2

Coordinate systems

The practical description of dynamical systems involves a variety of coordinates systems. While the Cartesian coordinates discussed in section 2.1 are probably the most commonly used, many problems are more easily treated with special coordinate systems. The differential geometry of curves is studied in section 2.2 and leads to the concept of path coordinates, treated in section 2.3. Similarly, the differential geometry of surfaces is investigated in section 2.4 and leads to the concept of surface coordinates, treated in section 2.5. Finally, the differential geometry of three-dimensional maps is studied in section 2.6 and leads to orthogonal curvilinear coordinates developed in section 2.7.

2.1 Cartesian coordinates

The simplest way to represent the location of a point in three-dimensional space is to make use of a reference frame, $\mathcal{F} = [\mathbf{O}, \mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)]$, consisting of an orthonormal basis $\mathcal{I}$ with its origin and point **O**, as described in section 1.2.2. The time-dependent position vector of point **P** is represented by its *Cartesian coordinates*, $x_1(t)$, $x_2(t)$, and $x_3(t)$, resolved along unit vectors, $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$, respectively,

$$\underline{r}(t) = x_1(t)\bar{\imath}_1 + x_2(t)\bar{\imath}_2 + x_3(t)\bar{\imath}_3, \qquad (2.1)$$

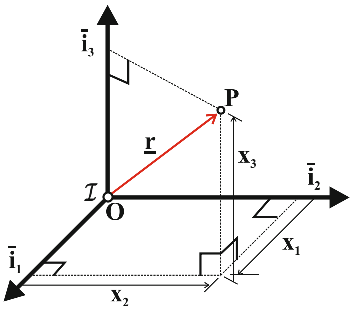

Fig. 2.1. Cartesian coordinate system.

where t denotes time. Figure 2.1 depicts the situation: Cartesian coordinate $x_1 = \bar{\imath}_1^T \underline{r}$ is the projection of the position vector of point **P** along unit vector $\bar{\imath}_1$. Similarly, Cartesian coordinates x_1 and x_2 are the projections of the same position vector along unit vectors $\bar{\imath}_2$ and $\bar{\imath}_3$, respectively.

The components of the velocity vector are readily obtained by differentiating the expression for the position vector, eq. (2.1), to find

O. A. Bauchau, *Flexible Multibody Dynamics,*
DOI 10.1007/978-94-007-0335-3_2

$$\underline{v}(t) = \dot{x}_1(t)\bar{\imath}_1 + \dot{x}_2(t)\bar{\imath}_2 + \dot{x}_3(t)\bar{\imath}_3 = v_1(t)\bar{\imath}_1 + v_2(t)\bar{\imath}_2 + v_3(t)\bar{\imath}_3. \quad (2.2)$$

The Cartesian components of the velocity vector are simply the time derivatives of the corresponding Cartesian components of the position vector: $v_1(t) = \dot{x}_1(t)$, $v_2(t) = \dot{x}_2(t)$, and $v_3(t) = \dot{x}_3(t)$.

Finally, the acceleration vector is obtained by taking a time derivative of the velocity vector to find

$$\underline{a}(t) = \ddot{x}_1(t)\bar{\imath}_1 + \ddot{x}_2(t)\bar{\imath}_2 + \ddot{x}_3(t)\bar{\imath}_3 = a_1(t)\bar{\imath}_1 + a_2(t)\bar{\imath}_2 + a_3(t)\bar{\imath}_3. \quad (2.3)$$

Here again, the Cartesian components of the acceleration vector are simply the derivatives of the corresponding Cartesian components of the velocity vector, or the second derivatives of the position components: $a_1(t) = \dot{v}_1(t) = \ddot{x}_1(t)$, $a_2(t) = \dot{v}_2(t) = \ddot{x}_2(t)$, and $a_3(t) = \dot{v}_3(t) = \ddot{x}_3(t)$.

Cartesian coordinates are simple to manipulate and are the most commonly used coordinate system in computational applications that deal with problems presenting arbitrary topologies. On the other hand, several other coordinate systems, such as those discussed in the rest of this chapter, are often used because they can ease the solution process for specific problems. In such cases, a specific coordinate system is used solve a specific problem. For instance, polar coordinates are very efficient to describe the behavior of a particle constrained to move along a circular path.

2.2 Differential geometry of a curve

This section investigates the differential geometry of a curve, leading to the concept of path coordinates. Both intrinsic and arbitrary parameterizations will be considered. Frenet's triad is defined and its derivatives evaluated.

2.2.1 Intrinsic parameterization

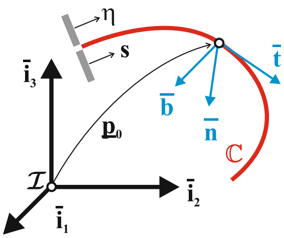

Fig. 2.2. Configuration of a curve in space.

Figure 2.2 depicts a curve, denoted $\mathbb{C}$, in three-dimensional space. A curve is the locus of the points generated by a single parameter, such that the position vector, $\underline{p}_0$, of such points can be written as

$$\underline{p}_0 = \underline{p}_0(s), \quad (2.4)$$

where s is the parameter that generates the curve. If parameter s is the *curvilinear coordinate* that measures length along the curve, it is said to define the *intrinsic parameterization* or *natural parameterization* of the curve.

Frenet's triad

A differential element of length, $\mathrm{d}s$, along the curve is written as $\mathrm{d}s^2 = \mathrm{d}\underline{p}_0^T \mathrm{d}\underline{p}_0$, and in follows that $(\mathrm{d}\underline{p}_0/\mathrm{d}s)^T \, (\mathrm{d}\underline{p}_0/\mathrm{d}s) = 1$. The *unit tangent vector* to the curve is defined as

$$\bar{t} = \frac{\mathrm{d}\underline{p}_0}{\mathrm{d}s}. \tag{2.5}$$

By construction, this is a unit vector because $\bar{t}^T\bar{t} = 1$.

Taking a derivative of this relationship with respect to the curvilinear coordinate leads to $\bar{t}^T\mathrm{d}\bar{t}/\mathrm{d}s = 0$. Vector $\mathrm{d}\bar{t}/\mathrm{d}s$ is normal to the tangent vector. The *unit normal vector* to the curve is defined as

$$\bar{n} = \rho\frac{\mathrm{d}\bar{t}}{\mathrm{d}s}, \tag{2.6}$$

where ρ is the *radius of curvature* of the curve, such that

$$\frac{1}{\rho} = \|\frac{\mathrm{d}\bar{t}}{\mathrm{d}s}\|. \tag{2.7}$$

The quantity $1/\rho$ is the *curvature* of the curve, and ρ its radius of curvature. The two unit vector, $\bar{t}$ and $\bar{n}$, are said to form the *osculating plane of the curve*.

An orthonormal triad is now constructed by defining the *binormal vector,* $\bar{b}$, as the cross product of the tangent by the normal vectors,

$$\bar{b} = \widetilde{t}\,\bar{n}. \tag{2.8}$$

The unit tangent, normal, and binormal vectors form an orthonormal triad, called *Frenet's triad*, depicted in fig. 2.2.

Derivatives of Frenet's triad

First, the derivative of the normal vector is resolved in Frenet's triad as $\mathrm{d}\bar{n}/\mathrm{d}s = \alpha\bar{t} + \beta\bar{n} + \gamma\bar{b}$, where α, β, and γ are unknown coefficients. Pre-multiplying this relationship by $\bar{n}^T$ yields $\beta = \bar{n}^T\mathrm{d}\bar{n}/\mathrm{d}s = 0$, because $\bar{n}$ is a unit vector. Pre-multiplying by $\bar{t}^T$ yields $\alpha = \bar{t}^T\mathrm{d}\bar{n}/\mathrm{d}s = -\bar{n}^T\mathrm{d}\bar{t}/\mathrm{d}s = -1/\rho$, where eq. (2.6) was used. Finally, pre-multiplying by $\bar{b}^T$ yields $\gamma = \bar{b}^T\mathrm{d}\bar{n}/\mathrm{d}s = 1/\tau$. Combining all these results yields

$$\frac{\mathrm{d}\bar{n}}{\mathrm{d}s} = -\frac{1}{\rho}\bar{t} + \frac{1}{\tau}\bar{b}, \tag{2.9}$$

where τ is the *radius of twist* of the curve, defined as

$$\frac{1}{\tau} = \bar{b}^T\frac{\mathrm{d}\bar{n}}{\mathrm{d}s}. \tag{2.10}$$

Next, the derivative of the binormal vector is resolved in Frenet's triad as $\mathrm{d}\bar{b}/\mathrm{d}s = \alpha\bar{t} + \beta\bar{n} + \gamma\bar{b}$, where α, β, and γ are unknown coefficients. Pre-multiplying this relationship by $\bar{b}^T$ yields $\gamma = \bar{b}^T\mathrm{d}\bar{b}/\mathrm{d}s = 0$, because $\bar{b}$ is a unit vector. Pre-multiplying by $\bar{t}^T$ yields $\alpha = \bar{t}^T\mathrm{d}\bar{b}/\mathrm{d}s = -\bar{b}^T\mathrm{d}\bar{t}/\mathrm{d}s = -\bar{b}^T\bar{n}/\rho = 0$. Finally, pre-multiplying by $\bar{n}^T$ yields $\beta = \bar{n}^T\mathrm{d}\bar{b}/\mathrm{d}s = -\bar{b}^T\mathrm{d}\bar{n}/\mathrm{d}s = -1/\tau$, where eq. (2.10) was used. Combining all these results yields

$$\frac{\mathrm{d}\bar{b}}{\mathrm{d}s} = -\frac{1}{\tau}\bar{n}. \tag{2.11}$$

It follows that the *twist of the curve* can also be written as

$$\frac{1}{\tau} = \|\frac{\mathrm{d}\bar{b}}{\mathrm{d}s}\|. \tag{2.12}$$

If the binormal vector has a constant direction at all points along the curve, $\mathrm{d}\bar{b}/\mathrm{d}s = 0$, and the curve entirely lies in the plane defined by vectors $\bar{t}$ and $\bar{n}$, *i.e.*, the osculating plane is the same at all points of the curve. The curve is then a *planar curve*, and eq. (2.12) implies that $1/\tau = 0$, *i.e.*, the twist of the curve vanishes.

