On substituting this expression into (2.31), we get

$$\varepsilon_{\alpha\beta} = \frac{1}{2}|\overset{\circ}{\mathbf{r}}_\alpha||\overset{\circ}{\mathbf{r}}_\beta|\left((1+\delta_\alpha)(1+\delta_\beta)\cos\psi_{\alpha\beta} - \cos\overset{\circ}{\psi}_{\alpha\beta}\right). \tag{2.36}$$

Consider the case when $\alpha = \beta$, then $\psi_{\alpha\beta} = \overset{\circ}{\psi}_{\alpha\beta} = 0$ and

$$\varepsilon_{\alpha\alpha} = \frac{1}{2}|\overset{\circ}{\mathbf{r}}_\alpha|^2\left((1+\delta_\alpha)^2 - 1\right) = \frac{\overset{\circ}{g}_{\alpha\alpha}}{2}\left((1+\delta_\alpha)^2 - 1\right). \tag{2.37}$$

Let coordinates X^i be coincident with Cartesian coordinates x^i, then $\overset{\circ}{g}_{\alpha\beta} = \delta_{\alpha\beta}$; and for infinitesimal values of the relative elongation, when $\delta_\alpha \ll 1$, we obtain

$$\varepsilon_{\alpha\alpha} \approx \delta_\alpha, \tag{2.38}$$

i.e. $\varepsilon_{\alpha\alpha}$ is coincident with the relative elongation.

In general, $\varepsilon_{\alpha\alpha}$ is a nonlinear function of corresponding elongations.

Consider $\alpha \neq \beta$ and assume that $X^i = x^i$, then $\overset{\circ}{\psi}_{\alpha\beta} = \pi/2$; and from (2.36) we get

$$\varepsilon_{\alpha\beta} = \frac{1}{2}|\overset{\circ}{\mathbf{r}}_\alpha||\overset{\circ}{\mathbf{r}}_\beta|(1+\delta_\alpha)(1+\delta_\beta)\cos\psi_{\alpha\beta} = \\ = \frac{1}{2}\sqrt{\overset{\circ}{g}_{\alpha\alpha}}\sqrt{\overset{\circ}{g}_{\beta\beta}}\,(1+\delta_\alpha)(1+\delta_\beta)\sin\chi_{\alpha\beta} = \frac{1}{2}(1+\delta_\alpha)(1+\delta_\beta)\sin\chi_{\alpha\beta}, \quad (2.39)$$

where $\chi_{\alpha\beta} = \overset{\circ}{\psi}_{\alpha\beta} - \psi_{\alpha\beta} = (\pi/2) - \psi_{\alpha\beta}$ is the change of the angle between basis vectors $\mathbf{r}_\alpha$ and $\mathbf{r}_\beta$. For small relative elongations, when $\delta_\alpha \ll 1$, and small angles $\chi_{\alpha\beta} \ll 1$, from (2.39) we get

$$\varepsilon_{\alpha\beta} \approx \chi_{\alpha\beta}/2, \quad (2.40)$$

i.e. $\varepsilon_{\alpha\beta}$ is a half of the misalignment angle of the basis vectors.

1.2.6. Transformation of an Oriented Surface Element. In actual configuration $\mathcal{K}$ consider a smooth surface Σ, which contains two coordinate lines X^α and X^β.

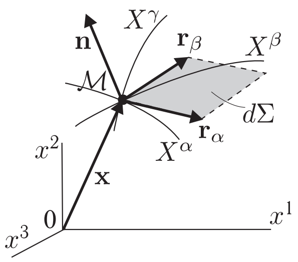

Figure 1.11. Introduction of oriented surface element $\mathbf{n}\,d\Sigma$

Then we can introduce the normal $\mathbf{n}$ to the surface Σ as follows:

$$\mathbf{n} = \frac{1}{\sqrt{\widetilde{g}}}\mathbf{r}_\alpha \times \mathbf{r}_\beta. \quad (2.41)$$

Here $\widetilde{g} = \det\,(\widetilde{g}_{\alpha\beta})$, and $\widetilde{g}_{\alpha\beta}$ is the two-dimensional matrix of the surface ($\alpha, \beta = 1, 2$):

$$\widetilde{g}_{\alpha\beta} = \mathbf{r}_\alpha \cdot \mathbf{r}_\beta \quad (2.42)$$

(it is not to be confused with the metric matrix $g_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j$).

In configuration $\mathcal{K}$ consider a surface element $d\Sigma$ constructed on elementary radius-vectors $d\mathbf{x}_\alpha$, which are directed along local basis vectors, i.e. $d\mathbf{x}_\alpha = \mathbf{r}_\alpha dX^\alpha$ (Figure 1.11). The value

$$d\Sigma = \sqrt{\widetilde{g}}\;\, dX^\alpha\; dX^\beta \quad (2.43)$$

is called the *area of the surface element* $d\Sigma$ constructed on vectors $d\mathbf{x}_\alpha$ and $d\mathbf{x}_\beta$. Then formula (2.41) takes the form

$$\mathbf{n}\; d\Sigma = \mathbf{r}_\alpha\; dX^\alpha \times \mathbf{r}_\beta\; dX^\beta = d\mathbf{x}_\alpha \times d\mathbf{x}_\beta, \quad (2.44)$$

where $\mathbf{n}\,d\Sigma$ is called the *oriented surface element*.

Show that the normal $\mathbf{n}$ defined by formula (2.41) is a unit vector. According to the property (1.14b) of the vector product of basis vectors and the results of Exercise 1.1.13, we can rewrite equation (2.44) in the form

$$\mathbf{n}\,d\Sigma = \mathbf{r}_\alpha \times \mathbf{r}_\beta\, dX^\alpha\, dX^\beta = \sqrt{g}\,\epsilon_{\alpha\beta\gamma}\mathbf{r}^\gamma\; dX^\alpha\; dX^\beta = \\ = (1/\sqrt{g}\,)\epsilon^{ijk}g_{\alpha i}g_{\beta j}\mathbf{r}_k\; dX^\alpha\; dX^\beta \quad (2.45)$$

(there is no summation over α, β). Thus,

$$\mathbf{n}\, d\Sigma \cdot \mathbf{n}\, d\Sigma = \sqrt{g}\ \epsilon_{\alpha\beta\gamma}\mathbf{r}^{\gamma} \cdot \frac{1}{\sqrt{g}}\ \epsilon^{ijk} g_{\alpha i} g_{\beta j}\mathbf{r}_k)(dX^{\alpha}\, dX^{\beta})^2 =$$
$$= (\epsilon_{\alpha\beta k}\epsilon^{ijk} g_{\alpha i} g_{\beta j})(dX^{\alpha} dX^{\beta})^2 = (\delta^i_{\alpha}\delta^j_{\beta} - \delta^i_{\beta}\delta^j_{\alpha}) g_{\alpha i} g_{\beta j}(dX^{\alpha} dX^{\beta})^2 =$$
$$= (g_{\alpha\alpha} g_{\beta\beta} - g^2_{\alpha\beta})(dX^{\alpha} dX^{\beta})^2 = \widetilde{g}(dX^{\alpha} dX^{\beta})^2 = d\Sigma^2. \quad (2.46)$$

Hence, $\mathbf{n} \cdot \mathbf{n} = 1$.

The surface element $d\Sigma$ in $\mathcal{K}$ corresponds to the surface element $d\overset{\circ}{\Sigma}$ in $\overset{\circ}{\mathcal{K}}$, which is constructed on elementary radius-vectors $d\overset{\circ}{\mathbf{x}}_{\alpha}$ and $d\overset{\circ}{\mathbf{x}}_{\beta}$:

$$\overset{\circ}{\mathbf{n}}\, d\overset{\circ}{\Sigma} = \overset{\circ}{\mathbf{r}}_{\alpha} dX^{\alpha} \times \overset{\circ}{\mathbf{r}}_{\beta}\, dX^{\beta} = \overset{\circ}{\mathbf{r}}_{\alpha} \times \overset{\circ}{\mathbf{r}}_{\beta}\, dX^{\alpha}\, dX^{\beta}. \quad (2.47)$$

Here $\overset{\circ}{\mathbf{n}}$ is the unit normal to $d\overset{\circ}{\Sigma}$.

Since $\mathbf{r}^{\gamma} = \mathbf{F}^{-1\mathrm{T}} \cdot \overset{\circ}{\mathbf{r}}{}^{\gamma}$, we get

$$\mathbf{n}\, d\Sigma = \sqrt{g}\ \epsilon_{\alpha\beta\gamma}\mathbf{F}^{-1\mathrm{T}} \cdot \overset{\circ}{\mathbf{r}}{}^{\gamma}\, dX^{\alpha}\, dX^{\beta} = \sqrt{g/\overset{\circ}{g}}\ \mathbf{F}^{-1\mathrm{T}} \cdot \overset{\circ}{\mathbf{r}}_{\alpha} \times \overset{\circ}{\mathbf{r}}_{\beta}\, dX^{\alpha}\, dX^{\beta} =$$
$$= \sqrt{g/\overset{\circ}{g}}\ \mathbf{F}^{-1\mathrm{T}} \cdot \overset{\circ}{\mathbf{n}}\, d\overset{\circ}{\Sigma}. \quad (2.48)$$

Thus, we have proved the following theorem.

Theorem 1.7. *The oriented surface elements* $\overset{\circ}{\mathbf{n}}\, d\overset{\circ}{\Sigma}$ *and* $\mathbf{n}\, d\Sigma$ *in* $\overset{\circ}{\mathcal{K}}$ *and* $\mathcal{K}$ *are connected by the relation*

$$\mathbf{n}\, d\Sigma = \sqrt{g/\overset{\circ}{g}}\ \overset{\circ}{\mathbf{n}} \cdot \mathbf{F}^{-1}\, d\overset{\circ}{\Sigma} = \sqrt{g/\overset{\circ}{g}}\ \mathbf{F}^{-1\mathrm{T}} \cdot \overset{\circ}{\mathbf{n}}\, d\overset{\circ}{\Sigma}. \quad (2.49)$$

With the help of the deformation measures we can derive formulae connecting the normals $\mathbf{n}$ and $\overset{\circ}{\mathbf{n}}$ to the surface element containing the same material points in $\mathcal{K}$ and $\overset{\circ}{\mathcal{K}}$.

Multiplying the equation (2.49) by itself and taking the formula $\mathbf{n} \cdot \mathbf{n} = 1$ into account, we get

$$d\Sigma^2 = \frac{g}{\overset{\circ}{g}}(\overset{\circ}{\mathbf{n}} \cdot \mathbf{F}^{-1} \cdot \mathbf{F}^{-1\mathrm{T}} \cdot \overset{\circ}{\mathbf{n}})\, d\overset{\circ}{\Sigma}{}^2 = \frac{g}{\overset{\circ}{g}}(\overset{\circ}{\mathbf{n}} \cdot \mathbf{G}^{-1} \cdot \overset{\circ}{\mathbf{n}})\, d\overset{\circ}{\Sigma}{}^2. \quad (2.50)$$

Thus,

$$d\Sigma/d\overset{\circ}{\Sigma} = \sqrt{g/\overset{\circ}{g}}\ (\overset{\circ}{\mathbf{n}} \cdot \mathbf{G}^{-1} \cdot \overset{\circ}{\mathbf{n}})^{1/2}. \quad (2.51)$$

On the other hand, expressing $\overset{\circ}{\mathbf{n}}$ from (2.49) and then multiplying the obtained relation by itself, we obtain

$$d\overset{\circ}{\Sigma}{}^2 = \frac{\overset{\circ}{g}}{g}\ (\mathbf{n} \cdot \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} \cdot \mathbf{n})\, d\Sigma^2 = \frac{\overset{\circ}{g}}{g}(\mathbf{n} \cdot \mathbf{g}^{-1} \cdot \mathbf{n})\, d\Sigma^2, \quad (2.52)$$

Thus, we find that

$$d\overset{\circ}{\Sigma}/d\Sigma = \sqrt{\overset{\circ}{g}/g}\ \ (\mathbf{n} \cdot \mathbf{g}^{-1} \cdot \mathbf{n})^{1/2}. \quad (2.53)$$

On introducing the notation

$$\overset{\circ}{k} = (\overset{\circ}{\mathbf{n}} \cdot \mathbf{G}^{-1} \cdot \overset{\circ}{\mathbf{n}})^{1/2} = (\overset{\circ}{\mathbf{n}} \cdot \mathbf{F}^{-1} \cdot \mathbf{F}^{-1\mathrm{T}} \cdot \overset{\circ}{\mathbf{n}})^{1/2},$$
$$k = (\mathbf{n} \cdot \mathbf{g}^{-1} \cdot \mathbf{n})^{1/2} = (\mathbf{n} \cdot \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} \cdot \mathbf{n})^{1/2}, \tag{2.54}$$

relations (2.52) take the form

$$d\overset{\circ}{\Sigma}/d\Sigma = \sqrt{\overset{\circ}{g}/g}\;\; k = \sqrt{\overset{\circ}{g}/g}\;\; (1/\overset{\circ}{k}). \tag{2.55}$$

Thus, we get

$$k = 1/\overset{\circ}{k}. \tag{2.56}$$

Substitution of (2.52) and (2.53) into (2.49) gives the desired relations

$$\overset{\circ}{k}\mathbf{n} = \mathbf{F}^{-1\mathrm{T}} \cdot \overset{\circ}{\mathbf{n}}, \qquad k\overset{\circ}{\mathbf{n}} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{n}. \tag{2.57}$$

1.2.7. Representation of the Inverse Metric Matrix in terms of Components of the Deformation Tensor. Components g_{ij} of the metric matrix are connected to components of the deformation tensor ε_{ij} by relation (2.2). In continuum mechanics, one often needs to know the expression of the inverse metric matrix g^{ij} in terms of ε_{ij} (but not in terms of ε^{ij}). To derive this relation, we should use the connection between components of a matrix and its inverse (see [12]):

$$g^{ij} = \frac{1}{2g}\epsilon^{imn}\epsilon^{jkl}g_{mk}g_{nl}. \tag{2.58}$$

For $\overset{\circ}{g}{}^{ij}$, we have the similar formula

$$\overset{\circ}{g}{}^{ij} = \frac{1}{2\overset{\circ}{g}}\epsilon^{imn}\epsilon^{jkl}\overset{\circ}{g}_{mk}\overset{\circ}{g}_{nl}. \tag{2.59}$$

On substituting the relations (2.2) between g_{mn}, $\overset{\circ}{g}_{mn}$ and ε_{mn} into (2.59), we get

$$g^{ij} = \frac{1}{2g}\epsilon^{imn}\epsilon^{jkl}(\overset{\circ}{g}_{mk} + 2\varepsilon_{mk})(\overset{\circ}{g}_{nl} + 2\varepsilon_{nl}) =$$
$$= \frac{1}{2g}\epsilon^{imn}\epsilon^{jkl}(\overset{\circ}{g}_{mk}\overset{\circ}{g}_{nl} + 2\overset{\circ}{g}_{mk}\varepsilon_{nl} + 1\overset{\circ}{g}_{nl}\varepsilon_{mk} + 4\varepsilon_{mk}\varepsilon_{nl}). \tag{2.60}$$

Removing the parentheses, modify four summands in (2.60) in the following way. The first summand with taking formula (2.59) into account gives the matrix $\overset{\circ}{g}{}^{ij}(\overset{\circ}{g}/g)$. To transform the second and the third summands, we should use the formulae

$$(1/\sqrt{\overset{\circ}{g}}\,)\epsilon^{jkl} = \sqrt{\overset{\circ}{g}}\;\; \epsilon_{tsp}\overset{\circ}{g}{}^{jt}\overset{\circ}{g}{}^{ks}\overset{\circ}{g}{}^{lp}. \tag{2.61}$$

$$(1/\overset{\circ}{g})\epsilon^{imn}\epsilon^{jkl}\overset{\circ}{g}_{mk} = \epsilon^{imn}\epsilon_{tsp}\overset{\circ}{g}{}^{jt}\overset{\circ}{g}{}^{ks}\overset{\circ}{g}{}^{lp}\overset{\circ}{g}_{mk} = \epsilon^{imn}\epsilon_{tmp}\overset{\circ}{g}{}^{jt}\overset{\circ}{g}{}^{lp} =$$
$$= (\delta^i_t\delta^n_p - \delta^i_p\delta^n_t)\overset{\circ}{g}{}^{jt}\overset{\circ}{g}{}^{lp} = \overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{nl} - \overset{\circ}{g}{}^{jn}\overset{\circ}{g}{}^{il}. \tag{2.62}$$

$$(1/\overset{\circ}{g})\epsilon^{imn}\epsilon^{jkl}\overset{\circ}{g}_{mk}\varepsilon_{nl} = (\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{nl} - \overset{\circ}{g}{}^{jn}\overset{\circ}{g}{}^{il})\varepsilon_{nl}. \tag{2.63}$$

Formula (2.61) follows from the relation

$$\sqrt{g}\ \epsilon_{ijk} = (1/\sqrt{g}\,)\ \epsilon^{mnl} g_{mi} g_{nj} g_{lk}$$

(see [12]), and relationship (2.62) has been obtained by using formula (2.61) and the properties of the Levi-Civita symbols (see Exercise 1.1.13).

On substituting formula (2.63) into (2.60), we get

$$g^{ij} = \frac{\overset{\circ}{g}}{g}\Big(\overset{\circ}{g}{}^{ij} + 2(\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{nl} - 2\overset{\circ}{g}{}^{il}\overset{\circ}{g}{}^{jn})\varepsilon_{nl} + \frac{2}{\overset{\circ}{g}}\epsilon^{imn}\epsilon^{jkl}\varepsilon_{mk}\varepsilon_{nl}\Big). \tag{2.64}$$

Finally, we should express the determinant $g = \det\ (\mathrm{g_{ij}})$ in terms of ε_{ij}. To do this, we multiply relation (2.64) by g_{ij} and take formula (2.2) into account:

$$3g = \overset{\circ}{g}\Big(\overset{\circ}{g}{}^{ij} + 2(\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{nl} - \overset{\circ}{g}{}^{il}\overset{\circ}{g}{}^{jn})\varepsilon_{nl} + \frac{2}{\overset{\circ}{g}}\epsilon^{imn}\epsilon^{jpl}\varepsilon_{mp}\varepsilon_{nl}\Big)(\overset{\circ}{g}_{ij} + 2\varepsilon_{ij}). \tag{2.65}$$

Thus, we get

$$\begin{aligned} 3g = \overset{\circ}{g}(3 + 4\overset{\circ}{g}{}^{nl}\varepsilon_{nl} + (2/\overset{\circ}{g})\epsilon^{imn}\epsilon^{jpl}\overset{\circ}{g}_{ij}\varepsilon_{mp}\varepsilon_{nl} + {}\\ + 2\overset{\circ}{g}{}^{ij}\varepsilon_{ij} + 4(\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{nl} - \overset{\circ}{g}{}^{il}\overset{\circ}{g}{}^{jn})\varepsilon_{nl}\varepsilon_{ij} + (4/\overset{\circ}{g})\epsilon^{imn}\epsilon^{jpl}\varepsilon_{ij}\varepsilon_{mp}\varepsilon_{nl}\Big). \end{aligned} \tag{2.66}$$

Modifying the third summand on the right-hand side by formula (2.63) and introducing the notation

$$I_{1\varepsilon} = \overset{\circ}{g}{}^{nl}\varepsilon_{nl}, \quad I_{2\varepsilon} = \frac{1}{2}(\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{nl} - \overset{\circ}{g}{}^{il}\overset{\circ}{g}{}^{jn})\varepsilon_{ij}\varepsilon_{nl}, \quad I_{3\varepsilon} = \det\ (\varepsilon_{\mathrm{ij}}\overset{\circ}{\mathrm{g}}{}^{\mathrm{ik}}), \tag{2.67}$$

from (2.66) we get the desired formula

$$g = \overset{\circ}{g}(1 + 2I_{1\varepsilon} + 4I_{2\varepsilon} + 8I_{3\varepsilon}). \tag{2.68}$$

Here we have taken account of formula (2.58) for the matrix determinant and also the relation $I_{3\varepsilon} = (1/\overset{\circ}{g})\det\ (\varepsilon_{\mathrm{ij}})$. Thus, we have proved the following theorem.

Theorem 1.8. *The inverse metric matrix g^{ij} is expressed in terms of components ε_{ij} of the deformation tensor and $\overset{\circ}{g}{}^{ij}$ by formulae (2.64) and (2.68).*

Formulae (2.64) and (2.68) allow us to find the expression of contravariant components ε^{ij} of the deformation tensor in terms of ε_{ij}. It follows from (2.2) that

$$\varepsilon^{ij} = (1/2)(\overset{\circ}{g}{}^{ij} - g^{ij}) = \overset{\circ}{g}{}^{ij}\frac{g - \overset{\circ}{g}}{2g} - \frac{\overset{\circ}{g}}{g}(\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{nl} - \overset{\circ}{g}{}^{il}\overset{\circ}{g}{}^{jn})\varepsilon_{nl} - \frac{1}{g}\epsilon^{imn}\epsilon^{jkl}\varepsilon_{mk}\varepsilon_{nl}. \tag{2.69}$$

Substitution of formulae (2.26) or (2.27) into (2.64) and (2.69) gives the expression for components ε^{ij} in terms of components of the displacement vector $\overset{\circ}{u}_i$ or u_i.

Exercises for 1.2

Exercise 1.2.1. Using the results of Exercise 1.1.1, show that the deformation gradient $\mathbf{F}$ and its inverse $\mathbf{F}^{-1}$ for the problem on a beam in tension (see Example 1.1) have the forms

$$\mathbf{F} = \mathbf{F}^{\mathrm{T}} = \sum_{\alpha=1}^{3} k_\alpha \bar{\mathbf{e}}_\alpha \otimes \bar{\mathbf{e}}_\alpha, \qquad \mathbf{F}^{-1} = \mathbf{F}^{-1\mathrm{T}} = \sum_{\alpha=1}^{3} \frac{1}{k_\alpha} \bar{\mathbf{e}}_\alpha \otimes \bar{\mathbf{e}}_\alpha.$$

For this problem, the deformation tensors are determined by the formulae

$$\mathbf{C} = \mathbf{\Lambda} = \frac{1}{2}\sum_{\alpha=1}^{3}(k_\alpha^2 - 1)\bar{\mathbf{e}}_\alpha \otimes \bar{\mathbf{e}}_\alpha, \qquad \mathbf{A} = \mathbf{\Lambda} = \frac{1}{2}\sum_{\alpha=1}^{3}(1 - k_\alpha^{-2})\bar{\mathbf{e}}_\alpha \otimes \bar{\mathbf{e}}_\alpha,$$

the deformation measures are determined as follows:

$$\mathbf{G} = \mathbf{g}^{-1} = \sum_{\alpha=1}^{3} k_\alpha^2 \bar{\mathbf{e}}_\alpha \otimes \bar{\mathbf{e}}_\alpha, \qquad \mathbf{g} = \mathbf{G}^{-1} = \sum_{\alpha=1}^{3} k_\alpha^{-2} \bar{\mathbf{e}}_\alpha \otimes \bar{\mathbf{e}}_\alpha,$$

and components of the deformation tensor take the forms

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(k_\alpha^2 - 1)\delta_{\alpha\beta}, \qquad \varepsilon^{\alpha\beta} = \frac{1}{2}(1 - k_\alpha^{-2})\delta_{\alpha\beta}.$$

Exercise 1.2.2. Using the results of Exercise 1.1.2, show that for the problem on a simple shear we have the following formulae for the deformation gradient:

$$\mathbf{F} = \overset{\circ}{F}{}^{ij} \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j = \mathbf{E} + a\bar{\mathbf{e}}_1 \otimes \bar{\mathbf{e}}_2, \qquad \mathbf{F}^{\mathrm{T}} = \mathbf{E} + a\bar{\mathbf{e}}_2 \otimes \bar{\mathbf{e}}_1,$$

$$\mathbf{F}^{-1} = \mathbf{E} - a\bar{\mathbf{e}}_1 \otimes \bar{\mathbf{e}}_2, \qquad \mathbf{F}^{-1\mathrm{T}} = \mathbf{E} - a\bar{\mathbf{e}}_2 \otimes \bar{\mathbf{e}}_1,$$

i.e.

$$\overset{\circ}{F}{}^{ij} = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \det \mathbf{F} = 1,$$

for the deformation tensors:

$$\mathbf{C} = (a/2)\mathbf{O}_3 + (a^2/2)\bar{\mathbf{e}}_2^2, \qquad \mathbf{A} = (a/2)\mathbf{O}_3 - (a^2/2)\bar{\mathbf{e}}_2^2,$$

$$\mathbf{\Lambda} = (a/2)\mathbf{O}_3 - (a^2/2)\bar{\mathbf{e}}_1^2, \qquad \mathbf{J} = (a/2)\mathbf{O}_3 + (a^2/2)\bar{\mathbf{e}}_1^2,$$

and for components of the deformation tensor:

$$(\varepsilon_{ij}) = \begin{pmatrix} 0 & a/2 & 0 \\ a/2 & a^2/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (\varepsilon^{ij}) = \begin{pmatrix} 0 & a/2 & 0 \\ a/2 & -a^2/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here we have introduced the notation

$$\mathbf{O}_3 \equiv \bar{\mathbf{e}}_1 \otimes \bar{\mathbf{e}}_2 + \bar{\mathbf{e}}_2 \otimes \bar{\mathbf{e}}_1, \qquad \bar{\mathbf{e}}_\alpha^2 \equiv \bar{\mathbf{e}}_\alpha \otimes \bar{\mathbf{e}}_\alpha, \qquad \alpha = 1, 2, 3.$$

Exercise 1.2.3. Using the formulae from Example 1.3 (see paragraph 1.1.1), show that for the problem on rotation of a beam with extension, the deformation gradient has the form

$$\mathbf{F} = F_0{}^i{}_j \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}^j = \cos\varphi \sum_{\alpha=1}^{2} k_\alpha \bar{\mathbf{e}}_\alpha \otimes \bar{\mathbf{e}}_\alpha + k_3 \bar{\mathbf{e}}_3 \otimes \bar{\mathbf{e}}_3 + \sin\varphi k_2 (\bar{\mathbf{e}}_2 \otimes \bar{\mathbf{e}}_1 - \bar{\mathbf{e}}_1 \otimes \bar{\mathbf{e}}_2).$$

Exercise 1.2.4. Using formulae (1.36), (2.18)–(2.25) and the results of Exercise 1.1.7, show that local basis vectors are connected to displacements by the relations

$$\mathbf{r}_i = \overset{\circ}{\mathbf{r}}_k(\delta_i^k + \overset{\circ}{\nabla}{}^k \overset{\circ}{u}_i), \qquad \overset{\circ}{\mathbf{r}}_i = \mathbf{r}_k(\delta_i^k - \nabla_i u^k).$$

Exercise 1.2.5. Using formulae (2.37) and (2.39), show that the physical components of the deformation tensor $\widehat{\overset{\circ}{\varepsilon}}_{\alpha\beta} = \varepsilon_{\alpha\beta}/\sqrt{\overset{\circ}{g}_{\alpha\alpha}\overset{\circ}{g}_{\beta\beta}}$ are connected to relative elongations δ_α and angles $\chi_{\alpha\beta}$ by the relations

$$\widehat{\overset{\circ}{\varepsilon}}_{\alpha\alpha} = \frac{1}{2}((1+\delta_\alpha)^2 - 1), \qquad \widehat{\overset{\circ}{\varepsilon}}_{\alpha\beta} = \frac{1}{2}(1+\delta_\alpha)(1+\delta_\beta)\sin\chi_{\alpha\beta}.$$

Exercise 1.2.6. Show that in the basis $\widetilde{\mathbf{r}}_i$ of curvilinear coordinate system $\widetilde{X}^i$ the expression of the tensor $\mathbf{F}^{-1}$ in terms of $\boldsymbol{\nabla}\otimes\mathbf{u}$ can be rewritten in the form similar to (2.22)–(2.25):

$$\mathbf{u} = \widetilde{u}^k\widetilde{\mathbf{r}}_k, \qquad \mathbf{F}^{-1} = (\widetilde{F}^{-1})^k{}_i\,\widetilde{\mathbf{r}}_k\otimes\widetilde{\mathbf{r}}^i, \qquad (\widetilde{F}_{-1})^k{}_i = \delta_i^k - \widetilde{\nabla}_i\widetilde{u}^k.$$

Exercise 1.2.7. Using formula (1.34), show that the following relationships hold:

$$|d\mathbf{x}|^2 = d\overset{\circ}{\mathbf{x}}\cdot\mathbf{G}\cdot d\overset{\circ}{\mathbf{x}}, \qquad |d\overset{\circ}{\mathbf{x}}|^2 = d\mathbf{x}\cdot\mathbf{g}\cdot d\mathbf{x}.$$

1.3. Polar Decomposition

1.3.1. Theorem on Polar Decomposition. According to (1.36), the tensor $\mathbf{F}$ can be considered as a tensor of the linear transformation from the basis $\overset{\circ}{\mathbf{r}}_i$ to the basis $\mathbf{r}_i$. Since the vectors $\overset{\circ}{\mathbf{r}}_i$ and $\mathbf{r}_i$ are linearly independent, the tensor $\mathbf{F}$ is nonsingular. Then for this tensor the following theorem is valid.

