Chapter 2
Mathematical tools

With my full philosophical rucksack I can only climb slowly up the mountain of mathematics.
– Ludwig Wittgenstein

In this chapter we discuss certain mathematical tools which are used extensively in the following chapters. Some of these concepts and methods are part of the standard baggage taught in undergraduate and graduate courses, while others enter the toolbox of more advanced researchers. These mathematical methods are very useful in formulating ETGs and in finding analytical solutions. We begin by studying conformal transformations, which allow for different representations of scalar-tensor and $f(R)$ theories of gravity, in addition to being useful in GR. We continue by discussing variational principles in GR, which are the basis for presenting ETGs in the following chapters. We close the chapter with a discussion of Noether symmetries, which are used elsewhere in this book to obtain analytical solutions.

2.1 Conformal transformations

A mathematical tool that has proved very useful in alternative gravitational theories as well as in GR is that of conformal transformations (see [467, 472, 769] for reviews). The idea is to perform a conformal rescaling of the spacetime metric $g_{\mu\nu} \to \tilde{g}_{\mu\nu}$. Often a scalar field is present in the theory and the metric rescaling is accompanied by a (nonlinear) redefinition of this field $\phi \to \tilde{\phi}$. New dynamical variables $\left(\tilde{g}_{\mu\nu}, \tilde{\phi}\right)$ are thus obtained. The scalar field redefinition serves the purpose of casting the kinetic energy density of this field in canonical form. The new set of variables $\left(\tilde{g}_{\mu\nu}, \tilde{\phi}\right)$ is called the *Einstein conformal frame*, while $\left(g_{\mu\nu}, \phi\right)$ constitute the *Jordan frame*. When a scalar degree of freedom ϕ is present in the theory, as in scalar-tensor or $f(R)$ gravity, it generates the transformation to the Einstein frame in the sense that the rescaling is completely determined by a function of ϕ. In principle, infinitely many conformal frames could be introduced, giving rise to as many representations of the theory. From the physical point of view, these different representations have been the subject of many debates and misinterpretations, which will be discussed later. For the moment we expose the mathematical technique.

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Let the pair $(M, g_{\mu\nu})$ be a spacetime, with M a smooth manifold of dimension $n \geq 2$ and $g_{\mu\nu}$ a Lorentzian or Riemannian metric on M. The point-dependent rescaling of the metric tensor

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \,, \tag{2.1}$$

where the *conformal factor* $\Omega(x)$ is a nowhere vanishing, regular[1] function, is called a *Weyl* or *conformal* transformation. Due to this metric rescaling, the lengths of spacelike and timelike intervals and the norms of spacelike and timelike vectors are changed, while null vectors and null intervals of the metric $g_{\mu\nu}$ remain null in the rescaled metric $\tilde{g}_{\mu\nu}$. The light cones are left unchanged by the transformation (2.1) and the spacetimes $(M, g_{\mu\nu})$ and $(M, \tilde{g}_{\mu\nu})$ exhibit the same causal structure; the converse is also true [1139]. A vector that is timelike, spacelike, or null with respect to the metric $g_{\mu\nu}$ has the same character with respect to $\tilde{g}_{\mu\nu}$, and *vice-versa*.

In the Arnowitt-Deser-Misner (ADM) [54] decomposition of the metric

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\left(N^2 - N_i N^i\right) dt^2 + 2N_j\, dt\, dx^j + h_{ij} dx^i\, dx^j \tag{2.2}$$

using the lapse function N and the shift vector N^i, the transformation properties of these quantities follow from Eq. (2.1):

$$\tilde{N} = \Omega\, N \,, \qquad \tilde{N}^i = N^i \,, \qquad \tilde{h}_{ij} = \Omega^2\, h_{ij} \,. \tag{2.3}$$

The ADM mass of an asymptotically flat spacetime [54] does not change under the conformal transformation and scalar field redefinition [282].

The transformation properties of various geometrical quantities are useful [1065, 1139]. We list them here, leaving their proof to the reader as an exercise:

$$\tilde{g}^{\mu\nu} = \Omega^{-2}\, g^{\mu\nu} \,, \qquad \tilde{g} = \Omega^{2n}\, g \tag{2.4}$$

for the inverse metric and the metric determinant,

$$\tilde{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + \Omega^{-1}\left(\delta^\alpha_\beta \nabla_\gamma \Omega + \delta^\alpha_\gamma \nabla_\beta \Omega - g_{\beta\gamma} \nabla^\alpha \Omega\right) \tag{2.5}$$

for the Christoffel symbols,

$$\begin{aligned}\widetilde{R_{\alpha\beta\gamma}}^{\delta} = {} & R_{\alpha\beta\gamma}{}^{\delta} + 2\,\delta^\delta_{[\alpha} \nabla_{\beta]} \nabla_\gamma (\ln \Omega) - 2g^{\delta\sigma} g_{\gamma[\alpha} \nabla_{\beta]} \nabla_\sigma (\ln \Omega) \\ & + 2\nabla_{[\alpha} (\ln \Omega)\, \delta^\delta_{\beta]} \nabla_\gamma (\ln \Omega) - 2\nabla_{[\alpha} (\ln \Omega)\, g_{\beta]\gamma}\, g^{\delta\sigma}\, \nabla_\sigma (\ln \Omega) \\ & - 2 g_{\gamma[\alpha} \delta^\delta_{\beta]}\, g^{\sigma\rho} \nabla_\sigma (\ln \Omega)\, \nabla_\rho (\ln \Omega)\end{aligned} \tag{2.6}$$

[1] See [171, 172, 180] for the possibility of continuation beyond singular points of the conformal factor.

