

## Chapter 2

# Water Movement and Solute Transport in Unsaturated Porous Media

The unsaturated zone, also termed the vadose zone, is the portion of the subsurface above the groundwater table. It contains air as well as water in the pores. This zone is also high in organic matter and clay, which promotes sorption, biological degradation and transformation of contaminants. In industrial or agricultural areas, where the ground surface is contaminated by hazardous wastes or fertilizers and pesticides, the unsaturated zone may be thought of as a buffer zone, which provides protection to the underlying aquifers. Unsaturated zone is often regarded as a filter removing undesirable substances before they affect aquifers, and the hydrogeologic properties of unsaturated zone are the most important factor for groundwater deterioration induced by surface contamination (Stephens 1996; Selker et al. 1999).

On the other hand, from hydrologic viewpoints, unsaturated zone is a zone that to a large degree controls the transmission of water to aquifers, as well as to the land surface, to water on the surface, and to the atmosphere. Processes in the unsaturated zone may be a controlling factor in the distribution of the atmospheric precipitation between runoff, infiltration, and evapotranspiration.

Therefore the knowledge and understanding of water flow and solute transport in the unsaturated zone is becoming increasingly important especially in terms of water resources planning and management, water quality management and the mitigation of groundwater pollution. The presence of different phases (soil/sediment matrix, water and air) results in many different physical and chemical processes taking place. These processes are often complex and require simplifying assumptions to provide achievable and verifiable results in terms of water and solute solution profiles in the shallow sediments (Selker et al. 1999; Gandola et al. 2001).

The assessment of groundwater protection against pollution requires the joint consideration of the processes of moisture and solute transfer under incomplete saturation. The rate of flow (infiltration) and the distribution of moisture in the vertical profile are functions (often nonstationary), the knowledge of which is a prerequisite for the correct solution of mass transfer problems. Strictly speaking, the presence of air in the subsurface material as an independent phase requires the consideration of multiphase flow models (Bear and Cheng 2010). The behavior of the system in the case of multiphase flow is governed by the relationship between capillary pressure  $P^c$ , volumetric water content  $\theta$  (or saturation  $S$ ), and the hydraulic conductivity  $k$ :  $P^c - \theta - k$ , which is referred to as the basic relationship.

However, the formulation and solution of many problems can be simplified, if only the motion of the liquid phase is considered, with the assumption that ideal (without resistance) counterflow of pore air exists, such that its pressure in any point is always constant and equal to the atmospheric pressure, both within the wetting zone and between the infiltration front and groundwater table. Therefore, hereafter we will consider only two migration “components”: the moisture and the solute. The presence of air will be accounted for indirectly through the transfer coefficient and storage characteristics.

Quantification of flow and transport in unsaturated *fractured rocks* requires often conceptualizing these processes in terms of discrete fracture network (DFN) or stochastic fracture-continuum (SFC) models (Neuman 2005). The DFN approach considers random networks as interconnected line or plane segments embedded in a low permeable matrix. SFC approach conceptualizes heterogeneous fracture media with a stochastic spatial distribution of fracture permeability and geometric parameters. Theoretical analysis and numerical modeling show discrete features of flow paths in individual fractures: overall gravity driven flow in the fracture network may focus in a few paths. Similar to sub-horizontal flow in aquifers (Sect. 1.1.1) the flux that occurs in the system of focused flow paths can be up to ten times more than the areally distributed infiltration flux prescribed at the inflow boundary (Pruess 1999; Liu et al. 2002; Zhang et al. 2004). It means that water can flow rapidly through fracture networks along localized preferential pathways, moreover under the natural conditions most such seepage may proceed in an unsteady and episodic manner. The mentioned phenomena are subject of special investigations related to development of sophisticated models of fracture networks based on statistical information derived from field-measured fracture data.

This chapter reviews only the fundamental mechanisms governing flow and transport in unsaturated zone represented by homogeneous single porosity shallow sediments employing macroscale continuum concept. An important attribute of the subsurface transport models is the link between fracture surface and matrix processes unified by the matrix potential (Chap. 7).

## 2.1 Basic Soil-Water Movement and Infiltration Models

Models for wetting-front migration and water saturation distribution behind the front provide the basis for mass-conservative coupling of fluid flow and solute transport in soils and unsaturated geological media. The main focus is detailed analysis of typical one-dimensional infiltration problems formulated here for the use in a further study of transport phenomena in homogeneous environment, ignoring the soil heterogeneity and preferential flow (Sect. 2.2). Also, the models discussed in this chapter can potentially be expanded to studying solute transfer processes in the unsaturated zone represented by structured soils with macropores or fractured formations exhibiting a dual porosity behavior (Chap. 7).

### 2.1.1 Governing Functions and Parameters

#### 2.1.1.1 Soil Water Potential

The total status of energy for water in partially saturated porous material is described by the total moisture potential (Nielsen 1991; Weight 2008),  $\varphi_t$ ,

$$\varphi_t = \varphi_g + \varphi_m, \quad (2.1)$$

where  $\varphi_g = gz$  and  $\varphi_m = P/\rho_w$  are the gravitational and matrix (or soil water) potentials [ $L^2T^{-2}$ ],  $g$  is the acceleration due to gravity [ $LT^{-2}$ ],  $z > 0$  is the elevation above an arbitrary datum [ $L$ ],  $\rho_w$  is water density [ $ML^{-3}$ ],  $P$  is the hydrostatic pressure [ $ML^{-1}T^{-2}$ ]. The gravitational potential presents a potential energy due to the vertical location of the liquid in the elementary volume of the unsaturated porous media. The matrix potential is an integral characteristic of the manifestation of capillary-adsorption forces caused by the presence of interphase boundaries in the pore space (the rock matrix, the water and air phases).

The total potential,  $\varphi_t$ , in the vadose zone can be converted to hydraulic (or piezometric) head (as is typically made in subsurface hydrology) by dividing by the acceleration due to gravity:

$$h = z + \psi, \quad (2.2)$$

where  $\psi$  is the *pore water (capillary) pressure head* [ $L$ ]. Thus, the hydraulic head is the sum of the pressure head,  $\psi = P/\rho_w g \leq 0$ , and the position,  $z$  (the  $z$ -axis is directed vertically upward). If the air forms a continuous phase, the hydrostatic pressure in water ( $P_w$ ) is less than the pressure in the air ( $P_g$ ). In this case, the pressure  $P$ , reduced to the atmospheric pressure is  $P_w - P_g < 0$ , therefore, the pressure head

$$\psi = \frac{P}{\rho_w g} \equiv -\frac{P^c}{\rho_w g} \quad (2.3)$$

( $P^c$  is the capillary pressure) is also negative. Function  $\psi$  characterizes the work against the capillary-adsorption forces performed to withdraw water from unsaturated porous medium and hence is a potential characteristic. Considering that saturation at the phreatic water surface is complete and  $\psi = 0$ , we have  $h = z$ .

Moisture transport in the aeration zone is largely determined by the functional relationship between its three major characteristics

$$\psi - \theta - k.$$

This relationship, which is commonly nonlinear, largely determines the specific approach to be used to solve the equations describing the migration of water and chemicals dissolved in it in the aeration zone represented by unconsolidated sedimentary materials (often referred to as soil) and rocks.

### 2.1.1.2 Water Retention Curves

Pressure head is a function of the moisture content of subsurface materials,  $\theta$ :  $\psi = f(\theta)$ . This function is referred to as the *water retention curve*, and is also called the *soil moisture characteristic*. The empirical expressions fitted to data of physical experiments and relating the characteristics  $\psi$  and  $\theta$  are very diverse. Thus, the shape of water retention curves can be characterized by several models. The relationships of the form

$$\psi = f(\theta) \text{ or } \theta = f^{-1}(\psi) \quad (2.4)$$

are commonly nonlinear.

