Chapter 2
The Net Energy Balance

In this section we see what the terms in the Navier–Stokes equation contribute to the production and dissipation of energy in a fluid. Consider a fluid contained in a volume V whose boundary is the surface S. We prescribe that no material shall cross S and we thus have the boundary condition that the velocity component normal to S is zero everywhere on S, i.e.,

$$\boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} = 0 \quad \text{on } S, \tag{2.1}$$

where $\mathrm{d}\boldsymbol{S}$ is a vector normal to the surface and of arbitrary magnitude. If we multiply (1.1) by ρu_i we obtain, after integrating both sides over V,

$$\rho \int_V u_i \frac{\partial u_i}{\partial t} \mathrm{d}V + \rho \int_V u_i u_j \frac{\partial u_i}{\partial x_j} \mathrm{d}V = - \int_V u_i \frac{\partial p}{\partial x_i} \mathrm{d}V + \nu\rho \int_V u_i \nabla^2 u_i \mathrm{d}V. \tag{2.2}$$

Here $\mathrm{d}V$ is the element of volume in V, ρ has been treated as constant, and the solenoidal character of u_i will be assumed throughout. We then consider the four terms of (2.2) separately:

(a) **The First Term on the Left-Hand Side:**

$$\rho \int_V u_i \frac{\partial u_i}{\partial t} \mathrm{d}V = \frac{\rho}{2} \int_V \frac{\partial (u_i u_i)}{\partial t} \mathrm{d}V = \frac{\rho}{2} \int_V \frac{\partial |\boldsymbol{u}|^2}{\partial t} \mathrm{d}V = \frac{\partial \mathcal{T}}{\partial t},$$

where $\mathcal{T}$ is the kinetic energy contained in the volume.

(b) **The Non-Linear Term:** In (1.1) the term $\partial(u_i u_j)/\partial x_i$ arises as the changing velocity of a mass element arising from its changing position in the velocity field; it is known as the inertial term. We have,

$$\rho \int_V u_i u_j \frac{\partial u_i}{\partial x_j} \mathrm{d}V = \frac{1}{2}\rho \int_V u_j \frac{\partial (u_i u_j)}{\partial x_j} \mathrm{d}V = \frac{1}{2}\rho \int_V \frac{\partial (u_j |\boldsymbol{u}|^2)}{\partial x_j} \mathrm{d}V,$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_2, © Springer Science+Business Media B.V. 2011

since u_j is solenoidal. Then, by the divergence theorem,

$$\begin{aligned}\rho\int_V u_i u_j \frac{\partial u_i}{\partial x_j}\mathrm{d}V &= \frac{1}{2}\rho\int_V \mathrm{div}\,(\boldsymbol{u}|\boldsymbol{u}|^2)\mathrm{d}V\\ &= \frac{1}{2}\rho\int_S |\boldsymbol{u}|^2\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{S}\\ &= 0 \quad \text{by (2.1).}\end{aligned}$$

(c) **The Pressure Term:**

$$\int_V u_i\frac{\partial p}{\partial x_i}\mathrm{d}V = \int_V \frac{\partial(pu_i)}{\partial x_i}\mathrm{d}V = \int_S p\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{S} = 0,$$

by (2.1).

(d) **The Dissipation Term:** We have the lemma

$$u_i\nabla^2 u_i = -|\mathrm{curl}\,\boldsymbol{u}|^2 + \mathrm{div}\,(\boldsymbol{u}\times\mathrm{curl}\,\boldsymbol{u}),$$

proved at the end of this section. Then, in view of this identity,

$$\begin{aligned}\rho\nu\int_V u_i\nabla^2 u_i\mathrm{d}V &= \mu\int_V\Big[-|\mathrm{curl}\,\boldsymbol{u}|^2 + \mathrm{div}\,(\boldsymbol{u}\times\mathrm{curl}\,\boldsymbol{u})\Big]\mathrm{d}V\\ &= -\mu\int_V|\mathrm{curl}\,\boldsymbol{u}|^2\mathrm{d}V + \mu\int_S(\boldsymbol{u}\times\mathrm{curl}\,\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{S}.\end{aligned}$$

Let $\boldsymbol{\omega} = \mathrm{curl}\,\boldsymbol{u}$. Then, on gathering the various terms of (2.2) we obtain

$$\frac{\partial\mathcal{T}}{\partial t} = -\mu\int_V|\boldsymbol{\omega}|^2\mathrm{d}V + \mu\int_S(\boldsymbol{u}\times\mathrm{curl}\,\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{S}. \tag{2.3}$$

The first term on the right-hand side is the viscous dissipation of the vorticity, i.e., $-\mu|\boldsymbol{\omega}|^2$ is the rate of dissipation of energy per unit volume. The stationary state requires $\partial\mathcal{T}/\partial t = 0$, and so the energy input must be balanced by $\int_V(-\mu|\boldsymbol{\omega}|^2)\mathrm{d}V$. This implies that the small scale motion predominates in the dissipation, as will become clear below.

