Chapter 2
Quantum Fields

2.1 Relativistic Inner Product

In this Chapter we start a systematic discussion of a quantum theory in external classical background fields. First, we introduce a specific inner product and define quantization conditions based on this product.

The method of quantization which we present below is not a substitute for more profound methods, like the full Hamiltonian analysis or the so-called BRST approach, but it allows one to arrive faster to the results in the lowest order of the perturbation theory on non-trivial backgrounds. In this section, we shall be rather sloppy with a mathematical side of the statements, ignore all the functional analysis issues, for example, and simply use a finite-dimensional intuition in infinite-dimensional spaces of fields.

Let φ be a non-interacting field on a Lorentzian space-time $\mathscr{M}$. We call φ a dynamical variable to distinguish it from the background. It is assumed that φ belongs to a section in a fiber bundle over $\mathscr{M}$. We further assume that $\mathscr{M}$ is a globally hyperbolic space-time. The dimensionality n of $\mathscr{M}$ is not fixed ($n \geq 2$).

Let us choose on $\mathscr{M}$ some coordinate system $x^\mu \equiv (t, x^k)$, where $k = 1, \dots, d$ and $d = n - 1$. Here, we assume existence of a foliation $\mathscr{M}$ by spatial sections, so that at least locally the space-time manifold looks as a direct product of a one-dimensional "time" and a d-dimensional "space". As we saw (see Sect. 1.6), small fluctuations φ obey linearized equations of motion of the form

$$P(\partial_t, \partial_k)\varphi(t, x^k) = 0, \tag{2.1}$$

where for integer spin fields $P(\partial_t, \partial_k)$ is a second order hyperbolic type partial differential operator. For spin 1/2 fields $P(\partial_t, \partial_k)$ is a first order operator, see (1.74).

Let f_1 and f_2 be a pair of solutions to (2.1). We are going to introduce a so-called *relativistic* inner product, $\langle f_1, f_2\rangle$, between these solutions. The product is constructed through a conserved current corresponding to some global symmetry. Since the inner product and the current must depend on *two* fields instead of one, we have to double the number of fields in the quadratic form of the action. This is

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done in the following way. Consider first the case of a complex field and write the quadratic action which generates the linearized equations (2.1) as

$$I_2[\varphi] = \int d^n x \sqrt{-g}\, \varphi^* P \varphi, \tag{2.2}$$

where all vector or gauge indexes (if any) are suppressed. An example of a functional which can be brought to this form (after integrating by parts) is the scalar action (1.68). Although (2.2) is a rather typical form of the quadratic action it is not universal. In some cases, for instance, for an action of small fluctuations in the $(\varphi\varphi^*)^2$ model, there may also appear $(\varphi^*)^2$ and φ^2 terms. We shall comment how to deal with such models in the end of this section.

Next, we go from (2.2) to a sesquilinear form

$$I[f_1, f_2] = \int d^n x \sqrt{-g}\; f_1^* P f_2 \equiv \int d^n x \sqrt{-g} L(f_1^*, f_2) \tag{2.3}$$

such that $I_2[\varphi] = I[\varphi, \varphi]$. We assume that the operator P is at least formally self-adjoint (see Sect. 3.1), therefore the form is Hermitian. The sesquilinear form $I[f_1, f_2]$ is linear in the second argument and antilinear in the first. The quantities $I[f_1, f_2]$, $L(f_1^*, f_2)$ can be considered as a field theory action and a Lagrange density, respectively. Indeed, one gets for f_k the same equations of motions (2.1) by requiring that variations $I[f_1, f_2]$ over f_k have to vanish. If P is a second order operator we shall always assume that second derivatives in $I[f_1, f_2]$ are eliminated by integrating by parts and $L(f_1^*, f_2)$ contains at most first derivatives of f_1 and f_2.

The functional $I[f_1, f_2]$ has an obvious global symmetry $\{f_1, f_2\} \to \{e^{i\alpha} f_1, e^{i\alpha} f_2\}$ which implies the existence of a conserved current. To derive this current, consider an infinitesimal version of the transformations

$$\delta_\alpha f_1^* = -i\alpha f_1^*, \qquad \delta_\alpha f_2 = i\alpha f_2 \tag{2.4}$$

and assume for a moment that the transformation parameter α depends on the coordinates, $\alpha \to \alpha(x)$. Then, transformations (2.4) are no longer symmetries of the action. Nevertheless, the variation of (2.3) vanishes on constant α and, hence, is proportional to the derivative of α,

$$\begin{aligned}\delta_\alpha I[f_1, f_2] &= -\int d^n x \sqrt{-g} (\partial_\mu \alpha) \cdot j^\mu[f_1, f_2] \\ &= \int d^n x \sqrt{-g} \alpha \cdot \nabla_\mu j^\mu[f_1, f_2]\end{aligned} \tag{2.5}$$

for some current j^μ. Next, suppose that f_1 and f_2 are solutions to the classical field equations. Then, any infinitesimal variation of the action vanishes, including the one given in (2.4) with arbitrary local parameter α. In other words, on shell $\delta_\alpha I = 0$ for any alpha, and the current j^μ is conserved

$$\nabla_\mu j^\mu(f_1, f_2) = 0. \tag{2.6}$$

The arguments presented above also provide us with a method to compute the conserved current. For bosonic theories with actions depending on the first derivatives at most, one can easily show, that

$$j^{\mu}(f_1, f_2)\alpha = \frac{\partial L(f_1^*, f_2)}{\partial f_{2,\mu}}\delta_\alpha f_2 + \frac{\partial L(f_1^*, f_2)}{\partial f_{1,\mu}^*}\delta_\alpha f_1^* \tag{2.7}$$

or

$$j^{\mu}(f_1, f_2) = i\left(\frac{\partial L[f_1^*, f_2]}{\partial f_{2,\mu}} f_2 - \frac{\partial L[f_1^*, f_2]}{\partial f_{1,\mu}^*} f_1^*\right). \tag{2.8}$$

These statements are a particular case of what is known as the Noether theorem for global symmetries. We shall also deal with the Noether theorem for local symmetries in Chap. 8. The current $j^{\mu}(f_1, f_2)$ is called the Noether current.

Let us take a space-like hypersurface Σ in $\mathscr{M}$ and construct the following inner product between the classical solutions

$$\langle f_1, f_2 \rangle = \int_{\Sigma} d\Sigma^{\mu}\, j_{\mu}(f_1, f_2), \tag{2.9}$$

which is called the *relativistic inner product*. The product is linear in the second argument f_2 and anti-linear in f_1, and is Hermitian, $\langle f_1, f_2\rangle = \langle f_2, f_1\rangle^*$. Here $d\Sigma^{\mu} = n^{\mu} \det h\, d^d x$, n^{μ} is a unit future directed vector orthogonal to Σ, $\det h\, d^d x$ is the invariant measure on Σ. The continuity property (2.6) ensures that the product $\langle f_1, f_2\rangle$ does not depend on local deformations of Σ.

We call $\langle f_1, f_2\rangle$ a relativistic product to distinguish it from another inner product between sections of the fiber bundles defined in Sect. 1.5. Let us give now a couple of examples.

Charged Scalar Field For the model described by the action (1.68) we assume that the gauge field is a background field, and φ is dynamical, and obtain

$$j_{\mu}(f_1, f_2) = i(f_1^* D_{\mu} f_2 - (D_{\mu} f_1)^* f_2). \tag{2.10}$$

The continuity equation (2.6) can be checked directly. It follows from the identity $\nabla^{\mu}(f_1^* D_{\mu} f_2) = (D^{\mu} f_1)^* D_{\mu} f_1 + f_1^*(D^{\mu} D_{\mu} f_2)$ and Eq. (1.69). The relativistic product constructed from this current is called the Klein-Gordon product.

