2 Mathematical Foundations

Chapter 2 includes notation, definitions, and a number of identities that are applied subsequently in later Chapters of the text. Emphasized are differential-geometric aspects of kinematics of finite deformations, linear connections and covariant differentiation, the deformation gradient of continuum mechanics, time derivatives and rate kinematics, theorems of Gauss and Stokes, and compatibility conditions.

As will be clear in later Chapters, from the standpoint of crystals with defects, a geometric approach is advantageous for dealing with incompatible material configurations not globally homeomorphic to, i.e., configurations topologically inequivalent to, three-dimensional Euclidean space (Eckart 1948; Kondo 1949, 1964; Bilby et al. 1955; Truesdell and Noll 1965; Noll 1967; Teodosiu 1967a, b). Detailed descriptions invoking formalisms of tensor algebra and tensor calculus on differential manifolds have been devoted elsewhere to elasticity theory (Marsden and Hughes 1983; Yavari et al. 2006) and general nonlinear continuum mechanics (Van der Giessen and Kollmann 1996; Stumpf and Hoppe 1997; Clayton et al. 2005; Epstein and Elzanowski 2007). In many instances, the differential-geometric approach is favored over conventional Cartesian formulations for the former's generality in terms of available choices of coordinates (e.g., curvilinear coordinates in the former versus rectangular coordinates in the latter) and representation of spaces with non-vanishing curvature (e.g., curved surfaces such as shells). Compact, coordinate-free (i.e., component-free) representations are possible for many mathematical expressions and identities, and have become popular among many authors in recent literature. However, coordinate-based representations are frequently exercised in this text for clarity of presentation and for drawing comparisons with other treatments from historic and more recent literature. Hence, the index notation is used often in Chapter 2, especially in the context of geometric objects such as connections, torsion, curvature, and important properties and mathematical identities for these objects.

Included in Chapter 2 is only that content deemed relevant and necessary for development of theories of material behavior in later Chapters. Comprehensive supplementary treatment of topics in differential geometry can be found in historical texts of Eisenhart (1926) and Schouten (1954),

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while more modern presentations can be found in monographs of Boothby (1975) and Kosinski (1993). In-depth descriptions of continuum mechanics in the setting of general curvilinear coordinates and finite deformations can be found in a number of texts, including those of Truesdell and Toupin (1960), Eringen (1962), Sedov (1966), Malvern (1969), Wang and Truesdell (1973), and Marsden and Hughes (1983).

2.1 Geometric Description of a Deformable Body

Section 2.1 addresses the following topics: terminology of geometric spaces in the context of continuum mechanics; configurations of a body subjected to large deformations; manifolds and associated tangent and cotangent spaces; coordinate systems; basis vectors and their reciprocal vectors; and metric tensors.

2.1.1 Terminology

The definitions given immediately below are free of notation and hence are somewhat qualitative. More precise mathematical formulae and supplementary figures follow in Section 2.1.2 and later in Section 2.2.

A configuration denotes a time-dependent realization of a body. A body is said to consist of a number of material particles, each encompassing a representative set of atoms or molecules pertinent to the scale of resolution afforded by the continuum description. A configuration may be actual or virtual (i.e., real or fictitious). In finite deformation continuum mechanics, the terms reference, initial, undeformed, or Lagrangian configuration most often refer to a description of the body at zero time, though broader definitions enabling multiple and evolving reference configurations are possible. Similarly, the current, spatial, deformed, or Eulerian configuration usually corresponds to the current instant of time. In the absence of discontinuities, reference and current configurations are holonomic to one another, implying that current coordinates of a material particle can be written as single-valued functions of reference coordinates of that particle, and vice-versa. In contrast, as will be demonstrated in Chapter 3, a virtual intermediate, relaxed, or unloaded configuration is often introduced, for example to describe crystals with distributions of defects or those undergoing large inelastic deformations. Such an intermediate configuration is anholonomic when its "coordinates" cannot be prescribed as single-valued functions of reference or current coordinates of material particles. The term placement

(Noll 1967; Maugin 1993) has also been used to refer to a configuration of a deformable body.

A connection is a rank three construct that enables evaluation of the covariant derivative of vectors and tensors of higher rank. The covariant derivative operation defines the connection coefficients, also called Christoffel symbols of the connection in the context of Riemannian geometry. The content of this book only deals with linear connections, also called affine connections. Nonlinear connections can arise in more generalized spaces such as Finsler spaces (Rund 1959; Bejancu 1990) and are not addressed in this text.

A metric tensor, or simply a metric, is a rank two covariant tensor that defines the scalar product of contravariant vectors, and consequently, the squared length of a vector. A metric tensor that is both symmetric and positive definite is called a Riemannian metric tensor[1]; non-Riemannian metrics are not considered explicitly in this text. Metric tensors perform other consequential functions, such as raising and lowering of vector and tensor indices and defining scalar products of higher-order tensors.

A particular configuration can be assigned more than one connection, just as it can be assigned more than one metric tensor. The pair of {configuration, connection} or triplet of {configuration, connection, metric tensor} can often be classified as one or more of the following five types of geometric spaces: Euclidean, anholonomic, non-metric, Cartan, and/or Riemannian.

For a space to be classified as Euclidean or non-Euclidean, it must include a configuration, a metric tensor, and a connection. A Euclidean space satisfies three requirements: (i) the torsion tensor of the connection vanishes, (ii) the covariant derivative of the metric tensor vanishes, and (iii) the Riemann-Christoffel curvature tensor constructed from the connection coefficients vanishes. A Euclidean n-space permits at each location a transformation from local, possibly curvilinear n-dimensional coordinates to a global n-dimensional Cartesian coordinate system. In continuum mechanics, both the reference and current configurations are typically viewed as three-dimensional Euclidean spaces, with the motion acting as a diffeomorphism, i.e., a differentiable homeomorphism, or a differentiable one-to-one, invertible mapping, between these two configurations (Stumpf and Hoppe 1997). Notice the distinction between Euclidean

[1] A non-Riemannian metric tensor need not be positive definite. Such metrics can arise in more general geometric settings such as Finsler spaces, or for example Minkowski's spacetime (Rund 1959; Synge 1960) wherein the determinant of the metric in 4-space can be negative in sign. A non-symmetric fundamental tensor was suggested in Einstein's unified field theory (Einstein 1945; Schouten 1954).

and Cartesian: a general coordinate system in the former, for example spherical coordinates in three dimensions, admits a global transformation to a more specific kind of coordinate system in the latter, i.e., coordinate axes consisting of three constant orthonormal basis vectors.

An anholonomic space is a configuration associated with a non-integrable, two-point deformation map. For example, in multiplicative elastoplasticity theory, since elastic and plastic deformation maps (i.e., tangent maps) taken individually are generally non-integrable or anholonomic functions of current and reference coordinates, respectively, the corresponding intermediate configuration is generally anholonomic. Continuous coordinates on such an anholonomic space do not exist; rather, anholonomic coordinates can be regarded as discontinuous, multi-valued functions of holonomic coordinates of the current or reference configuration. Anholonomicity is related to Cartan's torsion tensor of a special connection constructed from the non-integrable, two-point deformation map. Anholonomic coordinates are analyzed at length by Schouten (1954), and to a lesser extent, by Ericksen (1960).

Designation of a general space as metric or non-metric requires that the configuration be assigned both a linear connection and a metric tensor; i.e., one must examine the triad of {configuration, connection, metric tensor}. In a non-metric space, the covariant derivative of the metric tensor taken with respect to the connection is nonzero. On the other hand, in a metric space the covariant derivative of the metric tensor vanishes identically.

A configuration with a connection admitting a non-vanishing torsion tensor is labeled a Cartan space, sometimes called a non-symmetric space. Only the pair {configuration, connection} need be considered to enable labeling a space as Cartan or non-Cartan. A space with vanishing torsion is called a symmetric space.

For a space to be labeled as Riemannian or non-Riemannian, it must include the pair {configuration, connection}—a metric is not needed for such a designation as defined herein. A Riemannian space is defined here as a configuration with connection coefficients whose components yield a nonzero Riemann-Christoffel curvature tensor[2]. A space with non-vanishing curvature is necessarily non-Euclidean. For example, a global two-dimensional Cartesian coordinate system cannot be used to parameterize a shell unless the shell is flat. Conversely, a space with vanishing curvature is non-Riemannian and is said to be flat, and the connection in a non-Riemannian space is said to be integrable.

[2] The mathematical field of study traditionally referred to as Riemannian geometry considers metric spaces with vanishing torsion, but in general non-vanishing curvature (Eisenhart 1926; Schouten 1954).

Euclidean spaces by definition exclude the other four types of (non-Euclidean) spaces: anholonomic, non-metric, Cartan, and Riemannian spaces. On the other hand, the four types of non-Euclidean spaces are not mutually exclusive. The above terminology is not always consistent in the literature (Schouten 1954; Bilby et al. 1955; Kondo 1964; Noll 1967; Marsden and Hughes 1983; Steinmann 1996; Clayton et al. 2005). However, definitions given here are deemed as those used either most frequently or most logically for describing kinematics of deformable crystalline solids, as will become clear later in Chapter 3.

2.1.2 Manifolds, Coordinates, and Metrics

Denoted by $\chi_t(\mathcal{B}):\mathcal{B} \to \mathrm{E}^3$ is a smooth, invertible, time-dependent embedding of a material body $\mathcal{B}$ into three-dimensional Euclidean space E^3. The configuration of body $\mathcal{B}$ at time t is denoted by $B_t = \chi_t(\mathcal{B})$, henceforth written simply as B for $t > 0$. Initially, i.e., at $t = 0$, material particles $\mathcal{X} \in \mathcal{B}$ are said to occupy reference configuration $B_0 = \chi_0(\mathcal{B})$ and are assigned reference coordinates $X^A = \chi_0^A(\mathcal{X})$. Particles of material mapped to current configuration B are assigned spatial coordinates $x^a = \chi_t^a(\mathcal{X})$. A description of the motion of all material particles is furnished by the continuous, invertible mapping $\varphi = \chi_t \circ \chi_0{}^{-1}: B_0 \to B$. The $\circ$ operator denotes the composition, such that for two functions f and g, $(f \circ g)(X) = f(g(X))$. The inverse obeys $f \circ f^{-1} = i$, with i the identity operator. The local motion for a material particle identified by particular reference coordinates $X^A = \chi_0^A(\mathcal{X})$ is written φ_X. Restricting the motion so that no two material particles occupy the same spatial location may limit the domain of $\varphi(X, t)$ to open regions of the body, e.g., when two points at different referential locations of the boundary of a body may come into contact as a result of deformation. Mappings χ_0, χ_t, and φ can be applied pointwise or globally and are not vector fields. For this reason, and since these mappings may operate without explicit introduction of coordinate basis vectors, boldface notation is not used to symbolically represent these functions. As indicated implicitly already and as will be made clear later, contravariant indices (i.e., indices in the upper position) are appended to such mappings as needed when coordinate systems are involved.

A point in space occupied by the body in the reference configuration is denoted by $X \in B_0$, while a point in space occupied by the body in the current configuration is denoted by $x \in B$. Functions depending on position in the reference configuration are denoted by $f(X)$, while functions depending on position in the current configuration are written $f(x)$. No distinction is made here between functions depending on material particle, e.g., a material description $f(X)$ (Malvern 1969), and those depending on the reference coordinates X^A of that particle, e.g., a Lagrangian description $f(\mathbf{X})$. Because the reference configuration is embedded in Euclidean space, a unique vector of coordinates **X** can always be assigned to each material particle at *X*. Similar arguments hold for the spatial description, since at any given time *t*, each location *x* can be assigned a unique vector of coordinates **x**. Thus the notation for a function of spatial position, $f(x)$, is hereafter used interchangeably with $f(\mathbf{x})$.

In indicial notation, time-dependent components of the motion and its inverse, respectively, are expressed in functional form as

$$x^a = x^a(X,t), \; X^A = X^A(x,t). \tag{2.1}$$

The first of (2.1) implies that spatial coordinates x^a of each material particle depend upon the choice of point *X*, or equivalently the material particle located at that point in the reference configuration, and time *t*. The second of (2.1) assigns a set of reference coordinates X^A to a material particle that occupies spatial location *x* at time *t*. Spatial coordinates x^a of a material particle corresponding to location *X* will generally change with time as a result of motion, and spatial locations *x* occupied by the body at one instance of time may not coincide with those occupied at a different instance.

The manifold concept, in the context of differential geometry, is now formally introduced (Boothby 1975). An *n*-manifold is a set $\mathcal{M}$ such that for each point $P \in \mathcal{M}$ there is a subset $\mathcal{U}$ of $\mathcal{M}$ containing *P*, and a one-to-one mapping called a chart or coordinate system from $\mathcal{U}$ onto an open set in the *n*-dimensional space of real numbers $\mathbb{R}^n$. Multiple charts or coordinate systems may be introduced on (regions of) a given manifold. Transformations between different coordinate systems over (regions of) the manifold are assumed to be infinitely differentiable if the manifold is smooth; i.e., such changes in coordinates are of continuity class C^∞ for smooth manifolds. A collection of charts covering $\mathcal{M}$ is called an atlas. As will be discussed in Section 2.2.4, the chart of a smooth *n*-manifold can be embedded in *n*-dimensional Euclidean space E^n if and only if the curvature tensor of its corresponding Levi-Civita connection vanishes.

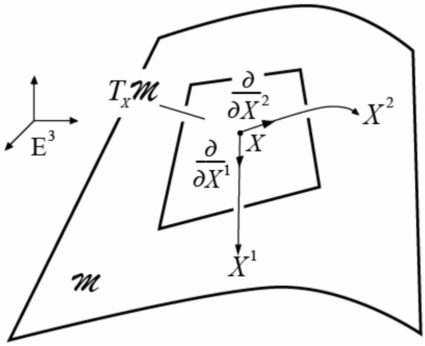

Fig. 2.1 Two-dimensional manifold embedded in 3-D Euclidean space

The tangent space of $\mathcal{M}$ at P is the n-dimensional space of contravariant vectors emanating from P, written as $T_P\mathcal{M}$. The collection of base points P and tangent vectors at all P comprises the tangent bundle $T\mathcal{M}$ (i.e., the tangent bundle is the union of tangent spaces over all points P on $\mathcal{M}$), and the map $\pi_{\mathcal{M}}$ from a tangent vector to its base point is called that vector's projection. Inverse mappings $\pi_{\mathcal{M}}^{-1}$ comprise sections of the tangent bundle. Similarly, the cotangent space of $\mathcal{M}$ at P, written as $T_P^*\mathcal{M}$, is the n-dimensional space of covariant vectors—also called covectors, one-forms, reciprocal vectors, or dual vectors—emanating from P. The cotangent bundle, denoted by $T^*\mathcal{M}$, is defined analogously to the tangent bundle, i.e., the cotangent bundle is the union of cotangent spaces over manifold $\mathcal{M}$. Shown in Fig. 2.1 is a two-dimensional manifold $\mathcal{M}$ with non-vanishing curvature embedded in E^3, with a point X described by a pair of coordinates $\mathbf{X} = (X^1, X^2)$. Tangent space $T_X\mathcal{M}$ is also shown.

Each time-dependent configuration of a deformable body can be regarded as a manifold, with locations of particles of material in that configuration identified in a one-to-one manner with points $\mathcal{X}$ of $\mathcal{B}$; the latter itself can also be viewed as a manifold since charts of smooth coordinates (e.g., $\mathcal{X}^a$) can, in principle, be introduced to cover $\mathcal{B}$. Referential locations X and spatial locations x can be associated with base points on corresponding manifolds B_0 and B, respectively. Tangent spaces to reference and current manifolds at points X and x are written $T_X B_0$ and $T_x B$, respectively. Body $\mathcal{B}$, configurations B_0 and B, and tangent spaces $T_X B_0$ and $T_x B$ are illustrated in Fig. 2.2. Reference and current tangent bundles are

$TB_0 = \cup_{X \in B_0} T_X B_0$ and $TB = \cup_{x \in B} T_x B$, respectively. Analogously, cotangent spaces are written as $T_X^* B_0$ and $T_x^* B$, with $T^* B_0 = \cup_{X \in B_0} T_X^* B_0$ and $T^* B = \cup_{x \in B} T_x^* B$ the corresponding cotangent bundles in reference and current configurations, respectively.

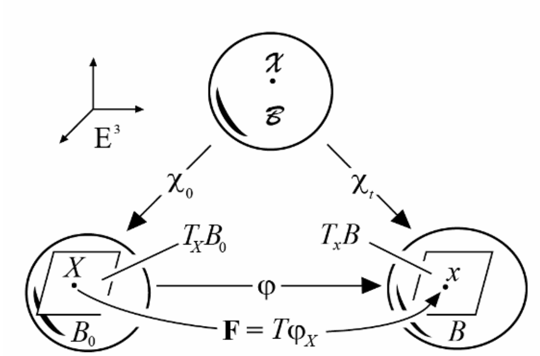

Fig. 2.2 Configurations, mappings, and tangent spaces

Natural or holonomic basis vectors in reference and current configurations are defined, respectively, by

$$\mathbf{G}_A = \frac{\partial}{\partial X^A} \in T_X B_0, \ \mathbf{g}_a = \frac{\partial}{\partial x^a} \in T_x B, \tag{2.2}$$

and are tangent to local coordinate curves at X or x; for example, see Fig. 2.1 in which $\mathcal{M}$ can represent the reference configuration B_0. Sometimes the basis vectors in (2.2) are called covariant basis vectors because their indices occupy lower positions, though this can cause confusion because these vectors act as the basis for general contravariant vectors. Basis vectors in (2.2) are sometimes written as $\mathbf{G}_A = \partial_A \mathbf{X} = \mathbf{X}_{,A}$ and $\mathbf{g}_a = \partial_a \mathbf{x} = \mathbf{x}_{,a}$. Dual or reciprocal bases to (2.2) are written as

$$\mathbf{G}^A = \mathbf{d}X^A \in T_X^* B_0, \ \mathbf{g}^a = \mathbf{d}x^a \in T_x^* B. \tag{2.3}$$

Basis vectors (2.2) and their duals (2.3) satisfy the orthonormality relations

$$\left\langle \mathbf{G}^A, \mathbf{G}_B \right\rangle_{B_0} = \delta^A_{.B}, \ \left\langle \mathbf{g}^a, \mathbf{g}_b \right\rangle_B = \delta^a_{.b}. \tag{2.4}$$

In (2.4), dual pairings (i.e., scalar product of vector and covector) in reference and spatial configurations correspond to the respective operations $\langle . , . \rangle_{B_0} : T_X^* B_0 \times T_X B_0 \to \mathbb{R}$ and $\langle . , . \rangle_B : T_x^* B \times T_x B \to \mathbb{R}$. The dual basis vectors are sometimes called contravariant basis vectors because their indices occupy upper positions. Also introduced in (2.4), Kronecker delta symbols satisfy

$$\delta^A_{.B} = \begin{cases} 1 \ \forall \ A = B, \\ 0 \ \forall \ A \neq B, \end{cases} \quad \delta^a_{.b} = \begin{cases} 1 \ \forall \ a = b, \\ 0 \ \forall \ a \neq b. \end{cases} \tag{2.5}$$

Location of the placeholder (index) denoted by the period is arbitrary in the special symbols defined in (2.5), but such placeholders are not always arbitrary for general tensor-valued quantities defined later in this book. Henceforward, subscripts on scalar product operations as in (2.4) denoting configuration(s) of arguments in angled brackets are omitted since the appropriate configuration(s) can always be inferred from the arguments. Summation over repeated indices produces $\delta^A_{.A} = \delta^a_{.a} = \delta^1_{.1} + \delta^2_{.2} + \delta^3_{.3} = 3$.

From (2.2)-(2.4), $T_X B_0$ is the linear vector space of all contravariant vectors $\mathbf{V} = V^A \mathbf{G}_A$ emanating from point $X \in B_0$. Cotangent space $T^*_X B_0$ is the linear vector space of all one-forms $\boldsymbol{\alpha} = \alpha_A \mathbf{G}^A$ emanating from point $X \in B_0$. In the context of dual products, vectors and one-forms are linear functions, $\mathbf{V}: T^*_X B_0 \to \mathbb{R}$ and $\boldsymbol{\alpha}: T_X B_0 \to \mathbb{R}$, and correspondingly

$$\langle \boldsymbol{\alpha}, \mathbf{V} \rangle = \boldsymbol{\alpha}(\mathbf{V}) = \alpha_A V^A = V^A \alpha_A = \mathbf{V}(\boldsymbol{\alpha}) = \langle \mathbf{V}, \boldsymbol{\alpha} \rangle, \tag{2.6}$$

invoking the symmetry property of the scalar product operation. Similar arguments hold for vectors and one-forms on respective tangent and cotangent spaces referred to current configuration B. Contravariant component V^A of vector $\mathbf{V}$ can be obtained via the scalar product operation as $\langle \mathbf{V}, \mathbf{G}^A \rangle = \langle V^B \mathbf{G}_B, \mathbf{G}^A \rangle = V^B \delta^A_{.B} = V^A$, where (2.4) has been used.

Let $\mathbf{G}: (\mathbf{V} \in T_X B_0, \mathbf{W} \in T_X B_0) \to \mathbb{R}$ be a symmetric, positive definite bilinear form assigning a real number to any two vectors $\mathbf{V}$ and $\mathbf{W}$ in $T_X B_0$. Object $\mathbf{G}$ is called a metric tensor, or simply a metric. Since $\mathbf{G}$ is symmetric, its components obey $G_{AB} = G_{(AB)}$, where indices in parentheses are symmetric: $2A_{(AB)} = A_{AB} + A_{BA}$ for arbitrary second-order tensor $\mathbf{A}$. Metric $\mathbf{G}$ maps a contravariant vector $\mathbf{V}$ into its associated covariant vector $\mathbf{V}_\#$:

$$\mathbf{V}_\# = \mathbf{G}\mathbf{V} = \left(G_{AB} \mathbf{G}^A \otimes \mathbf{G}^B\right)\left(V^B \mathbf{G}_B\right) = V_A \mathbf{G}^A, \ V_A = G_{AB} V^B. \tag{2.7}$$

The outer product (also called tensor product) of vectors and/or covectors, corresponding to a juxtaposition of indices, is denoted by $\otimes$ and satisfies the identity $(\mathbf{G}^A \otimes \mathbf{G}^B)\mathbf{G}_C = \mathbf{G}^A \langle \mathbf{G}^B, \mathbf{G}_C \rangle$.