The derivatives of Frenet's triad can be expressed in a compact manner by combining eqs. (2.6), (2.9), and (2.11),

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{\begin{matrix}\bar{t}\\ \bar{n}\\ \bar{b}\end{matrix}\right\} = \begin{bmatrix} 0 & 1/\rho & 0\\ -1/\rho & 0 & 1/\tau\\ 0 & -1/\tau & 0\end{bmatrix}\left\{\begin{matrix}\bar{t}\\ \bar{n}\\ \bar{b}\end{matrix}\right\}. \tag{2.13}$$

2.2.2 Arbitrary parameterization

The previous section has developed a representation of a curve based on its natural or intrinsic parameterization. In many instances, however, this parameterization is difficult to obtain; instead, the curve is defined in terms of a single parameter, η, that does not measure length along the curve, see fig. 2.2. The position vector of a point on the curve is now $\underline{p}_0 = \underline{p}_0(\eta)$. The derivatives of the position vector with respect to parameter η will be denoted as

$$\underline{p}_1 = \frac{\mathrm{d}\underline{p}_0}{\mathrm{d}\eta}, \quad \underline{p}_2 = \frac{\mathrm{d}^2\underline{p}_0}{\mathrm{d}\eta^2}, \quad \underline{p}_3 = \frac{\mathrm{d}^3\underline{p}_0}{\mathrm{d}\eta^3}, \quad \underline{p}_4 = \frac{\mathrm{d}^4\underline{p}_0}{\mathrm{d}\eta^4}.$$

A similar notation will be used for the tangent and normal vectors,

$$\bar{t}_i = \frac{\mathrm{d}^i\bar{t}}{\mathrm{d}\eta^i}, \quad \bar{n}_i = \frac{\mathrm{d}^i\bar{n}}{\mathrm{d}\eta^i}.$$

The differential element of length along the curve can be written as $\mathrm{d}s^2 = (\mathrm{d}\underline{p}_0/\mathrm{d}\eta)^T(\mathrm{d}\underline{p}_0/\mathrm{d}\eta)\ \mathrm{d}\eta^2$. The ratio of the increment in length along the curve, $\mathrm{d}s$, to the increment in parameter value, $\mathrm{d}\eta$, is then

$$\frac{\mathrm{d}s}{\mathrm{d}\eta} = \sqrt{\underline{p}_1^T\underline{p}_1} = p_1. \tag{2.14}$$

Notation $(\cdot)'$ will be used to indicate a derivative with respect to η, and hence, $\mathrm{d}/\mathrm{d}s = (\cdot)'/p_1$. The unit tangent vector to the curve is evaluated with the help of eq. (2.5) as

$$\bar{t} = \frac{\underline{p}_1}{p_1} \tag{2.15}$$

Next, the derivative of the tangent vector is found as

$$\bar{t}_1 = \frac{p_1\, \underline{p}_2 - \underline{p}_1\, (\underline{p}_1^T \underline{p}_2)/p_1}{p_1^2} = \frac{1}{p_1}(1 - \bar{t}\,\bar{t}^T)\underline{p}_2 = \frac{1}{p_1}\left[\underline{p}_2 - (\bar{t}^T\underline{p}_2)\bar{t}\right]. \qquad (2.16)$$

From eq. (2.7), the radius of curvature now becomes

$$\frac{1}{\rho} = \|\frac{\mathrm{d}\bar{t}}{\mathrm{d}s}\| = \frac{1}{p_1}\|\bar{t}_1\|.$$

It follows that $\|\bar{t}_1\| = t_1 = p_1/\rho$. For a straight line, the tangent vector has a fixed direction in space, $\bar{t}_1 = 0$. It follows that for a straight line $1/\rho = 0$, *i.e.*, its radius of curvature is infinite. The curve's curvature is found to be

$$\frac{1}{\rho} = \frac{\sqrt{p_1^2 p_2^2 - (\underline{p}_2^T \underline{p}_1)^2}}{p_1^3} \qquad (2.17)$$

Higher-order derivatives of the tangent vector are found in a similar manner

$$\bar{t}_2 = \frac{1}{p_1}\left[\underline{p}_3 - (\bar{t}^T\underline{p}_3 + \bar{t}_1^T\underline{p}_2)\bar{t} - 2(\bar{t}^T\underline{p}_2)\bar{t}_1\right],$$

and

$$\bar{t}_3 = \frac{1}{p_1}\left[\underline{p}_4 - (\bar{t}^T\underline{p}_4 + 2\bar{t}_1^T\underline{p}_3 + \bar{t}_2^T\underline{p}_2)\bar{t} - 3(\bar{t}^T\underline{p}_3 + \bar{t}_1^T\underline{p}_2)\bar{t}_1 - 3(\bar{t}^T\underline{p}_2)\bar{t}_2\right].$$

Next, the normal vector defined in eq. (2.6) becomes

$$\bar{n} = \frac{\bar{t}_1}{\|\bar{t}_1\|} = \frac{1}{t_1}\bar{t}_1. \qquad (2.18)$$

For a straight line, $\bar{t}_1 = 0$, and hence, the normal vector is not defined. In fact, any vector normal to a straight line is a normal vector. The derivative of the normal vector with respect to η then follows as

$$\bar{n}_1 = \frac{1}{t_1}\left[\bar{t}_2 - (\bar{n}^T\bar{t}_2)\bar{n}\right]. \qquad (2.19)$$

The second-order derivative is then

$$\bar{n}_2 = \frac{1}{t_1}\left[\bar{t}_3 - (\bar{n}^T\bar{t}_3 + \bar{n}_1^T\bar{t}_2)\bar{n} - 2(\bar{n}^T\bar{t}_2)\bar{n}_1\right]. \qquad (2.20)$$

The binormal vector is readily expressed as

$$\bar{b} = \widetilde{t}\,\bar{n} = \frac{1}{t_1}\widetilde{t}\,\bar{t}_1 = \frac{\rho}{p_1^3}\widetilde{p}_1\,\underline{p}_2. \qquad (2.21)$$

Because the normal vector is not defined for a straight line, the binormal vector is not defined in that case. In fact, any vector normal to a straight line is a binormal vector.

The derivative of the binormal vector becomes

$$\bar{b}_1 = (\frac{\rho}{p_1^3})' \widetilde{p}_1 \underline{p}_2 + \frac{\rho}{p_1^3} \widetilde{p}_1 \underline{p}_3. \tag{2.22}$$

Using eq. (2.10), the twist of the curve is found to be

$$\frac{1}{\tau} = -\frac{1}{p_1} \bar{n}^T \bar{b}_1 = -\frac{\rho}{p_1^5} \left[p_1^2 \underline{p}_2^T - (\underline{p}_1^T \underline{p}_2) \underline{p}_1^T \right] \bar{b}_1.$$

Finally, introducing eq. (2.22) leads to

$$\frac{1}{\tau} = -\frac{\rho^2}{p_1^6} \, \underline{p}_2^T \widetilde{p}_1 \underline{p}_3. \tag{2.23}$$

The twist of the curve is closely related to the volume defined by vectors $\underline{p}_1$, $\underline{p}_2$, and $\underline{p}_3$. Note that a straight line has a vanishing twist, $1/\tau = 0$.

Derivatives of the binormal vector are more easily expressed as $\bar{b}_1 = \widetilde{t}_1 \bar{n} + \widetilde{t} \bar{n}_1 = \widetilde{t} \bar{n}_1$, and $\bar{b}_2 = \widetilde{t}_1 \bar{n}_1 + \widetilde{t} \bar{n}_2 = \widetilde{n} \bar{t}_2 + \widetilde{t} \bar{n}_2$, where eqs. (2.18) and (2.19) were used.

Example 2.1. The helix

Figure 2.3 depicts a helix, which is a three-dimensional curve defined by the following position vector

$$\underline{p}_0(\eta) = a \cos\eta \, \bar{\imath}_1 + a \sin\eta \, \bar{\imath}_2 + k\eta \, \bar{\imath}_3, \tag{2.24}$$

where a and k are two parameters defining the shape of the curve. The derivatives of the position vector are $\underline{p}_1 = -a \sin\eta \, \bar{\imath}_1 + a \cos\eta \, \bar{\imath}_2 + k \, \bar{\imath}_3$, $\underline{p}_2 = -a \cos\eta \, \bar{\imath}_1 - a \sin\eta \, \bar{\imath}_2$, and $\underline{p}_3 = a \sin\eta \, \bar{\imath}_1 - a \cos\eta \, \bar{\imath}_2$. The curvature and twist of the helix are found with the help of eqs. (2.17) and (2.23), respectively, as

$$\frac{1}{\rho} = \frac{a}{a^2 + k^2}, \quad \frac{1}{\tau} = \frac{k}{a^2 + k^2}.$$

Note that both curvature and twist are constant along the helix. The unit tangent vector is evaluated with the help of eq. (2.15) as

$$\bar{t} = \frac{1}{\sqrt{a^2 + k^2}} \underline{p}_1 = \frac{1}{\sqrt{a^2 + k^2}} (-a \sin\eta \bar{\imath}_1 + a \cos\eta \bar{\imath}_2 + k \bar{\imath}_3). \tag{2.25}$$

The ratio between an increment in length along the curve and the increment in the parameter value is then $\mathrm{d}s = \sqrt{a^2 + k^2} \, \mathrm{d}\eta$, see eq. (2.14). Next, the derivative of the tangent vector is computed with the help of eq. (2.16) as $\bar{t}_1 = \underline{p}_2 / p_1$ and the normal vector then follows as

$$\bar{n} = -\cos\eta \, \bar{\imath}_1 - \sin\eta \, \bar{\imath}_2.$$

Finally, the binormal vector found from eq. (2.21)

$$\bar{b} = \frac{1}{\sqrt{a^2 + k^2}} \left[k \sin\eta \, \bar{\imath}_1 - k \cos\eta \, \bar{\imath}_2 + a \, \bar{\imath}_3 \right].$$

The derivatives of Frenet's triad are found with the help of eq. (2.13) as

$$\frac{\mathrm{d}\bar{t}}{\mathrm{d}s} = \frac{a}{a^2 + k^2} \bar{n}, \quad \frac{\mathrm{d}\bar{n}}{\mathrm{d}s} = -\frac{a}{a^2 + k^2} \bar{t} + \frac{k}{a^2 + k^2} \bar{b}, \quad \frac{\mathrm{d}\bar{b}}{\mathrm{d}s} = -\frac{k}{a^2 + k^2} \bar{n}.$$

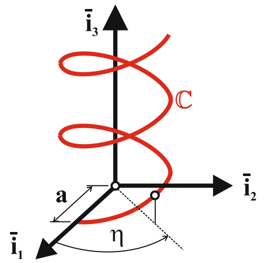

Fig. 2.3. Configuration of a helix in three-dimensional space.

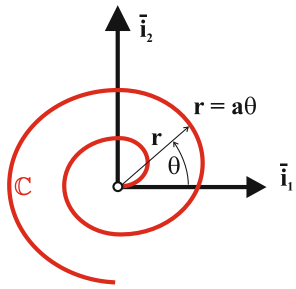

Fig. 2.4. Configuration of a planar linear spiral.

Example 2.2. The linear spiral
Figure 2.4 depicts a linear spiral, which is a planar curve defined by the following position vector

$$\underline{p}_0 = a\theta \cos\theta \,\bar{\imath}_1 + a\theta \sin\theta \,\bar{\imath}_2, \tag{2.26}$$

where a is a parameter defining the shape of the curve. The derivatives of the position vector are $\underline{p}_1 = a\left[(\cos\theta - \theta\sin\theta)\bar{\imath}_1 + (\sin\theta + \theta\cos\theta)\bar{\imath}_2\right]$, $\underline{p}_2 = a\left[-(2\sin\theta + \theta\cos\theta)\bar{\imath}_1 + (2\cos\theta - \theta\sin\theta)\bar{\imath}_2\right]$. It is readily verified that $p_1^2 = a^2(1+\theta^2)$, $p_2^2 = a^2(4+\theta^2)$ and $\underline{p}_1^T\underline{p}_2 = a^2\theta$. The curvature of the linear spiral is found with the help of eq. (2.17)

$$\frac{a}{\rho} = \frac{2+\theta^2}{(1+\theta^2)^{3/2}}.$$

Note that the curvature varies along the spiral. Of course, the twist is zero since the curve is planar. The unit tangent vector is evaluated with the help of eq. (2.15) as

$$\bar{t} = \frac{(\cos\theta - \theta\sin\theta)\bar{\imath}_1 + (\sin\theta + \theta\cos\theta)\bar{\imath}_2}{\sqrt{1+\theta^2}}.$$

Finally, the normal vector becomes

$$\bar{n} = \frac{-\left[2\sin\theta + \theta\cos\theta(2+\theta^2)\right]\bar{\imath}_1 + \left[2\cos\theta - \theta\sin\theta(2+\theta^2)\right]\bar{\imath}_2}{\sqrt{4+\theta^2(2+\theta^2)^2}}.$$

Example 2.3. Using polar coordinates to represent curves
Cams play an important role in numerous mechanical systems: cam-follower pairs typically transform the rotary motion of the cam into a desirable motion of the follower. Figure 2.5 depicts a typical cam whose outer shape is defined by a curve. It is convenient to define this curve using the polar coordinate system indicated on the figure: for each angle α, the distance from point **O** to point **P** is denoted r. The complete curve is then defined by function $r = r(\alpha)$; angle α provides an arbitrary parameterization of the curve. If $r(\alpha)$ is a periodic function of angle α, the curve will be a closed curve, as expected for a cam.