Spinor Fields Consider the model (1.74). By repeating the computations presented above and taking care of the order of fields, one arrives at

$$j_{\mu}(\psi_1, \psi_2) = -i\bar{\psi}_1 \gamma_{\mu} \psi_2. \tag{2.11}$$

For the case when the field φ is *real*, instead of (2.2) one has the functional

$$I_2[\varphi] = \frac{1}{2}\int d^n x \sqrt{-g}\, \varphi P \varphi. \tag{2.12}$$

The corresponding complex Hermitian sesquilinear form is

$$I_R[f_1, f_2] = \frac{1}{2}\int d^n x \sqrt{-g}\, f_1^* P f_2. \tag{2.13}$$

(Notice the difference between real and complex field actions in the coefficient $1/2$.) The Noether current which is used to construct the relativistic product is given by (2.8). Examples involving real fields are as follows.

Vector Fields For the model described by equation of motion (1.76) after the replacement $f_k \to A_k^\mu$ one finds

$$j_\mu(A_1, A_2) = i(A_1^\nu)^* F_{\mu\nu}(A_2) - i A_2^\nu (F_{\mu\nu}(A_1))^*. \tag{2.14}$$

The same product holds for the gauge potential in the Maxwell theory.

Linearized Yang-Mills Theory For the theory described by the linearized equations (1.79)

$$j_\mu(A_1, A_2) = -2i \operatorname{Tr}((A_1^\nu)^+ G_{\mu\nu}(A_2) - (G_{\mu\nu}(A_1))^+ A_2^\nu), \tag{2.15}$$

where the tensor $G_{\mu\nu}$ is defined in (1.80).

The following useful permutation property of the product can be now inferred from (2.10), (2.11), (2.14), (2.15):

$$\langle f_1, f_2 \rangle = \pm \langle f_2^*, f_1^* \rangle. \tag{2.16}$$

Here the plus sign in the r.h.s. corresponds to the (non-Grassmann) Dirac fields and the minus sign stands for scalar and vector fields. The sign, thus, depends on whether the spin is integer or half-odd-integer. For spinor fields the star operation in (2.16) can be replaced with the charge conjugation defined in (1.66), see Exercise 2.7.

As we have already mentioned above, not any quadratic action depending on complex fields can be represented through a sesquilinear form. To overcome this difficulty one has to introduce independent real fields as real and imaginary parts of the original complex field, diagonalize the action and then complexify it as we have just described. As a result, one obtains a conserved current, but the number of complex degrees of freedom is twice the original one.

2.2 Quantization and Single-Particle Excitations

To set the stage for the quantization we start with anti-linear functionals acting on classical solutions. Each such functional can be constructed as $\varphi[f] = \langle f, \varphi \rangle$, where φ is some *fixed solution* to the classical equations. It is allowed to multiply these functionals calculated for several solutions with different arguments. In this way, one obtains multilinear functionals. One can as well define complex conjugated linear functionals $\varphi^+[f] = \langle \varphi, f \rangle = (\varphi[f])^*$ and introduce 'real' functionals which obey the restriction $\langle \varphi, f \rangle = \langle \varphi^*, f \rangle$. With the help of (2.16) the reality condition can be written as

$$\varphi^+[f] = \pm \varphi[f^*], \tag{2.17}$$

where the plus or minus signs correspond to spin $1/2$ or spins 0 and 1, respectively. The star operation for spin $1/2$ fields denotes the charge conjugation (1.66). In the case of spin $1/2$ fields, the classical solutions will be considered as commuting (classical) spinors, while the functionals φ will anticommute before the quantization.

Quantization means that to each classical solution f one puts into a correspondence an operator $\varphi[f]$ and its Hermitian conjugate $\varphi^+[f]$. These operators act on vector spaces, the so-called Fock spaces discussed below. The operators $\varphi[f]$, $\varphi^+[f]$ are operator-valued distributions, an analog of classical functionals defined above. Thus, they are denoted by the same letter. Like the classical functionals, $\varphi[f]$, $\varphi^+[f]$ are, respectively, anti-linear or linear in their arguments. It is also required that operators preserve symmetry properties of the classical functionals.

The operators are required to obey the following *quantization conditions*:

$$[\varphi[f_1], \varphi^+[f_2]]_\pm \equiv \varphi[f_1]\varphi^+[f_2] \pm \varphi^+[f_2]\varphi[f_1] = \hbar\langle f_1, f_2\rangle, \tag{2.18}$$

where the parameter $\hbar$ is the Planck constant. Starting with Sect. 2.6 we shall put $\hbar = 1$. For integer spin fields one uses the commutator $[,]_-$ and says that the fields obey the Bose statistics, for half-odd-integer spins one uses anti commutator $[,]_+$ which implies the Fermi statistics. The quantization condition (2.18) is fully covariant, it does not depend on the choice of coordinates and the Cauchy surface used. The features of a particular model which is quantized are encoded in the relativistic product and properties of the classical solutions f_k.

Classically, the bosonic field functionals commute, while the fermionic ones anticommute. Quantization means that we *deform* these simple (anti-)commutation relations by adding a non-zero right hand side to relation (2.18). The Plank constant $\hbar$ plays the role of a deformation parameter.

One can define Hermitian operators by condition (2.17). These operators corresponds to real fields. Quantization in this case is determined by the same rule (2.18).

As a next step one has to consider the two problems: to find an operator analog of a local field and to describe elementary field excitations. The second task is motivated by the fact that a free field theory can be interpreted as a system of infinitely many oscillators. We have to find a way how to decouple different oscillations and introduce the corresponding creation and annihilation operators by following the quantum mechanical example.

To solve the two problems we need a basis which brings the relativistic inner product to a canonical form. In this section, it is convenient to consider models where *the relativistic product is non-degenerate*, i.e. if $\langle f_1, f_2\rangle = 0$ for all f_2, then $f_1 = 0$. An important example of theories with the degenerate product are gauge theories. They will be considered in Sect. 2.3.

When the relativistic product is non-degenerate it can be diagonalized by introducing a basis $\{f_A\}$, so that $\langle f_A, f_B\rangle = \lambda_A\delta_{AB}$. Because of the hermiticity of the inner product, the eigenvalues λ_A are real. By a suitable rescaling one can make these eigenvalues equal to ± 1. This yields a set of modes $\{f_i^{(+)}, f_j^{(-)}\}$ which satisfies the following conditions:

$$\langle f_i^{(+)}, f_j^{(-)}\rangle = 0, \tag{2.19}$$

$$\langle f_i^{(\pm)}, f_j^{(\pm)}\rangle = \pm\delta_{ij}, \quad \text{for Bose statistics}, \tag{2.20}$$

$$\langle f_i^{(\pm)}, f_j^{(\pm)}\rangle = \delta_{ij}, \quad \text{for Fermi statistics}. \tag{2.21}$$

Here δ_{ij} is the Kronecker symbol if i, j are discrete indices, and it is a delta-function if i, j take continuous values.

We call $f_i^{(\pm)}$ the *single-particle* modes. In the case of Bose fields the relativistic product is not positive-definite, and the modes $f^{(+)}$, $f^{(-)}$ have positive or negative norm, respectively. In the case of Fermi fields the product is positive, see Exercise 2.7. The division on "+" and "−" modes in this case is related to other properties, for example, to the sign of the frequency carried by the mode in stationary or asymptotically stationary space-times, see details in Sect. 2.5. In certain cases the "−" spin $1/2$ modes can be also defined as charge conjugated "+" modes, see below.

Any solution to field equations (2.1) can be uniquely represented as a linear combination of $f_i^{(+)}$ and $f_j^{(-)}$,

$$f(x) = \sum_i c_i f_i^{(+)}(x) + \sum_j d_j f_j^{(-)}(x), \tag{2.22}$$

where c_i and d_j are some complex numbers which can be determined with the help of the normalization conditions (2.19)–(2.21), $c_i = \langle f_i^{(+)}, f\rangle$, $d_j = \mp\langle f_j^{(-)}, f\rangle$. If i and j take continuous values the sums in (2.22) correspond to integrals.

