Chapter 2
Linear Wave Phenomena

A few simple examples of the linearised boundary and initial-boundary value problems formulated in the previous chapter will be solved by the Fourier or Laplace transform method. Through these simple examples, basic wave phenomena or terminologies in water waves will be introduced. These are *phase velocity, dispersion relation, group velocity, wave fronts*, to name a few.

Of particular importance is the asymptotic behaviour of the free surface elevation for large values of relevant spaces and for time variables. This behaviour can be best obtained by the method of stationary phase (see Sect. 9.1). In this connection, the method of characteristics for treating first-order non-linear partial differential equations for the phase function is employed. Hence a brief summary of the concept of characteristics is included in Sect. 9.2.

A systematic derivation of oscillatory source singularity functions is presented for the disturbance below the free surface with and without current in Sects. 2.3 and 2.7.2. In Sect. 2.4 we derive for the steady case the field for a pressure disturbance at the free surface and for a point source below the free surface in Sect. 2.7.1. These source functions are often called Green functions and are used in numerical codes. One may derive different formulations for the functions as is shown.

2.1 Travelling Plane Waves

2.1.1 Plane Waves

It is easy to obtain travelling plane waves. As in Chap. 1 for small amplitude waves the linearised problem is defined by (1.32). For simplicity we restrict ourselves to the situation where $U = 0$. We consider two cases according to the water depth. We begin with the infinite depth. In this case the boundary value problem (1.32) consists of the Laplace equation

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \tag{2.1}$$

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together with the surface conditions

$$\varphi_{tt} + g\varphi_y = 0 \quad \text{at } y = 0 \tag{2.2}$$

and the condition at infinity

$$\varphi_y \to 0 \quad \text{as } y \to -\infty. \tag{2.3}$$

We seek a solution $\varphi(x, y, z, t)$ of (2.1)–(2.3) in the form

$$\varphi(x, y, z, t) = A\mathrm{e}^{\mathrm{i}(\alpha x+\beta z)+ky+\mathrm{i}\omega t}, \tag{2.4}$$

where α, β, k, ω and A are constants. Clearly (2.3) will be satisfied if k is positive. Substituting (2.4) into (2.1) and (2.2) we obtain

$$k = \alpha^2 + \beta^2 \quad \text{and} \quad -\omega^2 + gk = 0. \tag{2.5}$$

Set $\alpha = -k\cos\theta$ and $\beta = -k\sin\theta$ which clearly satisfy the first equation of (2.5) for any k. The second one gives that $k = \frac{\omega^2}{g}$ which is known as the *dispersion relation*—a relation between wave number k and frequency ω. Then the potential function has the form

$$\varphi(x, y, z, t) = A\exp\left\{-\mathrm{i}\omega\left[\frac{\omega}{g}(x\cos\theta + z\sin\theta) - t\right] + \frac{\omega^2}{g}y\right\}, \tag{2.6}$$

and consequently the water height is given by

$$\eta(x, z, t) = -\frac{1}{g}\varphi_t = -A\frac{\mathrm{i}\omega}{g}\exp\left\{-\mathrm{i}\omega\left[\frac{\omega}{g}(x\cos\theta + z\sin\theta) - t\right]\right\} \tag{2.7}$$

through use of (1.33). This formula represents plane waves.

For $\theta = 0$, we have plane waves travelling along the x-axis, independent of the z-coordinate:

$$\eta(x, t) = -\frac{\mathrm{i}\omega}{g}A\mathrm{e}^{-\mathrm{i}(\frac{\omega^2}{g}x-\omega t)} = A_1\mathrm{e}^{-\mathrm{i}\frac{\omega^2}{g}(x-ct)}, \tag{2.8}$$

where $c = \frac{g}{\omega}$ is the velocity of the wave (or *phase velocity*) and $A_1 = -\frac{\mathrm{i}\omega}{g}A$ is the amplitude of the wave. The real part of (2.8) corresponds to the real values wave height.

We now consider a wave train consisting of two plane waves in the x-direction with slightly different frequencies ω and $\omega + \delta\omega$. The total wave height may be written as

$$\begin{aligned}\eta(x, t) &= A_1\cos(kx - \omega t) + A_2\cos((k + \delta k)x - (\omega + \delta\omega)t \\ &= A(x, t)\cos(kx - \omega t + \theta(x, t))),\end{aligned} \tag{2.9}$$

where the amplitude function $A(x,t)$ and the phase function $\theta(x,t)$ are slowly varying functions. They can be written as

$$A(x,t) = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\delta kx - \delta\omega t)} \quad \text{and}$$
$$\tan\theta(x,t) = \frac{A_2\sin(\delta kx - \delta\omega t)}{A_1 + A_2\cos(\delta kx - \delta\omega t)}. \tag{2.10}$$

The amplitude moves with the velocity $\frac{\delta\omega}{\delta k}$. It will be shown in Sect. 2.1.2 that the wave energy is proportional to the square of the amplitude, hence we may expect that the energy moves with a velocity

$$c_g = \lim_{\delta\omega\to 0}\frac{\delta\omega}{\delta k} = \frac{\mathrm{d}\omega}{\mathrm{d}k}. \tag{2.11}$$

This velocity c_g is called the *group velocity*.

The corresponding problem for finite water depth can be treated in the same way. We write

$$\varphi(x,y,z,t) = \hat{\varphi}(x,y,z)\mathrm{e}^{\mathrm{i}\omega t}.$$

Then in this case we have from (2.2) the surface condition

$$\hat{\varphi}_y = \frac{\omega^2}{g}\hat{\varphi} \quad \text{at } y = 0, \tag{2.12}$$

while the condition at infinity (2.3) is replaced by the boundary condition (1.32). In terms of $\hat{\varphi}$ we have

$$\hat{\varphi}_y = 0 \quad \text{at } y = -h. \tag{2.13}$$

For travelling waves in the direction of the x-axis, i.e., $\hat{\varphi} = \hat{\varphi}(x,y)$, a simple manipulation by the method of separation of variables leads to the solution

$$\varphi(x,y,t) = A\cosh[k(y+h)]\mathrm{e}^{-\mathrm{i}(kx-\omega t)}, \tag{2.14}$$

where the wave number k and the frequency ω are related by the dispersion relation

$$\omega^2 = gk\tanh(kh). \tag{2.15}$$

Waves with a different wave number travel with a different phase velocity c which is defined by

$$c = \frac{\omega}{k} = \sqrt{\frac{g\tanh(kh)}{k}}. \tag{2.16}$$

Note that for kh small, since $\tanh(kh) = kh + \mathrm{O}((kh)^3)$, we have $c = \sqrt{gh}$ which is the case without dispersion. Observe again that if we let $h \to \infty$, we recover the case of infinite depth, (2.5). The dispersion causes a wave pattern, which at a certain place x and time t is a superposition of harmonic waves to be distorted at other places, because the components travel with different velocities. In the case of dispersion, it is difficult to determine the concept of 'wave speed'.

2.1.2 Wave Energy Transport

For the description of plane waves it is sufficient to restrict the considerations to the one-dimensional case. We represent at $t = 0$ the water height $\eta(x, 0)$ by the real integral

$$\eta(x) = \int_0^\infty C(k)\cos(kx)\,\mathrm{d}k + \int_0^\infty S(k)\sin(kx)\,\mathrm{d}k \tag{2.17}$$

with

$$C(k) = \frac{1}{\pi}\int_0^\infty \eta(x)\cos(kx)\,\mathrm{d}x, \quad \text{and}$$
$$S(k) = \frac{1}{\pi}\int_0^\infty \eta(x)\sin(kx)\,\mathrm{d}x.$$

Since $C(k)$ and $S(k)$ are respectively even and odd functions, setting

$$A(k) = \frac{1}{2}(C(k) + \mathrm{i}S(k)), \tag{2.18}$$

we can rewrite $\eta(x)$ as a complex integral

$$\eta(x) = \int_{-\infty}^\infty A(k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k. \tag{2.19}$$

A simple calculation shows that

$$\eta(x) = 2\Re\int_0^\infty A(k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k = \int_{-\infty}^\infty A^*(k)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k, \tag{2.20}$$

where $A^*(k)$ is the complex conjugate of $A(k)$.

For an understanding of the wave dispersion phenomenon, it is necessary to consider the energy propagation in the wave (linearised approximation). If the function $\eta(x)$ belongs to L^2, i.e., $\int_{-\infty}^\infty \eta(x)^2\,\mathrm{d}x$ exists, the potential energy is given by

$$E = \frac{1}{2}\rho g\int_{-\infty}^\infty \eta(x)^2\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty\left(\int_{-\infty}^\infty A(k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k\right)\left(\int_{-\infty}^\infty A^*(k')\mathrm{e}^{\mathrm{i}k'x}\,\mathrm{d}k'\right)\mathrm{d}x$$

from (2.19) and (2.20). The latter integral can now be calculated by making use of the Fourier inversion theorem and the fact that $\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}(k'-k)x}\,\mathrm{d}x = 2\pi\delta(k'-k)$.

This gives

$$E = \frac{1}{2}\rho g 2\pi\int_{-\infty}^\infty |A(k)|^2\,\mathrm{d}k. \tag{2.21}$$

Hence from (2.18) we have

$$E = \frac{\rho g\pi}{4}\int_{-\infty}^\infty \{C(k)^2 + S(k)^2\}\,\mathrm{d}k. \tag{2.22}$$

If the dispersion relation $\omega = \omega(k)$ is known (for convenience we extend the definition of $\omega(-k) = -\omega(k)$), then we can compute the water height η at any arbitrary time t as follows:

$$\eta(x,t) = \int_{-\infty}^{\infty} A(k)\mathrm{e}^{\mathrm{i}(\omega t - kx)}\,\mathrm{d}k = \int_{-\infty}^{\infty} A(k)\mathrm{e}^{-\mathrm{i}(\omega t - kx)}\,\mathrm{d}k \tag{2.23}$$

in terms of the phase velocity $c = \omega/k$. Here it is assumed that the initial conditions are such that the wave propagates only in the projection of the positive x-axis.