Proof of the Lemma

To show

$$u_i\nabla^2 u_i = -|\mathrm{curl}\,\boldsymbol{u}|^2 + \mathrm{div}\,(\boldsymbol{u}\times\mathrm{curl}\,\boldsymbol{u}),$$

consider

$$\mathrm{curl}_i\boldsymbol{u} = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}, \qquad |\mathrm{curl}\,\boldsymbol{u}|^2 = \epsilon_{ijk}\epsilon_{imn}u_{k,j}u_{n,m}. \tag{2.4}$$

But

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{2.5}$$

Combining (2.4) and (2.5) leads to

$$|\mathrm{curl}\,\boldsymbol{u}|^2 = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})u_{k,j}u_{n,m}$$
$$= u_{i,j}u_{i,j} - u_{i,j}u_{j,i}.$$

Further,

$$(\boldsymbol{u} \times \mathrm{curl}\,\boldsymbol{u})_i = \epsilon_{ijk}u_j\,\epsilon_{kmn}u_{n,m}$$
$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})u_j u_{n,m} = u_j u_{j,i} - u_j u_{i,j}, \tag{2.6}$$

whence

$$\mathrm{div}\,(\boldsymbol{u} \times \mathrm{curl}\,\boldsymbol{u}) = u_{j,i}u_{j,i} + u_j u_{j,ii} - u_{j,i}u_{i,j} - u_j u_{i,ji}.$$

Now,

$$u_j u_{i,ji} = u_j u_{i,ij} = u_j \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_i}\right) = 0,$$

by (1.2). Then,

$$\mathrm{div}\,(\boldsymbol{u} \times \mathrm{curl}\,\boldsymbol{u}) - |\mathrm{curl}\,\boldsymbol{u}|^2 = u_i \nabla^2 u_i,$$

and the lemma is proved. □

Returning to the example of concentric rotating cylinders, we note that the second term on the right of (2.3) must be the energy introduced by the cylinders per unit time and must be balanced by the dissipation term if $\partial\mathcal{T}/\partial t$ is to be zero. Let $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ be unit vectors in the r, θ, z directions, respectively. Then, the velocity (at least in one solution) is given by $v_r = v_z = 0$ and $v_\theta = v$, where v is given by (1.3). Then

$$\mathrm{curl}\,\boldsymbol{u} = \boldsymbol{k}\left[\frac{1}{r}\frac{\partial(rv)}{\partial r}\right] = 2A\boldsymbol{k},$$

whence

$$\boldsymbol{u} \times \mathrm{curl}\,\boldsymbol{u} = \boldsymbol{i}2Av,$$

and

$$(\boldsymbol{u} \times \mathrm{curl}\,\boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{S} = 2Av\mathrm{d}S,$$

so that

$$\int (\boldsymbol{u} \times \mathrm{curl}\,\boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{S} = \int_0^l \int_0^{2\pi} \Big[2Av\Big]_{R_1}^{R_2} r\,\mathrm{d}\theta\,\mathrm{d}z$$
$$= 4\pi l A\left[R_2\left(AR_2 + \frac{B}{R_2}\right) - R_1\left(AR_1 + \frac{B}{R_1}\right)\right]$$
$$= 4\pi l A^2(R_2^2 - R_1^2).$$

Also, since

$$|\mathrm{curl}\,\boldsymbol{u}|^2 = \left[\frac{1}{r}\frac{\partial(rv)}{\partial r}\right]^2 = (2A)^2,$$

$$\begin{aligned}\int_V |\mathrm{curl}\,\boldsymbol{u}|^2 \mathrm{d}V &= 4A^2 \int_V \mathrm{d}V = 4A^2(\pi l R_2^2 - \pi l R_1^2) \\ &= 4\pi l A^2 (R_2^2 - R_1^2),\end{aligned}$$

and $\partial\mathcal{T}/\partial t = 0$, which will hold on the average, even when instability occurs.

Springer

http://www.springer.com/978-94-007-0116-8

The Theory of Turbulence
Subrahmanyan Chandrasekhar's 1954 Lectures
Spiegel, E.A. (Ed.)
2011, XVII, 117 p., Softcover
ISBN: 978-94-007-0116-8