for the Riemann tensor,

$$\begin{aligned}\tilde{R}_{\alpha\beta} = & R_{\alpha\beta} - (n-2)\nabla_\alpha\nabla_\beta(\ln\Omega) - g_{\alpha\beta}\, g^{\rho\sigma}\,\nabla_\sigma\nabla_\rho(\ln\Omega) \\ & + (n-2)\,\nabla_\alpha(\ln\Omega)\nabla_\beta(\ln\Omega) \\ & - (n-2)\, g_{\alpha\beta}\, g^{\rho\sigma}\,\nabla_\rho(\ln\Omega)\nabla_\sigma(\ln\Omega)\end{aligned} \tag{2.7}$$

for the Ricci tensor, and

$$\tilde{R} \equiv \tilde{g}^{\alpha\beta}\,\tilde{R}_{\alpha\beta} = \Omega^{-2}\left[R - 2\,(n-1)\,\Box\,(\ln\Omega) - (n-1)\,(n-2)\,\frac{g^{\alpha\beta}\nabla_\alpha\Omega\,\nabla_\beta\Omega}{\Omega^2}\right] \tag{2.8}$$

for the Ricci scalar. In the case of $n = 4$ spacetime dimensions, the transformation property of the Ricci scalar can be written as

$$\begin{aligned}\tilde{R} &= \Omega^{-2}\left[R - \frac{6\Box\Omega}{\Omega}\right] \\ &= \Omega^{-2}\left[R - \frac{12\Box(\sqrt{\Omega})}{\sqrt{\Omega}} + \frac{3g^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta\Omega}{\Omega^2}\right].\end{aligned} \tag{2.9}$$

The Weyl tensor ${C_{\alpha\beta\gamma}}^{\delta}$ with the last index contravariant is conformally invariant,

$$\widetilde{{C_{\alpha\beta\gamma}}^{\delta}} = {C_{\alpha\beta\gamma}}^{\delta}, \tag{2.10}$$

but the same tensor with indices raised or lowered with respect to ${C_{\alpha\beta\gamma}}^{\delta}$ is not. This property explains the name *conformal tensor* sometimes used for ${C_{\alpha\beta\gamma}}^{\delta}$ [749]. If the original metric $g_{\alpha\beta}$ is Ricci-flat (*i.e.*, $R_{\alpha\beta} = 0$), the conformally transformed metric $\tilde{g}_{\alpha\beta}$ is not (cf. Eq. (2.7)). In the conformally transformed world the conformal factor Ω plays the role of an effective form of matter and this fact has consequences for the physical interpretation of the theory. A vacuum metric in the Jordan frame is not such in the Einstein frame, and the interpretation of what is matter and what is gravity becomes frame-dependent [1035]. However, if the Weyl tensor vanishes in one frame, it also vanishes in the conformally related frame. Conformally flat metrics are mapped into conformally flat metrics, a property used in cosmology when mapping FLRW universes (which are conformally flat) into each other. In particular, de Sitter spaces with scale factor $a(t) = a_0\, \exp(H_0 t)$ and a constant scalar field as the material source are mapped into similar de Sitter spaces.

Since, in general, tensorial quantities are not invariant under conformal transformations, neither are the tensorial equations describing geometry and physics. An equation involving a tensor field ψ is said to be *conformally invariant* if there exists a number w (the *conformal weight* of ψ) such that, if ψ is a solution of a tensor equation with the metric $g_{\mu\nu}$ and the associated geometrical quantities, $\tilde{\psi} \equiv \Omega^w \psi$ is a solution of the corresponding equation with the metric $\tilde{g}_{\mu\nu}$ and the associated geometry.

In addition to geometric quantities, one needs to consider the behavior of common forms of matter under conformal transformations. It goes without saying that most forms of matter or fields are not conformally invariant: invariance under conformal transformations is a very special property. In general, the covariant conservation equation for a (symmetric) stress-energy tensor $T^{(m)}_{\alpha\beta}$ representing ordinary matter,

$$\nabla^\beta \, T^{(m)}_{\alpha\beta} = 0 \tag{2.11}$$

is not conformally invariant [1139]. The conformally transformed $\tilde{T}^{(m)}_{\alpha\beta}$ satisfies the equation

$$\tilde{\nabla}^\beta \, \tilde{T}^{(m)}_{\alpha\beta} = -\tilde{T}^{(m)} \, \tilde{\nabla}_\alpha \left(\ln \Omega\right) \, . \tag{2.12}$$

Clearly, the conservation equation (2.11) is conformally invariant only for a matter component that has vanishing trace $T^{(m)}$ of the energy-momentum tensor. This feature is associated with light-like behavior; examples are the electromagnetic field and a radiative fluid with equation of state $P^{(m)} = \rho^{(m)}/3$. Unless $T^{(m)} = 0$, Eq. (2.12) describes an exchange of energy and momentum between matter and the scalar field Ω, reflecting the fact that matter and the geometric factor Ω are directly coupled in the Einstein frame description.

Since the geodesic equation ruling the motion of free particles in GR can be derived from the conservation equation (2.11) (*geodesic hypothesis*), it follows that timelike geodesics of the original metric $g_{\alpha\beta}$ are not geodesics of the rescaled metric $\tilde{g}_{\alpha\beta}$ and *vice-versa*. Particles in free fall in the world $\left(M, g_{\alpha\beta}\right)$ are subject to a force proportional to the gradient $\tilde{\nabla}^\alpha \Omega$ in the rescaled world $\left(M, \tilde{g}_{\alpha\beta}\right)$ – this is often identified as a fifth force acting on all massive particles and, therefore, it can be said that no massive test particles exist in the Einstein frame. The stress-energy tensor definition in terms of the matter action $S^{(m)} = \int d^4x \, \sqrt{-g} \, \mathscr{L}^{(m)}$,

$$\tilde{T}^{(m)}_{\alpha\beta} = \frac{-2}{\sqrt{-\tilde{g}}} \, \frac{\delta\left(\sqrt{-\tilde{g}} \; \mathscr{L}^{(m)}\right)}{\delta \tilde{g}^{\alpha\beta}} \, , \tag{2.13}$$

together with the rescaling (2.1) of the metric, yields

$$\tilde{T}_{\alpha\beta}{}^{(m)} = \Omega^{-2} \, T^{(m)}_{\alpha\beta} \, , \quad \widetilde{T_\alpha{}^{\beta}}{}^{(m)} = \Omega^{-4} \, T_\alpha{}^{\beta\,(m)} \, , \quad \tilde{T}^{\alpha\beta} = \Omega^{-6} \, T^{\alpha\beta\,(m)} \, , \tag{2.14}$$

and

$$\tilde{T}^{(m)} = \Omega^{-4} \, T^{(m)} \, . \tag{2.15}$$

The last equation makes it clear that the trace vanishes in the Einstein frame if and only if it vanishes in the Jordan frame.