The inverse of the metric tensor $\mathbf{G}^{-1}$, or simply the inverse metric, maps covectors $\boldsymbol{\alpha}$ into associated vectors $\boldsymbol{\alpha}^\#$:

$$\boldsymbol{\alpha}^\# = \mathbf{G}^{-1}\boldsymbol{\alpha} = \left(G^{AB} \mathbf{G}_A \otimes \mathbf{G}_B\right)\left(\alpha_B \mathbf{G}^B\right) = \alpha^A \mathbf{G}_A, \ \alpha^A = G^{AB} \alpha_B. \tag{2.8}$$

The standard notational convention of denoting components of $\mathbf{G}^{-1}$ with G^{AB} (i.e., $(\mathbf{G}^{-1})^{AB} = G^{AB}$) is followed henceforth.

On vector space $T_X B_0$, metric tensor $\mathbf{G}$ enables evaluation of the inner product or dot product of vectors:

$$\mathbf{V} \bullet \mathbf{W} = \langle \mathbf{V}, \mathbf{G}\mathbf{W} \rangle = V^A G_{AB} W^B . \tag{2.9}$$

Covariant component V_A of vector $\mathbf{V}$ can be obtained via use of the dot product operation as follows: $\mathbf{V} \bullet \mathbf{G}_A = V^B \mathbf{G}_B \bullet \mathbf{G}_A = V^B G_{BA} = V_A$. Similarly to (2.9), the inverse of the metric tensor, i.e., its contravariant form, enables evaluation of the inner product or dot product of covectors:

$$\boldsymbol{\alpha} \bullet \boldsymbol{\beta} = \langle \boldsymbol{\alpha}, \mathbf{G}^{-1} \boldsymbol{\beta} \rangle = \alpha_A G^{AB} \beta_B . \tag{2.10}$$

In the current configuration B, the metric tensor associated with spatial coordinates $\mathbf{x}$ is denoted by $\mathbf{g} : (\mathbf{v} \in T_x B, \mathbf{w} \in T_x B) \to \mathbb{R}$, with inverse $\mathbf{g}^{-1}$. Equations analogous to (2.7)-(2.10) apply for spatial metric $\mathbf{g}$ and its inverse in the current configuration.

From (2.4), (2.9), and (2.10), it is implied that matrix components of metric tensors and their inverses in reference and current configurations are given, respectively, by

$$G_{AB} = \mathbf{G}_A \bullet \mathbf{G}_B , \; G^{AB} = \mathbf{G}^A \bullet \mathbf{G}^B ; \tag{2.11}$$

$$g_{ab} = \mathbf{g}_a \bullet \mathbf{g}_b , \; g^{ab} = \mathbf{g}^a \bullet \mathbf{g}^b . \tag{2.12}$$

Metrics and their inverses are related, by definition, as follows:

$$G^{AB} G_{BC} = \delta^A_{.C} , \; g^{ab} g_{bc} = \delta^a_{.c} . \tag{2.13}$$

Relationships between basis vectors and their reciprocal vectors are

$$\mathbf{G}^A = G^{AB} \mathbf{G}_B , \; \mathbf{G}_A = G_{AB} \mathbf{G}^B , \; \mathbf{g}^a = g^{ab} \mathbf{g}_b , \; \mathbf{g}_a = g_{ab} \mathbf{g}^b . \tag{2.14}$$

In addition to their role in scalar product operations, metric tensors (or their inverses) may be used to lower (or raise) indices of tensors of arbitrarily higher order. For example, let $\mathbf{A} = A^{AB} \mathbf{G}_A \otimes \mathbf{G}_B$ be a generic contravariant tensor of order two. The fully covariant representation of $\mathbf{A}$ is

$$\mathbf{A}_\# = G_{AC} G_{BD} A^{CD} \mathbf{G}^A \otimes \mathbf{G}^B = A_{AB} \mathbf{G}^A \otimes \mathbf{G}^B . \tag{2.15}$$

When considering multiple configurations, it often becomes necessary to express components of vectors or tensors introduced in one configuration, with one set of coordinates, with respect to bases in another configuration with a different set of coordinates. For example, a vector $\mathbf{V}$ defined in a parallel manner on $T_X B_0$ and $T_x B$ is written (Eringen 1962)

$$\mathbf{V} = V^A \mathbf{G}_A (X) = V^a \mathbf{g}_a (x) . \tag{2.16}$$

Taking the inner product with dual basis vectors in each configuration, it follows that

$$V^B = V^A \left\langle \mathbf{G}_A, \mathbf{G}^B \right\rangle = V^a \left\langle \mathbf{g}_a, \mathbf{G}^B \right\rangle = V^a g^B_{.a}, \quad \mathbf{g}_a = g_a^{.A} \mathbf{G}_A; \tag{2.17}$$

$$V^b = V^a \left\langle \mathbf{g}_a, \mathbf{g}^b \right\rangle = V^A \left\langle \mathbf{G}_A, \mathbf{g}^b \right\rangle = V^A g^b_{.A}, \quad \mathbf{G}_A = g_A^{.a} \mathbf{g}_a; \tag{2.18}$$

where mixed-variant components of shifters, examples of two-point tensors, are defined as

$$g^A_{.a}(x, X) = \left\langle \mathbf{G}^A, \mathbf{g}_a \right\rangle = g_a^{.A}, \quad g^a_{.A}(x, X) = \left\langle \mathbf{g}^a, \mathbf{G}_A \right\rangle = g_A^{.a}. \tag{2.19}$$

Similarly, fully covariant and fully contravariant components of shifter tensors are, respectively,

$$g_{Aa}(x, X) = \mathbf{G}_A \bullet \mathbf{g}_a = g_{aA}, \quad g^{Aa}(x, X) = \mathbf{G}^A \bullet \mathbf{g}^a = g^{aA}. \tag{2.20}$$

From (2.14), it follows that components of the shifters are raised and lowered by components of metric tensors of the corresponding configuration:

$$g_{Aa} = g_{ab} g^b_{.A} = G_{AB} g^B_{.a} = g_{ab} G_{AB} g^{Bb}. \tag{2.21}$$

Moreover, summation over one set of indices leads to the identities

$$g^A_{.b} g^a_{.A} = \delta^a_{.b}, \quad g^A_{.a} g^a_{.B} = \delta^A_{.B}. \tag{2.22}$$

It follows that $\det(g^A_{.a}) = 1/\det(g^a_{.A}) = \sqrt{\det(g_{ab})/\det(G_{AB})}$, where det is the usual determinant of a second-order tensor. Shifters can also be used to express components of vectors and tensors introduced in one frame of coordinates with respect to a different coordinate frame in the same configuration via parallel transport (Toupin 1956; Ericksen 1960). Differentiation indices cannot usually be shifted; e.g., generally $\partial_a \neq g^A_{.a} \partial_A$.

Components of metric tensors on Euclidean spaces can be defined equivalently in terms of transformations to Cartesian coordinates and the dot product of vectors. For example, let X^A denote general curvilinear coordinates on B_0, and let $Z^A = Z^A(X)$ denote Cartesian coordinates on B_0. Since tangent vectors to Z^A are orthonormal by definition,

$$G_{AB} = \frac{\partial}{\partial X^A} \bullet \frac{\partial}{\partial X^B} = \frac{\partial Z^C}{\partial X^A} \frac{\partial Z^D}{\partial X^B} \left(\frac{\partial}{\partial Z^C} \bullet \frac{\partial}{\partial Z^D} \right) = \frac{\partial Z^C}{\partial X^A} \frac{\partial Z^D}{\partial X^B} \delta_{CD}, \tag{2.23}$$

with δ_{CD} covariant Kronecker delta symbols. From (2.23), the determinant of the matrix of metric tensor components is always non-negative: $\det(G_{AB}) = [\det(\partial Z^A / \partial X^B)]^2 \geq 0$. Similarly for the inverse metric tensor, $\det(G^{AB}) = 1/\det(G_{AB}) = [\det(\partial X^A / \partial Z^B)]^2 \geq 0$. The metric tensor (or its inverse) may have zero determinant along certain singular points or curves (e.g., at $R = 0$ in cylindrical or spherical coordinates) but is non-singular over any volume (Malvern 1969). In Cartesian space, the metric is simply $G_{AB} = \delta_{AB}$, and distinction between covariant and contravariant indices is not necessary. Analogously for a shifter, components can be computed via

$$g_{Aa} = \frac{\partial}{\partial X^A} \bullet \frac{\partial}{\partial x^a} = \frac{\partial Z^B}{\partial X^A} \frac{\partial z^b}{\partial x^a} \left(\frac{\partial}{\partial Z^B} \bullet \frac{\partial}{\partial z^b} \right) = \frac{\partial Z^B}{\partial X^A} \frac{\partial z^b}{\partial x^a} \delta_{Bb}, \tag{2.24}$$

where x^a and $z^a = z^a(x)$ denote curvilinear and Cartesian coordinates, respectively, in spatial configuration B, and δ_{Bb} reduces to Kronecker's delta when coincident Cartesian coordinate frames for Z^A and z^a are prescribed, respectively, in reference and current configurations.

Generally non-vanishing components of metric tensors for several common three-dimensional coordinate systems are listed in Table 2.1. Metric tensor components listed in Table 2.1 can be applied towards representations in either of the reference or current configurations.

Table 2.1 Metric tensors for common three-dimensional coordinate systems

Coordinate system	Non-vanishing components of G_{AB}
Cartesian: $(X^1, X^2, X^3) \to (X, Y, Z)$	$G_{XX} = G_{YY} = G_{ZZ} = 1$
Cylindrical: $(X^1, X^2, X^3) \to (R, \theta, Z)$	$G_{RR} = G_{ZZ} = 1$, $G_{\theta\theta} = R^2$
$X = R\cos\theta,\ Y = R\sin\theta,\ Z = Z$	
$R \geq 0,\ \theta \in (-\pi, \pi]$	
Spherical: $(X^1, X^2, X^3) \to (R, \theta, \varphi)$	$G_{RR} = 1$, $G_{\theta\theta} = R^2$, $G_{\varphi\varphi} = R^2 \sin^2\theta$
$X = R\sin\theta\cos\varphi,\ Y = R\sin\theta\sin\varphi,\ Z = R\cos\theta$	
$R \geq 0,\ \theta \in [0, \pi],\ \varphi \in (-\pi, \pi]$	

2.2 Linear Connections

The following topics are addressed in Section 2.2: the definition of a generic linear connection, its covariant derivative, and connection coefficients; torsion and curvature of a connection; identities from differential geometry describing properties of the connection and its torsion and curvature; and a special kind of connection called the Levi-Civita connection.

2.2.1 The Covariant Derivative and Connection Coefficients

A linear connection, also often called an affine connection, on a manifold B_0 induces an operation ∇ that assigns to two vector fields $\mathbf{V}, \mathbf{W} \in TB_0$ a third vector field $\nabla_{\mathbf{V}} \mathbf{W} \in TB_0$, called the covariant derivative of $\mathbf{W}$ along $\mathbf{V}$, such that

$$\begin{aligned}&\text{(i) } \nabla_{\mathbf{V}}\mathbf{W} \text{ is linear in both } \mathbf{V} \text{ and } \mathbf{W};\\&\text{(ii) } \nabla_{f\mathbf{V}}\mathbf{W} = f\nabla_{\mathbf{V}}\mathbf{W} \text{ for scalar function } f;\\&\text{(iii) } \nabla_{\mathbf{V}}(f\,\mathbf{W}) = f\nabla_{\mathbf{V}}\mathbf{W} + \langle \mathbf{V}, \mathbf{D}f\rangle \mathbf{W}.\end{aligned} \tag{2.25}$$

The derivative of f is $\mathbf{D}f$, and in coordinates the derivative of f in the direction of $\mathbf{V}$ is $\langle \mathbf{V}, \mathbf{D}f\rangle = V^A f_{,A} = V^A\partial_A f = V^A \partial f/\partial X^A$. The covariant derivative of $\mathbf{W}$ along $\mathbf{V}$ is written in indicial notation as

$$\nabla_{\mathbf{V}}\mathbf{W} = (V^B\partial_B W^A + \Gamma^{..A}_{BC}V^B W^C)\mathbf{G}_A = (V^B W^A_{,B} + \Gamma^{..A}_{BC}V^B W^C)\mathbf{G}_A, \tag{2.26}$$

where the notation $(\,.\,)_{,B} = \partial_B(\,.\,) = \partial(\,.\,)/\partial X^B$ is used interchangeably throughout this text for partial coordinate differentiation. The n^3 coefficients of the connection in n-dimensional space are written $\Gamma^{..A}_{BC}$. Connection coefficients do not follow conventional coordinate transformation laws for third-order tensors. Consider a coordinate transformation $X^A \to \hat{X}^{\hat{A}}$. Since vector field $\nabla_{\mathbf{V}}\mathbf{W}$ of (2.26) transforms conventionally under a change of basis as (Schouten 1954)

$$\begin{aligned}(\nabla_V W)^A &= V^B\frac{\partial W^A}{\partial X^B} + \Gamma^{..A}_{BC}V^B W^C\\
&= \hat{V}^{\hat{B}}\frac{\partial X^B}{\partial \hat{X}^{\hat{B}}}\frac{\partial}{\partial X^B}\left(\hat{W}^{\hat{C}}\frac{\partial X^A}{\partial \hat{X}^{\hat{C}}}\right) + \Gamma^{..A}_{BC}\hat{V}^{\hat{B}}\frac{\partial X^B}{\partial \hat{X}^{\hat{B}}}\hat{W}^{\hat{C}}\frac{\partial X^C}{\partial \hat{X}^{\hat{C}}}\\
&= \hat{V}^{\hat{B}}\frac{\partial}{\partial \hat{X}^{\hat{B}}}\left(\hat{W}^{\hat{C}}\frac{\partial X^A}{\partial \hat{X}^{\hat{C}}}\right) + \Gamma^{..A}_{BC}\frac{\partial X^B}{\partial \hat{X}^{\hat{B}}}\frac{\partial X^C}{\partial \hat{X}^{\hat{C}}}\hat{V}^{\hat{B}}\hat{W}^{\hat{C}}\\
&= \hat{V}^{\hat{B}}\frac{\partial \hat{W}^{\hat{C}}}{\partial \hat{X}^{\hat{B}}}\frac{\partial X^A}{\partial \hat{X}^{\hat{C}}} + \left(\Gamma^{..A}_{BC}\frac{\partial X^B}{\partial \hat{X}^{\hat{B}}}\frac{\partial X^C}{\partial \hat{X}^{\hat{C}}} + \frac{\partial}{\partial \hat{X}^{\hat{B}}}\frac{\partial X^A}{\partial \hat{X}^{\hat{C}}}\right)\hat{V}^{\hat{B}}\hat{W}^{\hat{C}}\\
&= \frac{\partial X^A}{\partial \hat{X}^{\hat{A}}}\hat{V}^{\hat{B}}\frac{\partial \hat{W}^{\hat{A}}}{\partial \hat{X}^{\hat{B}}} + \frac{\partial X^A}{\partial \hat{X}^{\hat{A}}}\left(\Gamma^{..D}_{BC}\frac{\partial \hat{X}^{\hat{A}}}{\partial X^D}\frac{\partial X^B}{\partial \hat{X}^{\hat{B}}}\frac{\partial X^C}{\partial \hat{X}^{\hat{C}}}\right.\\
&\qquad\left. + \frac{\partial \hat{X}^{\hat{A}}}{\partial X^C}\frac{\partial}{\partial \hat{X}^{\hat{B}}}\frac{\partial X^C}{\partial \hat{X}^{\hat{C}}}\right)\hat{V}^{\hat{B}}\hat{W}^{\hat{C}}\\
&= \frac{\partial X^A}{\partial \hat{X}^{\hat{A}}}\left(\hat{V}^{\hat{B}}\frac{\partial \hat{W}^{\hat{A}}}{\partial \hat{X}^{\hat{B}}} + \hat{\Gamma}^{..\hat{A}}_{\hat{B}\hat{C}}\hat{V}^{\hat{B}}\hat{W}^{\hat{C}}\right) = \frac{\partial X^A}{\partial \hat{X}^{\hat{A}}}(\nabla_{\hat{V}}\hat{W})^{\hat{A}},\end{aligned} \tag{2.27}$$

transformation formulae for the connection coefficients are deduced as

$$\begin{aligned}\hat{\Gamma}^{\hat{A}}_{\hat{B}\hat{C}} &= \frac{\partial X^B}{\partial \hat{X}^{\hat{B}}}\frac{\partial X^C}{\partial \hat{X}^{\hat{C}}}\frac{\partial \hat{X}^{\hat{A}}}{\partial X^A}\Gamma^{..A}_{BC} + \frac{\partial \hat{X}^{\hat{A}}}{\partial X^C}\frac{\partial}{\partial \hat{X}^{\hat{B}}}\frac{\partial X^C}{\partial \hat{X}^{\hat{C}}} \\ &= \frac{\partial X^B}{\partial \hat{X}^{\hat{B}}}\frac{\partial X^C}{\partial \hat{X}^{\hat{C}}}\left(\frac{\partial \hat{X}^{\hat{A}}}{\partial X^A}\Gamma^{..A}_{BC} - \frac{\partial}{\partial X^B}\frac{\partial \hat{X}^{\hat{A}}}{\partial X^C}\right),\end{aligned} \tag{2.28}$$

with the second equality in (2.28) following readily from the identity $(\partial_C \hat{X}^{\hat{A}} \partial_{\hat{C}} X^C)_{,\hat{B}} = (\delta^{\hat{A}}_{.\hat{C}})_{,\hat{B}} = 0$. Furthermore, the covariant derivative of a vector field, $\nabla \mathbf{W}(X)$, is a $\left\{\begin{smallmatrix}1\\1\end{smallmatrix}\right\}$ tensor field, expressed in components as

$$\begin{aligned}\nabla \mathbf{W}(X) &= (\partial_B W^A + \Gamma^{..A}_{BC} W^C)\mathbf{G}_A \otimes \mathbf{G}^B \\ &= (W^A_{.,B} + \Gamma^{..A}_{BC} W^C)\mathbf{G}_A \otimes \mathbf{G}^B.\end{aligned} \tag{2.29}$$

An affine connection on tangent bundle TB_0 enables parallel transport of vectors across different tangent spaces in TB_0. A vector is said to undergo parallel transport with respect to a connection with covariant derivative ∇ along paths for which its covariant derivative vanishes. For example, a vector $\mathbf{W}$ is considered to be parallel along a curve $\boldsymbol{\lambda}(t)$ if $\nabla_{\mathbf{V}} \mathbf{W} = 0$, where $\mathbf{V} = \partial \boldsymbol{\lambda} / \partial t$ is tangent to the curve parameterized by t. A vector $\mathbf{W}$ is then parallel transported along $\boldsymbol{\lambda}$ if it is extended to a parallel vector field $\mathbf{W}(t)$ for all values of t.

The covariant derivative is applied to covector fields and to tensor fields of higher order as follows (Schouten 1954):

$$\begin{aligned}\nabla_N A^{A...F}{}_{G...M} = {} & A^{A...F}{}_{G...M,N} \\ & + \Gamma^{..A}_{NR} A^{RB...F}{}_{G...M} + ... + \Gamma^{..F}_{NR} A^{A...ER}{}_{G...M} \\ & - \Gamma^{..R}_{NG} A^{A...F}{}_{RH...M} - ... - \Gamma^{..R}_{NM} A^{A...F}{}_{G...LR},\end{aligned} \tag{2.30}$$

where the index of covariant differentiation is a subscript immediately following the ∇-operator. Notice that the first covariant index of the connection coefficients corresponds to that of the differentiation. The covariant derivative of an absolute or true scalar function is defined in the same way as its ordinary partial derivative: $\nabla_N A = \partial_N A = A_{,N}$. Hence the covariant derivative of a constant scalar vanishes identically. Another useful identity is $\nabla_N \delta^A_{.B} = \delta^A_{.B,N} + \Gamma^{..A}_{NC}\delta^C_{.B} - \Gamma^{..C}_{NB}\delta^A_{.C} = \delta^A_{.B,N} + \Gamma^{..A}_{NB} - \Gamma^{..A}_{NB} = \delta^A_{.B,N} = 0$. From the linearity property of the connection, the covariant derivative of a sum of objects is equal to the sum of the covariant derivatives of these objects. Covariant differentiation obeys the product rule of Leibniz; e.g., $\nabla_C(V^A W^B) = V^A \nabla_C W^B + W^B \nabla_C V^A$. Analogous to terminology for parallel transport of vectors, a tensor is said to be parallel transported along a curve if its covariant derivative vanishes as the tensor is dragged along the curve.

2.2.2 Torsion and Curvature

The torsion tensor **T** of a connection is defined by the operation

$$2\mathbf{T}(\mathbf{V},\mathbf{W}) = \nabla_{\mathbf{V}}\mathbf{W} - \nabla_{\mathbf{W}}\mathbf{V} - [\mathbf{V},\mathbf{W}], \tag{2.31}$$

where the Lie bracket of vector fields **V** and **W** on TB_0 is

$$[\mathbf{V},\mathbf{W}] = \left(V^B W^A_{.,B} - W^B V^A_{.,B}\right)\mathbf{G}_A . \tag{2.32}$$

From (2.26), (2.31), and (2.32), the torsion tensor is

$$\mathbf{T} = \Gamma^{..A}_{[BC]}\mathbf{G}^B \otimes \mathbf{G}^C \otimes \mathbf{G}_A , \tag{2.33}$$

where pairs of indices in square brackets are anti-symmetric, e.g., for a second-rank tensor $2A_{[AB]} = A_{AB} - A_{BA}$. A connection is torsion-free when its torsion tensor vanishes, or equivalently, when its connection coefficients are symmetric in covariant indices. The torsion tensor of a linear connection on a manifold is often called Cartan's torsion, by association with geometer E. Cartan (Cartan 1922). A connection with vanishing torsion is said to be symmetric. Sometimes twice the quantity in (2.31) and (2.33) is used as the definition of the torsion (Marsden and Hughes 1983; Clayton et al. 2004a, b, 2005, 2006, 2008). One may verify that the torsion transforms like a true tensor of order $\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}$ under a change of holonomic coordinate basis, by direct substitution of (2.33) into (2.28).