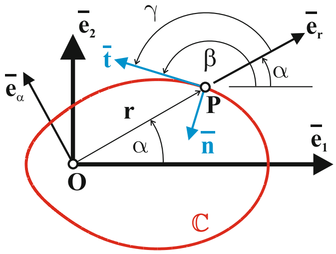

Fig. 2.5. Configuration of a cam.

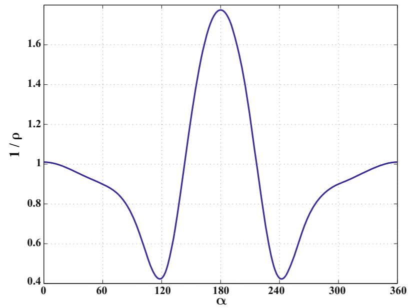

Fig. 2.6. Curvature distribution for the cam.

Vectors $\underline{p}_0$, $\underline{p}_1$, and $\underline{p}_2$ now become

$$\underline{p}_0 = rC_\alpha\,\bar{e}_1 + rS_\alpha\,\bar{e}_2, \tag{2.27a}$$

$$\underline{p}_1 = (r'C_\alpha - rS_\alpha)\,\bar{e}_1 + (r'S_\alpha + rC_\alpha)\,\bar{e}_2, \tag{2.27b}$$

$$\underline{p}_2 = (r''C_\alpha - 2r'S_\alpha - rC_\alpha)\,\bar{e}_1 + (r''S_\alpha + 2r'C_\alpha - rS_\alpha)\,\bar{e}_2, \tag{2.27c}$$

where the notation $(\cdot)'$ indicates a derivative with respect to α, $S_\alpha = \sin\alpha$, and $C_\alpha = \cos\alpha$. It then follows that $p_1^2 = r^2 + r'^2$ and $p_2^2 = (r'' - r)^2 + 4r'^2$. The various properties of the curve can then be evaluated; for instance, eqs. (2.15) and (2.17) yield the tangent vector and curvature along the curve, respectively.

The curve depicted in fig. 2.5 is defined by the following equation, $r(\alpha) = 1.0 + 0.5\cos\alpha + 0.15\cos 2\alpha$ and fig. 2.6 shows the curvature distribution as a function of angle α.

Figure 2.5 shows the unit tangent vector, $\bar{t}$, at point **P** of the curve and defines angles $\beta = (\bar{e}_1, \bar{t})$ and $\gamma = (\bar{e}_r, \bar{t})$; note that $\gamma = \beta - \alpha$. The unit tangent vector can now be written as $\bar{t} = C_\beta \bar{e}_1 + S_\beta \bar{e}_2 = \underline{p}_1/p_1$, where the second equality follows from eq. (2.15). Pre-multiplying this relationship by $\bar{e}_1^T$ and $\bar{e}_2^T$ yields $p_1 C_\beta = r'C_\alpha - rS_\alpha$ and $p_1 S_\beta = r'S_\alpha + rC_\alpha$, respectively. Solving these two equations for r and r' and using elementary trigonometric identities then leads to

$$r = p_1 \sin(\beta - \alpha) = p_1 S_\gamma, \tag{2.28a}$$

$$r' = p_1 \cos(\beta - \alpha) = p_1 C_\gamma, \tag{2.28b}$$

where $S_\gamma = \sin\gamma$, and $C_\gamma = \cos\gamma$. The quotient of these two equations then yields the following relationship

$$\mathrm{d}\alpha = \tan\gamma \frac{\mathrm{d}r}{r}. \tag{2.29}$$

The derivative of the unit tangent vector with respect to the curvilinear coordinate along the curve is $\mathrm{d}\bar{t}/\mathrm{d}s = (-S_\gamma \bar{e}_1 + C_\gamma \bar{e}_2)\mathrm{d}\beta/\mathrm{d}s$, and the curvature is then $1/\rho = |\mathrm{d}\beta/\mathrm{d}s|$. If the curve is convex, which is generally the case for cams, angle β is a monotonically increasing function of s, and hence, $1/\rho = \mathrm{d}\beta/\mathrm{d}s$. The chain rule for derivatives implies $\mathrm{d}\beta = (1/\rho)(\mathrm{d}s/\mathrm{d}\alpha)(\mathrm{d}\alpha/\mathrm{d}r)\mathrm{d}r$ and introducing eqs. (2.14), (2.28a), and (2.29) then yields

$$\mathrm{d}\beta = \frac{\mathrm{d}r}{\rho C_\gamma}. \tag{2.30}$$

It is left to the reader to verify that eq. (2.30) yields an alternative, simplified expression for the curvature of the cam

$$\frac{1}{\rho} = \frac{2r'^2 - rr'' + r^2}{p_1^3}. \tag{2.31}$$

Finally, an increment in angle γ can be expressed as $\mathrm{d}\gamma = \mathrm{d}\beta - \mathrm{d}\alpha$ and introducing eqs. (2.30) and (2.29) yields

$$\mathrm{d}\gamma = \left(\frac{1}{\rho C_\gamma} - \frac{\tan\gamma}{r}\right)\mathrm{d}r. \tag{2.32}$$

2.3 Path coordinates

Consider a particle moving along a curve such that its position, $s(t)$, is a given function of time. The velocity vector, $\underline{v}$, of the particle is then

$$\underline{v} = \frac{\mathrm{d}\underline{p}_0}{\mathrm{d}t} = \frac{\mathrm{d}\underline{p}_0}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = v\bar{t}, \tag{2.33}$$

where $v = \mathrm{d}s/\mathrm{d}t$ is the *speed of the particle*, Clearly, the velocity vector of the particle is along the tangent to the curve.

Next, the particle acceleration vector, $\underline{a}$, becomes

$$\underline{a} = \frac{\mathrm{d}\underline{v}}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}t}\bar{t} + v\frac{\mathrm{d}\bar{t}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \dot{v}\bar{t} + \frac{v^2}{\rho}\bar{n}. \tag{2.34}$$

The acceleration vector is contained in the osculating plane, and can be written as $\underline{a} = a_t\bar{t} + a_n\bar{n}$, where a_t and a_n are the tangential and normal components of acceleration, respectively. The tangential component of acceleration, $a_t = \dot{v}$, simply measures the change in particle speed. The normal component, $a_n = v^2/\rho$, is always directed towards the center of curvature since v^2/ρ is a positive number. This normal acceleration is clearly related to the curvature of the path; in fact, when the path is a straight line, $1/\rho = 0$, and the normal acceleration vanishes.

2.3.1 Problems

Problem 2.1. Prove identity
Prove that $1/\rho = p_2/p_1^2 \,|\sin\alpha|$, where $p_2 = \|\underline{p}_2\|$ and α is the angle between vectors $\underline{p}_1$ and $\underline{p}_2$.

Problem 2.2. Study of a curve
Consider the following spatial curve: $\underline{p}_0 = a(\eta + \sin\eta)\bar{\imath}_1 + a(1+\cos\eta)\bar{\imath}_2 + a(1-\cos\eta)\bar{\imath}_3$, where $a > 0$ is a given parameter. *(1)* Find the tangent, normal, and binormal vectors for this curve. *(2)* Determine the curvature, radius of curvature, and twist of the curve. Is this a planar curve? Is the tangent vector defined at all points of the curve?

Problem 2.3. Study of a curve
Consider the following spatial curve: $\underline{p}_0 = \rho(\cos\alpha\eta)(\cos\eta)\bar{\imath}_1 + \rho(\cos\alpha\eta)(\sin\eta)\bar{\imath}_2 + \rho(\sin\alpha\eta)\bar{\imath}_3$, where $\rho > 0$ and α are given parameters. *(1)* Find the tangent, normal, and binormal vectors for this curve. *(2)* Determine the curvature, radius of curvature, and twist of the curve.

Problem 2.4. Short questions
(1) A particle of mass m is sliding along a planar curve. Find the component of the particle's acceleration vector along the binormal vector of Frenet's triad. *(2)* A particle of mass m is sliding along a three-dimensional curve. Find the component of the particle's acceleration vector along the binormal vector of Frenet's triad. *(3)* State the criterion used to ascertain whether a curve is planar or three-dimensional.

Problem 2.5. Study of a curve defined in polar coordinates
The outer surface of a cam is specified by the following curve defined in polar coordinates, $r(\alpha) = 1.0 - 0.5\cos\alpha + 0.18\cos 2\alpha$. *(1)* Plot the curve. *(2)* Plot the curvature distribution for $\alpha \in [0, 2\pi]$.

2.4 Differential geometry of a surface

This section investigates the differential geometry of surfaces, leading to the concept of surface coordinates. The differential geometry of surfaces is more complex than that of curves. The first and second metric tensors of surfaces are introduced first, and the analysis of the curvature of surfaces leads to the concept of lines of curvatures and associated principal radii of curvature. Finally, the base vectors and their derivatives are evaluated, leading to Gauss' and Weingarten's formulæ.

2.4.1 The first metric tensor of a surface

Figure 2.7 depicts a surface, denoted $\mathbb{S}$, in three-dimensional space. A surface is the locus of the points generated by two parameters, η_1 and η_2, such that the position vector, $\underline{p}_0$, of such points can be written as

$$\underline{p}_0 = \underline{p}_0(\eta_1, \eta_2). \tag{2.35}$$

If η_2 is kept constant, $\eta_2 = c_2$, $\underline{p}_0 = \underline{p}_0(\eta_1, c_2)$ defines a curve embedded into the surface; such curve is called an "η_1 curve." Figure 2.7 shows a grid of such curves for various values of c_2. Similarly, "η_2 curves" can be defined, corresponding to $\underline{p}_0 = \underline{p}_0(c_1, \eta_2)$; a grid of η_2 curves obtained for different constant c_1 is also shown on the figure. In general, parameters η_1 and η_2 do not measure length along these embedded curves, and hence, they do not define intrinsic parameterizations of the curves.

The *surface base vectors* are defined as follows

$$\underline{a}_1 = \frac{\partial \underline{p}_0}{\partial \eta_1}, \quad \underline{a}_2 = \frac{\partial \underline{p}_0}{\partial \eta_2}, \tag{2.36}$$

and are shown in fig. 2.7. Clearly, vectors $\underline{a}_1$ and $\underline{a}_2$ are tangent to the η_1 and η_2 curves that intersect at point **P**, respectively.

Consequently, they lie in the plane tangent to the surface at this point. Since η_1 and η_2 do not form an intrinsic parameterization, vectors $\underline{a}_1$ and $\underline{a}_2$ are not unit tangent vectors. Furthermore, these two vectors are not, in general, orthogonal to each other.