Theorem 1.9 (on the polar decomposition). *Any nonsingular second-order tensor* $\mathbf{F}$ *can be represented as the scalar product of two second-order tensors:*

$$\mathbf{F} = \mathbf{O}\cdot\mathbf{U} \quad \textit{or} \quad \mathbf{F} = \mathbf{V}\cdot\mathbf{O}. \tag{3.1}$$

Here $\mathbf{U}$ *and* $\mathbf{V}$ *are the symmetric and positive-definite tensors,* $\mathbf{O}$ *is the orthogonal tensor, and each of the decompositions (3.1) is unique.*

▼ Prove the existence of the decomposition (3.1) in the constructive way, i.e. we should construct the tensors $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{O}$. To do this, consider the contractions of the tensor $\mathbf{F}$ with its transpose: $\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}$ and $\mathbf{F}\cdot\mathbf{F}^{\mathrm{T}}$. Both the tensors are symmetric, because

$$(\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F})^{\mathrm{T}} = \mathbf{F}^{\mathrm{T}}\cdot(\mathbf{F}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{F}^{\mathrm{T}}\cdot\mathbf{F} \ \text{ and } \ (\mathbf{F}\cdot\mathbf{F}^{\mathrm{T}})^{\mathrm{T}} = (\mathbf{F}^{\mathrm{T}})^{\mathrm{T}}\cdot\mathbf{F}^{\mathrm{T}} = \mathbf{F}\cdot\mathbf{F}^{\mathrm{T}}, \tag{3.2}$$

and positive-definite:

$$\mathbf{a}\cdot(\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F})\cdot\mathbf{a} = (\mathbf{a}\cdot\mathbf{F}^{\mathrm{T}})\cdot(\mathbf{F}\cdot\mathbf{a}) = (\mathbf{F}\cdot\mathbf{a})\cdot(\mathbf{F}\cdot\mathbf{a}) = \mathbf{b}\cdot\mathbf{b} = |\mathbf{b}|^2 > 0 \tag{3.3}$$

for any non-zero vector $\mathbf{a}$, where $\mathbf{b} = \mathbf{F}\cdot\mathbf{a}$. Since any symmetric positive-definite tensor has three real positive eigenvalues [12], eigenvalues of tensors $\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}$

and $\mathbf{F}\cdot\mathbf{F}^{\mathrm{T}}$ can be denoted as $\overset{\circ}{\lambda}{}_{\alpha}^{2}$ and λ_{α}^{2}. These tensors are diagonal in their eigenbases, i.e. they have the following forms:

$$\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}=\sum_{\alpha=1}^{3}\overset{\circ}{\lambda}{}_{\alpha}^{2}\overset{\circ}{\mathbf{p}}_{\alpha}\otimes\overset{\circ}{\mathbf{p}}_{\alpha},\qquad \mathbf{F}\cdot\mathbf{F}^{\mathrm{T}}=\sum_{\alpha=1}^{3}\lambda_{\alpha}^{2}\mathbf{p}_{\alpha}\otimes\mathbf{p}_{\alpha}. \tag{3.4}$$

Here $\overset{\circ}{\mathbf{p}}_{\alpha}$ are the eigenvectors of the tensor $\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}$ and $\mathbf{p}_{\alpha}$ — of the tensor $\mathbf{F}\cdot\mathbf{F}^{\mathrm{T}}$, which are real-valued and orthonormal:

$$\overset{\circ}{\mathbf{p}}_{\alpha}\cdot\overset{\circ}{\mathbf{p}}_{\beta}=\delta_{\alpha\beta},\qquad \mathbf{p}_{\alpha}\cdot\mathbf{p}_{\beta}=\delta_{\alpha\beta}. \tag{3.5}$$

The right-hand sides of (3.4) are the squares of certain tensors $\mathbf{U}$ and $\mathbf{V}$ defined as

$$\mathbf{U}=\sum_{\alpha=1}^{3}\overset{\circ}{\lambda}_{\alpha}\overset{\circ}{\mathbf{p}}_{\alpha}\otimes\overset{\circ}{\mathbf{p}}_{\alpha},\quad \overset{\circ}{\lambda}_{\alpha}>0;\qquad \mathbf{V}=\sum_{\alpha=1}^{3}\lambda_{\alpha}\mathbf{p}_{\alpha}\otimes\mathbf{p}_{\alpha},\quad \lambda_{\alpha}>0. \tag{3.6}$$

Here signs at λ_{α} are always chosen positive.

In this case, the following relations are valid:

$$\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}=\mathbf{U}^{2},\qquad \mathbf{F}\cdot\mathbf{F}^{\mathrm{T}}=\mathbf{V}^{2}. \tag{3.7}$$

The constructed tensors $\mathbf{V}$ and $\mathbf{U}$ are symmetric due to formula (3.6) and positive-definite, because for any nonzero vector $\mathbf{a}$ we have

$$\mathbf{a}\cdot\mathbf{U}\cdot\mathbf{a}=\sum_{\alpha=1}^{3}\overset{\circ}{\lambda}_{\alpha}\mathbf{a}\cdot\overset{\circ}{\mathbf{p}}_{\alpha}\otimes\overset{\circ}{\mathbf{p}}_{\alpha}\cdot\mathbf{a}=\sum_{\alpha=1}^{3}\overset{\circ}{\lambda}_{\alpha}(\mathbf{a}\cdot\overset{\circ}{\mathbf{p}}_{\alpha})^{2}>0, \tag{3.8}$$

as $\overset{\circ}{\lambda}_{\alpha}>0$. In a similar way, we can prove that the tensor $\mathbf{V}$ is positive-definite.

Both the tensors $\mathbf{V}$ and $\mathbf{U}$ are nonsingular, because, under the conditions of the theorem, the tensor $\mathbf{F}$ is nonsingular. And from (3.7) we get

$$(\det\,\mathbf{U})^{2}=\det\,\mathbf{U}^{2}=\det\,(\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F})=(\det\,\mathbf{F})^{2}\neq 0. \tag{3.9}$$

Then there exist inverse tensors $\mathbf{U}^{-1}$ and $\mathbf{V}^{-1}$, with the help of which we can construct two more new tensors

$$\overset{\circ}{\mathbf{O}}=\mathbf{F}\cdot\mathbf{U}^{-1},\qquad \mathbf{O}=\mathbf{V}^{-1}\cdot\mathbf{F}, \tag{3.10}$$

which are orthogonal. Indeed,

$$\overset{\circ}{\mathbf{O}}{}^{\mathrm{T}}\cdot\overset{\circ}{\mathbf{O}}=(\mathbf{F}\cdot\mathbf{U}^{-1})^{\mathrm{T}}\cdot(\mathbf{F}\cdot\mathbf{U}^{-1})=\mathbf{U}^{-1}\cdot\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}\cdot\mathbf{U}^{-1}=\mathbf{U}^{-1}\cdot\mathbf{U}^{2}\cdot\mathbf{U}^{-1}=\mathbf{E}. \tag{3.11}$$

According to [12], this means that the tensor $\overset{\circ}{\mathbf{O}}$ is orthogonal. In a similar way, we can show that the tensor $\mathbf{O}$ is orthogonal as well.

Thus, we have really constructed the tensors $\mathbf{U}$ and $\overset{\circ}{\mathbf{O}}$, and also $\mathbf{V}$ and $\mathbf{O}$, the product of which, due to (3.10), gives the tensor $\mathbf{F}$:

$$\mathbf{F}=\overset{\circ}{\mathbf{O}}\cdot\mathbf{U}=\mathbf{V}\cdot\mathbf{O}. \tag{3.12}$$

Here $\mathbf{U}$ and $\mathbf{V}$ are symmetric, positive-definite tensors, $\mathbf{O}$ and $\overset{\circ}{\mathbf{O}}$ are orthogonal tensors.

Show that each of the decompositions (3.12) is unique. By contradiction, let there be one more resolution, for example

$$\mathbf{F} = \overset{\circ}{\widetilde{\mathbf{O}}} \cdot \widetilde{\mathbf{U}}. \tag{3.13}$$

But then

$$\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} = \widetilde{\mathbf{U}}^2 = \mathbf{U}^2, \tag{3.14}$$

hence, $\widetilde{\mathbf{U}} = \mathbf{U}$, because the decomposition of the tensor $\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}$ for its eigenbasis is unique. Signs at $\overset{\circ}{\lambda}_\alpha$ and $\overset{\circ}{\widetilde{\lambda}}_\alpha$ are chosen positive by the condition. The coincidence of $\mathbf{U}$ and $\widetilde{\mathbf{U}}$ leads to the fact that $\overset{\circ}{\widetilde{\mathbf{O}}}$ and $\overset{\circ}{\mathbf{O}}$ are coincident as well, because

$$\overset{\circ}{\widetilde{\mathbf{O}}} = \mathbf{F} \cdot \widetilde{\mathbf{U}}^{-1} = \mathbf{F} \cdot \mathbf{U}^{-1} = \overset{\circ}{\mathbf{O}}. \tag{3.15}$$

This has proved uniqueness of the decomposition (3.12). We can verify uniqueness of the decomposition $\mathbf{F} = \mathbf{V} \cdot \mathbf{O}$ in a similar way.

Finally, we must show that the orthogonal tensors $\overset{\circ}{\mathbf{O}}$ and $\mathbf{O}$ are coincident, i.e. formula (3.1) follows from (3.12). To do this, we construct the tensor

$$\mathbf{F} \cdot \overset{\circ}{\mathbf{O}}{}^{\mathrm{T}} = \overset{\circ}{\mathbf{O}} \cdot \mathbf{U} \cdot \overset{\circ}{\mathbf{O}}{}^{\mathrm{T}}. \tag{3.16}$$

Due to (3.12), this tensor satisfies the following relationship:

$$\overset{\circ}{\mathbf{O}} \cdot \mathbf{U} \cdot \overset{\circ}{\mathbf{O}}{}^{\mathrm{T}} = \mathbf{V} \cdot \mathbf{O} \cdot \overset{\circ}{\mathbf{O}}{}^{\mathrm{T}}. \tag{3.17}$$

The tensor $\mathbf{O} \cdot \overset{\circ}{\mathbf{O}}{}^{\mathrm{T}}$ is orthogonal, because

$$(\mathbf{O} \cdot \overset{\circ}{\mathbf{O}}{}^{\mathrm{T}})^{\mathrm{T}} \cdot (\mathbf{O} \cdot \overset{\circ}{\mathbf{O}}{}^{\mathrm{T}}) = \overset{\circ}{\mathbf{O}} \cdot \mathbf{O}^{\mathrm{T}} \cdot \mathbf{O} \cdot \overset{\circ}{\mathbf{O}}{}^{\mathrm{T}} = \overset{\circ}{\mathbf{O}} \cdot \overset{\circ}{\mathbf{O}}{}^{\mathrm{T}} = \mathbf{E}. \tag{3.18}$$

Then the relationship (3.17) can be considered as the polar decomposition of the tensor $\overset{\circ}{\mathbf{O}} \cdot \mathbf{U} \cdot \overset{\circ}{\mathbf{O}}{}^{\mathrm{T}}$. This tensor is symmetric, because

$$(\overset{\circ}{\mathbf{O}} \cdot \mathbf{U} \cdot \overset{\circ}{\mathbf{O}}{}^{\mathrm{T}})^{\mathrm{T}} = (\overset{\circ}{\mathbf{O}}{}^{\mathrm{T}})^{\mathrm{T}} \cdot (\overset{\circ}{\mathbf{O}} \cdot \mathbf{U})^{\mathrm{T}} = \overset{\circ}{\mathbf{O}} \cdot \mathbf{U} \cdot \overset{\circ}{\mathbf{O}}{}^{\mathrm{T}}. \tag{3.19}$$

Then the formal equality

$$\overset{\circ}{\mathbf{O}} \cdot \mathbf{U} \cdot \overset{\circ}{\mathbf{O}}{}^{\mathrm{T}} = \overset{\circ}{\mathbf{O}} \cdot \mathbf{U} \cdot \overset{\circ}{\mathbf{O}}{}^{\mathrm{T}} \tag{3.20}$$

is one more polar decomposition. However, as was shown above, the polar decomposition is unique; hence, the following relationships must be satisfied:

$$\mathbf{V} = \overset{\circ}{\mathbf{O}} \cdot \mathbf{U} \cdot \overset{\circ}{\mathbf{O}}{}^{\mathrm{T}} \quad \text{and} \quad \mathbf{O} \cdot \overset{\circ}{\mathbf{O}}{}^{\mathrm{T}} = \mathbf{E}. \tag{3.21}$$

Thus, the orthogonal tensors $\mathbf{O}$ and $\overset{\circ}{\mathbf{O}}$ are coincident: $\mathbf{O} = \overset{\circ}{\mathbf{O}}$. ▲

The tensors $\mathbf{U}$ and $\mathbf{V}$ are called the *right and left stretch tensors*, respectively, and $\mathbf{O}$ is the *rotation tensor accompanying the deformation*.

The tensor $\mathbf{F}$ has nine independent components, the tensor $\mathbf{O}$ — three independent components, and each of the tensors $\mathbf{U}$ and $\mathbf{V}$ — six independent components.

Remark 1. Since the rotation tensor $\mathbf{O}$ is unique in the polar decomposition, from formula (3.21) we get that the stretch tensors $\mathbf{U}$ and $\mathbf{V}$ are connected by means of the tensor $\mathbf{O}$:

$$\mathbf{V} = \mathbf{O}\cdot\mathbf{U}\cdot\mathbf{O}^{\mathrm{T}}, \quad \mathbf{U} = \mathbf{O}^{\mathrm{T}}\cdot\mathbf{V}\cdot\mathbf{O}. \quad \square \tag{3.21a}$$

Theorem 1.10. *The Cauchy–Green and Almansi deformation tensors can be expressed in terms of the stretch tensors* $\mathbf{U}$ *and* $\mathbf{V}$ *as follows:*

$$\mathbf{C} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{E}), \qquad \mathbf{A} = \frac{1}{2}(\mathbf{E} - \mathbf{V}^{-2}),$$
$$\boldsymbol{\Lambda} = \frac{1}{2}(\mathbf{E} - \mathbf{U}^{-2}), \qquad \mathbf{J} = \frac{1}{2}(\mathbf{V}^2 - \mathbf{E}). \tag{3.22}$$

▼ To see this, let us substitute the polar decomposition (3.1) into (2.5), and then we get the relationships (3.22). ▲

1.3.2. Eigenvalues and Eigenbases.

Theorem 1.11. *Eigenvalues of the tensors* $\mathbf{U}$ *and* $\mathbf{V}$ *defined by (3.6) are coincident:*

$$\lambda_\alpha = \overset{\circ}{\lambda}_\alpha, \quad \alpha = 1, 2, 3, \tag{3.23}$$

and eigenvectors $\overset{\circ}{\mathbf{p}}_\alpha$ *and* $\mathbf{p}_\alpha$ *are connected by the rotation tensor* $\mathbf{O}$*:*

$$\mathbf{p}_\alpha = \mathbf{O}\cdot\overset{\circ}{\mathbf{p}}_\alpha. \tag{3.23a}$$

▼ To prove the theorem, we use the definition (3.6) and the first formula of (3.21a):

$$\mathbf{V} = \sum_{\alpha=1}^{3}\lambda_\alpha\mathbf{p}_\alpha\otimes\mathbf{p}_\alpha = \mathbf{O}\cdot\mathbf{U}\cdot\mathbf{O}^{\mathrm{T}} = \sum_{\alpha=1}^{3}\overset{\circ}{\lambda}_\alpha\mathbf{O}\cdot\overset{\circ}{\mathbf{p}}_\alpha\otimes(\mathbf{O}\cdot\overset{\circ}{\mathbf{p}}_\alpha) = \sum_{\alpha=1}^{3}\overset{\circ}{\lambda}_\alpha\mathbf{p}'_\alpha\otimes\mathbf{p}'_\alpha,$$

where

$$\mathbf{p}'_\alpha = \mathbf{O}\cdot\overset{\circ}{\mathbf{p}}_\alpha.$$

According to the relationship, we have obtained two different eigenbases of the tensor $\mathbf{V}$ and two sets of eigenvalues, that is impossible. Therefore,

$$\mathbf{p}'_\alpha = \mathbf{O}\cdot\overset{\circ}{\mathbf{p}}_\alpha = \mathbf{p}_\alpha \ \text{ and } \ \lambda_\alpha = \overset{\circ}{\lambda}_\alpha,$$

as was to be proved. ▲

Due to (3.5), both the eigenbases are orthogonal. Therefore, reciprocal vectors of the eigenbases do not differ from $\mathbf{p}_\alpha$ and $\overset{\circ}{\mathbf{p}}_\alpha$:

$$\mathbf{p}_\alpha = \mathbf{p}^\alpha, \qquad \overset{\circ}{\mathbf{p}}_\alpha = \overset{\circ}{\mathbf{p}}{}^\alpha. \tag{3.24}$$

The important problem for applications is to determine λ_α, $\mathbf{p}_\alpha$ and $\overset{\circ}{\mathbf{p}}_\alpha$ by the given deformation gradient $\mathbf{F}$. To solve the problem, one should use the following method.

1) Construct the tensor $\mathbf{U}^2 = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}$ (or $\mathbf{V}^2 = \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}}$) and find its components in some basis being suitable for a considered problem; for example, in the Cartesian basis $\bar{\mathbf{e}}_i$:

$$\mathbf{U}^2 = (\bar{U}^2)^i{}_j \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}^j \quad \text{and} \quad \mathbf{V}^2 = (\bar{V}^2)^i{}_j \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}^j.$$

2) Find eigenvalues of the matrix $(\bar{U}^2)^i{}_j$ by solving the characteristic equation

$$\det\,(\mathbf{U}^2 - \lambda_\alpha^2 \mathbf{E}) = 0, \tag{3.25}$$

which in the basis $\bar{\mathbf{e}}_i$ takes the form

$$\det\,((\mathrm{U}^2)^{\mathrm{i}}{}_{\mathrm{j}} - \lambda_\alpha^2 \delta_{\mathrm{j}}^{\mathrm{i}}) = 0. \tag{3.25a}$$

3) Find eigenvectors $\overset{\circ}{\mathbf{p}}_\alpha$ of the tensor $\mathbf{U}$ and eigenvectors $\mathbf{p}_\alpha$ of the tensor $\mathbf{V}$ from the equations

$$\mathbf{U}^2 \cdot \overset{\circ}{\mathbf{p}}_\alpha = \lambda_\alpha^2 \overset{\circ}{\mathbf{p}}_\alpha, \quad \mathbf{V}^2 \cdot \mathbf{p}_\alpha = \lambda_\alpha^2 \mathbf{p}_\alpha, \tag{3.26}$$

written, for example, in the basis $\bar{\mathbf{e}}_i$:

$$((\bar{U}^2)^i{}_j - \lambda_\alpha^2 \delta_j^i)\overset{\circ}{\widehat{Q}}{}^j{}_\alpha = 0, \quad ((\bar{V}^2)^i{}_j - \lambda_\alpha^2 \delta_j^i)\widehat{Q}^j{}_\alpha = 0, \tag{3.26a}$$

where $\widehat{Q}^j{}_\alpha$ and $\overset{\circ}{\widehat{Q}}{}^j{}_\alpha$ are Jacobian matrices of the eigenvectors:

$$\overset{\circ}{\mathbf{p}}_\alpha = \overset{\circ}{\widehat{Q}}{}^j{}_\alpha \bar{\mathbf{e}}_j, \qquad \mathbf{p}_\alpha = \widehat{Q}^j{}_\alpha \bar{\mathbf{e}}_j. \tag{3.27}$$

To determine the matrices $\overset{\circ}{\widehat{Q}}{}^j{}_\alpha$ and $\widehat{Q}^j{}_\alpha$, one should consider only independent equations of the system (3.26a) and the normalization conditions (3.5):

$$|\mathbf{p}_\alpha| = 1, \qquad |\overset{\circ}{\mathbf{p}}_\alpha| = 1, \tag{3.28}$$

which are equivalent to the quadratic equations

$$\overset{\circ}{\widehat{Q}}{}^i{}_\alpha \overset{\circ}{\widehat{Q}}{}^j{}_\alpha \delta_{ij} = 1, \quad \widehat{Q}^i{}_\alpha \widehat{Q}^j{}_\alpha \delta_{ij} = 1. \tag{3.28a}$$

4) Write the dyadic products (3.6) and find resolutions of the tensors $\mathbf{U}$ and $\mathbf{V}$ for the eigenbases; for example, for the Cartesian basis $\bar{\mathbf{e}}_i$:

$$\mathbf{U} = \sum_{\alpha=1}^{3} \lambda_\alpha \overset{\circ}{\widehat{Q}}{}^i{}_\alpha \overset{\circ}{\widehat{Q}}{}^j{}_\alpha \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j, \quad \mathbf{V} = \sum_{\alpha=1}^{3} \lambda_\alpha \widehat{Q}^i{}_\alpha \widehat{Q}^j{}_\alpha \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j.$$

Exercises 1.3.2–1.3.4 show examples of determination of the tensors $\mathbf{U}$ and $\mathbf{V}$.

Remark 2. Notice that a solution of the quadratic equations (3.28a) may be not unique due to the choice of signs of matrix components $\overset{\circ}{\widehat{Q}}{}^i{}_\alpha$ and $\widehat{Q}^i{}_\alpha$, this ambiguity is resolved by applying one more additional condition, namely the condition of coincidence of the vectors $\overset{\circ}{\mathbf{p}}_\alpha$ and $\mathbf{p}_\alpha$ when $t \to 0_+$:

$$t \to 0_+ \quad \Rightarrow \quad \mathbf{p}_\alpha(t) = \overset{\circ}{\mathbf{p}}_\alpha(t), \qquad \alpha = 1,\ 2,\ 3.$$

For the matrix $\overset{\circ}{\widehat{Q}}{}^i{}_\alpha$, the ambiguity of the sign choice remains. However, if there is a field of eigenvectors $\overset{\circ}{\mathbf{p}}_\alpha(\mathbf{x}, t)$, then this ambiguity may be retained only at one point $\mathbf{x}_0$ at one time, for example, $t = 0$; and for the remaining $\mathbf{x}$ and t, a sign at $\widehat{Q}^i{}_\alpha$ is chosen from the continuity condition of the vector field $\overset{\circ}{\mathbf{p}}_\alpha(\mathbf{x}, t)$ (for continuous motions). If the eigenvector field $\overset{\circ}{\mathbf{p}}_\alpha(\mathbf{x}_0, 0)$ contains the vectors $\bar{\mathbf{e}}_\alpha$, then the remaining ambiguity is resolved by the condition $\overset{\circ}{\mathbf{p}}_\alpha(\mathbf{x}_0, 0) = \bar{\mathbf{e}}_\alpha$.

The ambiguity of a solution of the system (3.26a), (3.28a) may also appear, if at some time t_1 at a point $\mathbf{x}$ the eigenvalues $\lambda_\alpha(t_1)$ prove to be triple. In this case, values of the matrices $\widehat{Q}^i{}_\alpha(t_1)$ and $\overset{\circ}{\widehat{Q}}{}^i{}_\alpha(t_1)$ are determined, as a rule, by passage to the limit:

$$\widehat{Q}^i{}_\alpha(t_1) = \lim_{t \to t_1} \widehat{Q}^i{}_\alpha(t), \qquad \overset{\circ}{\widehat{Q}}{}^i{}_\alpha(t_1) = \lim_{t \to t_1} \overset{\circ}{\widehat{Q}}{}^i{}_\alpha(t), \quad \alpha = 1, 2, 3.$$

In the case of double eigenvalues λ_α, these formulae are applied only to their corresponding matrix components $\widehat{Q}^i{}_\alpha$ and $\overset{\circ}{\widehat{Q}}{}^i{}_\alpha$. □

1.3.3. Representation of the Deformation Tensors in Eigenbases.

Theorem 1.12. *In the tensor bases $\mathbf{p}_\alpha \otimes \mathbf{p}_\beta$ and $\overset{\circ}{\mathbf{p}}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\beta$, the Cauchy–Green tensors $\mathbf{C}$ and $\mathbf{J}$, the Almansi tensors $\mathbf{A}$ and $\mathbf{\Lambda}$, and the deformation measures $\mathbf{G}$, $\mathbf{g}^{-1}$ and $\mathbf{G}^{-1}$, $\mathbf{g}$ have the diagonal form:*

$$\mathbf{C} = \sum_{\alpha=1}^{3} \frac{1}{2}(\lambda_\alpha^2 - 1)\overset{\circ}{\mathbf{p}}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\alpha, \qquad \mathbf{\Lambda} = \sum_{\alpha=1}^{3} \frac{1}{2}(1 - \lambda_\alpha^{-2})\overset{\circ}{\mathbf{p}}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\alpha, \tag{3.29a}$$

$$\mathbf{A} = \sum_{\alpha=1}^{3} \frac{1}{2}(1 - \lambda_\alpha^{-2})\mathbf{p}_\alpha \otimes \mathbf{p}_\alpha, \qquad \mathbf{J} = \sum_{\alpha=1}^{3} \frac{1}{2}(\lambda_\alpha^2 - 1)\mathbf{p}_\alpha \otimes \mathbf{p}_\alpha;$$

and

$$\mathbf{G} = \sum_{\alpha=1}^{3} \lambda_\alpha^2 \overset{\circ}{\mathbf{p}}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\alpha, \qquad \mathbf{G}^{-1} = \sum_{\alpha=1}^{3} \lambda_\alpha^{-2}\ \overset{\circ}{\mathbf{p}}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\alpha, \tag{3.29b}$$

$$\mathbf{g}^{-1} = \sum_{\alpha=1}^{3} \lambda_\alpha^2 \mathbf{p}_\alpha \otimes \mathbf{p}_\alpha, \qquad \mathbf{g} = \sum_{\alpha=1}^{3} \lambda_\alpha^{-2}\ \mathbf{p}_\alpha \otimes \mathbf{p}_\alpha.$$

▼ On substituting formulae (3.6) into (3.22), we get (3.29a). Formulae (3.29b) follow from (3.29a) and (2.7), (2.8). ▲

Similarly to formulae (3.29), we can introduce new deformation tensors by determining their components with respect to the bases $\overset{\circ}{\mathbf{p}}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\beta$ or $\mathbf{p}_\alpha \otimes \mathbf{p}_\beta$ as follows:

$$\overset{\circ}{\mathbf{M}} = \sum_{\alpha=1}^{3} f(\lambda_\alpha)\overset{\circ}{\mathbf{p}}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\beta, \qquad \mathbf{M} = \sum_{\alpha=1}^{3} f(\lambda_\alpha)\mathbf{p}_\alpha \otimes \mathbf{p}_\beta, \tag{3.30}$$

where $f(\lambda_\alpha)$ is a function of λ_α. If $f(1) = 0$, then we get the deformation tensors; and if $f(1) = 1$, then we get the deformation measures.

Among the tensors (3.30), the *logarithmic deformation tensors and measures*

$$\overset{\circ}{\mathbf{H}} = \sum_{\alpha=1}^{3} \lg\, \lambda_\alpha \overset{\circ}{\mathbf{p}}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\beta, \qquad \mathbf{H} = \sum_{\alpha=1}^{3} \lg\, \lambda_\alpha \mathbf{p}_\alpha \otimes \mathbf{p}_\beta, \tag{3.31}$$
$$\overset{\circ}{\mathbf{H}}_1 = \overset{\circ}{\mathbf{H}} + \mathbf{E}, \qquad \mathbf{H}_1 = \mathbf{H} + \mathbf{E},$$

are the most widely known; they are called the *right* and *left Hencky tensors*, and also the *right* and *left Hencky measures*, respectively.

With the help of the eigenvectors $\mathbf{p}_\alpha$ and $\overset{\circ}{\mathbf{p}}_\alpha$ we can form the mixed dyads

$$\sum_{\alpha=1}^{3} \mathbf{p}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\alpha = \sum_{\alpha=1}^{3} \mathbf{p}_\alpha \otimes \mathbf{p}_\alpha \cdot \mathbf{O} = (\sum_{\alpha=1}^{3} \mathbf{p}_\alpha \otimes \mathbf{p}^\alpha) \cdot \mathbf{O} = \mathbf{E} \cdot \mathbf{O}. \tag{3.32}$$

Here we have used the properties (3.23a) and (3.24), and the representation of the unit tensor $\mathbf{E}$ in an arbitrary mixed dyadic basis.

Thus, the rotation tensor $\mathbf{O}$ accompanying the deformation can be expressed in the eigenbasis as follows:

$$\mathbf{O} = \sum_{\alpha=1}^{3} \mathbf{p}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\alpha = \mathbf{p}_i \otimes \overset{\circ}{\mathbf{p}}{}^i. \tag{3.33}$$

On substituting (3.33) and (3.6) into (3.1) and taking (3.5) into account, we get the following expression of the deformation gradient in the tensor eigenbasis:

$$\mathbf{F} = \mathbf{O} \cdot \mathbf{U} = \sum_{\alpha=1}^{3} \mathbf{p}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\alpha \cdot \sum_{\beta=1}^{3} \overset{\circ}{\lambda}_\beta \overset{\circ}{\mathbf{p}}_\beta \otimes \overset{\circ}{\mathbf{p}}_\beta = \sum_{\alpha=1}^{3} \lambda_\alpha \mathbf{p}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\alpha. \tag{3.34}$$

According to (3.34), the transpose $\mathbf{F}^{\mathrm{T}}$ and inverse $\mathbf{F}^{-1}$ gradients are expressed as follows:

$$\mathbf{F}^{\mathrm{T}} = \sum_{\alpha=1}^{3} \lambda_\alpha \overset{\circ}{\mathbf{p}}_\alpha \otimes \mathbf{p}_\alpha, \qquad \mathbf{F}^{-1} = \sum_{\alpha=1}^{3} \lambda_\alpha^{-1} \overset{\circ}{\mathbf{p}}_\alpha \otimes \mathbf{p}_\alpha. \tag{3.35}$$

1.3.4. Geometrical Meaning of Eigenvalues. Vectors of eigenbases $\overset{\circ}{\mathbf{p}}_\alpha$ and $\mathbf{p}_\alpha$ are connected by the transformation (3.23a). In $\overset{\circ}{\mathcal{K}}$ take elementary radius-vectors $d\overset{\circ}{\mathbf{x}}_\alpha$ oriented along the eigenbasis vectors $\overset{\circ}{\mathbf{p}}_\alpha$, then in $\mathcal{K}$ they correspond to radius-vectors $d\mathbf{x}_\alpha$:

$$d\overset{\circ}{\mathbf{x}}_\alpha = \overset{\circ}{\mathbf{p}}_\alpha |d\overset{\circ}{\mathbf{x}}_\alpha|, \qquad d\mathbf{x}_\alpha = \mathbf{F} \cdot d\overset{\circ}{\mathbf{x}}_\alpha. \tag{3.36}$$

Substitution of (3.34) into (3.36) yields

$$d\mathbf{x}_\alpha = \sum_{\beta=1}^{3} \lambda_\beta \mathbf{p}_\beta \otimes \overset{\circ}{\mathbf{p}}_\beta \cdot \overset{\circ}{\mathbf{p}}_\alpha |d\overset{\circ}{\mathbf{x}}_\alpha| = \lambda_\alpha |d\mathbf{x}_\alpha| \mathbf{p}_\alpha, \tag{3.37}$$

i.e. the elementary radius-vectors $d\mathbf{x}_\alpha$ in $\mathcal{K}$ will be also oriented along the corresponding eigenbasis vectors $\mathbf{p}_\alpha$.