Perfect fluids. Now let us consider the stress-energy tensor of a perfect fluid,[2]

$$T^{(m)}_{\alpha\beta} = \left(P^{(m)} + \rho^{(m)} \right) u_\alpha u_\beta + P^{(m)} g_{\alpha\beta} : \tag{2.16}$$

the corresponding tensor in the rescaled world is

$$\tilde{T}^{(m)}_{\alpha\beta} = \left(\tilde{P}^{(m)} + \tilde{\rho}^{(m)} \right) \tilde{u}_\alpha \tilde{u}_\beta + \tilde{P}^{(m)} \, \tilde{g}_{\alpha\beta} \, , \tag{2.17}$$

where the four-velocity $\tilde{u}^\mu$ of the fluid satisfies

$$\tilde{g}_{\alpha\beta} \, \tilde{u}^\alpha \tilde{u}^\beta = -1 . \tag{2.18}$$

Together with the metric rescaling (2.1), this normalization gives the transformation properties of the fluid four-velocity and of its inverse, which are widely used in the literature,

$$\tilde{u}^\mu = \Omega^{-1} \, u^\mu \, , \qquad \tilde{u}_\mu = \Omega \, u_\mu . \tag{2.19}$$

By comparing Eqs. (2.14) and (2.17) and using Eq. (2.19), one obtains

$$\left(\tilde{P}^{(m)} + \tilde{\rho}^{(m)} \right) \tilde{u}_\alpha \tilde{u}_\beta + \tilde{P}^{(m)} \, \tilde{g}_{\alpha\beta} = \Omega^{-2} \left[\left(P^{(m)} + \rho^{(m)} \right) u_\alpha u_\beta \right. \\ \left. + \, P^{(m)} \, g_{\alpha\beta} \right] , \tag{2.20}$$

and the transformation properties of the energy density and pressure of the fluid under the conformal transformation (2.1) are

$$\tilde{\rho}^{(m)} = \Omega^{-4} \, \rho^{(m)} \, , \qquad \tilde{P}^{(m)} = \Omega^{-4} \, P^{(m)} \, . \tag{2.21}$$

If, in the Jordan frame, the fluid has a barotropic equation of state of the form

$$P^{(m)} = (\gamma - 1) \, \rho^{(m)} \tag{2.22}$$

with γ =constant, then the same equation of state is valid in the Einstein frame thanks to the relations (2.21) between $\rho^{(m)}$, $P^{(m)}$ and their conformal cousins $\tilde{\rho}^{(m)}$ and $\tilde{P}^{(m)}$. However, this property does not hold true for a more general barotropic equation of state $P = P(\rho)$ which is not of the form (2.22).

[2] See [317] for the transformation properties of an imperfect fluid under a conformal transformation.

In the case of FLRW metrics the usual Jordan frame conservation equation for a fluid

$$\frac{d\rho^{(m)}}{dt} + 3H\left(P^{(m)} + \rho^{(m)}\right) = 0 \tag{2.23}$$

is modified in the Einstein frame to

$$\frac{d\tilde{\rho}^{(m)}}{dt} + 3\tilde{H}\left(\tilde{P}^{(m)} + \tilde{\rho}^{(m)}\right) = \left(3\tilde{P}^{(m)} - \tilde{\rho}^{(m)}\right)\frac{d\,(\ln \Omega)}{dt}\,, \tag{2.24}$$

as follows from Eq. (2.12).

Let us now review some fundamental fields:

The Klein-Gordon field. The source-free Klein-Gordon equation $\Box\phi = 0$ in the absence of self-interactions is not conformally invariant. However, its generalization

$$\Box\phi - \frac{n-2}{4(n-1)}\,R\,\phi = 0 \tag{2.25}$$

for $n \geq 2$ is conformally invariant [298, 898, 1139]. It is reasonable to allow for the possibility that the scalar ϕ acquires a mass or other potential at high energies and, accordingly, in particle physics and in cosmology it is customary to introduce a potential energy density $V(\phi)$ for the Klein-Gordon scalar. We have already discussed how a non-minimal coupling between ϕ and the Ricci curvature arises. Taking both of these into account, the relevant equation for ϕ becomes

$$\Box\phi - \xi\;R\,\phi - \frac{dV}{d\phi} = 0\,, \tag{2.26}$$

where ξ is the dimensionless coupling constant. The introduction of non-minimal coupling with $\xi \neq 0$ makes the theory a scalar-tensor one.

Equation (2.26) is conformally invariant in four spacetime dimensions if $\xi = 1/6$ and $V = 0$ or $V = \lambda\phi^4$ [205, 898, 1139]. Even a constant potential V, equivalent to a cosmological constant, corresponds to an effective mass for the scalar (not to be identified with a real mass [464]) which breaks conformal invariance [762].

Although unintuitive, it is not difficult to understand why a quartic potential preserves conformal invariance on the basis of dimensional considerations. Conformal invariance corresponds to the absence of a characteristic length (or mass) scale in the physics. In general, the potential $V(\phi)$ contains dimensional parameters (such as the mass m in $V = m^2\phi^2/2$) but, when $V = \lambda\phi^4$, the dimension of V (a mass to the fourth power) is carried by ϕ^4 and the self-coupling constant λ is dimensionless, *i.e.*, there is no scale associated to V in this case.

The Maxwell field. In four spacetime dimensions the Maxwell equations are conformally invariant, while the equation satisfied by the electromagnetic four-potential A^μ,

$$\Box A_\mu - R^\nu{}_\mu A_\nu = -4\pi j_\mu \tag{2.27}$$

(where j^μ is the four-current) is not [112, 353]. However, this quantity is gauge-dependent and is not an observable. As already discussed, quantum corrections to classical electrodynamics, including the generation of mass terms and the conformal anomaly, break the conformal invariance.

Higher spin fields. The conditions for conformal invariance of fields of arbitrary spin in general spacetime dimensions are varied and, generally, complicated; we refer the reader to [623].

2.2 Variational principles in General Relativity

Variational principles are used to formulate the equations of motion of particles and fields in theoretical physics, and GR is no exception. We first discuss the variational principle for test particles and then the one leading to the Einstein equations.

