Preface

Massive penetration of operators in physics started with the birth of Quantum Mechanics. The mathematical methods used in quantum theory, being rather elementary at the very beginning, rapidly achieved a high level of sophistication. It was realized soon that relevant information on the operators may be collected in the objects called the *spectral functions*, which depend on the operator in question and on a real or complex parameter. In the context of quantum field theory (QFT) the most frequently used spectral function is the *heat kernel*. This function was first applied to the problems of quantum physics by Fock [111] already in 1937, and then by Schwinger in his seminal paper [225]. In 1960's DeWitt [77–80] put the heat kernel as a corner stone for his method of calculating quantum corrections. About 10 years later, Dowker and Critchley [92] and Hawking [155] suggested a regularization scheme of quantum field theory based on another prominent spectral function, the *zeta-function*. All these works determined for the rest of the century a landscape of methods of quantum field theory based on geometrical properties of operators. This has led to very interesting and important developments in practically all areas of QFT, ranging from quantum gravity, to anomalies, strings, and the Casimir effect. Apart form powerful technical tools this approach provided the physicists with a new rather adequate language to describe complicated phenomena, and with a new, very fruitful, point of view on quantum effects in general.

The main idea of the approach is easy to understand even without knowing precise definitions. Consider a QFT on some curved space. The spectrum of quantum fluctuations is defined by the spectrum of an operator, say, a Laplacian, on this space. The aim then is two-fold. On one hand, it is to relate quantities of interest in QFT, such as a ground state energy, to functions of the spectrum of this operator, i.e., to spectral functions. On the other hand, one looks for a connection of the spectral functions to geometric characteristics of the space. It is desirable to have sufficiently general relations operating with essential ingredients of the problem, such as geometric invariants of the manifold, and not depending on inessential features, for instance, on a particular choice of coordinates.

Interestingly, similar problems were being solved in mathematics at about the time. Kac [165] put the problem in the following way: Can one hear the shape of a drum? In other words, knowing the acoustic eigenfrequencies, what can one say about the geometry? Probably the first step in this direction was done long ago by H. Weyl [257, 258] who found a relation between asymptotic distribution of eigenvalues of a Laplacian and the volume of the manifold. Minakshisundaram and Pleijel [189] derived more detailed relations involving other geometric invariants. A firm basis for such kind of calculations was developed by Seeley [226–228], and a powerful technique for actual computations was suggested by Gilkey [132].

By mid 1980's the methods related to spectral functions, mainly to the heat kernel, became standard in QFT, especially in curved space, but also in all cases when non-trivial geometry and topology were essential. The famous Birrell and Davies book [37] could be found on the desk of practically everyone even remotely connected with quantum gravity, often accompanied by a more technical review by Barvinsky and Vilkovisky [26]. New challenges appeared very soon. Among them there were spectral problems with boundaries stemming from quantum cosmology, strings, and the Casimir effect. Then appeared problems with various types of singularities following from the brane world scenario and black hole physics. Finally, the 21st century put spectral problems on noncommutative spaces in the center of interest.

Of course, many good books appeared meanwhile. Some of them are listed below as recommended literature. There is, however, a gap, which we would like to fill in by the present work. We were aiming at writing a text starting with the level of an advanced textbook, i.e., containing all basic information, especially on the mathematical side, and gradually reaching rather advanced physical topics. We tried to make the book as selfcontained as possible to be useful for both active researchers and graduate students. Inclusion of more than a hundred exercises with their solutions makes it possible to use this material in lecture courses on physical applications of the spectral theory.

These aims determined the choice of the material and the style of the presentation. The exposition of main mathematical methods is very detailed, though not always reaching the depth and generality of specialized research monographs. In applications, instead of studying one particular area in all detail, we took examples from various fields, including finite temperature field theory, anomalies, quantum solitons, strings, and noncommutative field theories. In each case, we demonstrate how the use of general methods allows to achieve interesting and important results in an elegant and relatively easy manner. All applications are taken from active areas of research. We organized this material to prepare the reader to work further on his/her own in any specific area of QFT.

This book is organized as follows. Part I contains some basic information and serves to settle notations, but not only. Chapter 1 devoted to differential geometry, contains some less standard material on boundaries and singularities. Chapter 2 introduces main notions of QFT basing on relativistic inner products rather than on usual operator quantization. This facilitates applications to the problems in the rest of this book and, in particular, is more convenient in relation to free fields theories in classical backgrounds.

Part II is devoted to mathematical foundations, namely, to the spectral geometry. Chapter 3 explains main properties of operators. Chapters 4, 5 are the central Chapters for this book. Chapter 4 is an introduction to the heat equation and asymptotic properties of the heat kernel expansion. It is organized so that to present briefly a variety of techniques for computation of the heat coefficients on different base manifolds. Chapter 5 contains definitions of main spectral functions, lists their properties, and methods of computation. It defines zeta-functions and determinants of differential operators, explains their transformation properties and the merit of the index theorem. Much space is devoted to variations of the determinants, which will later serve as a basis for calculations of quantum anomalies. Chapter 6 deals with non-linear spectral problems, for which the "eigenvalues" enter the operator itself.

Part III contains applications to various problems in physics. The chapters in this part are relatively independent, except Chap. 7 which introduces the effective action, a notion used many times later on. We use the spectral geometry methods to reproduce a number of known QFT results which are derived usually with the help of Feynman diagrams. Among them are one-loop effective potential and beta functions in gauge theories. In Chap. 8 we turn to the quantum anomalies and calculate almost all known types of anomalies, including gravitational and parity anomalies, for two dimensional models. In Chap. 9 we consider the methods of calculations of the vacuum energy, with the quantum corrections to the kink mass being the principal example. Applications to string theory are contained in Chap. 10 where we derive the Born-Infeld action for open strings and come to noncommutativity of the coordinates of string endpoints. Chapter 11 is devoted to spectral geometry and field theory on noncommutative manifolds, which is studied by using the same universal tools.

Each chapter contains exercises. Some of them are included for pure pedagogical reasons, others are interesting as a complementary material. In any case, exercises are an integral part of this book. We encourage the reader at least to look at their formulations. Solutions to all exercises are given in Part IV.

Not to distract the attention of the reader we avoided references in the main text unless absolutely necessary. Instead, we added sections with literature remarks at the end of each chapter. Because of a vast volume of material we were not able to mention all relevant references. Instead we tried to give a starting point for a literature search.

Here we like to mention several general sources. For more mathematics we recommend the monographs by Gilkey [133, 134] and the one by Kirsten [169], which contains also an analysis of physical applications. The review paper [243] gives an overview of the heat kernel methods in QFT. There are numerous research monographs treating various aspects of applications of spectral functions to QFT and quantum gravity [18, 53, 100, 101, 103–105]. A recent elementary introduction into quantum physics in curved spaces including some of the heat kernel methods is [194]. A very detailed discussion of quantum anomalies may be found in the book by Bertlmann [35]. For a long time the zeta-function techniques were applied to calculation of the Casimir effect. Modern status of this area is described in [45, 187].

We benefited from numerous discussions with our friends and colleagues, and we are grateful to all of them. Special thanks go to J. Buchbinder and H. Grosse for a number of valuable remarks. This work was made possible due to generous support from various grant agencies. In particular, D.V.V. was supported by DFG while in Germany, and by FAPESP and CNPq while in Brazil.

http://www.springer.com/978-94-007-0204-2

Operators, Geometry and Quanta
Methods of Spectral Geometry in Quantum Field Theory
Fursaev, D.; Vassilevich, D.
2011, XVI, 288 p., Hardcover
ISBN: 978-94-007-0204-2