Denote lengths of the vectors $d\overset{\circ}{\mathbf{x}}_\alpha$ and $d\mathbf{x}_\alpha$ by $d\overset{\circ}{s}_\alpha$ and ds, respectively, and derive relations between them:

$$ds_\alpha^2 = d\mathbf{x}_\alpha \cdot d\mathbf{x}_\alpha = d\overset{\circ}{\mathbf{x}}_\alpha \cdot \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} \cdot d\overset{\circ}{\mathbf{x}}_\alpha = |d\overset{\circ}{\mathbf{x}}_\alpha|^2 \overset{\circ}{\mathbf{p}}_\alpha \cdot \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} \cdot \overset{\circ}{\mathbf{p}}_\alpha = \\ = d\overset{\circ}{s}{}^2_\alpha \overset{\circ}{\mathbf{p}}_\alpha \cdot \mathbf{G} \cdot \overset{\circ}{\mathbf{p}}_\alpha = d\overset{\circ}{s}{}^2_\alpha \lambda_\alpha^2. \quad (3.38)$$

Here we have used equations (3.29b) and (3.36). Formula (3.38) proves the following theorem.

Theorem 1.13. *Eigenvalues λ_α (principal stretches) are the elongation ratios for material fibres oriented along the principal (eigen-) directions:*

$$\lambda_\alpha = ds_\alpha / d\overset{\circ}{s}_\alpha. \quad (3.39)$$

1.3.5. Geometric Picture of Transformation of a Small Neighborhood of a Point of a Continuum. In $\overset{\circ}{\mathcal{K}}$, consider a small neighborhood of the material point $\mathcal{M}$ contained in a continuum; then every point $\mathcal{M}'$, connected to $\mathcal{M}$ by the elementary radius-vector $d\overset{\circ}{\mathbf{x}}$ (Figure 1.12), will be connected to the same point $\mathcal{M}$ by radius-vector $d\mathbf{x}$ in $\mathcal{K}$. These radius-vectors are related as follows:

$$d\mathbf{x} = \mathbf{F} \cdot d\overset{\circ}{\mathbf{x}}. \quad (3.40)$$

The relation can be considered as the transformation of arbitrary radius-vector $d\overset{\circ}{\mathbf{x}}$ into $d\mathbf{x}$.

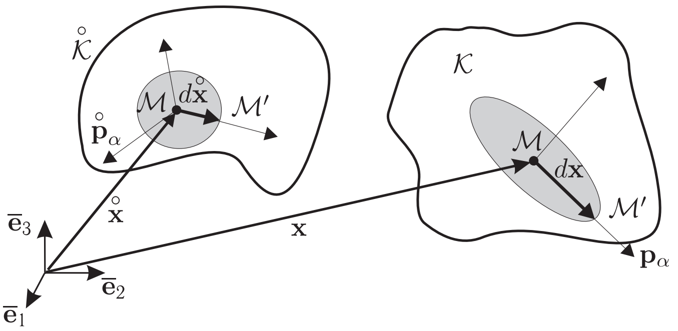

Figure 1.12. Transformation of a small neighborhood of the point contained in a continuum

Rewrite the relation (3.40) in Cartesian coordinates:

$$dx^i = \bar{F}^i_m d\overset{\circ}{x}{}^m, \quad (3.41)$$

where $\bar{F}^i_m$ are components of the deformation gradient with respect to the Cartesian basis (see Exercise 1.1.5):

$$\bar{F}^i_m = (\partial x^i / \partial \overset{\circ}{x}{}^m), \quad (3.42)$$

which depend only on coordinates $\overset{\circ}{x}{}^m$ of the point $\mathcal{M}$, but they are independent of coordinates $d\overset{\circ}{x}{}^m$ of its neighboring points $\mathcal{M}'$. Therefore the transformation

(3.41) is a linear transformation of coordinates $d\overset{\circ}{x}{}^m$ into dx^i, i.e. this is an affine transformation.

As follows from the general properties of affine transformations, straight lines and planes contained in a small neighborhood in $\overset{\circ}{\mathcal{K}}$ will be straight lines and planes in actual configuration $\mathcal{K}$. Parallel straight lines and planes are transformed into parallel straight lines and planes. Therefore if a small neighborhood in $\overset{\circ}{\mathcal{K}}$ is chosen to be a parallelogram, then in $\mathcal{K}$ the neighborhood will be a parallelogram as well (although angles between its edges, edge lengths and orientation of planes in space may change).

Since a second-order surface in $\overset{\circ}{\mathcal{K}}$ (and, in general, a surface specified by an algebraic expression of arbitrary n-th order) is transformed into a surface of the same order in $\mathcal{K}$, a small spherical neighborhood in $\overset{\circ}{\mathcal{K}}$ is transformed into an ellipsoid in actual configuration $\mathcal{K}$ (Figure 1.12).

As follows from formula (2.34), the ratio of lengths $ds_\alpha/d\overset{\circ}{s}_\alpha$ of an arbitrary vector (or of elementary radius-vector $d\mathbf{x}$ in $\mathcal{K}$ and $\overset{\circ}{\mathcal{K}}$) is independent of the initial length $d\overset{\circ}{s}_\alpha$ of the vector (because the relative elongation δ_α is independent of $d\overset{\circ}{s}_\alpha$).

According to the polar decomposition (3.1), the transformation (3.40) from $\overset{\circ}{\mathcal{K}}$ to $\mathcal{K}$ can always be represented as the superposition of two transformations:

$$d\mathbf{x} = \mathbf{O} \cdot d\overset{\circ}{\mathbf{x}}{}', \qquad d\overset{\circ}{\mathbf{x}}{}' = \mathbf{U} \cdot d\overset{\circ}{\mathbf{x}}, \tag{3.43}$$

realized with the help of the stretch tensor $\mathbf{U}$ and the rotation tensor $\mathbf{O}$, or

$$d\mathbf{x} = \mathbf{V} \cdot d\mathbf{x}', \qquad d\mathbf{x}' = \mathbf{O} \cdot d\overset{\circ}{\mathbf{x}}. \tag{3.44}$$

The stretch tensor $\mathbf{U}$, which has three eigendirections $\overset{\circ}{\mathbf{p}}_\alpha$, transforms a small neighborhood of the point $\mathcal{M}$ with compressing or extending the neighborhood along these three directions $\overset{\circ}{\mathbf{p}}_\alpha$. The tensor $\mathbf{O}$ rotates the neighborhood deformed along $\overset{\circ}{\mathbf{p}}_\alpha$ as a rigid whole until the direction of $\overset{\circ}{\mathbf{p}}_\alpha$ becomes the direction of $\mathbf{p}_\alpha$. If one use the left stretch tensor $\mathbf{V}$, so rotation of axes $\overset{\circ}{\mathbf{p}}_\alpha$ in $\overset{\circ}{\mathcal{K}}$ till their coincidence with $\mathbf{p}_\alpha$ is first realized, and then compression or tension of the neighborhood occurs along the direction $\mathbf{p}_\alpha$. The result will be the same as for $\mathbf{U}$.

If a point $\mathcal{M}_\alpha$ is connected to $\mathcal{M}$ by radius-vector $d\overset{\circ}{\mathbf{x}}_\alpha$ oriented along the eigendirection $\overset{\circ}{\mathbf{p}}_\alpha$ (which is unknown before deformation), then in $\mathcal{K}$ the point $\mathcal{M}_\alpha$ will be connected to $\mathcal{M}$ by radius-vector $d\mathbf{x}_\alpha$ oriented along the corresponding eigendirection $\mathbf{p}_\alpha$.

If a small neighborhood of point $\mathcal{M}$ is chosen to be a sphere in $\overset{\circ}{\mathcal{K}}$ (see Figure 1.12), then in $\mathcal{K}$ the sphere becomes an ellipsoid with principal axes oriented along the eigendirections $\mathbf{p}_\alpha$.

Thus, the transformation of a small neighborhood of every point $\mathcal{M}$ contained in a continuum under deformation can always be represented as a superposition of tension/compression along eigendirections and rotation of the neighborhood as a rigid whole, and also displacement as a rigid whole.

Exercises for 1.3

Exercise 1.3.1. Using the formula (3.21a), show that the following relations between $\mathbf{V}$ and $\mathbf{U}$ hold:

$$\mathbf{V}^m = \mathbf{O} \cdot \mathbf{U}^m \cdot \mathbf{O}^{\mathrm{T}}, \qquad \mathbf{U}^m = \mathbf{O}^{\mathrm{T}} \cdot \mathbf{V}^m \cdot \mathbf{O}$$

for all integer m (positive and negative).

Exercise 1.3.2. Using the results of Exercises 1.1.1 and 1.2.1, show that for the problem on tension of a beam, eigenvalues λ_α are

$$\lambda_\alpha = k_\alpha, \qquad \alpha = 1, 2, 3.$$

The stretch tensors $\mathbf{U}$ and $\mathbf{V}$ are coincident and have the form

$$\mathbf{U} = \mathbf{V} = \sum_{\alpha=1}^{3} k_\alpha \mathbf{e}_\alpha \otimes \mathbf{e}_\alpha,$$

and eigenvectors $\overset{\circ}{\mathbf{p}}_\alpha$ and $\mathbf{p}_\alpha$ coincide with $\mathbf{e}_\alpha$:

$$\overset{\circ}{\mathbf{p}}_\alpha = \mathbf{p}_\alpha = \mathbf{e}_\alpha, \qquad \alpha = 1, 2, 3.$$

The rotation tensor $\mathbf{O}$ for this problem is the unit one: $\mathbf{O} = \mathbf{E}$.

Exercise 1.3.3. Using the results of Exercises 1.1.2, 1.2.2 and Remark 2, show that for the problem on simple shear (see Example 1.2 from paragraph 1.1.1), the tensors $\mathbf{U}^2$ and $\mathbf{V}^2$ are expressed as follows:

$$\mathbf{U}^2 = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} = \mathbf{E} + a\mathbf{O}_3 + a^2 \mathbf{e}_i^2 = (\bar{U}^2)^i{}_j \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}^j,$$

$$\mathbf{V}^2 = \mathbf{E} + a\mathbf{O}_3 + a^2 \mathbf{e}_i^2 = (\bar{V}^2)^i{}_j \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}^j,$$

$$(\bar{U}^2)^i{}_j = \begin{pmatrix} 1 & a & 0 \\ a & 1+a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (\bar{V}^2)^i{}_j = \begin{pmatrix} 1+a^2 & a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

eigenvalues λ_α are

$$\lambda_\alpha^2 = 1 + b_\alpha |a|, \quad \alpha = 1, 2; \qquad \lambda_3 = 1,$$

$$b_1 = \frac{a}{2} + \sqrt{1 + a^2/4}\,, \qquad b_2 = \frac{a}{2} - \sqrt{1 + a^2/4}\,,$$

eigenvectors $\overset{\circ}{\mathbf{p}}_\alpha$ and $\mathbf{p}_\alpha$ $(a > 0)$ are

$$\overset{\circ}{\mathbf{p}}_\alpha = \frac{1}{\sqrt{1 + b_\alpha^2}} (\bar{\mathbf{e}}_1 + b_\alpha \bar{\mathbf{e}}_2), \quad \overset{\circ}{\mathbf{p}}_3 = \bar{\mathbf{e}}_3,$$

$$\mathbf{p}_1 = \frac{1}{\sqrt{1 + b_1^2}}\, (b_1 \bar{\mathbf{e}}_1 + \bar{\mathbf{e}}_2), \quad \mathbf{p}_2 = -\frac{1}{\sqrt{1 + b_2^2}}\, (b_2 \bar{\mathbf{e}}_1 + \bar{\mathbf{e}}_2), \quad \mathbf{p}_3 = \bar{\mathbf{e}}_3,$$

the stretch tensors $\mathbf{U}$ and $\mathbf{V}$ are

$$\mathbf{U} = \bar{U}^i{}_j \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}^j = U_0 \bar{\mathbf{e}}_1^2 + U_1 \mathbf{O}_3 + U_2 \bar{\mathbf{e}}_2^2 + \bar{\mathbf{e}}_3^2,$$

$$\mathbf{V} = \bar{V}^i{}_j\,\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}^j = U_2\bar{\mathbf{e}}_1^2 + U_1\mathbf{O}_3 + U_0\bar{\mathbf{e}}_2^2 + \bar{\mathbf{e}}_3^2,$$

$$\bar{U}^i{}_j = \begin{pmatrix} U_0 & U_1 & 0 \\ U_1 & U_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \bar{V}^i{}_j = \begin{pmatrix} U_2 & U_1 & 0 \\ U_1 & U_0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$U_\beta = \frac{b_1^\beta\sqrt{1+b_1a}}{1+b_1^2} + \frac{b_2^\beta\sqrt{1+b_2a}}{1+b_2^2}, \qquad \beta = 0, 1, 2,$$

and the rotation tensor $\mathbf{O}$ has the form

$$\mathbf{O} = \bar{O}^i{}_j\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}^j = \cos\varphi(\bar{\mathbf{e}}_1^2 + \bar{\mathbf{e}}_2^2) + \sin\varphi(\bar{\mathbf{e}}_1 \otimes \bar{\mathbf{e}}_2 - \bar{\mathbf{e}}_2 \otimes \bar{\mathbf{e}}_1),$$

$$\bar{O}^i{}_j = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\cos\varphi = \frac{b_1}{1+b_1^2} - \frac{b_2}{1+b_2^2}, \qquad \sin\varphi = \frac{b_1^2}{1+b_1^2} - \frac{b_2^2}{1+b_2^2}.$$

Show that functions $b_1(a)$ and $b_2(a)$ satisfy the following relationships:

$$b_1 + b_2 = a, \qquad b_1b_2 = -1, \qquad b_1^2 + b_2^2 = 2 + a^2.$$

Show that at $a = 0$ for the considered problem the following equations really hold:

$$b_1 = 1, \qquad b_2 = -1, \qquad \lambda_1 = \lambda_2 = \lambda_3 = 1,$$

$$\overset{\circ}{\mathbf{p}}_1 = \mathbf{p}_1 = \frac{1}{\sqrt{2}}(\bar{\mathbf{e}}_1 + \bar{\mathbf{e}}_2), \qquad \overset{\circ}{\mathbf{p}}_2 = \mathbf{p}_2 = \frac{1}{\sqrt{2}}(\bar{\mathbf{e}}_1 - \bar{\mathbf{e}}_2).$$

Exercise 1.3.4. Using the results of Exercise 1.2.3, show that for the problem on rotation of a beam with tension (see Example 1.3 from paragraph 1.1.1), eigenvalues λ_α have the form

$$\lambda_\alpha = k_\alpha, \qquad \alpha = 1, 2, 3,$$

and eigenvectors

$$\overset{\circ}{\mathbf{p}}_\alpha = \bar{\mathbf{e}}_\alpha, \qquad \mathbf{p}_\alpha = \mathbf{O}_0 \cdot \bar{\mathbf{e}}_\alpha, \qquad \alpha = 1, 2, 3.$$

Using formulae from Exercise 1.1.3 and data from Example 1.3, show that tensors $\mathbf{U}$, $\mathbf{V}$, $\mathbf{O}$, and also $\mathbf{C}$, $\mathbf{A}$, $\mathbf{\Lambda}$ and $\mathbf{J}$ have the form

$$\mathbf{U} = \mathbf{U}_0 = \sum_{\alpha=1}^{3} k_\alpha\bar{\mathbf{e}}_\alpha \otimes \bar{\mathbf{e}}_\alpha, \qquad \mathbf{O} = \mathbf{O}_0 = O_0{}^i{}_j\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}^j,$$

$$\mathbf{V} = \mathbf{O}_0 \cdot \mathbf{U}_0 \cdot \mathbf{O}_0^{\mathrm{T}} = V_0\bar{\mathbf{e}}_1^2 + V_1\mathbf{O}_3 + V_2\bar{\mathbf{e}}_2^2 + k_3\bar{\mathbf{e}}_3^2 = V_0{}^i{}_j\,\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}^j,$$

$$V_0{}^i{}_j = \begin{pmatrix} V_0 & V_1 & 0 \\ V_1 & V_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix},$$

$$V_1 = k_1\cos^2\varphi + k_2\sin^2\varphi, \qquad V_1 = (k_1 - k_2)\cos\varphi\ \sin\varphi,$$

$$V_2 = k_1\sin^2\varphi + k_2\cos^2\varphi,$$

$$\mathbf{C} = \frac{1}{2}(\mathbf{U}_0^2 - \mathbf{E}) = \sum_{\alpha=1}^{3}\frac{1}{2}(k_\alpha^2 - 1)\bar{\mathbf{e}}_\alpha \otimes \bar{\mathbf{e}}_\alpha,$$

$$\mathbf{\Lambda} = \frac{1}{2}(\mathbf{E} - \mathbf{U}_0^{-2}) = \sum_{\alpha=1}^{3} \frac{1}{2}(1 - k_\alpha^{-2})\bar{\mathbf{e}}_\alpha \otimes \bar{\mathbf{e}}_\alpha,$$

$$\mathbf{A} = \frac{1}{2}(\mathbf{E} - \mathbf{V}^{-2}) = \frac{1}{2}(\delta^{ij} - g^{ij})\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j, \qquad \mathbf{J} = \frac{1}{2}(\mathbf{V}^2 - \mathbf{E}) = \frac{1}{2}(g_{ij} - \delta_{ij})\bar{\mathbf{e}}^i \otimes \bar{\mathbf{e}}^j,$$

where metric matrices g_{ij} and g^{ij} are determined by formulae from Exercise 1.1.3.

We should take into consideration that the tensors $\mathbf{C}$ and $\mathbf{\Lambda}$ do not feel the beam rotation — they are coincident with the corresponding tensors for the problem on pure tension of the beam. Show that if we change the sequence of transformations (i.e. we first rotate and then extend the beam), then the tensors $\mathbf{A}$ and $\mathbf{J}$ do not feel the rotation.

1.4. Rate Characteristics of Continuum Motion

1.4.1. Velocity. The *velocity* (vector) *of the motion* of a material point $\mathcal{M}$ with Lagrangian coordinates X^i is determined as the partial derivative of the radius-vector $\mathbf{x}(X^i, t)$ with respect to time at fixed values of X^i:

$$\mathbf{v}(X^i, t) = \frac{\partial \mathbf{x}}{\partial t}(X^i, t)\Big|_{X^i}. \tag{4.1}$$

Velocity components $\bar{v}^i$ with respect to the basis $\bar{\mathbf{e}}_i$ have the form

$$\mathbf{v} = \bar{v}^i \bar{\mathbf{e}}_i = \frac{\partial x^i}{\partial t}\bar{\mathbf{e}}_i, \qquad \bar{v}^i = \frac{\partial x^i}{\partial t}(X^j, t). \tag{4.2}$$

1.4.2. Total Derivative of a Tensor with respect to Time. Any vector field $\mathbf{a}(\mathbf{x}, t)$ (and also scalar or tensor field) varying with time, which describes some physical process in a continuum, can be expressed in both Eulerian and Lagrangian descriptions with the help of the motion law (1.3):

$$\mathbf{a}(\mathbf{x}, t) = \mathbf{a}(\mathbf{x}(X^j, t), t). \tag{4.3}$$

Determine the derivative of the function with respect to time at fixed X^i (i.e. for a fixed point $\mathcal{M}$):

$$\frac{\partial \mathbf{a}}{\partial t}\Big|_{X^i} = \frac{\partial \mathbf{a}}{\partial t}\Big|_{x^i} + \frac{\partial \mathbf{a}}{\partial x^j}\frac{\partial x^j}{\partial t}\Big|_{X^i}. \tag{4.4}$$

Definition 1.1. *The partial derivative of a varying vector field* $\mathbf{a}$ *(4.3) with respect to time* t *at fixed coordinates* X^i *is called the t o t a l derivative of the function (4.3) with respect to time:*

$$\dot{\mathbf{a}} \equiv \frac{d\mathbf{a}}{dt} = \frac{\partial \mathbf{a}}{\partial t}\Big|_{X^i}. \tag{4.5}$$

According to formulae (4.2), (1.11) and (1.23), the second summand on the right-hand side of (4.4) can be rewritten as follows:

$$\frac{\partial \mathbf{a}}{\partial x^j}\frac{\partial x^j}{\partial t} = \bar{v}^j P^k_{\ j}\frac{\partial \mathbf{a}}{\partial X^k} = \bar{v}^i \bar{\mathbf{e}}_i \cdot \bar{\mathbf{e}}^j P^k_{\ j} \otimes \frac{\partial \mathbf{a}}{\partial X^k} = \mathbf{v} \cdot \mathbf{r}^k \otimes \frac{\partial \mathbf{a}}{\partial X^k} = \mathbf{v} \cdot \boldsymbol{\nabla} \otimes \mathbf{a}. \tag{4.6}$$

Then the relationship (4.4) yields

$$\frac{d\mathbf{a}}{dt} = \frac{\partial \mathbf{a}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \otimes \mathbf{a}, \tag{4.7}$$

where we have introduced the notation for the partial derivative with respect to time which will be widely used below:

$$\frac{\partial \mathbf{a}}{\partial t} = \frac{\partial \mathbf{a}}{\partial t}(x^i, t)\Big|_{x^i}. \tag{4.8}$$

In formula (4.7) the vector $\mathbf{a}$ is considered as a function $\mathbf{a}(x^j, t)$. It is evident that if $\mathbf{a}$ is considered as a function of (X^j, t), then from the definition (4.5) we get

$$\frac{d\mathbf{a}}{dt}(x^i, t) = \frac{\partial}{\partial t}\mathbf{a}(X^j, t)\Big|_{X^i}. \tag{4.9}$$

The total derivative $(d\mathbf{a}/dt)$ is also called the *material (substantial, individual) derivative* with respect to time, $(\partial\mathbf{a}/\partial t)$ in (4.7) is the *partial (local) derivative* with respect to time, and $\mathbf{v}\cdot\nabla\otimes\mathbf{a}$ is the *convective derivative*.

The material derivative $d\mathbf{a}/dt$ characterizes a change of the vector field $\mathbf{a}$ in a fixed material point $\mathcal{M}$, the local derivative determines a change of values of $\mathbf{a}$ in time at a fixed point $\mathbf{x}$ in space, and from (4.6) we get that the convective derivative characterizes a change of the field due to transfer of the material particle $\mathcal{M}$ from a point $\mathbf{x}$ to a point $\mathbf{x}+\mathbf{v}dt$ in space.

If we choose the vector $\mathbf{v}$ as $\mathbf{a}$, then the relationship between the displacement $\mathbf{u}$ and the velocity $\mathbf{v}$ vectors has the form

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\otimes\mathbf{u}. \tag{4.10}$$

Similarly to formula (4.5), we can define the *total derivative* of the nth-order *tensor* ${}^n\boldsymbol{\Omega}$ with respect to time:

$${}^n\dot{\boldsymbol{\Omega}} = \frac{d}{dt}{}^n\boldsymbol{\Omega}(x^i, t) = \frac{\partial\,{}^n\boldsymbol{\Omega}}{\partial t}(X^i, t)\Big|_{X^i}. \tag{4.11}$$

Theorem 1.14. *The total derivative (4.11) of a varying tensor field* ${}^n\boldsymbol{\Omega}(x^i, t)$ *can be written as a sum of local and convective derivatives:*

$$\frac{d}{dt}\,{}^n\boldsymbol{\Omega} = \frac{\partial\,{}^n\boldsymbol{\Omega}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\otimes{}^n\boldsymbol{\Omega}. \tag{4.12}$$

▼ Proof of the theorem is similar to the proof of the relationship (4.7). Details are left as Exercise 1.4.6. ▲

Let us consider now the question on components of the total derivative tensor.

Theorem 1.15. *Components of the total derivative tensor* ${}^n\dot{\boldsymbol{\Omega}}$ *are connected with the corresponding components of a tensor* ${}^n\boldsymbol{\Omega}$ *with respect to stationary bases* $\overset{\circ}{\mathbf{r}}_i$, $\bar{\mathbf{e}}_i$ *and* $\widetilde{\mathbf{r}}_i$ *and a moving basis* $\mathbf{r}_i$ *as follows:*

$$\frac{d}{dt}\overset{\circ}{\Omega}{}^{i_1\ldots i_n} = \frac{\partial}{\partial t}\overset{\circ}{\Omega}{}^{i_1\ldots i_n}(X^i, t)\Big|_{X^i}, \tag{4.13}$$

$$\frac{d}{dt}\bar{\Omega}^{i_1\ldots i_n} = \frac{\partial}{\partial t}\bar{\Omega}^{i_1\ldots i_n}(x^i, t) + \bar{v}^k\frac{\partial}{\partial x^k}\bar{\Omega}^{i_1\ldots i_n}(x^i, t), \tag{4.14}$$

$$\frac{d}{dt}\widetilde{\Omega}^{i_1\ldots i_n} = \frac{\partial}{\partial t}\widetilde{\Omega}^{i_1\ldots i_n}(\widetilde{X}^i, t) + \widetilde{v}^k\widetilde{\nabla}_k\widetilde{\Omega}^{i_1\ldots i_n}(\widetilde{X}^i, t), \tag{4.15}$$

$$\frac{d}{dt}\Omega^{i_1\ldots i_n} = \frac{\partial}{\partial t}\Omega^{i_1\ldots i_n}(X^i,t) + \sum_{\alpha=1}^{n}(\Omega^{i_1\ldots k\ldots i_n}\nabla_k v^{i_\alpha})(X^i,t), \tag{4.16}$$

where

$$\begin{aligned}{}^n\dot{\boldsymbol{\Omega}} &= \frac{d}{dt}\bar{\Omega}^{i_1\ldots i_n}(x^i,t)\bar{\mathbf{e}}_{i_1}\otimes\ldots\otimes\bar{\mathbf{e}}_{i_n} = \frac{d}{dt}\overset{\circ}{\Omega}{}^{i_1\ldots i_n}(X^i,t)\overset{\circ}{\mathbf{r}}_{i_1}\otimes\ldots\otimes\overset{\circ}{\mathbf{r}}_{i_n} = \\ &= \frac{d}{dt}\widetilde{\Omega}^{i_1\ldots i_n}(\widetilde{X}^i,t)\widetilde{\mathbf{r}}_{i_1}\otimes\ldots\otimes\widetilde{\mathbf{r}}_{i_n} = \frac{d}{dt}\Omega^{i_1\ldots i_n}(X^i,t)\mathbf{r}_{i_1}\otimes\ldots\otimes\mathbf{r}_{i_n}.\end{aligned} \tag{4.17}$$

In formula (4.16), the component $\Omega^{i_1\ldots k\ldots i_n}$ as the αth superscript has index k in place of i_α.

▼ To prove the theorem, we resolve the tensor ${}^n\boldsymbol{\Omega}$ for different bases:

$$\begin{aligned}{}^n\boldsymbol{\Omega} &= \bar{\Omega}^{i_1\ldots i_n}(x^i,t)\bar{\mathbf{e}}_{i_1}\otimes\ldots\otimes\bar{\mathbf{e}}_{i_n} = \overset{\circ}{\Omega}{}^{i_1\ldots i_n}(X^i,t)\overset{\circ}{\mathbf{r}}_{i_1}\otimes\ldots\otimes\overset{\circ}{\mathbf{r}}_{i_n} = \\ &= \widetilde{\Omega}^{i_1\ldots i_n}(\widetilde{X}^i,t)\widetilde{\mathbf{r}}_{i_1}\otimes\ldots\otimes\widetilde{\mathbf{r}}_{i_n} = \Omega^{i_1\ldots i_n}(X^i,t)\mathbf{r}_{i_1}\otimes\ldots\otimes\mathbf{r}_{i_n},\end{aligned} \tag{4.18}$$

and choose arguments of components of the tensor ${}^n\boldsymbol{\Omega}$ as in formula (4.18).

Then, substituting the resolution (4.18) for the basis $\overset{\circ}{\mathbf{r}}_i$ into the definition (4.11), we get the expression (4.13), because $d\overset{\circ}{\mathbf{r}}_i/dt = 0$.

On substituting the resolution (4.18) for the basis $\bar{\mathbf{e}}_i$ into the relationship (4.12), we obtain

$${}^n\dot{\boldsymbol{\Omega}} = \frac{\partial\bar{\Omega}^{i_1\ldots i_n}}{\partial t}\bar{\mathbf{e}}_{i_1}\otimes\ldots\otimes\bar{\mathbf{e}}_{i_n} + \bar{v}^k\bar{\mathbf{e}}_k\cdot\frac{\partial}{\partial x^m}\bar{\Omega}^{i_1\ldots i_n}\bar{\mathbf{e}}^m\otimes\bar{\mathbf{e}}_{i_1}\otimes\ldots\otimes\bar{\mathbf{e}}_{i_n}. \tag{4.18a}$$

It is evident that formula (4.14) follows from (4.18a).

In a similar way, substituting the resolution (4.18) for the basis $\widetilde{\mathbf{r}}_i$ into the relation (4.12) and using the property (1.61) of nabla-operators $\boldsymbol{\nabla}$ and $\widetilde{\boldsymbol{\nabla}}$, and also the equation $\partial\widetilde{\mathbf{r}}_i/\partial t = 0$ (for the stationary basis $\widetilde{\mathbf{r}}_i$ see paragraph 1.1.7), we get

$${}^n\dot{\boldsymbol{\Omega}} = \frac{\partial\widetilde{\Omega}^{i_1\ldots i_n}}{\partial t}\widetilde{\mathbf{r}}_{i_1}\otimes\ldots\otimes\widetilde{\mathbf{r}}_{i_n} + \widetilde{v}^k\widetilde{\mathbf{r}}_k\cdot\widetilde{\nabla}_m\widetilde{\Omega}^{i_1\ldots i_n}\widetilde{\mathbf{r}}^m\otimes\widetilde{\mathbf{r}}_{i_1}\otimes\ldots\otimes\widetilde{\mathbf{r}}_{i_n}; \tag{4.19}$$

and formula (4.15) follows from (4.19) at once.