Local field operators can be defined by analogy with (2.22). First one introduces the operators

$$a_i = \varphi[f_i^{(+)}], \qquad b_i^+ = \mp\varphi[f_i^{(-)}], \tag{2.23}$$

called the *annihilation* and *creation* operators, respectively. In the definition of b_i^+ the minus sign corresponds to the Bose statistics, the plus sign is for the Fermi statistics. By using (2.22), (2.23) and the assumption that $f_i^{(\pm)}$ is a complete set of modes the operator functionals $\varphi[f]$ can be represented as

$$\varphi[f] = \sum_i c_i a_i + \sum_j d_j b_j^+. \tag{2.24}$$

The *local* operator of a quantized field is then defined as

$$\varphi(x) = \sum_i a_i f_i^{(+)}(x) + \sum_j b_j^+ f_j^{(-)}(x). \tag{2.25}$$

This formula together with (2.24) allows one to write the operator functionals in the form, $\varphi[f] = \langle f, \varphi\rangle$, where the local operator (2.25) appears as an argument in the product. The important feature of the quantized field operator $\varphi(x)$ is that it is a formal solution to the field equations (2.1). Due to this property, Eq. (2.25) is a key formula for computing quantum averages.

The creation and annihilation operators (2.23) solve the problem of an oscillator representation of a free field theory. Indeed, by using (2.18) and normalization conditions (2.19)–(2.21) one arrives at the following commutation relations:

$$[a_i, a_j^+]_\pm = \hbar\delta_{ij}, \qquad [b_i, b_j^+]_\pm = \hbar\delta_{ij} \tag{2.26}$$

(commutators between a_i and b_j vanish). Apart from a different meaning of the indices i and j, these commutators are identical to those appearing in quantum mechanics of harmonic oscillator.

Formal polynomials of creation and annihilation operators modulo relation (2.26) form an associative algebra. It can be represented by linear operators acting on the Fock space, which can be introduced in the following way. First, one takes a special vector $|0\rangle$ such that

$$a_i|0\rangle = b_i|0\rangle = 0, \tag{2.27}$$

for all annihilation operators. It is called the vacuum vector or the ground state. Other vectors which constitute a basis in the Fock space are obtained by acting on $|0\rangle$ by all possible monomials of the creation operators,

$$|i_1, \dots i_k, j_1 \dots j_p\rangle = C_{i_1,\dots i_k, j_1 \dots j_p}(a^+_{i_1})^{n_1} \dots (a^+_{i_k})^{n_k}(b^+_{j_1})^{m_1} \dots (b^+_{j_p})^{m_p}|0\rangle, \tag{2.28}$$

where $C_{i_1,\dots i_k, j_1 \dots j_p}$ are normalization coefficients. These states describe fields excitations with a fixed number of quanta.

There can be infinitely many different ways to specify field excitations and to choose a set of single-particle modes $f_i^{(\pm)}$. This also implies that the ground state is not universal. The ground state with respect to one set of quanta may look as a state containing quanta defined in a different way. Indices i and j describe quantum numbers such as, for example, spin of the quanta and the momentum in a certain frame of reference. Thus, the choice of modes is determined by physical characteristics of the systems which are measured. The different sets of creation and annihilation operators are related to each other by unitary transformations called the Bogoliubov transformations, see Exercise 2.2.

At the end of this section a comment on quantum theory of real fields is in order. The equations of motion for real fields are invariant with respect to the complex conjugation or the charge conjugation (as in case of spin 1/2 fields). By taking into account (2.16) and conditions (2.19)–(2.21) one can conclude that $(f_j^{(-)})^* = f_j^{(+)}$. Since the corresponding operators are Hermitian, Eqs. (2.17) and (2.23) show that operators a_i and b_i coincide and just one set of these operators, say a_i and a_i^+, is used in this case.

2.3 Comments on Gauge Fields

Consider now theories with a degenerate relativistic inner product. The degeneracy means that there are classical solutions ξ for which the product with any other solution f vanishes identically, $\langle f, \xi \rangle \equiv 0$. Such a situation happens in theories where gauge fields are dynamical variables and we call ξ gauge modes. The examples are the Maxwell and Yang-Mills models (the model (1.75) for $M = 0$ and the model (1.77), respectively). In both models the classical action and equations of motion are invariant with respect to the gauge transformations $\delta_\xi f = \xi$. For Yang-Mills fields and other fields with non-linear dynamics this property applies to small perturbations which are described by linear equations.

The gauge modes are unphysical degrees of freedom because they do not contribute to physical quantities. In contrast, one can define *physical modes* as solutions f with a non-vanishing norm $\langle f, f\rangle \neq 0$. Modes related by a gauge transformation, f and $f_\xi = f + \xi$, are physically equivalent. One says that they belong the same orbit of the gauge group.

Consider classical functionals introduced in Sect. 2.2. In the case of gauge theories, let us require that $\varphi[f]$ acts on a set of all physical modes and does not vanish identically on this set. By their definition, the functionals $\varphi[f]$ are gauge invariant, $\varphi[f] = \varphi[f_\xi]$, thus, one can also say that they are defined on the orbits of the gauge group.

When going to quantum theory one replaces classical functionals with operator functionals $\varphi[f]$ also acting on the orbits. Introduction of the gauge invariant operators is justified because the r.h.s. of the commutation relations (2.18) is gauge invariant. Such quantization approach can be called "quantization in physical modes".

Instead of working with an orbit it is more convenient to choose one of its representatives, a particular mode by requiring that the mode obeys certain conditions. This is called a gauge fixing procedure. The fact that gauge conditions eliminate the gauge freedom implies that their solutions intersect each orbit of the gauge group in exactly one point. Generically, such conditions cannot be chosen globally on the whole space of the fields, but, since we are working with small fluctuations only, it is not a problem.

Let us illustrate the method by using a pure Maxwell theory. The gauge modes here have the simple form, $\xi_\mu = \partial_\mu \lambda$, where the gauge parameter λ is a sufficiently smooth function. For any potential A_μ there is a gauge parameter λ such that after the corresponding transformation the potential satisfies the so-called *Lorentz condition* $\nabla^\mu A_\mu = 0$. This condition does not eliminate the gauge freedom completely because it is invariant under the transformations where the gauge parameter is a solution to equation $\nabla^2 \lambda = 0$. This extra freedom is fixed by requiring that some components of the potential are vanishing, for example, that $A_0 = 0$ (on classical solutions). This means that the number of physical degrees of freedom of a photon in n dimensions is $n-2$. For a theory in Minkowski space-time the above conditions can be written as $A_0 = \partial^i A_i = 0$. This confirms the fact that the physical degrees of a photon are two polarizations orthogonal to the spatial momentum.

Once the gauge is fixed and physical modes are chosen one can proceed as in Sect. 2.2. In particular one can introduce (gauge invariant) creation and annihilation operators by Eq. (2.23), require decomposition (2.24), and finally define local field operators (2.25) in the given gauge.

In the rest of the book this procedure will be implied but not actually used. One just needs spectra of physical modes to calculate corresponding spectral functions, see Chap. 7, and show that calculation of physical quantities does not depend on the choice of the gauge conditions. "Quantization in physical modes" can be related to standard methods and attributes of quantum gauge theories, such as the Faddeev-Popov quantization etc., which are more convenient in interacting theories. We shall briefly comment on this in Sect. 7.8. More intuition on gauge models can be acquired from Exercises 2.6, 2.9, 2.10.