The total potential energy is conserved; the wave only changes the distribution of the energy along the x-axis. In fact we have

$$\begin{aligned} E(t) &= \frac{1}{2}\rho g \int_{-\infty}^{\infty} |\eta(x,t)|^2\,\mathrm{d}x \\ &= \frac{1}{2}\rho g \int_{-\infty}^{\infty} \mathrm{d}x \left(\int_{-\infty}^{\infty} A(k)\mathrm{e}^{\mathrm{i}(\omega t - kx)}\,\mathrm{d}k\right)\left(\int_{-\infty}^{\infty} A^*(k')\mathrm{e}^{-\mathrm{i}(\omega' t - k'x)}\,\mathrm{d}k'\right) \end{aligned}$$

with $\omega' = \omega(k')$. The latter integral follows from (2.23) and can be calculated similarly according to the Fourier inversion theorem. We find again

$$E(t) = \rho g \pi \int_{-\infty}^{\infty} |A(k)|^2\,\mathrm{d}k. \tag{2.24}$$

Now we are going to find a measure for the velocity of the energy propagation and calculate to this end the location of the *centre of gravity* $\bar{x}(t)$ of the first moment of the energy, which is defined by

$$\bar{x}(t) = \frac{\int_{-\infty}^{\infty} x|\eta(x,t)|^2\,\mathrm{d}x}{\int_{-\infty}^{\infty} |\eta(x,t)|^2\,\mathrm{d}x}, \tag{2.25}$$

provided both integrals exist. Here the denominator has been shown to be a constant in time and can be calculated easily from (2.24). The numerator, however, requires some investigation. Equation (2.23) yields

$$\begin{aligned} &\int_{-\infty}^{\infty} x|\eta(x,t)^2|\,\mathrm{d}x \\ &\quad= \int_{-\infty}^{\infty} x\,\mathrm{d}x \int_{-\infty}^{\infty} A(k)\mathrm{e}^{\mathrm{i}(\omega t - kx)}\,\mathrm{d}k \int_{-\infty}^{\infty} A^*(k')\mathrm{e}^{-\mathrm{i}(\omega' t - k'x)}\,\mathrm{d}k' \\ &\quad= \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} A(k)\,\mathrm{d}\big(\mathrm{e}^{\mathrm{i}(\omega t - kx)}\big) \int_{-\infty}^{\infty} A^*(k')\mathrm{e}^{-\mathrm{i}(\omega' t - k'x)}\,\mathrm{d}k' \\ &\qquad+ t\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} A(k)\frac{\mathrm{d}\omega(k)}{\mathrm{d}k} \int_{-\infty}^{\infty} A^*(k')\mathrm{e}^{\mathrm{i}(\omega t - kx) - \mathrm{i}(\omega' t - k'x)}\,\mathrm{d}k\,\mathrm{d}k' \\ &\quad:= \mathscr{J}_1 + \mathscr{J}_2. \end{aligned}$$

Integrating by parts and taking account the fact that $A(k) \to 0$, as $k \to \pm\infty$ in view of Bessel's inequality, we find

$$\mathscr{J}_1 = -\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \frac{\mathrm{d}A(k)}{\mathrm{d}k} \int_{-\infty}^{\infty} A^*(k')\mathrm{e}^{\mathrm{i}(\omega t - kx) - \mathrm{i}(\omega' t - k'x)}\,\mathrm{d}k\,\mathrm{d}k'.$$

Then by the Fourier inversion formula, we obtain

$$\mathscr{J}_1 = -2\pi\mathrm{i}\int_{-\infty}^{\infty} \frac{\mathrm{d}A(k)}{\mathrm{d}k} A^*(k)\,\mathrm{d}k;$$

$$\mathscr{J}_2 = 2\pi t\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega(k)}{\mathrm{d}k} |A(k)|^2\,\mathrm{d}k.$$

Adding $\mathscr{J}_1$ and $\mathscr{J}_2$, we have

$$\int_{-\infty}^{\infty} x|\eta(x,t)^2|\,\mathrm{d}x = 2\pi\left\{-\mathrm{i}\int_{-\infty}^{\infty} \frac{\mathrm{d}A(k)}{\mathrm{d}k} A^*(k)\,\mathrm{d}k + t\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega(k)}{\mathrm{d}k} |A(k)|^2\,\mathrm{d}k\right\}. \tag{2.26}$$

We define, as a mean value of a quantity $\psi(k)$ in the k-domain,

$$\bar{\psi} = \frac{\int_{-\infty}^{\infty} \psi(k)|A(k)|^2\,\mathrm{d}k}{\int_{-\infty}^{\infty} |A(k)|^2\,\mathrm{d}k}, \tag{2.27}$$

and remark that the first term in (2.26) determines the position of $\bar{x}(t)$ for $t = 0$. Hence we find

$$\bar{x}(t) = \bar{x}(0) + t\,\overline{\frac{\mathrm{d}\omega}{\mathrm{d}k}}, \tag{2.28}$$

i.e. the centre of gravity propagates with a velocity which is equal to the mean velocity of $\frac{\mathrm{d}\omega}{\mathrm{d}k}$. Here $\frac{\mathrm{d}\omega}{\mathrm{d}k}$ is called the *group velocity*; hence the mean value of the group velocity $\overline{\frac{\mathrm{d}\omega}{\mathrm{d}k}}$ is a measure for the speed of propagation of the energy.

The significance of this result becomes clear when we consider an amplitude spectrum $A(k)$, which extends only over a narrow wave number band:

$$\eta(x,t) = \int_{k_0-\varepsilon}^{k_0+\varepsilon} A(k)\mathrm{e}^{-\mathrm{i}(kx - \omega(k)t}\,\mathrm{d}k \quad \varepsilon > 0. \tag{2.29}$$

The centre of gravity satisfies

$$\bar{x}(t; k_0, \varepsilon) = \bar{x}(0; k_0, \varepsilon) + t\overline{\omega'(k_0, \varepsilon)}, \tag{2.30}$$

where $\overline{\omega'(k_0, \varepsilon)}$ is now the mean value of $\omega'(k) = \frac{\mathrm{d}\omega}{\mathrm{d}k}$ over the narrow band $[k_0 - \varepsilon, k_0 + \varepsilon]$. For small values of ε we simply replace $\overline{\omega'(k_0, \varepsilon)}$ by $\omega'(k_0)$.

For small values of t, one can make a more accurate analysis of the motion as follows. Expanding $\omega(k)$ in the form

$$\omega(k) = \omega(k_0) + (k - k_0)\omega'(k_0) + \frac{(k - k_0)^2}{2}\tilde{\omega}''(k),$$

and substituting into (2.29), we may write

$$\eta(x,t) = \int_{k_0-\varepsilon}^{k_0+\varepsilon} A(k)\mathrm{e}^{-\mathrm{i}\{(k_0x-\omega(k_0)t)+(k-k_0)(x-\omega'(k_0)t)-\frac{(k-k_0)^2}{2}\tilde{\omega}''(k)t\}}\,\mathrm{d}k$$
$$= \mathrm{e}^{-\mathrm{i}(k_0x-\omega(k_0)t)} \int_{k_0-\varepsilon}^{k_0+\varepsilon} A(k)\mathrm{e}^{-\mathrm{i}(k-k_0)(x-\omega'(k_0)t)}\,\mathrm{d}k + R, \tag{2.31}$$

where

$$R = \mathrm{e}^{-\mathrm{i}(k_0x-\omega(k_0)t)} \int_{k_0-\varepsilon}^{k_0+\varepsilon} A(k)\mathrm{e}^{-\mathrm{i}(x-\omega'(k_0)t)(k-k_0)} \cdot \left\{\exp\left[\frac{\mathrm{i}(k-k_0)^2}{2}\tilde{\omega}''(k)t\right]-1\right\}\mathrm{d}k.$$

Using the inequality $|\mathrm{e}^{\mathrm{i}u}-1| \le |u|$, we find an estimate of the remainder

$$|R| \le \int_{k_0-\varepsilon}^{k_0+\varepsilon} |A(k)|\frac{(k-k_0)^2}{2}|\tilde{\omega}''(k)t|\,\mathrm{d}k$$
$$\le \frac{1}{3}\Big(\max_{|k-k_0|<\varepsilon}|A(k)|\Big)\Big(\max_{|k-k_0|<\varepsilon}|\omega''(k)|\Big)\varepsilon^3 t,$$

which shows that for not too large values of t, the first term of (2.30) gives a good approximation of η. Assuming, for small ε, $A(k)$ to be constant $A(k_0)$ over the interval, we can integrate:

$$A(k_0)\mathrm{e}^{-\mathrm{i}(k_0x-\omega(k_0)t)} \int_{k_0-\varepsilon}^{k_0+\varepsilon} \mathrm{e}^{-\mathrm{i}(k-k_0)(x-\omega'(k_0)t)}\,\mathrm{d}k$$
$$= A(k_0)\mathrm{e}^{-\mathrm{i}k_0(x-\frac{\omega(k_0)}{k_0}t)}\frac{2\sin[(x-\omega'(k_0)t)\varepsilon]}{x-\omega'(k_0)t}.$$

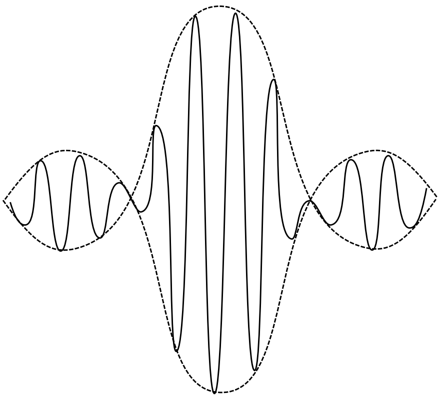

Fig. 2.1 Wave train

Hence we have, for small ε and t not too large,

$$\eta(x,t) \cong A(k_0)\mathrm{e}^{-\mathrm{i}k_0(x-\frac{\omega(k_0)}{k_0}t)}\frac{2\sin[(x-\omega'(k_0)t)\varepsilon]}{x-\omega'(k_0)t} \tag{2.32}$$

as shown in Fig. 2.1.

This represents a modulated wave; the amplitude moves with the group velocity $\omega'(k_)$ (the dotted enveloping curves) while the phase moves with the phase velocity $\omega(k_0)/k_0$ (the inscribed solid curves).

2.2 Cylindrical Waves

The boundary value problem for a cylindrical wave, at zero speed, $U = 0$, is defined by the same equations in (1.32) for small amplitude waves. For harmonic oscillations we put

$$\begin{aligned} \varphi(x,y,z,t) &= \hat{\varphi}(x,y,z)\mathrm{e}^{\mathrm{i}\omega t}; \\ \eta(x,z,t) &= \hat{\eta}(x,z)\mathrm{e}^{\mathrm{i}\omega t}. \end{aligned} \tag{2.33}$$

Then the potential function $\hat{\varphi}(x,z,t)$ satisfies the Laplace equation

$$\hat{\varphi}_{xx} + \hat{\varphi}_{yy} + \hat{\varphi}_{zz} = 0 \tag{2.34}$$

and the surface equation

$$\hat{\varphi}_y = \frac{\omega^2}{g}\hat{\varphi} \quad \text{for } y = 0. \tag{2.35}$$

For infinite depth, we have again the condition

$$\hat{\phi} \text{ finite} \quad \text{for } y \to -\infty. \tag{2.36}$$

Since the problem now is axially symmetric, it is natural to make use of cylindrical coordinates $x = r\cos\theta$, $z = r\sin\theta$, $y = y$. Thus (2.34) reads

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\hat{\varphi}}{\partial r}\right) + \frac{\partial^2\hat{\varphi}}{\partial y^2} = 0. \tag{2.37}$$

We introduce dimensionless coordinates

$$\bar{r} = \frac{r\omega^2}{g}, \qquad \bar{y} = \frac{y\omega^2}{g}.$$

The transform leaves the differential equation (2.37) invariant, but the boundary condition (2.35) becomes

$$\hat{\varphi} = \hat{\varphi}_{\bar{y}} \quad \text{for } \bar{y} = 0. \tag{2.38}$$

We solve this problem by the method of separation of variables and assume that

$$\hat{\varphi}(\bar{r}, \bar{y}) = \mathrm{e}^{\lambda \bar{y}} R(\bar{r}),$$

where $R(\bar{r})$ is a solution of the ordinary differential equation

$$\frac{1}{\bar{r}} \frac{\mathrm{d}}{\mathrm{d}\bar{r}} \left(\bar{r} \frac{\mathrm{d}R}{\mathrm{d}\bar{r}} \right) + \lambda^2 R = 0.$$

The boundary condition (2.38) gives that $\lambda = 1$. Thus we obtain

$$\hat{\varphi}(\bar{r}, \bar{y}) = \mathrm{e}^{\bar{y}} \left\{ A H_0^{(1)}(\bar{r}) + B H_0^{(2)}(\bar{r}) \right\}, \tag{2.39}$$

where $H_0^{(i)}$ are Hankel functions of order zero, and A and B are constants to be determined from the radiation condition as follows.