The Riemann-Christoffel curvature tensor associated with a linear connection with covariant derivative ∇, $\mathbf{R}$: $\overset{*}{T_X}B_0 \times T_X B_0 \times T_X B_0 \times T_X B_0 \to \mathbb{R}$, is a $\left\{\begin{smallmatrix}1\\3\end{smallmatrix}\right\}$ tensor with component representation

$$\begin{aligned} R^{...A}_{BCD} &= \Gamma^{..A}_{CD,B} - \Gamma^{..A}_{BD,C} + \Gamma^{..A}_{BE}\Gamma^{..E}_{CD} - \Gamma^{..A}_{CE}\Gamma^{..E}_{BD} \\ &= 2\partial_{[B}\Gamma^{..A}_{C]D} + 2\Gamma^{..A}_{[B|E|}\Gamma^{..E}_{C]D}, \end{aligned} \tag{2.34}$$

where indices in vertical bars are excluded from the anti-symmetry operation. Order and placement of indices used in the definition of **R** vary among authors (Schouten 1954; Fosdick 1966; Marsden and Hughes 1983; Clayton et al. 2005); conventions adopted in this book for the Riemann-Christoffel curvature tensor and quantities derived from it follow those of Schouten (1954) and Minagawa (1979). The Riemann-Christoffel curvature tensor transforms like a true tensor of order $\left\{\begin{smallmatrix}1\\3\end{smallmatrix}\right\}$ under a change of holonomic coordinates (Schouten 1954). From definition (2.34), **R** is always anti-symmetric in the first two covariant indices. Definitions, in indicial components, of several quantities constructed from the Riemann-Christoffel curvature tensor include its fully covariant version

$$R_{BCDA} = R_{BCD}^{...E} G_{EA}, \tag{2.35}$$

called simply the curvature tensor by Eringen (1962) and the Riemann tensor by Fosdick (1966), though again the placement of indices varies among authors. Components of the Ricci curvature are

$$R_{CD} = R_{ACD}^{...A}. \tag{2.36}$$

Recalling that n is the dimensionality of the space, scalar curvature κ is defined as

$$\kappa = \frac{1}{n(n-1)} R_{AB} G^{AB} = \frac{1}{n(n-1)} R, \tag{2.37}$$

with R the trace of the Ricci curvature. For $n = 2$, R of (2.37) is equivalent to the Gaussian curvature. Finally, Einstein's tensor $\boldsymbol{\theta}$ has components

$$4\theta^{AB} = \varepsilon^{ACD} \varepsilon^{BEF} R_{CDEF}, \tag{2.38}$$

where components of the permutation tensor ε^{ACD} are introduced formally later in (2.64).

The skew second covariant derivatives of a contravariant vector $\mathbf{V}$ and covariant vector $\boldsymbol{\alpha}$ can be expressed as (Schouten 1954)

$$\nabla_{[B}\nabla_{C]} V^A = \frac{1}{2} R_{BCD}^{...A} V^D - T_{BC}^{..D} \nabla_D V^A, \tag{2.39}$$

$$\nabla_{[B}\nabla_{C]} \alpha_D = -\frac{1}{2} R_{BCD}^{...A} \alpha_A - T_{BC}^{..A} \nabla_A \alpha_D. \tag{2.40}$$

Skew differential operator $\nabla_{[B}\nabla_{C]}$ obeys the product rule of Leibniz, e.g.,

$$\nabla_{[B}\nabla_{C]}(V^A \alpha_D) = \alpha_D \nabla_{[B}\nabla_{C]} V^A + V^A \nabla_{[B}\nabla_{C]} \alpha_D. \tag{2.41}$$

2.2.3 Identities for the Connection Coefficients and Curvature

Coefficients of an arbitrary linear connection can be written in the form (Schouten 1954)

$$\Gamma_{BC}^{..A} = \left\{ {}_{BC}^{..A} \right\} + T_{BC}^{..A} - T_{C.B}^{.A.} + T_{.BC}^{A..} + \frac{1}{2}(M_{BC}^{..A} + M_{C.B}^{.A.} - M_{.BC}^{A..}), \tag{2.42}$$

where for a symmetric, three times differentiable, and invertible but otherwise arbitrary second-rank tensor with components G_{AB} and inverse components G^{AB}, the quantities

$$\left\{ {}^{..A}_{BC} \right\} = \frac{1}{2} G^{AD} (G_{CD,B} + G_{BD,C} - G_{BC,D}) \tag{2.43}$$

are called Christoffel symbols[3] of the tensor G_{AB}. Christoffel symbols in (2.43) are symmetric in covariant indices. Also in (2.42), the third-order object

$$M^{..A}_{BC} = G^{AD} M_{BCD} = -G^{AD} \nabla_B G_{CD} = G_{CD} \nabla_B G^{AD}, \tag{2.44}$$

with the final equality following from identity $\nabla_B (G_{CD} G^{AD}) = \nabla_B \delta^A_{.C} = 0$. The covariant derivative of G_{AB} is, from (2.30) and coefficients in (2.42),

$$\nabla_A G_{BC} = G_{BC,A} - \Gamma^{..D}_{AB} G_{DC} - \Gamma^{..D}_{AC} G_{BD} = -M_{ABC}. \tag{2.45}$$

Symmetry conditions $M_{ABC} = M_{A(BC)}$ follow immediately from conditions $G_{AB} = G_{BA}$. When $\nabla_A G_{BC} = 0$, or equivalently when $M_{ABC} = 0$, the connection is said to be metric with respect to G_{AB}. In that case, covariant differentiation via ∇ and raising (or lowering) of indices via G^{AB} (or G_{AB}) or commute. In general, G_{AB} of (2.43)-(2.45) need not be the metric tensor used to define scalar products of vectors. However, when the connection of (2.42) is metric, and when G_{AB} is in fact the metric tensor of the space with connection (2.42), then G_{AB} is called the fundamental tensor of the space (Schouten 1954). In the particular case of Riemannian geometry, by definition the connection is simultaneously symmetric ($T^{..A}_{BC} = 0$) and metric ($M^{..A}_{BC} = 0$), leading to

$$\Gamma^{..A}_{BC} = \left\{ {}^{..A}_{BC} \right\} \text{ (Riemannian geometry).} \tag{2.46}$$

Returning now to the general case in (2.42) with possibly non-vanishing torsion and non-metric connection, the Riemann-Christoffel curvature tensor of (2.34) exhibits the following properties (Schouten 1954):

$$R^{...A}_{(BC)D} = 0, \tag{2.47}$$

$$R^{...A}_{[BCD]} = 2\nabla_{[B} T^{..A}_{CD]} - 4T^{..E}_{[BC} T^{..A}_{D]E}, \tag{2.48}$$

$$R_{AB(CD)} = \nabla_{[A} M_{B]CD} + T^{..E}_{AB} M_{ECD}, \tag{2.49}$$

$$\nabla_{[E} R^{...A}_{BC]D} = 2T^{..F}_{[EB} R^{...A}_{C]FD}, \tag{2.50}$$

where anti-symmetry over three indices is expressed as

$$6A_{[ABC]} = A_{ABC} + A_{BCA} + A_{CAB} - A_{BAC} - A_{CBA} - A_{ACB}. \tag{2.51}$$

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[3] In tensor analysis, $\left\{ {}^{..A}_{BC} \right\}$ are often labeled Christoffel symbols of the second kind, and $[BC,A] = G_{AD} \left\{ {}^{..D}_{BC} \right\}$ are often labeled Christoffel symbols of the first kind.

Relation (2.50) is often called Bianchi's identity. For a symmetric connection the right sides of (2.48) and (2.50) vanish, and for a metric connection the right side of (2.49) vanishes. For a Riemannian connection of the type (2.46) that is both symmetric and metric,

$$R_{(BC)DA} = 0\,,\ R_{[BCD]A} = 0\,,\ R_{BC(DA)} = 0\,,\ R_{BCDA} = R_{DABC}\,, \tag{2.52}$$

and the number of independent components of $\mathbf{R}$ is $n^2(n^2-1)/12$, for example one independent component for a two-dimensional space and six independent components for a three-dimensional space. In Riemannian geometry, the Ricci tensor of (2.36) and Einstein's tensor of (2.38) are both symmetric and satisfy

$$\theta_{AB} = R_{AB} - \frac{1}{2}RG_{AB}\,,\ \nabla_A \theta^A_{.B} = \nabla_A\left(R_{CB}G^{AC} - \frac{1}{2}R\delta^A_{.B}\right) = 0\,. \tag{2.53}$$

A space with connection for which the covariant derivative of the Ricci tensor of (2.36) vanishes, i.e., for which $\nabla_A R_{CD} = 0$, is called a Ricci space. A space in Riemannian geometry—that is a symmetric space with connection and metric in which (2.46) applies—in which the Ricci curvature and metric differ only by a scalar factor as

$$R_{CD} = \frac{1}{n}RG_{CD} = (n-1)\kappa G_{CD} \tag{2.54}$$

is called an Einstein space. From vanishing of the right side of Bianchi's identity (2.50) for a symmetric space, it follows that the scalar curvature of an Einstein space is constant, i.e., $\nabla_A \kappa = 0$ (Schouten 1954).

2.2.4 The Levi-Civita Connection

For a smooth manifold B_0 with metric tensor $\mathbf{G} = (\mathbf{G}_A \bullet \mathbf{G}_B)\mathbf{G}^A \otimes \mathbf{G}^B$, there is a unique affine connection with the covariant derivative operator $\overset{\mathbf{G}}{\nabla}$ on B_0 that is torsion-free ($\overset{\mathbf{G}}{\mathbf{T}} = 0$) and metric ($\overset{\mathbf{G}}{\nabla}\mathbf{G} = 0$), i.e., for which parallel transport preserves the dot products of vectors. It is called the Levi-Civita connection (Marsden and Hughes 1983). The Levi-Civita connection is a particular example of (2.46) and hence is often called the Riemannian connection (Hou and Hou 1997). In this book the term Levi-Civita connection is reserved for the particular Riemannian connection whose curvature tensor vanishes as discussed below. Coefficients of $\overset{\mathbf{G}}{\nabla}$ are defined as

$$\overset{G}{\Gamma}{}^{..A}_{BC} = \frac{1}{2}G^{AD}(G_{BD,C} + G_{CD,B} - G_{BC,D}) = \overset{G}{\Gamma}{}^{..A}_{CB}\,. \tag{2.55}$$

Superscript **G** (or G) of the Levi-Civita connection and its corresponding covariant derivative or gradient operator is not subject to the summation convention. The Riemann-Christoffel curvature tensor formed by inserting Christoffel symbols of Levi-Civita connection (2.55) into definition (2.34) is denoted by $\overset{\mathbf{G}}{\mathbf{R}}$. This curvature tensor can be expressed completely in terms of metric $\mathbf{G}$ and its first and second partial derivatives with respect to coordinates X^A. A space B_0 with metric **G** and having $\overset{\mathbf{G}}{\mathbf{R}} = 0$ is called flat. One may show (Schouten 1954) that $\overset{\mathbf{G}}{\mathbf{R}} = 0$ if and only if one may assign parallel orthonormal coordinate basis vectors at each point $X \in B_0$ such that $G_{AB} \to \delta_{AB}$. Thus, the curvature tensor vanishes identically and the space is flat when B_0 is Euclidean. In fact, $\overset{\mathbf{G}}{\mathbf{R}} = 0$ are compatibility conditions for the existence of connection coefficients $\overset{G}{\Gamma}{}^{..A}_{BC}$ derived from a Euclidean metric tensor $\mathbf{G}$ via (2.55) (Schouten 1954; Ciarlet 1998). Henceforward in this book, space B_0 with metric **G** and associated connection (2.55) is always assumed Euclidean, meaning that $\overset{\mathbf{G}}{\mathbf{R}} = 0$ by definition. Thus, the notation $\overset{G}{\Gamma}{}^{..A}_{BC}$ is used to denote components of $\left\{ {}^{..A}_{BC} \right\}$ of (2.46) when $G_{AB} = \mathbf{G}_A \bullet \mathbf{G}_B$ and when the Riemann Christoffel curvature tensor formed from $\left\{ {}^{..A}_{BC} \right\}$ vanishes. On the other hand, on a curved surface—for example a two-dimensional shell parameterized by a pair of coordinates $X^A : \mathcal{M} \to \mathbb{R}^2$ embedded in E^3 such as shown in Fig. 2.1—the single independent component of the curvature from $\left\{ {}^{..A}_{BC} \right\}$ does not vanish.

In terms of Levi-Civita connection (2.55), partial coordinate derivatives of natural basis vectors and dual basis vectors are

$$\mathbf{G}^A_{\ ,B} = -\overset{G}{\Gamma}{}^{..A}_{BC}\mathbf{G}^C, \ \mathbf{G}_{A,B} = \overset{G}{\Gamma}{}^{..C}_{BA}\mathbf{G}_C = \mathbf{G}_{B,A}, \tag{2.56}$$

and partial derivatives of components of the metric tensor are

$$G_{AB,C} = (\mathbf{G}_A \bullet \mathbf{G}_B)_{,C} = \overset{G}{\Gamma}{}^{..D}_{CB}G_{AD} + \overset{G}{\Gamma}{}^{..D}_{CA}G_{BD}. \tag{2.57}$$

Viewed another way, (2.56) and (2.57) imply that the basis vectors, dual basis vectors, and metric tensor are all parallel (i.e., have vanishing covariant derivatives) with respect to the Levi-Civita connection. Since the curvature and torsion of the connection (2.55) vanish by definition, (2.39) implies that skew second covariant derivatives vanish, i.e., $\overset{G}{\nabla}_{[B}\overset{G}{\nabla}_{C]}V^A = 0$

for a vector field with components $V^A(X)$. From the second of (2.56), $\mathbf{G}_{[A,B]} = 0$. However, in general $\partial_{[B} \overset{G}{\nabla}_{C]} V^A \neq 0$, meaning that partial and covariant differentiation do not always commute. From the first of (2.4) and the second of (2.56), referential Christoffel symbols of (2.55) can also be computed as $\overset{G}{\Gamma}{}^{..C}_{BA} = \overset{G}{\Gamma}{}^{..D}_{BA} \delta^C_{.D} = \overset{G}{\Gamma}{}^{..D}_{BA} \langle \mathbf{G}_D, \mathbf{G}^C \rangle = \langle \mathbf{G}_{A,B}, \mathbf{G}^C \rangle$.

In the context of geometrically nonlinear continuum mechanics, (2.55)-(2.57) may be formulated in spatial configuration B by replacing reference coordinates X^A with current coordinates x^a, replacing reference metric **G** with spatial metric **g**, and replacing reference configuration basis (co)vectors with current configuration basis (co)vectors. Christoffel symbols of the second kind for covariant derivative operator $\overset{\mathbf{g}}{\nabla}$ on B are

$$\overset{g}{\Gamma}{}^{..a}_{bc} = \frac{1}{2} g^{ad} (g_{bd,c} + g_{cd,b} - g_{bc,d}) = \overset{g}{\Gamma}{}^{..a}_{bc}, \tag{2.58}$$

and the curvature tensor derived from $\overset{g}{\Gamma}{}^{..a}_{bc}$ using (2.34) is denoted by $\overset{\mathbf{g}}{\mathbf{R}}$. Configuration B with connection (2.58) and metric **g** is always assumed a Euclidean space. Since the set $\{B, \overset{g}{\Gamma}{}^{..a}_{bc}, g_{ab}\}$ constitutes a Euclidean space, torsion and curvature tensors formed from (2.58) vanish identically. Analogs of (2.56) and (2.57) in the current configuration are

$$\mathbf{g}^a_{,b} = -\overset{g}{\Gamma}{}^{..a}_{bc} \mathbf{g}^c, \quad \mathbf{g}_{a,b} = \overset{g}{\Gamma}{}^{..c}_{ba} \mathbf{g}_c = \mathbf{g}_{b,a}; \tag{2.59}$$

$$g_{ab,c} = (\mathbf{g}_a \bullet \mathbf{g}_b)_{,c} = \overset{g}{\Gamma}{}^{..d}_{cb} g_{ad} + \overset{g}{\Gamma}{}^{..d}_{ca} g_{bd}; \tag{2.60}$$

and $\overset{g}{\nabla}_{[b} \overset{g}{\nabla}_{c]} V^a = 0$ for a spatial vector field with components $V^a(x)$. From the symmetry properties evident in (2.58), $\mathbf{g}_{[a,b]} = 0$. From the second of (2.4) and the second of (2.59), spatial Christoffel symbols of (2.58) can also be computed as $\overset{g}{\Gamma}{}^{..c}_{ba} = \overset{g}{\Gamma}{}^{..d}_{ba} \delta^c_{.d} = \overset{g}{\Gamma}{}^{..d}_{ba} \langle \mathbf{g}_d, \mathbf{g}^c \rangle = \langle \mathbf{g}_{a,b}, \mathbf{g}^c \rangle$.

In terms of transformation formulae to Cartesian coordinates introduced in (2.23) and (2.24), coefficients in (2.55) and (2.58) satisfy

$$\overset{G}{\Gamma}{}^{..A}_{BC} = \frac{\partial^2 Z^D}{\partial X^B \partial X^C} \frac{\partial X^A}{\partial Z^D}, \quad \overset{g}{\Gamma}{}^{..a}_{bc} = \frac{\partial^2 z^d}{\partial x^b \partial x^c} \frac{\partial x^a}{\partial z^d}. \tag{2.61}$$

Levi-Civita connection coefficients for cylindrical and spherical coordinate systems are listed in Table 2.2, where the superscript G has been dropped for brevity.

Since metric tensor components $G_{AB} = \delta_{AB} = \text{constant}$ in Cartesian coordinates, Christoffel symbols obtained from G_{AB} all vanish by (2.55) in Cartesian coordinates. So long as basis vectors $\mathbf{G}_A$ are spatially constant (but not necessarily orthogonal), Christoffel symbols formed from $G_{AB} = \mathbf{G}_A \bullet \mathbf{G}_B$ vanish identically, and covariant differentiation via $\overset{G}{\nabla}_A$ and partial differentiation via $\partial_A = \partial / \partial X^A$ are equivalent operations.

Table 2.2 Connection coefficients for common coordinate systems

Coordinate system	Nonzero components of $\Gamma^{..A}_{BC}$
Cartesian: $(X^1, X^2, X^3) \to (X, Y, Z)$	None
Cylindrical: $(X^1, X^2, X^3) \to (R, \theta, Z)$ $X = R\cos\theta,\ Y = R\sin\theta,\ Z = Z$	$\Gamma^{..\theta}_{R\theta} = \Gamma^{..\theta}_{\theta R} = 1/R$, $\Gamma^{..R}_{\theta\theta} = -R$
Spherical: $(X^1, X^2, X^3) \to (R, \theta, \varphi)$ $X = R\sin\theta\cos\varphi,\ Y = R\sin\theta\sin\varphi,\ Z = R\cos\theta$	$\Gamma^{..\theta}_{R\theta} = \Gamma^{..\theta}_{\theta R} = \Gamma^{..\varphi}_{R\varphi} = \Gamma^{..\varphi}_{\varphi R} = 1/R$, $\Gamma^{..R}_{\theta\theta} = -R$, $\Gamma^{..R}_{\varphi\varphi} = -R\sin^2\theta$, $\Gamma^{..\theta}_{\varphi\varphi} = -\sin\theta\cos\theta$, $\Gamma^{..\varphi}_{\theta\varphi} = \Gamma^{..\varphi}_{\varphi\theta} = \cot\theta$

2.3 Notation, Differential Operators, and Other Identities

To avoid frequent writing of the gradient operator for covariant derivatives, the following compact notation is used henceforward for covariant differentiation with respect to associated Levi-Civita connections with coefficients in (2.55) and (2.58):

$$\overset{G}{\nabla}_N A(X)^{A\ldots H}{}_{I\ldots M} = A^{A\ldots H}{}_{I\ldots M;N}\,, \tag{2.62}$$

$$\overset{g}{\nabla}_n B(x)^{a\ldots h}{}_{i\ldots m} = B^{a\ldots h}{}_{i\ldots m;n}\,, \tag{2.63}$$

where $\mathbf{A}(X)$ and $\mathbf{B}(x)$ are tensors of arbitrary order referred to reference and current coordinate systems, respectively. From (2.55)-(2.60), it follows that $\mathbf{G}^A_{;B} = 0$, $\mathbf{G}_{A;B} = 0$, $G_{AB;C} = 0$, $\mathbf{g}^a_{;b} = 0$, $\mathbf{g}_{a;b} = 0$, and $g_{ab;c} = 0$.

Contravariant and covariant components of third rank permutation tensors on configurations B_0 and B, respectively, are defined by

$$\varepsilon^{ABC} = \sqrt{G}^{\,-1} e^{ABC},\ \varepsilon_{ABC} = \sqrt{G}\, e_{ABC}\,; \tag{2.64}$$

$$\varepsilon^{abc} = \sqrt{g}^{\,-1} e^{abc},\ \varepsilon_{abc} = \sqrt{g}\, e_{abc}\,; \tag{2.65}$$

where henceforth the standard abbreviated notation $G = \det \mathbf{G} = \det(G_{AB})$ and $g = \det \mathbf{g} = \det(g_{ab})$ is used for determinants of metric tensors. Recall also the compact notation used in this text for reciprocals of square roots: $\sqrt[-1]{G} = 1/\sqrt{G}$ and $\sqrt[-1]{g} = 1/\sqrt{g}$. Permutation symbols, also often called Levi-Civita symbols, satisfy

$$e^{ABC} = e_{ABC} = \begin{cases} 0 \text{ when any two indices are equal} \\ +1 \text{ for } ABC = 123, 231, 312 \\ -1 \text{ for } ABC = 132, 213, 321; \end{cases} \tag{2.66}$$

$$e^{abc} = e_{abc} = \begin{cases} 0 \text{ when any two indices are equal} \\ +1 \text{ for } abc = 123, 231, 312 \\ -1 \text{ for } abc = 132, 213, 321. \end{cases} \tag{2.67}$$

Permutation tensors and symbols in (2.64)-(2.67) are anti-symmetric in all pairs and triplets of indices, e.g., $\varepsilon^{ABC} = \varepsilon^{[AB]C} = \varepsilon^{A[BC]} = \varepsilon^{[A|B|C]} = \varepsilon^{[ABC]}$ (refer to (2.51)). Covariant derivatives of permutation tensors in (2.64) and (2.65) with respect to Levi-Civita connections vanish since $G_{;A} = 0$ and $g_{;a} = 0$ (Malvern 1969). However, in general, $G_{,A} \neq 0$ and $g_{,a} \neq 0$ since $G(X)$ and $g(x)$ are not invariant under general changes of coordinates and hence are not absolute scalars. Identities for symbols of (2.66) are tabulated in Table 2.3. Analogous identities apply for spatial symbols e^{abc} and e_{abc} of (2.67) and for the permutation tensors of (2.64) and (2.65).