2.4 Canonical Quantization

In quantum mechanics one imposes canonical commutation relations

$$[q, \pi] = i\hbar \tag{2.29}$$

between the canonical coordinates q and their respective momenta π. The general scheme of quantization of free fields introduced above is equivalent to canonical quantization. The canonical momenta π are defined by the variational derivative

$$\pi(x) = \frac{\delta \mathscr{L}}{\delta \dot{\varphi}(x)} \tag{2.30}$$

of the Lagrangian $\mathscr{L}$. Here $\dot{\varphi}$ denotes the time derivative and, therefore, the definition of the momenta depends on the choice of the coordinate system. If t is a time coordinate the Lagrangian in this system is defined as the density of the classical action, $I = \int dt\, \mathscr{L}$.

Let us demonstrate equivalence of the two quantization procedures for the scalar field model (1.68) in Minkowski space-time. For a complex scalar field there are two sets of canonical coordinates and conjugate momenta, $\varphi, \pi = \dot{\varphi}^+$ and $\varphi^+, \pi^+ = \dot{\varphi}$. Let us fix an inertial frame of reference with the coordinates $x^\mu = (t, \mathbf{x})$ and choose Σ as a constant time hypersurface $t = const$. On Σ

$$\langle f_1, f_2 \rangle = i(f_1, \dot{f}_2) - i(\dot{f}_1, f_2), \tag{2.31}$$

where (f_1, f_2) is an inner product in the Hilbert space L^2 on Σ

$$(f_1, f_2) \equiv \int d^d x\, f_1^*(\mathbf{x}) f_2(\mathbf{x}). \tag{2.32}$$

From (2.31) one gets

$$\varphi(f_k) = i(f_k, \dot{\varphi}) - i(\dot{f}_k, \varphi) = i(f_k, \pi^+) - i(\dot{f}_k, \varphi). \tag{2.33}$$

It should be emphasized that $f_k(t, \mathbf{x})$ and $\dot{f}_k(t, \mathbf{x})$ at t fixed represent independent variables, the Cauchy data for the solutions $f_k(x)$. If one chooses $f_1 = \dot{f}_2 = 0$, Eqs. (2.18) and (2.33) imply the commutation rules

$$[(\dot{f}_1, \varphi), (f_2^*, \pi)] = i\hbar(\dot{f}_1, f_2) \quad \text{and} \quad [\varphi(\mathbf{x}), \pi(\mathbf{y})] = i\hbar\delta(\mathbf{x} - \mathbf{y}), \tag{2.34}$$

which coincide with (2.29). In the same way one gets other commutators between canonical variables.

In Minkowski space it is easy to construct normalized "modes" $f_i^{(\pm)}$. One of such examples is the so-called plane waves

$$f_{\mathbf{p}}^{(+)}(x) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}(2\pi)^{d/2}} e^{-i\omega_{\mathbf{p}} t + i\mathbf{p}\mathbf{x}}, \tag{2.35}$$

$f_{\mathbf{p}}^{(-)}(x) = (f_{\mathbf{p}}^{(+)}(x))^*$. The vector $\mathbf{p} \in \mathbb{R}^d$ is the momentum of the mode, $\omega_{\mathbf{p}} = (\mathbf{p}^2 + m^2)^{1/2}$ is the energy. It can be checked, that (2.35) are properly normalized,

$$\langle f_{\mathbf{p}}^{(\pm)}, f_{\mathbf{k}}^{(\pm)} \rangle = \pm\delta(\mathbf{p} - \mathbf{k}). \tag{2.36}$$

The operators

$$a^+(\mathbf{p}) = -\langle f_{\mathbf{p}}^{(-)}, \varphi^+\rangle, \qquad b^+(\mathbf{p}) = -\langle f_{\mathbf{p}}^{(-)}, \varphi\rangle \tag{2.37}$$

are creation operators for particles and anti-particles with fixed energies and momenta.

In the solutions (2.35), a transition from one inertial frame to another generates a covariant transformation of quantities $(\omega_{\mathbf{p}}, \mathbf{p})$ as components of a four vector p^μ. This means that (2.35) is a universal set of modes (plane waves) for all inertial observers. This also implies that for all such observers the vacuum state $|0\rangle$ is unique.

2.5 Quantum Theory on Stationary Backgrounds

Suppose that external classical background fields are stationary, i.e., there is a coordinate system $x^\mu = (t, x^i)$ where the background fields do not depend on the time coordinate t. In this case the energy of an isolated system is conserved. There are two definitions of the energy which can be found in the literature. One is determined in terms of the stress-energy tensor (1.22),

$$E = \int_\Sigma T_{\mu\nu} t^\mu \, d\Sigma^\nu, \tag{2.38}$$

where t^μ is the Killing vector field which generates translations along the time coordinate t. The integral is taken over a space-like surface Σ (which can be chosen as a surface of constant time). Another definition of the energy is known as the *canonical energy* or the *Hamiltonian*,

$$H = \sum_i \int d^d x \, \dot{\varphi}_i \frac{\delta L}{\delta \dot{\varphi}_i} - L, \tag{2.39}$$

where L is the Lagrangian of the system and φ_i is a set of dynamical variables together with its time derivatives $\dot{\varphi}_i = \partial_t \varphi_i$. Definition (2.39) implies that the Lagrangian does not contain time derivatives higher than the first order. It can be shown [122] that E and H differ by a surface term which vanishes under the suitable boundary conditions, see an example in Exercise 2.11.

If the background is stationary the classical canonical energy $H[f]$ computed for a solution f to the equation of motion (2.1) can be represented as

$$H[f] = \frac{i}{2}(\langle f, \dot{f}\rangle + \langle f^+, \dot{f}^+\rangle). \tag{2.40}$$

For theories with a real (Hermitian) fields one finds

$$H[f] = \frac{i}{2}\langle f, \dot{f}\rangle. \tag{2.41}$$

We leave the proof of these statements in different models for Exercise 2.8.

An important property of the theory on a stationary background is that the time variable is separated. As a consequence, one can introduce a special set of solutions to (2.1) which are the eigenfunctions of the operator $i\partial_t$,

$$i\partial_t f_i^{(\pm)}(x) = \pm\omega_i^{(\pm)} f_i^{(\pm)}(x). \tag{2.42}$$

We assume that $\omega_i^{(\pm)} > 0$. Thus, "+" and "−" modes are eigenfunctions of $i\partial_t$ with positive or negative eigenvalues, respectively. The numbers $\omega_i^{(\pm)}$ determine the spectrum of single-particle excitations and are called the single-particle energies.

The spectrum of single-particle energies is determined by an eigenvalue problem which follows from (2.1). For integer spin fields the operator $P(\partial_t, \partial_k)$ is a second order partial differential operator. For these fields (2.1) is reduced to

$$(P_0\omega^2 + P_1\omega + P_2) f_\omega(x^k) = 0, \tag{2.43}$$

where P_k is a k-th order differential operator. For spin 1/2 fields the problem like (2.43) is obtained by taking the square of the Dirac equation (1.74). The operators P_k do not commute between each other in general. Equation (2.43) is a non-linear spectral problem which is discussed in Chap. 6.

The normalization constant in the relativistic product is chosen such that the energies of elementary field excitations (described by $f_i^{(\pm)}$) coincide with frequencies of the modes. To see this for complex fields, we first use (2.40) and (2.42) to get

$$H[f_i^{(\pm)}] = \pm\omega_i^{(\pm)} \langle f_i^{(\pm)}, f_i^{(\pm)} \rangle. \tag{2.44}$$

Then the cases of Bose and Fermi statistics are considered separately.

Bose Statistics If the normalization condition (2.20) is satisfied, Eq. (2.44) yields

$$H[f_i^{(\pm)}] = \omega_i^{(\pm)}. \tag{2.45}$$

This equation implies that $H[f] \geq 0$, which may not be the case in general. In static space-times (when the Killing field t^μ is orthogonal to constant time hypersurfaces) one can guarantee positivity of H for systems whose stress-energy tensor satisfies the so-called *weak energy condition* [156]. The condition requires that $T_{\mu\nu}u^\mu u^\nu \geq 0$ for any time-like vector u^μ.