The simplest one is the exponential function of the form:

$$\bar{\theta}(\psi) = e^{\alpha\psi} (\psi \leq 0), \bar{\theta} = \frac{\theta - \theta_r}{\theta_s - \theta_r}, \quad (2.5)$$

where  $\bar{\theta}$  is the normalized volumetric water content,  $\theta_s$  is the saturated volumetric water content (maximum water content),  $\theta_r$  is the residual volumetric water content,  $\alpha$  is the sorptive number, regarded as the reciprocal of the capillary length scale [ $L^{-1}$ ],  $\alpha \sim 1/h_c$  ( $h_c$  is an adduced capillary height). Sometimes this coefficient is called the desaturation coefficient which represents the desaturation rate of the soil-water characteristic curve (Ng and Menzies 2007); the value  $\alpha$  is related to the soil grain size distribution.

With respect to the applicability of Eq. 2.5, it should be noted: (1) the relative degree of water saturation (effective water saturation),  $\bar{S}$ , which indicates the percentage of the voids filled with water, is often used in place of the normalized water content,  $\bar{\theta}$ ; (2) the difference between the saturated soil water content and the residual soil water content  $\Delta\theta = \theta_s - \theta_r$  in (2.5) is called *soil moisture capacity*; (3) for the subsequent expert estimates, the parameter  $\alpha$  may be assumed to vary within the most likely range from 0.5 (clays and loams) to  $10 \text{ m}^{-1}$  (sand type of sediments), thus, the greater the clay content, the smaller the value of  $\alpha$ . The relationship (2.5), which has been thoroughly studied before (Gardner 1958), is often referred to as the *Gardner formula* in the scientific literature.

Widely used in the practice is the exponential *Brook–Corey function* (Brooks and Corey 1964; Corey 1977):

$$\bar{\theta}(\psi) = \begin{cases} \left( \frac{\psi_d}{\psi} \right)^\lambda & \psi < \psi_d, \\ 1 & \psi \geq \psi_d, \end{cases} \quad (2.6)$$

where  $\psi_d \equiv \psi_{ge}$  is the air entry suction head at which air enters the capillary-porous body (i.e., the minimal capillary pressure the nonwetting phase must overpass to enter the pores occupied by the wetting phase) [L],  $\lambda (> 0)$  is a pore-size distribution index [–].

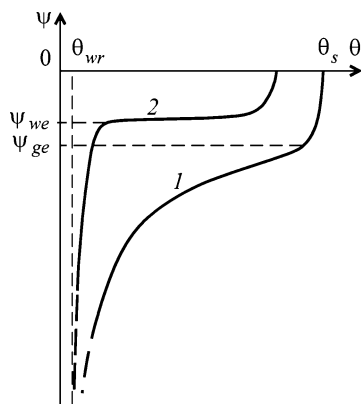
Equally popular in terms of citation in the scientific literature is the *van Genuchten formula* (Van Genuchten 1980)

$$\bar{\theta}(\psi) = \begin{cases} \frac{1}{[1 + |\alpha\psi|^n]^m} & \psi < 0, \\ 1 & \psi \geq 0, \end{cases} \quad (2.7)$$

which in some cases yields more accurate (as compared with Eq. 2.6) description of experimental data. Here  $\alpha$ ,  $n$  and  $m$  are empirical parameters;  $\alpha$  [ $L^{-1}$ ] approximately equals the inverse of the air entry value for small  $m/n$ , while for large  $m/n$  this parameter roughly equals the inverse of the pressure head at the inflection point (Van Genuchten and Nielsen 1985);  $n$  ( $>1$ ) is a parameter related to pore size distribution index [-]. To obtain a relatively simple predictive close-form analytical expression for the unsaturated hydraulic conductivity, Van Genuchten (1980) assumed a unique relationship between the parameters  $m$  and  $n$ ,  $m = 1 - 1/n$ . This relationship between  $m$  and  $n$  implies a capillary pressure saturation curve with saturation in the low capillary pressure range to a greater extent than measurements (Corey 1994).

The relationship (2.7) is attractive in that, owing to its flexibility, it can describe a wider range of experimental curves  $\psi = f(\theta)$ , obtained for different types of sedimentary deposits. As compared with Eqs. 2.5 and 2.6, Eq. 2.7, better approximates the character of the curve  $\psi = f(\theta)$  during the initial period of sample draining, when the suction head abruptly changes against a relatively small drop in saturation. Unlike (2.6), function (2.7) is continuous. However, an advantage of (2.6) is the fact that the parameter  $\psi_d$  in it has a clear physical meaning.

The relationship between  $\psi$  and  $\theta$  is ambiguous: when a completely saturated sediment is being drained, the function  $\psi(\theta)$  is such that each value  $\psi$  is associated with the maximum moisture content; while during saturation of a dry rock sediment, the same moisture content values will be associated with suction head with lesser absolute values. This reflects the effect of hysteresis of retention functions (Fig. 2.1). In general the hysteresis is caused by a change of the energy status of water when



**Fig. 2.1** Typical retention curves  $\psi(\theta)$ . 1 – drainage curve, 2 – wetting curve.  $\psi_{we}$  is the water entry suction head,  $\psi_{ge}$  is the air entry suction head,  $\theta_{wr}$  is the residual water content

a wetting process is switched to a drying process or vice versa. The change in the energy status can be measured by a change in potential  $\psi$ . The hysteresis phenomena are of particular importance in the description of unsteady waves of moisture content during time-varying infiltration, as well as during the analysis of the instability of the wetting front (Nielsen et al. 1986). In this chapter, such phenomena are considered in their most general form.

The ability of soils and rocks to yield or to gain moisture with changing pressure head is characterized by the parameter

$$C(\psi) = d\theta/d\psi > 0, \quad (2.8)$$

called *specific moisture capacity* [ $L^{-1}$ ]. It can be evaluated by differentiating the retention function  $\theta(\psi)$ . The specific moisture capacity can be defined precisely as the volume of water released from or taken into storage, per unit volume of vadoze zone, per unit change in pressure head (Stephens 1996).

### 2.1.1.3 Darcy's Law and Hydraulic Conductivity of Unsaturated Sediments and Rocks

Under conditions of vertical water movement through unsaturated single porosity subsurface materials, Darcy's law, also appropriately referred to as the Darcy–Buckingham equation (Narasimhan 2005; Raats and Genuchten 2006), is expressed as

$$q = -k(h) \frac{\partial h}{\partial z} = -k(\psi) \left( \frac{\partial \psi}{\partial z} + 1 \right), \quad (2.9)$$

where  $q$  is the *specific flux*,  $k(\psi)$  is some coefficient of proportionality between the specific flux (Darcy velocity) and the gradient of hydraulic head which is called *unsaturated soil hydraulic conductivity*.

Considering the existence of a nonlinear relationship between  $\psi$  and  $\theta$ , we can suppose that the hydraulic conductivity of an unsaturated soil or rock,  $k$ , is not constant and strongly depends on the volumetric water content. It varies from the value  $k(\theta_0) = k_s$ , which corresponds to the hydraulic conductivity at complete porous material saturation ( $\theta = \theta_s$ ), to zero when the pores are dry (more exactly, the porous material at residual saturation  $\theta = \theta_r$ ). The nonlinearity of the function

$$k = k(\theta) \text{ or } k = k(\psi)$$

is supported by a vast body of experimental data. A number of equations for the function  $k$  have been suggested by different researchers. Many of the theoretical and experimental findings can be rearranged to an exponential form of the *Averjanov (1950) equation*:

$$k(\bar{\theta}) = k_s \bar{\theta}^n, \quad (2.10)$$

where the exponent  $n$  has quite a wide range of values. Some of the pore macroscopic models predict  $n = 2-4$  (Brutsaert 1967, 1968; Corey 1994; Leong and Rahardjo 1997). The coefficient  $n$  was found to be dependent on pore-size distribution and pore disconnectedness.