Table 2.3 Identities for permutation symbols

Identity	Identity
$e^{ABC} e_{DEF} = \det \begin{bmatrix} \delta^A_{.D} & \delta^A_{.E} & \delta^A_{.F} \\ \delta^B_{.D} & \delta^B_{.E} & \delta^B_{.F} \\ \delta^C_{.D} & \delta^C_{.E} & \delta^C_{.F} \end{bmatrix}$	$e^{ABC} e_{ADE} = \delta^B_{.D}\delta^C_{.E} - \delta^B_{.E}\delta^C_{.D}$
$e^{ABC} e_{ABE} = 2\delta^C_{.E}$	$e^{ABC} e_{ABC} = 2\delta^A_{.A} = 6$

Several mathematical operations are now introduced in reference coordinates. Analogous definitions apply for spatial coordinates. The trace of a second-order tensor $\mathbf{A}$ is defined as summation over its mixed-variant indices:

$$\mathrm{tr}\mathbf{A} = A^A_{.A} = A^{AB} G_{AB} = A_{AB} G^{AB} . \tag{2.68}$$

The transpose is identified as a horizontal switch of indices:

$$A^T_{AB} = A_{BA}\,,\ (A^T)^{AB} = A^{BA}\,. \tag{2.69}$$

The gradient of a scalar function f is equivalent to its partial or covariant derivative:

$$\overset{\mathbf{G}}{\nabla} f = f_{;A}\mathbf{G}^A = f_{,A}\mathbf{G}^A\,. \tag{2.70}$$

From (2.56) and (2.62), the gradient of a contravariant vector field $\mathbf{V}$ is

$$\begin{aligned}\overset{\mathbf{G}}{\nabla}\mathbf{V} &= \mathbf{V}_{,B}\otimes\mathbf{G}^B = (V^A\mathbf{G}_A)_{,B}\otimes\mathbf{G}^B\\ &= (V^A_{.,B}\mathbf{G}_A + V^A\mathbf{G}_{A,B})\otimes\mathbf{G}^B\\ &= V^A_{.,B}\mathbf{G}_A\otimes\mathbf{G}^B + V^A\overset{G}{\Gamma}{}^{..C}_{BA}\mathbf{G}_C\otimes\mathbf{G}^B\\ &= (V^A_{.,B} + V^C\overset{G}{\Gamma}{}^{..A}_{BC})\mathbf{G}_A\otimes\mathbf{G}^B = V^A_{;B}\mathbf{G}_A\otimes\mathbf{G}^B\,.\end{aligned} \tag{2.71}$$

The divergence of a contravariant vector field $\mathbf{V}$ satisfies

$$\begin{aligned}\left\langle\overset{\mathbf{G}}{\nabla},\mathbf{V}\right\rangle &= \mathrm{tr}\,\overset{\mathbf{G}}{\nabla}\mathbf{V} = \left\langle\mathbf{V}_{,A},\mathbf{G}^A\right\rangle\\ &= \left\langle(V^B\mathbf{G}_B)_{,A},\mathbf{G}^A\right\rangle = V^A_{;A} = \sqrt{G}^{-1}\left(\sqrt{G}V^A\right)_{,A}.\end{aligned} \tag{2.72}$$

The final equality in (2.72) follows from the identity (Eringen 1962)

$$\left(\ln\sqrt{G}\right)_{,A} = \overset{G}{\Gamma}{}^{..B}_{AB} = \overset{G}{\Gamma}{}^{..B}_{BA}\,. \tag{2.73}$$

The curl of a covariant vector field $\boldsymbol{\alpha}$ satisfies

$$\overset{\mathbf{G}}{\nabla}\times\boldsymbol{\alpha} = \mathbf{G}^A\times(\alpha_B\mathbf{G}^B)_{,A} = \mathbf{G}^A\times\mathbf{G}^B\alpha_{B;A} = \varepsilon^{ABC}\alpha_{C;B}\mathbf{G}_A\,, \tag{2.74}$$

where the vector cross product of two contravariant or covariant vectors, respectively, is

$$\mathbf{V}\times\mathbf{W} = \varepsilon_{ABC}V^BW^C\mathbf{G}^A\,,\ \boldsymbol{\alpha}\times\boldsymbol{\beta} = \varepsilon^{ABC}\alpha_B\beta_C\mathbf{G}_A\,. \tag{2.75}$$

Since the Levi-Civita connection is symmetric, the covariant derivative in (2.74) can be replaced with a partial derivative:

$$\varepsilon^{ABC}\alpha_{C;B} = \varepsilon^{ABC}\alpha_{C,B} - \varepsilon^{ABC}\overset{G}{\Gamma}{}^{..A}_{[BC]}\alpha_A = \varepsilon^{ABC}\alpha_{C,B} = \varepsilon^{ABC}\partial_B\alpha_C\,. \tag{2.76}$$

The Laplacian of a scalar function f is computed in general curvilinear coordinates as

$$\overset{\mathbf{G}}{\nabla}{}^2 f = \left(G^{AB}f_{,A}\right)_{;B} = G^{AB}\left(f_{,A}\right)_{;B} = \sqrt{G}^{-1}\left(\sqrt{G}G^{AB}f_{,A}\right)_{,B}\,. \tag{2.77}$$

A number of other identities can be derived immediately from definitions in (2.68)-(2.77). For example,

$$\left\langle\overset{\mathbf{G}}{\nabla},\overset{\mathbf{G}}{\nabla}\times\boldsymbol{\alpha}\right\rangle = \varepsilon^{ABC}\alpha_{C;BA} = \varepsilon^{ABC}\alpha_{C;(BA)} = 0\,, \tag{2.78}$$

$$\vec{\nabla}\times\vec{\nabla} f = \varepsilon^{ABC} f_{;CB}\mathbf{G}_A = \varepsilon^{ABC} f_{;(CB)}\mathbf{G}_A = 0\,. \tag{2.79}$$

Many others can be found in books on vector calculus, electromagnetism (Stratton 1941; Jackson 1999), or continuum mechanics (Malvern 1969).

Two mathematical operations for second- and higher-order tensors are defined next: the generalized dual product and the double-dot product. These operations will become particularly useful in Chapter 4 (and in subsequent Chapters) for defining energetic quantities in the context of the continuum thermodynamics of deformable bodies. The generalized dual product written for rank two tensors extends (2.4) to second-order tensors, providing a scalar product of two such quantities. For example,

$$\langle \mathbf{A},\mathbf{B}\rangle = A^{AB}B_{BA} \quad \left\{\forall \mathbf{A}\in T_X B_0 \times T_X B_0, \mathbf{B}\in T_X^* B_0 \times T_X^* B_0\right\}, \tag{2.80}$$

$$\langle \mathbf{a},\mathbf{b}\rangle = a^{ab}b_{ba} \quad \left\{\forall \mathbf{a}\in T_x B \times T_x B, \mathbf{b}\in T_x^* B \times T_x^* B\right\}, \tag{2.81}$$

$$\langle \mathbf{C},\mathbf{D}\rangle = C^{a}_{.A}D^{A}_{.a} \quad \left\{\forall \mathbf{C}\in T_x B \times T_X^* B_0, \mathbf{D}\in T_X B_0 \times T_x^* B\right\}. \tag{2.82}$$

In (2.82), **C** and **D** are examples of two-point tensors. The double-dot product, denoted by boldface colon **:**, implies summation over two sets of adjacent indices of second- and higher-order tensors. As with the generalized dual product, the double-dot product may be applied to quantities defined on the tangent and/or cotangent spaces of one or more configurations, i.e., two-point tensors. For example,

$$\mathbf{A}:\mathbf{B} = A^{AB}B_{AB} \quad \left\{\forall \mathbf{A}\in T_X B_0 \times T_X B_0, \mathbf{B}\in T_X^* B_0 \times T_X^* B_0\right\}, \tag{2.83}$$

$$\mathbf{a}:\mathbf{b} = a^{ab}b_{ab} \quad \left\{\forall \mathbf{a}\in T_x B \times T_x B, \mathbf{b}\in T_x^* B \times T_x^* B\right\}, \tag{2.84}$$

$$\mathbf{C}:\mathbf{D} = C^{aA}D_{aA} \quad \left\{\forall \mathbf{C}\in T_x B \times T_X B_0, \mathbf{D}\in T_x^* B \times T_x^* B_0\right\}, \tag{2.85}$$

$$\mathbf{E}:\mathbf{F} = E^{abcd}F_{cd}\,\mathbf{g}_a \otimes \mathbf{g}_b \left\{\forall \mathbf{E}\in T_x B \times T_x B \times T_x B \times T_x B, \mathbf{F}\in T_x^* B \times T_x^* B\right\}. \tag{2.86}$$

Notice from (2.86) that when tensors of rank greater than two are involved, the double-dot product operation does not yield a scalar as the result. In situations where there is no chance for confusion, e.g., scalar products of mixed contravariant-covariant pairs along the lines of (2.82), the dual product and double-dot product notations may be used interchangeably. In terms of the trace operation of (2.68),

$$\langle \mathbf{A},\mathbf{B}\rangle = \mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{A}^T\mathbf{B}^T),\ \mathbf{A}:\mathbf{B} = \mathrm{tr}(\mathbf{AB}^T) = \mathrm{tr}(\mathbf{A}^T\mathbf{B}). \tag{2.87}$$

2.4 Physical Components

Components of vectors and tensors expressed in general curvilinear coordinates generally do not all exhibit the same dimensional units. In general curvilinear systems, basis vectors need not be dimensionless. Physical components of these objects can be introduced so that all components exhibit the same dimensions, and so that all basis vectors are dimensionless and of unit length. Developments that follow in Section 2.4 are expressed with respect to the reference configuration; analogous expressions apply for spatial coordinates.

Physical components are referred to dimensionless basis vectors $\mathbf{E}_A$ obtained by normalizing basis vectors of a general curvilinear coordinate system by their lengths:

$$\mathbf{E}_A = \frac{\mathbf{G}_A}{(\mathbf{G}_A \bullet \mathbf{G}_A)^{1/2}} = \mathbf{G}_A \sqrt[-1]{G_{\underline{AA}}}\,, \tag{2.88}$$

where underlined indices are not subject to the summation convention. An arbitrary vector $\mathbf{V} \in T_X B_0$ can be written as

$$\mathbf{V} = V^A \mathbf{G}_A = V^A \sqrt{G_{\underline{AA}}}\,\mathbf{E}_A = V^{\langle A \rangle} \mathbf{E}_A\,, \tag{2.89}$$

where contravariant physical components of **V** are

$$V^{\langle A \rangle} = V^A \sqrt{G_{\underline{AA}}}\,. \tag{2.90}$$

Covariant components satisfy

$$V_A = G_{AB} V^B = G_{AB} \sqrt[-1]{G_{\underline{BB}}}\, V^{\langle B \rangle}\,. \tag{2.91}$$

Physical components of a tensor of higher order can be defined by considering the way the tensor represents phenomena in physics; however, transformation formulae for higher-order tensors from general curvilinear components to physical components often involve lengthy manipulations (Malvern 1969). Listed next are useful expressions for general orthogonal curvilinear coordinates, cylindrical coordinates, and spherical coordinates. Other kinds of curvilinear coordinate systems—elliptic, parabolic, bipolar, spheroidal, paraboloidal, and ellipsoidal—are addressed by Stratton (1941).

2.4.1 General Orthogonal Coordinates

General orthogonal coordinates are defined by the requirement that at each point X, the coordinate curves X^A passing through X are mutually orthogonal. This requirement results in a diagonal metric tensor

$$G_{AB} = \mathbf{G}_A \bullet \mathbf{G}_B = \begin{cases} G_{\underline{AA}} & \forall A = B, \\ 0 & \forall A \neq B. \end{cases} \tag{2.92}$$

Thus

$$d\mathbf{X} \bullet d\mathbf{X} = dX^A G_{AB} dX^B = G_{11}(dX^1)^2 + G_{22}(dX^2)^2 + G_{33}(dX^3)^2 , \tag{2.93}$$

$$G_{\underline{AA}} = 1/G^{\underline{AA}} , \tag{2.94}$$

$$G = G_{11}G_{22}G_{33} , \; G^{-1} = G^{11}G^{22}G^{33} . \tag{2.95}$$

Non-vanishing Christoffel symbols are

$$\overset{G}{\Gamma}{}^{..B}_{\underline{AA}} = -\frac{1}{2G_{\underline{BB}}} G_{\underline{AA},B} , \; \overset{G}{\Gamma}{}^{..B}_{\underline{B}A} = \overset{G}{\Gamma}{}^{..B}_{A\underline{B}} = (\ln\sqrt{G_{\underline{BB}}})_{,A} , \; \overset{G}{\Gamma}{}^{..A}_{\underline{AA}} = (\ln\sqrt{G_{\underline{AA}}})_{,A} , \tag{2.96}$$

and otherwise

$$\overset{G}{\Gamma}{}^{..A}_{BC} = 0 \; (A \neq B \neq C) . \tag{2.97}$$

In contravariant physical components of orthogonal coordinates, the gradient and Laplacian of a scalar function $f(X)$ and the divergence and curl of a vector field $\mathbf{V}(X)$ are, respectively (Eringen 1962)

$$\overset{\mathbf{G}}{\nabla} f = \sqrt[-1]{G_{11}} f_{,1}\mathbf{E}_1 + \sqrt[-1]{G_{22}} f_{,2}\mathbf{E}_2 + \sqrt[-1]{G_{33}} f_{,3}\mathbf{E}_3 , \tag{2.98}$$

$$\begin{aligned} \overset{\mathbf{G}}{\nabla}{}^2 f = \sqrt[-1]{G} \Big[& (\sqrt[-1]{G_{11}} \sqrt{G_{22}G_{33}} f_{,1})_{,1} \\ & + (\sqrt[-1]{G_{22}} \sqrt{G_{33}G_{11}} f_{,2})_{,2} + (\sqrt[-1]{G_{33}} \sqrt{G_{11}G_{22}} f_{,3})_{,3} \Big], \end{aligned} \tag{2.99}$$

$$\left\langle \overset{\mathbf{G}}{\nabla}, \mathbf{V} \right\rangle = \sqrt[-1]{G} \Big[(\sqrt{G_{22}G_{33}} V^{\langle 1 \rangle})_{,1} + (\sqrt{G_{33}G_{11}} V^{\langle 2 \rangle})_{,2} + (\sqrt{G_{11}G_{22}} V^{\langle 3 \rangle})_{,3} \Big], \tag{2.100}$$

$$\begin{aligned} \overset{\mathbf{G}}{\nabla} \times \mathbf{V} = & \sqrt[-1]{G_{22}G_{33}} \Big[(\sqrt{G_{33}} V^{\langle 3 \rangle})_{,2} - (\sqrt{G_{22}} V^{\langle 2 \rangle})_{,3} \Big] \mathbf{E}_1 \\ & + \sqrt[-1]{G_{33}G_{11}} \Big[(\sqrt{G_{11}} V^{\langle 1 \rangle})_{,3} - (\sqrt{G_{33}} V^{\langle 3 \rangle})_{,1} \Big] \mathbf{E}_2 \\ & + \sqrt[-1]{G_{11}G_{22}} \Big[(\sqrt{G_{22}} V^{\langle 2 \rangle})_{,1} - (\sqrt{G_{11}} V^{\langle 1 \rangle})_{,2} \Big] \mathbf{E}_3 . \end{aligned} \tag{2.101}$$

2.4.2 Cylindrical Coordinates

Cylindrical coordinates $(X^1, X^2, X^3) \to (R, \theta, Z)$ are a particular kind of orthogonal curvilinear coordinates. Transformation formulae to Cartesian coordinates and metric tensor components are listed in Table 2.1, and Christoffel symbols are listed in Table 2.2. In particular, a squared increment of distance is

$$d\mathbf{X}\bullet d\mathbf{X} = dX^A G_{AB} dX^B = G_{RR}(dR)^2 + G_{\theta\theta}(d\theta)^2 + G_{ZZ}(dZ)^2 \\ = (dR)^2 + R^2(d\theta)^2 + (dZ)^2. \tag{2.102}$$

In physical components of cylindrical coordinates, the gradient and Laplacian of a scalar $f(X)$ and the divergence and curl of a vector field $\mathbf{V}(X)$ are, respectively,

$$\overset{\mathbf{G}}{\nabla} f = f_{,R}\mathbf{E}_R + R^{-1} f_{,\theta}\mathbf{E}_\theta + f_{,Z}\mathbf{E}_Z, \tag{2.103}$$

$$\overset{\mathbf{G}}{\nabla}{}^2 f = f_{,RR} + R^{-1} f_{,R} + R^{-2} f_{,\theta\theta} + f_{,ZZ}, \tag{2.104}$$

$$\left\langle \overset{\mathbf{G}}{\nabla}, \mathbf{V} \right\rangle = R^{-1}(RV_R)_{,R} + R^{-1} V_{\theta,\theta} + V_{Z,Z}, \tag{2.105}$$

$$\overset{\mathbf{G}}{\nabla} \times \mathbf{V} = \left[R^{-1} V_{Z,\theta} - V_{\theta,Z} \right]\mathbf{E}_R + \left[V_{R,Z} - V_{Z,R} \right]\mathbf{E}_\theta \\ + R^{-1}\left[(RV_\theta)_{,R} - V_{R,\theta} \right]\mathbf{E}_Z. \tag{2.106}$$

By convention, indices of physical components of a vector in cylindrical coordinates are written in the subscript position, with angled brackets on indices omitted: $V^{\langle 1\rangle} = V^{\langle R\rangle} = V_R$, $V^{\langle 2\rangle} = V^{\langle \theta\rangle} = V_\theta$, and $V^{\langle 3\rangle} = V^{\langle Z\rangle} = V_Z$.

2.4.3 Spherical Coordinates

Spherical coordinates $(X^1, X^2, X^3) \to (R, \theta, \varphi)$ are a second particular kind of orthogonal curvilinear coordinates. Transformation formulae to Cartesian coordinates and metric tensor components are listed in Table 2.1; Christoffel symbols are listed in Tables 2.2. In particular, a squared increment of distance is

$$d\mathbf{X}\bullet d\mathbf{X} = dX^A G_{AB} dX^B = G_{RR}(dR)^2 + G_{\theta\theta}(d\theta)^2 + G_{\varphi\varphi}(d\varphi)^2 \\ = (dR)^2 + R^2(d\theta)^2 + R^2 \sin^2\theta (d\varphi)^2. \tag{2.107}$$

In physical components of spherical coordinates, the gradient and Laplacian of a scalar $f(X)$ and the divergence and curl of a vector field $\mathbf{V}(X)$ are computed, respectively, as follows:

$$\overset{\mathbf{G}}{\nabla} f = f_{,R}\mathbf{E}_R + R^{-1} f_{,\theta}\mathbf{E}_\theta + (R\sin\varphi)^{-1} f_{,\varphi}\mathbf{E}_\varphi, \tag{2.108}$$

$$\overset{\mathbf{G}}{\nabla}{}^2 f = R^{-2}(R^2 f_{,R})_{,R} + (R^2\sin\theta)^{-1}(f_{,\theta}\sin\theta)_{,\theta} + (R\sin\theta)^{-2} f_{,\varphi\varphi}, \tag{2.109}$$

$$\left\langle \overset{\mathbf{G}}{\nabla}, \mathbf{V} \right\rangle = R^{-2}(R^2 V_R)_{,R} + (R\sin\theta)^{-1}\left[(V_\theta \sin\theta)_{,\theta} + V_{\varphi,\varphi} \right], \tag{2.110}$$

$$\overset{\mathbf{G}}{\nabla}\times\mathbf{V}=(R\sin\theta)^{-1}\left[(V_\varphi\sin\theta)_{,\theta}-V_{\theta,\vartheta}\right]\mathbf{E}_R \\ +R^{-1}\left[(\sin\theta)^{-1}V_{R,\varphi}-(RV_\varphi)_{,R}\right]\mathbf{E}_\theta \\ +R^{-1}\left[(RV_\theta)_{,R}-V_{R,\theta}\right]\mathbf{E}_\varphi. \tag{2.111}$$

By convention, indices of physical components of a vector in spherical coordinates are written in the subscript position, with angled brackets on indices omitted: $V^{\langle 1\rangle}=V^{\langle R\rangle}=V_R$, $V^{\langle 2\rangle}=V^{\langle\theta\rangle}=V_\theta$, and $V^{\langle 3\rangle}=V^{\langle\varphi\rangle}=V_\varphi$.

2.5 The Deformation Gradient

The deformation gradient is a fundamental descriptor of kinematics of deformable bodies, and is of particular importance in the context of nonlinear solid mechanics. In Section 2.5, the definition and interpretation of the deformation gradient are provided, followed by definitions and identities for key quantities derived from the deformation gradient.

2.5.1 Fundamentals

Recall from Section 2.1.2 that motion from the reference configuration to the current configuration of a deformable body is denoted by continuous, invertible, one-to-one function $\varphi=\chi_t\circ\chi_0{}^{-1}:B_0\to B$. The deformation gradient field is defined as the tangent of φ, mapping vectors in TB_0 to vectors in TB, or locally at material point X, $\mathbf{F}(X)=T\varphi_X:T_XB_0\to T_xB$. A visual interpretation is provided in Fig. 2.2. When coordinate systems x^a and X^a are introduced on B_0 and B, respectively, such that $x^a=\varphi^a(X^A,t)$, then deformation gradient $\mathbf{F}$ can be written as

$$\mathbf{F}=F^a_{.A}\mathbf{g}_a\otimes\mathbf{G}^A=\frac{\partial\varphi^a}{\partial X^A}\mathbf{g}_a\otimes\mathbf{G}^A\in T_xB\times T^*_XB_0. \tag{2.112}$$

In components, the deformation gradient is often written[4] $F^a_{.A}(X,t)=x^a_{,A}$. At any particular time t, spatial coordinates x^a occupied by the body are assumed one-to-one functions of X^A and (usually) differential of class C^r ($r\geq 1$) with respect to X^A. Also at any time t, reference locations occupied by the body with coordinates X^A are assumed one-to-one and of

[4] Partial differentiation proceeds as $\partial_A(\cdot)=(\cdot)_{,A}=(\cdot)_{,a}\partial_Ax^a=(\cdot)_{,a}x^a_{,A}=\partial_a(\cdot)F^a_{.A}$.

class C^r with respect to spatial coordinates x^a. Thus $\det(x^a_{,A}) \neq 0$. If $r = 0$ across certain singular surfaces within an otherwise "smooth" body, then **F** may be discontinuous across such surfaces. Because it is referred to (possibly) distinct coordinate systems in different configurations, **F** is said to be a two-point tensor or double tensor (Ericksen 1960) whose components each transform like those of a vector (upper index) or covector (lower index) under transformations of only one set of coordinates. The deformation gradient operates on an arbitrary vector $\mathbf{V} = V^B \mathbf{G}_B \in T_X B_0$ as

$$\begin{aligned}\mathbf{FV} &= F^a_{.A} \mathbf{g}_a \otimes \mathbf{G}^A \left(V^B \mathbf{G}_B \right) \\ &= F^a_{.A} V^B \mathbf{g}_a \left\langle \mathbf{G}^A, \mathbf{G}_B \right\rangle = F^a_{.A} V^A \mathbf{g}_a \in T_x B,\end{aligned} \tag{2.113}$$

where the first of (2.4) has been used. From (2.113) it is clear why the deformation gradient is called a tangent map: **F** maps a reference vector **V** that is a linear combination of tangent basis vectors at a point X on the reference configuration to a spatial vector **FV** that is a linear combination of tangent basis vectors at a point x on the current configuration.