As is well known, for large values of $\bar{r}$ we have

$$H_0^{(1)}(\bar{r}) \approx \sqrt{\frac{2}{\pi \bar{r}}} \mathrm{e}^{\mathrm{i}(\bar{r} - \frac{\pi}{4})}, \quad \text{and}$$
$$H_0^{(2)}(\bar{r}) \approx \sqrt{\frac{2}{\pi \bar{r}}} \mathrm{e}^{-\mathrm{i}(\bar{r} - \frac{\pi}{4})}.$$

With time dependence $\mathrm{e}^{\mathrm{i}\omega t}$, only the solution

$$\varphi(\bar{r}, \bar{y}, \bar{t}) = B \mathrm{e}^{\bar{y}} H_0^{(2)}(\bar{r}) \mathrm{e}^{\mathrm{i}\omega t} \tag{2.40}$$

represents outgoing waves. For large values of $\bar{r}$ it behaves as

$$\varphi(\bar{r}, \bar{y}, \bar{t}) \approx B \mathrm{e}^{\bar{y}} \sqrt{\frac{2}{\pi \bar{r}}} \mathrm{e}^{-\mathrm{i}(\bar{r} - \omega t - \frac{\pi}{4})},$$

and the phase is defined by

$$\bar{r} - \omega t = \frac{\omega^2}{g} \left(r - \frac{g}{\omega} t \right),$$

which gives $\frac{g}{\omega}$ for the *phase velocity*.

The water height η is given by

$$\eta(\bar{r}, t) = -\frac{\mathrm{i}\omega}{g} B \mathrm{e}^{\bar{y}} H_0^{(2)}(\bar{r}) \mathrm{e}^{\mathrm{i}\omega t} \tag{2.41}$$

from (2.33), (1.33) and (2.40). Here it is understood that either the real or the imaginary part of the right-hand side of (2.41) is to be taken. We usually take the real part. This is an example of centred outgoing waves. The solution is obviously singular at $r = 0$ and $\forall y$. In the next Sect. 2.3 we will see that the far field of an harmonic point singularity has such a far-field behaviour.

2.3 Harmonic Source Singularity

It is of interest to determine the field disturbance of the free surface due to an harmonic singularity in a point below or at the free surface. As will be shown in Chap. 3, many methods to solve the problem of diffraction of waves by an object we make use of a distribution of singularities at the surface of the object. Here we will determine the field generated by such a singularity. As an example we treat the finite water depth case. The singularity is written as a *Dirac* δ-function in the right-hand side of the Laplace equation

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = \delta(x - x_0, y - y_0, z - z_0)\mathrm{e}^{\mathrm{i}\omega t}. \tag{2.42}$$

If we introduce $\varphi(x, y, z, t) = \hat{\varphi}(x, y, z)\mathrm{e}^{\mathrm{i}\omega t}$, the boundary value problem to be solved becomes

$$\begin{aligned} &\hat{\varphi}_{xx} + \hat{\varphi}_{yy} + \hat{\varphi}_{zz} = \delta(x - x_0, y - y_0, z - z_0), \\ &\hat{\varphi}_y = 0 \qquad \text{at } y = -h, \\ &\hat{\varphi}_y = \frac{\omega^2}{g}\hat{\varphi} \quad \text{at } y = 0. \end{aligned} \tag{2.43}$$

This formulation is not complete. We must add a condition at large horizontal distance from the source. The solution must fulfil the *radiation condition*. The disturbance for large values of $R = \sqrt{(x - x_0)^2 + (z - z_0)^2}$ may only consist of outgoing waves. The solution must have the form

$$\varphi(x, y, z, t) \approx A(R, y)\mathrm{e}^{-\mathrm{i}(kR - \omega t)}, \tag{2.44}$$

where the amplitude function tends to zero if $R \to 0$.

There are several ways to solve this problem. We shall employ the method of Fourier transform to obtain a solution. We introduce the following exponential transform of $\hat{\varphi}$ with respect to the x and z coordinates

$$\phi(\alpha, y, \beta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\alpha x + \beta z)}\hat{\varphi}(x, y, z)\,\mathrm{d}x\,\mathrm{d}z. \tag{2.45}$$

The inverse transform is

$$\hat{\varphi}(x, y, z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(\alpha x + \beta z)}\phi(\alpha, y, \beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{2.46}$$

We introduce the transform in the Laplace equation and the boundary conditions for $\hat{\varphi}$ and obtain an ordinary differential equation for ϕ with appropriate boundary conditions

$$\begin{aligned} &\phi_{yy} - (\alpha^2 + \beta^2)\phi = \mathrm{e}^{\mathrm{i}(\alpha x_0 + \beta z_0)}\delta(y - y_0), \\ &\phi_y = 0 \qquad \text{at } y = -h, \\ &\phi_y = \frac{\omega^2}{g}\phi \quad \text{at } y = 0. \end{aligned} \tag{2.47}$$

The singularity in the right-hand side of the differential equation can be replaced by the following conditions for the function $\phi(\alpha, \beta, y)$:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}(\phi_y(\alpha, y_0+\varepsilon, \beta) - \phi_y(\alpha, y_0-\varepsilon, \beta)) = \mathrm{e}^{\mathrm{i}(\alpha x_0+\beta z_0)},\\
&\lim_{\varepsilon\to 0}(\phi(\alpha, y_0+\varepsilon, \beta) - \phi(\alpha, y_0-\varepsilon, \beta)) = 0.
\end{aligned} \tag{2.48}$$

The solution of the problem is written as ϕ^+ for $y_0 < y \le 0$ and ϕ^- for $-h < y < y_0$. A convenient choice of the solution is

$$\begin{aligned}
\phi^+ &= A\cosh(k(y+h)) + B\sinh(k(y+h)),\\
\phi^- &= C\cosh(k(y+h)).
\end{aligned}$$

Here k is defined as the distance to the origin in the Fourier space which is the positive root of $k^2 = \alpha^2 + \beta^2$. With this choice the bottom condition is fulfilled automatically. The constants A, B and C are determined by the condition at the free surface $y = 0$ together with the conditions at $y = y_0$. After some manipulations we find the solution for $y_0 < y \le 0$,

$$\phi^+ = -\frac{\cosh(k(y_0+h))\{\nu\sinh(ky) + k\cosh(ky)\}}{k\{k\sinh(kh) - \nu\cosh(kh)\}}\mathrm{e}^{\mathrm{i}(\alpha x_0+\beta z_0)}, \tag{2.49}$$

and for $-h < y < y_0$,

$$\phi^- = -\frac{\cosh(k(y+h))\{\nu\sinh(ky_0) + k\cosh(ky_0)\}}{k\{k\sinh(kh) - \nu\cosh(kh)\}}\mathrm{e}^{\mathrm{i}(\alpha x_0+\beta z_0)}, \tag{2.50}$$

where $\nu = \frac{\omega^2}{g}$. We now apply the inverse transform given by (2.46) to ϕ^+

$$\begin{aligned}
\hat{\varphi}^+(x, y, z) = \frac{-1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} &\mathrm{e}^{-\mathrm{i}(\alpha(x-x_0)+\beta(z-z_0))}\\
&\cdot\frac{\cosh(k(y_0+h))\{\nu\sinh(ky) + k\cosh(ky)\}}{k\{k\sinh(kh) - \nu\cosh(kh)\}}\,\mathrm{d}\alpha\,\mathrm{d}\beta.
\end{aligned} \tag{2.51}$$

It is convenient to introduce polar coordinates, both in the physical space and the Fourier space. We introduce

$$x - x_0 = R\cos\theta, \qquad z - z_0 = R\sin\theta \tag{2.52}$$

and

$$\alpha = k\cos\vartheta, \qquad \beta = k\sin\vartheta. \tag{2.53}$$

The solution can then be written as

$$\begin{aligned}
\hat{\varphi}^+(x, y, z) = \frac{-1}{4\pi^2}\int_0^{2\pi}\int_0^{\infty} &\mathrm{e}^{-\mathrm{i}kR\cos(\vartheta-\theta)}\\
&\cdot\frac{\cosh(k(y_0+h))\{\nu\sinh(ky) + k\cosh(ky)\}}{k\sinh(kh) - \nu\cosh(kh)}\,\mathrm{d}\vartheta\,\mathrm{d}k.
\end{aligned} \tag{2.54}$$

The integration with respect to ϑ can be carried out by making use of the following definition of the Bessel function J_0:

$$J_0(kR) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kR\cos(\vartheta-\theta)}\,\mathrm{d}\vartheta. \tag{2.55}$$

Hence, if we follow the same procedure for $\hat{\varphi}^-$, we obtain

$$\begin{aligned}\hat{\varphi}^+(x,y,z) &= \frac{-1}{2\pi}\int_0^{\infty}\frac{\cosh(k(y_0+h))\{\nu\sinh(ky)+k\cosh(ky)\}}{k\sinh(kh)-\nu\cosh(kh)}J_0(kR)\,\mathrm{d}k,\\ \hat{\varphi}^-(x,y,z) &= \frac{-1}{2\pi}\int_0^{\infty}\frac{\cosh(k(y+h))\{\nu\sinh(ky_0)+k\cosh(ky_0)\}}{k\sinh(kh)-\nu\cosh(kh)}J_0(kR)\,\mathrm{d}k.\end{aligned} \tag{2.56}$$

Until this point the radiation condition is not used. We will see that to define a proper inverse transform it has to be used. The integrands of the functions $\hat{\varphi}^{+,-}$ each have a singularity for a real value of the denominator. Hence, the integrals are not well defined. The equation $k\sinh(kh)-\nu\cosh(kh)=0$ has one real root together with an infinite number of purely imaginary roots. From the theory of Fourier integral we know that the contour of integration has to pass, in the complex k-plane, above or below the singularity. The choice is determined by the radiation condition. A way to determine the correct choice is to introduce a small artificial damping in the fluid. If we assume the far field to be of the form $\mathrm{e}^{-\mathrm{i}(kR-\omega t)}$ we see that the only choice for vanishing waves is to introduce a complex wave number of the form $k=\bar{k}-\mathrm{i}\tilde{k}$. The negative imaginary part of the wave number may be generated by some artificial, non-physical, damping. This indicates that the singularity on the real axis must be passed above. Representation (2.56) for $\hat{\varphi}$ consists of different forms depending on whether y is larger or smaller than y_0. This might be not practical. One may obtain a single expression if we use some lemmas from the theory of complex functions. We use the following lemma for analytic functions $f(z)$ and $g(z)$, while the function $f(z)$ has simple zeros z_i in the complex plane. If we define $f(z)=z\sinh(zh)-\nu\cosh(zh)$ and $g(z)=\cosh(zp)\{\nu\sinh(zq)+z\cosh(zq)$ respectively, then for $|z|\to\infty$ the function $\frac{g(z)}{f(z)}\to 0$ fast enough and we have

$$\frac{g(z)}{f(z)} = \frac{g(0)}{f(0)} + \sum_i g(z_i)\gamma_i\left(\frac{1}{z-z_i}+\frac{1}{z_i}\right) \quad \text{with } \gamma_i = \frac{1}{f_z(z_i)}, \tag{2.57}$$

which is an expansion of $\frac{g(z)}{f(z)}$ in rational fractions of z, see [21], Sect. 7.4.