According to the tests of Brooks and Corey (1964) one may assume the following equation for unsaturated conductivity

$$k(\psi) = \begin{cases} k_s \left( \frac{\psi_d}{\psi} \right)^m & \psi < \psi_d, \\ k_s & \psi \geq \psi_d. \end{cases} \quad (2.11)$$

Since there exists a relationship between  $\bar{\theta}$  and  $\psi_d/\psi$  in the form of Eq. 2.6, the unsaturated hydraulic conductivity can be represented as the power function (2.10). Experiment and theory demonstrate that  $n$  may be estimated from  $\lambda$  as  $n = (2 + 3\lambda)/\lambda$ . In equality (2.11) coefficient  $m$  can also be represented as a function of  $\lambda$ .

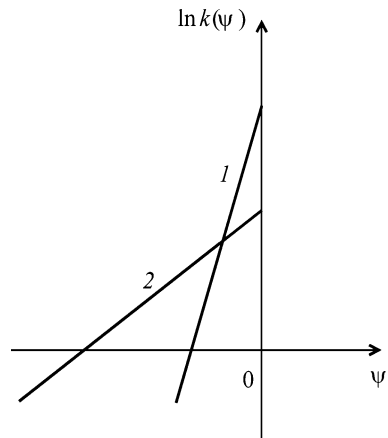
Substituting (2.5) into (2.10) we obtain the expression for the unsaturated hydraulic conductivity

$$k(\psi) = k_s e^{n\alpha\psi} \quad (\psi \leq 0), \quad (2.12)$$

suggested by Gardner (1958). Taking the logarithms of the left and the right parts of (2.12), we rewrite it as

$$\ln k(\psi) = \ln k_s + n\alpha\psi. \quad (2.12a)$$

Plots of (2.12a) in Fig. 2.2 reflect an interesting feature of the functional relationship  $k(\psi)$  for two types of soils of mostly sand and clay composition (Gelhar 1993; Pease and Stormont 1996). When the moisture content is high (and, accordingly,  $|\psi|$  is small), the ability of sand subsurface material to transfer water is also greater than that for clay-enriched sediments. Conversely, when the moisture content is low ( $|\psi|$  is large), clay-enriched sediments, when not saturated, may be more permeable for water.



**Fig. 2.2** Functional relationship  $\ln k(\psi)$ . 1 and 2 are sand and clay-enriched sediments, respectively

This effect, resulting from the difference between the absolute values of parameters (sands have high  $k_s$  and  $\alpha$ , while compacted clayey materials have low  $k_s$  and  $\alpha$ ), should be taken into account in the studies of moisture migration in complexes of aerated stratified sediments as well as sediments (rocks) with dual-porosity structure, the fractures serve as high-permeability elements. Thus, it is clear that during dry periods (with low precipitation), moisture in the aeration zone presented by fractured-porous rocks would mostly migrate through low-permeability pore blocks, because the estimated moisture transfer coefficient for them will be larger than for considerably drained (dewatered) fractures. In rainy periods, when fissures are filled with water, they should contribute most to moisture transfer (Chap. 7).

In general there are two major approaches to obtain the hydraulic conductivity function for an unsaturated subsurface material: (a) empirical fitting of equations to measured lab or field data, (b) statistical models which can be used to predict the hydraulic conductivity function when the saturated hydraulic conductivity  $k_s$  and the soil-water characteristic curves are available. In particular, a statistical model analysis results in the following relationship for hydraulic conductivity for the van Genuchten retention curve (Van Genuchten 1980):

$$k(\bar{\theta}) = k_s \sqrt{\bar{\theta}} \left( 1 - (1 - \bar{\theta}^{1/m})^m \right)^2, \quad (2.13)$$

where  $k_s$  is the hydraulic conductivity at saturation [ $\text{LT}^{-1}$ ]; the unsaturated hydraulic properties are approximated reasonably well with the more restricted case where  $m = 1 - 1/n$  (Van Genuchten and Nielsen 1985). Equation 2.13 was obtained by substituting Eq. 2.7 into an expression derived by Mualem (1976).

Taking into account hysteresis in the water retention functions (Sect. 2.1.1.2), one may expect the relationship between unsaturated hydraulic conductivity and pressure head also to be hysteretic. Obviously, problems in which hysteresis may be important to consider involve periods of both wetting and drying, such as can occur during infiltration and subsequent redistribution of a square-wave pulse of infiltrated water (Stephens 1996). Such finite pulse of water cannot accurately be modeled by assuming as input parameters either the wetting or drying unsaturated hydraulic conductivity curves.

#### 2.1.1.4 Soil Water Diffusivity

Expression (2.9) for moisture flux can be represented in the alternative form:

$$-q = k(\theta) \frac{d\psi}{d\theta} \frac{\partial \theta}{\partial z} + k(\theta) = D(\theta) \frac{\partial \theta}{\partial z} + k(\theta), \quad (2.14)$$

where

$$D(\theta) = \frac{k(\theta)}{d\theta/d\psi}, \quad d\theta/d\psi > 0. \quad (2.14a)$$



The representation (2.14) has an advection–dispersion form: the first term in the right part of the latter identity (2.14) reflects the diffusivity of moisture flux at the expense of capillary-adsorption forces; therefore, the parameter  $D(\theta)$  (2.14a) is referred to as *soil water (moisture) diffusivity* [ $L^2T^{-1}$ ]; water diffusivity does not exist near saturation (i.e., as  $D(\theta)$  approaches  $\infty$ ) since  $C(\psi)$  approaches zero; the second term in the right part of (2.14) is the gravity-induced (advection) transfer of moisture under unit gradient.

Similar to  $k(\theta)$ , function  $D(\theta)$  can be formulated in terms of macroscopic pore models (Corey 1994; Singh 1997). Combining Eqs. 2.5, 2.10 with Eq. 2.14a and Eqs. 2.6, 2.11 with Eq. 2.14a gives

$$D(\bar{\theta}) = \frac{k_s}{\alpha \Delta \theta} \bar{\theta}^{n-1} \text{ and } D(\bar{\theta}) = \frac{k_s |\Psi_d|}{\lambda \Delta \theta} \bar{\theta}^{\frac{m-\lambda-1}{\lambda}} \quad (2.15)$$

for Averjanov and Gardner, Brooks–Corey models, respectively. Thus, the soil water diffusivity may be described as a power function of relative water content; the difference  $\Delta \theta = (\theta_s - \theta_r)$  is the *moisture capacity*, a parameter which appears in many analytical formulas related to water and solute transport in unsaturated zone.

The nonlinear soil water diffusivity as a power function of relative water content

$$D(\theta) = D_s \bar{\theta}^\gamma \quad (2.16)$$

has been used over the last few decades by a number of soil physicists; here  $D_s = D(\theta_s)$  (saturated soil water diffusivity) and  $\gamma$  are constants. Some researchers employed the exponential diffusivity law for function  $D(\theta)$ , postulating (Parlange 1980; Singh 1997)

$$D(\theta) = ae^{b\bar{\theta}}, \quad (2.17)$$

where  $a$  and  $b$  are empirical coefficients.

Formulas (2.16) and (2.17) illustrate the basic difficulty of water movement in soil, i.e.  $D$  increases very rapidly with  $\theta$ , corresponding to a profile where most of change in  $\theta$  occurs in a narrow zone  $\theta^0 \leq \theta \leq \theta_s$ . That is totally different from a molecular diffusion process (Sect. 1.1.2) with near-constant diffusivity. Hence standard linearization techniques are inapplicable to calculate the profile  $\theta(z)$  based on a given single value of  $D$  (Parlange 1980).