Finally, substituting the resolution (4.18) for the moving basis $\mathbf{r}_i$ into the definition of the total derivative (4.12), we obtain

$${}^n\dot{\boldsymbol{\Omega}} = \frac{\partial\Omega^{i_1\ldots i_n}}{\partial t}\Big|_{X^i}\mathbf{r}_{i_1}\otimes\ldots\otimes\mathbf{r}_{i_n} + \sum_{\alpha=1}^{n}\Omega^{i_1\ldots i_\alpha\ldots i_n}\mathbf{r}_{i_1}\otimes\ldots\frac{\partial\mathbf{r}_{i_\alpha}}{\partial t}\Big|_{X^i}\ldots\otimes\mathbf{r}_{i_n}. \tag{4.20}$$

Due to the definition (1.10) of local bases vectors and the definition (4.1) of the velocity, we have

$$\frac{\partial\mathbf{r}_{i_\alpha}}{\partial t}(X^j,t)\Big|_{X^j} = \frac{\partial^2\mathbf{x}}{\partial t\partial X^{i_\alpha}}\Big|_{X^j} = \frac{\partial\mathbf{v}}{\partial X^{i_\alpha}}\Big|_{X^j} = \nabla_{i_\alpha}v^k\mathbf{r}_k. \tag{4.21}$$

On substituting (4.21) into (4.20) and then collecting components at the same elements of the polyadic basis, we derive the formula (4.16). ▲

It should be noted that arguments of the resolutions (4.18) and of the derivatives of tensor components (4.13)–(4.16) have been chosen in the specific way.

1.4.3. Differential of a Tensor.

Definition 1.2. *For a tensor field ${}^n\boldsymbol{\Omega}(x^i,t)$, the following object*

$$d\,{}^n\boldsymbol{\Omega} = \frac{d\,{}^n\boldsymbol{\Omega}}{dt}dt \tag{4.22}$$

is called the d i f f e r e n t i a l *of a tensor field (or the differential of a tensor)* ${}^n\boldsymbol{\Omega}(x^i,t)$.

According to formula (4.12) for the total derivative of a tensor with respect to time, we get that the differential of a tensor can be written in the form

$$d\,{}^n\boldsymbol{\Omega}(x^i,t) = \Big(\frac{\partial\,{}^n\boldsymbol{\Omega}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\otimes{}^n\boldsymbol{\Omega}\Big)\,dt. \tag{4.23}$$

According to (4.10), the relation (4.23) takes the form

$$d\,{}^n\boldsymbol{\Omega} = \frac{\partial\,{}^n\boldsymbol{\Omega}}{\partial t}dt + d\mathbf{x}\cdot\boldsymbol{\nabla}\otimes{}^n\boldsymbol{\Omega} \tag{4.24}$$

When a tensor field is *stationary* (i.e. $\partial\,{}^n\boldsymbol{\Omega}/\partial t = 0$), the differential of the tensor field has the form

$$\widehat{d}\,{}^n\boldsymbol{\Omega} = d\mathbf{x}\cdot\boldsymbol{\nabla}\otimes{}^n\boldsymbol{\Omega}. \tag{4.25}$$

For stationary tensor fields $\widehat{d}\,{}^n\boldsymbol{\Omega} = d\,{}^n\boldsymbol{\Omega}$, but in general these differentials are not coincident.

According to Theorem 1.15, components of the tensor $d\,{}^n\boldsymbol{\Omega}$ with respect to the fixed basis $\mathring{\mathbf{r}}_i$ are written as follows:

$$d\,{}^n\boldsymbol{\Omega} = d\mathring{\Omega}^{j_1\ldots j_n}\mathring{\mathbf{r}}_{j_1}\otimes\ldots\otimes\mathring{\mathbf{r}}_{j_n}, \qquad d\mathring{\Omega}^{j_1\ldots j_n} = \frac{d\,\mathring{\Omega}^{j_1\ldots j_n}}{dt}dt. \tag{4.26}$$

From (4.22) and (4.7) we get the following expression for the *differential of a vector*:

$$d\mathbf{a}(X^i,t) = \frac{d\mathbf{a}}{dt}dt = \Big(\frac{\partial\mathbf{a}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\otimes\mathbf{a}\Big)\,dt, \tag{4.27}$$

and from (4.25) we have

$$d\widehat{\mathbf{a}} = (\mathbf{v}\cdot\boldsymbol{\nabla}\otimes\mathbf{a})dt = (\boldsymbol{\nabla}\otimes\mathbf{a})^{\mathrm{T}}\cdot d\mathbf{x}. \tag{4.28}$$

In particular, if $\mathbf{a} = \mathring{\mathbf{x}}$, then, by formulae (4.28) and (1.35a), we obtain

$$d\widehat{\mathring{\mathbf{x}}} = (\boldsymbol{\nabla}\otimes\mathring{\mathbf{x}})^{\mathrm{T}}\cdot d\mathbf{x} = \mathbf{F}^{-1}\cdot d\mathbf{x}, \tag{4.29}$$

or

$$d\mathbf{x} = \mathbf{F}\cdot d\widehat{\mathring{\mathbf{x}}}. \tag{4.30}$$

On comparing formulae (4.30) with (1.34), we find that the elementary radius-vector $d\mathring{\mathbf{x}}$, introduced in paragraph 1.1 and connecting two infinitesimally close material points $\mathcal{M}$ and $\mathcal{M}'$, coincides with the vector $d\widehat{\mathring{\mathbf{x}}}$ in the notation (4.25).

1.4.4. Properties of Derivatives with respect to Time. Let us establish now important properties of partial and total derivatives of vector fields with respect to time.

Theorem 1.16. *The partial derivative of the vector product of basis vectors with respect to time has the form*

$$\frac{\partial}{\partial t}(\mathbf{r}_\alpha \times \mathbf{r}_\beta) = \frac{\partial \mathbf{r}_\alpha}{\partial t} \times \mathbf{r}_\beta + \mathbf{r}_\alpha \times \frac{\partial \mathbf{r}_\beta}{\partial t}. \tag{4.31}$$

▼ Determine the derivative of the vector product of two local basis vectors with respect to time:

$$\frac{\partial}{\partial t}(\mathbf{r}_\alpha \times \mathbf{r}_\beta) = \frac{\partial}{\partial t}(Q^i{}_\alpha \bar{\mathbf{e}}_i \times Q^j{}_\beta \bar{\mathbf{e}}_j) = \frac{\partial}{\partial t}(Q^i{}_\alpha Q^j{}_\beta)\bar{\mathbf{e}}_i \times \bar{\mathbf{e}}_j =$$

$$= \frac{\partial Q^i{}_\alpha}{\partial t}\bar{\mathbf{e}}_i \times Q^j{}_\beta \bar{\mathbf{e}}_j + Q^i{}_\alpha \bar{\mathbf{e}}_i \times \frac{\partial Q^j{}_\beta}{\partial t}\bar{\mathbf{e}}_j.$$

With use of relation (1.10) we really get (4.31). ▲

Theorem 1.17. *For arbitrary continuously differentiable vector fields* $\mathbf{a}(\mathbf{x}, t) = \bar{a}^i(x^k, t)\bar{\mathbf{e}}_i$ *and* $\mathbf{b}(\mathbf{x}, t) = \bar{b}^i(x^k, t)\bar{\mathbf{e}}_i$, *we have the formulae*

$$\frac{\partial}{\partial t}(\mathbf{a} \times \mathbf{b}) = \frac{\partial \mathbf{a}}{\partial t} \times \mathbf{b} + \mathbf{a} \times \frac{\partial \mathbf{b}}{\partial t}, \tag{4.32}$$

$$\frac{\partial}{\partial t}(\mathbf{a} \otimes \mathbf{b}) = \frac{\partial \mathbf{a}}{\partial t} \otimes \mathbf{b} + \mathbf{a} \otimes \frac{\partial \mathbf{b}}{\partial t}, \tag{4.33}$$

$$\frac{\partial}{\partial t}(\mathbf{a} \cdot \mathbf{b}) = \frac{\partial \mathbf{a}}{\partial t} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{\partial \mathbf{b}}{\partial t}. \tag{4.34}$$

▼ A proof is similar to the proof of Theorem 1.16. ▲

Theorem 1.18. *The total derivatives of the vector and scalar products of two arbitrary vector fields* $\mathbf{a}(\mathbf{x}, t)$ *and* $\mathbf{b}(\mathbf{x}, t)$ *with respect to time have the forms*

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}, \tag{4.35}$$

$$\frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \frac{d\mathbf{a}}{dt} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\mathbf{b}}{dt}. \tag{4.36}$$

▼ To prove formula (4.35), one should use the property of the total derivative (4.7):

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \frac{\partial}{\partial t}(\mathbf{a} \times \mathbf{b}) + \mathbf{v} \cdot \boldsymbol{\nabla} \otimes (\mathbf{a} \times \mathbf{b}).$$

Modify the first summand by formula (4.32) and the second summand — by the formula $\boldsymbol{\nabla} \otimes (\mathbf{a} \times \mathbf{b}) = (\boldsymbol{\nabla} \otimes \mathbf{a}) \times \mathbf{b} - (\boldsymbol{\nabla} \otimes \mathbf{b}) \times \mathbf{a}$ [12], then we get

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \frac{\partial \mathbf{a}}{\partial t} \times \mathbf{b} - \frac{\partial \mathbf{b}}{\partial t} \times \mathbf{a} + \mathbf{v} \cdot (\boldsymbol{\nabla} \otimes \mathbf{a}) \times \mathbf{b} - \mathbf{v} \cdot (\boldsymbol{\nabla} \otimes \mathbf{b}) \times \mathbf{a}.$$

Collecting the first summand with the third one and the second summand with the fourth one, and using the property (4.7) of the total derivative of a vector, we obtain

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} - \frac{d\mathbf{b}}{dt} \times \mathbf{a} = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}.$$

Formula (4.36) can be proved in a similar way. ▲

Theorem 1.19. *The total derivative of $\sqrt{g}$ with respect to time is connected to the divergence of the velocity* $\mathbf{v}$ *by*

$$\frac{d}{dt}\sqrt{g} = \sqrt{g}\ \nabla_i v^i = \sqrt{g}\ \boldsymbol{\nabla}\cdot\mathbf{v}. \tag{4.37}$$

▼ Let us differentiate the second relation of (1.15) with taking formula (4.31) into account:

$$\frac{d}{dt}\sqrt{g} = \frac{d}{dt}\mathbf{r}_1\cdot(\mathbf{r}_2\times\mathbf{r}_3) = \frac{\partial^2\mathbf{x}}{\partial t\partial X^1}\cdot(\mathbf{r}_2\times\mathbf{r}_3)+ \\ +\mathbf{r}_1\cdot\left(\frac{\partial^2\mathbf{x}}{\partial t\partial X^2}\times\mathbf{r}_3\right)+\mathbf{r}_1\cdot\left(\mathbf{r}_2\times\frac{\partial^2\mathbf{x}}{\partial t\partial X^3}\right). \tag{4.38}$$

Since

$$\frac{\partial^2\mathbf{x}}{\partial t\partial X^i} = \frac{\partial\mathbf{v}}{\partial X^i} = \nabla_i\mathbf{v} = \nabla_i v^j\mathbf{r}_j,$$

we get

$$\frac{d}{dt}\sqrt{g} = \nabla_1\mathbf{v}\sqrt{g}\ \cdot\mathbf{r}^1 + \mathbf{r}_1\cdot(\nabla_2\mathbf{v}\times\mathbf{r}_3)+\mathbf{r}_1\cdot(\mathbf{r}_2\times\nabla_3\mathbf{v}). \tag{4.39}$$

Here we have used the relations from Exercise 1.1.14.

According to the definition of the vector product (0.2), we obtain

$$\mathbf{r}_1\cdot\nabla_2\mathbf{v}\times\mathbf{r}_3 = \mathbf{r}_1\cdot\sqrt{g}\ \epsilon_{ijk}\nabla_2 v^i\delta_3^j\mathbf{r}^k = \sqrt{g}\ \epsilon_{i31}\nabla_2 v^i = \sqrt{g}\ \nabla_2 v^2. \tag{4.40}$$

On substituting (4.40) into (4.39), we really get formula (4.37). ▲

1.4.5. The Velocity Gradient, the Deformation Rate Tensor and the Vorticity Tensor. Consider elementary radius-vectors $d\overset{\circ}{\mathbf{x}}$ and $d\mathbf{x}$ connecting two infinitesimally close points $\mathcal{M}$ and $\mathcal{M}'$ in configurations $\overset{\circ}{\mathcal{K}}$ and $\mathcal{K}$, respectively. Determine the velocity of the point $\mathcal{M}'$ relative to the configuration connected to the point $\mathcal{M}$. To do this, determine the velocity differential $\widehat{d}\mathbf{v}$:

$$\widehat{d}\mathbf{v} = \frac{\partial}{\partial t}d\mathbf{x} = \frac{\partial^2\mathbf{x}}{\partial X^i\partial t}dX^i = \frac{\partial^2\mathbf{x}}{\partial X^i\partial t}\otimes\overset{\circ}{\mathbf{r}}{}^i\cdot d\overset{\circ}{\mathbf{x}} = \left(\overset{\circ}{\mathbf{r}}{}^i\otimes\frac{\partial\mathbf{v}}{\partial X^i}\right)^{\mathrm{T}}\cdot d\overset{\circ}{\mathbf{x}} = \left(\overset{\circ}{\boldsymbol{\nabla}}\otimes\mathbf{v}\right)^{\mathrm{T}}\cdot d\overset{\circ}{\mathbf{x}}. \tag{4.41}$$

Here we have used the second equation of (1.33), the definition of the gradient (1.24) and formula (4.1). In a similar way, using the first equation of (1.33): $dX^i = \mathbf{r}^i\cdot d\mathbf{x}$, we get one more expression for the vector $\widehat{d}\mathbf{v}$:

$$\widehat{d}\mathbf{v} = (\boldsymbol{\nabla}\otimes\mathbf{v})^{\mathrm{T}}\cdot d\mathbf{x}. \tag{4.42}$$

The second-order tensor $(\boldsymbol{\nabla}\otimes\mathbf{v})^{\mathrm{T}}$ is called the *velocity gradient*, which connects the relative velocity $\widehat{d}\mathbf{v}$ of an elementary radius-vector $d\mathbf{x}$ to the vector $d\mathbf{x}$ itself:

$$\widehat{d}\mathbf{v} = \mathbf{L}\cdot d\mathbf{x}, \qquad \mathbf{L} = (\boldsymbol{\nabla}\otimes\mathbf{v})^{\mathrm{T}}. \tag{4.43}$$

Just as any second-order tensor (see [12]), the tensor $\mathbf{L}$ can be represented by a sum of the symmetric tensor $\mathbf{D}$ and the skew-symmetric tensor $\mathbf{W}$:

$$\mathbf{L} = \mathbf{D} + \mathbf{W}. \tag{4.44}$$

The symmetric *deformation rate tensor* $\mathbf{D}$ is determined as follows:

$$\mathbf{D} = \frac{1}{2}(\boldsymbol{\nabla} \otimes \mathbf{v} + \boldsymbol{\nabla} \otimes \mathbf{v}^{\mathrm{T}}). \tag{4.45}$$

This tensor has six independent components.

The skew-symmetric *vorticity tensor* $\mathbf{W}$ is determined as follows:

$$\mathbf{W} = \frac{1}{2}(\boldsymbol{\nabla} \otimes \mathbf{v}^{\mathrm{T}} - \boldsymbol{\nabla} \otimes \mathbf{v}). \tag{4.46}$$

Since the tensor $\mathbf{W}$ is skew-symmetric and has three independent components, we can put the tensor $\mathbf{W}$ in correspondence with the *vorticity vector* $\boldsymbol{\omega}$ connected to the tensor (see [12]) as follows:

$$\boldsymbol{\omega} = \frac{1}{2}\mathbf{W} \cdot\cdot\, \boldsymbol{\epsilon}, \qquad \mathbf{W} = \boldsymbol{\omega} \times \mathbf{E}. \tag{4.47}$$

where $\boldsymbol{\epsilon}$ is the *Levi-Civita tensor*, which has the third order (see [12]). This tensor is determined as follows:

$$\boldsymbol{\epsilon} = \frac{1}{\sqrt{g}}\epsilon^{ijk}\mathbf{r}_i \otimes \mathbf{r}_j \otimes \mathbf{r}_k. \tag{4.48}$$

On substituting (4.44)–(4.47) into (4.42), we prove the following theorem.

Theorem 1.20 (Cauchy–Helmholtz). *The velocity $\mathbf{v}(\mathcal{M}')$ of an arbitrary point $\mathcal{M}'$ in a neighborhood of the material point $\mathcal{M}$ consists of the translational motion velocity $\mathbf{v}(\mathcal{M})$ of the point $\mathcal{M}$, the velocity $\boldsymbol{\omega} \times d\mathbf{x}$ of rotation as a rigid whole and the deformation rate $\mathbf{D} \cdot d\mathbf{x}$, i.e.*

$$\widehat{d}\mathbf{v} = \boldsymbol{\omega} \times d\mathbf{x} + \mathbf{D} \cdot d\mathbf{x} \tag{4.49}$$

or

$$\mathbf{v}(\mathcal{M}') = \mathbf{v}(\mathcal{M}) + \boldsymbol{\omega} \times d\mathbf{x} + \mathbf{D} \cdot d\mathbf{x} + o\,(|d\mathbf{x}|). \tag{4.49a}$$

Example 1.4. Determine the tensor $\mathbf{L}$ for the problem on tension of a beam (see Example 1.1), substituting (4.2) into (4.43):

$$\mathbf{L}^{\mathrm{T}} = \bar{\mathbf{e}}^i \frac{\partial}{\partial X^i} \otimes \mathbf{v} = \bar{\mathbf{e}}^i \otimes \frac{\partial}{\partial X^i} \sum_{\alpha=1}^{3} \dot{k}_\alpha X^\alpha \bar{\mathbf{e}}_\alpha = \sum_{\alpha=1}^{3} \dot{k}_\alpha \bar{\mathbf{e}}_\alpha \otimes \bar{\mathbf{e}}_\alpha = \mathbf{L}.$$

Since the velocity gradient $\mathbf{L}$ in this case proves to be a symmetric tensor, from (4.45) and (4.46) it follows that

$$\mathbf{D} = \mathbf{L}, \qquad \mathbf{W} = \mathbf{0}.$$

Thus, in this case $\boldsymbol{\omega} = \mathbf{0}$. $\square$

Example 1.5. Determine the tensor $\mathbf{L}$ for the problem on simple shear (see Example 1.2), substituting formula (4.2) into (4.43):

$$\mathbf{L}^{\mathrm{T}} = \bar{\mathbf{e}}^i \otimes \frac{\partial \mathbf{v}}{\partial X^i} = \frac{\partial \bar{v}^j}{\partial X^i}\bar{\mathbf{e}}^i \otimes \bar{\mathbf{e}}_j = \frac{\partial \bar{v}^1}{\partial X^2}\bar{\mathbf{e}}_2 \otimes \bar{\mathbf{e}}_1 = \dot{a}\bar{\mathbf{e}}_2 \otimes \bar{\mathbf{e}}_1.$$

According to formulae (4.45) and (4.46), we get

$$\mathbf{D} = (\dot{a}/2)(\bar{\mathbf{e}}_1 \otimes \bar{\mathbf{e}}_2 + \bar{\mathbf{e}}_2 \otimes \bar{\mathbf{e}}_1),$$

$$\mathbf{W} = (\dot{a}/2)(\bar{\mathbf{e}}_1 \otimes \bar{\mathbf{e}}_2 - \bar{\mathbf{e}}_2 \otimes \bar{\mathbf{e}}_1) = (\dot{a}/2)(\delta_1^i \delta_2^j - \delta_2^i \delta_1^j)\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j.$$

Using formula (4.47), we determine the vorticity vector

$$\boldsymbol{\omega} = \frac{1}{2}\mathbf{W} \cdot\cdot\, \boldsymbol{\epsilon} = \frac{\dot{a}}{4}(\delta_1^i \delta_2^j - \delta_2^i \delta_1^j)\epsilon_{jik}\bar{\mathbf{e}}^k = \frac{\dot{a}}{4}(\epsilon_{21k} - \epsilon_{12k})\bar{\mathbf{e}}^k = -\frac{\dot{a}}{2}\bar{\mathbf{e}}^3,$$

which is orthogonal to the shear plane. □

1.4.6. Eigenvalues of the Deformation Rate Tensor. Just as any symmetric tensor, the deformation rate tensor $\mathbf{D}$ has three orthonormal real-valued eigenvectors and three real positive eigenvalues (see [12]). Denote the eigenvectors by $\mathbf{q}_\alpha$ (these vectors, in general, are not coincident with $\mathbf{p}_\alpha$) and the eigenvalues — by D_α. Then the tensor $\mathbf{D}$ can be resolved for its dyadic eigenbasis as follows:

$$\mathbf{D} = \sum_{\alpha=1}^{3} D_\alpha \mathbf{q}_\alpha \otimes \mathbf{q}_\alpha, \qquad \mathbf{q}_\alpha \cdot \mathbf{q}_\beta = \delta_{\alpha\beta}. \tag{4.50}$$

Take in the actual configuration $\mathcal{K}$ an elementary radius-vector $d\mathbf{x}_\alpha$, connecting points $\mathcal{M}$ and $\mathcal{M}'$, so that the vector is oriented along the eigenvector $\mathbf{q}_\alpha$ of the tensor $\mathbf{D}$; then, similarly to (3.36), we can write

$$d\mathbf{x}_\alpha = \mathbf{q}_\alpha |d\mathbf{x}_\alpha|, \qquad |d\mathbf{x}_\alpha| = (d\mathbf{x}_\alpha \cdot d\mathbf{x}_\alpha)^{1/2}. \tag{4.51}$$

Apply the Cauchy–Helmholtz theorem (4.49) to the elementary radius-vector:

$$\widehat{d}\mathbf{v}_\alpha = \boldsymbol{\omega} \times d\mathbf{x}_\alpha + \mathbf{D} \cdot d\mathbf{x}_\alpha. \tag{4.52}$$

Multiplying the left and right sides of the equation by $d\mathbf{x}_\alpha$ and taking account of the property of the mixed derivative $d\mathbf{x}_\alpha \cdot (\boldsymbol{\omega} \times d\mathbf{x}_\alpha) = 0$, we get

$$d\mathbf{x}_\alpha \cdot \widehat{d}\mathbf{v}_\alpha = d\mathbf{x}_\alpha \cdot \mathbf{D} \cdot d\mathbf{x}_\alpha. \tag{4.53}$$

Substituting in place of $\mathbf{D}$ its expression (4.50) and in place of $d\mathbf{x}_\alpha$ their expressions (4.51), we obtain

$$d\mathbf{x}_\alpha \cdot \widehat{d}\mathbf{v}_\alpha = |d\mathbf{x}_\alpha|^2 \sum_{\beta=1}^{3} D_\beta \mathbf{q}_\alpha \cdot \mathbf{q}_\beta \otimes \mathbf{q}_\beta \cdot \mathbf{q}_\alpha = D_\alpha |\mathbf{q}_\alpha|^2. \tag{4.54}$$

Here we have used the property (4.50) of orthonormal vectors $\mathbf{q}_\alpha$.

Modify the scalar product on the left-hand side as follows:

$$d\mathbf{x}_\alpha \cdot \widehat{d}\mathbf{v}_\alpha = d\mathbf{x}_\alpha \cdot \frac{\partial}{\partial t} d\mathbf{x}_\alpha = \frac{1}{2}\frac{\partial}{\partial t}(d\mathbf{x}_\alpha \cdot d\mathbf{x}_\alpha) = |d\mathbf{x}_\alpha| \frac{\partial}{\partial t}|d\mathbf{x}_\alpha|. \tag{4.55}$$

On comparing (4.54) with (4.55), we obtain the following theorem.

Theorem 1.21. *Eigenvalues D_α of the deformation rate tensor $\mathbf{D}$ are the rates of relative elongations of elementary material fibres oriented along the eigenvectors $\mathbf{q}_\alpha$:*

$$D_\alpha = \frac{1}{|d\mathbf{x}_\alpha|}\frac{\partial}{\partial t}|d\mathbf{x}_\alpha|. \tag{4.56}$$

1.4.7. Resolution of the Vorticity Tensor for the Eigenbasis of the Deformation Rate Tensor. Modify the right-hand side of (4.52) as follows:

$$\widehat{d}\mathbf{v}_\alpha = \boldsymbol{\omega} \times d\mathbf{x}_\alpha + \mathbf{D} \cdot d\mathbf{x}_\alpha = (\boldsymbol{\omega} \times \mathbf{q}_\alpha + D_\alpha \mathbf{q}_\alpha)|d\mathbf{x}_\alpha|, \tag{4.57}$$

and the left-hand side of (4.52) with taking (4.56) into account:

$$\widehat{d}\mathbf{v}_\alpha = \frac{\partial}{\partial t} d\mathbf{x}_\alpha = \frac{\partial}{\partial t}(|d\mathbf{x}_\alpha|\mathbf{q}_\alpha) = \frac{\partial |d\mathbf{x}_\alpha|}{\partial t}\mathbf{q}_\alpha + |d\mathbf{x}_\alpha| \frac{\partial \mathbf{q}_\alpha}{\partial t} = |d\mathbf{x}_\alpha| \left(D_\alpha \mathbf{q}_\alpha + \frac{\partial \mathbf{q}_\alpha}{\partial t} \right). \tag{4.58}$$

On comparing (4.57) with (4.58), we get the following theorem.

Theorem 1.22. *The vorticity tensor* $\mathbf{W}$ *(or the vorticity vector* $\boldsymbol{\omega}$*) connects the rate of changing the eigenvectors* $\mathbf{q}_\alpha$ *to the vectors* $\mathbf{q}_\alpha$ *themselves:*

$$\dot{\mathbf{q}}_\alpha = \frac{\partial \mathbf{q}_\alpha}{\partial t} = \boldsymbol{\omega} \times \mathbf{q}_\alpha = \mathbf{W} \cdot \mathbf{q}_\alpha. \tag{4.59}$$

Using formula (4.59), we can resolve the tensor $\mathbf{W}$ for the eigenbasis $\mathbf{q}_\alpha$ of the deformation rate tensor as follows:

$$\mathbf{W} = \sum_{\alpha=1}^{3} \dot{\mathbf{q}}_\alpha \otimes \mathbf{q}_\alpha = \dot{\mathbf{q}}_i \otimes \mathbf{q}^i. \tag{4.60}$$

1.4.8. Geometric Picture of Infinitesimal Transformation of a Small Neighborhood of a Point. If in configuration $\mathcal{K}$ at time t we consider an elementary radius-vector $d\mathbf{x}$ connecting two infinitesimally close material points $\mathcal{M}$ and $\mathcal{M}'$, then for infinitesimal time dt the radius-vector is transformed into radius-vector $d\mathbf{x}'$ in configuration $\mathcal{K}(t+dt)$ (Figure 1.13):

$$d\mathbf{x} = \mathbf{x}'(t) - \mathbf{x}(t), \qquad d\mathbf{x}' = \mathbf{x}'(t+dt) - \mathbf{x}(t+dt), \tag{4.61}$$

where $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ are radius-vectors of the points $\mathcal{M}$ and $\mathcal{M}'$ in configuration $\mathcal{K}(t)$, respectively; and $\mathbf{x}(t+dt)$ and $\mathbf{x}'(t+dt)$ — in configuration $\mathcal{K}(t+dt)$. Displacements of points $\mathcal{M}$ and $\mathcal{M}'$ for infinitesimal time are defined by the velocity vectors $\mathbf{v}(\mathcal{M})$ and $\mathbf{v}(\mathcal{M}')$, respectively:

$$\mathbf{x}(t+dt) - \mathbf{x}(t) = \mathbf{v}(\mathcal{M})\, dt, \qquad \mathbf{x}'(t+dt) - \mathbf{x}'(t) = \mathbf{v}(\mathcal{M}')\, dt. \tag{4.62}$$

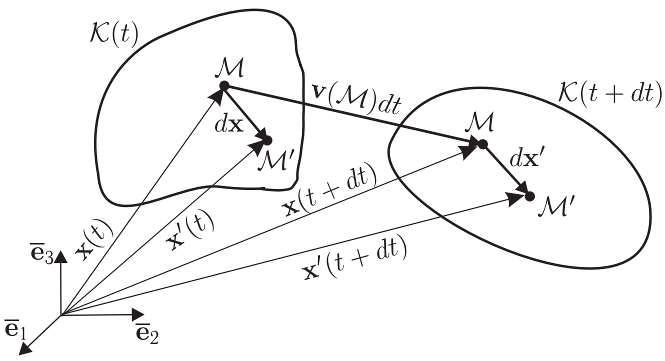

Figure 1.13. Infinitesimal transformation of an elementary radius-vector

Formulae (4.61) and (4.62) and simple geometric relations (see Figure 1.13) give

$$\mathbf{v}(\mathcal{M}')dt - \mathbf{v}(\mathcal{M})dt = d\mathbf{x}' - d\mathbf{x}. \tag{4.63}$$

On substituting (4.63) into (4.49a), we obtain the relation between elementary radius-vectors $d\mathbf{x}'$ and $d\mathbf{x}$:

$$d\mathbf{x}' = d\mathbf{x} + dt\boldsymbol{\omega} \times d\mathbf{x} + dt\mathbf{D} \cdot d\mathbf{x} + dt\ o(|d\mathbf{x}|). \tag{4.64}$$

The relation (4.64) can be considered as the transformation of coordinates $dx^i \to dx'^i$ in a small neighborhood of the point contained in a continuum. Since $dt\boldsymbol{\omega}$ and $dt\mathbf{D}$ are independent of $d\mathbf{x}$ and $d\mathbf{x}'$, so the transformation is linear, i.e. affine. The relation (4.64) can be represented as a superposition of two transformations up to an accuracy of $o(|d\mathbf{x}|)$:

$$d\mathbf{x}'' = \mathbf{A}_D \cdot d\mathbf{x}, \qquad \mathbf{A}_D = \mathbf{E} + dt\mathbf{D}, \tag{4.65}$$

$$d\mathbf{x}' = \mathbf{Q}_\omega \cdot d\mathbf{x}'', \qquad \mathbf{Q}_\omega = \mathbf{E} + dt\boldsymbol{\omega} \times \mathbf{E}. \tag{4.66}$$

The tensor $\mathbf{A}_D$ is symmetric and has three eigendirections, which are coincident with the eigendirections $\mathbf{q}_\alpha$ of the deformation rate tensor $\mathbf{D}$.