2.2.1 Geodesics

In GR the spacetime metric is related to geodesic motion because the Equivalence Principle requires that the motion of a point-like body in free fall be described by the geodesic equation. The latter can be derived from the variational principle

$$\delta S = \delta \int_A^B ds = 0 \,, \tag{2.28}$$

where ds is the line element and A and B are the initial and final points along the spacetime trajectory, respectively. The line element is written as

$$ds = \left| g_{\alpha\beta} dx^\alpha dx^\beta \right|^{1/2} = \left| g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \right|^{1/2} ds \,, \tag{2.29}$$

with s playing the role of an affine parameter, and from which it follows that

$$g_{\alpha\beta} u^\alpha u^\beta = -1 \,, \tag{2.30}$$

where $u^\alpha = \dfrac{dx^\alpha}{ds}$ is the four-velocity of the particle. Substitution into Eq. (2.28) yields

$$\delta S = \delta \int_A^B \left| g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \right|^{1/2} ds = 0 \,. \tag{2.31}$$

By performing this variation, one obtains

$$\delta S = \int_A^B \frac{1}{2\sqrt{|g_{\alpha\beta}u^\alpha u^\beta|}} \left[g_{\alpha\beta,\lambda}\delta x^\lambda \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} + 2g_{\alpha\beta}\frac{d}{ds}\left(\delta x^\alpha\right)\frac{dx^\beta}{ds}\right] ds = 0\,. \tag{2.32}$$

The second term in square brackets is $g_{\alpha\beta}\delta\left(\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right)$ as a consequence of the fact that $\delta\left(ds\right) = d\left(\delta s\right)$, hence

$$g_{\alpha\beta}\delta\left(\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right) = g_{\alpha\beta}\frac{dx^\alpha}{ds}\delta\left(\frac{dx^\beta}{ds}\right) + g_{\alpha\beta}\frac{dx^\beta}{ds}\delta\left(\frac{dx^\alpha}{ds}\right) = 2g_{\alpha\beta}\frac{dx^\beta}{ds}\frac{d}{ds}\left(\delta x^\alpha\right)\,. \tag{2.33}$$

Using $g_{\alpha\beta}u^\alpha u^\beta = -1$, it is

$$\delta S = \int_A^B \frac{1}{2}\left[g_{\alpha\beta,\lambda}\delta x^\lambda \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} + 2g_{\alpha\beta}\frac{dx^\beta}{ds}\frac{d}{ds}\left(\delta x^\alpha\right)\right] ds = 0 \tag{2.34}$$

and integration by parts of the second term yields

$$\begin{aligned}\delta S =& \int_A^B \frac{1}{2}\left(g_{\alpha\beta,\lambda}\delta x^\lambda \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right) ds + \left[g_{\alpha\beta}\frac{dx^\beta}{ds}\delta x^\alpha\right]_A^B \\ &- \int_A^B \frac{d}{ds}\left(g_{\alpha\beta}\frac{dx^\beta}{ds}\right)\delta x^\alpha ds = 0\,.\end{aligned} \tag{2.35}$$

By imposing that, at the endpoints, it is $\delta x^\alpha\left(A\right) = \delta x^\alpha\left(B\right) = 0$, the second term vanishes and

$$\begin{aligned}\delta S =& \int_A^B \frac{1}{2}\left(g_{\alpha\beta,\lambda}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\delta x^\lambda\right) ds - \int_A^B \left(g_{\alpha\beta}\frac{d^2x^\beta}{ds^2} + g_{\alpha\beta,\lambda}\frac{dx^\lambda}{ds}\frac{dx^\beta}{ds}\right)\delta x^\alpha ds \\ =& \, 0\,.\end{aligned} \tag{2.36}$$

This equation can be written as

$$\delta S = \int_A^B \left[\left(\frac{1}{2}g_{\alpha\beta,\lambda} - g_{\lambda\beta,\alpha}\right)\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} - g_{\lambda\beta}\frac{d^2x^\beta}{ds^2}\right]\delta x^\lambda ds = 0\,. \tag{2.37}$$

This integral vanishes for all variations δx^λ with fixed endpoints if

$$g_{\lambda\beta}\frac{d^2x^\beta}{ds^2} = \left(\frac{1}{2}g_{\alpha\beta,\lambda} - g_{\lambda\beta,\alpha}\right)u^\alpha u^\beta\,. \tag{2.38}$$

Since

$$g_{\lambda\beta,\alpha}u^{\alpha}u^{\beta} = g_{\lambda\alpha,\beta}u^{\beta}u^{\alpha} = \frac{1}{2}\left(g_{\lambda\beta,\alpha} + g_{\lambda\alpha,\beta}\right)u^{\alpha}u^{\beta}\,, \tag{2.39}$$

whereas

$$g_{\lambda\beta}\,\frac{d^2x^{\beta}}{ds^2} = \frac{1}{2}\left(g_{\alpha\beta,\lambda} - g_{\lambda\beta,\alpha} - g_{\lambda\alpha,\beta}\right)u^{\alpha}u^{\beta} \tag{2.40}$$

we have

$$\{\lambda,\alpha\beta\} = \frac{1}{2}\left(g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda}\right) \tag{2.41}$$

and

$$g_{\lambda\beta}\frac{d^2x^{\beta}}{ds^2} + \{\lambda,\alpha\beta\}\,u^{\alpha}u^{\beta} = 0\,. \tag{2.42}$$

Multiplying by $g^{\lambda\tau}$ and remembering that

$$g^{\lambda\tau}g_{\lambda\beta} = \delta^{\tau}_{\beta}\,, \qquad g^{\lambda\tau}\{\lambda,\alpha\beta\} = \Gamma^{\tau}_{\alpha\beta}\,, \tag{2.43}$$

one has

$$\frac{d^2x^{\tau}}{ds^2} + \Gamma^{\tau}_{\alpha\beta}u^{\alpha}u^{\beta} = 0\,, \tag{2.44}$$

which is the geodesic equation describing the free fall motion of a point-like body in the gravitational field $\Gamma^{\tau}_{\alpha\beta}$.

2.2.2 *Field equations*

The Einstein equations or the gravitational field equations of any ETG can be derived from a variational principle. Of course, the description is more involved than for point particles because we are discussing a field theory, *i.e.*, a distributed physical system with an infinite number of degrees of freedom. We illustrate the derivation of the Einstein field equations *in vacuo* as the starting point.