### 2.1.2 Continuity Equation and its Major Representations

The equation of continuity (or mass conservation) is

$$\frac{\partial \theta \rho_w}{\partial t} = \nabla \cdot (\rho_w \mathbf{q}). \quad (2.18)$$

Assuming that the density of water  $\rho_w$  remains constant, the equation of continuity (2.18) for 1D water flow in  $z$ direction becomes:

$$\frac{\partial \theta}{\partial t} + \frac{\partial q}{\partial z} = 0. \quad (2.19)$$

Expressing the infiltration rate in terms of the hydraulic head function (2.2), we come to the equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left( k(h) \frac{\partial h}{\partial z} \right), \quad (2.20)$$

which is of a fundamental character and is often referred to as *Richards equation*. Clearly, it is based on the continuity equation and Darcy's law (Richards 1931). The Richards equation for soil moisture movement holds a very important place in modern theoretical and applied hydrology, in particular providing a basis for studies of global water cycle and the assessment of groundwater protection against the industrial contamination. The Richards equation (2.20) allows further transformations (representations), whose specific features are determined by the character of problems to be solved and the form of the available parametric (experimental) functions  $\psi - \theta - k$ .

Thus, setting  $\partial \theta / \partial t = (d\theta/d\psi) \partial \psi / \partial t$ , we obtain the  $\psi$ -based formulation of Richards equation (pressure head-based Richards equation):

$$\text{or} \quad C(\psi) \frac{\partial \psi}{\partial t} = \frac{\partial}{\partial z} \left( k(\psi) \frac{\partial \psi}{\partial z} \right) + \frac{dk(\psi)}{d\psi} \frac{\partial \psi}{\partial z}, \quad (2.21)$$

where  $C(\psi) = d\theta/d\psi > 0$  is defined as the specific moisture capacity function (2.8). The coefficients of Eq. 2.21 involve two highly non-linear functions related to the soil water potential.

Once the flux  $q$  in the continuity Eq. 2.19 is determined, from Eq. 2.14 we obtain the so-called  $\theta$ -based formulation of Richards equation (2.20) (Gardner 1958; Vanderborght et al. 2005):

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left( D(\theta) \frac{\partial \theta}{\partial z} \right) + \frac{dk(\theta)}{d\theta} \frac{\partial \theta}{\partial z}. \quad (2.22)$$

The dependence of  $D$  and  $k$  on  $\theta$  renders Eq. 2.22 highly nonlinear.

The moisture-based Eq. 2.22 is appropriate for describing flow in unsaturated homogeneous media only such that  $\psi(\theta)$  is single-valued. This equation can not be used to describe water flow under saturated condition (Corey 1994). Therefore the pressure-based form (2.21) more suitable for variably-saturated soil (including fully saturated conditions) and spatially heterogeneous soils.

Finally, the "mixed" Richards equation has the form:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left( k(\psi) \frac{\partial \psi}{\partial z} \right) + \frac{\partial k(\psi)}{\partial z}. \quad (2.23)$$

Koo and Suh (2001) note that the use of the mixed Eq. 2.23 for the numerical solution of moisture transfer problems (especially, two- and three-dimensional ones) allows one to avoid the shortcomings of the implementation of algorithms based on the two previous representations of the equation in question.

To make the presentation of material general, we give here a three-dimensional form of the basic equation of mass transport in the aeration zone:

$$\frac{\partial \theta}{\partial t} = \nabla \cdot [\mathbf{k}(\psi) \cdot \nabla(\psi + z)], \quad (2.24)$$

where  $\mathbf{k}(\psi)$  is a second-order tensor of the moisture transfer; the  $z$ -axis is directed vertically upward.

In conclusion we make a comment. The functional relationships  $\psi - \theta$  and  $k - \psi(\theta)$  used in the mathematical formulation of moisture transfer problems are evaluated from experiments carried out under the conditions close to thermodynamic equilibrium because of the long duration of experiments: to obtain experimental curves similar to those graphically represented in Fig. 2.1 commonly requires several days or weeks. Therefore, the parameters that characterize the kinetic stage of the process, preceding the attainment of  $\psi - \theta$  capillary equilibria, commonly are not involved in the interpretation procedures. However, the processes in nature that accompany soil drainage or saturation often take a few hours or days. Therefore, the classic description of such processes on the basis of the Richards equation, which makes no allowance for the real dynamics (kinetics) of the establishment of capillary equilibrium  $\psi - \theta$ , can be not quite adequate. Such effects were called *dynamic* (Hassanizadeh et al. 2002).

The correct use of experimental equilibrium relationships may require in some cases the inclusion into the model of some kinetic parameters to allow one to more thoroughly study the unsteady moisture transfer in the cases when the rates of changes in the moisture content function (or saturation  $S = \theta/\phi$ ) are large. Relationships between the static ( $\psi_{\text{stat}} \equiv \psi = \psi(S)$ ) and dynamic ( $\psi_{\text{din}}$ ) suction pressures have been proposed in the form of the kinetic equation (Hassanizadeh et al. 2002):

$$\psi_{\text{din}} = \psi(S) - \frac{\tau}{\phi \rho g} \left( \frac{\partial \theta}{\partial t} \right), \quad (2.25)$$

where  $\tau$  is a relaxation factor of the system [ $\text{ML}^{-1}\text{T}^{-1}$ ], which can vary depending on the saturation of the medium; its values vary within the orders of  $10^4$ – $10^7$  kg/ms.

The equality (2.25) allows us to transform the initial equations of moisture transfer, thus taking into account the dynamic effects. The need for this can arise in the analysis of significantly unstable moisture flows in fine-grained media with high entry capillary pressure, as well as in the description of the instability of moisture transfer front. However, the traditional formulation of the problem of moisture transfer in porous medium under the effect of gravity (advection) and adsorption-capillary forces (sometimes, conventionally referred to as diffusion) is sufficient in the overwhelming majority of practically significant cases.

### 2.1.3 Particular Solutions for Moisture Migration and Their Analysis

#### 2.1.3.1 Problem Formulation

The moisture transfer equations given above are essentially nonlinear, since their coefficients depend on the functions sought for ( $\psi$  or  $\theta$ ). This creates apparent difficulties for the search of their solutions in a closed analytical form. The original differential equations become more complicated because of the inclusion of additional source terms to account for moisture losses from the transition zones (pores, macropores, and fissures) due to evapotranspiration (evaporation and consumption by plant roots) and the saturation of porous rock blocks in contact with those zones (Chap. 7).

The search for solutions of the Richards equation, which are adapted to real situations, by numerical methods is a complex problem (Van Dam et al. 2004). This is due not only to the strong nonlinearity of the process being simulated, but also to very wide variation ranges of the control variables (functions). For example, with moderate moisture content of near-surface sedimentary deposits in the aeration zone, the suction pressure  $\psi$  can reach  $-10^6$  cm. Heavy precipitation or accidental spills of a liquid can increase porous material moistening within a few minutes, resulting in a pressure drop to zero. At the same time, the values of the moisture transfer coefficient increase by 5–8 orders of magnitude. The modeling of such events requires the time steps to be very small and very strict convergence criteria for the solution of the Richards equation to be met (Van Dam et al. 2004).

In view of the further practical application, the mathematical formulation of the problem should reflect the specific hydrogeological and geomorphological conditions of individual areas, the direction of the process (whether the aeration zone is being drained or saturated – because of the hysteresis of parameter functions), and the regime and rate of infiltration recharge. One should also take into account that, although Eqs. 2.21 and 2.22 have been derived from similar physical assumptions, their application conditions do not coincide, i.e., they are not completely equivalent (Ravi and Williams 1998). However, notwithstanding the restrictions for the use of Eq. 2.22, it is convenient for the analysis of moisture migration in homogeneous media.