So just as the tensor $\mathbf{U}$, the tensor $\mathbf{A}_D$ transforms a small neighborhood of a point $\mathcal{M}$ by extending or compressing the neighborhood along the principal directions $\mathbf{q}_\alpha$. The material segments $|d\mathbf{x}''_\alpha|$ oriented along the eigendirections $\mathbf{q}_\alpha$ retain their orientation under the transformations (4.65), but their lengths vary as follows:

$$d\mathbf{x}_\alpha = |d\mathbf{x}_\alpha|\mathbf{q}_\alpha, \qquad d\mathbf{x}''_\alpha = (1 + D_\alpha dt|d\mathbf{x}_\alpha|)\mathbf{q}_\alpha = (1 + D_\alpha dt)d\mathbf{x}_\alpha.$$

The tensor $\mathbf{Q}_\omega$ (4.66) is orthogonal up to an accuracy of values $\sim (dt)^2$, because

$$\mathbf{Q}_\omega \cdot \mathbf{Q}_\omega^{\mathrm{T}} = (\mathbf{E} + dt\mathbf{W}) \cdot (\mathbf{E} + dt\mathbf{W}^{\mathrm{T}}) = \mathbf{E} - (dt)^2\mathbf{W}^2. \tag{4.67}$$

Here we have taken into account that the vorticity tensor $\mathbf{W}$ is skew-symmetric.

Thus, the transformation (4.66) determined by the tensor $\mathbf{Q}_\omega$ is rotation of the $\mathcal{M}$-point neighborhood as a rigid whole for infinitesimal time dt.

The vorticity vector $\boldsymbol{\omega}$ forming the tensor $\mathbf{Q}_\omega$ can be considered as instantaneous angular rate of rotation of the small neighborhood as a rigid whole, or as instantaneous angular rate of rotation of the eigentrihedron $\mathbf{q}_\alpha$ of the deformation rate tensor relative to the fixed basis $\bar{\mathbf{e}}_i$. This fact will be considered in detail in paragraph 1.5.7.

On uniting the properties of the transformations (4.65) and (4.66), we can make the following conclusion.

Theorem 1.23. *The infinitesimal transformation of a small neighborhood of the point contained in a continuum is a superposition of tension-compression of the neighborhood along the eigendirections $\mathbf{q}_\alpha$ and rotation of the axes $\mathbf{q}_\alpha$ as a rigid whole about the axis with the direction vector $\boldsymbol{\omega}$.*

Thus, we have the certain analogy between the eigendirections $\overset{\circ}{\mathbf{p}}_\alpha$ of the tensor $\mathbf{U}$ and the directions $\mathbf{q}_\alpha$ of the tensor $\mathbf{D}$: elementary material fibres oriented along $\overset{\circ}{\mathbf{p}}_\alpha$ and along $\mathbf{q}_\alpha$ remain mutually orthogonal and undergo only tension-compression. The axes $\overset{\circ}{\mathbf{p}}_\alpha$ remain mutually orthogonal under any finite transformations from $\overset{\circ}{\mathcal{K}}$ to $\mathcal{K}$, but $\mathbf{q}_\alpha$ — only under infinitesimal transformations from $\mathcal{K}(t)$ to $\mathcal{K}(t+dt)$.

1.4.9. Kinematic Meaning of the Vorticity Vector. Just as any orthogonal tensor, the orthogonal tensor $\mathbf{Q}_\omega$ of infinitesimal rotation from $\mathcal{K}(t)$ to $\mathcal{K}(t+dt)$ can be represented in the form (see [12])

$$\mathbf{Q}_\omega = \mathbf{E}\cos(d\varphi) + (1-\cos(d\varphi))\mathbf{e}\otimes\mathbf{e} - \mathbf{e}\times\mathbf{E}\sin(d\varphi), \tag{4.68}$$

where $d\varphi$ is the infinitesimal angle of rotation of the trihedron $\mathbf{q}_\alpha$ about the axis with the direction vector $\mathbf{e}$. Since values of $d\varphi$ are infinitesimal, we have

$$\mathbf{Q}_\omega = \mathbf{E} - \mathbf{e}\times\mathbf{E}d\varphi. \tag{4.69}$$

On comparing (4.69) with (4.66), we get

$$\boldsymbol{\omega} = -\frac{d\varphi}{dt}\mathbf{e}, \qquad |\boldsymbol{\omega}| = \frac{d\varphi}{dt}, \tag{4.70}$$

i.e. the vorticity vector $\boldsymbol{\omega}$ is really oriented along the instantaneous rotation axis $\mathbf{e}$, and the length $|\boldsymbol{\omega}|$ is equal to the instantaneous angular rate of rotation of the trihedron $\mathbf{q}_\alpha$ of the deformation rate tensor.

Let us consider now the question: relative to what system the vorticity vector $\boldsymbol{\omega}$ defines the rotation rate.

To answer the question, we introduce another orthogonal rotation tensor

$$\mathbf{O}_W = \mathbf{q}_i\otimes\bar{\mathbf{e}}_i, \tag{4.71}$$

which transforms the Cartesian trihedron $\bar{\mathbf{e}}_i$ as a rigid whole into the orthonormal trihedron $\mathbf{q}_i$:

$$\mathbf{q}_i = \mathbf{O}_W\cdot\bar{\mathbf{e}}_i. \tag{4.72}$$

The tensor $\mathbf{O}_W$ is a function of time t, because $\mathbf{q}_i = \mathbf{q}_i(t)$.

According to (4.60) and (4.72), the vorticity tensor $\mathbf{W}$ takes the form

$$\mathbf{W} = \dot{\mathbf{q}}_i\otimes\mathbf{q}_i = \dot{\mathbf{O}}_W\cdot\mathbf{O}_W^{\mathrm{T}}. \tag{4.73}$$

With the help of (4.73) we can represented the orthogonal tensor $\mathbf{Q}_\omega$ as follows:

$$\mathbf{Q}_\omega = \mathbf{E} + dt\mathbf{W} = \mathbf{E} + dt\dot{\mathbf{O}}_W\cdot\mathbf{O}_W^{\mathrm{T}}. \tag{4.74}$$

Thus, at each time t two orthogonal tensors $\mathbf{O}_W$ and $\mathbf{Q}_\omega$ connect local neighborhoods of a point $\mathcal{M}$ in the reference $\overset{\circ}{\mathcal{K}}$ and actual configurations $\mathcal{K}(t+dt)$. If in $\mathcal{K}$ we consider an elementary radius-vector $d\mathbf{x}''$, then in $\overset{\circ}{\mathcal{K}}$ we find its corresponding radius-vector $d\overset{\circ}{\mathbf{x}}{}''$ obtained with the help of the rotation tensor $\mathbf{O}_W$, and in $\mathcal{K}(t+dt)$ — radius-vector $d\mathbf{x}'$:

$$d\mathbf{x}'' = \mathbf{O}_W\cdot d\overset{\circ}{\mathbf{x}}{}'', \qquad d\mathbf{x}' = \mathbf{Q}_\omega\cdot d\mathbf{x}''.$$

A fixed observer connected to the Cartesian trihedron $\bar{\mathbf{e}}_i$ sees both the transformations: finite rotation for time t, which is described by the tensor $\mathbf{O}_W$, and instantaneous rotation of a local neighborhood for time dt, which is described by the infinitesimal rotation tensor $\mathbf{Q}_\omega$

Thus, the vorticity vector $\boldsymbol{\omega}$ is the vector of instantaneous angular rate of rotation of the trihedron $\mathbf{q}_\alpha$ relative to the trihedron $\bar{\mathbf{e}}_i$.

Comparing (4.66) with (4.74) (or (4.73) with (4.47)), we get

$$\dot{\mathbf{O}}_W \cdot \mathbf{O}_W^{\mathrm{T}} = \boldsymbol{\omega} \times \mathbf{E}. \tag{4.75}$$

1.4.10. Tensor of Angular Rate of Rotation (Spin). In paragraph 1.3 we have introduced the tensor $\dot{\mathbf{O}}_W \cdot \mathbf{O}_W^{\mathrm{T}}$, where $\mathbf{O}_W$ is the orthogonal tensor of rotation. Such tensor $\boldsymbol{\Omega} = \dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}}$ can be set up for any orthogonal tensor $\mathbf{Q}$ depending on time t.

The tensor $\dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}}$ is skew-symmetric, because

$$\dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}} + \mathbf{Q} \cdot \dot{\mathbf{Q}}^{\mathrm{T}} = (\mathbf{Q} \cdot \mathbf{Q}^{\mathrm{T}})^{\bullet} = (\mathbf{E})^{\bullet} = 0, \tag{4.76}$$

i.e.

$$\boldsymbol{\Omega}^{\mathrm{T}} = (\dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{Q} \cdot \dot{\mathbf{Q}}^{\mathrm{T}} = -\dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}} = -\boldsymbol{\Omega}. \tag{4.77}$$

. This tensor characterizes the angular rate of rotation of the orthonormal trihedron $\mathbf{h}_i$ formed with the help of $\mathbf{Q}$:

$$\mathbf{h}_i = \mathbf{Q} \cdot \bar{\mathbf{e}}_i, \tag{4.78}$$

relative to the Cartesian trihedron $\bar{\mathbf{e}}_i$.

Indeed, with the help of the tensor $\dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}}$ we can form the tensor (4.74):

$$\mathbf{Q}_\omega = \mathbf{E} + dt\dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}}, \tag{4.79}$$

which, according to (4.67), is the orthogonal tensor of infinitesimal rotation; and this tensor can be represented in the form (4.68) or (4.69):

$$\mathbf{Q}_\omega = \mathbf{E} - d\varphi \mathbf{e} \times \mathbf{E}, \tag{4.80}$$

where $d\varphi$ is the infinitesimal angle of rotation of the trihedron $\mathbf{h}_i$ about the axis with the direction vector $\mathbf{e}$. Comparing (4.79) with (4.80), we get the expression

$$\dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}} = -\frac{d\varphi}{dt}\mathbf{e} \times \mathbf{E}, \tag{4.81}$$

which makes clear the sentence that the tensor $\dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}}$ characterizes the instantaneous angular rate $d\varphi/dt$ of rotation of the trihedron $\mathbf{h}_i$ about the axis $\mathbf{e}$.

The tensor $\dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}}$ is called the *tensor of angular rate of rotation* or the *spin*.

Expressing the tensor $\mathbf{Q}$ from (4.78) in terms of the bases $\mathbf{h}_i$ and $\bar{\mathbf{e}}_i$ ($\mathbf{h}_\alpha = \mathbf{h}^\alpha$, $\alpha = 1, 2, 3$, as the vectors are orthonormal):

$$\mathbf{Q} = \mathbf{h}^i \otimes \bar{\mathbf{e}}_i, \tag{4.82}$$

we get another representation of the spin:

$$\dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}} = \dot{\mathbf{h}}_i \otimes \mathbf{h}^i. \tag{4.83}$$

Thus, we have proved the following theorem.

Theorem 1.24. *The spin connects the rates $\dot{\mathbf{h}}_i$ and the vectors $\mathbf{h}_i$ defined by formula (4.78) as follows:*

$$\dot{\mathbf{h}}_i = (\dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}}) \cdot \mathbf{h}_i. \tag{4.84}$$

Since the spin $\dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}}$ is a skew-symmetric tensor, we can introduce the corresponding vorticity vector $\boldsymbol{\omega}_h$:

$$\dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}} = \boldsymbol{\omega}_h \times \mathbf{E} \tag{4.85}$$

(from formulae (4.81) and (4.85) it follows that $\boldsymbol{\omega}_h = -(d\varphi/dt)\mathbf{e}$); then formula (4.84) takes the form

$$\dot{\mathbf{h}}_i = \boldsymbol{\omega}_h \times \mathbf{h}_i. \tag{4.86}$$

Resolving the vector $\boldsymbol{\omega}_h$ for the orthonormal basis $\mathbf{h}_i$: $\boldsymbol{\omega}_h = \omega^j{}_h \mathbf{h}_j$, we get one more representation of formula (4.84):

$$\dot{\mathbf{h}}_i = \omega^j{}_h \mathbf{h}_j \times \mathbf{h}_i = \epsilon_{jik} \omega^j{}_h \mathbf{h}^k. \tag{4.87}$$

This formula can also be rewritten in the form

$$\dot{\mathbf{h}}_\alpha = \omega_h^\beta \mathbf{h}_\gamma, \qquad \alpha \neq \beta \neq \gamma \neq \alpha. \tag{4.88}$$

Taking different orthogonal tensors (or orthonormal bases) as $\mathbf{Q}$ (or $\mathbf{h}_i$), we obtain different spins.

1) If we choose eigenvectors of the stretch tensor $\mathbf{U}$ as $\mathbf{h}_i$, i.e. $\mathbf{h}_i = \mathring{\mathbf{p}}_i$, then, according to (4.83), the corresponding spin $\boldsymbol{\Omega}_U$ takes the form

$$\boldsymbol{\Omega}_U = \dot{\mathbf{O}}_U \cdot \mathbf{O}_U^{\mathrm{T}} = \mathring{\mathbf{p}}_i^{\bullet} \otimes \mathring{\mathbf{p}}^i, \qquad \mathbf{O}_U = \mathring{\mathbf{p}}^i \otimes \bar{\mathbf{e}}_i, \tag{4.89}$$

and formulae (4.84) yields

$$\dot{\mathring{\mathbf{p}}}_i = \boldsymbol{\Omega}_U \cdot \mathring{\mathbf{p}}_i. \tag{4.90}$$

2) If $\mathbf{h}_i = \mathbf{p}_i$, then the corresponding spin $\boldsymbol{\Omega}_V$ and the rotation tensor $\mathbf{O}_V$ have the forms

$$\boldsymbol{\Omega}_V = \dot{\mathbf{O}}_V \cdot \mathbf{O}_V^{\mathrm{T}} = \dot{\mathbf{p}}_i \otimes \mathbf{p}^i, \qquad \mathbf{O}_V = \mathbf{p}_i \otimes \bar{\mathbf{e}}_i, \tag{4.91}$$

$$\dot{\mathbf{p}}_i = \boldsymbol{\Omega}_V \cdot \mathbf{p}_i. \tag{4.92}$$

3) If $\mathbf{h}_i = \mathbf{q}_i$, then the corresponding spin $\boldsymbol{\Omega}_W$ coincides with the vorticity tensor $\mathbf{W}$ (see (4.73)):

$$\boldsymbol{\Omega}_W = \dot{\mathbf{O}}_W \cdot \mathbf{O}_W^{\mathrm{T}} = \dot{\mathbf{q}}_i \otimes \mathbf{q}_i = \mathbf{W}, \tag{4.93}$$

$$\dot{\mathbf{q}}_i = \boldsymbol{\Omega}_W \cdot \mathbf{q}_i. \tag{4.94}$$

4) If we take the rotation tensor $\mathbf{O}$ accompanying deformation as $\mathbf{Q}$, then, as shown in (3.23a), the tensor $\mathbf{O}$ connects two moving bases $\mathbf{p}_i$ and $\mathring{\mathbf{p}}_i$:

$$\mathbf{p}_i = \mathbf{O} \cdot \mathring{\mathbf{p}}_i. \tag{4.95}$$

The tensor $\mathbf{O}$ can be expressed in terms of $\mathbf{O}_V$ and $\mathbf{O}_U$ as follows:

$$\mathbf{O} = \mathbf{p}_i \otimes \mathring{\mathbf{p}}_i = \mathbf{p}_i \otimes \bar{\mathbf{e}}_i \cdot \bar{\mathbf{e}}_j \otimes \mathring{\mathbf{p}}_j = \mathbf{O}_V \cdot \mathbf{O}_U^{\mathrm{T}}. \tag{4.96}$$

The corresponding spin $\boldsymbol{\Omega}$ has the form

$$\begin{aligned} \boldsymbol{\Omega} = \dot{\mathbf{O}} \cdot \mathbf{O}^{\mathrm{T}} &= (\dot{\mathbf{O}}_V \cdot \mathbf{O}_U^{\mathrm{T}} + \mathbf{O}_V \cdot \dot{\mathbf{O}}_U^{\mathrm{T}}) \cdot \mathbf{O}_U \cdot \mathbf{O}_V^{\mathrm{T}} = \\ &= \dot{\mathbf{O}}_V \cdot \mathbf{O}_V^{\mathrm{T}} + \mathbf{O}_V \cdot \dot{\mathbf{O}}_U^{\mathrm{T}} \cdot \mathbf{O}_U \cdot \mathbf{O}_V^{\mathrm{T}} = \boldsymbol{\Omega}_V - \mathbf{O}_V \cdot \boldsymbol{\Omega}_U \cdot \mathbf{O}_V^{\mathrm{T}}. \end{aligned} \tag{4.97}$$

Unlike the cases 1)–3), the spin tensor $\mathbf{\Omega}$ characterizes the angular rate of rotation of the trihedron $\mathbf{p}_i$ relative to the moving trihedron $\overset{\circ}{\mathbf{p}}_i$, but not relative to the trihedron $\bar{\mathbf{e}}_i$ being fixed.

Therefore, for the cases 1)–3) the spins characterize the total angular rate, and for the case 4) – the relative rate.

1.4.11. Relationships between Rates of Deformation Tensors and Velocity Gradients. In continuum mechanics, one often needs the relations between rates of the deformation tensors (and also measures) and the velocity gradients $\mathbf{L} = (\nabla \otimes \mathbf{v})^{\mathrm{T}}$ and

$$\overset{\circ}{\mathbf{L}} = (\overset{\circ}{\nabla} \otimes \mathbf{v})^{\mathrm{T}}. \tag{4.98}$$

Let us derive these relations.

Theorem 1.25. *The rates of varying the gradient* $\dot{\mathbf{F}}$ *and the inverse gradient* $(\mathbf{F}^{-1})^{\bullet}$ *are connected to* $\mathbf{L}$ *and* $\overset{\circ}{\mathbf{L}}$ *by the relations*

$$\dot{\mathbf{F}} = \mathbf{L}\cdot\mathbf{F}, \qquad \dot{\mathbf{F}} = \overset{\circ}{\mathbf{L}},$$
$$(\mathbf{F}^{-1})^{\bullet} = -\mathbf{F}^{-1}\cdot\mathbf{L}, \qquad (\mathbf{F}^{-1})^{\bullet} = -\mathbf{F}^{-1}\cdot\mathbf{F}^{-1\mathrm{T}}\cdot\overset{\circ}{\mathbf{L}}. \tag{4.99}$$

▼ Differentiating the relationships (1.35a) with respect to time t and taking the definition of the velocity (4.1) into account, we get

$$\dot{\mathbf{F}}^{\mathrm{T}} = \overset{\circ}{\nabla}\otimes\mathbf{v} = \mathbf{F}^{\mathrm{T}}\cdot\nabla\otimes\mathbf{v}, \qquad \dot{\mathbf{F}} = (\nabla\otimes\mathbf{v})^{\mathrm{T}}\cdot\mathbf{F}. \tag{4.99a}$$

According to the definitions of tensors $\mathbf{L}$ (4.43) and $\overset{\circ}{\mathbf{L}}$ (4.98), from (4.99a) we obtain formulae (4.99).

Differentiating the identity $(\mathbf{F}\cdot\mathbf{F}^{-1})^{\bullet} = \dot{\mathbf{E}} = 0$, we find that $\dot{\mathbf{F}}\cdot\mathbf{F}^{-1} = -\mathbf{F}\cdot(\mathbf{F}^{-1})^{\bullet}$; whence we get

$$(\mathbf{F}^{-1})^{\bullet} = -\mathbf{F}^{-1}\cdot\dot{\mathbf{F}}\cdot\mathbf{F}^{-1}. \tag{4.100}$$

On substituting the first two formulae of (4.99) into (4.100), we obtain

$$(\mathbf{F}^{-1})^{\bullet} = -\mathbf{F}^{-1}\cdot(\nabla\otimes\mathbf{v})^{\mathrm{T}}, \qquad (\mathbf{F}^{-1\mathrm{T}})^{\bullet} = -(\nabla\otimes\mathbf{v})\cdot\mathbf{F}^{-1\mathrm{T}}, \tag{4.101}$$

i.e. the third and the fourth relationships of (4.99) hold as well. ▲

According to formulae (4.45), (4.99a) and (4.101), we find that the rate of the deformation gradient is connected to the deformation rate tensor $\mathbf{D}$ by the relations

$$\mathbf{D} = \frac{1}{2}(\dot{\mathbf{F}}\cdot\mathbf{F}^{-1} + \mathbf{F}^{-1\mathrm{T}}\cdot\dot{\mathbf{F}}^{\mathrm{T}}), \qquad \mathbf{D} = -\frac{1}{2}(\mathbf{F}\cdot\dot{\mathbf{F}}^{-1} + \dot{\mathbf{F}}^{-1\mathrm{T}}\cdot\mathbf{F}^{\mathrm{T}}). \tag{4.102}$$

Here and below we will use the notation $\dot{\mathbf{F}}^{-1} \equiv (\mathbf{F}^{-1})^{\bullet}$.

Theorem 1.26. *The rates of the deformation tensors* $\dot{\mathbf{C}}$, $\dot{\mathbf{A}}$, $\dot{\mathbf{\Lambda}}$, $\dot{\mathbf{J}}$ *and deformation measures* $\dot{\mathbf{G}}$, $\dot{\mathbf{g}}$, $(\mathbf{G}^{-1})^{\bullet}$ *and* $(\mathbf{g}^{-1})^{\bullet}$ *are connected to the velocity gradients* $\mathbf{L}$ *and* $\overset{\circ}{\mathbf{L}}$ *by the relationships*

$$\begin{cases} \dot{\mathbf{C}} = \mathbf{F}^{\mathrm{T}}\cdot\mathbf{D}\cdot\mathbf{F}, & \dot{\mathbf{A}} = \mathbf{D} - \mathbf{L}^{\mathrm{T}}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{L}, \\ \dot{\mathbf{\Lambda}} = \mathbf{F}^{-1}\cdot\mathbf{D}\cdot\mathbf{F}^{-1\mathrm{T}}, & \dot{\mathbf{J}} = \mathbf{D} + \mathbf{L}\cdot\mathbf{J} + \mathbf{J}\cdot\mathbf{L}^{\mathrm{T}}, \end{cases} \tag{4.103}$$

and

$$\begin{cases} \dot{\mathbf{G}} = 2\mathbf{F}^{\mathrm{T}} \cdot \mathbf{D} \cdot \mathbf{F}, & \dot{\mathbf{g}} = -\mathbf{L}^{\mathrm{T}} \cdot \mathbf{g} - \mathbf{g} \cdot \mathbf{L}, \\ (\mathbf{G}^{-1})^{\bullet} = -2\mathbf{F}^{-1} \cdot \mathbf{D} \cdot \mathbf{F}^{-1\mathrm{T}}, & (\mathbf{g}^{-1})^{\bullet} = \mathbf{L} \cdot \mathbf{g}^{-1} + \mathbf{g}^{-1} \cdot \mathbf{L}^{\mathrm{T}}; \end{cases} \tag{4.104}$$

and also

$$\begin{cases} \dot{\mathbf{C}} = (1/2)(\mathbf{F}^{\mathrm{T}} \cdot \overset{\circ}{\mathbf{L}} + \overset{\circ}{\mathbf{L}}{}^{\mathrm{T}} \cdot \mathbf{F}), \quad \dot{\mathbf{J}} = (1/2)(\overset{\circ}{\mathbf{L}} \cdot \mathbf{F}^{\mathrm{T}} + \mathbf{F}^{\mathrm{T}} \cdot \overset{\circ}{\mathbf{L}}), \\ \dot{\mathbf{A}} = (1/2)((\mathbf{E} - 2\mathbf{A}) \cdot \overset{\circ}{\mathbf{L}} \cdot \mathbf{F}^{-1} + \mathbf{F}^{-1\mathrm{T}} \cdot \overset{\circ}{\mathbf{L}}{}^{\mathrm{T}} \cdot (\mathbf{E} - 2\mathbf{A})), \\ \dot{\mathbf{\Lambda}} = (1/2)(\mathbf{F}^{-1} \cdot \overset{\circ}{\mathbf{L}} \cdot (\mathbf{E} - 2\mathbf{\Lambda}) + (\mathbf{E} - 2\mathbf{\Lambda}) \cdot \overset{\circ}{\mathbf{L}}{}^{\mathrm{T}} \cdot \mathbf{F}^{-1\mathrm{T}}), \end{cases} \tag{4.105}$$

and

$$\begin{cases} \dot{\mathbf{G}} = \mathbf{F}^{\mathrm{T}} \cdot \overset{\circ}{\mathbf{L}} + \overset{\circ}{\mathbf{L}}{}^{\mathrm{T}} \cdot \mathbf{F}, \quad \dot{\mathbf{g}} = -(\mathbf{g} \cdot \overset{\circ}{\mathbf{L}} \cdot \mathbf{F}^{-1} + \mathbf{F}^{-1\mathrm{T}} \cdot \overset{\circ}{\mathbf{L}}{}^{\mathrm{T}} \cdot \mathbf{g}), \\ (\mathbf{G}^{-1})^{\bullet} = -(\mathbf{F}^{-1} \cdot \overset{\circ}{\mathbf{L}} \cdot \mathbf{G}^{-1} + \mathbf{G}^{-1} \cdot \overset{\circ}{\mathbf{L}}{}^{\mathrm{T}} \cdot \mathbf{F}^{-1\mathrm{T}}), \\ (\mathbf{g}^{-1})^{\bullet} = \overset{\circ}{\mathbf{L}} \cdot \mathbf{F}^{\mathrm{T}} + \mathbf{F}^{\mathrm{T}} \cdot \overset{\circ}{\mathbf{L}}. \end{cases} \tag{4.106}$$

▼ To prove formula (4.103), we must differentiate the relationships (2.5) with respect to t and apply formulae (4.99):

$$\dot{\mathbf{C}} = \frac{1}{2}(\dot{\mathbf{F}}^{\mathrm{T}} \cdot \mathbf{F} + \mathbf{F}^{\mathrm{T}} \cdot \dot{\mathbf{F}}) = \frac{1}{2}(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{L}^{\mathrm{T}} \cdot \mathbf{F} + \mathbf{F}^{\mathrm{T}} \cdot \mathbf{L} \cdot \mathbf{F}) = \\ = \frac{1}{2}\mathbf{F}^{\mathrm{T}} \cdot (\mathbf{L}^{\mathrm{T}} + \mathbf{L}) \cdot \mathbf{F} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{D} \cdot \mathbf{F}, \tag{4.107}$$

$$\dot{\mathbf{A}} = \frac{1}{2}(\dot{\mathbf{F}}^{-1\mathrm{T}} \cdot \mathbf{F}^{-1} + \mathbf{F}^{-1\mathrm{T}} \cdot \dot{\mathbf{F}}^{-1}) = \\ = \frac{1}{2}(\mathbf{L}^{\mathrm{T}} \cdot \mathbf{F}^{-1\mathrm{T}} \cdot \mathbf{F}^{-1} + \mathbf{F}^{-1\mathrm{T}} \cdot \mathbf{F}^{-1} \cdot \mathbf{L}) = \\ = \frac{1}{2}(\mathbf{L}^{\mathrm{T}} \cdot (\mathbf{E} - 2\mathbf{A}) + (\mathbf{E} - 2\mathbf{A}) \cdot \mathbf{L}) = \mathbf{D} - \mathbf{L}^{\mathrm{T}} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{L}, \tag{4.108}$$

$$\dot{\mathbf{\Lambda}} = -\frac{1}{2}(\dot{\mathbf{F}}^{-1} \cdot \mathbf{F}^{-1\mathrm{T}} + \mathbf{F}^{-1} \cdot \dot{\mathbf{F}}^{-1\mathrm{T}}) = \frac{1}{2}(\mathbf{F}^{-1} \cdot \mathbf{L} \cdot \mathbf{F}^{-1\mathrm{T}} + \\ + \mathbf{F}^{-1} \cdot \mathbf{L}^{\mathrm{T}} \cdot \mathbf{F}^{-1\mathrm{T}}) = \frac{1}{2}\mathbf{F}^{-1} \cdot (\mathbf{L} + \mathbf{L}^{\mathrm{T}}) \cdot \mathbf{F}^{-1\mathrm{T}} = \mathbf{F}^{-1} \cdot \mathbf{D} \cdot \mathbf{F}^{-1\mathrm{T}}, \tag{4.109}$$

$$\dot{\mathbf{J}} = \frac{1}{2}(\dot{\mathbf{F}} \cdot \mathbf{F}^{\mathrm{T}} + \mathbf{F} \cdot \dot{\mathbf{F}}^{\mathrm{T}}) = \frac{1}{2}(\mathbf{L} \cdot \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} + \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} \cdot \mathbf{L}^{\mathrm{T}}) = \\ = \frac{1}{2}(\mathbf{L} \cdot (\mathbf{E} + 2\mathbf{J}) + (\mathbf{E} + 2\mathbf{J}) \cdot \mathbf{L}^{\mathrm{T}}) = \mathbf{D} + \mathbf{L} \cdot \mathbf{J} + \mathbf{J} \cdot \mathbf{L}^{\mathrm{T}}. \tag{4.110}$$

Formulae (4.104) follow from (4.103), if we have used the connections (2.7) and (2.8) between the deformation tensors and measures.