The integrands of both integrals in the expression for $\hat{\varphi}(x,y,z)$ (2.56) has infinitely many simple poles $k=\pm k_i$ $(i=0,1,2,\ldots)$ in the complex k-plane. We have

$$k_i\sinh(k_ih)-\nu\cosh(k_ih)=0. \tag{2.58}$$

The positive real zero is k_0, while the positive imaginary roots are $k_i=\mathrm{i}\kappa_i$ $(i=1,2,\ldots)$, see Fig. 2.2.

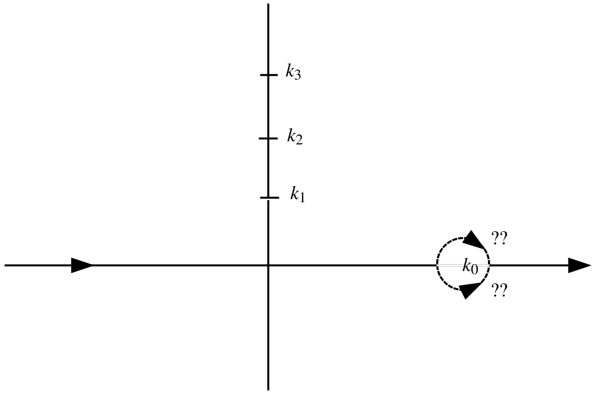

Fig. 2.2 The singularities in the complex k-plane

According to (2.57) we may write

$$\frac{g(k)}{k\sinh(kh)-\nu\cosh(kh)}=\sum_{i=0}^{\infty}g(k_i)\left(\frac{\alpha_i^+}{k-k_i}+\frac{\alpha_i^-}{k+k_i}\right), \tag{2.59}$$

where we used the fact that in our case $g(k)$ is antisymmetric and $g(0)=0$ and where α_i is defined as

$$\alpha_i^{\pm}=\frac{\pm k_i}{(\nu+k_i^2h-\nu^2h)\cosh(k_ih)}. \tag{2.60}$$

If we work out the integrands of (2.56) we find one expression for $\hat{\varphi}(x,y,z)$, valid for $-h<y\le 0$. We obtain

$$\hat{\varphi}(x,y,z)=\frac{-1}{2\pi}\sum_{i=0}^{\infty}\frac{k_i^2-\nu^2}{\nu+k_i^2h-\nu^2h}\cosh(k_i(y+h))\cosh(k_i(y_0+h))$$
$$\cdot\int_0^{\infty}\left(\frac{1}{k-k_i}-\frac{1}{k+k_i}\right)J_0(kR)\,\mathrm{d}k. \tag{2.61}$$

The integral in the right hand side can, by introducing $k=-k^*$ in the second part, be rewritten as

$$\mathscr{I}(k_i)=\frac{1}{2}\int_{-\infty}^{\infty}\frac{H_0^{(1)}(kR)}{k-k_i}\,\mathrm{d}k+\frac{1}{2}\int_{-\infty}^{\infty}\frac{H_0^{(2)}(kR)}{k-k_i}\,\mathrm{d}k. \tag{2.62}$$

Due to the asymptotic behaviour of the Hankel functions we may close the first integral in the upper half of the complex k plane, while the second one may be closed in the lower half. In this way the contributions of the contours at $|k|\to\infty$ tend to zero. If the path of integration in (2.62) passes the singularity $k=k_0$ in the upper plane we obtain the following result for $i=0$:

$$\mathscr{I}(k_0)=-\pi\mathrm{i}H_0^{(2)}(k_0R), \tag{2.63}$$

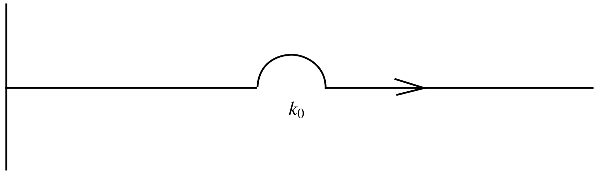

Fig. 2.3 Line of integration

and for $i = 1, 2, \ldots$

$$\mathscr{J}(k_i) = \pi \mathrm{i} H_0^{(1)}(\mathrm{i}\kappa_i R) = 2K_0(\kappa_i R), \tag{2.64}$$

where $K_0(z)$ is the modified Bessel function. The contribution of $H_0^{(2)}(k_0 R)$ represents an outgoing circular wave, while the contribution of each $K_0(\kappa_i R)$ is exponential decaying for large values of R. This confirms the right choice of the contour of integration, see Fig. 2.3. We notice that the use of an artificial damping to shift k_0 actually is not the only way to find the correct contour of integration. If one chooses the contour to pass underneath k_0 the wavy behaviour is described by $H_0^{(1)}(k_0 R)$, describing an incoming circular wave field. Waves travelling towards the source clearly which disobey the radiation condition.

The expression for the total field now becomes

$$\varphi(x, y, z, t) = \mathrm{e}^{\mathrm{i}\omega t}\hat{\varphi}(x, y, z)$$

with

$$\begin{aligned}\hat{\varphi}(x, y, z) = {} & \frac{\mathrm{i}(k_0^2 - \nu^2)}{2(\nu + k_0^2 h - \nu^2 h)} \cosh(k_0(y+h)) \cosh(k_0(y_0+h)) H_0^{(2)}(k_0 R) \\ & + \frac{1}{\pi} \sum_{i=1}^{\infty} \frac{\kappa_i^2 + \nu^2}{\nu - \kappa_i^2 h - \nu^2 h} \cos(\kappa_i(y+h)) \cos(\kappa_i(y_0+h)) \\ & \times K_0(\kappa_i R).\end{aligned} \tag{2.65}$$

If we take the real part of (2.65) and multiply it with -4π we have the famous result of F. John. The different factor originates from the normalisation of the point source. This formulation can be used to compute the disturbance due to a unit point source at finite difference from the source. However, the series does not converge close to the source. This was to be expected, because of the singular, $\frac{-1}{4\pi r}$, behaviour of $\hat{\varphi}$, where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + z - z_0)^2}$ is the distance to the singularity.

We expect to find a useful solution near the singularity if we write it as

$$\hat{\varphi}(x, y, z) = -\frac{1}{4\pi r} - \frac{1}{4\pi \tilde{r}} + \psi(x, y, z), \tag{2.66}$$

where $\tilde{r} = \sqrt{(x-x_0)^2 + (y+2h+y_0)^2 + (z-z_o)^2}$ is the distance to the mirror image, with respect to the bottom, of the source point. For $\psi(x, y, z)$ we obtain the following problem:

$$\begin{aligned}
&\psi_{xx}+\psi_{yy}+\psi_{zz}=0, && \\
&\psi_y=0 && \text{at } y=-h, \\
&\psi_y-\nu\psi=\frac{1}{4\pi}\left\{\frac{\partial}{\partial y}\left(\frac{1}{r}+\frac{1}{\tilde{r}}\right)-\nu\left(\frac{1}{r}+\frac{1}{\tilde{r}}\right)\right\} && \text{at } y=0. \\
&\qquad := g(x,z;x_0,y_0,z_0) &&
\end{aligned} \tag{2.67}$$

We apply the double Fourier transform to the function ψ,

$$\Psi(\alpha,y,\beta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(\alpha x+\beta z)}\psi(x,y,z)\,\mathrm{d}x\,\mathrm{d}z \tag{2.68}$$

and introduce polar coordinates (2.56) in the Fourier space. The ordinary differential equation and boundary conditions for Ψ become

$$\begin{aligned}
&\Psi_{yy}-k^2\Psi=0, && \\
&\Psi_y=0 && \text{at } y=-h, \\
&\Psi_y-\nu\Psi=G(k;x_0,y_0,z_0) && \text{at } y=0.
\end{aligned} \tag{2.69}$$

We make use of the known transform of $\frac{-1}{4\pi r}$, the point source for an infinite fluid where no free surface is present

$$\mathscr{F}\left(\frac{-1}{4\pi r}\right)=\frac{-1}{2k}\mathrm{e}^{\mathrm{i}(\alpha x_0+\beta z_0)-k|y-y_0|}. \tag{2.70}$$

This formula can be obtained by means of the double Fourier transform to the Laplace equation, as before, in the case of an infinite fluid. If we apply this formula to $g(x,z;x_0,y_0,z_0)$ we obtain

$$G(k;x_0,y_0,z_0)=-\frac{k+\nu}{k}\mathrm{e}^{-kh}\cosh(k(y_0+h))\mathrm{e}^{\mathrm{i}(\alpha x_0+\beta z_0)} \tag{2.71}$$

and the solution of (2.69) becomes

$$\Psi(\alpha,y,\beta)=-\frac{k+\nu}{k}\mathrm{e}^{-kh}\frac{\cosh(k(y+h))\cosh(k(y_0+h))}{k\sinh(kh)-\nu\cosh(kh)}\mathrm{e}^{\mathrm{i}(\alpha x_0+\beta z_0)}. \tag{2.72}$$

The inverse Fourier transform is defined as

$$\psi(x,y,z)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi(\alpha,y,\beta)\mathrm{e}^{-\mathrm{i}(\alpha x+\beta z)}\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{2.73}$$

With the introduction of polar coordinates in the physical (2.52) and Fourier (2.53) space we obtain with the use of (2.55) the total field

$$\varphi(x,y,z,t)=\mathrm{e}^{\mathrm{i}\omega t}\hat{\varphi}(x,y,z)$$

with

$$\hat{\varphi}(x, y, z) = -\frac{1}{4\pi r} - \frac{1}{4\pi \tilde{r}} - \frac{1}{2\pi}\int_0^\infty \frac{(k+\nu)\mathrm{e}^{-kh}\cosh(k(y+h))\cosh(k(y_0+h))}{k\sinh(kh) - \nu\cosh(kh)} J_0(kR)\,\mathrm{d}k. \tag{2.74}$$