In the ideal case, the initial condition ( $t = 0$ ) should specify the initial distribution of moisture or the suction pressure:

$$\theta(z, t = 0) = \theta^0(z), \quad \psi(z, t = 0) = \psi^0(z). \quad (2.26)$$

In the case of infiltration problems, the boundary conditions (the  $z$ -axis is directed vertically downward,  $z = 0$  on the surface) can be of the first, second, or third type, respectively:

$$\theta(z = 0, t) = \theta_0(t), \quad \psi(z = 0, t) = \psi_0(t), \quad (2.27a)$$

$$q(z = 0, t) = q_0(t), \quad (2.27b)$$

$$k(\theta) - D(\theta) \frac{\partial \theta}{\partial z} = q_0(t), \quad k(\psi) \left( 1 - \frac{\partial \psi}{\partial z} \right) = q_0(t). \quad (2.27c)$$

The correct specification of boundary conditions on the upper boundary is often problematic when groundwater level varies considerably and the land surface can be ponded in some moments. For example, after water level in the aquifer drops below this surface and the near-surface soil layer becomes dry, the suction pressure abruptly rises, creating difficulties with the convergence of numerical solutions of the Richards equation describing this process. Generally, effective procedures for “switching” boundary conditions from the first to the second type and back should be among the necessary options of mathematical models intended for studying such situations.

In the strict formulation, the boundary condition on the lower boundary of the aeration zone should specify the conjunction with the water-saturated rock formations, i.e., the aquifer (drainage condition). However, the process can be sometimes considered as taking place in a semi-infinite medium. In such cases, it is better to assume the  $z$ -axis to be directed vertically downward. Accordingly, the gravitational (advective) terms in the initial Eqs. 2.21, 2.22, and 2.23 change their signs to the opposite. A complete analysis of water redistribution in soil should take into account capillary hysteresis effects.

The problem of search for analytical solutions of the Richards equation is the subject of a vast body of scientific literature (Philip 1957a, b; Youngs 1957; Broadbridge and White 1988; Broadbridge 1990; Barry and Sander 1991; Ross and Parlange 1994; Salvucci 1996; Parlange et al. 1997, 1999; Wang et al. 2003; Wiltshire and El-Kafri 2004). The final analytical solutions are complicated, especially when the initial conditions are heterogeneous and the input (boundary) functions have a “burst” character (i.e., when the hysteresis of the parametric function  $\psi - \theta$  manifests itself in full measure). Therefore, infiltration processes are most often described by numerical models, which allow the hysteresis of soil-water retention functions to be taken into account. However, some useful results can be obtained with simplified physical models, neglecting certain mechanisms of moisture transfer at some process stages.

Although studying water motion in the unsaturated zone requires the use of unsteady state problems, the steady state formulation of the problem may be of interest as well. We will start our analysis from this case.

### 2.1.3.2 Steady-State Water Flow Models

Formula (2.9) yields two consequences reflecting some specific features of the behavior of function  $\psi$  in different conditions.

1. In the absence of moisture transfer in the aeration zone, when we can assume  $q = 0$ , the gradient  $\partial \psi / \partial z = -1$  ( $\partial h / \partial z = 0$ ), or  $\psi = z_0 - z$ , i.e., the suction

height is numerically equal to the elevation above the reference plane ( $z = z_0$ ), for example, above the groundwater level; if  $z_0 = 0$ , then  $\psi = -z$ . This fact implies that the moisture distribution in the capillary zone should correspond to the soil water retention function  $\theta(z) = \theta(\psi)$ .

2. In the case of *steady-state moisture transfer* (water-application rate  $q = \text{const}$ ) the distribution of the suction pressure depends on the form of function  $k(\psi)$  and is determined by the integral equality:

$$z - z_0 = - \int_0^{\psi} \frac{d\psi}{1 + q/k(\psi)}, \quad (2.28)$$

where the integral in (2.28) can have an analytical representation in algebraic functions when  $k(\psi)$  are available (Gardner 1958).

In the general steady-state formulation, the profile of the suction pressure  $\psi(z)$  can be calculated by applying the method of successive direct fitting (Ho 2001): with known pressure  $\psi(z_i)$  in point  $z_i$ , we search for the pressure  $\psi(z_{i+1})$  in the upstream point  $z_{i+1}$  by integrating function (2.28) over the interval  $[\psi(z_i), \psi(z_{i+1})]$  ( $\psi(z_{i+1})$  is the sought-for upper integration limit), i.e.:

$$z_{i+1} = z_i - \int_{\psi(z_i)}^{\psi(z_{i+1})} \frac{d\psi}{1 + q/k(\psi)}. \quad (2.29)$$

Moreover, the solution (2.29) can be extended to the case of steady-state moisture transfer in a stratified system by choosing the values of  $\psi(z)$  from  $\psi(z_1)$  to  $\psi(z_n)$  by calculation through all lithological beds with the value of function  $k(\psi)$  corresponding to the layer with a number, say,  $\Omega$ ,  $\Omega = 1, 2, \dots, m$  ( $m$  is the number of layers) used to calculate the integral function (Ho 2001):

$$z_{i+1}^{\Omega} = z_i^{\Omega} - \int_{\psi^{\Omega}(z_i)}^{\psi^{\Omega}(z_{i+1})} \frac{d\psi}{1 + q/k^{\Omega}(\psi)}. \quad (2.30)$$

When the calculation process reaches an interface between two layers, the pressure at the bottom of each layer (with bottom-upward numeration) is taken equal to the pressure at the top of the underlying layer, i.e., the boundary condition for each lithological contact has the form:

$$z_i^{\Omega+1} = z_i^{\Omega}; \psi(z_i^{\Omega+1}) = \psi(z_i^{\Omega}). \quad (2.31)$$

The calculated profile  $\psi(z)$  allows one, taking into account the functional form of the relationship  $\psi(\theta)$ , to determine the distribution of moisture over the depth; hence, given the Darcy velocity ( $q$ ), one can find variations in the real migration velocity  $u(z) = q/\theta(z)$ .

### 2.1.3.3 A Water Absorption Model (Moisture Movement Without Gravity)

If the gravity component is neglected, as is reasonable in the case of infiltration into relatively dry soils at the initial stages of moisture front formation, the Eq. 2.22 can be reduced to

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left( D(\theta) \frac{\partial \theta}{\partial z} \right), \quad (2.32)$$

known as *moisture diffusivity equation* (Bear and Cheng 2010).

As is proved in the theory of differential equations (Polyanin et al. 2005), the Eq. 2.32 has a self-similar solution in the form

$$\theta(z, t) = \theta(\lambda), \text{ where } \lambda = \frac{z}{\sqrt{t}} \quad (2.33)$$

is the so-called *Boltzmann transformation*, combining the variables  $z$  and  $t$  (Philip 1957a; Lagan 2001). The profiles of self-similar solutions in different time moments can be obtained from one another by a similarity transformation. Each value  $\lambda$  is associated with a single value  $\theta$ , and the conditions  $\lambda = \text{const}$  and  $z \sim \sqrt{t}$  hold for constant moisture content.

The substitution of the new variable  $\lambda = z/\sqrt{t}$  into the Eq. 2.32 allows this equation to be transformed into the ordinary differential equation

$$\frac{\lambda}{2} \frac{d\theta}{d\lambda} + \frac{d}{d\lambda} \left( D(\theta) \frac{d\theta}{d\lambda} \right) = 0, \quad (2.34)$$

with the following boundary conditions:

$$\theta(\lambda = 0) = \theta_0, \theta(\lambda \rightarrow \infty) = \theta^0. \quad (2.35)$$

The transition from (2.32) to (2.34) implied:

$$\frac{\partial}{\partial z} = \frac{\partial \lambda}{\partial z} \frac{\partial}{\partial \lambda} = \frac{1}{\sqrt{t}} \frac{d}{d\lambda}, \frac{\partial}{\partial t} = \frac{\partial \lambda}{\partial t} \frac{\partial}{\partial \lambda} = -\frac{1}{2} \frac{\lambda}{t} \frac{d}{d\lambda}. \quad (2.36)$$

The integration of (2.34) yields the integro-differential equation (Bruce and Klute 1956; Philip 1957a)

$$\int_{\theta^0}^{\theta} \lambda d\theta = -2D(\theta) \frac{d\theta}{d\lambda}, \quad (2.37)$$

which can be easily solved by iteration numerical methods.