The energy operator is constructed from its classical analog H when classical fields are replaced with corresponding operators. Substitution of (2.25) in (2.40) and using commutation relations (2.26) yields the quantum Hamiltonian in the following form:

$$H = \sum_i \omega_i^{(+)} a_i^+ a_i + \sum_j \omega_j^{(-)} b_j^+ b_j + E_0. \tag{2.46}$$

The constant E_0 in the r.h.s. of (2.46) is given by an infinite series

$$E_0 = \frac{\hbar}{2}\sum_i \omega_i^{(+)} + \frac{\hbar}{2}\sum_j \omega_j^{(-)}. \tag{2.47}$$

The result, as expected, is equivalent to the energy of an infinite number of harmonic oscillators. In field theory, the series (2.47) diverge and require a regularization (a cutoff) at large frequencies $\omega_i^{(\pm)}$. One finds with the help of (2.27) that the ground state is the eigenvector of the energy operator, $H|0\rangle = E_0|0\rangle$. For this reason E_0 is called the energy of zero-point fluctuations or the vacuum energy. The vacuum energy will be a special subject of Chap. 9.

Hermitian Bose fields are considered in the same way. In this case there is a single sort of creation and annihilation operators, say a_i^+, a_i and the single type of frequencies, $\omega_i = \omega_i^+ = \omega_i^-$. Therefore,

$$H = \sum_i \omega_i a_i^+ a_i + E_0, \tag{2.48}$$

$$E_0 = \frac{\hbar}{2} \sum_i \omega_i. \tag{2.49}$$

To get (2.48) one has to use Eq. (2.41) for the energy.

Fermi Statistics If the normalization condition (2.21) is satisfied, it follows from (2.44) that

$$H[f_i^{(\pm)}] = \pm\omega_i^{(\pm)}. \tag{2.50}$$

Thus, the classical energy is negative for modes with negative frequencies. On the quantum level contributions of negative and positive frequency modes to the energy have equal forms and signs because of Fermi statistics. When one uses anti-commutation relations (2.26) the energy operator looks as follows:

$$H = \sum_i \omega_i^{(+)} a_i^+ a_i + \sum_j \omega_j^{(-)} b_j^+ b_j + E_0, \tag{2.51}$$

$$E_0 = -\frac{\hbar}{2} \sum_i \omega_i^{(+)} - \frac{\hbar}{2} \sum_j \omega_j^{(-)}. \tag{2.52}$$

The negative constant E_0 is the vacuum energy.

Relation to Classical Mechanics We finish this section with the following comment. The relativistic product (2.9) is a structure which appears already in the classical mechanics for a finite number of degrees of freedom. Consider a system of N variables $q_k(t)$ whose evolution is described by the Hamilton equations. One can find the corresponding canonical momenta $p_k(t)$ and define the following symplectic form [224]:

$$\Omega(q_1, p_1; q_2, p_2) = i \sum_{k=1}^{N} (q_{1,k}(t) p_{2,k}(t) - p_{1,k}(t) q_{2,k}(t)), \tag{2.53}$$

where $(q_{i,k}, p_{i,k})$ are solutions to the Hamilton equations for the given system. One can show that $\partial_t \Omega(q_1, p_1; q_2, p_2) = 0$ and check that the canonical energy computed on a solution $q_k(t)$ can be written as [224]

$$H[q] = i\Omega(q, \partial_t q). \tag{2.54}$$

Thus, (2.53) is an analog of product (2.9), while (2.54) is an analog of (2.40).

2.6 Green's Functions

In this section we introduce a number of the so-called two-point Green's functions. Consider as an example a scalar field φ in the Minkowski space-time. The equation of motion is, see (1.69),

$$(-\partial_\mu \partial^\mu + m^2)\varphi(x) = 0. \tag{2.55}$$

With the help of local field operators (2.25) one can define the following functions:

$$G^+(x, x') = \langle 0|\varphi(x)\varphi^+(x')|0\rangle, \tag{2.56}$$

$$G^-(x, x') = \langle 0|\varphi^+(x')\varphi(x)|0\rangle, \tag{2.57}$$

$$iG(x, x') = [\varphi(x), \varphi^+(x')] = G^+(x, x') - G^-(x, x'), \tag{2.58}$$

$$G^{(1)}(x, x') = \langle 0|\varphi(x)\varphi^+(x') + \varphi^+(x')\varphi(x)|0\rangle = G^+(x, x') + G^-(x, x'), \tag{2.59}$$

$$iG_F(x, x') = \theta(t - t')G^+(x, x') + \theta(t' - t)G^-(x, x'), \tag{2.60}$$

$$G_R(x, x') = -\theta(t - t')G(x, x'), \tag{2.61}$$

$$G_A(x, x') = \theta(t' - t)G(x, x'). \tag{2.62}$$

Here $\theta(x)$ is a step function, $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. The names of the functions are the following: G^+ and G^- are the Wightman functions, G is the Pauli-Jordan function, $G^{(1)}$ is the Hadamard function, G_F is the Feynman function (or the Feynman propagator), G_R, G_A are retarded and advanced Green's functions, respectively.

Since the field operators obey (2.55) the Green's functions are solutions to similar homogeneous or inhomogeneous equations. For instance, it follows from (2.58) that

$$(-\partial_\mu \partial^\mu + m^2)G(x, x') = 0, \tag{2.63}$$

where the differential operator acts either on the argument x or x'. The same equation holds for G^-, G^+, and $G^{(1)}$. For the Feynman function one finds

$$(-\partial_\mu \partial^\mu + m^2)G_F(x, x') = \delta^{(n)}(x - x'), \tag{2.64}$$

where $\delta^{(n)}(x - x') = \delta(t - t')\delta(\mathbf{x} - \mathbf{x}')$. To get the r.h.s. of (2.64) one has to take into account canonical commutation relation (2.34), see Exercise 2.12.

Equations (2.25), (2.26) can be used to rewrite the Green's function in terms of single-particle modes. For instance, for the Wightman and the Pauli-Jordan functions one gets

$$G^+(x, x') = \langle 0|\varphi(x)\varphi^+(x')|0\rangle = \sum_i f_i^{(+)}(x)(f_i^{(+)}(x'))^*, \tag{2.65}$$

$$G^-(x, x') = \langle 0|\varphi^+(x')\varphi(x)|0\rangle = \sum_j (f_j^{(-)}(x'))^* f_j^{(-)}(x), \tag{2.66}$$

$$iG(x, x') = \sum_i f_i^{(+)}(x)(f_i^{(+)}(x'))^* - \sum_j (f_j^{(-)}(x'))^* f_j^{(-)}(x). \tag{2.67}$$

Other Green's functions can be expressed similarly.

The same representations, (2.65)–(2.67), hold for theories in arbitrary background fields. The Green's functions (except for the Pauli-Jordan function which is determined by the commutator) depend on the choice of the vacuum state.

By using (2.65)–(2.67) one can show that the Green's functions above have singularities on the light cone $(x-x')^{\mu}(x-x')_{\mu}=0$. There may be singularities of different types: power or logarithmic singularities, delta-function-like singularities or discontinuities.