Let us consider

$$\delta\int d\Omega\,\sqrt{-g}\,\mathscr{L} = 0\,, \tag{2.45}$$

where $\sqrt{-g}\,d\Omega$ is the invariant volume element and $\mathscr{L}$ is the desired Lagrangian density. In fact, under the coordinate transformation $\overline{x}^{\alpha} \to x^{\alpha} = x^{\alpha}\left(\overline{x}^{\mu}\right)$, where $\overline{x}^{\mu}$ are the "initial" local coordinates, we have

$$d\Omega = J\,d\overline{\Omega}\,, \qquad J = \det\left(\frac{\partial x^{\alpha}}{\partial\overline{x}^{\mu}}\right), \tag{2.46}$$

with J the Jacobian determinant of the transformation. Moreover, we have

$$\overline{g}_{\alpha\beta} = \text{diag}\ (-1, 1, 1, 1)\,, \tag{2.47}$$

$$\overline{g}_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial \overline{x}^{\alpha}} \frac{\partial x^{\nu}}{\partial \overline{x}^{\beta}}\, g_{\mu\nu}\,, \tag{2.48}$$

$\overline{g} = -1 = J^2 g$ and, therefore,

$$d\overline{\Omega} = \frac{d\Omega}{J} = \sqrt{-g}\, d\Omega\,. \tag{2.49}$$

Since we want the Euler-Lagrange equations deriving from the variational principle to be of second order, the Lagrangian must be quadratic in the first order derivatives of $g_{\mu\nu}$. These first order derivatives contain the Christoffel symbols, which are not coordinate-invariant. Then we have to choose for the Lagrangian density $\mathscr{L}$ expressions containing higher order derivatives and, *a priori*, this brings the danger that the field equations could become of order higher than second (we will discuss in detail this point for ETGs). The obvious choice of Hilbert and Einstein for the Lagrangian density $\mathscr{L}$ was the Ricci scalar curvature R. The variational principle is then

$$\delta \int d\Omega\, \sqrt{-g}\, R = 0\,. \tag{2.50}$$

The relations

$$\delta g = g\, g^{\mu\nu} \delta g_{\mu\nu} = -g\, g_{\mu\nu} \delta g^{\mu\nu} \tag{2.51}$$

yield

$$\delta\left(\sqrt{-g}\right) = -\frac{\delta g}{2\sqrt{-g}} = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\delta g^{\alpha\beta}\,, \tag{2.52}$$

from which it follows that

$$\begin{aligned} &\int \left[\left(\delta\sqrt{-g}\right) R + \sqrt{-g}\, R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}\, g^{\mu\nu}\delta R_{\mu\nu}\right] d\Omega \\ &\quad = \int \sqrt{-g}\, \delta g^{\mu\nu}\left[R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}\right] d\Omega + \int \sqrt{-g}\, g^{\mu\nu}\delta R_{\mu\nu} d\Omega = 0\,. \end{aligned} \tag{2.53}$$

The second integral can be evaluated in the local inertial frame, obtaining

$$R_{\mu\nu}(0) = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu}\,, \tag{2.54}$$

$$\delta R_{\mu\nu}(0) = \frac{\partial}{\partial x^{\alpha}}\left(\delta\Gamma^{\alpha}_{\mu\nu}\right) - \frac{\partial}{\partial x^{\nu}}\left(\delta\Gamma^{\alpha}_{\mu\alpha}\right)\,, \tag{2.55}$$

$$g^{\mu\nu}(0)\delta R_{\mu\nu}(0) = g^{\mu\nu}(0)\frac{\partial}{\partial x^\alpha}\left(\delta\Gamma^\alpha_{\mu\nu}\right) - g^{\mu\nu}(0)\frac{\partial}{\partial x^\nu}\left(\delta\Gamma^\alpha_{\mu\alpha}\right)$$
$$= g^{\mu\nu}(0)\frac{\partial}{\partial x^\rho}\left(\delta\Gamma^\rho_{\mu\nu}\right) - g^{\mu\rho}(0)\frac{\partial}{\partial x^\rho}\left(\delta\Gamma^\alpha_{\mu\alpha}\right)$$
$$= \frac{\partial}{\partial x^\rho}\left[g^{\mu\nu}(0)\delta\Gamma^\rho_{\mu\nu} - g^{\mu\rho}(0)\delta\Gamma^\alpha_{\mu\alpha}\right]. \tag{2.56}$$

Then, we can write

$$g^{\mu\nu}(0)\delta R_{\mu\nu}(0) = \frac{\partial W^\rho}{\partial x^\rho}, \qquad W^\rho = g^{\mu\nu}(0)\delta\Gamma^\rho_{\mu\nu} - g^{\mu\rho}(0)\delta\Gamma^\alpha_{\mu\alpha}. \tag{2.57}$$

The second integral in Eq. (2.53) can be discarded since its argument is a pure divergence; in fact, in general coordinates it is

$$\int \sqrt{-g}\, g^{\mu\nu}\delta R_{\mu\nu} d\Omega = \int \sqrt{-g}\,\frac{\partial W^\rho}{\partial x^\rho}\, d\Omega$$
$$= \int \sqrt{-g}\; W^\rho{}_{;\rho} d\Omega = \int \frac{\partial}{\partial x^\rho}\left(\sqrt{-g}\, W^\rho\right) d\Omega = 0, \tag{2.58}$$

and then

$$\int \sqrt{-g}\,\delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}\right) d\Omega = 0\,, \tag{2.59}$$

from which we obtain the vacuum field equations of GR

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = 0 \tag{2.60}$$

as Euler-Lagrange equations of the Hilbert-Einstein action. *Vice-versa*, starting from Eq. (2.60) and retracing the previous steps in inverse order (*i.e.*, integrating the Einstein equations), one can re-obtain the Hilbert-Einstein action (2.50), thus demonstrating the equivalence between this action and the field equations (2.60). Introducing matter fields as sources is straightforward, producing Eqs. (1.8) as a result.