Equation (2.37) shows that Boltzmann variable  $\lambda(\theta)$  linearizes Eq. 2.32 to give the solution for diffusivity at water content  $\theta'$ , in other words,  $D(\theta')$  can be quantified if  $\lambda(\theta)$  is measured

$$D(\theta') = -\frac{1}{2} \left( \frac{d\lambda}{d\theta} \right)_{\theta=\theta'} \int_{\theta^0}^{\theta'} \lambda(\theta) d\theta; \quad (2.38)$$

the Eq. 2.38 is known as the [Bruce and Klute \(1956\)](#) equation ([Stephens 1996](#); [Selker et al. 1999](#)).

The function

$$S \equiv S(\theta_0, \theta^0) = \int_{\theta^0}^{\theta_0} \lambda(\theta) d\theta, \quad (2.39)$$

which is determined by the left part of Eq. 2.37 for  $\theta = \theta_0$  and called by [Philip \(1957b\)](#) the *soil water sorptivity* [ $LT^{-1/2}$ ], is often an important component of the theoretical analysis of vertical infiltration in the absence of the effect of gravity. If moisture content  $\theta$  at  $z = 0$  is maintained at zero suction by free water being made available in excess at  $z = 0$ , then  $\theta = \theta_s$ , that is the most commonly discussed condition ([Philip 1957a](#); [Williams et al. 1998](#)).

The sorptivity  $S$  quantifies the effect of capillarity on liquid movement in a soil. The water sorptivity of a soil is a property identified by [Philip \(1957b\)](#) as the critical quantity governing the infiltration at *early time*. For liquid infiltration into soil,  $S$  is a parameter that integrates several factors describing the capacity of the soil to imbibe water at early time, such as initial dryness of the soil and water content behind the infiltrating front.

Two major characteristics of the infiltration process, describing its *dynamics* (see Sect. 2.1.3.6) are related with this characteristic: *the cumulative infiltration* (the accumulated volume of the imbibed liquid per unit surface area),  $I(t)$  [L], and *the infiltration rate* (or *infiltration capacity*) at time  $t$  (the volume of liquid that seeps from the unit area of the surface per unit time)  $i(t)$  [ $LT^{-1}$ ]. The sorptivity is a parameter that accounts for the effect of capillary forces on the unsteady infiltration process, which governs the variations of soil moisture content from the background (initial) value  $\theta^0$  to the boundary value  $\theta_0$ , i.e., the parameter  $S$  characterizes the state of the aeration zone (or soil) under specific landscape–climatic conditions.

At the initial stages of infiltration:

$$I(t) = \int_{\theta^0}^{\theta_0} z(\theta, t) d\theta = S\sqrt{t}, \quad i(t) = \frac{dI(t)}{dt} = \frac{1}{2} \frac{S}{\sqrt{t}}. \quad (2.40)$$

Attempts were made in some works to derive analytical expressions for parameter  $S$ , which would agree with calculated and experimental data ([Brutsaert 1976](#)). In order to find a relationship between the sorptivity and diffusivity, [Wang et al. \(2006\)](#) used the assumption of [Parlange \(1971a\)](#) to change Eq. 2.32 to the following

$$\frac{\partial z}{\partial t} + \frac{\partial}{\partial \theta} \left[ \frac{D(\theta)}{\partial z / \partial \theta} \right] = 0. \quad (2.41)$$



Assuming  $\theta_0 = \theta_s$  and integrating (2.41) for the gradient function

$$i = -D(\theta) \frac{\partial \theta}{\partial z} \approx -D(\theta_s) \left( \frac{\partial \theta}{\partial z} \right)_{z=0}, \quad (2.42)$$

one can get (Wang et al. 2006)

$$i = \frac{1}{\sqrt{2t}} \left[ \int_{\theta^0}^{\theta_s} \theta D(\theta) d\theta \right]^{1/2}. \quad (2.43)$$

Comparing Eq. 2.43 with Eq. 2.40 yields

$$S \equiv S(\theta_s, \theta^0) = \left[ 2 \int_{\theta^0}^{\theta_s} \theta D(\theta) d\theta \right]^{1/2}. \quad (2.44)$$

Equation (2.44) defines the relationship between  $S$  and  $D$  and coincides with a result obtained by Parlange (1971a). For the generalized exponential model (2.16), Eq. 2.44 is transformed as

$$S = \frac{\sqrt{2}}{\sqrt{\gamma+2}} [D_s ((\theta_s)^{\gamma+2} - (\theta^0)^{\gamma+2})]^{1/2}. \quad (2.45)$$

An approximate solution of (2.32) can be obtained in the form of a *traveling wave solution*:

$$\theta(z, t) = \theta(\eta), \quad \eta = z - u^w t, \quad (2.46)$$

where  $u^w$  is a constant propagation velocity of the moisture wave. The solution (2.46) has a self-similar front of moisture content, and solution profiles in different moments can be obtained from one another by a shift transformation (Polyanin et al. 2005). The shape front observed at any time repeats the shape front observed at a previous moment but with a shift along the  $z$ -axis, depending on the wave velocity  $u^w$ .

Substituting Eq. 2.46 into the initial Eq. 2.32 yields, as well as the Boltzmann transformation, the ordinary differential equation

$$\frac{d}{d\eta} \left( D(\theta) \frac{d\theta}{d\eta} \right) + u^w \frac{d\theta}{d\eta} = 0. \quad (2.47)$$

Its integration with constant boundary conditions yields

$$-\int_{\theta^0}^{\theta} \frac{D(\theta) d\theta}{u^w (\theta_0 - \theta)} = \eta, \quad (2.48)$$

where  $\theta^0 = \theta(z = \infty)$ ,  $\theta_0 = \theta(z = -\infty)$ .

Consider a possible transformation of this solution in the case of low initial moisture content, when  $\theta^0 = \theta_r$ . Let the dependence of functions  $\psi$  and  $k$  on the relative moisture content ( $\bar{\theta}$ ) be determined by (2.5) and (2.10), while  $u^w = k(\theta_0)/(\theta_0 - \theta_r)$ . Then, considering the functional form of the soil water diffusivity  $D(\theta)$  (2.15), we obtain:

$$\int_0^{\bar{\theta}} \frac{\bar{\theta}^k}{1 - \bar{\theta}} d\bar{\theta} = -\bar{\eta}, \quad k = n - 1, \quad \bar{\eta} = \frac{\alpha\eta}{\beta}, \quad \bar{\theta} = \frac{\theta - \theta_r}{\theta_0 - \theta_r}, \quad (2.49)$$

$$\beta = \frac{\theta_s - \theta_r}{\theta_0 - \theta_r}, \quad u^w = \frac{k(\theta_0)}{\theta_0 - \theta_r} = \frac{k_s}{\beta^{n-1}\Delta\theta}, \quad \Delta\theta = \theta_s - \theta_r. \quad (2.49a)$$

The integration of (2.49) yields the solution of the problem as the following implicit function of  $\bar{\theta}$ :

$$\sum_{m=0}^{k-1} \frac{\bar{\theta}^{k-m}}{k-m} + \ln(1 - \bar{\theta}) = \bar{\eta}, \quad (2.50)$$

for example, at  $n = 3$  ( $k = 2$ )

$$\frac{\bar{\theta}^2}{2} + \bar{\theta} + \ln(1 - \bar{\theta}) = \bar{\eta}. \quad (2.51)$$

Similar solution was obtained for the soil water diffusivity (2.15) from the Brooks–Corey model.