It is instructive to give explicit expressions for Green's functions for a massless scalar field in a four-dimensional Minkowski space-time. The modes are defined by (2.35) with $d=3$ and $m=0$. Due to translation invariance of the Minkowski space-time the Green's functions depend on the difference of the arguments, $G(x,x')=G(0,x'-x)\equiv G(x'-x)$. A straightforward computation yields for the Wightman functions, see Exercise 2.16,

$$G^{\pm}(x)=\frac{1}{4\pi^2 s^2}\pm\frac{i}{4\pi}\varepsilon(t)\delta(s^2), \tag{2.68}$$

where $x=(t,\mathbf{x})$, $s^2=s^2(x)\equiv -t^2+\mathbf{x}^2$ is the invariant interval between x and 0, and $\varepsilon(t)=\theta(t)-\theta(-t)$ is the sign function. With the help of (2.58), (2.59), (2.60), and (2.68) one gets the following expressions for the Pauli-Jordan, the Hadamard, and the Feynman functions:

$$G(x)=\frac{1}{2\pi}\varepsilon(t)\delta(s^2), \tag{2.69}$$

$$G^{(1)}(x)=\frac{1}{2\pi^2 s^2}, \tag{2.70}$$

$$G_F(x)=-\frac{i}{4\pi^2 s^2}-\frac{1}{4\pi}\delta(s^2). \tag{2.71}$$

The Pauli-Jordan function (2.69) vanishes outside $s=0$. For massive fields it vanishes under weaker conditions, if the interval is space-like, $s^2(x)>0$. This means that the field operators in causally disconnected points commute. Such a property holds in general, see Exercise 2.18.

2.7 Computation of Averages

The two-point Green's functions play an important role in physical applications. They are used in perturbation methods in quantum theories of interacting fields, see discussion in Sect. 7.7. Here we describe how the Green's functions can be used to find expectation values of operators corresponding to physical observables.

As an example, consider the vacuum expectation value for the stress-energy tensor of the scalar field discussed in the previous section. The classical stress-energy tensor in this model is, see (1.70),

$$T_{\mu\nu}=2\partial_{\mu}\varphi^*\partial_{\nu}\varphi-\eta_{\mu\nu}(\partial_{\sigma}\varphi^*\partial^{\sigma}\varphi+m^2\varphi^*\varphi). \tag{2.72}$$

In quantum theory the stress-energy tensor becomes an operator which is obtained from (2.72) by replacing classical fields with the corresponding operators. The vacuum average $\langle 0|T_{\mu\nu}|0\rangle$ suffers from divergences which appear in the averages of products of field at coinciding points, like in $\langle 0|\varphi(x)\varphi(x)|0\rangle$. Such averages are related to the Wightman function (2.56) which, as we have seen already, is singular when its arguments coincide.

To deal with the divergences one uses the so-called point-splitting method. For example, the regularized average of the stress-energy tensor can be defined as

$$\langle 0|T_{\mu\nu}(x)|0\rangle \equiv \lim_{x'\to x} \langle 0|2\partial_\mu \varphi^+(x')\partial_\nu \varphi(x) - \eta_{\mu\nu}(\partial_\sigma \varphi^+(x')\partial^\sigma \varphi(x) + m^2\varphi^+(x')\varphi(x))|0\rangle. \tag{2.73}$$

Here x and x' are close points, such that $x - x'$ is not light-like. This expression can be also written in terms of the Wightman function

$$\langle 0|T_{\mu\nu}(x)|0\rangle = \lim_{x'\to x} [2\partial'_\mu \partial_\nu - \eta_{\mu\nu}(\partial'_\sigma \partial^\sigma + m^2)]G^-(x', x), \tag{2.74}$$

where $\partial_\mu = \partial/\partial x^\mu$ and $\partial'_\mu = \partial/\partial(x')^\mu$. In the limit $x' = x$ the singularities of the Wightman function result in singularities of the average (2.73). The singular terms, however, can be separated from the finite ones and subtracted. The physical justification for this operation, which is called a renormalization, is explained in Sect. 7.5. The example of computation based on formula (2.74) is given in Exercise 9.1 to Chap. 9.

For non-interacting fields the point-splitting method is quite general. Consider a local classical quantity $\mathscr{O}$ which is, like the stress-energy tensor or gauge currents, a quadratic polynomial of the field variables φ and its derivatives up to the second order. It can be written as a coincidence limit

$$\mathscr{O}(x) = \lim_{x\to x'} D_{AB}(x, x')\varphi^A(x)\varphi^B(x'), \tag{2.75}$$

where $D_{AB}(x, x')$ is a bi-differential operator, A and B are field indices. In quantum theory the average value of the observable $\mathscr{O}$ is determined by using (2.75)

$$\langle \mathscr{O}(x)\rangle = \lim_{x\to x'} D_{AB}(x, x')\langle \varphi^A(x)\varphi^B(x')\rangle, \tag{2.76}$$

where, as before, the correlator $\langle \varphi^A(x)\varphi^B(x')\rangle$ can be expressed in terms of a two-point Green's function. The physical quantity is obtained from (2.75) after subtracting the divergent parts. The operator $D_{AB}(x, x')$ corresponding to a given $\mathscr{O}$ may be non-unique. It is not a problem if different definitions after subtracting divergences yield the same result.

There is an alternative method of computing the averages of operators which is based on using the effective action and is our main interest. We shall return to this issue in Chap. 7.

2.8 Quasinormal Modes

In constructing a quantum theory along the lines of previous sections one may encounter solutions to wave equations (2.1) which look similar to the single-particle modes but have nothing to do with quantum excitations. One type of such modes has a vanishing norm. This may happen because the modes have zero frequency or they are related to pure gauge degrees of freedom, see Exercise 2.6.

In this section we describe another type of classical solutions, the so-called quasinormal modes. Although these modes have complex frequencies and are not normalizable they carry important information about physical properties of the system. As an example we consider a two-dimensional scalar field model with the wave equation

$$(\partial_t^2 - \partial_x^2 + V(x))\varphi(t,x) = 0. \tag{2.77}$$

It is assumed that $-\infty < x < \infty$ and the "background field" is described by a "potential" $V(x)$. We suppose that $V(x)$ is a smooth bounded function with a compact support, such that $V(x) = 0$ if $|x| > b > 0$.

The spectrum of single-particle energies related to eigenvalues of the operator $-\partial_x^2 + V$ has a continuous part, and it is the only part if $V(x) > 0$. How can complex frequency modes appear in this problem? Suppose that a solution to (2.77) is determined at some initial moment, say at $t = 0$, by the Cauchy data, $\varphi(0,x)$, $\partial_t\varphi(0,x)$, which have a compact support. The quasinormal modes appear when one studies asymptotic of φ at late times.

Let us start with construction of a general solution to (2.77). We use the Laplace transform

$$\chi(\lambda,x) = \int_0^\infty e^{-\lambda t}\varphi(t,x)\,dt, \tag{2.78}$$

which enables us to represent the solution in the integral form

$$\varphi(t,x) = \frac{1}{2\pi i}\int_C e^{t\lambda}\chi(\lambda,x)\,d\lambda. \tag{2.79}$$

The contour C in the complex plane goes parallel to the imaginary axis such that $\Re\,\lambda = a > 0$. The function $\chi(\lambda,x)$ is defined through a one-dimensional problem

$$(\lambda^2 - \partial_x^2 + V(x))\chi(\lambda,x) = j(\lambda,x) \tag{2.80}$$

with a "source" determined by the Cauchy data,

$$j(\lambda,x) \equiv \partial_t\varphi(0,x) + \lambda\varphi(0,x). \tag{2.81}$$

To get (2.80) from (2.77) one has to start with the Laplace transform of $\partial_t^2\varphi$ and integrate by parts

$$\int_0^\infty e^{-\lambda t}\partial_t^2\varphi(t,x)\,dt = \lambda^2\chi(\lambda,x) - j(\lambda,x). \tag{2.82}$$