Formulae (4.105) follow from (4.107)–(4.110), if we have gone from $\mathbf{L}$ to $\overset{\circ}{\mathbf{L}}$:

$$\mathbf{L} = \overset{\circ}{\mathbf{L}} \cdot \mathbf{F}^{-1}. \tag{4.111}$$

Using the connections (2.7) and (2.8) between the deformation tensors and measures, from (4.105) we get formulae (4.106). ▲

Relations between the tensors $\dot{\mathbf{U}}$, $\dot{\mathbf{V}}$ and the velocity gradients $\mathbf{L}$ and $\overset{\circ}{\mathbf{L}}$ are more complicated. To derive them, we should use representations of $\mathbf{L}$ and $\overset{\circ}{\mathbf{L}}$ in terms of the eigenbasis vectors.

Theorem 1.27. *The following expressions for the velocity gradients hold:*

$$\overset{\circ}{\mathbf{L}}{}^{\mathrm{T}} = \overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{v} = \frac{\partial}{\partial t}\overset{\circ}{\boldsymbol{\nabla}} \otimes \mathbf{x} = \sum_{\alpha=1}^{3} \dot{\lambda}_\alpha \overset{\circ}{\mathbf{p}}_\alpha \otimes \mathbf{p}_\alpha + \boldsymbol{\Omega}_U \cdot \mathbf{F}^{\mathrm{T}} - \mathbf{F}^{\mathrm{T}} \cdot \boldsymbol{\Omega}_V, \qquad (4.112)$$

$$\mathbf{L} = (\boldsymbol{\nabla} \otimes \mathbf{v})^{\mathrm{T}} = \sum_{\alpha,\beta=1}^{3} \left(\frac{\dot{\lambda}_\alpha}{\lambda_\alpha}\delta_{\alpha\beta} + \frac{\lambda_\alpha}{\lambda_\beta}\overset{\circ}{\Omega}_{U\beta\alpha} \right) \mathbf{p}_\alpha \otimes \mathbf{p}_\beta + \boldsymbol{\Omega}_V, \qquad (4.113)$$

where $\overset{\circ}{\Omega}_{U\alpha\beta}$ *are components of the tensor* $\boldsymbol{\Omega}_U$ *with respect to the eigenbasis* $\overset{\circ}{\mathbf{p}}_\alpha$*:*

$$\boldsymbol{\Omega}_U = \overset{\circ}{\mathbf{p}}{}^{\bullet}_i \otimes \overset{\circ}{\mathbf{p}}{}^i = \sum_{\alpha=1}^{3} \overset{\circ}{\Omega}_{U\alpha\beta} \overset{\circ}{\mathbf{p}}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\beta, \qquad \overset{\circ}{\Omega}_{U\alpha\beta} = \overset{\circ}{\mathbf{p}}_\alpha \cdot \boldsymbol{\Omega}_U \cdot \overset{\circ}{\mathbf{p}}_\beta = \overset{\circ}{\mathbf{p}}{}^{\bullet}_\alpha \cdot \overset{\circ}{\mathbf{p}}_\beta. \quad (4.114)$$

▼ To prove formula (4.112), we should consider the first formula of (4.99a) and substitute into this formula the expression (3.35) for $\mathbf{F}$ in the eigenbasis:

$$\overset{\circ}{\mathbf{L}}{}^{\mathrm{T}} = \dot{\mathbf{F}}^{\mathrm{T}} = \sum_{\alpha=1}^{3} (\dot{\lambda}_\alpha \overset{\circ}{\mathbf{p}}_\alpha \otimes \mathbf{p}_\alpha + \lambda_\alpha (\overset{\circ}{\mathbf{p}}{}^{\bullet}_\alpha \otimes \mathbf{p}_\alpha + \overset{\circ}{\mathbf{p}}_\alpha \otimes \dot{\mathbf{p}}_\alpha)). \qquad (4.115)$$

Using formulae (4.90) and (4.92), from (4.115) we derive the relationship (4.112).

To prove formula (4.113), we use formulae (4.112) and (4.111), having expressed $\mathbf{F}^{-1}$ in the form (3.35); then we get

$$\mathbf{L} = \overset{\circ}{\mathbf{L}} \cdot \mathbf{F}^{-1} = (\sum_{\alpha=1}^{3} \dot{\lambda}_\alpha \mathbf{p}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\alpha + \lambda_\alpha (\mathbf{p}_\alpha \otimes \overset{\circ}{\mathbf{p}}{}^{\bullet}_\alpha + \dot{\mathbf{p}}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\alpha)) \cdot \sum_{\beta=1}^{3} \lambda_\beta^{-1} \overset{\circ}{\mathbf{p}}_\beta \otimes \mathbf{p}_\beta =$$

$$= \sum_{\alpha,\beta=1}^{3} \Big(\frac{\dot{\lambda}_\alpha}{\lambda_\beta}\delta_{\alpha\beta}\mathbf{p}_\alpha \otimes \mathbf{p}_\alpha + \frac{\lambda_\alpha}{\lambda_\beta}(\overset{\circ}{\mathbf{p}}{}^{\bullet}_\alpha \cdot \overset{\circ}{\mathbf{p}}_\beta)\mathbf{p}_\alpha \otimes \mathbf{p}_\beta \Big) + \sum_{\alpha=1}^{3} \dot{\mathbf{p}}_\alpha \otimes \mathbf{p}_\alpha. \quad (4.116)$$

Here we have taken into account that the vectors $\overset{\circ}{\mathbf{p}}_\alpha$ are orthonormal. Using formulae (4.90) and (4.92), from (4.116) we derive the formula (4.113). ▲

From formula (4.113) it follows that the deformation rate tensor $\mathbf{D}$ and the vorticity tensor can be represented in terms of the eigenbasis $\mathbf{p}_\alpha$ as follows:

$$\mathbf{D} = \sum_{\alpha,\beta=1}^{3} \left(\frac{\dot{\lambda}_\alpha}{\lambda_\alpha}\delta_{\alpha\beta} + \frac{1}{2}\left(\frac{\lambda_\alpha}{\lambda_\beta} - \frac{\lambda_\beta}{\lambda_\alpha} \right) \overset{\circ}{\Omega}_{U\beta\alpha} \right) \mathbf{p}_\alpha \otimes \mathbf{p}_\beta, \qquad (4.117)$$

$$\mathbf{W} = \frac{1}{2}\sum_{\alpha,\beta=1}^{3}\left(\frac{\lambda_\alpha}{\lambda_\beta} + \frac{\lambda_\beta}{\lambda_\alpha}\right)\overset{\circ}{\Omega}_{U\beta\alpha}\mathbf{p}_\alpha \otimes \mathbf{p}_\beta + \mathbf{\Omega}_V. \qquad (4.118)$$

Here we have taken into account that the tensors $\mathbf{\Omega}_U$ and $\mathbf{\Omega}_V$ are skew-symmetric.

Denote components of the tensor $\mathbf{D}$ with respect to the basis $\mathbf{p}_\alpha$ by $D_{\alpha\beta}$:

$$\mathbf{D} = \sum_{\alpha,\beta=1}^{3} D_{\alpha\beta}\mathbf{p}_\alpha \otimes \mathbf{p}_\beta, \quad D_{\alpha\beta} = \mathbf{p}_\alpha \cdot \mathbf{D} \cdot \mathbf{p}_\beta. \qquad (4.119)$$

Then from (4.117) and (4.119) we get that diagonal components of the deformation rate tensor $D_{\alpha\alpha}$ with respect to the eigenbasis $\mathbf{p}_\alpha$ determine the relative rates of lengthening the material fibres oriented along the eigenvectors $\mathbf{p}_\alpha$ (compare with formula (4.56)):

$$D_{\alpha\alpha} = \dot{\lambda}_\alpha/\lambda_\alpha = d\dot{s}_\alpha/ds_\alpha, \qquad \alpha = 1, 3; \qquad (4.120)$$

and off-diagonal components $D_{\alpha\beta}$ are connected to $\overset{\circ}{\Omega}_{U\alpha\beta}$ by the relations

$$D_{\alpha\beta} = \frac{1}{2}\Big(\frac{\lambda_\alpha^2 - \lambda_\beta^2}{\lambda_\alpha\lambda_\beta}\Big)\overset{\circ}{\Omega}_{U\beta\alpha}, \qquad \alpha \neq \beta. \qquad (4.121)$$

From formulae (4.119) and (4.121) we can express the components $\overset{\circ}{\Omega}_{U\alpha\beta}$ in terms of components of the deformation rate tensor:

$$\overset{\circ}{\Omega}_{U\alpha\beta} = \frac{2\lambda_\alpha\lambda_\beta}{\lambda_\beta^2 - \lambda_\alpha^2}D_{\alpha\beta}, \quad \alpha \neq \beta; \quad \overset{\circ}{\Omega}_{U\alpha\alpha} = 0. \qquad (4.122)$$

The diagonal components $\overset{\circ}{\Omega}_{U\alpha\alpha}$ are equal to zero, because the tensor $\mathbf{\Omega}_U$ is skew-symmetric.

On substituting the relationships (4.122) into (4.118), we find the expression for components $\overset{\circ}{\Omega}_{V\alpha\beta}$ of the tensor $\mathbf{\Omega}_V$ with respect to the basis $\mathbf{p}_\alpha$ in terms of the tensors $\mathbf{W}$ and $\mathbf{D}$ (and, hence, in terms of the velocity gradient $\mathbf{L}$):

$$\overset{\circ}{\Omega}_{V\alpha\beta} = \mathbf{p}_\alpha \cdot \mathbf{\Omega}_V \cdot \mathbf{p}_\alpha = \mathbf{p}_\alpha \cdot \mathbf{W} \cdot \mathbf{p}_\beta - \frac{\lambda_\alpha^2 + \lambda_\beta^2}{\lambda_\alpha^2 - \lambda_\beta^2}D_{\alpha\beta}, \quad \alpha \neq \beta, \qquad (4.123)$$

where

$$\mathbf{\Omega}_V = \sum_{\alpha,\beta=1}^{3} \Omega_{V\alpha\beta}\mathbf{p}_\alpha \otimes \mathbf{p}_\beta. \qquad (4.124)$$

Remark. The expressions (4.122) and (4.123) are valid only if the eigenvalues are not multiple: $\lambda_\alpha \neq \lambda_\beta$ ($\alpha \neq \beta$; $\alpha, \beta = 1, 2, 3$). If within the interval $[t_1, t_2]$ all three eigenvalues are coincident: $\lambda_\alpha = \lambda$ ($\alpha = 1, 2, 3$), then the stretch tensors are spherical: $\mathbf{U} = \lambda\overset{\circ}{\mathbf{p}}_i \otimes \overset{\circ}{\mathbf{p}}{}^i = \lambda\mathbf{E}$, $\mathbf{V} = \lambda\mathbf{E}$, and the eigenbases are not uniquely defined: as $\overset{\circ}{\mathbf{p}}_i$ and $\mathbf{p}_i$ we can take any orthonormal triple of vectors. In particular, one of the bases can be taken as fixed $\forall t \in [t_1, t_2]$, for example, $\overset{\circ}{\mathbf{p}}_i$ can be chosen as coincident with $\overset{\circ}{\mathbf{p}}_i(t_1)$; and the second basis $\mathbf{p}_i$ can depend on time t. In this case $\overset{\circ}{\mathbf{p}}{}_i^{\bullet} \equiv 0\ \forall t \in [t_1, t_2]$, and from (4.114) and (4.118) it follows that within the considered time interval:

$$\overset{\circ}{\Omega}_{U\alpha\beta} = 0, \quad \boldsymbol{\Omega}_U = \mathbf{0}, \tag{4.125}$$
$$\boldsymbol{\Omega}_V = \mathbf{W}, \quad \Omega_{V\alpha\beta} = \mathbf{p}_\alpha \cdot \mathbf{W} \cdot \mathbf{p}_\beta, \quad \alpha, \beta = 1, 2, 3.$$

These relationships take the place of formulae (4.122), (4.123) in this case.

If within the time interval $[t_1, t_2]$ only two of three eigenvalues are coincident, for example, $\lambda_\alpha = \lambda_\beta$, then their corresponding eigenvectors $\overset{\circ}{\mathbf{p}}_\alpha$ and $\overset{\circ}{\mathbf{p}}_\beta$ are not uniquely defined as well: only their orthogonality to the vector $\overset{\circ}{\mathbf{p}}_\gamma$, corresponding to the third eigenvalue λ_γ, is given. Then we can extend the definition of $\overset{\circ}{\mathbf{p}}_\alpha$ and $\overset{\circ}{\mathbf{p}}_\beta$ so that $\overset{\circ}{\mathbf{p}}{}_\alpha^{\bullet} \cdot \mathbf{p}_\beta = 0\ \forall t \in [t_1, t_2]$. In this case it follows from (4.114) that the only component $\overset{\circ}{\Omega}_{U\alpha\beta}$ vanishes, but $\overset{\circ}{\Omega}_{U\alpha\gamma} \neq 0$ and $\overset{\circ}{\Omega}_{U\beta\gamma} \neq 0$.

It follows from (4.118) that the component $\Omega_{V\alpha\beta}$ is determined by the formula

$$\Omega_{V\alpha\beta} = \mathbf{p}_\alpha \cdot \mathbf{W} \cdot \mathbf{p}_\beta, \quad \overset{\circ}{\Omega}_{U\alpha\beta} = 0. \tag{4.126}$$

The remaining components $\overset{\circ}{\Omega}_{U\alpha\gamma}$, $\overset{\circ}{\Omega}_{U\beta\gamma}$ and $\Omega_{V\alpha\gamma}$, $\Omega_{V\beta\gamma}$ are determined by formulae (4.122) and (4.123).

If the situation with multiple roots appears only at some time t, then the values of $\overset{\circ}{\Omega}_{U\alpha\beta}(t)$ and $\Omega_{V\alpha\beta}(t)$ can be determined by passing to the limit. □

Substituting formulae (4.114) and (4.118) into equation (4.97), and taking expressions (4.89) and (4.91) for $\mathbf{O}_U$ and $\mathbf{O}_V$ into consideration, we obtain the representation of the spin $\boldsymbol{\Omega}$ in the basis $\mathbf{p}_\alpha$:

$$\boldsymbol{\Omega} = \mathbf{W} - \frac{1}{2}\sum_{\alpha,\beta=1}^{3} \frac{\lambda_\alpha^2 + \lambda_\beta^2}{\lambda_\alpha\lambda_\beta} \overset{\circ}{\Omega}_{U\beta\alpha} \mathbf{p}_\alpha \otimes \mathbf{p}_\beta -$$
$$- \sum_{\alpha,\beta=1}^{3} \overset{\circ}{\Omega}_{U\alpha\beta} (\mathbf{p}_i \otimes \bar{\mathbf{e}}^i) \cdot \overset{\circ}{\mathbf{p}}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\beta \cdot (\bar{\mathbf{e}}^j \otimes \mathbf{p}_j). \tag{4.127}$$

Introducing the notation for direction cosines

$$\overset{\circ}{l}_{\alpha\beta} = \overset{\circ}{\mathbf{p}}_\alpha \cdot \bar{\mathbf{e}}_\beta, \quad l_{\alpha\beta} = \mathbf{p}_\alpha \cdot \bar{\mathbf{e}}_\beta, \tag{4.128}$$

substituting (4.122) into (4.127) and collecting like terms, we obtain the following expression of the spin $\boldsymbol{\Omega}$ in terms of $\mathbf{W}$ and $\mathbf{D}$ (i.e. in terms of $\mathbf{L}$):

$$\boldsymbol{\Omega} = \mathbf{W} + \widetilde{\boldsymbol{\Omega}}, \tag{4.129}$$
$$\widetilde{\boldsymbol{\Omega}} = \sum_{\gamma,\rho=1}^{3} \widetilde{\Omega}_{\gamma\rho} \mathbf{p}_\gamma \otimes \mathbf{p}_\rho, \quad \widetilde{\Omega}_{\gamma\rho} = \sum_{\alpha,\beta=1}^{3} \widetilde{\Omega}_{\alpha\beta},$$

$$\widetilde{\Omega}_{\gamma\rho} = \sum_{\alpha,\beta=1}^{3} \frac{1}{\lambda_\beta^2 - \lambda_\alpha^2}((\lambda_\alpha^2 + \lambda_\beta^2)\delta_{\alpha\gamma}\delta_{\beta\rho} - 2\lambda_\alpha\lambda_\beta l_{\alpha\gamma}l_{\beta\rho})D_{\alpha\beta}.$$

Theorem 1.28. *Rates of the deformation measures* $\dot{\mathbf{U}}$, $(\mathbf{U}^{-1})^\bullet$ *and* $\dot{\mathbf{V}}$, $(\mathbf{V}^{-1})^\bullet$ *are connected to the velocity gradient* $\mathbf{L}$ *by the formulae*

$$\begin{aligned}
\dot{\mathbf{U}} &= \frac{1}{2}(\mathbf{F}^{\mathrm{T}} \cdot (\mathbf{D} + \widetilde{\mathbf{\Omega}}) \cdot \mathbf{O} + \mathbf{O}^{\mathrm{T}} \cdot (\mathbf{D} + \widetilde{\mathbf{\Omega}}^{\mathrm{T}}) \cdot \mathbf{F}),\\
(\mathbf{U}^{-1})^\bullet &= -\frac{1}{2}(\mathbf{F}^{-1} \cdot (\mathbf{D} - \widetilde{\mathbf{\Omega}}) \cdot \mathbf{O} + \mathbf{O}^{\mathrm{T}} \cdot (\mathbf{D} - \widetilde{\mathbf{\Omega}}^{\mathrm{T}}) \cdot \mathbf{F}^{-1\mathrm{T}}),\\
\dot{\mathbf{V}} &= \frac{1}{2}((\mathbf{L} + \mathbf{\Omega}) \cdot \mathbf{V} + \mathbf{V} \cdot (\mathbf{L}^{\mathrm{T}} + \mathbf{\Omega}^{\mathrm{T}})),\\
(\mathbf{V}^{-1})^\bullet &= \frac{1}{2}((\mathbf{\Omega} - \mathbf{L}^{\mathrm{T}}) \cdot \mathbf{V}^{-1} + \mathbf{V}^{-1} \cdot (\mathbf{\Omega}^{\mathrm{T}} - \mathbf{L})).
\end{aligned} \tag{4.130}$$

▼ Let us express the tensors $\mathbf{V}$ and $\mathbf{U}$ from the polar decomposition (3.1):

$$\mathbf{U} = \mathbf{O}^{\mathrm{T}} \cdot \mathbf{F}, \qquad \mathbf{V} = \mathbf{F} \cdot \mathbf{O}^{\mathrm{T}}. \tag{4.131}$$

Since $\mathbf{U}$ and $\mathbf{V}$ are symmetric tensors, these expressions can be rewritten in the symmetrized form

$$\mathbf{U} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{O} + \mathbf{O}^{\mathrm{T}} \cdot \mathbf{F}), \qquad \mathbf{V} = \frac{1}{2}(\mathbf{F} \cdot \mathbf{O}^{\mathrm{T}} + \mathbf{O} \cdot \mathbf{F}^{\mathrm{T}}). \tag{4.132}$$

Let us differentiate these relationships:

$$\begin{aligned}
\dot{\mathbf{U}} &= \frac{1}{2}(\dot{\mathbf{F}}^{\mathrm{T}} \cdot \mathbf{O} + \mathbf{F}^{\mathrm{T}} \cdot \dot{\mathbf{O}} + \dot{\mathbf{O}}^{\mathrm{T}} \cdot \mathbf{F} + \mathbf{O}^{\mathrm{T}} \cdot \dot{\mathbf{F}}),\\
\dot{\mathbf{V}} &= \frac{1}{2}(\dot{\mathbf{F}} \cdot \mathbf{O}^{\mathrm{T}} + \mathbf{F} \cdot \dot{\mathbf{O}}^{\mathrm{T}} + \dot{\mathbf{O}} \cdot \mathbf{F}^{\mathrm{T}} + \mathbf{O} \cdot \dot{\mathbf{F}}^{\mathrm{T}}).
\end{aligned} \tag{4.133}$$

On substituting formulae (4.99) and the expression (4.97) for the spin $\mathbf{\Omega}$ into (4.133), we obtain

$$\begin{aligned}
\dot{\mathbf{U}} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{L}^{\mathrm{T}} \cdot \mathbf{O} + \mathbf{F}^{\mathrm{T}} \cdot \dot{\mathbf{O}} \cdot \mathbf{O}^{\mathrm{T}} \cdot \mathbf{O} + \mathbf{O}^{\mathrm{T}} \cdot \mathbf{O} \cdot \dot{\mathbf{O}}^{\mathrm{T}} \cdot \mathbf{F} + \mathbf{O}^{\mathrm{T}} \cdot \mathbf{L} \cdot \mathbf{F}) =\\
= \frac{1}{2}(\mathbf{F}^{\mathrm{T}} \cdot (\mathbf{L}^{\mathrm{T}} + \mathbf{\Omega}) \cdot \mathbf{O} + \mathbf{O}^{\mathrm{T}} \cdot (\mathbf{L} + \mathbf{\Omega}^{\mathrm{T}}) \cdot \mathbf{F}),
\end{aligned} \tag{4.134}$$

$$\begin{aligned}
\dot{\mathbf{V}} = \frac{1}{2}(\mathbf{L} \cdot \mathbf{F} \cdot \mathbf{O}^{\mathrm{T}} + \mathbf{F} \cdot \mathbf{O}^{\mathrm{T}} \cdot \mathbf{O} \cdot \dot{\mathbf{O}}^{\mathrm{T}} + \dot{\mathbf{O}} \cdot \mathbf{O}^{\mathrm{T}} \cdot \mathbf{O} \cdot \mathbf{F}^{\mathrm{T}} + \mathbf{O} \cdot \mathbf{F}^{\mathrm{T}} \cdot \mathbf{L}^{\mathrm{T}}) =\\
= \frac{1}{2}((\mathbf{L} + \mathbf{\Omega}) \cdot \mathbf{V} + \mathbf{V} \cdot (\mathbf{L}^{\mathrm{T}} + \mathbf{\Omega}^{\mathrm{T}})).
\end{aligned}$$

Taking formulae (4.129) and (4.44) into consideration, we find that

$$\mathbf{L}^{\mathrm{T}} + \mathbf{\Omega} = \mathbf{D} + \mathbf{W}^{\mathrm{T}} + \mathbf{W} + \widetilde{\mathbf{\Omega}} = \mathbf{D} + \widetilde{\mathbf{\Omega}}. \tag{4.135}$$

Substituting (4.135) into (4.134), we really get the first and the third formulae of (4.130).

The remaining two formulae in (4.130) can be proved in a similar way. ▲

In continuum mechanics the *deformation tensors* $\mathbf{B}$ and $\mathbf{Y}$ are applied, which have no explicit expression; they are defined by their derivatives and initial values:

$$\dot{\mathbf{B}} = \frac{1}{2}(\dot{\mathbf{U}} \cdot \mathbf{U}^{-1} + \mathbf{U}^{-1} \cdot \dot{\mathbf{U}}), \quad \mathbf{B}(0) = \mathbf{0}, \tag{4.136}$$

$$\dot{\mathbf{Y}} = \frac{1}{2}(\dot{\mathbf{V}} \cdot \mathbf{V}^{-1} + \mathbf{V}^{-1} \cdot \dot{\mathbf{V}}), \quad \mathbf{Y}(0) = \mathbf{0}. \tag{4.137}$$

After substitution of the expressions (4.131), formula (4.136) takes the form

$$\begin{aligned}\dot{\mathbf{B}} &= \frac{1}{2}((\dot{\mathbf{O}}^{\mathrm{T}} \cdot \mathbf{F} + \mathbf{O}^{\mathrm{T}} \cdot \dot{\mathbf{F}}) \cdot \mathbf{U}^{-1} + \mathbf{U}^{-1} \cdot (\dot{\mathbf{F}}^{\mathrm{T}} \cdot \mathbf{O} + \mathbf{F}^{\mathrm{T}} \cdot \dot{\mathbf{O}})) = \\ &= \frac{1}{2}(\mathbf{O}^{\mathrm{T}} \cdot (\boldsymbol{\Omega} + \mathbf{L}) \cdot \mathbf{F} \cdot \mathbf{U}^{-1} + \mathbf{U}^{-1} \cdot \mathbf{F}^{\mathrm{T}} \cdot (\mathbf{L}^{\mathrm{T}} + \boldsymbol{\Omega}^{\mathrm{T}}) \cdot \mathbf{O}) = \\ &= \frac{1}{2}(\mathbf{O}^{\mathrm{T}} \cdot (\boldsymbol{\Omega} + \mathbf{L}) \cdot \mathbf{O} + \mathbf{O}^{\mathrm{T}} \cdot (\mathbf{L}^{\mathrm{T}} + \boldsymbol{\Omega}^{\mathrm{T}}) \cdot \mathbf{O}),\end{aligned} \tag{4.138}$$

and formula (4.137) is rewritten as follows:

$$\begin{aligned}\dot{\mathbf{Y}} &= \frac{1}{2}((\dot{\mathbf{F}} \cdot \mathbf{O}^{\mathrm{T}} + \mathbf{F} \cdot \dot{\mathbf{O}}^{\mathrm{T}}) \cdot \mathbf{V}^{-1} + \mathbf{V}^{-1} \cdot (\dot{\mathbf{O}} \cdot \mathbf{F}^{\mathrm{T}} + \mathbf{O} \cdot \dot{\mathbf{F}}^{\mathrm{T}})) = \\ &= \frac{1}{2}(\mathbf{L} \cdot \mathbf{F} \cdot \mathbf{O}^{\mathrm{T}} \cdot \mathbf{V}^{-1} + \mathbf{F} \cdot \mathbf{O}^{\mathrm{T}} \cdot \boldsymbol{\Omega}^{\mathrm{T}} \cdot \mathbf{V}^{-1} + \mathbf{V}^{-1} \cdot \boldsymbol{\Omega} \cdot \mathbf{O} \cdot \mathbf{F}^{\mathrm{T}} + \\ &+ \mathbf{V}^{-1} \cdot \mathbf{O} \cdot \mathbf{F}^{\mathrm{T}} \cdot \mathbf{L}^{\mathrm{T}}) = \frac{1}{2}(\mathbf{L} + \mathbf{V} \cdot \boldsymbol{\Omega} \cdot \mathbf{V}^{-1} + \mathbf{V}^{-1} \cdot \boldsymbol{\Omega} \cdot \mathbf{V} + \mathbf{L}^{\mathrm{T}}).\end{aligned} \tag{4.139}$$

Finally, we get the following expressions for $\dot{\mathbf{B}}$ and $\dot{\mathbf{Y}}$:

$$\dot{\mathbf{B}} = \mathbf{O}^{\mathrm{T}} \cdot \mathbf{D} \cdot \mathbf{O}, \tag{4.140}$$

$$\dot{\mathbf{Y}} = \mathbf{D} + \frac{1}{2}(\mathbf{V} \cdot \boldsymbol{\Omega}^{\mathrm{T}} \cdot \mathbf{V}^{-1} + \mathbf{V}^{-1} \cdot \boldsymbol{\Omega} \cdot \mathbf{V}). \tag{4.141}$$

1.4.12. Trajectory of a Material Point, Streamline and Vortex Line. Having fixed coordinates X^i of a material point $\mathcal{M}$ in the motion law (1.3), we get the parametric equation of a certain curve, where time t is a parameter:

$$x^i = x^i(X^k, t), \qquad 0 \leqslant t \leqslant t'. \tag{4.142}$$

The origin of the curve at $t = 0$ is a point with Cartesian coordinates $\overset{\circ}{x}{}^i(X^k)$ of the material point $\mathcal{M}$ in $\overset{\circ}{\mathcal{K}}$, and the end of the curve at $t = t'$ is a point with Cartesian coordinates $x^i(X^k, t')$ of the point $\mathcal{M}$ in $\mathcal{K}(t')$ (Figure 1.14). The curve (4.142) is called the *trajectory of the point* $\mathcal{M}$ in the Cartesian coordinate system $O\bar{\mathbf{e}}_i$.

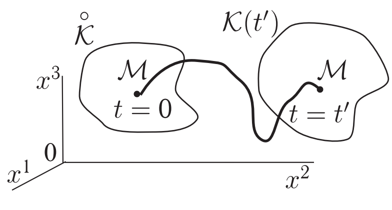

Figure 1.14. Trajectory of material point $\mathcal{M}$

In the spatial description, the trajectory (4.142) at fixed X^k is a solution of the kinematic equation (4.10):

$$dx^i/dt = \bar{v}^i(x^j, t), \qquad 0 < t \leqslant t' \tag{4.143}$$

with the initial condition

$$t = 0: \quad x^i = \overset{\circ}{x}{}^i.$$

Here $\bar{v}^i(x^j, t)$ are the velocity components with respect to the Cartesian basis $\bar{\mathbf{e}}_i$, which are assumed to be known.

Let a field of velocities $\mathbf{v}(x^j, t) = \bar{v}^i \bar{\mathbf{e}}_i$ be given. Fix a time t, and take a point $\mathcal{M}_1$ with Eulerian coordinates x_1^i and Lagrangian coordinates X_1^k. Then a *streamline* passing through the point $\mathcal{M}_1$ is the curve

$$x^i = x^i(X^k, \tau), \quad \tau_1 \leqslant \tau \leqslant \tau_2, \tag{4.144}$$

which at its every point x^i has a tangent being parallel to the velocity $\mathbf{v}(x^i, t)$ at the considered point and at the considered time. The equation of the streamline has the form

$$dx^i/d\tau = \bar{v}^i(x^j, t), \quad \tau_1 < \tau \leqslant \tau_2, \tag{4.145}$$
$$\tau = \tau_1: \quad x^i = x_1^i.$$

Thus, the trajectory of a material point and the streamline are described, in general, by different equations, and so they are not coincident.