The use of the solution (2.37), as well as of many other solutions obtained for the moisture transport equation in the closed analytical form, is restricted by the assumption that the initial condition is homogeneous ( $\theta^0 = \text{const}$ ). In the case where  $\theta^0 = \theta(z) \neq \text{const}$ , the following approach can be recommended for obtaining approximate solutions (Witelski 1998). The presence of the initial water distribution in the soil is taken into account in the analytical solution by introducing a time-shift,  $\tau$ , which advances the wetting front:

$$\theta(z, t) = \begin{cases} \theta(z/\sqrt{t + \tau(t)}) & z < z_*(t), \\ \theta^0(z) & z \geq z_*(t), \end{cases} \quad (2.52)$$

where  $\theta[\xi(t + \tau)]$  is the solution obtained under the condition  $\theta^0 = 0$  with the use of Boltzmann transformation ( $\xi$ ), in which the duration of the process is increased by  $\tau(t)$ ;  $z_*(t)$  is the position of the interface or wetting front. The time shift,  $\tau(t)$ , is calculated from the initial moisture distribution and the average soil water diffusivity:

$$\tau(t) = \int_0^{z_*} z \theta^0(z) dz / \int_0^{\xi_*} \xi \theta(\xi) d\xi. \quad (2.53)$$

For degenerate diffusion problems, one can calculate a dynamic time-shift function yielding a good approximate solution for all times (Witelski 1998).

### 2.1.3.4 Gravity-Induced Moisture Advection

Consider another utmost case where water moves under gravity, and capillary forces can be neglected ( $\partial\psi/\partial z \ll 1$ ), i.e.

$$q = -k(\psi) \left( \frac{\partial\psi}{\partial z} - 1 \right) = k(\psi) > 0, \quad (2.54)$$

that is, the Darcy law (2.9), written for the case where the  $z$ -axis is directed vertically downward.

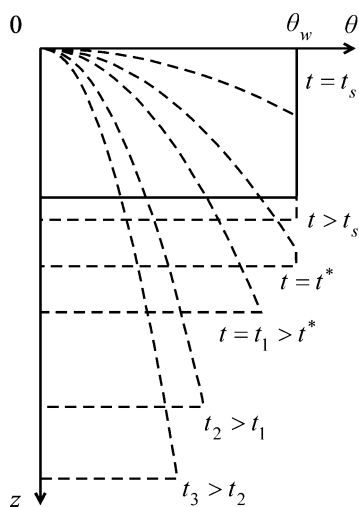
Let water input rate be less than the saturated hydraulic conductivity,  $q(z=0) = q_0 < k_s$ . It appears physically obvious that in this case the applied infiltration rate  $q_0$  is not sufficient to maintain water outflow from the recharge boundary under the effect of gravity with soils or rocks remaining fully water-saturated, i.e., the infiltration takes place under partial saturation of the pore space  $\theta = \theta_w < \theta_s$ , while the rest of it remains filled with air.

From the balance considerations, the propagation velocity of the *wetting front* (the interface between zones with volumetric moisture content  $\theta_w$  and  $\theta^0$ , Fig. 2.3) is determined by the kinematic equation

$$u^w = \frac{dz}{dt} = \frac{q_0}{\theta_w - \theta^0}. \quad (2.55)$$

Formula (2.55) is derived from the assumption that the moisture distribution is piston-like.

Given the analytical dependence of the unsaturated hydraulic conductivity on moisture content  $k = f(\theta)$ , the inverse function  $\theta = f^{-1}(k)$  can be found, allowing one to evaluate the moisture content  $\theta_w$  on the wetting front and behind



**Fig. 2.3** Moisture distribution in different moments for a square pulse of flux density input,  $\theta(z=0, t)$ . Solid lines correspond to a moisture profile at the end of infiltration  $t = t_s$ , dashed lines correspond to a desaturation profile for  $t > t_s$

it ( $z \leq u^w t$ )  $\theta_w = f^{-1}(q_0)$ , which enters the expression (2.55). Thus, for the Averjanov formula (2.10), which is widely used in hydrogeological calculations, we have:

$$\bar{\theta}_w = \left( \frac{q_0}{k_s} \right)^{\frac{1}{n}}. \quad (2.56)$$

Now the expression for the real velocity of vertical moisture transfer at  $\theta_r \ll \theta^0$  becomes

$$u^w = \frac{q_0}{\theta_s (q_0/k_s)^{1/n} - \theta^0}. \quad (2.57a)$$

If we assign  $\theta^0 = \theta_r$ , then

$$u^w = \frac{k_s}{\Delta\theta} \left( \frac{q_0}{k_s} \right)^{\frac{n-1}{n}} = \frac{k_s}{\Delta\theta} \bar{\theta}_w^{n-1}, \quad (2.57b)$$

$\Delta\theta = \theta_s - \theta_r$  (moisture capacity). Since  $n > 1$ , the flow velocity  $u^w$  increases with increasing soil saturation. This implies that the wetting front is of *self-sharpening* character.

The cessation of infiltration from the surface ( $q_0 = 0$ ) in moment  $t = t_s$  results in the formation of a *drying front* (Fig. 2.3). In Fig. 2.3a piston-like distribution is shown at the end of infiltration ( $t = t_s$ ). The expanding drying front ( $t > t_s$ ) forms where air replaces liquid at the interface between the dry upper soil and the top of the descending fluid phase. Because of the nonlinearity of the process, the drying front tends to disperse. An equation describing the spreading of the drying front results from the continuity Eq. 2.19 in which the specific flux is determined by (2.54):

$$\frac{\partial \theta}{\partial t} + \frac{dk(\theta)}{d\theta} \frac{\partial \theta}{\partial z} = 0. \quad (2.58)$$

Determining  $k(\theta)$  from (2.10), we obtain

$$\Delta\theta \frac{\partial \bar{\theta}}{\partial t} + nk_s \bar{\theta}^{n-1} \frac{\partial \bar{\theta}}{\partial z} = 0, \quad (2.59)$$

or

$$\frac{dt}{\Delta\theta} = \frac{dz}{nk_s \bar{\theta}^{n-1}} = \frac{d\bar{\theta}}{0}. \quad (2.60)$$

The integration of (2.60) with  $z(t - t_s = 0) = 0$  yields the characteristic solution

$$z(t) = u^d(\theta)(t - t_s) = \frac{nk_s \bar{\theta}^{n-1}}{\Delta\theta} (t - t_s) \text{ along the characteristic } \bar{\theta} = \text{const.} \quad (2.61)$$

For  $t > t_s$ , the moisture content profile is determined by two simultaneous processes (Fig. 2.3): drying at the top of the fluid pulse and continuous downward propagation of the wetting front at the bottom of the pulse (Lessoff and Indelman 2004). Since  $n > 1$ , as it follows from comparison of Eqs. 2.57b and 2.61, the drying front propagates faster than the wetting front and eventually overtakes the wetting front at the meeting time  $t^*$ , which can be determined from the simple equation  $u^w_{t^*} = u^d(t^* - t)$ , where  $u^w$  and  $u^d$  are determined from Eqs. 2.57b and 2.61 assuming in the latter  $\bar{\theta} = \bar{\theta}_w$ . In particular, from the above it follows:  $t^* = nt_s/(n - 1)$ .

After the wetting and dry fronts meet ( $t > t^*$ ), the front location obeys both (2.57b) and (2.61), whereas the common front saturation  $\bar{\theta}_f(t)$  satisfies the equation (Lessoff and Indelman 2004)

$$\frac{d\bar{\theta}_f}{dt} = \frac{1}{n} \frac{\bar{\theta}_f}{t - t_s}. \quad (2.62)$$

Since, according to (2.56),  $\bar{\theta}_f(t^*) = (q_0/k_s)^{1/n}$ , Eq. 2.62 has the solution

$$\bar{\theta}_f(t) = \left(\frac{q_0}{k_s}\right)^{\frac{1}{n}} \left(\frac{t^* - t_s}{t - t_s}\right)^{\frac{1}{n}}, \quad t^* = \frac{n}{n-1}t_s. \quad (2.63)$$

One may see that as time goes to infinity, the wetting front unboundedly moves downwards and the moisture content approaches the residual value  $\theta_r$  everywhere (Lessoff and Indelman 2004).