The solution to (2.80) can be written with the help of a Green's function $G_\lambda(x,y)$

$$\chi(\lambda,x)=\int_{-\infty}^{\infty}dy\,G_\lambda(x,y)j(\lambda,y), \tag{2.83}$$

$$(\lambda^2-\partial_x^2+V(x))G_\lambda(x,y)=\delta(x-y). \tag{2.84}$$

The differential operator in (2.84) does not depend on time. We shall describe a method how to construct $G_\lambda(x,y)$ in a way which differs from the procedure applicable to time-dependent Green's functions (2.65)–(2.67). Consider the homogeneous equation

$$(\lambda^2-\partial_x^2+V(x))f(\lambda,x)=0. \tag{2.85}$$

A pair of independent solutions to (2.85), f_k, can be determined by their asymptotic behavior at $x\to\pm\infty$. Because $V(x)=0$ for $|x|>b$ one can choose the following asymptotics

$$f_1(\lambda,x)\sim e^{-\lambda x},\quad x\gg b, \tag{2.86}$$

$$f_2(\lambda,x)\sim e^{+\lambda x},\quad x\ll -b. \tag{2.87}$$

The Wronskian of the system

$$W(\lambda)=\frac{1}{2}(f_2'(\lambda,x)f_1(\lambda,x)-f_1'(\lambda,x)f_2(\lambda,x)), \tag{2.88}$$

does not depend on the coordinate x and is not vanishing on independent solutions, $W(\lambda)\neq 0$. One can check (see Exercise 2.21) that a solution to (2.84) can be written as

$$\begin{aligned} G_\lambda(x,y)&=\frac{g_\lambda(x,y)}{W(\lambda)},\\ g_\lambda(x,y)&=\theta(x-y)f_1(\lambda,x)f_2(\lambda,y)+\theta(y-x)f_1(\lambda,y)f_2(\lambda,x). \end{aligned} \tag{2.89}$$

It can be shown that the Laplace transform $\chi(\lambda,x)$, see (2.78), is uniquely determined for $\Re\,\lambda>0$ by (2.83), (2.89) and is a bounded function provided that the Cauchy data (the "source" $j(\lambda,x)$) have a compact support.

The way how one can determine the late time behavior of the solution $\varphi(t,x)$ provided that its initial perturbation is localized in a finite region is the following. Consider representation (2.79) and use (2.83), (2.88) to get

$$\varphi(t,x)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}dy\int_C d\lambda\,e^{t\lambda}\frac{g_\lambda(x,y)}{W(\lambda)}j(\lambda,y). \tag{2.90}$$

Because $t>0$ one can add to the contour C a semicircle lying in left half of the complex plane. Then integration in (2.90) is equivalent to integration over a closed contour and can be performed by using the Cauchy theorem. If $W(\lambda)$ have complex zeros λ_k ($\Re\,\lambda_k<0$) the contour integral in (2.90) acquires contributions from the residues of $1/W(\lambda)$ at λ_k. At late t the main contribution to (2.90) is determined by the pole λ_0 with the smallest real part $|\Re\,\lambda_0|$. Therefore, at late t

$$\varphi(t,x)\simeq e^{t\lambda_0}\varphi_0(x), \tag{2.91}$$

where $\varphi_0(x)$ is some function.

It can be shown that for positive potentials $V(x)$ with a compact support the Wronskian always has a countable number of zeros. The idea behind finding these zeros is quite simple. If $W(\lambda) = 0$ the corresponding solutions, $f_k(\lambda, x)$, are not independent. Suppose that for a certain complex value λ ($\Re\lambda > 0$) the wave equation (2.77) allows for a solution, $\bar{f}(\lambda, x)$, which has both asymptotics, $\bar{f}(\lambda, x) \sim e^{\pm\lambda x}$ at $x \to \mp\infty$. It then follows from (2.86), (2.87) that λ is one of the zeros of the Wronskian.

Such solutions $\bar{f}(\lambda, x)$ are called the quasinormal modes. The complex numbers λ which are the zeros of the Wronskian are called the quasinormal spectrum. The ringing frequencies of a bell, which can be heard, are related to the quasinormal spectrum. The inverse of $|\Re\lambda_0|$ yields the lifetime of the main overtone which decays the last.

Let us emphasize once again that quasinormal frequencies despite their physical importance are not eigenvalues of the operator $-\partial_x^2 + V$ because the corresponding modes are not normalizable.

2.9 Literature Remarks

Commutation relation (2.18) of free fields on a gravitational background has been used by a number of authors, see e.g. a pioneering paper by Chernikov and Tagirov on quantum theory in de Sitter space-time [65]. An alternative way would be to postulate, by following DeWitt [77], the local commutation relations as

$$[\varphi(x), \varphi^+(x')] = iG(x, x'),$$

where $G(x, x')$ is the Pauli-Jordan function which can be defined in classical theory by Eq. (2.67). This scheme of quantization is manifestly covariant and is equivalent to (2.18) on globally hyperbolic space-times.

A quantization procedure of fields of different spins in Minkowski space-time along with properties and singularity structure of Green's functions is described in detail in the classical book by Bogoliubov and Shirkov [40]. Among the modern monographs on quantum field theory we mention the book by Peskin and Schroeder [205] and the book by Weinberg [253].

Quantization of gauge theories and constrained dynamics is presented in many books, see e.g. [108, 137]. We should note that a method of "quantization in physical modes" discussed in Sect. 2.3 may fail on some curved backgrounds, see e.g. [105, 242]. Although what is described in Sect. 2.3 and later in Sect. 7.8 is enough to demonstrate applications of the spectral theory to different problems with quantum gauge fields.

Quasinormal modes encode important characteristics of frequencies and lifetimes of gravitational waves emitted at late stages by a black hole after its perturbation, see more on this subject in the book by S. Chandrasekhar [62]. A recent review of quasinormal modes of stars and black holes can be found in [171, 173]. Possible role of quasinormal modes in quantum gravity theory is discussed in [160].

There are several reasons why we do not discuss in this book higher-spin fields (spins $3/2$, 2 and etc.). The main reason is that we study here the Lagrangian field theories while a problem of a Lagrangian formulation for higher spin fields is open in general and details in its resolution are still missing. Classical free Lagrangian higher spin field theories in Minkowski space-time were formulated in the middle of 70th of the last century. However, coupling of these fields to arbitrary external backgrounds or interaction among higher spin fields faces the problem of consistency. A consistent interacting massless spin 2 field theory is general relativity, non-contradictory interacting massless spin $3/2$ and 2 fields enter in supergravity. At present, there exists Lagrangian formulation for massless and massive arbitrary higher spin fields in anti de Sitter (AdS) space-time. Besides, spin $3/2$ and 2 field Lagrangian formulation exists in the Einstein space. As for general higher spin interaction, it seems that Lagrangian formulation should include an infinite tower of all higher spin fields (like in string theory) and it is not so clear how to quantize such theories. The higher spin fields in the AdS space can be quantized by standard methods and, in principle, the mathematical techniques which are considered in this book can be applied to study an effective action in this case. As far as we know, such a consideration has never been carried out in general, besides spin $3/2$ and 2 fields.

Recommended Exercises are 2.5, 2.6, 2.8, 2.10, 2.16.

2.10 Exercises

Exercise 2.1 Prove that the integral $Q = \int_\Sigma d\Sigma^\mu j_\mu$ on a hypersurface Σ does not change under smooth local transformations Σ if $\nabla^\mu j_\mu = 0$.

Exercise 2.2 Consider the following linear combination of single-particle modes $f_i^{(\pm)}$:

$$\tilde{f}_i^{(+)} = \sum_k \alpha_{ik}^{(+)} f_k^{(+)} + \sum_p \beta_{ip}^{(+)} f_p^{(-)}, \tag{2.92}$$

$$\tilde{f}_j^{(-)} = \sum_k \alpha_{jk}^{(-)} f_k^{(+)} + \sum_p \beta_{jp}^{(-)} f_p^{(-)}, \tag{2.93}$$

where $\alpha_{ik}^{(\pm)}$ and $\beta_{ip}^{(\pm)}$ are some complex numbers.

1) Find relations between $\alpha_{ik}^{(\pm)}$ and $\beta_{ip}^{(\pm)}$ which guarantee that $\tilde{f}_i^{(\pm)}$ form another set of single-particle modes which satisfy (2.19)–(2.21).
2) Find a transformation from creation and annihilation operators determined by modes $f_i^{(\pm)}$ to creation and annihilation operators determined by modes $\tilde{f}_i^{(\pm)}$ (this transformation is called the Bogoliubov transformation after N.N. Bogoliubov who introduced it in the theories of superfluidity and superconductivity).
3) Calculate the number of particles of the new sort in the vacuum state (2.27).