The deformation gradient provides the first-order approximation of the length and direction of a differential line element $d\mathbf{x} \in T_x B$ mapped to the current configuration from its referential representation $d\mathbf{X} \in T_X B_0$. For example, a Taylor-like series expansion can be written (Toupin 1964)

$$\begin{aligned} dx^a &= x^a(X') - x^a(X) \\ &= \underbrace{x^a_{,A}\Big|_X}_{F^a_{.A}} dX^A + \frac{1}{2!} \underbrace{x^a_{:AB}\Big|_X}_{F^a_{.A:B}} dX^A dX^B + \frac{1}{3!} \underbrace{x^a_{:ABC}\Big|_X}_{F^a_{.A:BC}} dX^A dX^B dX^C + \ldots, \end{aligned} \tag{2.114}$$

with $dX^A = X'^A - X^A$ an infinitesimal vector between reference points X and X'. Section D.1 of Appendix D contains a complete tensor derivation of (2.114), to second order in $d\mathbf{X}$. To first order in $d\mathbf{X}$,

$$d\mathbf{x} = \mathbf{F} d\mathbf{X}, \quad dx^a = F^a_{.A} dX^A = x^a_{,A} dX^A. \tag{2.115}$$

In (2.114), the total covariant derivative of the deformation gradient, i.e., the second-order position gradient, obeys (Eringen 1962; Toupin 1964)

$$\begin{aligned} x^a_{:AB} = F^a_{.A:B} &= F^a_{.A,B} - \overset{G}{\Gamma}{}^{..C}_{BA} F^a_{.C} + \overset{g}{\Gamma}{}^{..a}_{bc} F^c_{.A} F^b_{.B} \\ &= x^a_{,AB} - \overset{G}{\Gamma}{}^{..C}_{BA} x^a_{,C} + \overset{g}{\Gamma}{}^{..a}_{bc} x^c_{,A} x^b_{,B}. \end{aligned} \tag{2.116}$$

The total covariant derivative of a generic two-point or double tensor of arbitrary order, $\mathbf{A}(X, x)$, is defined as (Ericksen 1960; Eringen 1962)

$$\begin{aligned} (A^{A\ldots C\;a\ldots c}_{D\ldots F\;d\ldots f})_{:K} &= (A^{A\ldots C\;a\ldots c}_{D\ldots F\;d\ldots f})_{;K} + (A^{A\ldots C\;a\ldots c}_{D\ldots F\;d\ldots f})_{;k} F^k_{.K} \\ &= (A^{A\ldots C\;a\ldots c}_{D\ldots F\;d\ldots f})_{:k} F^k_{.K} = (A^{A\ldots C\;a\ldots c}_{D\ldots F\;d\ldots f})_{:k} x^k_{,K}. \end{aligned} \tag{2.117}$$

Partial covariant derivatives of **A** are found by applying the usual rules of covariant differentiation in (2.30) to only one index. For example,

$$A^{a}_{.A;B} = A^{a}_{.A,B} - \overset{G}{\Gamma}{}^{..C}_{BA} A^{a}_{.C}, \quad A^{a}_{.A;b} = A^{a}_{.A,b} + \overset{g}{\Gamma}{}^{..a}_{bc} A^{c}_{.A}. \tag{2.118}$$

In (2.116), regarding $x^{a}_{,A} = x^{a}_{,A}(X)$ leads to inclusion of only one second-order partial derivative in the total covariant derivative of **F** (Eringen 1962; Toupin 1964), though in general $F^{a}_{.A,b} = F^{a}_{.A,B} F^{-1B}_{\quad .b} \neq 0$. From the chain rule, $F^{a}_{.A:B} = F^{a}_{.A:b} F^{b}_{.B}$ and $F^{a}_{.A:b} = F^{a}_{.A:B} F^{-1B}_{\quad .b}$ (Ericksen 1960). Third-order position gradient $x^{a}_{:ABC}$ can be computed by taking the total covariant derivative of $F^{a}_{.A:B}$ via another iteration of (2.117). Partial covariant derivatives of two-point shifter tensors vanish: $g^{A}_{.a;b} = 0$, $g^{a}_{.A;B} = 0$, $g^{a}_{.A;b} = 0$, and $g^{A}_{.a;B} = 0$ (Toupin 1956; Ericksen 1960). For example, $g^{A}_{.a;b} = \mathbf{G}^{A} \bullet \mathbf{g}_{a;b} = 0$. Then from (2.117), total covariant derivatives of shifters must vanish: $g^{A}_{.a:b} = 0$, $g^{a}_{.A:B} = 0$, $g^{a}_{.A:b} = 0$, and $g^{A}_{.a:B} = 0$. Thus it follows that partial covariant differentiation and total covariant differentiation both commute with shifting, raising, and lowering of indices.

The dual map $\mathbf{F}^{*}$ of the two-point tensor $\mathbf{F}$ is defined by

$$\left\langle \mathbf{F}^{*}\boldsymbol{\alpha}, \mathbf{V} \right\rangle = \left\langle \boldsymbol{\alpha}, \mathbf{F}\mathbf{V} \right\rangle \;\forall\; \boldsymbol{\alpha} \in T^{*}_{x}B, \mathbf{V} \in T_{X}B_{0}. \tag{2.119}$$

From the properties of the mapping φ, deformation gradient tensor **F** is non-singular, and its inverse satisfies

$$\mathbf{F}^{-1}\mathbf{F} = \mathbf{1}_{\mathbf{0}}, \; X^{A}_{,a} x^{a}_{,B} = \delta^{A}_{.B}; \; \mathbf{F}\mathbf{F}^{-1} = \mathbf{1}, \; x^{a}_{,A} X^{A}_{,b} = \delta^{a}_{.b}; \tag{2.120}$$

where $\mathbf{1}_{\mathbf{0}} = \delta^{A}_{.B} \mathbf{G}_{A} \otimes \mathbf{G}^{B}$ and $\mathbf{1} = \delta^{a}_{.b} \mathbf{g}_{a} \otimes \mathbf{g}^{b}$ are identity tensors on B_0 and B. In components, the inverse $F^{-1A}_{\quad .a}(x,t) = X^{A}_{,a}$, and partial differentiation $\partial_{a}(.) = \partial_{A}(.) F^{-1A}_{\quad .a}$. The dual and inverse commute: $\mathbf{F}^{*-1} = \mathbf{F}^{-1*} = \mathbf{F}^{-*}$. In components, the dual, inverse, and dual-inverse of **F** are, respectively,

$$\mathbf{F}^{*} = F^{*.a}_{A} \, \mathbf{G}^{A} \otimes \mathbf{g}_{a} \in T^{*}_{X}B_{0} \times T_{x}B, \tag{2.121}$$

$$\mathbf{F}^{-1} = F^{-1\,A}_{\quad .a} \mathbf{G}_{A} \otimes \mathbf{g}^{a} \in T_{X}B_{0} \times T^{*}_{x}B, \tag{2.122}$$

$$\mathbf{F}^{-*} = F^{-*\,.A}_{\quad a} \mathbf{g}^{a} \otimes \mathbf{G}_{A} \in T^{*}_{x}B \times T_{X}B_{0}. \tag{2.123}$$

The transpose map $\mathbf{F}^{T}$ between metric vector spaces $(T_{X}B_{0}, \mathbf{G})$ and $(T_{x}B, \mathbf{g})$ is defined by the operation

$$(\mathbf{F}^{T}\mathbf{w}) \bullet \mathbf{V} = \mathbf{w} \bullet (\mathbf{F}\mathbf{V}) \;\forall\; \mathbf{w} \in T_{x}B, \mathbf{V} \in T_{X}B_{0}. \tag{2.124}$$

Upon inspection of (2.119) and (2.124), the relationship between the transpose and dual maps is apparent:

$$\mathbf{F}^T = \mathbf{G}^{-1}\mathbf{F}^*\mathbf{g}. \quad (2.125)$$

Notice that the transpose map depends on the metric tensors of each configuration, while the dual map is defined independently of the metric in either configuration. In index notation, the dual map corresponds to a horizontal switch of indices, while the transpose corresponds to a diagonal switch. In coordinates, transpose and inverse-transpose maps are written

$$\mathbf{F}^T = G^{AB} F^{*.b}_{B} g_{ba} \mathbf{G}_{\mathrm{A}} \otimes \mathbf{g}^a \in T_X B_0 \times T_x^* B, \quad (2.126)$$

$$\mathbf{F}^{-T} = g^{ab} F^{-*.B}_{b} G_{BA} \mathbf{g}_a \otimes \mathbf{G}^A \in T_x B \times T_X^* B_0. \quad (2.127)$$

Two specific kinds of coordinate representation of the motion (and hence the deformation gradient) are common: convected coordinate representations and Cartesian coordinate representations.

For one kind of representation in convected coordinates, basis vectors of the spatial frame are updated with the motion in such a way that numerical values of the coordinates of a material particle in reference and spatial descriptions always coincide:

$$x^a(X,t) = \delta^a_{.A} X^A, \; F^a_{.A} = \delta^a_{.A}, \; \mathbf{F} = \mathbf{g}_a(x,t) \otimes \mathbf{G}^A(X). \quad (2.128)$$

Alternatively, convected coordinates can be expressed in terms of spatial basis vectors fixed in time and updated reference basis vectors, again resulting in coincident numerical values of coordinates of a material particle:

$$x^a(X,t) = \delta^a_{.A} X^A, \; F^a_{.A} = \delta^a_{.A}, \; \mathbf{F} = \mathbf{g}_a(x) \otimes \mathbf{G}^A(X,t). \quad (2.129)$$

Because basis vectors for spatial and reference coordinates, respectively, in (2.128) and (2.129) evolve with time, associated metric tensors will also change with time as the body deforms. Convected coordinates of types (2.128) and (2.129) will not be used henceforward in this book for describing the relationship between reference and spatial coordinates of a material particle. Definition (2.112)—which is applicable in general curvilinear coordinates—is used this book, but it is henceforth restricted to non-deforming, inertial coordinate systems on B_0 and B, thus ruling out use of convected coordinates. Accordingly, natural basis vectors and metric tensors do not depend explicitly on time: in the reference configuration $\mathbf{G}_A = \mathbf{G}_A(X)$, $\mathbf{G}^A = \mathbf{G}^A(X)$, and hence $G_{AB} = G_{AB}(X)$. Likewise in the current configuration, $\mathbf{g}_a = \mathbf{g}_a(x)$, $\mathbf{g}^a = \mathbf{g}^a(x)$, and $g_{ab} = g_{ab}(x)$. Thus in the remainder of this book, holonomic basis vectors can change with position (X or x) as is the case for curvilinear coordinates, but the origin of each coordinate system for X^A and x^a remains fixed in both time and space.

The presentation of kinematics of deformable bodies simplifies considerably in Cartesian coordinates, also often called rectangular coordinates, rectilinear coordinates, or flat coordinates. If coincident Cartesian coordinate axes[5] are specified with $\mathbf{e}_A = \mathbf{G}_A = \delta^a_{.A}\mathbf{g}_a$, $\mathbf{e}^A = \mathbf{G}^A = \delta^A_{.a}\mathbf{g}^a$, $G_{AB} = \delta_{AB}$, $g_{ab} = \delta_{ab}$, $g^a_{.A} = \delta^a_{.A}$, and $g^A_{.a} = \delta^A_{.a}$, then (2.112) becomes, quite simply,

$$\mathbf{F} = F^A_{.B}\mathbf{e}_A \otimes \mathbf{e}^B . \tag{2.130}$$

Occasionally Cartesian representation (2.130) is used in this book, again assuming that the orientation of each basis vector remains fixed in time. Restriction of validity of a corresponding expression to Cartesian coordinates will be stated explicitly. In (2.130), notice $F^A_{.B} = \delta^A_{.a}x^a_{,B} \neq X^A_{,B} = \delta^A_{.B}$.

Push-forward and pull-back operations are now defined. The push forward of a scalar $f : B_0 \to \mathbb{R}$ is defined by $\varphi_* f = f \circ \varphi^{-1}$. The pull-back of a scalar $h : B \to \mathbb{R}$ is defined by $\varphi^* h = h \circ \varphi$. The push-forward $\varphi_*\mathbf{V}$ of a vector $\mathbf{V} \in T_X B_0$, the pull-back $\varphi^*\mathbf{w}$ of a vector $\mathbf{w} \in T_x B$, the push-forward $\varphi_*\boldsymbol{\alpha}$ of a covector $\boldsymbol{\alpha} \in T^*_X B_0$, and the pull-back $\varphi^*\boldsymbol{\beta}$ of a covector $\boldsymbol{\beta} \in T^*_x B$ are defined by

$$\varphi_*\mathbf{V} = T\varphi_X(\mathbf{V}) \circ \varphi^{-1} = \mathbf{F}\mathbf{V} \circ \varphi^{-1} \in T_x B , \tag{2.131}$$

$$\varphi^*\mathbf{w} = (T\varphi_X)^{-1}(\mathbf{w}) \circ \varphi = \mathbf{F}^{-1}\mathbf{w} \circ \varphi \in T_X B_0 , \tag{2.132}$$

$$\varphi_*\boldsymbol{\alpha} = (T\varphi_X)^{-*}(\boldsymbol{\alpha}) \circ \varphi^{-1} = \mathbf{F}^{-*}\boldsymbol{\alpha} \circ \varphi^{-1} \in T^*_x B , \tag{2.133}$$

$$\varphi^*\boldsymbol{\beta} = (T\varphi_X)^{*}(\boldsymbol{\beta}) \circ \varphi = \mathbf{F}^{*}\boldsymbol{\beta} \circ \varphi \in T^*_X B_0 . \tag{2.134}$$

By extension, the push-forward to the current configuration of arbitrary tensor $\mathbf{A}$ of rank $\left\{ {N \atop M} \right\}$ referred to the reference configuration is defined as

$$\begin{gathered} \varphi_*\mathbf{A}(\mathbf{w}_1,\ldots\mathbf{w}_M,\boldsymbol{\beta}_1,\ldots\boldsymbol{\beta}_N) = \mathbf{A}(\mathbf{F}^{-1}\mathbf{w}_1,\ldots\mathbf{F}^{-1}\mathbf{w}_M,\mathbf{F}^*\boldsymbol{\beta}_1,\ldots\mathbf{F}^*\boldsymbol{\beta}_N) \\ \forall\ \mathbf{w}_i \in T_x B,\ \boldsymbol{\beta}_i \in T^*_x B. \end{gathered} \tag{2.135}$$

Similarly, the pull-back from the spatial configuration of arbitrary tensor $\mathbf{b}$ of order $\left\{ {N \atop M} \right\}$ to the reference configuration is defined as

$$\begin{gathered} \varphi^*\mathbf{b}(\mathbf{V}_1,\ldots\mathbf{V}_M,\boldsymbol{\alpha}_1,\ldots\boldsymbol{\alpha}_N) = \mathbf{b}(\mathbf{F}\mathbf{V}_1,\ldots\mathbf{F}\mathbf{V}_M,\mathbf{F}^{-*}\boldsymbol{\alpha}_1,\ldots\mathbf{F}^{-*}\boldsymbol{\alpha}_N) \\ \forall\ \mathbf{V}_i \in T_X B_0,\ \boldsymbol{\alpha}_i \in T^*_X B_0. \end{gathered} \tag{2.136}$$

Composition with φ^{-1} and φ for respective push-forward and pull-back operations in (2.131)-(2.134) is often implied rather than written explicitly and is omitted henceforth when there is no chance of confusion.

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[5] This is a stricter requirement than non-coincident Cartesian coordinates, wherein the metric tensors reduce to Kronecker delta symbols, but the shifters represent rigid rotations independent of location.

Since reference and current configurations are embedded in Euclidean space, it is possible to introduce, for each material particle, a displacement function **u** with spatial components (Eringen 1962)

$$u^a = x^a - X^a + \xi^a, \tag{2.137}$$

where x^a are spatial components of the vector from the origin of a fixed Cartesian coordinate frame to the particle in the deformed body, $X^a = g^a_{.A} X^A$ are spatial components of the vector from the origin of a different fixed Cartesian coordinate frame to the same particle in the undeformed body, and ξ^a are spatial components of the vector extending from the origin of the Cartesian frame in the reference configuration to the origin of the Cartesian frame in the current configuration. Using shifters and metric tensors of Section 2.1.2,

$$u^A = g^A_{.a} u^a = g^{Aa} u_a = G^{AB} u_B. \tag{2.138}$$

Since ξ^a are components of a constant vector, deformation gradient components $F^a_{.A}(X,t)$ can be represented as (Toupin 1956; Suhubi and Eringen 1964)

$$\begin{aligned} F^a_{.A} = x^a_{.,A} &= (u^a + X^a - \xi^a)_{,A} = (u^B g^a_{.B})_{,A} + (X^B g^a_{.B})_{,A} - (\xi^B g^a_{.B})_{,A} \\ &= (u^B_{.,A} + X^B_{.,A} - \xi^B_{.,A}) g^a_{.B} + (X^B + u^B - \xi^B)(\mathbf{g}^a \bullet \mathbf{G}_{B,A}) \\ &= (u^B_{;A} + \delta^B_{.A}) g^a_{.B}. \end{aligned} \tag{2.139}$$

Similarly, components of the inverse deformation gradient, $F^{-1A}_{\ \ .a}(x,t)$, can be expressed as

$$F^{-1A}_{\ \ .a} = X^A_{.,a} = (\delta^b_{.a} - u^b_{;a}) g^A_{.b}. \tag{2.140}$$

It is emphasized that the deformation gradient and its inverse cannot be represented by (2.139) and (2.140) when reference and current configurations are non-Euclidean spaces, in which case (2.137) does not apply.

2.5.2 Derived Kinematic Quantities and Identities

The Jacobian determinant of **F** provides the relationship between a differential reference volume element $dV = \sqrt{G} dX^1 dX^2 dX^3 \subset B_0$ and its deformed counterpart in the current configuration $dv = \sqrt{g} dx^1 dx^2 dx^3 \subset B$:

$$J dV = dv. \tag{2.141}$$

In coordinates (Truesdell and Toupin 1960; Eringen 1962; Marsden and Hughes 1983; Dluzewski 1996), the Jacobian determinate is computed as follows:

$$\begin{aligned} J &= \frac{1}{6}\varepsilon_{abc}\varepsilon^{ABC}F^a_{.A}F^b_{.B}F^c_{.C} = \frac{1}{6}\sqrt{g/G}e_{abc}e^{ABC}F^a_{.A}F^b_{.B}F^c_{.C} \\ &= \frac{1}{6}\det\mathbf{F}\sqrt{g/G} = \frac{1}{6}\det(x^a_{,A})\sqrt{\det(g_{ab})/\det(G_{AB})}, \end{aligned} \tag{2.142}$$

where (2.64) and (2.65) have been used. For the inverse deformation[6], $6J^{-1} = \varepsilon^{abc}\varepsilon_{ABC}F^{-1A}_{\ \ .a}F^{-1B}_{\ \ .b}F^{-1C}_{\ \ .c} = \sqrt{G/g}\det\mathbf{F}^{-1}$. From (2.64)-(2.67) and Table 2.3, it follows that permutation tensors map between configurations via $\varepsilon^{abc} = J^{-1}\varepsilon^{ABC}F^a_{.A}F^b_{.B}F^c_{.C}$ and $\varepsilon_{abc} = J\varepsilon_{ABC}F^{-1A}_{\ \ .a}F^{-1B}_{\ \ .b}F^{-1C}_{\ \ .c}$. Jacobian $J(X,t)$, unlike $\det\mathbf{F}$, is an absolute scalar, invariant under coordinate transformations (Marsden and Hughes 1983). Since volume v remains positive and bounded, $0 < J < \infty$. When there is no deformation, e.g., when a body undergoes only rigid translation, then $\mathbf{F} = \mathbf{1} = g^a_{.A}\mathbf{g}_a \otimes \mathbf{G}^A$, and in that case $J = \sqrt{g/G}\det(g^a_{.A}) = 1$ follows from (2.142).

The following identity applies for the derivative of the determinant of a non-singular, second-order matrix $\mathbf{A}$ (Ericksen 1960; Thurston 1974):

$$\frac{\partial(\det\mathbf{A})}{\partial A^A_{.B}} = A^{-1B}_{\ \ .A}\det\mathbf{A}\,. \tag{2.143}$$

Applying (2.143) to J and J^{-1} produces the following identities:

$$\frac{\partial J}{\partial F^a_{.A}} = JF^{-1A}_{\ \ .a}, \quad \frac{\partial J^{-1}}{\partial F^{-1A}_{\ \ .a}} = J^{-1}F^a_{.A}, \tag{2.144}$$

as can be verified directly via inspection of (2.142). The total covariant derivative of the second of (2.144), using (2.73) and the chain rule (see the full derivation in Section D.2 of Appendix D), is

$$\begin{aligned} (\partial J^{-1}/\partial F^{-1A}_{\ \ .a})_{:a} &= (J^{-1}F^a_{.A})_{:a} = \tfrac{1}{\sqrt{g}}(\sqrt{g}J^{-1}F^a_{.A})_{,a} - J^{-1}F^a_{.C}\overset{G}{\Gamma}{}^{..C}_{BA}F^{-1B}_{\ \ .a} \\ &= J^{-1}F^{-1B}_{\ \ .a}\left[x^a_{,AB} - x^a_{,BA}\right] = 0. \end{aligned} \tag{2.145}$$

Analogously, the divergence of the first of (2.144) produces the identity

$$(\partial J/\partial F^a_{.A})_{:A} = (JF^{-1A}_{\ \ .a})_{:A} = \tfrac{1}{\sqrt{G}}(\sqrt{G}JF^{-1A}_{\ \ .a})_{,A} - J\overset{g}{\Gamma}{}^{..b}_{ba} = 0\,. \tag{2.146}$$

Relations (2.145) and (2.146) are often called Piola identities. Using the definition of the determinant and (2.139), the Jacobian satisfies

$$\begin{aligned} J &= \sqrt{g/G}\det(g^a_{.A})\det\left[u^A_{;B} + \delta^A_{.B}\right] = \det\left[u^A_{;B} + \delta^A_{.B}\right] \\ &\approx 1 + u^A_{;A} + (1/2)\left[(u^A_{;A})^2 - u^A_{;B}u^B_{;A}\right]. \end{aligned} \tag{2.147}$$

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[6] Recall that for two generic, non-singular square matrices $\mathbf{A}$ and $\mathbf{B}$ the following identities apply: $\det(\mathbf{AB}) = (\det\mathbf{A})(\det\mathbf{B})$ and $\det(\mathbf{A}^{-1}) = (\det\mathbf{A})^{-1}$.