If we introduce some artificial damping in the problem we observe that the contour of integration passes above the real pole in the integrand. This finally leads to the expression

$$\begin{aligned}\hat{\varphi}&(x, y, z)\\ &= -\frac{1}{4\pi r} - \frac{1}{4\pi \tilde{r}}\\ &\quad - \frac{1}{2\pi}⨍_0^\infty \frac{(k+\nu)\mathrm{e}^{-kh}\cosh(k(y+h))\cosh(k(y_0+h))}{k\sinh(kh) - \nu\cosh(kh)} J_0(kR)\,\mathrm{d}k\\ &\quad + \frac{\mathrm{i}}{2}\frac{(k_0+\nu)\mathrm{e}^{-k_0h}\sinh(k_0h)\cosh(k_0(y+h))\cosh(k_0(y_0+h))}{\nu h + \sinh^2(k_0h)} J_0(k_0R),\end{aligned} \tag{2.75}$$

where $⨍$ indicates the principal value of the integral. If we are interested in the deep water case we may obtain an expression for the source potential by using (2.75) for large values of h. We obtain for the limit $h \to \infty$,

$$\hat{\varphi}(x, y, z) = -\frac{1}{4\pi r} - \frac{1}{4\pi}⨍_0^\infty \frac{k+\nu}{k-\nu}\mathrm{e}^{k(y+y_0)} J_0(kR)\,\mathrm{d}k + \frac{\mathrm{i}}{2}\nu\mathrm{e}^{\nu(y+y_0)} J_0(\nu R). \tag{2.76}$$

This result may be rewritten as

$$\hat{\varphi}(x, y, z) = -\frac{1}{4\pi r} + \frac{1}{4\pi \bar{r}} - \frac{1}{4\pi}⨍_0^\infty \frac{2k}{k-\nu}\mathrm{e}^{k(y+y_0)} J_0(kR)\,\mathrm{d}k + \frac{\mathrm{i}}{2}\nu\mathrm{e}^{\nu(y+y_0)} J_0(\nu R), \tag{2.77}$$

where $\bar{r} = \sqrt{(x-x_0)^2 + (y+y_0)^2 + (z-z_0)^2}$ is the distance to the mirror point, with respect to the unperturbed free surface.

The contour of integration may be deformed to obtain different forms of (2.74). We can rewrite the integral as a contribution of the pole and an integral along the vertical axis of the complex k-plane. Instead of the way the solution is written in (2.76) one also may write the solution as the sum $-(\frac{1}{4\pi r} + \frac{1}{4\pi \bar{r}})$, where $-\frac{1}{4\pi \bar{r}}$ is the field of a singularity located at $(x_0, -y_0, z_0)$ in an infinite fluid, and use an integral expression for this term. There are more choices possible, they are sometimes used in the literature for different reasons.

2.4 The Moving Pressure Point

We consider the field generated by a pressure point disturbance at the free surface, moving in the direction of the positive x-axis. For small amplitude waves the linearised free surface condition is defined by (1.32). We suppose the bottom at infinity, $y=-\infty$. Hence the bottom condition is replaced by the condition that φ remains finite as $y\to-\infty$. We look for a very simple solution in a steady flow, for which everywhere at $y=0$ except at $x=z=0$ the pressure vanishes. By introducing the dimensionless coordinates

$$\bar{x}=\frac{xg}{U^2},\qquad \bar{y}=\frac{yg}{U^2},\qquad \bar{z}=\frac{zg}{U^2},$$

we can formulate the boundary value problem as follows;

$$\begin{aligned}
&\varphi_{\bar{x}\bar{x}}+\varphi_{\bar{y}\bar{y}}+\varphi_{\bar{z}\bar{z}}=0,\\
&\varphi_{\bar{x}\bar{x}}+\varphi_{\bar{y}}=0 \quad \text{at } \bar{y}=0,\ (\bar{x},\bar{z})\neq(0,0),\\
&\varphi \text{ finite} \qquad\quad\ \text{as } \bar{y}\to\infty.
\end{aligned}\tag{2.78}$$

We seek solutions of (2.78) by means of a Fourier transform with respect to $\bar{x}$,

$$\hat{\varphi}(\alpha,\bar{y},\bar{z})=\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\alpha\bar{x}}\varphi(\bar{x},\bar{y},\bar{z})\,\mathrm{d}\bar{x}\tag{2.79}$$

with its inverse transform

$$\varphi(\bar{x},\bar{y},\bar{z})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\alpha\bar{x}}\hat{\varphi}(\alpha,\bar{y},\bar{z})\,\mathrm{d}\bar{\alpha}.\tag{2.80}$$

This leads to the boundary value problem for $\hat{\varphi}(\alpha,\bar{y},\bar{z})$:

$$\begin{aligned}
&\hat{\varphi}_{\bar{y}\bar{y}}+\hat{\varphi}_{\bar{z}\bar{z}}-\alpha^2\hat{\varphi}=0,\\
&-\alpha^2\hat{\varphi}+\hat{\varphi}_{\bar{y}}=0 \quad \text{at } \bar{y}=0,\\
&\hat{\varphi} \text{ finite} \qquad\qquad\ \text{as } \bar{y}\to-\infty.
\end{aligned}\tag{2.81}$$

A simple solution of (2.81) can be found in the form

$$\hat{\varphi}=\mathrm{e}^{\alpha^2\bar{y}}F(\bar{z}),$$

where $F(\bar{z})$ satisfies the equation

$$(\alpha^4-\alpha^2)F+F_{\bar{z}\bar{z}}=0.$$

Consequently, we take as a possible solution

$$\varphi(\bar{x},\bar{y},\bar{z})=\frac{A}{2\pi}\int_{-\infty}^{\infty}\exp\left\{-\mathrm{i}\alpha\bar{x}+\alpha^2\bar{y}+\mathrm{i}\alpha(\alpha^2-1)^{\frac{1}{2}}\bar{z}\right\}\mathrm{d}\alpha,\tag{2.82}$$

for A being a constant. Note that $\varphi(\bar{x},\bar{y},\bar{z})$ is not defined for $\bar{x}=\bar{y}=\bar{z}=0$. From (1.33) we find the free surface elevation

$$\eta(\bar{x},\bar{z}) = \frac{A\mathrm{i}}{2\pi U}\lim_{\bar{y}\to 0}\int_{-\infty}^{\infty}\left(\alpha \mathrm{e}^{\alpha^2\bar{y}}\right)\exp\left\{\mathrm{i}\left(-\alpha\bar{x}+\alpha(\alpha^2-1)^{\frac{1}{2}}\bar{z}\right)\right\}\mathrm{d}\alpha \tag{2.83}$$

which apparently is infinite for $\bar{x}=\bar{z}=0$.

In order to get a better insight into the shape of the surface we shall evaluate this expression (2.83) for large values of $\bar{x}$ and $\bar{z}$; that is distances to the origin that are large compared to the reference length U^2/g. This evaluation is performed by the method of stationary phase (see Sect. 9.1).

We note that if we let $R=(\bar{x}^2+\bar{z}^2)^{\frac{1}{2}}$, $\bar{x}=R\cos\vartheta$ and $\bar{z}=R\sin\vartheta$, then for each fixed ϑ, (2.83) can be written in the form

$$\int_{-\infty}^{\infty} g(\alpha)\exp(\mathrm{i}Rf(\alpha))\,\mathrm{d}\alpha,$$

where

$$g(\alpha) := \frac{A\mathrm{i}}{2\pi U}\alpha \quad \text{and}$$

$$Rf(\alpha) := -\alpha\bar{x}+\alpha(\alpha^2-1)^{\frac{1}{2}}\bar{z}.$$

Hence the stationary points are solutions of the equation

$$\frac{\partial}{\partial\alpha}\left\{-\alpha\bar{x}+\alpha(\alpha^2-1)^{\frac{1}{2}}\bar{z}\right\}=0. \tag{2.84}$$

(cf. (9.13)).

Let α_0 be a solution of (2.84). We obtain therefore the asymptotic form of $\eta(\bar{x},\bar{z})$:

$$\eta(\bar{x},\bar{z}) \cong \frac{A\mathrm{i}}{\pi U}\alpha_0\sqrt{\frac{\pi\mathrm{i}\alpha_0(\alpha_0^2-1)^{\frac{3}{2}}}{2\bar{z}(2\alpha_0^2-3)}}\exp\left\{\mathrm{i}\left(-\alpha_0\bar{x}+\alpha_0(\alpha_0^2-1)^{\frac{1}{2}}\bar{z}\right)\right\} \tag{2.85}$$

The phase function is of the most importance. If we put

$$\psi = -\alpha_0\bar{x}+\alpha_0(\alpha_0^2-1)^{\frac{1}{2}}\bar{z}, \tag{2.86}$$

we obtain from (2.84)

$$-\bar{x}+\frac{2\alpha_0^2-1}{\sqrt{\alpha_0^2-1}}\bar{z}=0. \tag{2.87}$$

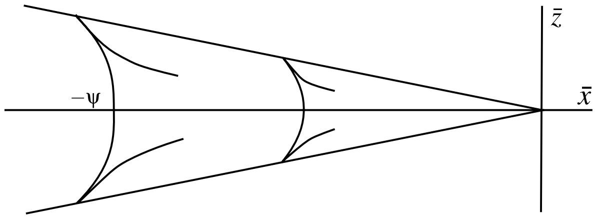

Fig. 2.4 Kelvin pattern

Setting $\alpha_0 = \frac{1}{\cos\theta}$, we obtain from (2.86) and (2.87) the equations

$$\begin{aligned} \bar{x} &= -\psi(2\cos\theta - \cos^3\theta) = -\frac{1}{4}\psi(5\cos\theta - \cos(3\theta)), \\ \bar{z} &= -\psi\cos^2\theta\sin\theta = -\frac{1}{4}\psi(\sin\theta + \sin(3\theta)) \end{aligned} \tag{2.88}$$

for the curves of constant phase ψ, which give the wave pattern. These curves are all similar with the origin as centre, and have wave cusps at $\bar{x} = \bar{z} = 0$ (or $\theta = \pi/2$) and at the points where $\frac{d\bar{x}}{d\theta} = \frac{d\bar{z}}{d\theta} = 0$. Since

$$\frac{d\bar{x}}{d\theta} = -\psi\sin\theta(2 - 3\cos^2\theta) \quad \text{and} \quad \frac{d\bar{z}}{d\theta} = -\psi\cos\theta(3\cos^2\theta - 2),$$

it follows that at the points

$$\bar{x} = -\psi\frac{4\sqrt{6}}{9}, \qquad \bar{z} = -\psi\frac{2\sqrt{3}}{9}$$

corresponding to $\cos\theta = \sqrt{2/3}$, there are cusps (it is understood that the expression (2.85) is not valid in the neighbourhood of cusps). A typical curve is shown in Fig. 2.4. We see that the curve intersects the $\bar{x}$-axis at the points $\bar{x} = -\psi$ (corresponding to $\theta = 0$). The cusps lie on a straight line, through the origin, which makes a fixed angle with the $\bar{x}$-axis. The pattern obtained this way is called the *Kelvin* wave pattern.