However, if the motion of a continuum is *steady-state* within time interval $t_1 \leqslant t \leqslant t'$, then in Eulerian description all the partial derivatives of all values, describing the motion, with respect to time vanish, in particular $\partial\mathbf{v}(x^i, t)/\partial t = 0$. So the trajectory equations (4.143) and the streamline equation (4.145) become coincident within the interval $t_1 \leqslant t \leqslant t'$, if they have at least one common point $\mathcal{M}_1$:

$$dx^i/d\tau = \bar{v}^i(x^j), \quad t_1 = \tau_1 < \tau \leqslant \tau_2 = t', \tag{4.146}$$
$$\tau = \tau_1: \quad x^i = x_1^i = x^i(X_1^k, t_1).$$

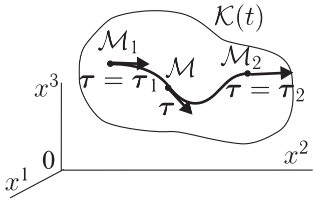

Figure 1.15. The streamline

In other words, in the steady-state motion a material point $\mathcal{M}$ moves along a streamline: at time $t = t_1$ its coordinates are the same as coordinates of point $\mathcal{M}_1$ at parameter value $\tau = \tau_1$, and at time $t = t_2$ they are the same as coordinates of point $\mathcal{M}_2$ at parameter value $\tau = \tau_2$ (Figure 1.15).

Multiplying the equation (4.145) by the basis vectors $\bar{\mathbf{e}}_i$, we can rewrite the streamline equation in the vector form

$$d\mathbf{x} = \mathbf{v}(\mathbf{x}, t)d\tau, \quad \tau_1 < \tau \leqslant \tau_2,$$
$$\mathbf{x} = \mathbf{x}_1, \quad \tau = \tau_1. \tag{4.147}$$

Let us define now a *vortex line* passing through a point $\mathcal{M}_1$: this is a curve, which at its every point x^i has a tangent being parallel to the vorticity vector $\boldsymbol{\omega}(x^j, t)$ at the considered point and at the fixed time t. The vortex line is described by the equation

$$d\mathbf{x} = \boldsymbol{\omega}(\mathbf{x}, t)d\tau, \quad \tau_1 < \tau \leqslant \tau_2, \tag{4.148}$$
$$\mathbf{x} = \mathbf{x}_1, \quad \tau = \tau_1.$$

1.4.13. Stream Tubes and Vortex Tubes. Consider a curve L in coordinates x^i and draw a streamline through each point of the curve L. If L is not a streamline itself, then we get a surface Σ_v, at each point of which the velocity $\mathbf{v}$ lies on the tangent plane to the surface. This surface is called the *stream surface*.

Let

$$f_v(x^i) = 0 \tag{4.149}$$

be the equation of the stream surface. Since the vector ∇f is normal to the surface Σ_v [12], so it is orthogonal to the velocity $\mathbf{v}$, i.e. we have the relation

$$\mathbf{v} \cdot \nabla f_v = 0, \tag{4.150}$$

which is a partial differential equation for determination of the function $f(x^i)$ by the known velocity field $\mathbf{v}(x^i, t)$ at fixed t.

If a curve L is closed, then the set of streamlines drawn through its points is called the *stream tube*.

Let a curve L be not a vortex line. Drawing a vortex line through each point of the curve L, we obtain the *vortex surface* Σ_ω, which is described by the equation $f_\omega(x^i) = 0$. This relation is a solution of the differential equation

$$\boldsymbol{\omega} \cdot \nabla f_\omega = 0. \tag{4.151}$$

If L is a closed curve, then the surface Σ_ω is called the *vortex tube*.

Exercises for 1.4.

Exercise 1.4.1. Show that the tensors $\mathbf{Q}_U$ and $\mathbf{Q}_V$ are orthogonal.

Exercise 1.4.2. Using formulae (4.104) and (2.57), show that for the coefficient $\overset{\circ}{k}$ determined by formula (2.54) we have the following relationship:

$$\overset{\circ}{k}{}^\bullet = \overset{\circ}{k}(\mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n}).$$

Exercise 1.4.3. Using formulae (2.57), (4.99) and the result of Exercise 1.4.2, show that a rate of changing the normal $\mathbf{n}$ is determined as follows:

$$\dot{\mathbf{n}} = \gamma \mathbf{n} - \mathbf{n} \cdot \mathbf{L}, \qquad \gamma = \mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n}.$$

Exercise 1.4.4. Using the results of Exercise 1.4.2, show that for the coefficient k determined by formula (2.54) we have the following equation:

$$\dot{k} = -k(\mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n}).$$

Exercise 1.4.5. Show that the transformations (4.65) of infinitesimal tension–compression (4.65) and infinitesimal rotation (4.66) are commutative up to terms of order $(dt)^2$:

$$\mathbf{Q}_\omega \cdot \mathbf{A}_D \cdot d\mathbf{x} = \mathbf{A}_\omega \cdot \mathbf{Q}_D \cdot d\mathbf{x},$$

while transformations of a small neighborhood determined by the tensors $\mathbf{O}$ and $\mathbf{U}$ or $\mathbf{O}$ and $\mathbf{V}$ are not commutative in general.

Exercise 1.4.6. Prove Theorem 1.14.

Exercise 1.4.7. Using the representation (3.6) for tensors $\mathbf{U}$ and $\mathbf{V}$ and formulae (4.90) and (4.92), show that rates of stretch tensors are expressed as follows:

$$\dot{\mathbf{U}} = \sum_{\alpha=1}^{3} \dot{\lambda}_\alpha \overset{\circ}{\mathbf{p}}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\alpha + \mathbf{\Omega}_U \cdot \mathbf{U} - \mathbf{U} \cdot \mathbf{\Omega}_U, \quad \dot{\mathbf{V}} = \sum_{\alpha=1}^{3} \dot{\lambda}_\alpha \mathbf{p}_\alpha \otimes \mathbf{p}_\alpha + \mathbf{\Omega}_V \cdot \mathbf{V} - \mathbf{V} \cdot \mathbf{\Omega}_V.$$

Exercise 1.4.8. Show that expressions for rates of the Hencky tensors (3.31) have the form

$$\overset{\circ}{\mathbf{H}}{}^{\bullet} = \sum_{\alpha=1}^{3} \frac{\dot{\lambda}_\alpha}{\lambda_\alpha} \overset{\circ}{\mathbf{p}}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\alpha + \mathbf{\Omega}_U \cdot \overset{\circ}{\mathbf{H}} - \overset{\circ}{\mathbf{H}} \cdot \mathbf{\Omega}_U, \qquad \mathbf{H}^{\bullet} = \sum_{\alpha=1}^{3} \frac{\dot{\lambda}_\alpha}{\lambda_\alpha} \mathbf{p}_\alpha \otimes \mathbf{p}_\alpha + \mathbf{\Omega}_V \cdot \mathbf{H} - \mathbf{H} \cdot \mathbf{\Omega}_V.$$

Exercise 1.4.9. Using representations (4.114) and (4.124) for tensors $\mathbf{\Omega}_U$ and $\mathbf{\Omega}_V$, and also equation (4.120) and the result of Exercise 1.4.8, show that rates of the Hencky tensors can be expressed in the form

$$\overset{\circ}{\mathbf{H}}{}^{\bullet} = \sum_{\alpha,\beta=1}^{3} D_{\alpha\beta} \overset{\circ}{\mathbf{p}}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\beta + \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{3} \Big(\frac{2\lambda_\alpha\lambda_\beta}{\lambda_\beta^2 - \lambda_\alpha^2} \lg\frac{\lambda_\beta}{\lambda_\alpha} - 1\Big) D_{\alpha\beta} \overset{\circ}{\mathbf{p}}_\alpha \otimes \overset{\circ}{\mathbf{p}}_\beta,$$

$$\dot{\mathbf{H}} = \mathbf{D} + \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{3} \Big(\frac{2\lambda_\alpha\lambda_\beta}{\lambda_\beta^2 - \lambda_\alpha^2} \lg\frac{\lambda_\beta}{\lambda_\alpha} - 1\Big) (\mathbf{p}_\alpha \cdot \mathbf{D} \cdot \mathbf{p}_\beta) \mathbf{p}_\alpha \otimes \mathbf{p}_\beta.$$

Exercise 1.4.10. Show that expressions for $\overset{\circ}{\mathbf{H}}{}^{\bullet}$ and $\dot{\mathbf{H}}$ derived in Exercise 1.4.9 can be rewritten as follows:

$$\overset{\circ}{\mathbf{H}}{}^{\bullet} = {}^4\overset{\circ}{\mathbf{X}}_H \cdot\cdot\, \mathbf{D}, \qquad \dot{\mathbf{H}} = {}^4\mathbf{X}_H \cdot\cdot\, \mathbf{D},$$

where the following fourth-order tensors are denoted:

$${}^4\overset{\circ}{\mathbf{X}}_H = X_{H\ ijkl} \overset{\circ}{\mathbf{p}}{}^i \otimes \overset{\circ}{\mathbf{p}}{}^j \otimes \mathbf{p}^k \otimes \mathbf{p}^l, \quad {}^4\mathbf{X}_H = X_{H\ ijkl} \mathbf{p}^i \otimes \mathbf{p}^j \otimes \mathbf{p}^k \otimes \mathbf{p}^l,$$

$$X_{H\ ijkl} = \begin{cases} \Big(\dfrac{2\lambda_\alpha\lambda_\beta}{\lambda_\beta^2 - \lambda_\alpha^2} \lg\dfrac{\lambda_\beta}{\lambda_\alpha}\Big) \Delta_{\alpha\beta kl}, & \alpha \neq \beta, \\ \Delta_{\alpha\beta kl}, & \alpha = \beta, \end{cases} \qquad \Delta_{\alpha\beta kl} = (1/2)(\delta_{\alpha k}\delta_{\beta l} + \delta_{\alpha l}\delta_{\beta k}).$$

Exercise 1.4.11. Show that relations (4.114) and (4.122) for $\mathbf{\Omega}_U$, equations (4.123) and (4.124) for $\mathbf{\Omega}_V$ and equation (4.129) for $\mathbf{\Omega}$ can be rewritten as follows:

$$\mathbf{\Omega}_U = {}^4\mathbf{\Omega}_U \cdot\cdot\, \mathbf{D}, \qquad \mathbf{\Omega}_V = {}^4\mathbf{\Omega}_V \cdot\cdot\, \mathbf{D} + \mathbf{W}, \qquad \mathbf{\Omega} = {}^4\widetilde{\mathbf{\Omega}} \cdot\cdot\, \mathbf{D} + \mathbf{W},$$

where

$${}^4\mathbf{\Omega}_U = \Omega_{Uijkl} \overset{\circ}{\mathbf{p}}{}^i \otimes \overset{\circ}{\mathbf{p}}{}^j \otimes \mathbf{p}^k \otimes \mathbf{p}^l, \qquad {}^4\mathbf{\Omega}_V = \Omega_{Vijkl} \mathbf{p}^i \otimes \mathbf{p}^j \otimes \mathbf{p}^k \otimes \mathbf{p}^l,$$

$${}^4\widetilde{\mathbf{\Omega}} = \widetilde{\Omega}_{ijkl} \mathbf{p}^i \otimes \mathbf{p}^j \otimes \mathbf{p}^k \otimes \mathbf{p}^l,$$

$$\Omega_{U\alpha\beta kl} = \begin{cases} \dfrac{2\lambda_\alpha\lambda_\beta}{\lambda_\beta^2 - \lambda_\alpha^2} \Delta_{\alpha\beta kl}, & \alpha \neq \beta, \\ 0, & \alpha = \beta, \end{cases} \qquad \Omega_{V\alpha\beta kl} = \begin{cases} \dfrac{\lambda_\alpha^2 + \lambda_\beta^2}{\lambda_\beta^2 - \lambda_\alpha^2} \Delta_{\alpha\beta kl}, & \alpha \neq \beta, \\ 0, & \alpha = \beta, \end{cases}$$

$$\widetilde{\Omega}_{\alpha\beta\gamma\rho} = \Omega_{V\alpha\beta\gamma\rho} - \frac{\lambda_\gamma\lambda_\rho}{\lambda_\rho^2 - \lambda_\gamma^2}(l_{\gamma\alpha}l_{\rho\beta} - l_{\rho\alpha}l_{\gamma\beta}), \quad \alpha \neq \beta, \quad \gamma \neq \rho,$$

if $\alpha = \beta$ (or $\gamma = \rho$), then the first (or the second) summand vanishes.

Exercise 1.4.12. Using the definitions (4.45) and (4.46) of the tensors $\mathbf{D}$ and $\mathbf{W}$, and also the properties of unit tensors, show that $\mathbf{D}$, $\mathbf{W}$ and $\mathbf{L}$ are connected by the formulae

$$\mathbf{D} = \boldsymbol{\Delta} \cdot\cdot\, \mathbf{L}, \quad \mathbf{W} = \widetilde{\boldsymbol{\Delta}} \cdot\cdot\, \mathbf{L}, \quad \widetilde{\boldsymbol{\Delta}} = (1/2)(\Delta_{\mathrm{III}} - \Delta_{\mathrm{II}}).$$

1.5. Co-rotational Derivatives

1.5.1. Definition of Co-rotational Derivatives. Besides the total derivative $d\mathbf{a}/dt$ introduced in paragraph 1.4.1 and partial derivative of vectors and tensors with respect to time $\partial\mathbf{a}/\partial t$, so-called *co-rotational derivatives* are of great importance in continuum mechanics. They determine rates of changing tensors relative to some *moving basis* $\mathbf{h}_i$, i.e. the relative rates.

Let in a actual configuration $\mathcal{K}(t)$ there be some moving bases $\mathbf{h}_i$ or $\mathbf{h}^i$ and arbitrary varying scalar $\psi(X^i, t)$, vector $\mathbf{a}(X^i, t)$ and second-order tensor $\mathbf{T}(X^i, t)$ fields with the following components with respect to the bases:

$$\mathbf{a} = a^i\mathbf{h}_i = a_i\mathbf{h}^i, \tag{5.1}$$

$$\mathbf{T} = T^{ij}\mathbf{h}_i \otimes \mathbf{h}_j = T_{ij}\mathbf{h}^i \otimes \mathbf{h}^j = T^i{}_j\mathbf{h}_i \otimes \mathbf{h}^j = T_i{}^j\mathbf{h}^i \otimes \mathbf{h}_j. \tag{5.2}$$

Since any scalar function $\psi(X^i, t)$ is not connected to any basis (moving or fixed), it is evident that the co-rotational derivative of the function must be coincident with the total derivative with respect to time:

$$\psi^h = \dot{\psi}. \tag{5.3}$$

For a vector $\mathbf{a}$ and a tensor $\mathbf{T}$ we can introduce *co-rotational derivatives* $\mathbf{a}^h$ and $\mathbf{T}^h$ as vectors or tensors, components of which with respect to the same basis $\mathbf{h}_i$ coincide with rates of changing vector $\mathbf{a}$ and tensor $\mathbf{T}$ components, respectively:

$$\mathbf{a}^h = \frac{da^i}{dt}\mathbf{h}_i, \qquad \mathbf{T}^h = \frac{d}{dt}T^{ij}\mathbf{h}_i \otimes \mathbf{h}_j. \tag{5.4}$$

If we consider the basis $\mathbf{h}^i$, then for the basis we can determine other co-rotational derivatives:

$$\mathbf{a}^H = \frac{da_i}{dt}\mathbf{h}^i, \tag{5.5}$$

$$\mathbf{T}^H = \frac{d}{dt}T_{ij}\mathbf{h}^i \otimes \mathbf{h}^j. \tag{5.6}$$

Thus, the co-rotational derivative $\mathbf{a}^h$ (or $\mathbf{T}^h$) determines the rate of varying a vector $\mathbf{a}$ (or a tensor $\mathbf{T}$) for an observer moving together with the basis $\mathbf{h}_i$. For the observer, the basis $\mathbf{h}_i$ is fixed, and hence in (5.4) the basis is not differentiated with respect to time. In a similar way, the derivatives $\mathbf{a}^H$ and $\mathbf{T}^H$ determine the rates of changing $\mathbf{a}$ and $\mathbf{T}$ for an observer moving together with the basis $\mathbf{h}^i$.

We can determine the co-rotational derivatives of a second-order tensor $\mathbf{T}$ in mixed moving dyadic bases $\mathbf{h}_i \otimes \mathbf{h}^j$ and $\mathbf{h}^i \otimes \mathbf{h}_j$, respectively:

$$\mathbf{T}^d = \frac{d}{dt}T^i{}_j \mathbf{h}_i \otimes \mathbf{h}^j, \qquad \mathbf{T}^D = \frac{d}{dt}T_i{}^j \mathbf{h}^i \otimes \mathbf{h}_j. \tag{5.7}$$

Since vector components a^i and a_i can always be expressed in the form

$$a^i = \mathbf{a} \cdot \mathbf{h}^i, \qquad a_i = \mathbf{a} \cdot \mathbf{h}_i, \tag{5.8}$$

and tensor components T_{ij}, T^{ij}, $T^i{}_j$ and $T_i{}^j$, with the help of the scalar product of (5.2) by $\mathbf{h}^i$ or $\mathbf{h}_j$, can be written as follows:

$$T^{ij} = \mathbf{h}^i \cdot \mathbf{T} \cdot \mathbf{h}^j, \qquad T_{ij} = \mathbf{h}_i \cdot \mathbf{T} \cdot \mathbf{h}_j, \qquad T^i{}_j = \mathbf{h}^i \cdot \mathbf{T} \cdot \mathbf{h}_j, \qquad T_i{}^j = \mathbf{h}_i \cdot \mathbf{T} \cdot \mathbf{h}^j, \tag{5.9}$$

so rates of changing vector and tensor components in (5.4), (5.5) and (5.7) can be represented in the explicit form:

$$\frac{da^i}{dt} = \frac{d\mathbf{a}}{dt} \cdot \mathbf{h}^i + \mathbf{a} \cdot \frac{d\mathbf{h}^i}{dt}, \qquad \frac{da_i}{dt} = \frac{d\mathbf{a}}{dt} \cdot \mathbf{h}_i + \mathbf{a} \cdot \frac{d\mathbf{h}_i}{dt}, \tag{5.10}$$

and also

$$\begin{aligned}
\frac{dT^{ij}}{dt} &= \mathbf{h}^i \cdot \frac{d\mathbf{T}}{dt} \cdot \mathbf{h}^j + \frac{d\mathbf{h}^i}{dt} \cdot \mathbf{T} \cdot \mathbf{h}^j + \mathbf{h}^i \cdot \mathbf{T} \cdot \frac{d\mathbf{h}^j}{dt}, \\
\frac{dT_{ij}}{dt} &= \mathbf{h}_i \cdot \frac{d\mathbf{T}}{dt} \cdot \mathbf{h}_j + \frac{d\mathbf{h}_i}{dt} \cdot \mathbf{T} \cdot \mathbf{h}_j + \mathbf{h}_i \cdot \mathbf{T} \cdot \frac{d\mathbf{h}_j}{dt}, \\
\frac{d}{dt}T^i{}_j &= \mathbf{h}^i \cdot \frac{d}{dt}\mathbf{T} \cdot \mathbf{h}_j + \frac{d\mathbf{h}^i}{dt} \cdot \mathbf{T} \cdot \mathbf{h}_j + \mathbf{h}^i \cdot \mathbf{T} \cdot \frac{d\mathbf{h}_j}{dt}, \\
\frac{d}{dt}T_i{}^j &= \mathbf{h}_i \cdot \frac{d}{dt}\mathbf{T} \cdot \mathbf{h}^j + \frac{d\mathbf{h}_i}{dt} \cdot \mathbf{T} \cdot \mathbf{h}^j + \mathbf{h}_i \cdot \mathbf{T} \cdot \frac{d}{dt}\mathbf{h}^j.
\end{aligned} \tag{5.11}$$

Here total derivatives $d\mathbf{a}/dt$ and $d\mathbf{T}/dt$ are determined by the rules (4.7) and (4.12), respectively. Rates of changing basis vectors $d\mathbf{h}^i/dt$ and $d\mathbf{h}_i/dt$ are defined by the choice of basis $\mathbf{h}_i$ or $\mathbf{h}^j$.

Taking different bases as $\mathbf{h}_i$ and $\mathbf{h}^j$, we get different co-rotational derivatives. Let us consider the most widely used bases.

1.5.2. The Oldroyd Derivative($\mathbf{h}_i = \mathbf{r}_i$). If we choose the general local vector basis $\mathbf{r}_i$ as $\mathbf{h}_i$, then the derivative $\mathbf{a}^h = \mathbf{a}^{\mathrm{Ol}}$ (or $\mathbf{T}^h = \mathbf{T}^{\mathrm{Ol}}$) determines the rate of changing $\mathbf{a}$ (or $\mathbf{T}$) relative to the Lagrangian coordinate system X^i moving together with the continuum. This derivative is called the *Oldroyd derivative*.

The derivative $d\mathbf{r}_i/dt$ is determined as follows:

$$\frac{d\mathbf{h}_i}{dt} = \frac{d\mathbf{r}_i}{dt} = \frac{\partial^2 \mathbf{x}}{\partial t \partial X^i} = \frac{\partial \mathbf{v}}{\partial X^i} = \mathbf{r}_i \cdot \mathbf{r}^j \otimes \frac{\partial \mathbf{v}}{\partial X^j} = \mathbf{r}_i \cdot \boldsymbol{\nabla} \otimes \mathbf{v} = (\boldsymbol{\nabla} \otimes \mathbf{v})^{\mathrm{T}} \cdot \mathbf{r}_i. \tag{5.12}$$

In this case, as a basis $\mathbf{h}^i$ we consider the reciprocal local basis $\mathbf{r}^i$, the derivative of which $d\mathbf{r}^i/dt$ with respect to time has the form

$$\frac{d}{dt}(\mathbf{r}^i \otimes \mathbf{r}_i) = \frac{d}{dt}\mathbf{E} = 0, \tag{5.13}$$

or

$$\frac{d\mathbf{r}^i}{dt} \otimes \mathbf{r}_i = -\mathbf{r}^i \otimes \frac{d\mathbf{r}_i}{dt} = -\mathbf{r}^i \otimes (\boldsymbol{\nabla} \otimes \mathbf{v})^{\mathrm{T}} \cdot \mathbf{r}_i. \tag{5.14}$$

Multiplying the equation by $\mathbf{r}^j$ from the right, we get

$$\frac{d\mathbf{h}^j}{dt} = \frac{d\mathbf{r}^j}{dt} = -\mathbf{r}^j \cdot (\boldsymbol{\nabla} \otimes \mathbf{v})^{\mathrm{T}} = -(\boldsymbol{\nabla} \otimes \mathbf{v}) \cdot \mathbf{r}^j. \tag{5.15}$$

On substituting the expressions (5.15) for the derivatives $d\mathbf{h}^i/dt$ and $d\mathbf{h}_j/dt$ into (5.10), we get the formula for the Oldroyd derivative in the basis $\mathbf{r}_i$:

$$\mathbf{a}^{\mathrm{Ol}} = \frac{da^i}{dt}\mathbf{r}_i = \frac{d\mathbf{a}}{dt} \cdot \mathbf{r}^i \otimes \mathbf{r}_i - \mathbf{a} \cdot (\boldsymbol{\nabla} \otimes \mathbf{v}) \cdot \mathbf{r}^i \otimes \mathbf{r}_i = \frac{d}{dt}\mathbf{a} - \mathbf{a} \cdot \boldsymbol{\nabla} \otimes \mathbf{v}, \tag{5.16}$$

$$\begin{aligned}\mathbf{T}^{\mathrm{Ol}} = \frac{d\mathbf{T}^{ij}}{dt}\mathbf{r}_i \otimes \mathbf{r}_j = \mathbf{r}_i \otimes \frac{dT^{ij}}{dt}\mathbf{r}_j = \mathbf{r}_i \otimes \mathbf{r}^i \cdot \frac{d}{dt}\mathbf{T} \cdot \mathbf{r}^j \otimes \mathbf{r}_j - \\ - \mathbf{r}_i \otimes \mathbf{r}^i \cdot (\boldsymbol{\nabla} \otimes \mathbf{v})^{\mathrm{T}} \cdot \mathbf{T} \cdot \mathbf{r}^j \otimes \mathbf{r}_j - \mathbf{r}_i \otimes \mathbf{r}^i \cdot \mathbf{T} \cdot (\boldsymbol{\nabla} \otimes \mathbf{v}) \cdot \mathbf{r}^j \otimes \mathbf{r}_j = \\ = \frac{d}{dt}\mathbf{T} - \mathbf{T} \cdot \boldsymbol{\nabla} \otimes \mathbf{v} - (\boldsymbol{\nabla} \otimes \mathbf{v})^{\mathrm{T}} \cdot \mathbf{T}.\end{aligned} \tag{5.17}$$

Here we have taken into account that $\mathbf{r}^i \otimes \mathbf{r}_i = \mathbf{E}$. Thus, we have proved the following theorem.

Theorem 1.29. *The Oldroyd derivative is related to the total derivative with respect to time as follows (for a vector* $\mathbf{a}$ *and for a tensor* $\mathbf{T}$*, respectively):*

$$\mathbf{a}^{\mathrm{Ol}} = \dot{\mathbf{a}} - \mathbf{a} \cdot \boldsymbol{\nabla} \otimes \mathbf{v}, \qquad \mathbf{T}^{\mathrm{Ol}} = \dot{\mathbf{T}} - \mathbf{T} \cdot \boldsymbol{\nabla} \otimes \mathbf{v} - (\boldsymbol{\nabla} \otimes \mathbf{v})^{\mathrm{T}} \cdot \mathbf{T}. \tag{5.18}$$

1.5.3. The Cotter–Rivlin Derivative ($\mathbf{h}^i = \mathbf{r}^i$). If we choose the reciprocal local basis $\mathbf{r}^i$ as a moving basis $\mathbf{h}^i$, then the derivative $\mathbf{a}^H$ (or $\mathbf{T}^H$) characterizes the rate of changing $\mathbf{a}$ (or $\mathbf{T}$) relative to the basis $\mathbf{r}^i$ moving together with the Lagrangian coordinate system X^i. This derivative is called the *Cotter-Rivlin derivative.*

Because of formulae (5.10) and (5.15), we get the following theorem.

Theorem 1.30. *The Cotter–Rivlin derivative is related to the total derivative as follows (for a vector* $\mathbf{a}$ *and for a tensor* $\mathbf{T}$*, respectively):*

$$\mathbf{a}^H \equiv \mathbf{a}^{\mathrm{CR}} = \frac{da_i}{dt}\mathbf{r}^i = \dot{\mathbf{a}} + (\boldsymbol{\nabla} \otimes \mathbf{v}) \cdot \mathbf{a}, \tag{5.19}$$

$$\mathbf{T}^H \equiv \mathbf{T}^{\mathrm{CR}} = \frac{dT_{ij}}{dt}\mathbf{r}^i \otimes \mathbf{r}^j = \dot{\mathbf{T}} + \boldsymbol{\nabla} \otimes \mathbf{v} \cdot \mathbf{T} + \mathbf{T} \cdot (\boldsymbol{\nabla} \otimes \mathbf{v})^{\mathrm{T}}. \tag{5.20}$$

1.5.4. Mixed Co-rotational Derivatives. Since any vector $\mathbf{a}$ is defined by its components with respect to a vector basis, for example, in a moving basis $\mathbf{h}_i$ or $\mathbf{h}^j$, so for the vector in the moving bases we can determine only two co-rotational derivatives: by Oldroyd and by Cotter–Rivlin.

Any second-order tensor $\mathbf{T}$ is defined by its components with respect to a dyadic basis. Therefore, besides the Oldroyd and Cotter–Rivlin derivatives, which specify the rates of changes of a tensor $\mathbf{T}$ in moving dyadic bases $\mathbf{r}_i \otimes \mathbf{r}_j$ and $\mathbf{r}^i \otimes \mathbf{r}^j$, by formulae (5.7) we can determine two more derivatives in moving mixed dyadic bases:

$$\mathbf{T}^d = \frac{dT^i{}_j}{dt}\mathbf{r}_i \otimes \mathbf{r}^j, \qquad \mathbf{T}^D = \frac{d}{dt}T_i{}^j\mathbf{r}^i \otimes \mathbf{r}_j. \tag{5.21}$$

On substituting the expressions (5.12) and (5.15) into (5.11), we get the following formulae for the rates of changing mixed components of the tensor $\mathbf{T}$:

$$\frac{d}{dt}T^i{}_j = \mathbf{r}^i \cdot \dot{\mathbf{T}} \cdot \mathbf{r}_j - \mathbf{r}^i \cdot (\boldsymbol{\nabla} \otimes \mathbf{v})^{\mathrm{T}} \cdot \mathbf{T} \cdot \mathbf{r}_j + \mathbf{r}^i \cdot \mathbf{T} \cdot (\boldsymbol{\nabla} \otimes \mathbf{v})^{\mathrm{T}} \cdot \mathbf{r}_j,$$

$$\frac{d}{dt}T_i{}^j = \mathbf{r}_i \cdot \dot{\mathbf{T}} \cdot \mathbf{r}^j + \mathbf{r}_i \cdot (\boldsymbol{\nabla} \otimes \mathbf{v}) \cdot \mathbf{T} \cdot \mathbf{r}^j - \mathbf{r}_i \cdot \mathbf{T} \cdot (\boldsymbol{\nabla} \otimes \mathbf{v}) \cdot \mathbf{r}^j. \tag{5.22}$$

Having substituted (5.22) into (5.21), we get the following theorem.
Theorem 1.31. *The mixed derivatives (5.21) are connected to the total derivative by the relations*

$$\mathbf{T}^d = \dot{\mathbf{T}} - \mathbf{L} \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{L}, \qquad \mathbf{T}^D = \dot{\mathbf{T}} + \mathbf{L}^{\mathrm{T}} \cdot \mathbf{T} - \mathbf{T} \cdot \mathbf{L}^{\mathrm{T}}. \tag{5.23}$$

The derivatives (5.21) are called the *left and right mixed co-rotational derivatives*, where $\mathbf{L} = (\nabla \otimes \mathbf{v})^{\mathrm{T}}$ is the velocity gradient (see (4.43)).