2.3 Adding torsion

Several questions of interest in modern physics could depend on the fact that GR is a classical theory that does not include ultraviolet quantum effects. Quantum effects should be considered in any theory dealing with gravity at a fundamental level and even in effective theories. Assuming a $\mathbf{U}_4$ manifold instead of the usual $\mathbf{V}_4$ (see below) is a straightforward generalization of GR which attempts to include fields with non-zero spin in the geometrical framework of GR. The Einstein-Cartan-Sciama-Kibble (ECSK) theory is one of the most serious attempts in this direction [584]. However, the mere inclusion of spin matter fields does not exhaust the role of torsion, which can give important contributions in any fundamental theory. For example, a torsion field appears in (super)string theories if we consider the fundamental string modes. One needs at least a scalar and two tensor modes, a symmetric and an antisymmetric one. In the low-energy limit, the latter is a torsion field [553].

Several attempts to unify gravity with electromagnetism have taken into account torsion in four- and higher-dimensional theories such as Kaluza-Klein models [700]. Any theory of gravity incorporating twistors needs to include torsion [601], while supergravity is the natural arena in which torsion, curvature, and matter fields enter on the same footing [888].

Several authors agree that curvature and torsion could play various roles in the cosmological dynamics at both early and late epochs. In fact, the interplay of curvature and torsion produces naturally repulsive contributions to the energy-momentum tensor, hence cosmological models become singularity-free and accelerating [385].

All these reasons suggest considering torsion in any comprehensive theory of gravity which takes into account non-gravitational fundamental interactions. However, in most papers in the literature, a clear distinction between the different kinds of torsion is not made. Usually torsion is simply related to the spin density of matter but, very often, it assumes more general meanings. There are more than one independent torsion tensors with different properties [240]. The problem of extending GR to actions more general than the Hilbert-Einstein one is naturally related to the consideration of torsion. In this section we illustrate the general features of torsion and the associated quantities defined in $\mathbf{U}_4$ spacetimes [584]. This formalism can be applied, in general, to any alternative theory of gravity.

The torsion tensor $S_{\mu\nu}{}^{\rho}$ is the antisymmetric part of the affine connection coefficients $\Gamma^{\rho}_{\mu\nu}$,

$$S_{\mu\nu}{}^{\rho} = \frac{1}{2}\left(\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu}\right) \equiv \Gamma^{\rho}_{[\mu\nu]} \,. \tag{2.61}$$

In GR it is postulated that $S_{\mu\nu}{}^{\rho} = 0$. It is a general convention to call $\mathbf{U}_4$ a four-dimensional spacetime manifold endowed with metric and torsion, while four-dimensional manifolds with metric and without torsion are labelled $\mathbf{V}_4$. In general, torsion occurs in linear combinations as the *contortion tensor*

$$K_{\mu\nu}{}^{\rho} = -S_{\mu\nu}{}^{\rho} - S^{\rho}{}_{\mu\nu} + S_{\nu}{}^{\rho}{}_{\mu} = -K_{\mu}{}^{\rho}{}_{\nu} \,, \tag{2.62}$$

and the *modified torsion tensor*

$$T_{\mu\nu}{}^{\rho} = S_{\mu\nu}{}^{\rho} + 2\delta_{[\mu}{}^{\rho} S_{\nu]} \,, \tag{2.63}$$

where $S_{\mu} \equiv S_{\mu\nu}{}^{\nu}$. According to these definitions, the affine connection can be written as

$$\Gamma^{\rho}_{\mu\nu} = \left\{{\rho \atop \mu\nu}\right\} - K_{\mu\nu}{}^{\rho} \,, \tag{2.64}$$

where $\left\{{\rho \atop \mu\nu}\right\}$ are the usual Christoffel symbols of the symmetric connection. The presence of torsion in the affine connection implies that the covariant derivatives of a scalar field ϕ do not commute, *i.e.*,

$$\tilde{\nabla}_{[\mu}\tilde{\nabla}_{\nu]}\phi = -S_{\mu\nu}{}^{\rho}\tilde{\nabla}_{\rho}\phi\,. \tag{2.65}$$

For a vector v^a and a covector w_a, the relations

$$(\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu} - \tilde{\nabla}_{\nu}\tilde{\nabla}_{\mu})v^{\rho} = R_{\mu\nu\alpha}{}^{\rho}v^{\alpha} - 2S_{\mu\nu}{}^{\alpha}\tilde{\nabla}_{\alpha}v^{\rho} \tag{2.66}$$

and

$$(\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu} - \tilde{\nabla}_{\nu}\tilde{\nabla}_{\mu})w_{\rho} = R_{\mu\nu\rho}{}^{\alpha}w_{\alpha} - 2S_{\mu\nu}{}^{\alpha}\tilde{\nabla}_{\alpha}w_{\rho} \tag{2.67}$$

hold. The torsion contribution to the Riemann tensor $R_{\mu\nu\rho}{}^{\sigma}$ is given explicitly by

$$R_{\mu\nu\rho}{}^{\sigma} = R_{\mu\nu\rho}{}^{\sigma}(\{\}) - \nabla_{\mu}K_{\nu\rho}{}^{\sigma} + \nabla_{\nu}K_{\mu\rho}{}^{\sigma} + K_{\mu\beta}{}^{\sigma}K_{nu\rho}{}^{\beta} - K_{\nu\beta}{}^{\sigma}K_{\mu\rho}{}^{\beta}\,, \tag{2.68}$$

where $R_{\mu\nu\rho}{}^{\sigma}(\{\})$ is the tensor generated by the Christoffel symbols. The symbols $\tilde{\nabla}_{\mu}$ and ∇_{μ} denote the covariant derivative operators with and without torsion, respectively. Using Eq. (2.68), the expressions for the Ricci tensor and the curvature scalar are

$$R_{\mu\rho} = R_{\mu\rho}(\{\}) - 2\nabla_{\mu}S_{\rho} + \nabla_{\nu}K_{\mu\rho}{}^{\nu} + K_{\mu\beta}{}^{\nu}K_{\nu\rho}{}^{\beta} - 2S_{\beta}K_{\mu\rho}{}^{\beta} \tag{2.69}$$

and

$$R = R(\{\}) - 4\nabla_{\mu}S^{\mu} + K_{\rho\beta\nu}K^{\nu\rho\beta} - 4S_{\mu}S^{\mu}\,. \tag{2.70}$$