Terms of order three in referential displacement gradients are omitted in the final approximation in (2.147).

Let oriented differential area elements referred to configurations B_0 and B be labeled $\mathbf{N}dS$ and $\mathbf{n}ds$, respectively, where unit normal covariant vectors $\mathbf{N} \in T_X^* B_0$ and $\mathbf{n} \in T_x^* B$. These area elements are related by the transformation formula

$$\mathbf{n}ds = J\varphi_* \mathbf{N}dS,\; n_a ds = JF^{-*.A}_{\ \ a} N_A dS = JF^{-1A}_{\ \ .a} N_A dS\,. \tag{2.148}$$

Relation (2.148) is called Nanson's formula (Malvern 1969) and represents the Piola transformation between differential forms $\mathbf{N}dS$ and $\mathbf{n}ds$ (Marsden and Hughes 1983). Let $n_a ds = \varepsilon_{abc} dx^{[b} dx^{c]} = \varepsilon_{abc} dx^b \wedge dx^c / 2$ and $N_A dS = \varepsilon_{ABC} dX^{[B} dX^{C]} = \varepsilon_{ABC} dX^B \wedge dX^C / 2$, where for two rank-one objects, the wedge product satisfies $(\alpha \wedge \beta)^{AB} = \alpha^A \wedge \beta^B \ = \alpha^A \beta^B - \alpha^B \beta^A$. Schouten (1954) calls $dx^a \wedge dx^b$ an infinitesimal bivector. From the above definitions of oriented differential area elements and the properties of the permutation tensor in Table 2.3, it follows that $dx^a \wedge dx^b = \varepsilon^{abc} n_c ds$ and $dX^A \wedge dX^B = \varepsilon^{ABC} N_C dS$. Notice that dx^a and dx^b are treated as components of distinct vectors in the definition of the area element; otherwise the dyad $dx^a dx^b$ would be symmetric. Notice also that the oriented area element $n_a ds = \varepsilon_{abc} dx^{[b} dx^{c]} = \varepsilon_{abc} dx^b dx^c$.

Decomposition of the deformation gradient into the product of two tensors—a rotation and a symmetric positive definite tensor called a stretch tensor—is always possible since $\mathbf{F}$ is non-singular. This separation into stretch and rotation is called the polar decomposition and is written as

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}\,, \tag{2.149}$$

where rotation $\mathbf{R} : T_X B_0 \to T_x B$ is a proper orthogonal two-point tensor, i.e., $\mathbf{R}^{-1} = \mathbf{R}^T$ and $\mathbf{R}$ has positive Jacobian determinant of unit magnitude:

$$\mathbf{R}^T\mathbf{R} = \mathbf{1}_0,\; \mathbf{R}\mathbf{R}^T = \mathbf{1},\; \varepsilon_{abc}\varepsilon^{ABC} R^a_{\ .A} R^b_{\ .B} R^c_{\ .C} = 6\,. \tag{2.150}$$

Right stretch tensor $\mathbf{U} : T_X B_0 \to T_X B_0$ and left stretch tensor $\mathbf{V} : T_x B \to T_x B$ satisfy the following symmetry conditions:

$$G_{AB} U^B_{\ .C} = G_{CB} U^B_{\ .A},\; g_{ab} V^b_{\ .c} = g_{cb} V^b_{\ .a}\,. \tag{2.151}$$

The stretch tensors in (2.151) are determined uniquely by the polar decomposition of $\mathbf{F}$, and are related to the right and left Cauchy-Green deformation tensors, $\mathbf{C}$ and $\mathbf{B}$ respectively, as

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2,\; \mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2\,. \tag{2.152}$$

Since $\det \mathbf{R} = \sqrt{G/g} > 0$ by convention (2.150), $J = \det \mathbf{U} = \det \mathbf{V} > 0$.

The covariant version of **C** is the pull-back of spatial metric **g**, and the contravariant version of **B** is the push-forward of reference metric $\mathbf{G}^{-1}$:

$$\mathbf{C} = \varphi^*(\mathbf{g}), \; C_{AB} = F^a_{.A} g_{ab} F^b_{.B} = C_{(AB)}; \tag{2.153}$$

$$\mathbf{B} = \varphi_*(\mathbf{G}^{-1}), \; B^{ab} = F^a_{.A} G^{AB} F^b_{.B} = B^{(ab)}. \tag{2.154}$$

The tensors defined in (2.153) and (2.154) are clearly symmetric. From (2.115) and (2.153), **C** assigns, to first order, the length of an infinitesimal line element after deformation:

$$|d\mathbf{x}| = |\mathbf{F}d\mathbf{X}| = \sqrt{\langle d\mathbf{X}, \mathbf{C}d\mathbf{X}\rangle}. \tag{2.155}$$

Furthermore, the right Cauchy-Green strain tensor

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{G}), \; E_{AB} = \frac{1}{2}(C_{AB} - G_{AB}) = E_{(AB)}, \tag{2.156}$$

provides a relationship for the difference in squared lengths of deformed and undeformed line elements as

$$d\mathbf{x} \bullet d\mathbf{x} - d\mathbf{X} \bullet d\mathbf{X} = 2\langle d\mathbf{X}, \mathbf{E}d\mathbf{X}\rangle. \tag{2.157}$$

A number of other finite strain measures not described here may be constructed from **F** or its inverse (Malvern 1969; Marsden and Hughes 1983). Notice that it is customary to represent the rotation **R** of (2.149) and (2.150) as a two-point tensor. If instead **R** is referred exclusively to reference coordinates, for example, then a shifter must be introduced into the deformation gradient, e.g., $F^a_{.A} = R^a_{.C} U^C_{.A} = g^a_{.B} R^B_{.C} U^C_{.A} = g^a_{.B} R^B_{.C} (C^{1/2})^C_{.A}$.

More kinematic identities emerge from straightforward differentiation[7]:

$$\frac{\partial C_{AB}}{\partial g_{ab}} = F^a_{.A} F^b_{.B}, \; \frac{\partial C_{AB}}{\partial F^a_{.C}} = 2 g_{ab} \delta^C_{.(A} F^b_{.B)}, \; \frac{\partial E_{AB}}{\partial C_{CD}} = \frac{1}{2} \delta^{(C}_{.(A} \delta^{D)}_{.B)}. \tag{2.158}$$

The following identity is derived from (2.143) and the symmetry of **C**:

$$\frac{\partial(\det \mathbf{C})}{\partial C_{AB}} = C^{-1BA} \det \mathbf{C} = C^{-1AB} \det \mathbf{C}. \tag{2.159}$$

Then because $\det \mathbf{C} = \det(C^A_{.B}) = \det(G^{AC} F^c_{.C} g_{cb} F^b_{.B}) = (g/G)(\det \mathbf{F})^2 = J^2$,

$$\begin{aligned} \frac{\partial J}{\partial E_{AB}} &= 2\frac{\partial J}{\partial C_{AB}} = \frac{1}{J}\frac{\partial J^2}{\partial C_{AB}} = \frac{1}{J}\frac{\partial(\det \mathbf{C})}{\partial C_{AB}} \\ &= JC^{-1AB} = JF^{-1A}_{.a} g^{ab} F^{-1B}_{.b} = JX^A_{.,a} g^{ab} X^B_{.,b}. \end{aligned} \tag{2.160}$$

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[7] Additionally, from (2.120), $\dfrac{\partial F^a_{.B}}{\partial F^{-1A}_{.b}} = -F^a_{.A} F^b_{.B}$ and $\dfrac{\partial F^{-1A}_{.b}}{\partial F^a_{.B}} = -F^{-1A}_{.a} F^{-1B}_{.b}$.

2.5.3 Linearization

It is instructive to consider the small deformation, i.e., geometrically linear or infinitesimal, kinematic description often applied in engineering practice. In the usual linear theory, displacements and displacement gradients are assumed small so that configurations B_0 and B nearly coincide, and the same coordinate system is used in both configurations. Thus the shifter degenerates to $g^{a}_{.A} = \delta^{a}_{.A}$, and partial coordinate differentiation with respect to either configuration is nearly the same operation, e.g., $\partial_a(.) \approx \delta^{A}_{.a}\partial_A(.)$. Since there is only one coordinate system, lower case indices are used in this book for representation of mathematical objects the linear theory, by default. Covariant differentiation with respect to the Levi-Civita connection is represented by $\overset{\mathbf{g}}{\nabla}$, Christoffel symbols by $\overset{g}{\Gamma}{}^{..a}_{bc}(x)$, and metric by $g_{ab}(x)$. In Euclidean space, spatial and reference coordinates are related in the linear theory by the vector addition rule

$$\mathbf{x} = \mathbf{X} + \mathbf{u}\,, \tag{2.161}$$

where $\mathbf{u}$ is the displacement vector. Components of the right Cauchy-Green strain tensor $\mathbf{E}(X,t)$ of (2.156) are related to spatial covariant derivatives of $\mathbf{u}(x,t)$ as (Eringen 1962)

$$2F^{-1A}_{.a}E_{AB}F^{-1B}_{.b} = u_{a;b} + u_{b;a} - u^{c}_{;a}u_{c;b}\,. \tag{2.162}$$

Components of the infinitesimal strain tensor $\boldsymbol{\varepsilon}(x,t)$ (also called the linear or small strain tensor) are defined as

$$\varepsilon_{ab} = u_{(a;b)} = \frac{1}{2}(u_{a;b} + u_{b;a})\,, \tag{2.163}$$

and clearly differ from those of finite strain tensor $\mathbf{E}$ by terms of order two in displacement gradients. Such terms are assumed negligibly small with respect to the displacement gradient itself in the linear theory. Volume changes in the linear theory are computed after omitting terms of orders two and higher in displacement gradients in (2.147) as

$$J \approx 1 + u^{a}_{;a} = 1 + \mathrm{tr}(\overset{\mathbf{g}}{\nabla}\mathbf{u})\,,\; J^{-1} \approx 1 - u^{a}_{;a} = 1 - \mathrm{tr}(\overset{\mathbf{g}}{\nabla}\mathbf{u})\,. \tag{2.164}$$

The skew rotation tensor $\boldsymbol{\Omega}$ is also introduced in the linear theory:

$$\Omega_{ab} = u_{[a;b]} = u_{[a,b]} - \overset{g}{\Gamma}{}^{..c}_{[ba]}u_c = u_{[a,b]}\,, \tag{2.165}$$

which itself can be reduced to a vector $\mathbf{w}$ via the axial transformation

$$w^a = -\frac{1}{2}\varepsilon^{abc}\Omega_{bc} = -\frac{1}{2}\varepsilon^{abc}u_{[b,c]}\,,\; \Omega_{ab} = -\varepsilon_{abc}w^c\,. \tag{2.166}$$

Tensor $\mathbf{\Omega}$, when acting on differential element $d\mathbf{x}$**,** produces the relative displacement $\mathbf{\Omega}d\mathbf{x} = \mathbf{w} \times d\mathbf{x}$, where $(\mathbf{w} \times d\mathbf{x})_a = \varepsilon_{abc} w^b dx^c$. When all components of $\mathbf{\Omega}$ are small compared to one radian, this relative displacement represents a true rotation, and in Cartesian coordinates, $\mathbf{R} = \mathbf{1} + \mathbf{\Omega}$ is the corresponding rotation tensor appearing in (2.149).

2.6 Velocities and Time Differentiation

Material velocity $\mathbf{V}(X,t)$ and spatial velocity $\mathbf{v}(x,t)$ satisfy the following definitions in direct and indicial notation, respectively:

$$\mathbf{V} = \frac{\partial \mathbf{x}}{\partial t} = \frac{\partial \varphi^a}{\partial t}\bigg|_X \mathbf{g}_a, \; V^a\left(X,t\right) = \frac{\partial}{\partial t}\varphi^a\left(X,t\right) = v^a\left(\varphi(X,t),t\right); \quad (2.167)$$

$$\mathbf{v}_t = \mathbf{V}_t \circ \varphi_t^{-1}, \; v^a\left(x,t\right) = V^a\left(\varphi^{-1}(x,t),t\right). \quad (2.168)$$

Recall the notation $x = \varphi(X,t)$ and $X = \varphi^{-1}(x,t)$ from Sections 2.1.2 and 2.5.1. Subscript t in the first of (2.168) denotes a spatial field quantity at a particular (fixed) time t. Notice that components of both velocity fields are referred to spatial coordinates. In each definition, the material particle at X is held constant during time differentiation, meaning that velocity is the time rate of change of position of a given material particle. The composition operation in (2.168) is often implicitly assumed rather than written explicitly, and is omitted later in the text when no confusion arises.

2.6.1 The Material Time Derivative

The material time derivative measures the rate of change of a quantity associated with given a material particle at X. Because spatial position x of a given particle may change with time, the material time derivative and the partial time derivative of a function expressed in terms of spatial position x can differ. The material time derivative of a differentiable scalar function f of time t and spatial position x is defined as

$$\begin{aligned} \dot{f}\left(x,t\right) &= \frac{df}{dt} = \frac{\partial f}{\partial t}\bigg|_x + \frac{\delta f}{\delta t} \\ &= \frac{\partial f}{\partial t}\bigg|_x + \frac{\partial f}{\partial x^a}\bigg|_t \frac{\partial x^a}{\partial t}\bigg|_X \\ &= \frac{\partial f}{\partial t}\bigg|_x + f_{;a} v^a, \end{aligned} \quad (2.169)$$

where the partial time derivative is taken with spatial coordinates **x** (i.e., position x) held constant, and the intrinsic derivative that accounts for convective changes of scalar function f resulting from the velocity field is

$$\frac{\delta f}{\delta t} = \overset{\mathbf{g}}{\nabla}_{\mathbf{v}} f = f_{;a} v^a = (f v^a)_{;a} - f v^a_{;a} = f_{,a} v^a . \tag{2.170}$$

It follows that for differentiable functions f and g, $d(f+g)/dt = \dot{f} + \dot{g}$ and $d(fg)/dt = \dot{f}g + f\dot{g}$. The material time derivative is applied in an analogous way to vectors and tensors of higher order as, for example,

$$\dot{f}^a(x,t) = \left.\frac{\partial f^a}{\partial t}\right|_x + f^a_{;b} v^b , \quad \dot{f}^{a...c}_{d...f}(x,t) = \left.\frac{\partial f^{a...c}_{d...f}}{\partial t}\right|_x + f^{a...c}_{d...f;k} v^k . \tag{2.171}$$

The material time derivative of a time-dependent function of position in the reference configuration, with all indices referred to reference coordinates[8], by definition equals its partial time derivative:

$$\dot{f}(X,t) = \left.\frac{\partial f}{\partial t}\right|_X , \quad \dot{f}^A(X,t) = \left.\frac{\partial f^A}{\partial t}\right|_X , \quad \dot{f}^{A...C}_{D...F}(X,t) = \left.\frac{\partial f^{A...C}_{D...F}}{\partial t}\right|_X . \tag{2.172}$$

One application of (2.171) for a spatial vector is the spatial acceleration:

$$\begin{aligned}
\mathbf{a}(x,t) = a^a \mathbf{g}_a = \dot{\mathbf{v}} &= \left.\frac{\partial \mathbf{v}}{\partial t}\right|_x + \overset{\mathbf{g}}{\nabla}_{\mathbf{v}} \mathbf{v} \\
&= \left.\frac{\partial v^a}{\partial t}\right|_x \mathbf{g}_a + (\mathbf{v}_{,b} \mathbf{g}^b)\mathbf{v} \\
&= \left.\frac{\partial v^a}{\partial t}\right|_x \mathbf{g}_a + [(v^c \mathbf{g}_c)_{,b} \mathbf{g}^b)][v^d \mathbf{g}_d] \\
&= \left.\frac{\partial v^a}{\partial t}\right|_x \mathbf{g}_a + \left(v^c_{.,b} \mathbf{g}_c + v^c \overset{g}{\Gamma}{}^{..a}_{bc} \mathbf{g}_a \right) v^b \\
&= \left[\left.\frac{\partial v^a}{\partial t}\right|_x + \left(v^a_{.,b} + \overset{g}{\Gamma}{}^{..a}_{bc} v^c \right) v^b \right] \mathbf{g}_a .
\end{aligned} \tag{2.173}$$

[8] For a function f of reference position with one or more indices referred to spatial coordinates, letting D/Dt denote the partial derivative with both **X** and **g** constant, the material time derivative is defined by (Truesdell and Toupin 1960)

$$\dot{f}^{A...C\ a...c}_{D...F\ d...f}(X,t) = \frac{D f^{A...C\ a...c}_{D...F\ d...f}}{Dt} + \left(\overset{g}{\Gamma}{}^{..a}_{bk} f^{A...C\ k...c}_{D...F\ d...f} + \ldots - \overset{g}{\Gamma}{}^{..k}_{bd} f^{A...C\ a...c}_{D...F\ k...f} - \ldots \right) V^b .$$

On the other hand, the material acceleration, i.e., the material time derivative of material velocity (2.167) referred to spatial coordinates, satisfies

$$\begin{aligned}\mathbf{A}(X,t) = A^a\mathbf{g}_a &= \left.\frac{\partial \mathbf{V}}{\partial t}\right|_X = \left.\frac{\partial (V^a \mathbf{g}_a)}{\partial t}\right|_X \\ &= \left.\frac{\partial V^a}{\partial t}\right|_X \mathbf{g}_a + V^a \left.\frac{\partial \mathbf{g}_a(x)}{\partial t}\right|_X \\ &= \left.\frac{\partial V^a}{\partial t}\right|_X \mathbf{g}_a + V^c \mathbf{g}_{c,b} \left.\frac{\partial x^b}{\partial t}\right|_X \\ &= \left(\left.\frac{\partial V^a}{\partial t}\right|_X + \overset{g}{\Gamma}{}^{..a}_{bc} V^c V^b\right)\mathbf{g}_a,\end{aligned} \tag{2.174}$$

noting that $(\partial V^a/\partial t)\big|_X = (\partial v^a/\partial t)\big|_x + v^a_{,b}v^b$ from (2.167). Expressed as material time derivatives, contravariant velocity components are written as $V^a(X,t) = v^a(x,t) = \dot{x}^a(X,t)$ and contravariant acceleration components are written as $A^a(X,t) = a^a(x,t) = \ddot{x}^a(X,t)$.

Using (2.118) and (2.171), the material time derivative of the deformation gradient $\mathbf{F} = F^a_{.A}\mathbf{g}_a \otimes \mathbf{G}^A = x^a_{,A}\mathbf{g}_a \otimes \mathbf{G}^A$ of (2.112), an example of a two-point tensor, is calculated in components as follows (Eringen 1962):

$$\begin{aligned}\dot{F}^a_{.A} &= \left.\frac{\partial F^a_{.A}}{\partial t}\right|_x + \frac{\delta F^a_{.A}}{\delta t} = \left.\frac{\partial F^a_{.A}}{\partial t}\right|_x + F^a_{.A;b}v^b \\ &= \left.\frac{\partial F^a_{.A}}{\partial t}\right|_x + \left(F^a_{.A,b} + \overset{g}{\Gamma}{}^{..a}_{bc}F^c_{.A}\right)v^b \\ &= \frac{DF^a_{.A}}{Dt} + \overset{g}{\Gamma}{}^{..a}_{bc}F^c_{.A}V^b = V^a_{,A} + \overset{g}{\Gamma}{}^{..a}_{bc}F^c_{.A}V^b \\ &= \left(v^a_{,c} + \overset{g}{\Gamma}{}^{..a}_{cb}v^b\right)F^c_{.A} = v^a_{;c}F^c_{.A}.\end{aligned} \tag{2.175}$$

The spatial velocity gradient tensor $\mathbf{L}(x,t)$, i.e., the covariant spatial derivative of the velocity vector $\mathbf{v}(x,t)$, is

$$\begin{aligned}\mathbf{L} = \mathbf{v}_{,b} \otimes \mathbf{g}^b &= v^a_{;b}\mathbf{g}_a \otimes \mathbf{g}^b = \dot{\mathbf{F}}\mathbf{F}^{-1}, \\ L^a_{.b} = v^a_{;b} &= (\dot{x}^a)_{;b} = \dot{F}^a_{.A}F^{-1A}_{\quad .b}.\end{aligned} \tag{2.176}$$

Noting that $d(F^a_{.A}F^{-1A}_{\quad .b})/dt = d(\delta^a_{.b})/dt = 0$, the material time derivative of the inverse deformation gradient satisfies $(\dot{F}^{-1})^A_{.a} = -F^{-1A}_{\quad .b}L^b_{.a} = -X^A_{,b}(\dot{x}^b)_{;a}$.

The covariant derivative of a metric tensor with respect to its Levi-Civita connection always vanishes identically; thus the material time derivative of spatial metric tensor with components $g_{ab}(x)$ vanishes:

$$\dot{g}_{ab}(x) = \frac{\partial g_{ab}}{\partial t}\bigg|_x + g_{ab;c}v^c = 0, \; \dot{g}^{ab}(x) = \frac{\partial g^{ab}}{\partial t}\bigg|_x + g^{ab}_{;c}v^c = 0. \quad (2.177)$$

Furthermore, $\dot{G}_{AB}(X) = (\partial G_{AB} / \partial t)\big|_X = 0$. Thus, raising or lowering of indices commutes with material time differentiation in either configuration.