The *first metric tensor of the surface,* $\underline{\underline{A}}$, is defined as

$$\underline{\underline{A}} = \begin{bmatrix} \underline{a}_1^T \underline{a}_1 & \underline{a}_1^T \underline{a}_2 \\ \underline{a}_2^T \underline{a}_1 & \underline{a}_2^T \underline{a}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}, \tag{2.37}$$

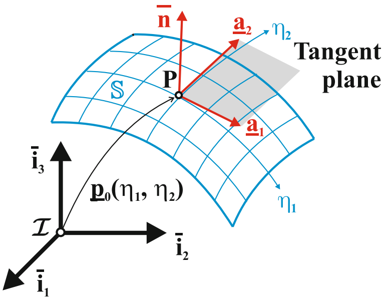

Fig. 2.7. The base vectors of a surface.

and its determinant is denoted $a = \det(\underline{\underline{A}})$. A differential element of length on the surface is found as

$$\mathrm{d}s^2 = \mathrm{d}\underline{p}_0^T \mathrm{d}\underline{p}_0 = \left(\underline{a}_1^T \mathrm{d}\eta_1 + \underline{a}_2^T \mathrm{d}\eta_2\right)\left(\underline{a}_1 \mathrm{d}\eta_1 + \underline{a}_2 \mathrm{d}\eta_2\right) = \mathrm{d}\underline{\eta}^T \underline{\underline{A}}\, \mathrm{d}\underline{\eta}. \tag{2.38}$$

where $\mathrm{d}\underline{\eta}^T = \left\{\mathrm{d}\eta_1, \mathrm{d}\eta_2\right\}$. Clearly, the first metric tensor is closely related to length measurements on the surface.

Because the base vectors define the plane tangent to the surface, the unit vector, $\bar{n}$, normal to the surface is readily found as

$$\bar{n} = \frac{\widetilde{a}_1 \underline{a}_2}{\|\widetilde{a}_1 \underline{a}_2\|} = \frac{\widetilde{a}_1 \underline{a}_2}{\sqrt{a}}. \tag{2.39}$$

The area of a differential element of the surface then becomes

$$\mathrm{d}a = \|\widetilde{a}_1 \underline{a}_2\, \mathrm{d}\eta_1 \mathrm{d}\eta_2\| = \|\widetilde{a}_1 \underline{a}_2\|\, \mathrm{d}\eta_1 \mathrm{d}\eta_2 = \sqrt{a}\, \mathrm{d}\eta_1 \mathrm{d}\eta_2. \tag{2.40}$$

2.4.2 Curve on a surface

Figure 2.8 depicts a curve, $\mathbb{C}$, entirely contained within surface $\mathbb{S}$. Let the curve be defined by its intrinsic parameter, s, the curvilinear variable along curve $\mathbb{C}$. The tangent vector, $\bar{t}$, to curve $\mathbb{C}$ is defined by eq. (2.5). This unit tangent vector clearly lies in the plane tangent to $\mathbb{S}$, and hence, it can be resolved along the base vectors, $\bar{t} = \lambda_1 \underline{a}_1 + \lambda_2 \underline{a}_2$.

Because $\bar{t}$ is a unit vector, it follows that

$$\bar{t}^T \bar{t} = \underline{\lambda}^T \, \underline{\underline{A}} \, \underline{\lambda} = 1, \tag{2.41}$$

where $\underline{\lambda}^T = \{\lambda_1, \lambda_2\}$. On the other hand, eq. (2.38) can be recast as

$$\frac{\mathrm{d}\underline{\eta}^T}{\mathrm{d}s} \underline{\underline{A}} \frac{\mathrm{d}\underline{\eta}}{\mathrm{d}s} = 1. \tag{2.42}$$

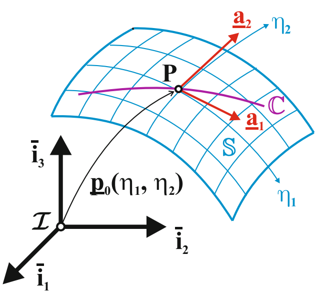

Fig. 2.8. A curve, $\mathbb{C}$, entirely contained within surface, $\mathbb{S}$

Because eqs. (2.41) and (2.42) must be identical for all curves on the surface,

$$\underline{\lambda} = \frac{\mathrm{d}\underline{\eta}}{\mathrm{d}s}. \tag{2.43}$$

This result is expected since $\mathrm{d}s$ is an increment of length along $\mathbb{C}$, and $\bar{t}$ is tangent to $\mathbb{C}$. Angles $\theta_1 = (\bar{t}, \underline{a}_1)$ and $\theta_2 = (\bar{t}, \underline{a}_2)$ can be obtained by expanding the dot products $\bar{t}^T \underline{a}_1$ and $\bar{t}^T \underline{a}_2$, respectively, to find

$$\left\{ \begin{matrix} \sqrt{a_{11}} \cos\theta_1 \\ \sqrt{a_{22}} \cos\theta_2 \end{matrix} \right\} = \underline{\underline{A}} \, \underline{\lambda}. \tag{2.44}$$

2.4.3 The second metric tensor of a surface

Consider once again a curve, $\mathbb{C}$, entirely contained within surface $\mathbb{S}$, as depicted in fig. 2.8. The unit tangent vector clearly lies in the plane tangent to the surface, but the curvature vector $\mathrm{d}\bar{t}/\mathrm{d}s$ will have components in and out of this tangent plane,

$$\frac{\mathrm{d}\bar{t}}{\mathrm{d}s} = \kappa_n \bar{n} + \kappa_g \bar{\rho}, \tag{2.45}$$

where κ_n is the *normal curvature*, κ_g the *geodesic curvature*, and $\bar{\rho}$ a unit vector belonging to the plane tangent to $\mathbb{S}$. The normal curvature can be evaluated as

$$\kappa_n = \bar{n}^T \frac{\mathrm{d}\bar{t}}{\mathrm{d}s} = -\bar{t}^T \frac{\mathrm{d}\bar{n}}{\mathrm{d}s} = -\frac{\mathrm{d}\underline{p}_0^T \mathrm{d}\bar{n}}{\mathrm{d}s^2}, \tag{2.46}$$

where the normality condition, $\bar{t}^T \bar{n} = 0$, was used. The numerator can be written as

$$\begin{aligned} -\mathrm{d}\underline{p}_0^T \mathrm{d}\bar{n} &= -\left(\underline{a}_1^T \mathrm{d}\eta_1 + \underline{a}_2^T \mathrm{d}\eta_2\right) \left(\frac{\partial \bar{n}}{\partial \eta_1} \mathrm{d}\eta_1 + \frac{\partial \bar{n}}{\partial \eta_2} \mathrm{d}\eta_2 \right) \\ &= -\left[\underline{a}_1^T \frac{\partial \bar{n}}{\partial \eta_1} \mathrm{d}\eta_1^2 + \underline{a}_2^T \frac{\partial \bar{n}}{\partial \eta_2} \mathrm{d}\eta_2^2 + \left(\underline{a}_1^T \frac{\partial \bar{n}}{\partial \eta_2} + \underline{a}_2^T \frac{\partial \bar{n}}{\partial \eta_1} \right) \mathrm{d}\eta_1 \mathrm{d}\eta_2 \right], \\ &= \left[\bar{n}^T \frac{\partial \underline{a}_1}{\partial \eta_1} \mathrm{d}\eta_1^2 + \bar{n}^T \frac{\partial \underline{a}_2}{\partial \eta_2} \mathrm{d}\eta_2^2 + \left(\bar{n}^T \frac{\partial \underline{a}_1}{\partial \eta_2} + \bar{n}^T \frac{\partial \underline{a}_2}{\partial \eta_1} \right) \mathrm{d}\eta_1 \mathrm{d}\eta_2 \right], \end{aligned}$$

where the orthogonality conditions, $\bar{n}^T \underline{a}_1 = 0$ and $\bar{n}^T \underline{a}_2 = 0$, were used to obtain the last equality.

The *second metric tensor of the surface* is defined as

$$\underline{\underline{B}} = \begin{bmatrix} \bar{n}^T \dfrac{\partial \underline{a}_1}{\partial \eta_1} & \bar{n}^T \dfrac{\partial \underline{a}_1}{\partial \eta_2} \\ \bar{n}^T \dfrac{\partial \underline{a}_2}{\partial \eta_1} & \bar{n}^T \dfrac{\partial \underline{a}_2}{\partial \eta_2} \end{bmatrix} = \begin{bmatrix} \bar{n}^T \dfrac{\partial^2 \underline{p}_0}{\partial \eta_1^2} & \bar{n}^T \dfrac{\partial^2 \underline{p}_0}{\partial \eta_1 \partial \eta_2} \\ \bar{n}^T \dfrac{\partial^2 \underline{p}_0}{\partial \eta_1 \partial \eta_2} & \bar{n}^T \dfrac{\partial^2 \underline{p}_0}{\partial \eta_2^2} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}, \tag{2.47}$$

and its determinant is denoted $b = \det(\underline{\underline{B}})$. The second equality shows that the second metric tensor is a symmetric tensor. It follows that $-\mathrm{d}\underline{p}_0^T \mathrm{d}\bar{n} = \mathrm{d}\underline{\eta}^T \underline{\underline{B}} \mathrm{d}\underline{\eta}$, and the normal curvature, eq. (2.46), becomes

$$\kappa_n = \frac{\mathrm{d}\underline{\eta}^T \underline{\underline{B}} \mathrm{d}\underline{\eta}}{\mathrm{d}s^2} = \frac{\mathrm{d}\underline{\eta}^T}{\mathrm{d}s} \underline{\underline{B}} \frac{\mathrm{d}\underline{\eta}}{\mathrm{d}s} = \underline{\lambda}^T \underline{\underline{B}}\, \underline{\lambda}. \tag{2.48}$$

2.4.4 Analysis of curvatures

Figure 2.9 shows a plane, $\mathcal{P}$, containing the normal, $\bar{n}$, to surface $\mathbb{S}$. Let curve $\mathbb{C}_n$ be at the intersection of plane $\mathcal{P}$ and surface $\mathbb{S}$. Because curve $\mathbb{C}_n$ is a planar curve, its curvature vector is in plane $\mathcal{P}$.

Next, let plane $\mathcal{P}$ rotate about $\bar{n}$. For each new orientation of the plane, a new curve, $\mathbb{C}_n$, is generated with its own normal curvature κ_n. The following problem will be investigated: what is the orientation of plane $\mathcal{P}$ that maximizes the normal curvature κ_n? In mathematical terms, the maximum value of $\kappa_n = \underline{\lambda}^T \underline{\underline{B}}\, \underline{\lambda}$ is sought, under the normality constraint, $\underline{\lambda}^T \underline{\underline{A}}\, \underline{\lambda} = 1$.

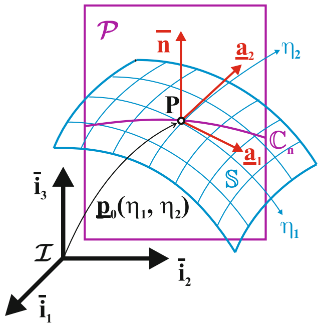

Fig. 2.9. Intersection of surface, $\mathbb{S}$, with plane, $\mathcal{P}$, that contains the normal to the surface.

This constrained maximization problem will be solved with the help of Lagrange's multiplier technique

$$\max_{\underline{\lambda},\mu} \left[\underline{\lambda}^T \underline{\underline{B}}\, \underline{\lambda} - \mu(\underline{\lambda}^T \underline{\underline{A}}\, \underline{\lambda} - 1) \right],$$

where μ is the Lagrange multiplier used to enforce the constraint. The solution of this problem implies $(\underline{\underline{B}} - \mu \underline{\underline{A}})\underline{\lambda} = 0$, and the normality condition $\underline{\lambda}^T \underline{\underline{A}}\, \underline{\lambda} = 1$. Pre-multiplying this equation by $\underline{\lambda}^T$ yields the physical interpretation of the Lagrange multiplier: $\underline{\lambda}^T \underline{\underline{B}}\, \underline{\lambda} - \mu\, \underline{\lambda}^T \underline{\underline{A}}\, \underline{\lambda} = 0$ or, in view of the normality constraint, $\mu = \underline{\lambda}^T \underline{\underline{B}}\, \underline{\lambda} = \kappa_n$, Hence, Lagrange's multiplier can be interpreted as the normal curvature itself.