Torsion can be decomposed with respect to the Lorentz group into three irreducible tensors

$$S_{\mu\nu}{}^{\rho} = {}^{T}S_{\mu\nu}{}^{\rho} + {}^{A}S_{\mu\nu}{}^{\rho} + {}^{V}S_{\mu\nu}{}^{\rho}\,, \tag{2.71}$$

where

$${}^{A}S_{\mu\nu}{}^{\rho} = g^{\rho\sigma}S_{[\mu\nu\sigma]} \tag{2.72}$$

is called the axial (or totally antisymmetric) torsion and

$${}^{T}S_{\mu\nu}{}^{\rho} = S_{\mu\nu}{}^{\rho} - {}^{A}S_{\mu\nu}{}^{\rho} - {}^{V}S_{\mu\nu}{}^{\rho} \tag{2.73}$$

is the traceless non-totally antisymmetric part of torsion. Torsion has 24 components, of which ${}^{T}S_{\mu\nu}$ has 16 components, ${}^{A}S_{\mu\nu}$ has 4, and ${}^{V}S_{\mu\nu}$ has the remaining 4. It is also

$${}^{V}S_{\mu\nu}{}^{\rho} = \frac{1}{3}\left(S_{\mu}\delta^{\rho}_{\nu} - S_{\nu}\delta^{\rho}_{\mu}\right). \tag{2.74}$$

It is clear that relating torsion to the spin density of matter is only one of its possible applications [240].

2.4 Noether symmetries

The celebrated theorem of Emmy Noether states that conserved quantities in the dynamics of a physical system are related to the existence of symmetries and cyclic variables in its Lagrangian [53, 774, 808]. Here we review the Noether symmetry approach for dynamical systems with a finite number of degrees of freedom. We will use it later to obtain exact solutions of ETGs.

Let $L\left(q^i, \dot{q}^i\right)$ be a canonical, non-degenerate, point-like Lagrangian satisfying

$$\frac{\partial L}{\partial \lambda} = 0\,, \qquad \det\left(H_{ij}\right) \equiv \det\left|\left|\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right|\right| \neq 0\,, \tag{2.75}$$

where H_{ij} is the Hessian matrix of L and an overdot denotes differentiation with respect to the affine parameter λ (which usually corresponds to the time t). In the Lagrangian formalism for point particles and rigid bodies, the Lagrangian L assumes the form

$$L = T\left(\mathbf{q}, \dot{\mathbf{q}}\right) - V\left(\mathbf{q}\right)\,, \tag{2.76}$$

where T and V are the kinetic and potential energies, respectively. T is a quadratic form of the $\dot{q}^i$. The Hamiltonian associated with L is

$$E_L \equiv \frac{\partial L}{\partial \dot{q}^i}\,\dot{q}^i - L\,, \tag{2.77}$$

it coincides with the total energy $T + V$, and is a constant of motion. Any smooth and invertible transformation[3] of the generalized coordinates $q^i \to Q^i(\mathbf{q})$ induces a transformation of the generalized velocities

$$\dot{q}^i \to \dot{Q}^i(\mathbf{q}) = \frac{\partial Q^i}{\partial q^j}\,\dot{q}^j\,. \tag{2.78}$$

We assume that the Jacobian matrix $\mathcal{J} = ||\partial Q^i/\partial q^j||$ of the coordinate transformation does not vanish. The Jacobian $\widetilde{\mathcal{J}}$ of the "induced" transformation is easily derived and $\mathcal{J} \neq 0$ implies that $\widetilde{\mathcal{J}} \neq 0$. In general, this transformation is local because the condition $\mathcal{J} \neq 0$ cannot be satisfied on the entire space but only in the neighbourhood of a given point. If the transformation is extended to the maximal submanifold on which $\mathcal{J} \neq 0$, problems can arise for the whole manifold due to the possibility of different topologies [808].

A point transformation $Q^i = Q^i(\mathbf{q})$ can depend on one or more parameters. Let us assume that a point transformation depends on a parameter ε, $Q^i = Q^i(\mathbf{q}, \varepsilon)$, and that it defines a one-parameter Lie group. For infinitesimal values of ε, the transformation is then generated by a vector field. Examples are the vector field $\partial/\partial x$

[3] Here we consider only point transformations.

associated with a translation along the x-axis, and the field $x(\partial/\partial y) - y(\partial/\partial x)$ associated with a rotation about the z-axis. In general, an infinitesimal point transformation is represented by a generic vector field on Q

$$\mathbf{X} = \alpha^i(\mathbf{q}) \frac{\partial}{\partial q^i} . \tag{2.79}$$

The induced transformation (2.78) is then represented by

$$\mathbf{X}^{(c)} = \alpha^i(\mathbf{q}) \frac{\partial}{\partial q^i} + \left(\frac{d}{d\lambda} \alpha^i(\mathbf{q}) \right) \frac{\partial}{\partial \dot{q}^j} . \tag{2.80}$$

The vector field $\mathbf{X}^{(c)}$ is called the *complete lift* of $\mathbf{X}$ [808]. A function $f(\mathbf{q}, \dot{\mathbf{q}})$ is invariant under the transformation $\mathbf{X}^{(c)}$ if

$$\pounds_{\mathbf{X}^{(c)}} f \equiv \alpha^i(\mathbf{q}) \frac{\partial f}{\partial q^i} + \left(\frac{d}{d\lambda} \alpha^i(\mathbf{q}) \right) \frac{\partial f}{\partial \dot{q}^j} = 0 , \tag{2.81}$$

where $\pounds_{\mathbf{X}^c} f$ is the Lie derivative of f along $\mathbf{X}^{(c)}$. If, in particular, $\pounds_{\mathbf{X}^{(c)}} L = 0$, then $\mathbf{X}^{(c)}$ is said to be a *symmetry* for the dynamics described by L.