2.6.2 The Lie Derivative

The Lie derivative of a differentiable but otherwise arbitrary function $f(x,t)$ on spatial manifold B taken with respect to the velocity field $\mathbf{v}(x,t)$ of (2.168) is computed by

$$\mathcal{L}_{\mathbf{v}} f = \varphi_* \left[\frac{d}{dt}(\varphi^* f) \right], \quad (2.178)$$

where φ^* and φ_* denote pull-back and push-forward operations with respect to the motion. The notation d/dt denotes a material time derivative, as implied already by (2.169). In this text, particular Lie derivative (2.178) is considered exclusively. However, more general definitions of Lie derivatives taken with respect to time-dependent vector fields, e.g., fields other than velocity $\mathbf{v}$, exist (Schouten 1954; Marsden and Hughes 1983). Lie derivatives are useful for positing constitutive equations in rate form because objective rates of second-rank tensors have objective Lie derivatives. The component representation of Lie derivative (2.178) for a scalar function f equals its material time derivative:

$$\mathcal{L}_{\mathbf{v}} f = \dot{f} = \frac{\partial f}{\partial t}\bigg|_X = \frac{\partial f}{\partial t}\bigg|_x + v^a f_{;a} = \frac{\partial f}{\partial t}\bigg|_x + (f v^a)_{;a} - f v^a_{;a}. \quad (2.179)$$

For vectors and tensors of higher order, the Lie derivative in components is

$$
\begin{aligned}
\mathcal{L}_{\mathbf{v}} f^{a\ldots h}{}_{i\ldots r} &= \left.\frac{\partial f^{a\ldots h}{}_{i\ldots r}}{\partial t}\right|_x + v^s f^{a\ldots h}{}_{i\ldots r;s} \\
&\qquad - v^a_{;s} f^{s\ldots h}{}_{i\ldots r} - \ldots - v^h_{;s} f^{a\ldots s}{}_{i\ldots r} \\
&\qquad + v^t_{;i} f^{s\ldots h}{}_{t\ldots r} + \ldots + v^t_{;r} f^{a\ldots s}{}_{i\ldots t} \\
&= \left.\frac{\partial f^{a\ldots h}{}_{i\ldots r}}{\partial t}\right|_x + v^s f^{a\ldots h}{}_{i\ldots r,s} \\
&\qquad - v^a_{.,s} f^{s\ldots h}{}_{i\ldots r} - \ldots - v^h_{.,s} f^{a\ldots s}{}_{i\ldots r} \\
&\qquad + v^t_{.,i} f^{s\ldots h}{}_{t\ldots r} + \ldots + v^t_{.,r} f^{a\ldots s}{}_{i\ldots t}
\end{aligned}
\tag{2.180}
$$

since terms involving Christoffel symbols in the covariant derivatives cancel by the symmetry of the Levi-Civita connection on B. For the scalar $J(X,t)$, a direct calculation with the chain rule, (2.144), and (2.176) gives

$$
\begin{aligned}
\mathcal{L}_{\mathbf{v}} J(F^a_{.A}(X,t), g(x), G(X)) = \dot{J} &= \frac{\partial J}{\partial F^a_{.A}} \dot{F}^a_{.A} = J F^{-1A}_{\quad .a} \dot{F}^a_{.A} \\
&= J L^a_{.a} = J D^a_{.a} = J v^a_{;a} = J(\dot{x}^a)_{;a}.
\end{aligned}
\tag{2.181}
$$

In components, the Lie derivative of the spatial metric tensor $g_{ab}(x)$ is

$$
(\mathcal{L}_{\mathbf{v}} g)_{ab} = \underbrace{\partial g_{ab} / \partial t\big|_x + v^c g_{ab;c}}_{=0} + v^c_{;a} g_{cb} + v^c_{;b} g_{ac} = 2L_{(ab)} = 2D_{ab}\,, \tag{2.182}
$$

where D_{ab} are components of the symmetric deformation rate tensor.

2.6.3 Rate Kinematics

Velocity gradient $\mathbf{L}$ provides, to first order, the difference $d\mathbf{v}$ in spatial velocities of two particles in the spatial frame separated by a small vector $d\mathbf{x}$. The material time derivative applied to (2.115) results in

$$
d\mathbf{v} = \mathbf{L} d\mathbf{x}\,,\ dv^a = d\dot{x}^a = \dot{F}^a_{.A} dX^A = \dot{F}^a_{.A} F^{-1A}_{\quad .b} dx^b = v^a_{;b} dx^b = L^a_{.b} dx^b\,. \tag{2.183}
$$

Deformation rate tensor $\mathbf{D}$ provides the material time derivative of the squared length of a differential line element $d\mathbf{x}$:

$$
\begin{aligned}
d\left(d\mathbf{x}\bullet d\mathbf{x}\right)/dt &= 2d\mathbf{x}\bullet d\mathbf{v} = 2L_{ab} dx^a dx^b \\
&= 2L_{(ab)} dx^a dx^b = 2D_{ab} dx^a dx^b
\end{aligned}
\tag{2.184}
$$

because $dx^a dx^b = dx^{(a} dx^{b)}$. Since from (2.115),

$$
d\left(d\mathbf{x}\bullet d\mathbf{x}\right)/dt = d(F^a_{.A} dX^A g_{ab} F^b_{.B} dX^B)/dt = \dot{C}_{AB} dX^A dX^B\,, \tag{2.185}
$$

time rates of right Cauchy-Green strain and deformation tensors satisfy

$$\dot{E}_{AB} = \frac{1}{2}\dot{C}_{AB} = F^{a}_{.A} D_{ab} F^{b}_{.B}, \tag{2.186}$$

relationships that can also be derived directly from (2.153), (2.156), (2.176), and (2.182), without resorting to use of first-order approximation (2.115). The skew covariant part of **L**, called the spin tensor or vorticity tensor and labeled **W**, provides the time rate of rotation of a differential line element $d\mathbf{x}$. In components,

$$W_{ab} = L_{[ab]} = L_{ab} - D_{ab} = v_{a;b} - v_{(a;b)} = v_{[a;b]} = v_{[a,b]}, \tag{2.187}$$

with the final equality following from the symmetry of the connection coefficients. Appealing to the polar decomposition in the first of (2.149) and (2.176), the velocity gradient can be written

$$\mathbf{L} = \dot{\mathbf{R}}\mathbf{R}^{T} + \mathbf{R}\dot{\mathbf{U}}\mathbf{U}^{-1}\mathbf{R}^{T}. \tag{2.188}$$

From time differentiation of (2.150), the covariant version of the first term on the right of (2.188) is always skew:

$$g_{ac}\dot{R}^{c}_{.A} R^{TA}_{\ \ .b} = -g_{bc}\dot{R}^{c}_{.A} R^{TA}_{\ \ .a}, \tag{2.189}$$

meaning that the spatial deformation rate is comprised of the symmetric part of the second term on the right of (2.188):

$$2D_{ab} = g_{ac} R^{c}_{.A} \dot{U}^{A}_{.B} U^{-1B}_{\ \ \ .C} R^{TC}_{\ \ .b} + g_{bc} R^{c}_{.A} \dot{U}^{A}_{.B} U^{-1B}_{\ \ \ .C} R^{TC}_{\ \ .a}. \tag{2.190}$$

Noting that $\dot{\mathbf{u}} = \mathbf{v}$, in the context of the geometrically linear theory of Section 2.5.3, the deformation rate is equivalent to the time rate of the small strain tensor, and the spin is equivalent to the rate of rotation:

$$\begin{aligned} D_{ab} &= v_{(a;b)} = \dot{u}_{(a;b)} = \dot{\varepsilon}_{ab}, \\ W_{ab} &= \dot{u}_{[a;b]} = \dot{u}_{[a,b]} = \dot{\Omega}_{ab} = -\varepsilon_{abc}\dot{w}^{c}. \end{aligned} \tag{2.191}$$

2.7 Theorems of Gauss and Stokes

Two fundamental theorems of vector calculus associated with integration are used frequently later in the text. The first, known as Gauss's theorem or Green's theorem, relates volume and surface integrals. A particular case is the well-known divergence theorem. The second is Stokes's theorem, and it relates surface and line integrals.

2.7.1 Gauss's Theorem

The generalized Gauss's theorem exhibits the form (Malvern 1969)

$$\int_V \mathbf{A} * \nabla\, dV = \int_S \mathbf{A} * \mathbf{N}\, dS, \tag{2.192}$$

where **A** is a scalar, vector, or tensor of arbitrary order that has continuous first derivatives with respect to local coordinates, ∇ is the covariant derivative operator with respect to the Levi-Civita connection of these coordinates, and **N** is the outward unit normal covariant vector to differential surface area element *dS*. Surface *S*, which encloses volume *V*, is required to be piecewise smooth and exhibit a topological outside and inside, such that **N** may be clearly assigned to point from the inside to the outside of *V* for each surface element *dS*. Volume *V* must be simply connected for a single continuous surface *S* to suffice[9]; otherwise, (2.192) may be applied over the union of disjoint surfaces completely enclosing a volume that is not simply connected. The $*$ operator—not to be confused with the push-forward φ_* or pull-back φ^* introduced in Section 2.5.1 that feature asterisks written in respective subscript or superscript positions—represents a general product that exhibits the distributive property. Examples include the dual product $\langle\ ,\ \rangle$, the dot product $\bullet$, the vector cross product $\times$, and the tensor (outer) product $\otimes$. The familiar divergence theorem of vector calculus is obtained from (2.192) when $*$ is the dual or dot product, e.g., for a spatial vector field $\hat{\mathbf{a}}(x,t)$,

$$\int_v \hat{a}^a_{;a}\, dv = \int_s \hat{a}^a n_a\, ds\,. \tag{2.193}$$

As a second example (Hill 1972, 1984; Clayton and McDowell 2003a), consider cases wherein the body is simply connected, enclosed by a single continuous surface. In (2.192), let $\mathbf{A} \to \mathbf{x}(X,t)$ and let *V* be the volume of the body in the reference configuration enclosed by surface *S*, with local outward normal **N**. The $*$ operator is chosen to be the outer product $\otimes$, giving

$$\int_V x^a_{;A} dV = \int_S x^a N_A dS\,. \tag{2.194}$$

Since (2.194) involves integration of a two-point tensor field, for the integrals to represent valid quantities at any location in Euclidean space[10], basis vectors $\mathbf{g}_a(x)$ and $\mathbf{G}_A(X)$ should be chosen as constant with respect to changes in position. This requirement leads to vanishing of the Christoffel

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[9] In many texts, integration over a closed surface in (2.192) or closed curve in (2.197) is delineated by the explicit notation $\oint$. Throughout this text, domains of integration are simply defined as they appear.

[10] Alternatively, the integral of a vector or tensor field can be defined in a valid manner at a single point in space by parallel transporting all position-dependent quantities within the integrand to that point using a shifter (Toupin 1956). This approach is pursued explicitly in Chapter 3 in the context of the Burgers circuit.

symbols in each configuration. This restriction does not require that basis vectors in the two configurations must coincide, nor does it require that basis vectors in each configuration must be orthogonal. In other words, (2.194) requires $g_{ab,c} = 0$ and $G_{AB,C} = 0$, but one can still have $g_{ab} \neq \delta_{ab}$, $G_{AB} \neq \delta_{AB}$, and $g^{a}_{.A} \neq \delta^{a}_{.A}$, as considered by Sedov (1966). Thus, (2.194) reduces to

$$\int_V x^a_{.,A} dV = \int_V F^a_{.A} dV = \int_S x^a N_A dS \, . \tag{2.195}$$

Dividing (2.195) by V gives the volume average of the deformation gradient in terms of spatial position and reference orientation of surface S:

$$V^{-1} \int_V \mathbf{F} dV = V^{-1} \int_S \mathbf{x} \otimes \mathbf{N} dS \, . \tag{2.196}$$

Strict application of (2.196) requires C^1-continuity of $\mathbf{x}(X,t)$.

2.7.2 Stokes's Theorem

The generalized Stokes's theorem is written (Malvern 1969)

$$\int_S (\mathbf{N} \times \nabla) * \mathbf{A} \, dS = \int_C d\mathbf{X} * \mathbf{A} \ , \tag{2.197}$$

where quantities introduced already in (2.192) have the same definitions, and where C is a closed curve with coordinates $\mathbf{X}$ encircling an oriented surface S with normal $\mathbf{N}$. Again, $\mathbf{A}$ must have continuous first derivatives with respect to coordinates corresponding to covariant derivative with gradient operator ∇. Surface S must be simply connected for a single curve C to suffice; otherwise the line integration must proceed over the collection of bounding curves interior and exterior to S. When path C of the line integral is taken in a counterclockwise sense, the positive direction of normal $\mathbf{N}$ is defined according to the usual right-hand rule of vector calculus. Stokes's theorem, like Gauss's theorem, can be applied in either configuration of the body. For example, when $\mathbf{A} \to \mathbf{v}(x,t)$ (the spatial velocity field) and $* \to \langle . \, , . \rangle_x$, the following equality applies:

$$\begin{aligned} \int_s \varepsilon^{abc} n_b \overset{g}{\nabla}_c v_a ds &= \int_s \varepsilon^{abc} v_{a;c} n_b ds = \int_s \varepsilon^{abc} (v_{a,c} - \overset{g}{\Gamma}{}^{..d}_{ca} v_d) n_b ds \\ &= -\int_s \varepsilon^{abc} v_{a,b} n_c ds = -\int_s v_{a,b} dx^a \wedge dx^b = \int_c v_a dx^a , \end{aligned} \tag{2.198}$$

where s and c denote, respectively, surfaces and bounding curves on a body in the spatial description. When the curl or skew gradient of $\mathbf{v}$ vanishes, i.e., when $v_{[a;c]} = v_{[a,c]} = W_{ac} = 0$, the integrand in (2.198) is identically zero, and the velocity field is said to be irrotational.

Consider now the deformation gradient in the context of Stokes's theorem. Let S and C denote, respectively, surfaces and bounding curves on a body in the reference description, let $\mathbf{A} \to \mathbf{F}(X,t)$, and let $* \to \langle . , . \rangle_X$. Then in indicial notation,

$$\begin{aligned}\int_S \varepsilon^{ABC} F^a_{.A;C} N_B dS &= -\int_S \varepsilon^{ABC} F^a_{.A,B} N_C dS \\ &= -\int_S F^a_{.A,B} dX^A \wedge dX^B = \int_C F^a_{.A} dX^A .\end{aligned} \tag{2.199}$$

where symmetry properties of the Christoffel symbols $\overset{G}{\Gamma}{}^{..A}_{BC} = \overset{G}{\Gamma}{}^{..A}_{(BC)}$ have been exploited. Likewise, in the spatial description,

$$\begin{aligned}\int_s \varepsilon^{abc} F^{-1A}_{.a;c} n_b ds &= -\int_s \varepsilon^{abc} F^{-1A}_{.a,b} n_c ds \\ &= -\int_s F^{-1A}_{.a,b} dx^a \wedge dx^b = \int_c F^{-1A}_{.a} dx^a .\end{aligned} \tag{2.200}$$

Restrictions on coordinate systems may apply since (2.199) and (2.200) involve integration of vector fields. Specifically, basis vectors $\mathbf{g}_a$ must be constant for all points x in the domain of integration in global equation (2.199), while basis vectors $\mathbf{G}_A$ must be constant for all points X in the domain of integration of (2.200). Otherwise, integrands in (2.199) and (2.200) must be parallel transported to a single point (x or X) using the appropriate shifter, and the integral then evaluated at that point (Toupin 1956). Furthermore, $\mathbf{F}(X,t)$ and $\mathbf{F}^{-1}(x,t)$ must have continuous first derivatives with respect to reference and spatial coordinates, respectively. From (2.115), since the line integral of position about a closed loop on the surface of or within a simply connected body vanishes,

$$0 = \int_c dx^a = \int_C F^a_{.A} dX^A = -\int_S \varepsilon^{ABC} F^a_{.A,B} N_C dS , \tag{2.201}$$

$$0 = \int_C dX^A = \int_c F^{-1A}_{.a} dx^a = -\int_s \varepsilon^{abc} F^{-1A}_{.a,b} n_c ds . \tag{2.202}$$

Since (2.201) and (2.202) must hold for any path within the body,

$$F^a_{.[A,B]} = F^a_{.[A;B]} = F^a_{.[A:B]} = 0 , \; F^{-1A}_{.[a,b]} = F^{-1A}_{.[a;b]} = F^{-1A}_{.[a:b]} = 0 . \tag{2.203}$$

As discussed further in Section 2.8, (2.203) can be interpreted as local compatibility conditions for the deformation gradient and its inverse.

2.8 Anholonomic Spaces and Compatibility

The following topics associated with compatibility, or lack thereof, of arbitrary tangent maps and linear connections are discussed in Section 2.8: anholonomic deformations and anholonomic configurations, strain compatibility, connection compatibility, and the Jacobian determinant.

2.8.1 Anholonomicity

Consider a field of covariant basis vectors $\tilde{\mathbf{g}}_\alpha$ $(\alpha = 1,2,3)$, spanning a tangent bundle $T\tilde{B}$ associated with arbitrary configuration $\tilde{B}$. By introducing the two-point map $\tilde{\mathbf{F}}(X,t) = \tilde{F}^\alpha_{.A}\,\tilde{\mathbf{g}}_\alpha \otimes \mathbf{G}^A$ that is assumed to be differentiable and invertible, generic vectors $\mathbf{V} \in T_X B_0$ are pushed forward to $T_{\tilde{x}}\tilde{B}$:

$$\begin{aligned}\tilde{\mathbf{F}}\mathbf{V} &= \tilde{F}^\alpha_{.A}\,\tilde{\mathbf{g}}_\alpha \otimes \mathbf{G}^A\left(V^B\mathbf{G}_B\right) = \tilde{F}^\alpha_{.A}V^B\tilde{\mathbf{g}}_\alpha\left\langle \mathbf{G}^A,\mathbf{G}_B\right\rangle \\ &= \tilde{F}^\alpha_{.A}V^B\delta^A_{.B}\tilde{\mathbf{g}}_\alpha = \tilde{F}^\alpha_{.A}V^A\tilde{\mathbf{g}}_\alpha \in T_{\tilde{x}}\tilde{B}.\end{aligned} \tag{2.204}$$

Basis vectors are tangent to globally continuous coordinate curves $\tilde{x}^\alpha$ (i.e., $\tilde{\mathbf{g}}_\alpha = \partial / \partial\tilde{x}^\alpha$ for some coordinate parameterization $\tilde{x}^\alpha(X,t)$) if and only if the following integrability conditions hold for $\tilde{\mathbf{F}}$ (Schouten 1954):

$$\tilde{F}^\alpha_{.A,B} = \tilde{F}^\alpha_{.B,A} \Leftrightarrow \frac{\partial^2\tilde{x}^\alpha}{\partial X^A \partial X^B} = \frac{\partial^2\tilde{x}^\alpha}{\partial X^B \partial X^A}. \tag{2.205}$$

If conditions (2.205) are not satisfied, then $\tilde{\mathbf{g}}_\alpha$ is called an anholonomic basis vector, the tangent map $\tilde{\mathbf{F}}$ is called an incompatible map, and $\tilde{B}$ is called an incompatible configuration or an anholonomic space[11]. In such a case, anholonomic coordinates $\tilde{x}^\alpha$, sometimes called non-holonomic coordinates (Stojanovitch 1969), are available as one-to-one functions of location X only over local patches of $\tilde{B}$, if at all. Anholonomic spaces can be interpreted as containing regions where coordinates $\tilde{x}^\alpha$ may be multivalued (overlaps) or undefined (holes) functions of X (Kondo 1964). From another perspective, when anholonomic, $\tilde{B}$ can be considered a nine-dimensional space (nine independent components of $\tilde{F}^\alpha_{.A}$) in contrast to

[11] When $\tilde{\mathbf{F}}$ does not have continuous first partial derivatives, (2.205) does not apply. For example, a piecewise linear field of coordinates $\tilde{x}^\alpha(X)$ can result in a piecewise constant deformation gradient $\tilde{x}^\alpha_{,A}(X)$ with discontinuities, as can occur physically in crystals in the context of deformation twinning (James 1981).

holonomic configuration B_0 parameterized by three independent coordinates X^A. Since the deformation gradient and its inverse are integrable, $F^a_{.[A,B]} = 0$ and $F^{-1A}_{.[a,b]} = 0$, as indicated in (2.203) in the context of Stokes's theorem.

Because conventional differentiation with respect to anholonomic coordinates does not apply, partial differentiation with respect to anholonomic $\tilde{x}^\alpha$ is defined in a special manner:

$$\begin{aligned}\partial_\alpha(\,.\,) = (\,.\,)_{,\alpha} = \frac{\partial(\,.\,)}{\partial\tilde{x}^\alpha} = \frac{\partial(\,.\,)}{\partial X^A}\tilde{F}^{-1A}_{.\alpha} \\ = \partial_A(\,.\,)\tilde{F}^{-1A}_{.\alpha} = \partial_a(\,.\,)\,F^a_{.A}\tilde{F}^{-1A}_{.\alpha}.\end{aligned} \tag{2.206}$$

The anholonomic object $\tilde{\boldsymbol{\kappa}}$ is introduced (Schouten 1954; Kondo 1964):

$$\tilde{\kappa}^{..\alpha}_{\beta\chi} = \tilde{F}^{-1A}_{.\beta}\tilde{F}^{-1B}_{.\chi}\partial_{[A}\tilde{F}^\alpha_{.B]} = \tilde{F}^{-1A}_{.\beta}\tilde{F}^{-1B}_{.\chi}\tilde{F}^\alpha_{.[B,A]} = \tilde{\kappa}^{..\alpha}_{[\beta\chi]}, \tag{2.207}$$

a geometric construct whose components vanish if and only if $\tilde{\mathbf{F}}$ is integrable. Consider the transformation law for the connection coefficients given by (2.28), but now applied with respect to a change from holonomic to anholonomic coordinates $X^A \to \tilde{x}^\alpha$, where (2.206) is used to define partial differentiation with respect to $\tilde{x}^\alpha$. Arbitrary connection coefficients $\Gamma^{..A}_{BC}$ on B_0 then transform to coefficients $\tilde{\Gamma}^{..\alpha}_{\beta\chi}$ on $\tilde{B}$ as

$$\begin{aligned}\tilde{\Gamma}^{..\alpha}_{\beta\chi} &= \tilde{F}^{-1B}_{.\beta}\tilde{F}^{-1C}_{.\chi}\tilde{F}^\alpha_{.A}\Gamma^{..A}_{BC} + \tilde{F}^{-1C}_{.\chi,\beta}\tilde{F}^\alpha_{.C} \\ &= \tilde{F}^{-1B}_{.\beta}\tilde{F}^{-1C}_{.\chi}\tilde{F}^\alpha_{.A}\Gamma^{..A}_{BC} - \tilde{F}^{-1B}_{.\beta}\tilde{F}^{-1C}_{.\chi}\tilde{F}^\alpha_{.C,B}.\end{aligned} \tag{2.208}$$

Torsion tensor $\mathbf{T}$ of (2.33) pushed forward to $\tilde{B}$ becomes, in components

$$\tilde{T}^{..\alpha}_{\beta\chi} = \tilde{F}^{-1B}_{.\beta}\tilde{F}^{-1C}_{.\chi}\tilde{F}^\alpha_{.A}T^{..A}_{BC} = \tilde{\Gamma}^{..\alpha}_{[\beta\chi]} + \tilde{\kappa}^{..\alpha}_{\beta\chi}, \tag{2.209}$$

implying that $\tilde{\Gamma}^{..\alpha}_{[\beta\chi]}$ need not vanish for covariant components of $\Gamma^{..A}_{BC}$ to be symmetric. Partial derivatives of basis vectors with respect to an arbitrary connection on $\tilde{B}$ are defined as analogs of (2.56):

$$\tilde{\mathbf{g}}^\alpha_{,\beta} = -\tilde{\Gamma}^{..\alpha}_{\beta\chi}\tilde{\mathbf{g}}^\chi, \quad \tilde{\mathbf{g}}_{\alpha,\beta} = \tilde{\Gamma}^{..\chi}_{\beta\alpha}\tilde{\mathbf{g}}_\chi. \tag{2.210}$$

Relations (2.210) are treated here as general postulates, applicable regardless of whether or not (2.208) is used to define connection coefficients on $\tilde{B}$. However, when (2.208) does specifically apply, skew partial derivatives of covariant basis vectors in (2.210) need not always vanish even if $\tilde{T}^{..\alpha}_{\beta\chi} = 0$, since

$$\tilde{\mathbf{g}}_{[\alpha,\beta]} = \tilde{\Gamma}^{..\chi}_{[\beta\alpha]}\tilde{\mathbf{g}}_\chi = (\tilde{T}^{..\chi}_{\beta\alpha} - \tilde{\kappa}^{..\chi}_{\beta\alpha})\tilde{\mathbf{g}}_\chi. \tag{2.211}$$

The expression for the Riemann-Christoffel curvature tensor of a linear connection, defined with respect to holonomic coordinates in B_0 as $R_{BCD}^{...A}$ in (2.34) and pushed forward to anholonomic coordinates $\tilde{x}^{\alpha}$, is (Schouten 1954; Kondo 1964)

$$\begin{aligned}\tilde{R}_{\beta\chi\delta}^{...\alpha} &= \tilde{F}^{-1B}_{.\beta}\tilde{F}^{-1C}_{.\chi}\tilde{F}^{-1D}_{.\delta}\tilde{F}^{\alpha}_{.A}R_{BCD}^{...A} \\ &= \frac{\partial\tilde{\Gamma}_{\chi\delta}^{..\alpha}}{\partial\tilde{x}^{\beta}} - \frac{\partial\tilde{\Gamma}_{\beta\delta}^{..\alpha}}{\partial\tilde{x}^{\chi}} + \tilde{\Gamma}_{\beta\varepsilon}^{..\alpha}\tilde{\Gamma}_{\chi\delta}^{..\varepsilon} - \tilde{\Gamma}_{\chi\varepsilon}^{..\alpha}\tilde{\Gamma}_{\beta\delta}^{..\varepsilon} + 2\tilde{\kappa}_{\beta\chi}^{..\varepsilon}\tilde{\Gamma}_{\varepsilon\delta}^{..\alpha} \\ &= 2\partial_{[\beta}\tilde{\Gamma}_{\chi]\delta}^{..\alpha} + 2\tilde{\Gamma}_{[\beta|\varepsilon|}^{..\alpha}\tilde{\Gamma}_{\chi]\delta}^{..\varepsilon} + 2\tilde{\kappa}_{[\beta\chi]}^{..\varepsilon}\tilde{\Gamma}_{\varepsilon\delta}^{..\alpha},\end{aligned} \tag{2.212}$$

which, upon comparison with the holonomic representation in (2.34), differs only by the rightmost term that includes the anholonomic object.