The condition for maximum normal curvature can now be written as $(\underline{\underline{B}} - \kappa_n \underline{\underline{A}})\underline{\lambda} = 0$. This set of homogeneous algebraic equations admits the trivial solution $\underline{\lambda} = 0$, but this solution violates the normality constraint. Non-trivial solutions correspond to the eigenpairs of the generalized eigenproblem $\underline{\underline{B}}\, \underline{\lambda} = \kappa_n \underline{\underline{A}}\, \underline{\lambda}$. Because $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are symmetric and $\underline{\underline{A}}$ is positive-definite, the eigenvalues are always real, and mutually orthogonal eigenvectors can be constructed.

The eigenvalues are the solution of the quadratic equation $\det(\underline{\underline{B}} - \kappa_n \underline{\underline{A}}) = 0$, or

$$\kappa_n^2 - 2\kappa_m \kappa_n + \frac{b}{a} = 0, \tag{2.49}$$

where $\kappa_m = (a_{11}b_{22} + a_{22}b_{11} - 2a_{12}b_{12})/2a$. The solutions of this quadratic equation are called the *principal curvatures*

$$\kappa_n^I, \kappa_n^{II} = \kappa_m \pm \sqrt{\kappa_m^2 - b/a}. \tag{2.50}$$

The *mean curvature* is defined as

$$\kappa_m = \frac{\kappa_n^I + \kappa_n^{II}}{2} = \frac{a_{11}b_{22} + a_{22}b_{11} - 2a_{12}b_{12}}{2a}, \tag{2.51}$$

and the *Gaussian curvature* as

$$\kappa_n^I \kappa_n^{II} = \frac{b}{a}. \tag{2.52}$$

When $b/a > 0$, the principal curvatures have the same sign, corresponding to a convex shape; when $b/a < 0$, the principal curvatures are of opposite sign, corresponding to a saddle shape; finally, when $b/a = 0$, one of the principal curvatures is zero, the surface $\mathbb{S}$ has zero curvature in one of the principal curvature directions.

2.4.5 Lines of curvature

A *line of curvature* of a surface is defined as a curve whose tangent vector always points along the principal curvature directions of the surface. Consider now a set of coordinates, η_1 and η_2, such that $a_{12} = b_{12} = 0$. It follows that $a = a_{11}a_{22}$, $b = b_{11}b_{22}$ and $\kappa_m = (b_{11}/a_{11} + b_{22}/a_{22})/2$. The principal curvatures then simply become

$$\kappa_n^I = \frac{b_{11}}{a_{11}}, \quad \kappa_n^{II} = \frac{b_{22}}{a_{22}}. \tag{2.53}$$

On the other hand, in view of eq. (2.41), η_1 or η_2 curves are characterized by $\underline{\lambda}^T = \left\{1/\sqrt{a_{11}}, 0\right\}$ or $\underline{\lambda}^T = \left\{0, 1/\sqrt{a_{22}}\right\}$, respectively. Their normal curvature then follows from eq. (2.48) as $\kappa_n = b_{11}/a_{11}$ and $\kappa_n = b_{22}/a_{22}$, respectively. It is now clear that when $a_{12} = b_{12} = 0$, the η_1 and η_2 curves are indeed the lines of curvatures. It is customary to introduce the *principal radii of curvature*, R_1 and R_2, defined as

$$\kappa_n^I = \frac{b_{11}}{a_{11}} = \frac{1}{R_1}, \quad \kappa_n^{II} = \frac{b_{22}}{a_{22}} = \frac{1}{R_2}. \tag{2.54}$$

2.4.6 Derivatives of the base vectors

At this point, the discussion will focus exclusively on surface parameterizations defining lines of curvatures. In this case, vectors $\underline{a}_1$, $\underline{a}_2$ and $\bar{n}$ form a set of mutually orthogonal vectors, although the first two are not necessarily unit vectors. An orthonormal triad can be constructed as follows

$$\bar{e}_1 = \frac{\underline{a}_1}{\|\underline{a}_1\|}, \quad \bar{e}_2 = \frac{\underline{a}_2}{\|\underline{a}_2\|}, \quad \bar{e}_3 = \bar{n}. \tag{2.55}$$

To interpret the meaning of these unit vectors, the chain rule for derivatives is used to write

$$\underline{a}_1 = \frac{\partial \underline{p}_0}{\partial \eta_1} = \frac{\partial \underline{p}_0}{\partial s_1}\frac{\mathrm{d}s_1}{\mathrm{d}\eta_1} = \frac{\mathrm{d}s_1}{\mathrm{d}\eta_1}\bar{e}_1,$$

where s_1 is the arc length measured along the η_1 curve. Because $\partial \underline{p}_0/\partial s_1 = \bar{e}_1$ is the unit tangent vector to the η_1 curve, see eq. (2.5), it follows that

$$\|\underline{a}_1\| = h_1 = \frac{\mathrm{d}s_1}{\mathrm{d}\eta_1}, \quad \|\underline{a}_2\| = h_2 = \frac{\mathrm{d}s_2}{\mathrm{d}\eta_2}. \tag{2.56}$$

Notation $h_1 = \|\underline{a}_1\|$ was introduced to simplify the writing. Clearly, h_1 is a *scale factor*, the ratio of the infinitesimal increment in length, $\mathrm{d}s_1$, to the infinitesimal increment in parameter η_1, $\mathrm{d}\eta_1$, along the curve.

It is interesting to compute the derivatives of the base vectors. To that effect, the following expression is considered

$$\frac{\partial^2 \underline{p}_0}{\partial \eta_1 \partial \eta_2} = \frac{\partial \underline{a}_1}{\partial \eta_2} = \frac{\partial \underline{a}_2}{\partial \eta_1} = \frac{\partial (h_1 \bar{e}_1)}{\partial \eta_2} = \frac{\partial (h_2 \bar{e}_2)}{\partial \eta_1}.$$

Expanding the derivatives leads to

$$\frac{\partial h_1}{\partial \eta_2}\bar{e}_1 + h_1\frac{\partial \bar{e}_1}{\partial \eta_2} = \frac{\partial h_2}{\partial \eta_1}\bar{e}_2 + h_2\frac{\partial \bar{e}_2}{\partial \eta_1}. \tag{2.57}$$

Pre-multiplying this relationship by $\bar{e}_1^T$ yields the following identity

$$\bar{e}_1^T\frac{\partial \bar{e}_2}{\partial \eta_1} = \frac{1}{h_2}\frac{\partial h_1}{\partial \eta_2}.$$

To obtain this result, the orthogonality of the base vectors, $\bar{e}_1^T\bar{e}_2 = 0$, was used; furthermore, $\bar{e}_1^T\partial\bar{e}_1/\partial\eta_2 = 0$, since $\bar{e}_1$ is a unit vector. In terms of intrinsic parameterization, this expression becomes

$$\bar{e}_1^T\frac{\partial \bar{e}_2}{\partial s_1} = -\bar{e}_2^T\frac{\partial \bar{e}_1}{\partial s_1} = \frac{1}{h_1}\frac{\partial h_1}{\partial s_2} = \frac{1}{T_1}, \tag{2.58}$$

where T_1 is the first *radius of twist* of the surface.

Next, eq. (2.57) is pre-multiplied $\bar{e}_2^T$ to yield

$$\bar{e}_2^T\frac{\partial \bar{e}_1}{\partial s_2} = -\bar{e}_1^T\frac{\partial \bar{e}_2}{\partial s_2} = \frac{1}{h_2}\frac{\partial h_2}{\partial s_1} = \frac{1}{T_2}, \tag{2.59}$$

where T_2 is the second *radius of twist* of the surface. Since the parameterization defines lines of curvatures, $b_{12} = 0$, and eq. (2.47) then implies

$$\bar{e}_2^T\frac{\partial \bar{n}}{\partial s_1} = \bar{n}^T\frac{\partial \bar{e}_2}{\partial s_1} = 0, \quad \bar{e}_1^T\frac{\partial \bar{n}}{\partial s_2} = \bar{n}^T\frac{\partial \bar{e}_1}{\partial s_2} = 0.$$

The definitions of the diagonal terms, b_{11} and b_{22}, of the second metric tensor, eq. (2.47), lead to

$$\bar{e}_1^T \frac{\partial \bar{n}}{\partial s_1} = -\bar{n}^T \frac{\partial \bar{e}_1}{\partial s_1} = -\frac{1}{R_1}, \quad \bar{e}_2^T \frac{\partial \bar{n}}{\partial s_2} = -\bar{n}^T \frac{\partial \bar{e}_2}{\partial s_2} = -\frac{1}{R_2},$$

where the principal radii of curvature, R_1 and R_2, were defined in eq. (2.54).

The derivatives of the surface base vector $\bar{e}_1$ can be resolved in the following manner

$$\frac{\partial \bar{e}_1}{\partial s_1} = c_1 \underline{e}_1 + c_2 \underline{e}_2 + c_3 \bar{n}, \tag{2.60}$$

where the unknown coefficients c_1, c_2, and c_3 are readily found by pre-multiplying the above relationship by $\bar{e}_1^T$, $\bar{e}_2^T$, and $\bar{n}^T$ to find

$$\frac{\partial \bar{e}_1}{\partial s_1} = -\frac{1}{T_1}\bar{e}_2 + \frac{1}{R_1}\bar{n}. \tag{2.61}$$

A similar development leads to

$$\frac{\partial \bar{e}_1}{\partial s_2} = \frac{1}{T_2}\bar{e}_2. \tag{2.62}$$

The derivatives of the surface base vector $\bar{e}_2$ are found in a similar manner

$$\frac{\partial \bar{e}_2}{\partial s_1} = \frac{1}{T_1}\bar{e}_1, \quad \frac{\partial \bar{e}_2}{\partial s_2} = -\frac{1}{T_2}\underline{e}_1 + \frac{1}{R_2}\bar{n}. \tag{2.63}$$

These results are known as *Gauss' formulæ*.

Proceeding in a similar fashion, the derivatives of the normal vector are resolved in the following manner

$$\frac{\partial \bar{n}}{\partial s_1} = -\frac{1}{R_1}\bar{e}_1, \quad \frac{\partial \bar{n}}{\partial s_2} = -\frac{1}{R_2}\bar{e}_2. \tag{2.64}$$

These results are known as *Weingarten's formulæ*.

Gauss' and Weingarten's formulæ can be combined to yield the derivatives of the base vectors in a compact manner as

$$\frac{\partial}{\partial s_1}\left\{\begin{matrix}\bar{e}_1\\ \bar{e}_2\\ \bar{n}\end{matrix}\right\} = \begin{bmatrix} 0 & -1/T_1 & 1/R_1 \\ 1/T_1 & 0 & 0 \\ -1/R_1 & 0 & 0 \end{bmatrix}\left\{\begin{matrix}\bar{e}_1\\ \bar{e}_2\\ \bar{n}\end{matrix}\right\}, \tag{2.65a}$$

$$\frac{\partial}{\partial s_2}\left\{\begin{matrix}\bar{e}_1\\ \bar{e}_2\\ \bar{n}\end{matrix}\right\} = \begin{bmatrix} 0 & 1/T_2 & 0 \\ -1/T_2 & 0 & 1/R_2 \\ 0 & -1/R_2 & 0 \end{bmatrix}\left\{\begin{matrix}\bar{e}_1\\ \bar{e}_2\\ \bar{n}\end{matrix}\right\}. \tag{2.65b}$$

These equations should be compared to the derivatives of Frenet's triad, eq. (2.13).