In order to fully flesh out the relation between Noether's theorem and cyclic variables, let us consider a Lagrangian L yielding the Euler-Lagrange equations

$$\frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{q}^j} \right) - \frac{\partial L}{\partial q^j} = 0 \tag{2.82}$$

and the vector field (2.80). By contracting Eq. (2.82) with the α^i's, one obtains

$$\alpha^j \left[\frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{q}^j} \right) - \frac{\partial L}{\partial q^j} \right] = 0 . \tag{2.83}$$

By using the fact that (as follows from Eq. (2.83))

$$\alpha^j \frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{q}^j} \right) = \frac{d}{d\lambda} \left(\alpha^j \frac{\partial L}{\partial \dot{q}^j} \right) - \left(\frac{d\alpha^j}{d\lambda} \right) \frac{\partial L}{\partial \dot{q}^j} , \tag{2.84}$$

one obtains that

$$\frac{d}{d\lambda} \left(\alpha^i \frac{\partial L}{\partial \dot{q}^i} \right) = \pounds_{\mathbf{X}} L . \tag{2.85}$$

For brevity, from now on we abuse notations when there is no possibility of confusion and we write $\mathbf{X}$ instead of $\mathbf{X}^{(c)}$. A straightforward consequence of Eq. (2.85) is the

Noether theorem:

If $\pounds_{\mathbf{X}} L = 0$*, then the function*

$$\Sigma_0 = \alpha^i \, \frac{\partial L}{\partial \dot{q}^i} \tag{2.86}$$

is a constant of motion.

A few remarks are in order. First, Eq. (2.86) can be expressed in a coordinate-independent way as the contraction of $\mathbf{X}$ with the Cartan one-form

$$\theta_L \equiv \frac{\partial L}{\partial \dot{q}^i} \, dq^i \, . \tag{2.87}$$

Given a generic vector field $\mathbf{Y} = y^i \partial/\partial x^i$ and a one-form $\beta = \beta_i dx^i$ it is, by definition,

$$i_{\mathbf{Y}} \beta = y^i \beta_i \tag{2.88}$$

and Eq. (2.86) can then be written as

$$i_{\mathbf{X}} \theta_L = \Sigma_0 \, . \tag{2.89}$$

Using a point transformation, the vector field $\mathbf{X}$ is rewritten as

$$\widetilde{\mathbf{X}} = \left(i_{\mathbf{X}} dQ^k \right) \frac{\partial}{\partial Q^k} + \left[\frac{d}{d\lambda} \left(i_x dQ^k \right) \right] \frac{\partial}{\partial \dot{Q}^k} \, , \tag{2.90}$$

hence $\widetilde{\mathbf{X}}'$ is still the lift of a vector field defined on the configuration space. If $\mathbf{X}$ is a symmetry and a point transformation is chosen such that

$$i_{\mathbf{X}} dQ^1 = 1 \, , \quad i_{\mathbf{X}} dQ^i = 0 \qquad (i \neq 1) \, , \tag{2.91}$$

it follows that

$$\widetilde{\mathbf{X}} = \frac{\partial}{\partial Q^1} \, , \qquad \frac{\partial L}{\partial Q^1} = 0 \, . \tag{2.92}$$

Therefore, Q^1 is a cyclic coordinate and the dynamics can be reduced [53, 774].

The coordinate transformation (2.91) is not unique and a clever choice can be very advantageous. Moreover, the solution of Eq. (2.91) is, in general, not defined on the entire space but only locally, as noted above. It is possible that multiple vector fields $\mathbf{X}$ are found, say $\mathbf{X}_1$ and $\mathbf{X}_2$. If these commute, $[\mathbf{X}_1, \mathbf{X}_2] = 0$, then two cyclic coordinates can be found by solving the system

$$i_{\mathbf{X}_1} dQ^1 = 1 \, , \quad i_{\mathbf{X}_2} dQ^2 = 1 \, , \quad i_{\mathbf{X}_1} dQ^i = 0 \;\; (i \neq 1) \, , \quad i_{\mathbf{X}_2} dQ^i = 0 \quad (i \neq 2) \, . \tag{2.93}$$

The transformed fields are then $\partial/\partial Q^1$ and $\partial/\partial Q^2$. If $\mathbf{X}_1$ and $\mathbf{X}_2$ do not commute, this procedure cannot be applied, as is clear from the fact that diffeomorphisms preserve the commutation relations. To proceed, let us note that the commutator $\mathbf{X}_3 = [\mathbf{X}_1, \mathbf{X}_2]$ is also a symmetry because

$$\pounds_{\mathbf{X_3}} L = \pounds_{\mathbf{X_1}} \pounds_{\mathbf{X_2}} L - \pounds_{\mathbf{X_2}} \pounds_{\mathbf{X_1}} L = 0\,. \tag{2.94}$$

If $\mathbf{X_3}$ does not depend on $\mathbf{X_1}$ and $\mathbf{X_2}$, the procedure is repeated until the vector fields close the Lie algebra. The usual way to treat this situation consists of performing a Legendre transformation to switch to the Hamiltonian formalism and to a Lie algebra of Poisson brackets. If a reduction to cyclic coordinates is sought for, this procedure can be achieved by:

1. choosing arbitrarily one of the symmetries or a linear combination of them and obtaining new coordinates as above. After the reduction, the new Lagrangian $\widetilde{L}(\mathbf{Q})$ is obtained.
2. Repeating the search for symmetries in this new space, performing a new reduction, and repeating this procedure until possible.
3. If the search for symmetries fails, another attempt is made with a different existing symmetry.

Let us now assume that L is of the form (2.76). Since $\mathbf{X}$ is of the form (2.80), $\pounds_{\mathbf{X}} L$ will consists of the sum of a second degree homogeneous polynomial in the velocities and of an inhomogeneous term in q^i. Since such a polynomial must vanish identically, all its coefficients vanish. If the configuration space has dimension n, one obtains $1 + \dfrac{n(n+1)}{2}$ partial differential equations; the system is then overdetermined and, if any solution exists, it must be expressed in terms of integration constants instead of boundary conditions. Clearly, an overall constant factor in the Lie vector $\mathbf{X}$ is irrelevant.

The Noether approach will be used in Chaps. 4 and 8 to obtain exact solutions with symmetries of ETGs.

2.5 Conclusions

Armed with the mathematical tools described in this chapter, we are now ready to explore in more detail the landscape of gravitational theories that lie beyond Einstein's GR. These theories are conveniently described in terms of their actions satisfying the variational principle, and the search for analytical solutions can be performed using Noether symmetries. In addition, general solutions in cosmology can be discussed using qualitative analysis, which is presented in Chap. 6.

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