2.5 Wave Fronts

In view of (2.7) and (2.23), we now consider the general representation for the free surface elevation:

$$\eta(x, z, t) = \int_{-\infty}^{\infty} A(k)e^{-i(kx\cos\theta + kz\sin\theta - \omega t)}\,dk. \tag{2.89}$$

In particular, we are interested in the asymptotic behaviour of η for large values of t. We apply the method of stationary phase, see Sect. 9.1, to (2.89). The method requires the determination of the value of k for which the phase

$$\Psi(x, z, k) = -\omega t + k(x\cos\theta + z\sin\theta)$$

$$= -t\left[k\left(\frac{x}{t}\cos\theta + \frac{z}{t}\right) - \omega(k)\right] \tag{2.90}$$

is stationary. (Here we consider Ψ as depending on the three parameters $\frac{x}{t}$, $\frac{z}{t}$ and t. For each pair of values of $\frac{x}{t}$ and $\frac{z}{t}$, the asymptotic expansion for η is considered for large t.) This leads to the consideration of solutions of the equations

$$\frac{\mathrm{d}\Psi}{\mathrm{d}k} = 0 \quad \text{or} \quad -\frac{\mathrm{d}\omega}{\mathrm{d}k}t + x\cos\theta + z\sin\theta = 0. \tag{2.91}$$

Let k_0 be any solution of (2.91). Then the approximate result for large t is

$$\eta(x,z,t) = A(k_0)\sqrt{\frac{2\pi}{t|\omega''(k_0)|}}\mathrm{e}^{-\mathrm{i}(k_0 x\cos\theta + k_0 z\sin\theta - \omega(k_0)t - \frac{\pi}{4}\operatorname{sgn}\omega''(k_0))} \tag{2.92}$$

provided that $\frac{\mathrm{d}^2\Psi(k_0)}{\mathrm{d}k^2} \neq 0$, i.e. $\omega''(k_0) \neq 0$.

The lines $\Psi =$ constant are lines of constant phase; these lines are called *wave fronts*. We can define a partial differential equation for the wave fronts from the dispersion relation $\omega = H(k)$. In fact, we can express k_0 in terms of x, z, t and θ from (2.92) so that differentiations of (2.91) (with $k = k_0$) with respect to these variables yield

$$\begin{aligned}
\Psi_x &= k_0\cos\theta + (x\cos\theta + z\sin\theta - \omega_0' t)\frac{\partial k_0}{\partial x} = k_0\cos\theta,\\
\Psi_z &= k_0\sin\theta + (x\cos\theta + z\sin\theta - \omega_0' t)\frac{\partial k_0}{\partial z} = k_0\sin\theta,\\
\Psi_t &= -\omega_0 + (x\cos\theta + z\sin\theta - \omega_0' t)\frac{\partial k_0}{\partial t} = -\omega_0,
\end{aligned} \tag{2.93}$$

with $\omega_0 = H(k_0)$. The first two equations of (2.93) imply that

$$k_0^2 = \Psi_x^2 + \Psi_z^2$$

with which the third one shows that the dispersion relation $\omega_0 = H(k_0)$ gives a partial differential equation for the phase function Ψ, the Hamilton-Jacobi equation

$$\begin{aligned}
&\Psi_t + H(\sqrt{\Psi_x^2 + \Psi_y^2}) = 0 \quad \text{or}\\
&\Psi_t + H(\sqrt{p^2 + q^2}) = 0,
\end{aligned} \tag{2.94}$$

where $p = \Psi_x = k_0\cos\theta$ and $q = \Psi_z = k_0\sin\theta$ are the conjugate variables to x and z, respectively. We have just seen that the wave fronts correspond to level curves of the Hamilton-Jacobi equation. But in wave phenomena one expects the dual concept of rays to appear also. The rays in the present case are the characteristics of the

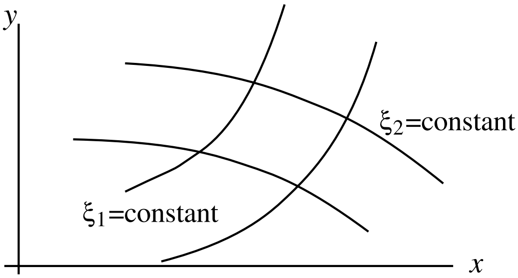

Fig. 2.5 Wave fronts

above Hamilton-Jacobi equation, i.e. the solutions of the system of ODE's:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\partial H}{\partial p}, \qquad & \frac{\mathrm{d}p}{\mathrm{d}t} &= -\frac{\partial H}{\partial x} = 0, \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial H}{\partial q}, \qquad & \frac{\mathrm{d}q}{\mathrm{d}t} &= -\frac{\partial H}{\partial z} = 0 \end{aligned} \tag{2.95}$$

(see Sect. 9.2 for a brief summary of the concepts of characteristics). From the (2.95) it is easy to see that in the x, z, t-space, the characteristics are straight lines for constant t as in (2.91).

2.6 Wave Patterns

In Sect. 2.5, the Hamilton-Jacobi equation (2.94) for the wave fronts was derived from the equations in a rather complicated way. At first we gave an exact solution η of the linearised problem (1.32), (1.33) with $U = 0$, to which we later applied an asymptotic expansion, which resulted in a first-order partial differential equation. The result obtained is more or less similar to the characteristic equation for hyperbolic equations, although the wave fronts are by no means characteristic surfaces for the equations, which do not even have real characteristics.

In order to give a direct derivation we first define a *wave front* on the two-dimensional x, y-plane as a curve along which a transverse derivative of the solution φ of the equation considered is much larger than the tangential derivative. This means that, introducing new coordinates ξ_1 transverse to the wave fronts and ξ_2 along the wave fronts (Fig. 2.5), we must have that $\varphi_{\xi_1} \gg \varphi_{\xi_2}$, i.e. there should exist a constant $K \gg 1$ such that $\varphi_{\xi_1} \approx K\varphi_{\xi_2}$. Here ξ_1 and ξ_2 are supposed to be functions of x and y with derivatives of order unity with respect to K. We introduce a new coordinate $s = K\xi_1$ such that

$$\varphi_s = \frac{1}{K}\varphi_{\xi_1} = \mathrm{O}(1). \tag{2.96}$$

We now illustrate this procedure by considering a simpler equation than the equation of water waves, the Klein-Gordon equation in dimensionless form

$$\varphi_{xx} - \varphi_{tt} - a^2\varphi = 0, \tag{2.97}$$

where a is a constant. We first derive the Hamilton-Jacobi equations for the phase function to the methods used in Sect. 2.5 and will refer to it as an indirect method. For solutions of the form $Ae^{\mathrm{i}(kx-\omega t)}$ we easily find the dispersion relation between k and ω,

$$\omega = \sqrt{a^2 + k^2} \doteqdot H(k) \tag{2.98}$$

which gives the Hamilton-Jacobi equation from (2.94) with $\Psi = J$:

$$J_t + H(J_x) = 0, \tag{2.99}$$

where $J_x = k$. The characteristics of (2.99) are solutions of the equations

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\partial H}{\partial J_x} = \frac{k}{\sqrt{a^2+k^2}}, \\ \frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{\mathrm{d}J_x}{\mathrm{d}t} = -\frac{\partial H}{\partial x} = 0, \end{aligned} \tag{2.100}$$

thus the characteristics are straight lines of the form

$$x - \frac{k}{\sqrt{a^2+k^2}} t = \text{constant}, \tag{2.101}$$

corresponding to the group velocity $\frac{\mathrm{d}H}{\mathrm{d}k} = \frac{k}{\sqrt{a^2+k^2}}$.

Now let us examine the above problem by the direct method. Using (2.96), a straightforward calculation shows that

$$\begin{aligned} \varphi_{xx} &= K^2 \varphi_{ss} \xi_{1x}^2 + K(2\varphi_{s\xi_2}\xi_{1x}\xi_{2x} + \xi_{1xx}\varphi_s) + \varphi_{\xi_2\xi_2}\xi_{2x}^2 + \varphi_{\xi_2}\xi_{2xx}, \\ \varphi_{tt} &= K^2 \varphi_{ss} \xi_{1t}^2 + K(2\varphi_{s\xi_2}\xi_{1t}\xi_{2t} + \xi_{1tt}\varphi_s) + \varphi_{\xi_2\xi_2}\xi_{2t}^2 + \varphi_{\xi_2}\xi_{2tt}. \end{aligned} \tag{2.102}$$

Substituting into (2.97) gives

$$\begin{aligned} K^2 \varphi_{ss}(\xi_{1x}^2 - \xi_{1t}^2) + K\{\varphi_s(\xi_{1xx} - \xi_{1tt}) + 2\varphi_{s\xi_2}(\xi_{1x}\xi_{2x} - \xi_{1t}\xi_{2t})\} \\ + \varphi_{\xi_2\xi_2}(\xi_{2x}^2 - \xi_{2t}^2) + \varphi_{\xi_2}(\xi_{2t} - \xi_{2tt}) - a^2\varphi = 0. \end{aligned} \tag{2.103}$$

As $K \to \infty$, we obtain the characteristic equation for (2.97). This is obvious because a characteristic would be a line along which the second derivative may be discontinuous. Now, regarding the constant a as a large number with respect to some reference length and identifying K with a, we have the equation

$$\varphi_{ss}(\xi_{1x}^2 - \xi_{1t}^2) - \varphi = 0, \tag{2.104}$$

to the first order of approximation. If we want this equation to represent the motion along the wave fronts, we must put the term $(\xi_{1x}^2 - \xi_{1t}^2)$ equal to a constant which we choose to be -1, i.e.,

$$\xi_{1x}^2 - \xi_{1t}^2 = -1. \tag{2.105}$$

Clearly, this gives immediately the Hamilton-Jacobi equation, $\xi_{1t} = \sqrt{1+\xi_{1x}^2}$, which reduces to (2.99) with ξ_1 replaced by $(-1/a)J$.

The same scheme can be applied to the problem of the moving singularity defined by the time independent form (1.32) and (1.33), i.e.:

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0,$$
$$U^2\varphi_{xx} + g\varphi_y = 0, \quad \text{for } y = 0.$$

In terms of the dimensionless variables $\overline{x} = \frac{x}{L}$, $\overline{y} = \frac{y}{L}$ and $\overline{z} = \frac{z}{L}$, we have

$$\begin{aligned}
&\varphi_{\overline{xx}} + \varphi_{\overline{yy}} + \varphi_{\overline{zz}} = 0,\\
&\varphi_{\overline{xx}} + \frac{gL}{U^2}\varphi_{\overline{y}} = 0, \quad \text{for } \overline{y} = 0.
\end{aligned} \tag{2.106}$$

Here L denotes a proper reference length.