Example 2.4. The spherical surface
The spherical surface in three-dimensional space depicted in fig. 2.10 is defined by following position vector $\underline{p}_0 = R\,(\sin\eta_1 \cos\eta_2\, \bar{\imath}_1 + \sin\eta_1 \sin\eta_2\, \bar{\imath}_2 + \cos\eta_1\, \bar{\imath}_3)$, where R is the radius of the sphere. The surface base vectors are readily evaluated as $\underline{a}_1 = \partial\underline{p}_0/\partial\eta_1 = R\,(\cos\eta_1 \cos\eta_2\, \bar{\imath}_1 + \cos\eta_1 \sin\eta_2\, \bar{\imath}_2 - \sin\eta_1\, \bar{\imath}_3)$, and $\underline{a}_2 = \partial\underline{p}_0/\partial\eta_2 = R\,(-\sin\eta_1 \sin\eta_2\, \bar{\imath}_1 + \sin\eta_1 \cos\eta_2\, \bar{\imath}_2)$.

The first metric tensor of the sphere now becomes

$$\underline{\underline{A}} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\eta_1 \end{bmatrix}.$$

Clearly, $h_1 = R$, $h_2 = R\sin\eta_1$, and $\sqrt{a} = R^2 \sin\eta_1$. The normal vector is then evaluated with the help of eq. (2.39), to find

$$\bar{n} = \frac{\widetilde{a}_1 \underline{a}_2}{\|\widetilde{a}_1 \underline{a}_2\|} = \sin\eta_1 \cos\eta_2 \bar{\imath}_1 + \sin\eta_1 \sin\eta_2 \bar{\imath}_2 + \cos\eta_1 \bar{\imath}_3.$$

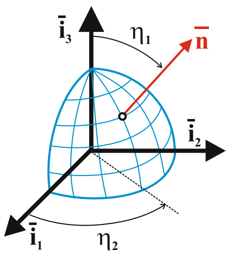

Fig. 2.10. Spherical surface configuration.

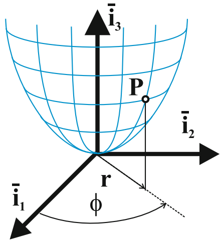

Fig. 2.11. Parabolic surface of revolution.

The second metric tensor of the spherical surface now follows from eq. (2.47)

$$\underline{\underline{B}} = \begin{bmatrix} -R & 0 \\ 0 & -R\sin^2\eta_1 \end{bmatrix}.$$

Note that since $a_{12} = 0$ and $b_{12} = 0$, the coordinates used here are lines of curvature for the spherical surface. The orthonormal triad to the surface is

$$\begin{aligned} \bar{e}_1 &= \cos\eta_1 \cos\eta_2\, \bar{\imath}_1 + \cos\eta_1 \sin\eta_2\, \bar{\imath}_2 - \sin\eta_1 \bar{\imath}_3, \\ \bar{e}_2 &= -\sin\eta_2\, \bar{\imath}_1 + \cos\eta_2\, \bar{\imath}_2, \\ \bar{n} &= \sin\eta_1 \cos\eta_2\, \bar{\imath}_1 + \sin\eta_1 \sin\eta_2\, \bar{\imath}_2 + \cos\eta_1\, \bar{\imath}_3. \end{aligned}$$

These expressions are readily inverted to find

$$\begin{aligned} \bar{\imath}_1 &= \cos\eta_1 \cos\eta_2\, \bar{e}_1 - \sin\eta_2\, \bar{e}_2 + \sin\eta_1 \cos\eta_2\, \bar{n}, \\ \bar{\imath}_2 &= \cos\eta_1 \sin\eta_2\, \bar{e}_1 + \cos\eta_2\, \bar{e}_2 + \sin\eta_1 \sin\eta_2\, \bar{n}, \\ \bar{\imath}_3 &= -\sin\eta_1\, \bar{e}_1 + \cos\eta_1\, \bar{n}. \end{aligned}$$

The mean curvature, eq. (2.51), and Gaussian curvature, eq. (2.52), are

$$\kappa_m = \frac{1}{2}\left(-\frac{R}{R^2} - \frac{R\sin^2\eta_1}{R^2\sin^2\eta_1}\right) = -\frac{1}{R}, \quad \kappa_n^I\,\kappa_n^{II} = \frac{R^2\sin^2\eta_1}{R^4\sin^2\eta_1} = \frac{1}{R^2}.$$

Finally, the principal curvatures, eq. (2.53), become

$$\kappa_n^I = -\frac{1}{R}, \quad \kappa_n^{II} = -\frac{1}{R}.$$

As expected, the principal radii of curvature $R_1 = R_2 = -R$ are equal to the radius of sphere. The twists of the surface now follow from eqs. (2.58) and (2.59)

$$\frac{1}{T_1} = \frac{1}{h_1h_2}\frac{\partial h_1}{\partial \eta_2} = 0, \quad \frac{1}{T_2} = \frac{1}{h_1h_2}\frac{\partial h_2}{\partial \eta_1} = \frac{\cos\eta_1}{R\sin\eta_1}. \tag{2.66}$$

2.4.7 Problems

Problem 2.6. The parabola of revolution

Figure 2.11 depicts a parabolic surface of revolution. It is defined by the following position vector $\underline{p}_0 = r\cos\phi\,\bar{\imath}_1 + r\sin\phi\,\bar{\imath}_2 + ar^2\bar{\imath}_3$, where $r \geq 0$ and $0 \leq \phi \leq 2\pi$. The following notation was used $\eta_1 = r$ and $\eta_2 = \phi$. *(1)* Find the first and second metric tensors of the surface. *(2)* Find the orthonormal triad $\bar{e}_1, \bar{e}_2$, and $\bar{n}$. *(3)* Find the mean curvature, the Gaussian curvature, and the principal radii of curvature of the surface. *(4)* Find the twists of the surface.

Problem 2.7. Jacobian of the transformation

Consider two parameterizations of a surface defined by coordinates (η_1, η_2) and $(\hat{\eta}_1, \hat{\eta}_2)$. Show that the base vectors in the two parameterizations are related as follows $\underline{\hat{a}}_1 = J_{11}\underline{a}_1 + J_{12}\underline{a}_2$ and $\underline{\hat{a}}_2 = J_{21}\underline{a}_1 + J_{22}\underline{a}_2$, where $\underline{\underline{J}}$ is the Jacobian of the coordinate transformation

$$\underline{\underline{J}} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \eta_1}{\partial \hat{\eta}_1} & \dfrac{\partial \eta_2}{\partial \hat{\eta}_1} \\ \dfrac{\partial \eta_1}{\partial \hat{\eta}_2} & \dfrac{\partial \eta_2}{\partial \hat{\eta}_2} \end{bmatrix}.$$

If $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are the first and second metric tensors in coordinate system (η_1, η_2) and $\underline{\underline{\hat{A}}}$ and $\underline{\underline{\hat{B}}}$ the corresponding quantities in coordinate system $(\hat{\eta}_1, \hat{\eta}_2)$, show that $\underline{\underline{\hat{A}}} = \underline{\underline{J}}\,\underline{\underline{A}}\,\underline{\underline{J}}^T$ and $\underline{\underline{\hat{B}}} = \underline{\underline{J}}\,\underline{\underline{B}}\,\underline{\underline{J}}^T$.

Problem 2.8. Finding the line of curvature system

Using the notations defined in problem 2.7, let (η_1, η_2) be a known coordinate system and $(\hat{\eta}_1, \hat{\eta}_2)$ the unknown line of curvature system. Find the Jacobian of the coordinate transformation that will bring (η_1, η_2) to the desired line of curvature system $(\hat{\eta}_1, \hat{\eta}_2)$. Show that the principal radii of curvature are

$$\frac{1}{R_1} = \frac{b_{11} + \gamma(2b_{12} + \gamma b_{22})}{a_{11} + \gamma(2a_{12} + \gamma a_{22})}, \quad \frac{1}{R_2} = \frac{b_{11} + \alpha(2b_{12} + \alpha b_{22})}{a_{11} + \alpha(2a_{12} + \alpha a_{22})}.$$

Hint: write the Jacobian as

$$\underline{\underline{J}} = \begin{bmatrix} 1 & \gamma \\ \alpha & 1 \end{bmatrix},$$

and compute the coefficients α and γ so as to enforce $\hat{a}_{12} = \hat{b}_{12} = 0$. The solution of the problem is $\alpha = C_\alpha/[\Delta/(1+\alpha\gamma)]$ and $\gamma = -C_\gamma/[\Delta/(1+\alpha\gamma)]$ where $C_\alpha = a_{22}b_{12} - b_{22}a_{12}$, $C_\gamma = a_{11}b_{12} - b_{11}a_{12}$, $\Delta = a_{11}b_{22} - b_{11}a_{22}$, and $\Delta/(1+\alpha\gamma) = \Delta/2 \pm \sqrt{(\Delta/2)^2 + C_\alpha C_\gamma}$.

2.5 Surface coordinates

A particle is moving on a surface and its position is given by the lines of curvature coordinates, $\eta_1(t)$ and $\eta_2(t)$. The velocity vector is computed with the help of the chain rule for derivatives

$$\underline{v} = \frac{\mathrm{d}\underline{p}_0}{\mathrm{d}t} = \frac{\partial \underline{p}_0}{\partial \eta_1}\dot{\eta}_1 + \frac{\partial \underline{p}_0}{\partial \eta_2}\dot{\eta}_2 = \dot{s}_1 \bar{e}_1 + \dot{s}_2 \bar{e}_2. \tag{2.67}$$

Note the close similarity between this expression and that obtained for path coordinates, eq. (2.33). The velocity vector is in the plane tangent to the surface, and the speed of the particle is $v = \sqrt{\dot{s}_1^2 + \dot{s}_2^2}$.

Next, the acceleration vector is computed as

$$\begin{aligned}\underline{a} &= \ddot{s}_1 \bar{e}_1 + \dot{s}_1 \dot{\bar{e}}_1 + \ddot{s}_2 \bar{e}_2 + \dot{s}_2 \dot{\bar{e}}_2 \\ &= \ddot{s}_1 \bar{e}_1 + \ddot{s}_2 \bar{e}_2 + \dot{s}_1 \left(\frac{\partial \bar{e}_1}{\partial s_1}\dot{s}_1 + \frac{\partial \bar{e}_1}{\partial s_2}\dot{s}_2 \right) + \dot{s}_2 \left(\frac{\partial \bar{e}_2}{\partial s_1}\dot{s}_1 + \frac{\partial \bar{e}_2}{\partial s_2}\dot{s}_2 \right).\end{aligned}$$

Introducing Gauss' formulae, eq. (2.61) to (2.63), then yields

$$\underline{a} = \left(\ddot{s}_1 + \frac{\dot{s}_1 \dot{s}_2}{T_1} - \frac{\dot{s}_2^2}{T_2} \right) \bar{e}_1 + \left(\ddot{s}_2 + \frac{\dot{s}_1 \dot{s}_2}{T_2} - \frac{\dot{s}_1^2}{T_1} \right) \bar{e}_2 + \left(\frac{\dot{s}_1^2}{R_1} + \frac{\dot{s}_2^2}{R_2} \right) \bar{n}. \tag{2.68}$$

Note here again the similarity between this expression and that obtained for path coordinates, eq. (2.34). The acceleration component along the normal to the surface is related to the principal radii of curvatures, R_1 and R_2. For a curve, the radius of curvature is always positive, see eq. (2.7), whereas for a surface, the radii of curvatures could be positive or negative, see eq. (2.54). Hence, the normal component of acceleration is not necessarily oriented along the normal to the surface.

The components of acceleration in the plane tangent to the surface are related to the second time derivative of the intrinsic parameters, as expected. Additional terms, however, associated with the surface radii of twist also appear. Clearly, the acceleration of a particle moving on the surface is affected by the surface radii of curvature and twist; the particle "feels" the curvatures and twists of the surface as it moves.