2.8.2 Strain Compatibility

A second interpretation of compatibility follows from consideration of the Riemann-Christoffel curvature tensor. From (2.43), Christoffel symbols of the second kind in the context of Riemannian geometry formed from the symmetric right Cauchy-Green deformation tensor $\mathbf{C}(X,t)$ are

$$\overset{C}{\Gamma}{}_{BC}^{..A} = \frac{1}{2}C^{-1AD}(C_{BD,C} + C_{CD,B} - C_{BC,D}) = \overset{C}{\Gamma}{}_{CB}^{..A}. \tag{2.213}$$

A curvature tensor $\overset{\mathbf{C}}{\mathbf{R}}$ can be constructed by substituting components $\overset{C}{\Gamma}{}_{BC}^{..A}$ into (2.34). Since $\mathbf{C} = \varphi^*(\mathbf{g})$, $\overset{\mathbf{C}}{\mathbf{R}} = \varphi^*(\overset{\mathbf{g}}{\mathbf{R}})$ follows from properties of the connection and curvature (Marsden and Hughes 1983). Thus, if the deformation tensor field $\mathbf{C}$ is derivable from a motion $\varphi(X,t)$, and since $\overset{\mathbf{g}}{\mathbf{R}} = 0$, it follows that $\overset{\mathbf{C}}{\mathbf{R}} = \varphi^*(0) = 0$ and $\mathbf{C}$ is compatible. Section D.3 of Appendix D contains an alternative derivation worked out in convected coordinates, demonstrating vanishing of the curvature tensor formed from $C_{AB} = F^a_{.A}g_{ab}F^b_{.B}$ noting that conditions $F^a_{.A} = x^a_{,A}$ and $F^a_{.A,B} = F^a_{.B,A}$ apply.

Notice that $\mathbf{C}$-compatibility does not require specification of a unique spatial configuration, since $\mathbf{C}$ is independent of the rotation tensor associated with the right polar decomposition in (2.149). Furthermore, even if $\mathbf{C}$ is compatible, integrability of an arbitrary field $\mathbf{F}$ generating the field $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ may be precluded by rotations $\mathbf{R}$ that do not arise from rigid-body transformations of φ. For example, consider the field $F^a_{.A} = R^a_{.A}$ with $R^a_{.A}R^{T\,A}_{\;\;.b} = \delta^a_{.b}$, such that $C_{AB} = F^a_{.A}g_{ab}F^b_{.B} = R^C_{.A}G_{CD}R^D_{.B} = G_{AB}$. Integrability of

F is violated by a rotation field for which $R^{a}_{.[A,B]} \neq 0$, even though the curvature tensor constructed from $C_{AB} = G_{AB}$ vanishes identically in this case. A material in such a condition is said to be in a state of contorted aleotropy (Noll 1967). The converse of the previous theorem has also been proven, albeit only locally (Eisenhart 1926). In other words, given a positive definite, symmetric, second-order tensor **C** whose curvature vanishes (i.e., $\overset{\mathbf{C}}{\mathbf{R}} = 0$) then at any point $X \in B_0$ there exists a neighborhood U_0 of X endowed with a mapping $\varphi : B_0 \supset U_0 \to U \subset B$ whose deformation tensor is **C**. A more extensive discussion of compatibility in terms of **C** is given by Fosdick (1966), who notes that vanishing of the Ricci tensor or the Einstein tensor constructed from $\overset{\mathbf{C}}{\mathbf{R}}$ is sufficient to ensure compatibility of **C**. One can consider **C**-compatibility an outcome of deformation gradient compatibility (i.e., **F**-compatibility): if $F^{a}_{.A}(X,t)$ is compatible, then it follows that $C_{AB}(X,t) = F^{a}_{.A} g_{ab} F^{b}_{.B}$ is also compatible, as demonstrated in Section D.3 of Appendix D.

It is emphasized that a Levi-Civita connection—a connection both torsion-free and metric with respect to $\tilde{\mathbf{g}}$, where the metric has components $\tilde{g}_{\alpha\beta} = \tilde{\mathbf{g}}_{\alpha} \bullet \tilde{\mathbf{g}}_{\beta}$—on anholonomic space $\tilde{B}$ may not exist, since the field of vectors $\tilde{\mathbf{g}}_{\alpha}$ may not be sufficiently smooth over all of $\tilde{B}$ to admit coordinate differentiation with respect to coordinates that may in fact be discontinuous or multi-valued, i.e., anholonomic coordinates. However, in anholonomic (e.g., intermediate or natural) configurations of elastoplasticity theory, each local volume element of material is often referred to an external system of coordinates with Euclidean metric tensor, typically taken as Kronecker's delta for convenience (Teodosiu 1967a, b; Simo and Ortiz 1985), though the assumption $\tilde{g}_{\alpha\beta} = \delta_{\alpha\beta}$ is not always necessary (Maugin 1995; Clayton et al. 2004a). This issue is discussed in more detail in Section 3.2.2. Notice also that the Riemann-Christoffel curvature tensor formed from the generally incompatible covariant deformation measure $\tilde{C}_{AB} = \tilde{F}^{\alpha}_{.A} \tilde{g}_{\alpha\beta} \tilde{F}^{\beta}_{.B}$ does not necessarily vanish unless $\tilde{x}^{\alpha}$ are holonomic, in contrast to the vanishing curvature tensor derived from connection (2.213) formed from the compatible deformation tensor **C**.

Compatibility equations for the small strain tensor of (2.163) are typically expressed somewhat differently than those for **C** (Malvern 1969; Mura 1982; Teodosiu 1982). The former are derived from differentiation of the right side of (2.163) and a sequence of indicial manipulations, leading to vanishing of the incompatibility tensor with components s^{cf} :

$$s^{cf} = s^{(cf)} = g^{-1} e^{abc} e^{def} \varepsilon_{bd;ae} = -g^{-1} e^{cab} e^{fed} \varepsilon_{bd;ae} = 0 . \tag{2.214}$$

The six independent equations in (2.214) ensure that symmetric tensor $\boldsymbol{\varepsilon}(x,t)$ with continuous second derivatives with respect to holonomic coordinates x^a is integrable; i.e., (2.214) ensures that a continuously differentiable displacement field $\mathbf{u}(x,t)$ exists such that (2.163) applies. Analogously to (2.203), $u_{a,[bc]} = 0$ from the commutative property the of the mixed second partial derivative with respect to holonomic coordinates, and vanishing of the curvature and torsion of the Levi-Civita connection results in $u_{a;[bc]} = 0$, as concluded from (2.40).

2.8.3 Connection Compatibility

Consider coefficients of a special linear connection formed by spatial differentiation of a smooth, possibly anholonomic tangent map with inverse $\bar{\mathbf{F}}^{-1}(x,t)$: $T_x B \to T_{\bar{x}} \bar{B}$, defined as follows:

$$\begin{aligned} \bar{\varGamma}^{..a}_{cb} &= \bar{F}^{a}_{.\alpha} \partial_c \bar{F}^{-1\alpha}_{.b} = \bar{F}^{a}_{.\alpha} \bar{F}^{-1\alpha}_{.b,c} \\ &= -\bar{F}^{a}_{.\alpha,c} \bar{F}^{-1\alpha}_{.b} = -\bar{F}^{a}_{.\alpha,\beta} \bar{F}^{-1\alpha}_{.b} \bar{F}^{-1\beta}_{.c} , \end{aligned} \tag{2.215}$$

where the third of (2.215) follows from $(\bar{F}^{a}_{.\alpha} \bar{F}^{-1\alpha}_{.b})_{,c} = \delta^{a}_{.b,c} = 0$ and the final equality follows from (2.206). Connections of the form (2.215) have special meaning in field theories of lattice defects (Bilby et al. 1955; Kroner 1960) and are said to exhibit the property of teleparallelism or absolute parallelism (Einstein 1928; Schouten 1954).

Partial differentiation of (2.215) yields

$$(\bar{F}^{-1\alpha}_{.a} \bar{\varGamma}^{..a}_{bc})_{,d} = \bar{F}^{-1\alpha}_{.c,bd} , \tag{2.216}$$

the left side of which is expanded as

$$\bar{F}^{-1\alpha}_{.a,d} \bar{\varGamma}^{..a}_{bc} + \bar{F}^{-1\alpha}_{.a} \bar{\varGamma}^{..a}_{bc,d} = \bar{F}^{-1\alpha}_{.a} (\bar{\varGamma}^{..a}_{de} \bar{\varGamma}^{..e}_{bc} + \bar{\varGamma}^{..a}_{bc,d}) . \tag{2.217}$$

Since the order of partial differentiation on the right side of (2.216) is arbitrary (Schouten 1954; Le and Stumpf 1996a),

$$\begin{aligned} 0 = 2\partial_{[b} \partial_{d]} \bar{F}^{-1\alpha}_{.c} &= \bar{F}^{-1\alpha}_{.a} (\bar{\varGamma}^{..a}_{dc,b} - \bar{\varGamma}^{..a}_{bc,d} + \bar{\varGamma}^{..a}_{be} \bar{\varGamma}^{..e}_{dc} - \bar{\varGamma}^{..a}_{de} \bar{\varGamma}^{..e}_{bc}) \\ &= \bar{F}^{-1\alpha}_{.a} \bar{R}^{...a}_{bdc} , \end{aligned} \tag{2.218}$$

where $\bar{R}^{...a}_{bdc}$ are components of the Riemann-Christoffel curvature tensor derived from $\bar{\varGamma}^{..a}_{cb}$ of (2.215) using definition (2.34). Upon multiplication of (2.218) through by $\bar{\mathbf{F}}$, it is evident that $\bar{R}_{cdb}^{\,..a} = 0$ are conditions ensuring satisfaction of (2.215). In that case (i.e., null curvature), the connection with coefficients $\bar{\Gamma}_{cb}^{\,.\,.a}$ is said to be integrable (Schouten 1954).

Secondarily, if the connection with coefficients in (2.215) is symmetric, then its torsion vanishes by definition, and it follows from arguments analogous to those accompanying (2.205) that $\bar{\mathbf{F}}^{-1}(x,t)$ is itself integrable:

$$\bar{T}_{cb}^{\,..a} = \bar{\Gamma}_{[cb]}^{\,..a} = \bar{F}_{.\alpha}^{a} \bar{F}_{.[b,c]}^{-1\alpha} = 0 \Leftrightarrow \bar{F}_{.a}^{-1\alpha} = \bar{x}_{.,a}^{\alpha} = \partial_a \bar{x}^{\alpha} \,. \tag{2.219}$$

When (2.219) applies,

$$\bar{\Gamma}_{bc}^{\,..a} = \frac{1}{2} \bar{C}^{-1ad} (\bar{C}_{bd,c} + \bar{C}_{cd,b} - \bar{C}_{bc,d}) \,, \; \bar{\nabla}_a \bar{C}_{bc} = 0 \,, \tag{2.220}$$

meaning that $\bar{\Gamma}_{cb}^{\,..a}$ are Levi-Civita coefficients formed from a metric tensor with components $\bar{C}_{ab}(x,t) = \bar{F}_{.a}^{-1\alpha} \bar{g}_{\alpha\beta} \bar{F}_{.b}^{-1\beta}$, and the set $\left\{\bar{B}, \bar{\Gamma}_{cb}^{\,..a}, \bar{C}_{ab}\right\}$ constitutes a Euclidean space since $\bar{R}_{bdc}^{\,...a} = 0$. Components of a second metric tensor on $\bar{B}$ are denoted by $\bar{g}_{\alpha\beta}$ and can be chosen as $\bar{g}_{\alpha\beta} = \delta_{\alpha\beta}$ for convenience. The first equality in (2.220) follows from direct calculation and (2.219) as

$$\begin{aligned} 2\bar{\Gamma}_{bc}^{\,..a} &= \bar{C}^{-1ad} (\bar{C}_{bd,c} + \bar{C}_{cd,b} - \bar{C}_{bc,d}) \\ &= 2\bar{F}_{.\alpha}^{a} \bar{F}_{.(c,b)}^{-1\alpha} + 2\bar{F}_{.\alpha}^{a} \bar{g}^{\alpha\beta} \bar{F}_{.\beta}^{d} \left[\bar{F}_{.b}^{-1\chi} \bar{F}_{.[d,c]}^{-1\delta} + \bar{F}_{.c}^{-1\chi} \bar{F}_{.[d,b]}^{-1\delta} \right] \bar{g}_{\chi\delta} \\ &= 2\bar{F}_{.\alpha}^{a} \bar{F}_{.(c,b)}^{-1\alpha}, \end{aligned} \tag{2.221}$$

and the second of (2.220) is verified by substituting the first of (2.220) into

$$\bar{\nabla}_a \bar{C}_{bc} = \bar{C}_{bc,a} - \bar{\Gamma}_{ab}^{\,..d} \bar{C}_{dc} - \bar{\Gamma}_{ac}^{\,..d} \bar{C}_{bd} = 0 \,. \tag{2.222}$$

2.8.4 The Jacobian Determinant for Anholonomic Mappings

The Jacobian determinant of anholonomic mapping $\tilde{\mathbf{F}}$ introduced in (2.204) provides, by definition, the relationship between a differential reference volume element $dV \subset B_0$ and its deformed counterpart in the anholonomic space, $d\tilde{V} \subset \tilde{B}$:

$$\tilde{J} dV = d\tilde{V} \,. \tag{2.223}$$

Analogously to (2.142), the function $\tilde{J}(X,t)$, regarded as a true scalar under coordinate transformations, is calculated in coordinates as

$$\tilde{J} = \frac{1}{6}\varepsilon_{\alpha\beta\chi}\varepsilon^{ABC}\tilde{F}^{\alpha}_{.A}\tilde{F}^{\beta}_{.B}\tilde{F}^{\chi}_{.C} = \frac{1}{6}\sqrt{\tilde{g}/G}e_{\alpha\beta\chi}e^{ABC}\tilde{F}^{\alpha}_{.A}\tilde{F}^{\beta}_{.B}\tilde{F}^{\chi}_{.C}$$
$$= \frac{1}{6}\det\tilde{\mathbf{F}}\sqrt{\tilde{g}/G} = \frac{1}{6}\det(\tilde{F}^{\alpha}_{.A})\sqrt{\det(\tilde{g}_{\alpha\beta})/\det(G_{AB})}, \tag{2.224}$$

where $\tilde{g} = \det(\tilde{g}_{\alpha\beta}) = \det(\tilde{\mathbf{g}}_{\alpha}\bullet\tilde{\mathbf{g}}_{\beta}) \geq 0$. Requiring that the volume of any element must remain positive and finite implies bounds $\infty > \tilde{J} > 0$. Differentiation of (2.224) produces an identity like (2.144):

$$\frac{\partial\tilde{J}}{\partial\tilde{F}^{\alpha}_{.A}} = \frac{1}{2}\varepsilon_{\alpha\beta\chi}\varepsilon^{ABC}\tilde{F}^{\beta}_{.B}\tilde{F}^{\chi}_{.C} = \tilde{J}F^{-1A}_{.\alpha}. \tag{2.225}$$

Taking the total covariant derivative of the first of (2.225) then gives

$$2(\tilde{J}\tilde{F}^{-1A}_{.\alpha})_{:A} = \varepsilon_{\alpha\beta\chi}\varepsilon^{ABC}(\tilde{F}^{\beta}_{.B:A}\tilde{F}^{\chi}_{.C} + \tilde{F}^{\beta}_{.B}\tilde{F}^{\chi}_{.C:A})$$
$$= \varepsilon_{\alpha\beta\chi}\varepsilon^{ABC}(\tilde{F}^{\beta}_{.[B:A]}\tilde{F}^{\chi}_{.C} + \tilde{F}^{\beta}_{.B}\tilde{F}^{\chi}_{.[C:A]}). \tag{2.226}$$

When (2.205) is not satisfied, meaning anholonomic object $\tilde{\boldsymbol{\kappa}}$ in (2.207) does not vanish, the right side of (2.226) may be nonzero and Piola identities such as those in (2.145) and (2.146) for holonomic mapping $\mathbf{F}$ and its inverse do not always hold for anholonomic mappings.

Let oriented differential area elements referred to configurations B_0 and $\tilde{B}$ be labeled $\mathbf{N}dS$ and $\tilde{\mathbf{n}}d\tilde{s}$, respectively. Analogously to (2.148), these elements are related by Nanson's formula or a Piola transformation:

$$\tilde{\mathbf{n}}d\tilde{s} = \tilde{J}\tilde{F}^{-*}\mathbf{N}dS, \quad \tilde{n}_{\alpha}d\tilde{s} = \tilde{J}\tilde{F}^{-*.A}_{\alpha}N_{A}dS = \tilde{J}\tilde{F}^{-1A}_{.\alpha}N_{A}dS. \tag{2.227}$$

In terms of the wedge product, $\tilde{n}_{\alpha}d\tilde{s} = \varepsilon_{\alpha\beta\chi}d\tilde{x}^{[\beta}d\tilde{x}^{\chi]} = \varepsilon_{\alpha\beta\chi}d\tilde{x}^{\beta}\wedge d\tilde{x}^{\chi}/2$ and $d\tilde{x}^{a}\wedge d\tilde{x}^{b} = \varepsilon^{\alpha\beta\chi}\tilde{n}_{\chi}d\tilde{s}$, with $d\tilde{x}^{a}$ and $d\tilde{x}^{b}$ components of two different infinitesimal vectors so their tensor product is not identically symmetric.

Relation (2.227) can be obtained directly from (2.224), noting that

$$6 = (\varepsilon_{\alpha\beta\chi})(\tilde{J}^{-1}\varepsilon^{ABC}\tilde{F}^{\alpha}_{.A}\tilde{F}^{\beta}_{.B}\tilde{F}^{\chi}_{.C})$$
$$= (\sqrt{\tilde{g}}e_{\alpha\beta\chi})(\tilde{J}^{-1}\sqrt[-1]{G}e^{ABC}\tilde{F}^{\alpha}_{.A}\tilde{F}^{\beta}_{.B}\tilde{F}^{\chi}_{.C})$$
$$= \varepsilon_{\alpha\beta\chi}\varepsilon^{\alpha\beta\chi} = \varepsilon_{ABC}\varepsilon^{ABC}$$
$$= (\tilde{J}^{-1}\sqrt{\tilde{g}}e_{\alpha\beta\chi}\tilde{F}^{\alpha}_{.A}\tilde{F}^{\beta}_{.B}\tilde{F}^{\chi}_{.C})(\sqrt[-1]{G}e^{ABC}). \tag{2.228}$$

Therefore, permutation tensors referred to configurations $\tilde{B}$ and B_0 satisfy

$$\varepsilon^{\alpha\beta\chi} = \tilde{J}^{-1}\varepsilon^{ABC}\tilde{F}^{\alpha}_{.A}\tilde{F}^{\beta}_{.B}\tilde{F}^{\chi}_{.C}, \quad \varepsilon_{ABC} = \tilde{J}^{-1}\varepsilon_{\alpha\beta\chi}\tilde{F}^{\alpha}_{.A}\tilde{F}^{\beta}_{.B}\tilde{F}^{\chi}_{.C}. \tag{2.229}$$

Letting $d\tilde{x}^{\alpha} = \tilde{F}^{\alpha}_{.A}dX^{A}$, it follows that an oriented area element in B_0 is

$$\begin{aligned} N_A dS &= \varepsilon_{ABC} dX^B dX^C = \varepsilon_{ABC} \tilde{F}^{-1B}_{.\varepsilon} \tilde{F}^{-1C}_{.\phi} d\tilde{x}^{\varepsilon} d\tilde{x}^{\phi} \\ &= \tilde{J}^{-1} \tilde{F}^{\alpha}_{.A} \varepsilon_{\alpha\beta\chi} d\tilde{x}^{\beta} d\tilde{x}^{\chi} = \tilde{J}^{-1} \tilde{F}^{\alpha}_{.A} \tilde{n}_{\alpha} d\tilde{s}, \end{aligned} \tag{2.230}$$

the inverse of which yields the final result in (2.227). The same approach can be used to derive (2.148) directly from (2.142), as demonstrated in Section D.2 of Appendix D.

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