Exercise 2.3 Consider free scalar field model (1.69) in a general gravitational and gauge background. Prove that the general quantization scheme presented in Sect. 2.1 coincides with the canonical quantization.

Exercise 2.4 Consider a theory of free quantum fields on a globally hyperbolic space-time $\mathscr{M}$. Prove that the quantization condition (2.18) implies that

$$[\varphi(x_1), \varphi^+(x_2)]_\pm = 0 \tag{2.94}$$

when points x_1 and x_2 are on a Cauchy surface Σ. By the definition of Σ (see Sect. 1.6) such points are casually independent.

Exercise 2.5 Consider a vector field action

$$I[A, g] = -\frac{1}{2} \int d^n x \sqrt{-g} (\nabla_\nu A_\mu \nabla^\nu A^\mu + R_{\mu\nu} A^\mu A^\nu + M^2 A_\mu A^\mu), \tag{2.95}$$

where $R_{\mu\nu}$ is the Ricci-tensor of the background metric. What is the difference between this model and model (1.75)? Why quantization of (2.95) yields a theory with unphysical properties?

Exercise 2.6 What is the difference between the massive and massless vector models (1.75)? Note that the massless model is the Maxwell theory in a vacuum. Identify physical degrees of freedom in the Maxwell theory.

Exercise 2.7 Prove that the norm $\langle \psi, \psi \rangle$ of c-number valued (non-Grassmann) spinor fields defined on a smooth space-like hypersurface by (2.9), (2.11) is positive. Prove the following property:

$$\langle \psi_1^c, \psi_2^c \rangle = \langle \psi_2, \psi_1 \rangle, \tag{2.96}$$

where ψ^c denotes a charge conjugated spinor, see (1.66). If ψ_1 and ψ_2 are Grassmann fields, a minus sign appears in the equation above.

Exercise 2.8 Derive formulae (2.40), (2.41) for the canonical energy in stationary backgrounds for models of scalar (1.68), spinor (1.73), vector (1.75), and non-Abelian gauge fields (1.79).

Exercise 2.9 Let ω_i be the spectrum of single-particle energies in a field model on a stationary background. One can define a spectral function

$$\Phi = \sum_i f(\omega_i), \tag{2.97}$$

where $f(x)$ is some smooth function which decays fast enough to ensure convergence of the series. One of the examples of the spectral function is the regularized vacuum energy, see (2.47), where $f(x) = x/2$ ($f(x) = 0$ for $x > a$ where a is a regularization parameter). Other examples are studied below.

Find a relation between the spectral functions of the vector models (1.75) and (2.95). Consider both massive and massless cases.

Exercise 2.10 Consider a theory of linear order perturbations A_μ of a $SU(N)$ gauge field over the background field B_μ, where B_μ is a static solution to Yang-Mills equations (1.78). The equations for the perturbations are (1.79).

By analogy with the Maxwell theory, see Exercise 2.6, study the single-particle spectrum ω_i of the perturbations in the Lorentz-like gauge $[D^\mu, A_\mu] = 0$. Find a representation of the spectral function (2.97) in this gauge in terms of the spectral functions of some unconstrained fields.

Exercise 2.11 Consider a model of a real scalar field with the so-called non-minimal coupling between the field and the curvature scalar

$$I = -\frac{1}{2}\int d^n x\,\sqrt{g}(\partial^\mu\varphi\partial_\mu\varphi + m^2\varphi^2 + \xi R\varphi^2). \tag{2.98}$$

By using definitions (2.38), (2.39) calculate the energy and the Hamiltonian for this model on a static background. Find the difference between the two quantities and show that it is reduced to a surface term.

Exercise 2.12 Prove the equation

$$(-\partial_\mu\partial^\mu + m^2)G_F(x, x') = \delta^{(n)}(x - x')$$

for the scalar Feynman function in Minkowski spacetime, see Sect. 2.6.

Exercise 2.13 Consider a massless field on a circle, i.e. field in two-dimensional space-time which obeys the periodic condition in spatial coordinate, $\varphi(t, x + l) = \varphi(t, x)$. Prove that the Wightman function for this model has the form

$$G^+(0, x^\mu) = -\frac{1}{4\pi}\left[\ln\left(-4\sin\frac{au}{2}\sin\frac{av}{2}\right) + \frac{i}{2}a(u + v)\right], \tag{2.99}$$

where $a = 2\pi/l$, $x^\mu = (t, x)$, $u = t - x$, $v = t + x$ and $\Im t = \epsilon > 0$.

Exercise 2.14 Get the following expression for the Wightman function of the massless two-dimensional field on an interval of the length l:

$$G^+(x^\mu, (x')^\mu) = \frac{1}{4\pi}\ln\left[\frac{\cos a\Delta t - \cos a(x + x')}{\cos a\Delta t - \cos a(x - x')}\right], \tag{2.100}$$

where $a = \pi/l$ and $\Delta t = t' - t$, $\Im t' > 0$. The boundary condition for the field is $\varphi(t, l) = \varphi(t, 0) = 0$.

Exercise 2.15 By using explicit expressions for the Wightman functions (2.99), (2.100) derive canonical commutation relation for the massless scalar field on a circle and on an interval.

Exercise 2.16 Prove expressions (2.68) for the Wightman functions of a massless scalar field in four-dimensional Minkowski space-time.

Exercise 2.17 Consider a massive scalar field in four-dimensional Minkowski space-time. Show that the Feynman, advanced and retarded Green's functions can be defined in the so-called momentum representation

$$\mathscr{G}(0,x)=\frac{1}{(2\pi)^4}\int d^4p\,\frac{e^{-ipx}}{p^2+m^2}, \tag{2.101}$$

where each of these functions is specified by a prescription how to pass the poles in the denominator. The following notations are used in (2.101): p is a four-dimensional momentum, $p=(p_0,\mathbf{p})$, $px=-p_0t+\mathbf{px}$, $p^2=-p_0^2+\mathbf{p}^2$.

Find also analogous prescription for the Wightman and Pauli-Jordan functions.

Exercise 2.18 By using results of Exercise 2.17 demonstrate the Lorentz invariance of Green's functions.

Exercise 2.19 Consider the Cauchy problem

$$(-\partial_\mu\partial^\mu+m^2)\varphi(x)=0, \tag{2.102}$$

$$\varphi(x)|_{t=0}=\varphi_1(\mathbf{x}), \qquad \dot\varphi(x)|_{t=0}=\varphi_2(\mathbf{x}). \tag{2.103}$$

Show that a solution to (2.102), (2.103) can be written with the help of the Pauli-Jordan function,

$$\varphi(x)=\int d\mathbf{y}[\partial_{t_y}G(x,y)|_{t_y=0}\varphi_1(\mathbf{y})-G(x,y)|_{t_y=0}\varphi_2(\mathbf{y})]. \tag{2.104}$$

Exercise 2.20 By using the point-splitting method, see (2.76), define the average value of the electric current for scalar and spinor field models (1.68), (1.73) in Minkowski spacetime. For the classical current use the definition (1.71).

Exercise 2.21 Consider Eq. (2.84) for a one-dimensional Green's function $G_\lambda(x,y)$. Let $f_k(\lambda,x)$ be two independent solutions to the homogeneous equation (2.85) and $W(\lambda)$ be their Wronskian (2.88). Demonstrate that the Green's function can be written as

$$G(x,y)=\frac{1}{W(\lambda)}(\theta(x-y)f_1(\lambda,x)f_2(\lambda,y)+\theta(x-y)f_1(\lambda,y)f_2(\lambda,x)). \tag{2.105}$$

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