2.6 Differential geometry of a three-dimensional mapping

This section investigates the differential geometry of mappings of the three-dimensional space onto itself. The differential geometry of such mappings is more complex than that of curves or surfaces. For simplicity, the analysis focuses on orthogonal mappings, leading to the definition of the curvatures of the coordinate system and orthogonal curvilinear coordinates. Two orthogonal curvilinear coordinate systems of great practical importance, the cylindrical and spherical coordinate systems are reviewed.

2.6.1 Arbitrary parameterization

Consider the following mapping of the three-dimensional space onto itself in terms of three parameters, η_1, η_2, and η_3,

$$\underline{p}_0(\eta_1, \eta_2, \eta_3) = x_1(\eta_1, \eta_2, \eta_3)\bar{\imath}_1 + x_2(\eta_1, \eta_2, \eta_3)\bar{\imath}_2 + x_3(\eta_1, \eta_2, \eta_3)\bar{\imath}_3. \tag{2.69}$$

This relationship defines a mapping between the parameters and the Cartesian coordinates

$$x_1 = x_1(\eta_1, \eta_2, \eta_3), \quad x_2 = x_2(\eta_1, \eta_2, \eta_3), \quad x_3 = x_3(\eta_1, \eta_2, \eta_3). \tag{2.70}$$

Let η_2 and η_3 be constants whereas η_1 only is allowed to vary: a general curve in three-dimensional space is generated. The analysis of section 2.2 would readily apply to this curve, called an "η_1 curve." Similarly, η_2 and η_3 curves could be defined.

Next, let η_1 be a constant, whereas η_2 and η_3 are allowed to vary: a general surface in three-dimensional space is generated. The analysis of section 2.4 would readily apply to this surface, called an "η_1 surface." Here again, η_2 and η_3 surfaces could be similarly defined.

A point in space with parameters (η_1, η_2, η_3) is at the intersection of three η_1, η_2, and η_3 curves, or at the intersection of three η_1, η_2, and η_3 surfaces. Furthermore, an η_1 curve forms the intersection of η_2 and η_3 surfaces.

The inverse mapping defines the parameters as functions of the Cartesian coordinates

$$\eta_1 = \eta_1(x_1, x_2, x_3), \quad \eta_2 = \eta_2(x_1, x_2, x_3), \quad \eta_3 = \eta_3(x_1, x_2, x_3). \tag{2.71}$$

It is assumed here that eqs. (2.70) and (2.71) define a *one to one* mapping, which implies that the *Jacobian* of the transformation,

$$\underline{\underline{J}} = \begin{bmatrix} \frac{\partial x_1}{\partial \eta_1} & \frac{\partial x_1}{\partial \eta_2} & \frac{\partial x_1}{\partial \eta_3} \\ \frac{\partial x_2}{\partial \eta_1} & \frac{\partial x_2}{\partial \eta_2} & \frac{\partial x_2}{\partial \eta_3} \\ \frac{\partial x_3}{\partial \eta_1} & \frac{\partial x_3}{\partial \eta_2} & \frac{\partial x_3}{\partial \eta_3} \end{bmatrix}, \tag{2.72}$$

has a non vanishing determinant at all points in space. Next, the *base vectors* associated with the parameters are defined as

$$\underline{g}_1 = \frac{\partial \underline{p}_0}{\partial \eta_1}, \quad \underline{g}_2 = \frac{\partial \underline{p}_0}{\partial \eta_2}, \quad \underline{g}_3 = \frac{\partial \underline{p}_0}{\partial \eta_3}. \tag{2.73}$$

For an arbitrary parameterization, the base vectors will not be unit vectors, nor will they be mutually orthogonal.

Consider the example of the cylindrical coordinate system defined by the following parameterization

$$x_1 = r\cos\theta,\ x_2 = r\sin\theta,\ x_3 = z,$$

where $r \geq 0$ and $0 \leq \theta < 2\pi$. The following notation was used: $\eta_1 = r$, $\eta_2 = \theta$ and $\eta_3 = z$. The inverse mapping is readily found as

$$r = \sqrt{x_1^2 + x_2^2},\ \theta = \tan^{-1}\frac{x_2}{x_1},\ z = x_3.$$

Figure 2.12 depicts this mapping; clearly, the familiar polar coordinates are used in the $(\bar{\imath}_1, \bar{\imath}_2)$ plane and z is the distance point **P** is above this plane. The Jacobian of the transformation becomes

$$J = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that $\det J = r$, and hence, vanishes at $r = 0$. Indeed, cylindrical coordinates are not defined at the origin since when $r = 0$, any angle θ maps to the same point, the origin.

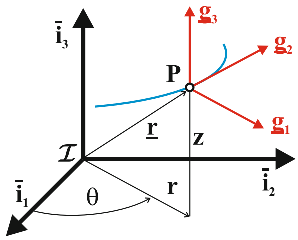

Fig. 2.12. The cylindrical coordinate system.

The base vectors of this coordinate system are $\underline{g}_1 = \cos\theta\,\bar{\imath}_1 + \sin\theta\,\bar{\imath}_2$, $\underline{g}_2 = -r\sin\theta\,\bar{\imath}_1 + r\cos\theta\,\bar{\imath}_2$, and $\underline{g}_3 = \bar{\imath}_3$. Note that $\underline{g}_1$ is a unit vector, since $\|\underline{g}_1\| = 1$, but $\underline{g}_2$ is not, $\|\underline{g}_2\| = r$. Also note that for cylindrical coordinates, $\underline{g}_2^T \underline{g}_3 = \underline{g}_1^T \underline{g}_3 = \underline{g}_1^T \underline{g}_2 = 0$, the base vectors are mutually orthogonal, as shown in fig. 2.12.

2.6.2 Orthogonal parameterization

When the base vectors associated with the parameterization are mutually orthogonal, the parameters define an *orthogonal parameterization* of the three-dimensional space. The rest of this section will be restricted to such parameterization. In this case, it is advantageous to define a set of orthonormal vectors

$$\bar{e}_1 = \frac{1}{\|\underline{g}_1\|}\,\underline{g}_1, \quad \bar{e}_2 = \frac{1}{\|\underline{g}_2\|}\,\underline{g}_2, \quad \bar{e}_3 = \frac{1}{\|\underline{g}_3\|}\,\underline{g}_3. \tag{2.74}$$

To interpret the meaning of these unit vectors, the chain rule for derivatives is used to write

$$\underline{g}_1 = \frac{\partial \underline{p}_0}{\partial \eta_1} = \frac{\partial \underline{p}_0}{\partial s_1}\frac{\mathrm{d}s_1}{\mathrm{d}\eta_1} = \bar{e}_1 \frac{\mathrm{d}s_1}{\mathrm{d}\eta_1}, \tag{2.75}$$

where s_1 is the arc length measured along the η_1 curve. Because $\partial \underline{p}_0/\partial s_1 = \bar{e}_1$ is the unit tangent to the η_1 curve, see eq. (2.5), it follows that

$$\|\underline{g}_1\| = h_1 = \frac{\mathrm{d}s_1}{\mathrm{d}\eta_1}, \quad \|\underline{g}_2\| = h_2 = \frac{\mathrm{d}s_2}{\mathrm{d}\eta_2}, \quad \|\underline{g}_3\| = h_3 = \frac{\mathrm{d}s_3}{\mathrm{d}\eta_3}. \tag{2.76}$$

Notation $h_1 = \|\underline{g}_1\|$ is introduced to simplify the notation. Clearly, h_1 is a *scale factor*, the ratio of the infinitesimal increment in length, $\mathrm{d}s_1$, to the infinitesimal increment in parameter η_1, $\mathrm{d}\eta_1$, along the curve.

2.6.3 Derivatives of the base vectors

Here again, the derivatives of the base vectors will be evaluated. To that effect, the following expression is considered

$$\frac{\partial^2 \underline{p}_0}{\partial \eta_1 \eta_2} = \frac{\partial \underline{g}_1}{\partial \eta_2} = \frac{\partial \underline{g}_2}{\partial \eta_1} = \frac{\partial (h_1 \bar{e}_1)}{\partial \eta_2} = \frac{\partial (h_2 \bar{e}_2)}{\partial \eta_1}. \tag{2.77}$$

Expanding the derivatives leads to

$$\frac{\partial h_1}{\partial \eta_2}\bar{e}_1 + h_1 \frac{\partial \bar{e}_1}{\partial \eta_2} = \frac{\partial h_2}{\partial \eta_1}\bar{e}_2 + h_2 \frac{\partial \bar{e}_2}{\partial \eta_1}. \tag{2.78}$$

Pre-multiplying this relationship by $\bar{e}_1^T$ yields the following identity

$$\bar{e}_1^T \frac{\partial \bar{e}_2}{\partial \eta_1} = \frac{1}{h_2} \frac{\partial h_1}{\partial \eta_2}. \tag{2.79}$$

To obtain this result, the orthogonality of the base vectors, $\bar{e}_1^T \bar{e}_2 = 0$, was used; furthermore, $\bar{e}_1^T \partial \bar{e}_1 / \partial \eta_2 = 0$, since $\bar{e}_1$ is a unit vector. Next, eq. (2.78) is pre-multiplied $\bar{e}_2^T$ to yield

$$\bar{e}_2^T \frac{\partial \bar{e}_1}{\partial \eta_2} = -\bar{e}_1^T \frac{\partial \bar{e}_2}{\partial \eta_2} = \frac{1}{h_1} \frac{\partial h_2}{\partial \eta_1}. \tag{2.80}$$

Finally, pre-multiplication by $\bar{e}_3^T$ leads to

$$h_1 \, \bar{e}_3^T \frac{\partial \bar{e}_1}{\partial \eta_2} = h_2 \, \bar{e}_3^T \frac{\partial \bar{e}_2}{\partial \eta_1}. \tag{2.81}$$

Since $\bar{e}_3^T \partial \bar{e}_2 / \partial \eta_1 = -\bar{e}_2^T \partial \bar{e}_3 / \partial \eta_1$, this result can be manipulated as follows

$$h_1 \, \bar{e}_3^T \frac{\partial \bar{e}_1}{\partial \eta_2} = -h_2 \, \bar{e}_2^T \frac{\partial \bar{e}_3}{\partial \eta_1} = -\frac{h_1 h_2}{h_3} \, \bar{e}_2^T \frac{\partial \bar{e}_1}{\partial \eta_3}, \tag{2.82}$$

where identity (2.81) was used with a permutation of the indices. Using the same identities once again leads to

$$h_1 \, \bar{e}_3^T \frac{\partial \bar{e}_1}{\partial \eta_2} = \frac{h_1 h_2}{h_3} \, \bar{e}_1^T \frac{\partial \bar{e}_2}{\partial \eta_3} = h_1 \, \bar{e}_1^T \frac{\partial \bar{e}_3}{\partial \eta_2} = -h_1 \, \bar{e}_3^T \frac{\partial \bar{e}_1}{\partial \eta_2}.$$

This result clearly implies

$$\bar{e}_3^T \frac{\partial \bar{e}_1}{\partial \eta_2} = 0. \tag{2.83}$$

The derivatives of the base vector can be resolved as

$$\frac{\partial \bar{e}_1}{\partial \eta_1} = c_1 \bar{e}_1 + c_2 \bar{e}_2 + c_3 \bar{e}_3,$$

where the unknown coefficients c_1, c_2, and c_3 are found by pre-multiplying this expression by $\bar{e}_1$, $\bar{e}_2$, and $\bar{e}_3$, respectively, and using identities (2.79), (2.80) and (2.83) to find

$$\frac{\partial \bar{e}_1}{\partial \eta_1} = -\frac{1}{h_2}\frac{\partial h_1}{\partial \eta_2}\bar{e}_2 - \frac{1}{h_3}\frac{\partial h_1}{\partial \eta_3}\bar{e}_3.$$