We are only interested in the *wave pattern*, hence in the lines of constant phase of η (which from (1.33) amounts to the same as for $\varphi_{\overline{x}}$ at $\overline{y} = 0$). We further remark that from the nature of (2.106) we know that the wave is only appreciable at the upper layer of the water. Hence we introduce the coordinates ξ_1 and ξ_2 in the x, z-plane, where the lines $\xi_1 =$ constant represent wave fronts, the derivative φ_{ξ_1} is large with respect to φ_{ξ_2} but the derivative $\varphi_{\overline{y}}$ must be of the same order of magnitude as φ_{ξ_1}. Therefore, we introduce a coordinate $s = K\xi_1$ and a coordinate $Y = K\overline{y}$ in terms of which we have

$$\varphi_{\overline{xx}} = K^2\varphi_{ss}\xi_{1\overline{x}}^2 + K(2\varphi_{s\xi_2}\xi_{1\overline{x}}\xi_{2\overline{x}} + \xi_{1\overline{xx}}\varphi_s) + \varphi_{\xi_2\xi_2}\xi_{2\overline{x}}^2 + \varphi_{\xi_2}\xi_{2\overline{xx}},$$
$$\varphi_{\overline{zz}} = K^2\varphi_{ss}\xi_{1\overline{z}}^2 + K(2\varphi_{s\xi_2}\xi_{1\overline{z}}\xi_{2\overline{z}} + \xi_{1\overline{zz}}\varphi_s) + \varphi_{\xi_2\xi_2}\xi_{2\overline{z}}^2 + \varphi_{\xi_2}\xi_{2\overline{zz}},$$

and

$$\varphi_{\overline{yy}} = K^2\varphi_{YY}.$$

From (2.106), we have then

$$K^2\varphi_{ss}(\xi_{1\overline{x}}^2 + \xi_{1\overline{z}}^2) + K^2\varphi_{YY} + \mathrm{O}(K) = 0, \tag{2.107}$$

together with the surface condition

$$K^2\varphi_{ss}\xi_{1\overline{x}}^2 + K\left(\frac{gL}{U^2}\right)\varphi_Y + \mathrm{O}(K) = 0, \quad \text{for } Y = 0. \tag{2.108}$$

This yields the first approximation

$$\begin{aligned}
&\varphi_{ss}(\xi_{1\overline{x}}^2 + \xi_{1\overline{z}}^2) + \varphi_{YY} = 0,\\
&\varphi_{ss}\xi_{1\overline{x}}^2 + \varphi_Y = 0, \quad \text{for } Y = 0,
\end{aligned} \tag{2.109}$$

where we identify K with $\frac{gL}{U^2}$.

Since $\xi_{1\bar{x}}^2 + \xi_{1\bar{z}}^2$ and $\xi_{1\bar{x}}^2$ are slowly varying variables, we introduce constants, α and β defined by

$$\alpha^2 = \xi_{1\bar{x}}^2 + \xi_{1\bar{z}}^2, \quad \text{and} \quad \beta = \xi_{1\bar{x}}^2.$$

This leads to the problem

$$\alpha^2 \varphi_{ss} + \varphi_{YY} = 0,$$
$$\beta \varphi_{ss} + \varphi_Y = 0, \quad \text{for } Y = 0,$$

which has a solution

$$\varphi = \mathrm{e}^{\mathrm{i}s + .\alpha Y}.$$

This solution which goes to zero as $Y \to \infty$ $(\alpha > 0)$ can satisfy the surface condition only if

$$\alpha = \beta$$

or

$$\xi_{1\bar{x}}^2 + \xi_{1\bar{z}}^2 = \xi_{1\bar{x}}^4. \tag{2.110}$$

It should be emphasised that these considerations are only valid to an order of magnitude of $1/K$. The present approach is a variation of the ray method in geometrical optics. Higher-order approximations can be derived in a similar manner.

The characteristic equations of the first-order partial differential equation (2.110) take the form

$$\dot{\bar{x}} = 4p^3 - 2p, \quad \dot{p} = 0,$$
$$\dot{\bar{z}} = -2q, \quad \dot{q} = 0,$$
$$\dot{\xi}_1 = (4p^4 - 2p^2 - 2q^2),$$

with $p = \xi_{1\bar{x}}$ and $q = \xi_{1\bar{z}}$, where the dot $\cdot$ notation denotes differentiation to some parameter, say, τ. Hence p and q are constants and we have the parametric equations for the rays,

$$\begin{aligned} \bar{x} &= 2p(2p^2 - 1)\tau, \\ \bar{z} &= -2q\tau, \\ \xi_1 &= (4p^4 - 2p^2 - 2q^2)\tau. \end{aligned} \tag{2.111}$$

From (2.110) we have

$$q = -p\sqrt{p^2 - 1}. \tag{2.112}$$

To eliminate τ from (2.111) by making use of (2.112), we finally obtain

$$\bar{x} = \xi_1 \frac{(2p^2 - 1)}{p^3}, \qquad \bar{z} = \xi_1 \frac{\sqrt{p^2 - 1}}{p^3},$$

which reduces to (2.88) if we set $p = -\frac{1}{\cos\theta}$. This shows that the curves $\xi_1 =$ constant are indeed the curves of constant phase.

2.7 Singularity in a Steady Current

2.7.1 Steady Singularity

As in the case of an oscillatory point source in still water it is useful to have the solution of a steady moving point source, or a point source in a steady current, available. For finite water depth the formulation becomes

$$\begin{aligned}
&\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = \delta(x - x_0, y - y_0, z - z_0), \\
&\varphi_y = 0 \qquad\qquad \text{at } y = -h, \\
&\upsilon\varphi_{xx} + \varphi_y = 0 \quad \text{at } y = 0,
\end{aligned} \tag{2.113}$$

where we introduced the notation $\upsilon = \frac{U^2}{g}$. To obtain a physically valid solution we have to add a far-field condition, comparable with the radiation condition in the oscillatory case. Here the requirement becomes that in front of the disturbance no wavy pattern is observed. In the downstream region a wavy disturbance may be present. In the deep water case it is similar to the disturbance of the moving pressure point. It is also noticed that a solution of (2.113) can not be unique, because we always may add an arbitrary constant. We make use of this fact later. We follow the same procedure as described before (2.66) to solve (2.113),

$$\varphi(x, y, z) = -\frac{1}{4\pi r} - \frac{1}{4\pi \tilde{r}} + \psi(x, y, z), \tag{2.114}$$

where $\tilde{r}$ denotes the distance to reflected, with respect to the bottom, source point.

For $\psi(x, y, z)$ we obtain the formulation

$$\begin{aligned}
&\psi_{xx} + \psi_{yy} + \psi_{zz} = 0, \\
&\psi_y = 0 && \text{at } \quad y = -h, \\
&\psi_y + \upsilon\psi_{xx} = \frac{1}{4\pi}\left\{\frac{\partial}{\partial y}\left(\frac{1}{r} + \frac{1}{\tilde{r}}\right) + \upsilon\frac{\partial^2}{\partial x^2}\left(\frac{1}{r} + \frac{1}{\tilde{r}}\right)\right\} && \text{at } y = 0. \\
&\qquad\qquad := r(x, z; x_0, y_0, z_0)
\end{aligned} \tag{2.115}$$

If we apply the double Fourier transform to the function ψ,

$$\Psi(\alpha, y, \beta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\alpha x + \beta z)}\psi(x, y, z)\,\mathrm{d}x\,\mathrm{d}z, \tag{2.116}$$

we obtain the following ordinary differential equation and boundary conditions for Ψ:

$$\begin{aligned} &\Psi_{yy} - (\alpha^2 + \beta^2)\Psi = 0, && \\ &\Psi_y = 0 && \text{at } y = -h, \\ &\Psi_y - \upsilon\alpha^2\Psi = R(\alpha, \beta; x_0, y_0, z_0) && \text{at } y = 0. \end{aligned} \tag{2.117}$$

Application of (2.70) leads to the following expression for $R(\alpha, \beta; x_0, y_0, z_0)$,

$$R(\alpha, \beta; x_0, y_0, z_0) = -\frac{k + \upsilon\alpha^2}{k} \mathrm{e}^{-kh} \cosh(k(y_0 + h)) \mathrm{e}^{\mathrm{i}(\alpha x_0 + \beta z_0)}, \tag{2.118}$$

where $k = \sqrt{\alpha^2 + \beta^2}$. The solution of (2.117) becomes

$$\Psi(\alpha, y, \beta) = -\frac{k + \upsilon\alpha^2}{k} \mathrm{e}^{-kh} \frac{\cosh(k(y + h)) \cosh(k(y_0 + h))}{k \sinh(kh) - \upsilon\alpha^2 \cosh(kh)} \mathrm{e}^{\mathrm{i}(\alpha x_0 + \beta z_0)}. \tag{2.119}$$

The inverse transform (2.73) of $\Psi(\alpha, y, \beta)$ becomes

$$\begin{aligned} \psi(x, y, z) = \frac{-1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} &\mathrm{e}^{-kh} \cosh(k(y + h)) \cosh(k(y_0 + h)) \\ &\cdot \frac{k + \upsilon\alpha^2}{k} \frac{\mathrm{e}^{-\mathrm{i}(\alpha(x - x_0) + \beta(z - z_0))}}{k \sinh(kh) - \upsilon\alpha^2 \cosh(kh)} \mathrm{d}\alpha\, \mathrm{d}\beta \end{aligned} \tag{2.120}$$

and after the introduction of polar coordinates in the Fourier plane (2.52), (2.53)

$$\begin{aligned} \psi(x, y, z) = \frac{-1}{4\pi^2} \int_0^{\infty} \int_0^{2\pi} &\mathrm{e}^{-kh} \cosh(k(y + h)) \cosh(k(y_0 + h)) \\ &\cdot \frac{(1 + k\upsilon \cos^2\vartheta) \mathrm{e}^{-\mathrm{i}k((x - x_0)\cos\vartheta + (z - z_0)\sin\vartheta)}}{\sinh(kh) - k\upsilon \cos^2\vartheta \cosh(kh)} \mathrm{d}k\, \mathrm{d}\vartheta. \end{aligned} \tag{2.121}$$

The integral is singular at $k = 0$. Therefor we make use of the fact that we may add a constant, with respect to x, y and z to the solution of (2.113). Hence a solution of (2.113) may be written as

$$\varphi(x, y, z) = -\frac{1}{4\pi r} - \frac{1}{4\pi \tilde{r}} + \tilde{\psi}(x, y, z) \tag{2.122}$$

with

$$\begin{aligned} \tilde{\psi}(x, y, z) = \frac{-1}{4\pi^2} \int_0^{\infty} \mathrm{d}k \int_0^{2\pi} \mathrm{d}\vartheta\, &\frac{\mathrm{e}^{-kh}}{\sinh(kh) - k\upsilon \cos^2\vartheta \cosh(kh)} \\ &\cdot \Big\{\cosh(k(y + h)) \cosh(k(y_0 + h))(1 + k\upsilon \cos^2\vartheta) \\ &\cdot \mathrm{e}^{-\mathrm{i}k((x - x_0)\cos\vartheta + (z - z_0)\sin\vartheta)} - 1\Big\}. \end{aligned} \tag{2.123}$$

This solution does not fulfil the condition that upstream ($x \to -\infty$) no wavy disturbance may be present. To obey this condition a path of integration along the singularity on the real k-axis has to be chosen. Depending on the sign of $\cos\vartheta$ the choice will be different.