It should be noticed that, unlike other co-rotational derivatives considered in this paragraph, the mixed derivatives $\mathbf{T}^d$ and $\mathbf{T}^D$ do not form a symmetric tensor when they are applied to a symmetric tensor $\mathbf{T}$. This fact explains a scarcer application of mixed derivatives in continuum mechanics.

1.5.5. The Derivative Relative to the Eigenbasis $\overset{\circ}{\mathbf{p}}_i$ of the Right Stretch Tensor. If we choose the eigenbasis $\overset{\circ}{\mathbf{p}}_i$ of the right stretch tensor $\mathbf{U}$ as a moving basis $\mathbf{h}_i$, then, since $\overset{\circ}{\mathbf{p}}_i$ are orthonormal, we get that $\mathbf{h}^i$ and $\mathbf{h}_j$ are coincident: $\mathbf{h}^\alpha = \mathbf{h}_\alpha$, $\alpha = 1, 2, 3$, and $|\mathbf{h}^i| = 1$. At every time, the moving coordinate system defined by the trihedron $\overset{\circ}{\mathbf{p}}_i$ executes an instantaneous rotation, which is characterized by the spin $\boldsymbol{\Omega}_U$ (4.89), and due to (4.90) we have

$$\frac{d\mathbf{h}_i}{dt} = \dot{\overset{\circ}{\mathbf{p}}}_i = \boldsymbol{\Omega}_U \cdot \overset{\circ}{\mathbf{p}}_i = \overset{\circ}{\mathbf{p}}_i \cdot \boldsymbol{\Omega}_U^{\mathrm{T}} = -\overset{\circ}{\mathbf{p}}_i \cdot \boldsymbol{\Omega}_U. \tag{5.24}$$

On substituting (5.24) into (5.11), we get

$$\mathbf{a}^h \equiv \mathbf{a}^U = \frac{da^i}{dt}\overset{\circ}{\mathbf{p}}_i = \dot{\mathbf{a}} \cdot \overset{\circ}{\mathbf{p}}{}^i \otimes \overset{\circ}{\mathbf{p}}_i + \mathbf{a} \cdot \boldsymbol{\Omega}_U \cdot \overset{\circ}{\mathbf{p}}{}^i \otimes \overset{\circ}{\mathbf{p}}_i = \dot{\mathbf{a}} + \mathbf{a} \cdot \boldsymbol{\Omega}_U, \tag{5.25}$$

$$\begin{aligned}\mathbf{T}^h \equiv \mathbf{T}^U = \frac{T^{ij}}{dt}\overset{\circ}{\mathbf{p}}_i \otimes \overset{\circ}{\mathbf{p}}_j = \overset{\circ}{\mathbf{p}}_i \otimes \frac{dT^{ij}}{dt}\overset{\circ}{\mathbf{p}}_j = \overset{\circ}{\mathbf{p}}{}^i \otimes \overset{\circ}{\mathbf{p}}_i \cdot \dot{\mathbf{T}} \cdot \overset{\circ}{\mathbf{p}}{}^j \otimes \overset{\circ}{\mathbf{p}}_j - \\ - \overset{\circ}{\mathbf{p}}{}^i \otimes \overset{\circ}{\mathbf{p}}_i \cdot \boldsymbol{\Omega}_U \cdot \mathbf{T} \cdot \overset{\circ}{\mathbf{p}}{}^j \otimes \overset{\circ}{\mathbf{p}}_j + \overset{\circ}{\mathbf{p}}{}^i \otimes \overset{\circ}{\mathbf{p}}_i \cdot \mathbf{T} \cdot \boldsymbol{\Omega}_U \cdot \overset{\circ}{\mathbf{p}}{}^j \otimes \overset{\circ}{\mathbf{p}}_j = \\ = \dot{\mathbf{T}} - \boldsymbol{\Omega}_U \cdot \mathbf{T} + \mathbf{T} \cdot \boldsymbol{\Omega}_U.\end{aligned} \tag{5.26}$$

The co-rotational derivative of a vector $\mathbf{a}$ (or a tensor $\mathbf{T}$) determined by (5.26) is called the *right derivative relative to the eigenbasis*.

Thus, we have proved the following theorem.
Theorem 1.32. *The right derivative relative to the eigenbasis is connected to the total derivative as follows (for a vector* $\mathbf{a}$ *and for a tensor* $\mathbf{T}$*, respectively):*

$$\mathbf{a}^U = \dot{\mathbf{a}} + \mathbf{a} \cdot \boldsymbol{\Omega}_U, \qquad \mathbf{T}^U = \dot{\mathbf{T}} - \boldsymbol{\Omega}_U \cdot \mathbf{T} + \mathbf{T} \cdot \boldsymbol{\Omega}_U. \tag{5.27}$$

1.5.6. The Derivative in the Eigenbasis ($\mathbf{h}_i = \mathbf{p}_i$) of the Left Stretch Tensor. Take the eigenbasis $\mathbf{p}_i$ of the left stretch tensor $\mathbf{V}$ as a moving basis $\mathbf{h}_i$ and define the following co-rotational derivatives

$$\mathbf{a}^H \equiv \mathbf{a}^V = \frac{da^i}{dt}\mathbf{p}_i, \qquad \mathbf{T}^H \equiv \mathbf{T}^V = \frac{dT^{ij}}{dt}\mathbf{p}_i \otimes \mathbf{p}_j, \tag{5.28}$$

called the *left derivatives in the eigenbasis*.

Theorem 1.33. *The left derivatives (5.28) in the eigenbasis are connected to the total derivative with respect to time by the following relations (for a vector* $\mathbf{a}$ *and for a tensor* $\mathbf{T}$*, respectively):*

$$\mathbf{a}^V = \dot{\mathbf{a}} - \boldsymbol{\Omega}_V \cdot \mathbf{a}, \qquad \mathbf{T}^V = \dot{\mathbf{T}} - \boldsymbol{\Omega}_V \cdot \mathbf{T} + \mathbf{T} \cdot \boldsymbol{\Omega}_V. \tag{5.29}$$

▼ A proof follows from (5.5), (5.10) and (5.11), because from (4.92) we have

$$\frac{d\mathbf{h}_i}{dt} = \dot{\mathbf{p}}_i = \boldsymbol{\Omega}_V \cdot \mathbf{p}_i. \tag{5.30}$$

Since the bases $\overset{\circ}{\mathbf{p}}_i$ and $\mathbf{p}_i$ are orthonormal, all the co-rotational derivatives relative to the mixed dyadic bases $\overset{\circ}{\mathbf{p}}_i \otimes \overset{\circ}{\mathbf{p}}{}^i$, $\mathbf{p}^i \otimes \mathbf{p}_i$ coincide with $\mathbf{T}^U$ or $\mathbf{T}^V$, respectively. ▲

1.5.7. The Jaumann Derivative ($\mathbf{h}_i = \mathbf{q}_i$). If we choose the eigenbasis of the deformation rate tensor as a moving basis $\mathbf{h}_i = \mathbf{q}_i$ (it should be noted that the basis $\mathbf{q}_i$ is also orthonormal and coincides with $\mathbf{q}^i$), then from (5.4) we get the *co-rotational Jaumann derivatives*:

$$\mathbf{a}^h \equiv \mathbf{a}^J = \frac{da^i}{dt}\mathbf{q}_i, \qquad \mathbf{T}^h \equiv \mathbf{T}^J = \frac{dT^{ij}}{dt}\mathbf{q}_i \otimes \mathbf{q}_j. \tag{5.31}$$

Theorem 1.34. *The Jaumann derivatives (5.31) are connected to the total derivatives with respect to time by the relations (for a vector* $\mathbf{a}$ *and for a tensor* $\mathbf{T}$*, respectively)*

$$\mathbf{a}^J = \dot{\mathbf{a}} + \mathbf{a} \cdot \mathbf{W}, \tag{5.32}$$

$$\mathbf{T}^J = \dot{\mathbf{T}} - \mathbf{W} \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{W}. \tag{5.33}$$

▼ According to the relationship (4.94), we get

$$\frac{d\mathbf{h}_i}{dt} = \dot{\mathbf{q}}_i = \mathbf{W} \cdot \mathbf{q}_i,$$

therefore, due to formulae (5.5) and (5.10) we find

$$\mathbf{a}^J = \dot{\mathbf{a}} \cdot \mathbf{q}_i \otimes \mathbf{q}^i + \mathbf{a} \cdot \mathbf{W} \cdot \mathbf{q}_i \otimes \mathbf{q}^i = \dot{\mathbf{a}} + \mathbf{a} \cdot \mathbf{W}.$$

In a similar way, we can prove the relation (5.33). ▲

1.5.8. Co-rotational Derivatives in a Moving Orthonormal Basis. Let $\mathbf{h}_i$ be a moving orthonormal basis. In this case we denote co-rotational derivatives by the following way: $\mathbf{a}^h \equiv \mathbf{a}^Q$ and $\mathbf{T}^h \equiv \mathbf{T}^Q$. Due to orthonormalization of the basis $\mathbf{h}_i$, the total derivatives of $\mathbf{a}$ and $\mathbf{T}$ with taking account of (5.1), (5.2) and (5.4) can be written as follows:

$$\dot{\mathbf{a}} \equiv \frac{d\mathbf{a}}{dt} = \frac{da^i}{dt}\mathbf{h}_i + a^i\frac{d\mathbf{h}_i}{dt} = \mathbf{a}^Q + a^i\boldsymbol{\omega}_h \times \mathbf{h}_i, \tag{5.34}$$

$$\dot{\mathbf{T}} = \frac{dT^{ij}}{dt}\mathbf{h}_i \otimes \mathbf{h}_j + T^{ij}\Big(\frac{d\mathbf{h}_i}{dt} \otimes \mathbf{h}_j + \mathbf{h}_i \otimes \frac{d\mathbf{h}_j}{dt}\Big) =$$
$$= \mathbf{T}^Q + T^{ij}\boldsymbol{\omega}_h \times \mathbf{h}_i \otimes \mathbf{h}_j - \mathbf{h}_i \otimes \mathbf{h}_j \times \boldsymbol{\omega}_h.$$

Here we have used formula (4.86) for derivative $\dot{\mathbf{h}}_i$ of the moving basis, where $\boldsymbol{\omega}_h$ is the *vorticity vector* giving a rotation of the basis $\mathbf{h}_i$ relative to the fixed basis $\bar{\mathbf{e}}_i$ (see (4.78) and (4.85)). With taking account of (5.1) and (5.2), formulae (5.34) can be written in the form

$$\mathbf{a}^Q = \dot{\mathbf{a}} - \boldsymbol{\omega}_h \times \mathbf{a}, \qquad \mathbf{T}^Q = \dot{\mathbf{T}} - \boldsymbol{\omega}_h \times \mathbf{T} + \mathbf{T} \times \boldsymbol{\omega}_h. \tag{5.35}$$

It should be noticed that if $\mathbf{a} = \boldsymbol{\omega}_h$, then

$$\dot{\boldsymbol{\omega}}_h = \boldsymbol{\omega}_h^h, \tag{5.35a}$$

because $\boldsymbol{\omega}_h \times \boldsymbol{\omega}_h = \mathbf{0}$ due to properties of the vector product.

1.5.9. Spin Derivative. Take an arbitrary orthonormal basis $\bar{\mathbf{h}}_i$ at a point $\mathcal{M}$ of a continuum in $\mathcal{K}$. The trihedron must have the only property that at any time t the basis $\bar{\mathbf{h}}_i$ rotates with the instantaneous angular rate, which is equal to the rotation rate of the trihedron $\mathbf{p}_i$ relative to the trihedron $\overset{\circ}{\mathbf{p}}_i$. As shown in 1.4.10, the instantaneous rotation of the trihedron is characterized by the spin tensor $\boldsymbol{\Omega} = \dot{\mathbf{O}} \cdot \mathbf{O}^{\mathrm{T}}$ determined by (4.97).

Then we can define the co-rotational derivative in the basis, which is called the *spin derivative* (of a vector $\mathbf{a}$ and of a tensor $\mathbf{T}$, respectively):

$$\mathbf{a}^h \equiv \mathbf{a}^S = \frac{da^i}{dt}\bar{\mathbf{h}}_i, \tag{5.36}$$

$$\mathbf{T}^h \equiv \mathbf{T}^S = \frac{dT^{ij}}{dt}\bar{\mathbf{h}}_i \otimes \bar{\mathbf{h}}_j. \tag{5.37}$$

Theorem 1.35. *The spin derivative is related to the total derivative with respect to time as follows (for a vector* $\mathbf{a}$ *and for a tensor* $\mathbf{T}$*, respectively):*

$$\mathbf{a}^S = \dot{\mathbf{a}} + \mathbf{a} \cdot \boldsymbol{\Omega}, \tag{5.38}$$

$$\mathbf{T}^S = \dot{\mathbf{T}} - \boldsymbol{\Omega} \cdot \mathbf{T} + \mathbf{T} \cdot \boldsymbol{\Omega}. \tag{5.39}$$

▼ A proof of Theorem 1.35 follows from (5.10), (5.11) and the relation

$$\dot{\bar{\mathbf{h}}}^i = \boldsymbol{\Omega} \cdot \bar{\mathbf{h}}^i, \tag{5.40}$$

which is a consequence of (4.84). The relation (5.39) follows from (5.11). ▲

1.5.10. Universal Form of the Co-rotational Derivatives. On comparing formulae (5.18), (5.19), (5.20), (5.27), (5.29), (5.33), (5.38) and (5.39), we can notice that all the representations of the co-rotational derivatives and also the total derivative with respect to time can be written in the universal form:

$$\mathbf{a}^h = \dot{\mathbf{a}} - \mathbf{Z}_h \cdot \mathbf{T}, \qquad \mathbf{T}^h = \dot{\mathbf{T}} - \mathbf{Z}_h \cdot \mathbf{T} - \mathbf{T} \cdot \mathbf{Z}_h^{\mathrm{T}}, \tag{5.41}$$
$$h = \{\ \cdot,\ \mathrm{Ol},\ \mathrm{CR},\ U,\ V,\ J,\ S\ \},$$

where tensors $\mathbf{Z}_h$ have the following representation for different h:

$$\mathbf{Z}_h = \{\ \mathbf{0},\ \mathbf{L},\ -\mathbf{L}^{\mathrm{T}},\ \boldsymbol{\Omega}_U,\ \boldsymbol{\Omega}_V,\ \mathbf{W},\ \boldsymbol{\Omega}\ \}, \tag{5.42}$$

$$h = \{\ \cdot,\ \mathrm{Ol},\ \mathrm{CR},\ U,\ V,\ J,\ S\ \}.$$

Since tensors $\boldsymbol{\Omega}_U$, $\boldsymbol{\Omega}_V$ and $\boldsymbol{\Omega}$ are linearly expressed in terms of $\mathbf{W}$ and $\mathbf{D}$ (see Exercise 1.4.11), so tensors $\mathbf{Z}_h$ can be written as linear functions of $\mathbf{W}$ and $\mathbf{D}$:

$$\mathbf{Z}_h = {}^4\mathbf{Z}_{Dh} \cdot\cdot\ \mathbf{D} + {}^4\mathbf{Z}_{Wh} \cdot\cdot\ \mathbf{W}. \tag{5.43}$$

Table 1.1 gives expressions for fourth-order tensors ${}^4\mathbf{Z}_{Dh}$ and ${}^4\mathbf{Z}_{Wh}$, where tensors ${}^4\boldsymbol{\Omega}_U$, ${}^4\boldsymbol{\Omega}_V$ and ${}^4\widetilde{\boldsymbol{\Omega}}$ are defined in Exercise 1.4.11.

Table 1.1. Expressions of tensors ${}^4\mathbf{Z}_{Dh}$, ${}^4\mathbf{Z}_{Wh}$ and ${}^4\mathbf{E}_h$ for different co-rotational derivatives

h	$\cdot$	Ol	CR	U	V	J	S
${}^4\mathbf{Z}_{Dh}$	$\mathbf{0}$	$\boldsymbol{\Delta}_{\mathrm{III}}$	$\boldsymbol{\Delta}_{-\mathrm{III}}$	${}^4\boldsymbol{\Omega}_U$	${}^4\boldsymbol{\Omega}_V$	$\mathbf{0}$	${}^4\widetilde{\boldsymbol{\Omega}}$
${}^4\mathbf{Z}_{Wh}$	$\mathbf{0}$	$\boldsymbol{\Delta}_{\mathrm{III}}$	$\boldsymbol{\Delta}_{\mathrm{III}}$	$\mathbf{0}$	$\boldsymbol{\Delta}_{\mathrm{III}}$	$\boldsymbol{\Delta}_{\mathrm{III}}$	$\boldsymbol{\Delta}_{\mathrm{III}}$
${}^4\mathbf{E}_h$	$\mathbf{0}$	$-2\boldsymbol{\Delta}$	$2\boldsymbol{\Delta}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

1.5.11. Relations between Co-rotational Derivatives of Deformation Rate Tensors and Velocity Gradient. In paragraph 1.4.11 we have derived the relationships between rates of deformation tensors and velocity gradient $\mathbf{L}$. Similar connections also exist between co-rotational derivatives of the tensors and $\mathbf{L}$. Let us establish them.

On substituting representations (4.103), (4.104) and (4.130) for rates $\dot{\mathbf{A}}$, $\dot{\mathbf{J}}$, $\dot{\mathbf{g}}$, $(\mathbf{g}^{-1})^{\bullet}$, $\dot{\mathbf{V}}$ and $(\mathbf{V}^{-1})^{\bullet}$ into formula (5.41), we get

$$\begin{gathered}
\mathbf{A}^h = \mathbf{D} - (\mathbf{Z}_h + \mathbf{L}^{\mathrm{T}}) \cdot \mathbf{A} - \mathbf{A} \cdot (\mathbf{Z}_h^{\mathrm{T}} + \mathbf{L}),\\
\mathbf{J}^h = \mathbf{D} - (\mathbf{Z}_h - \mathbf{L}) \cdot \mathbf{J} - \mathbf{J} \cdot (\mathbf{Z}_h^{\mathrm{T}} - \mathbf{L}^{\mathrm{T}}),\\
\mathbf{g}^h = -(\mathbf{Z}_h + \mathbf{L}^{\mathrm{T}}) \cdot \mathbf{g} - \mathbf{g} \cdot (\mathbf{Z}_h^{\mathrm{T}} + \mathbf{L}),\\
(\mathbf{g}^{-1})^h = -(\mathbf{Z}_h - \mathbf{L}) \cdot \mathbf{g}^{-1} - \mathbf{g}^{-1} \cdot (\mathbf{Z}_h^{\mathrm{T}} - \mathbf{L}^{\mathrm{T}}),\\
\mathbf{V}^h = -(\mathbf{Z}_h - \frac{1}{2}(\mathbf{L} + \boldsymbol{\Omega})) \cdot \mathbf{V} - \mathbf{V} \cdot (\mathbf{Z}_h^{\mathrm{T}} - \frac{1}{2}(\mathbf{L}^{\mathrm{T}} + \boldsymbol{\Omega}^{\mathrm{T}})),\\
(\mathbf{V}^{-1})^h = -(\mathbf{Z}_h - \frac{1}{2}(\boldsymbol{\Omega} - \mathbf{L}^{\mathrm{T}})) \cdot \mathbf{V}^{-1} - \mathbf{V}^{-1} \cdot (\mathbf{Z}_h^{\mathrm{T}} - \frac{1}{2}(\boldsymbol{\Omega}^{\mathrm{T}} - \mathbf{L})),\\
h = \{\ \cdot,\ \mathrm{Ol},\ \mathrm{CR},\ U,\ V,\ J,\ S\ \}.
\end{gathered} \tag{5.44}$$

From these relationships we can find the following expressions:

$$\begin{aligned}
(\mathbf{V} - \mathbf{E})^h = \mathbf{D} - {}^4\mathbf{E}_h \cdot\cdot\ \mathbf{D} - (\mathbf{Z}_h + \mathbf{Z}_h^{\mathrm{T}}) -&\\
- (\mathbf{Z}_h - \frac{1}{2}(\boldsymbol{\Omega} + \mathbf{L})) \cdot (\mathbf{V} - \mathbf{E}) - (\mathbf{V} - \mathbf{E}) \cdot (\mathbf{Z}_h^{\mathrm{T}} - \frac{1}{2}(\boldsymbol{\Omega}^{\mathrm{T}} + \mathbf{L}^{\mathrm{T}})),&
\end{aligned} \tag{5.45}$$

$$(\mathbf{E}-\mathbf{V}^{-1})^h = \mathbf{D} + {}^4\mathbf{E}_h \cdot\cdot\, \mathbf{D} + \mathbf{Z}_h + \mathbf{Z}_h^{\mathrm{T}} -$$
$$- (\mathbf{Z}_h - \frac{1}{2}(\boldsymbol{\Omega} - \mathbf{L}^{\mathrm{T}})) \cdot (\mathbf{E}-\mathbf{V}^{-1}) - (\mathbf{E}-\mathbf{V}^{-1}) \cdot (\mathbf{Z}_h^{\mathrm{T}} - \frac{1}{2}(\boldsymbol{\Omega}^{\mathrm{T}} - \mathbf{L})).$$

Here we have denoted the co-rotational derivative of the metric tensor by

$$\mathbf{E}^h = {}^4\mathbf{E}_h \cdot\cdot\, \mathbf{D}. \tag{5.46}$$

The tensor ${}^4\mathbf{E}_h$ differs from zero-tensor only when $h = \{\mathrm{CR}, \mathrm{Ol}\}$ (see Exercise 1.5.3), its expressions are given in Table 1.1 (see paragraph 1.5.10).

Since tensors $\mathbf{Z}_h$ and $\boldsymbol{\Omega}$ are linearly expressed in terms of $\mathbf{W}$ and $\mathbf{D}$ (see formulae (5.43), (4.129), (4.122)—(4.124)), so on the right-hand sides of equations (5.44) there are also linear functions of $\mathbf{W}$ and $\mathbf{D}$, their explicit expressions will be given in 3.2.22.

Exercises for 1.5.

Exercise 1.5.1. Show that the mixed co-rotational derivatives, the left and the right co-rotational derivatives relative to the eigenbasis and also the Jaumann and spin derivatives satisfy the differentiation rules of scalar products:

$$(\mathbf{A}\cdot\mathbf{B})^h = \mathbf{A}^h \cdot \mathbf{B} + \mathbf{A}\cdot\mathbf{B}^h, \qquad (\mathbf{a}\cdot\mathbf{A})^h = \mathbf{a}^h\cdot\mathbf{A} + \mathbf{a}\cdot\mathbf{A}^h,$$
$$(\psi\mathbf{A})^h = \psi^h\mathbf{A} + \psi\mathbf{A}^h, \qquad h = \{\cdot, d, D, U, V, J, S\},$$

and the Oldroyd and Cotter-Rivlin derivatives do not satisfy this rule.

Exercise 1.5.2. Show that for the co-rotational derivatives, the following rules of differentiation of scalar products of two vectors $\mathbf{a}$ and $\mathbf{b}$ and also of two tensors $\mathbf{T}$ and $\mathbf{B}$ remain valid:

$$(\mathbf{a}\cdot\mathbf{b})^h = (\mathbf{a}\cdot\mathbf{b})^{\bullet} = \mathbf{a}^h\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{b}^h, \qquad h = \{U, V, J, S\};$$
$$(\mathbf{T}\cdot\cdot\,\mathbf{B})^h = (\mathbf{T}\cdot\cdot\,\mathbf{B})^{\bullet} = \mathbf{T}^h\cdot\cdot\,\mathbf{B} + \mathbf{T}\cdot\cdot\,\mathbf{B}^h, \qquad h = \{d, D, U, V, J, S\}.$$

Exercise 1.5.3. Show that the following co-rotational derivatives of the unit tensor $\mathbf{E}$ give the zero-tensor:

$$\mathbf{E}^h = 0, \qquad h = \{\cdot, d, D, U, V, J, S\},$$

and the Oldroyd and Cotter–Rivlin derivatives of $\mathbf{E}$ are different from zero:

$$\mathbf{E}^{\mathrm{CR}} = 2\mathbf{D}, \qquad \mathbf{E}^{\mathrm{Ol}} = -2\mathbf{D}.$$

Exercise 1.5.4. Using formulae (4.103) and (5.20), show that the Cotter–Rivlin derivatives of the left Almansi deformation tensor $\mathbf{A}$ and of the left Almansi measure $\mathbf{g}$ have the form

$$\mathbf{A}^{\mathrm{CR}} = \mathbf{D}, \qquad \mathbf{g}^{\mathrm{CR}} = 0.$$

Exercise 1.5.5. Using formulae (4.103) and (5.18), show that the Oldroyd derivatives of the right Cauchy–Green tensor $\mathbf{J}$ and of the right deformation measure $\mathbf{g}^{-1}$ have the form

$$\mathbf{J}^{\mathrm{Ol}} = \mathbf{D}, \qquad (\mathbf{g}^{-1})^{\mathrm{Ol}} = 0.$$

Exercise 1.5.6. Using the expressions for the tensors $\mathbf{U}$ (3.6), $\mathbf{C}$ and $\boldsymbol{\Lambda}$ (3.29), and also $\mathbf{G}$ and $\mathbf{G}^{-1}$ (3.29), show that we can write the right derivative relative to the eigenbasis $\mathring{\mathbf{p}}_\alpha$ in the form

$$\mathbf{U}^U = \sum_{\alpha=1}^{3} \dot{\lambda}_\alpha \mathring{\mathbf{p}}_\alpha \otimes \mathring{\mathbf{p}}_\alpha, \qquad \mathbf{C}^U = \sum_{\alpha=1}^{3} \lambda_\alpha\dot{\lambda}_\alpha \mathring{\mathbf{p}}_\alpha \otimes \mathring{\mathbf{p}}_\alpha,$$

$$\mathbf{\Lambda}^U = \sum_{\alpha=1}^{3} \frac{\dot\lambda_\alpha}{\lambda_\alpha^3} \mathring{\mathbf{p}}_\alpha \otimes \mathring{\mathbf{p}}_\alpha, \qquad (\mathbf{U}^{-1})^U = \sum_{\alpha=1}^{3} \frac{\dot\lambda_\alpha}{\lambda_\alpha^2} \mathring{\mathbf{p}}_\alpha \otimes \mathring{\mathbf{p}}_\alpha,$$

$$\mathbf{G}^U = 2\mathbf{C}^U, \qquad (\mathbf{G}^{-1})^U = -2\mathbf{\Lambda}^U.$$

Exercise 1.5.7. Using the expressions for the tensors $\mathbf{V}$ (3.6), for $\mathbf{A}$ and $\mathbf{J}$ (3.29), and also for $\mathbf{g}$ and $\mathbf{g}^{-1}$ (3.29), show that we can write the left derivative relative to the eigenbasis $\mathbf{p}_\alpha$ in the form

$$\mathbf{V}^V = \sum_{\alpha=1}^{3} \dot\lambda_\alpha \mathbf{p}_\alpha \otimes \mathbf{p}_\alpha, \qquad \mathbf{A}^V = \sum_{\alpha=1}^{3} \frac{\dot\lambda_\alpha}{\lambda_\alpha^3} \mathbf{p}_\alpha \otimes \mathbf{p}_\alpha,$$

$$\mathbf{J}^V = \sum_{\alpha=1}^{3} \lambda_\alpha \dot\lambda_\alpha \mathbf{p}_\alpha \otimes \mathbf{p}_\alpha, \qquad \mathbf{g}^V = -2\mathbf{A}^V, \qquad (\mathbf{g}^{-1})^V = 2\mathbf{J}^V.$$

Exercise 1.5.8. Show that the Oldroyd and Jaumann derivatives of a second-order tensor $\mathbf{T}$ are connected by the relationship

$$\mathbf{T}^{\mathrm{Ol}} = \mathbf{T}^J - \mathbf{T}\cdot\mathbf{D} - \mathbf{D}\cdot\mathbf{T}.$$

Exercise 1.5.9. Show that if for an arbitrary symmetric tensor $\mathbf{T}$ its co-rotational derivatives are equal to zero:

$$\mathbf{T}^h = 0, \qquad h = \{d, D, U, V, J, S\},$$

then the first invariant of the tensor: $I_1(\mathbf{T}) = \mathbf{T}\cdot\cdot\,\mathbf{E}$ has its stationary value, i.e.

$$\dot I_1(\mathbf{T}) = 0.$$

Show that for the co-rotational Oldroyd and Cotter–Rivlin derivatives this statement is not valid.

Exercise 1.5.10. Using the results of Exercise 1.4.3, show that the co-rotational derivatives of the normal vector $\mathbf{n}$ satisfy the following relations:

$$\mathbf{n}^{\mathrm{CR}} = \gamma\mathbf{n}, \qquad \gamma = \mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n},$$

$$\mathbf{n}^{\mathrm{Ol}} = \gamma\mathbf{n} - \mathbf{n}\cdot\mathbf{L} - \mathbf{L}\cdot\mathbf{n}, \qquad \mathbf{n}^{\mathrm{J}} = \gamma\mathbf{n} - \mathbf{n}\cdot\mathbf{D}.$$

Exercise 1.5.11. Show that the following co-rotational derivatives of a symmetric tensor give a symmetric tensor:

$$\text{if } \mathbf{A} = \mathbf{A}^{\mathrm{T}}, \text{ then } (\mathbf{A}^h)^{\mathrm{T}} = \mathbf{A}^h, \quad h = \{U,\ V,\ J,\ S,\ \mathrm{Ol},\ \mathrm{CR}\},$$

and also a skew-symmetric tensor, if they are applied to a skew-symmetric tensor:

$$\text{if } \mathbf{B} = -\mathbf{B}^{\mathrm{T}}, \text{ then } (\mathbf{B}^h)^{\mathrm{T}} = -\mathbf{B}^h, \quad h = \{U,\ V,\ J,\ S,\ \mathrm{Ol},\ \mathrm{CR}\}.$$

The mixed co-rotational derivatives $h = d, D$ have no such properties.

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