Proceeding in a similar manner, the derivatives of base vector $\bar{e}_1$ with respect to η_2 and η_3 are found as

$$\frac{\partial \bar{e}_1}{\partial \eta_2} = \frac{1}{h_1}\frac{\partial h_2}{\partial \eta_1}\bar{e}_2, \qquad \frac{\partial \bar{e}_1}{\partial \eta_3} = \frac{1}{h_1}\frac{\partial h_3}{\partial \eta_1}\bar{e}_3.$$

Similar expression are readily found for the derivatives of the unit base vectors $\bar{e}_2$ and $\bar{e}_3$ through index permutations and are summarized as

$$\frac{\partial}{\partial s_1}\begin{Bmatrix}\bar{e}_1\\ \bar{e}_2\\ \bar{e}_3\end{Bmatrix} = \begin{bmatrix} 0 & 1/R_{13} & -1/R_{12}\\ -1/R_{13} & 0 & 0\\ 1/R_{12} & 0 & 0\end{bmatrix}\begin{Bmatrix}\bar{e}_1\\ \bar{e}_2\\ \bar{e}_3\end{Bmatrix}, \tag{2.84a}$$

$$\frac{\partial}{\partial s_2}\begin{Bmatrix}\bar{e}_1\\ \bar{e}_2\\ \bar{e}_3\end{Bmatrix} = \begin{bmatrix} 0 & 1/R_{23} & 0\\ -1/R_{23} & 0 & 1/R_{21}\\ 0 & -1/R_{21} & 0\end{bmatrix}\begin{Bmatrix}\bar{e}_1\\ \bar{e}_2\\ \bar{e}_3\end{Bmatrix}, \tag{2.84b}$$

$$\frac{\partial}{\partial s_3}\begin{Bmatrix}\bar{e}_1\\ \bar{e}_2\\ \bar{e}_3\end{Bmatrix} = \begin{bmatrix} 0 & 0 & -1/R_{32}\\ 0 & 0 & 1/R_{31}\\ 1/R_{32} & -1/R_{31} & \end{bmatrix}\begin{Bmatrix}\bar{e}_1\\ \bar{e}_2\\ \bar{e}_3\end{Bmatrix}, \tag{2.84c}$$

where the curvatures of the system were defined as

$$\frac{1}{R_{12}} = \frac{1}{h_1}\frac{\partial h_1}{\partial s_3}, \quad \frac{1}{R_{13}} = -\frac{1}{h_1}\frac{\partial h_1}{\partial s_2}, \tag{2.85a}$$

$$\frac{1}{R_{21}} = -\frac{1}{h_2}\frac{\partial h_2}{\partial s_3}, \quad \frac{1}{R_{23}} = \frac{1}{h_2}\frac{\partial h_2}{\partial s_1}, \tag{2.85b}$$

$$\frac{1}{R_{31}} = \frac{1}{h_3}\frac{\partial h_3}{\partial s_2}, \quad \frac{1}{R_{32}} = -\frac{1}{h_3}\frac{\partial h_3}{\partial s_1}. \tag{2.85c}$$

2.7 Orthogonal curvilinear coordinates

Consider a particle moving in three-dimension space. The position of this particle can be defined by eq. (2.69) in terms of an orthogonal parameterization of space. These parameter define a set of *orthogonal curvilinear coordinates* for the particle. The velocity vector is computed with the help of the chain rule for derivatives

$$\underline{v} = \frac{\mathrm{d}\underline{p}_0}{\mathrm{d}t} = \frac{\partial \underline{p}_0}{\partial s_1}\dot{s}_1 + \frac{\partial \underline{p}_0}{\partial s_2}\dot{s}_2 + \frac{\partial \underline{p}_0}{\partial s_3}\dot{s}_3 = \dot{s}_1\bar{e}_1 + \dot{s}_2\bar{e}_2 + \dot{s}_3\bar{e}_3. \tag{2.86}$$

The expression for the acceleration vector will involve term in $\ddot{s}_1\bar{e}_1$ and $\dot{s}_1\dot{\bar{e}}_1$, and similar terms for the other two indices. The latter term is further expanded using the chain rule for derivatives, and expressing the derivatives of the base vectors using eqs. (2.84) then yields

$$\begin{aligned}\underline{a} = &\left[\ddot{s}_1 - \dot{s}_2^2/R_{23} + \dot{s}_3^2/R_{32} - \dot{s}_1\dot{s}_2/R_{13} + \dot{s}_1\dot{s}_3/R_{12}\right]\bar{e}_1\\ &+ \left[\ddot{s}_2 + \dot{s}_1^2/R_{13} - \dot{s}_3^2/R_{31} + \dot{s}_1\dot{s}_2/R_{23} - \dot{s}_2\dot{s}_3/R_{21}\right]\bar{e}_2\\ &+ \left[\ddot{s}_3 - \dot{s}_1^2/R_{12} + \dot{s}_2^2/R_{21} - \dot{s}_1\dot{s}_3/R_{32} + \dot{s}_2\dot{s}_3/R_{31}\right]\bar{e}_3.\end{aligned} \quad (2.87)$$

Note here again the similarity between this expression and that obtained for path or surface coordinates, eqs. (2.34) or (2.68), respectively. The acceleration components in each direction involve the second time derivative of the intrinsic parameters, as expected. Additional terms, however, associated with the radii of curvature of the curvilinear coordinate system also appear.

2.7.1 Cylindrical coordinates

The *cylindrical coordinate system*, depicted in fig. 2.13, is an orthogonal curvilinear coordinate system defined as follows

$$\underline{p}_0 = r\cos\theta\,\bar{\imath}_1 + r\sin\theta\,\bar{\imath}_2 + z\,\bar{\imath}_3, \quad (2.88)$$

where $r \geq 0$ and $0 \leq \theta < 2\pi$. The following notation was used: $\eta_1 = r$, $\eta_2 = \theta$, and $\eta_3 = z$. Note that if $z = 0$, the cylindrical coordinate system reduces to coordinates r and θ in plane $(\bar{\imath}_1, \bar{\imath}_2)$ and are then often called polar coordinates.

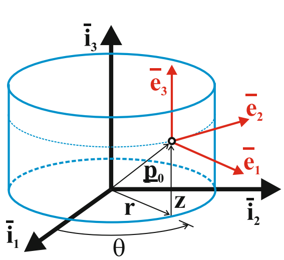

Fig. 2.13. The cylindrical coordinate system.

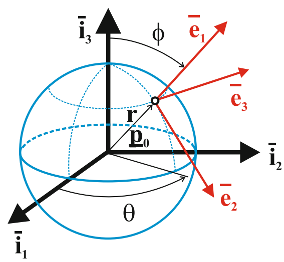

Fig. 2.14. The spherical coordinate system.

The following summarizes important formulæ in cylindrical coordinates. The scale factors are $h_1 = 1$, $h_2 = r$, and $h_3 = 1$. The curvatures of the cylindrical coordinate system all vanish, except that $R_{23} = r$. The base vectors expressed in terms of the Cartesian system are

$$\bar{e}_1 = \cos\theta\,\bar{\imath}_1 + \sin\theta\,\bar{\imath}_2, \quad (2.89a)$$

$$\bar{e}_2 = -\sin\theta\,\bar{\imath}_1 + \cos\theta\,\bar{\imath}_2, \quad (2.89b)$$

$$\bar{e}_3 = \bar{\imath}_3. \quad (2.89c)$$

The time derivatives of the based vectors resolved along this triad are

$$\dot{\bar{e}}_1 = \dot{\theta}\,\bar{e}_2, \quad (2.90a)$$

$$\dot{\bar{e}}_2 = -\dot{\theta}\,\bar{e}_1, \quad (2.90b)$$

$$\dot{\bar{e}}_3 = 0. \quad (2.90c)$$

Finally, the position, velocity, and acceleration vectors, resolved along the base vectors of the cylindrical coordinate system are

$$\underline{p}_0 = r\,\bar{e}_1 + z\,\bar{e}_3, \tag{2.91a}$$
$$\underline{v} = \dot{r}\,\bar{e}_1 + r\dot{\theta}\,\bar{e}_2 + \dot{z}\,\bar{e}_3, \tag{2.91b}$$
$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\,\bar{e}_1 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\bar{e}_2 + \ddot{z}\,\bar{e}_3. \tag{2.91c}$$

respectively.

2.7.2 Spherical coordinates

The *spherical coordinate system*, depicted in fig. 2.14, is an orthogonal curvilinear coordinate system defined as follows

$$\underline{p}_0 = r\sin\phi\cos\theta\,\bar{\imath}_1 + r\sin\phi\sin\theta\,\bar{\imath}_2 + r\cos\phi\,\bar{\imath}_3, \tag{2.92}$$

where $r \geq 0, 0 \leq \phi \leq \pi$, and $0 \leq \theta < 2\pi$. The following notation was used: $\eta_1 = r$, $\eta_2 = \phi$, and $\eta_3 = \theta$.

The following summarizes important formulæ in spherical coordinates. The scale factors are $h_1 = 1$, $h_2 = r$, and $h_3 = r\sin\phi$. The curvatures of the spherical coordinate system all vanish, except that $R_{23} = r$, $R_{31} = r\tan\phi$ and $R_{32} = -r$.

The base vectors expressed in terms of the Cartesian system are

$$\bar{e}_1 = \sin\phi\cos\theta\,\bar{\imath}_1 + \sin\phi\sin\theta\,\bar{\imath}_2 + \cos\phi\,\bar{\imath}_3, \tag{2.93a}$$
$$\bar{e}_2 = \cos\phi\cos\theta\,\bar{\imath}_1 + \cos\phi\sin\theta\,\bar{\imath}_2 - \sin\phi\,\bar{\imath}_3, \tag{2.93b}$$
$$\bar{e}_3 = -\sin\theta\,\bar{\imath}_1 + \cos\theta\,\bar{\imath}_2. \tag{2.93c}$$

The time derivatives of the based vectors resolved along this triad are

$$\dot{\bar{e}}_1 = \dot{\phi}\,\bar{e}_2 + \dot{\theta}\sin\phi\,\bar{e}_3, \tag{2.94a}$$
$$\dot{\bar{e}}_2 = -\dot{\phi}\,\bar{e}_1 + \dot{\theta}\cos\phi\,\bar{e}_3, \tag{2.94b}$$
$$\dot{\bar{e}}_3 = -\dot{\theta}(\sin\phi\,\bar{e}_1 + \cos\phi\,\bar{e}_2). \tag{2.94c}$$

Finally, the position, velocity, and acceleration vectors, resolved along the base vectors of the spherical coordinate system are

$$\underline{p}_0 = r\,\bar{e}_1, \tag{2.95a}$$
$$\underline{v} = \dot{r}\,\bar{e}_1 + r\dot{\phi}\,\bar{e}_2 + r\dot{\theta}\sin\phi\,\bar{e}_3, \tag{2.95b}$$
$$\underline{a} = (\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2\sin^2\phi)\,\bar{e}_1 + (r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2\sin\phi\cos\phi)\,\bar{e}_2$$
$$+ (r\ddot{\theta}\sin\phi + 2\dot{r}\dot{\theta}\sin\phi + 2r\dot{\phi}\dot{\theta}\cos\phi)\,\bar{e}_3. \tag{2.95c}$$

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Flexible Multibody Dynamics
Bauchau, O.A.
2011, XXII, 730 p., Hardcover
ISBN: 978-94-007-0334-6