We notice that for $\cos\vartheta > 0$ and $x - x_0 > 0$ we may close the integral with respect to k in the fourth quadrant of the complex k-plane. For $\cos^2\vartheta < \frac{h}{\upsilon}$ we find a simple pole on the real k-axis. This means that to obtain a wavy contribution this singularity on the real axis must be inside the contour. We obtain a contribution of the pole plus an integral along the negative imaginary axis. This integral represents an exponentially decaying contribution. If however $x - x_0 < 0$ we close the integral in the first quadrant and we obtain a contribution of an integral along the positive imaginary axis only.

Next we consider $\cos\vartheta < 0$ and $x - x_0 > 0$ and we may close the integral in the first quadrant of the complex k-plane. We obtain a contribution of the singularity on the real axis if we chose the pole inside the contour. Again the integral along the imaginary axis is exponentially decaying. If $x - x_0 < 0$ we may close the contour in the fourth quadrant. This gives rise to a decaying contribution only.

We may reformulate the integral part of the solution by splitting up the integration with respect to ϑ into four parts of length $\pi/2$ and to combine the integral. In this way we obtain

$$\begin{aligned}\tilde{\psi}(x,y,z) = -\frac{1}{\pi^2}\int_0^\infty \mathrm{d}k \int_0^{\frac{\pi}{2}} \mathrm{d}\vartheta \frac{\mathrm{e}^{-kh}}{\sinh(kh) - k\upsilon\cos^2\vartheta\cosh(kh)}\big\{\cosh(k(y+h))\\ \cdot\cosh(k(y_0+h))(1+k\upsilon\cos^2\vartheta)\cos((x-x_0)\\ \cdot\cos\vartheta)\cos((z-z_0)\sin\vartheta) - 1\big\}.\end{aligned} \tag{2.124}$$

In the Handbook of Physics [19], Wehausen gives further details of this expression.

To obtain an expression for the deep water case we let $h \to \infty$ in expression (2.123) and obtain.

$$\begin{aligned}\varphi(x,y,z) = -\frac{1}{4\pi r} + \frac{1}{4\pi\bar{r}} - \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}k \int_0^{\pi} \mathrm{d}\vartheta \frac{1}{1-k\upsilon\cos^2\vartheta}\\ \cdot\mathrm{e}^{k((y+y_0)-\mathrm{i}(x-x_0)\cos\vartheta)}\cos(k(z-z_0)\sin\vartheta)\end{aligned} \tag{2.125}$$

where we used (2.70) to obtain the contribution of a singularity at the point $(x_0, -y_0, z_0)$, hence $\bar{r}$ is defined as $\sqrt{(x-x_0)^2 + (y+y_0)^2 + (z-z_0)^2}$. The contour in the k-plane has to be chosen as before. For $\cos\vartheta > 0$ the contour passes the singularity in the upper plane, while for $\cos\vartheta < 0$ the contour passes the singularity in the lower plane. The contribution of the pole gives a far-field pattern comparable with the moving pressure point wave field described in Sect. 2.4.

2.7.2 Oscillating Singularity

The boundary value problem for the disturbance of a steady flow is described in (1.32). We consider a harmonic point source and assume that the potential function can be written as

$$\phi(x,y,z,t) = \mathrm{e}^{\mathrm{i}\omega t}\hat{\phi}(x,y,z).$$

The boundary value problem for the disturbance of a point source in (x_0, y_0, z_0) becomes

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = \delta(x-x_0, y-y_0, z-z_0)\mathrm{e}^{\mathrm{i}\omega t}. \tag{2.126}$$

If we introduce $\varphi(x,y,z,t) = \hat{\varphi}(x,y,z)\mathrm{e}^{\mathrm{i}\omega t}$, the boundary value problem to be solved becomes

$$\begin{aligned}
&\hat{\varphi}_{xx} + \hat{\varphi}_{yy} + \hat{\varphi}_{zz} = \delta(x-x_0, y-y_0, z-z_0), \\
&\hat{\varphi}_y = 0 && \text{at } y = -h, \\
&\upsilon\hat{\phi}_{xx} + 2\mathrm{i}\tau\hat{\varphi}_x - \nu\hat{\varphi} + \hat{\varphi}_y = 0 && \text{at } y = 0,
\end{aligned} \tag{2.127}$$

where we introduced the parameters $\nu = \omega^2/g$, $\upsilon = U^2/g$,and $\tau = (\omega U)/g$; notice that $\tau^2 = \nu\upsilon$.

$$\hat{\varphi}(x,y,z) = -\frac{1}{4\pi r} - \frac{1}{4\pi\tilde{r}} + \psi(x,y,z). \tag{2.128}$$

Introduction of the double Fourier transform leads to the following ordinary differential equation for the transform of ψ,

$$\begin{aligned}
&\Psi_{yy} - (\alpha^2+\beta^2)\Psi = 0, \\
&\Psi_y = 0 && \text{at } y = -h, \\
&\Psi_y - (\upsilon\alpha^2 + 2\tau\alpha + \nu)\Psi = S(\alpha,\beta;x_0,y_0,z_0) && \text{at } y = 0.
\end{aligned} \tag{2.129}$$

Application of (2.70) leads to the following expression for $S(\alpha,\beta;x_0,y_0,z_0)$,

$$S(\alpha,\beta;x_0,y_0,z_0) = -\frac{k+\upsilon\alpha^2+2\tau\alpha+\nu}{k}\mathrm{e}^{-kh}\cosh(k(y_0+h))\mathrm{e}^{\mathrm{i}(\alpha x_0+\beta z_0)}, \tag{2.130}$$

where $k = \sqrt{\alpha^2+\beta^2}$. The solution of (2.129) becomes

$$\begin{aligned}
\Psi(\alpha,y,\beta) = &-\frac{k+\upsilon\alpha^2+2\tau\alpha+\nu}{k}\mathrm{e}^{-kh} \\
&\cdot\frac{\cosh(k(y+h))\cosh(k(y_0+h))}{k\sinh(kh)-(\upsilon\alpha^2+2\tau\alpha+\nu)\cosh(kh)}\mathrm{e}^{\mathrm{i}(\alpha x_0+\beta z_0)}.
\end{aligned} \tag{2.131}$$

The inverse transform of $\Psi(\alpha,y,\beta)$ gives the solution of (2.127). The choice of the path of integration will be elucidated in the deep water case. Hence we consider the limit $h \to \infty$. We rewrite expression (2.131) in the form

$$\Psi(\alpha, y, \beta) = \left(\frac{1}{2} - \mathscr{L}(k, \alpha)\right) \frac{e^{k(y+y_0)} e^{i(\alpha x_0 + \beta z_0)}}{k}. \tag{2.132}$$

The function $\mathscr{L}(k, \alpha)$ becomes, in polar coordinates in the Fourier plane,

$$\lim_{h\to\infty} \mathscr{L}(k, \theta) = \frac{k}{k - (\upsilon\alpha^2 + 2\tau\alpha + \nu)} = \frac{gk}{gk - (\omega + kU\cos\theta)^2}. \tag{2.133}$$

Finally we obtain an expression for $\hat{\varphi}(x, y, z)$ where we still have to choose the proper path of integration in the complex k-plane

$$\varphi(x, y, z) = -\frac{1}{4\pi r} + \frac{1}{4\pi \bar{r}} - \frac{1}{2\pi^2} \int_0^\infty \int_0^\pi \frac{gk}{gk - (\omega + kU\cos\theta)^2}$$
$$\cdot e^{k((y+y_0) - i(x-x_0)\cos\vartheta)} \cos(k(z - z_0)\sin\vartheta)\, dk\, d\vartheta. \tag{2.134}$$

We will investigate the zeros of the denominator. The quadratic equation has two zeros,

$$gk^{\pm} = \frac{1 - 2\tau\cos\vartheta \pm \sqrt{1 - 4\tau\cos\vartheta}}{2\tau^2\cos^2\vartheta}\omega^2. \tag{2.135}$$

First of all we notice that, for values of ϑ for which we have

$$1 - 4\tau\cos\vartheta < 0, \tag{2.136}$$

we find no singularities of the integrand along the real k-axis. Hence for $\tau > 1/4$ we find a ϑ interval where the k-integral is regular for $0 \le \vartheta < \gamma$ with $\cos\gamma = 1/(4\tau)$. For $\tau < 1/4$ we have $\gamma = 0$. Next, to determine the contour of integration when two poles lie on the positive real axis we have to consider the condition in the far field. It is easy to show that for $\vartheta > \gamma$ both roots of the quadratic equation are situated on the positive real axis of the complex k-plane. It is convenient to consider the poles for small values of U and ω successively. In both cases τ becomes small, so we consider the two poles for $\tau \to 0$. We find

$$\lim_{\tau\to 0} k^- = \frac{\omega^2}{g} \quad \text{and} \quad \lim_{\tau\to 0} k^+ = \frac{g}{U^2\cos^2\vartheta}. \tag{2.137}$$

In the oscillatory case without current we have seen that the contour passes the pole in the first quadrant of the complex k-plane, for all values of ϑ. Actually we could carry out the ϑ integral in that case. Hence, we conclude that this is also the case for the singularity in k^-.

In the case of a steady source in a current we have seen that we have to consider the sign of $\cos\vartheta$ because in the downstream direction the far field shows a wavy character. Hence for $\cos\vartheta > 0$ the contour of integration passes k^+ in the first

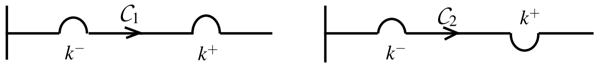

Fig. 2.6 Lines of integration

quadrant, while for $\cos\vartheta < 0$ the contour passes k^+ in the fourth quadrant of the complex k-plane (see Fig. 2.6). Finally the solution can be written as

$$\varphi(x, y, z) = -\frac{1}{4\pi r} + \frac{1}{4\pi \bar{r}} - \frac{1}{2\pi^2}\left\{\int_0^\infty \int_0^\gamma + \int_{\mathscr{C}_1} \int_\gamma^{\frac{\pi}{2}} + \int_{\mathscr{C}_2} \int_{\frac{\pi}{2}}^\pi \right\}$$
$$\cdot \frac{gk\mathrm{e}^{k((y+y_0)-\mathrm{i}(x-x_0)\cos\vartheta)}}{gk - (\omega + kU\cos\theta)^2} \cos(k(z - z_0)\sin\vartheta)\,\mathrm{d}k\,\mathrm{d}\vartheta. \qquad (2.138)$$

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