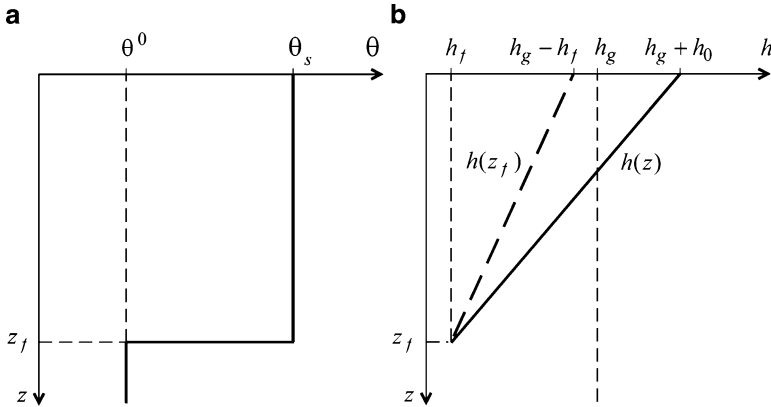
If the value of hydraulic conductivity  $k_s$  is small relative to the applied infiltration rate, i.e.,  $q_0/k_s > 1$ , a positive pore water pressure appears at the ground surface after certain time and the shallow soil becomes saturated (Ng and Menzies 2007). This is a result of the small  $k_s$ , which prevents some of the rainwater from infiltrating into the soil and results in the development of ponding on the ground surface (Ravi and Williams 1998; Ng and Menzies 2007).

### 2.1.3.5 Green–Ampt Model

When considering the piston-like (with an abrupt interface) motion of moisture in the aeration zone (Fig. 2.4a), we can try to take into account the joint effect of capillary-adsorption and gravitational components. Infiltration is hardly affected by gravity if the soil surface is ponded (even at the early stages of the process).

Suppose that the pressure in the liquid at the interface between the media ( $z = 0$ ) is determined by the atmospheric pressure and the pressure of the liquid layer covering the surface (Fig. 2.4b). This condition corresponds to the following expression written in hydraulic head functions:

$$h(0) = h_g + h_0, \quad (2.64)$$



**Fig. 2.4** The distribution of (a) volumetric moisture content and (b) the hydraulic head at piston-like motion of moisture in the aeration zone

where  $h_g$  is the head determined by the atmospheric pressure,  $h_0$  is the depth of the water layer. Now, the head on the moving infiltration-water front, whose position is determined by the current coordinate  $z_f(t)$ , will be

$$h(z_f) = h_g - h_f - z_f, \quad (2.65)$$

where  $h_f = -\psi_f$  is the capillary head at the moistening front – a characteristic, which, strictly speaking, should be related with the measurable physical parameters of the soil or rocks (Neuman 1976; Kao and Hunt 1996; Williams et al. 1998). An approximate expression for  $h_f$  can be also obtained from the phenomenological capillary model of porous medium (Kao and Hunt 1996).

Thus, the hydraulic gradient is:

$$\frac{dh}{dz} = -\frac{h(0) - h(z_f)}{z_f} = -\frac{h_0 + h_f + z_f}{z_f}. \quad (2.66)$$

The equation for the head function  $h(z)$  ( $0 \leq z < z_f$ ) (Fig. 2.4b):

$$h(z) = (h_g + h_0) + \frac{dh}{dz}z = (h_g + h_0) - \frac{h_0 + h_f + z_f}{z_f}z. \quad (2.67)$$

The migration velocity of the leading edge:

$$u^w = \frac{dz}{dt} = -\frac{k(\theta_s)}{\theta_s - \theta^0} \frac{dh}{dz} = \frac{k(\theta_s)}{\Delta\theta} \left( 1 + \frac{h_0 + h_f}{z_f} \right), \quad \Delta\theta = \theta_s - \theta^0, \quad k(\theta_s) \equiv k_s. \quad (2.68)$$

Integrating the ordinary differential Eq. 2.68 yields the following formula

$$t = \frac{\Delta\theta}{k_s} \left[ z_f - (h_0 + h_f) \ln \left( 1 + \frac{z_f}{h_0 + h_f} \right) \right], \quad (2.69)$$

known as the *Green–Ampt solution*. This solution, which was published in 1911 (Green and Ampt 1911), is among the fundamental solutions that have been in wide use for the analysis of the infiltration penetration of atmospheric water from the watered land surface and for the assessment of moisture transfer in the aeration zone.

However, this form of the solution is not convenient to use in the studies of infiltration characteristics, such as the rate of infiltration ( $i$ ) and the accumulation of moisture reserves in the aeration zone (specific accumulated infiltration volume  $I$ ). This requires one either to use special computational methods, such as iteration methods (Parlange et al. 2002), or to study the asymptotic solutions of the Eq. 2.69, which can be readily obtained in the form of finite analytical expressions, but are valid only for the description of the initial ( $t \rightarrow 0$ ) or, conversely, final and very long infiltration periods ( $t \rightarrow \infty$ ) (see below). Therefore, it is of interest to carry out an additional study of the Green–Ampt solution to obtain analytical relationships describing the behavior of functions  $i(t)$  and  $I(t)$  in the *explicit* form.

### 2.1.3.6 Modified and Explicit Forms of the Green–Ampt Model and Formulations for Infiltration Analysis

Rewrite Eqs. 2.68 and 2.69 as (Salvucci and Entekhabi 1994):

$$i = u^w \Delta\theta = k_s \left( 1 + \frac{(h_0 + h_f) \Delta\theta}{I} \right), \quad (2.70)$$

$$t = \frac{I}{k_s} - \frac{\Delta\theta(h_0 + h_f)}{k_s} \ln \left( 1 + \frac{I}{\Delta\theta(h_0 + h_f)} \right), \quad (2.71)$$

where  $i = \frac{dI}{dt}$ ,  $I = \Delta\theta z_f$ ,  $\Delta\theta = \theta_s - \theta^0$  (see Sect. 2.1.3.3). The substitution of function  $I$ , determined from the solution (2.70), into (2.71) yields the generalized expression

$$t = \frac{\chi}{\bar{i} - 1} - \chi \ln \left( \frac{\bar{i}}{\bar{i} - 1} \right), \quad \chi = \frac{\Delta\theta(h_0 + h_f)}{k_s} \equiv \frac{S^2}{2k_s^2}, \quad \bar{i} = \frac{i}{k_s} \quad (2.72)$$

relating the process time  $t$  with the dimensionless specific discharge,  $\bar{i}$ ;  $S^2 = 2k_s \Delta\theta(h_0 + h_f)$ .

The solution (2.72) can be differentiated with respect to  $t$ , yielding

$$\bar{i}(\bar{i} - 1)^2 + \chi \frac{d\bar{i}}{dt} = 0, \quad (2.73)$$

or, upon the introduction of the new dimensionless variable  $\tau = \frac{t}{t + \chi} = \frac{\bar{i}}{1 + \bar{i}}$

$\left( \bar{i} = \frac{t}{\chi} \right)$ , we obtain

$$\bar{i}(\bar{i} - 1)^2 + (1 - \tau)^2 \frac{d\bar{i}}{d\tau} = 0. \quad (2.74)$$

Thus, a new representation of the Green–Ampt equation (Salvucci and Entekhabi 1994) in the form of the ordinary differential Eq. 2.74, whose solution must satisfy the boundary condition

$$\bar{i} = \infty \text{ at } \tau = 0 \quad (2.75)$$

(since at  $t \rightarrow 0$ ,  $I = 0$  and hence  $\bar{i} \rightarrow \infty$ ).

Next it was proposed to approximate  $\bar{i}$  by the power series:

$$\bar{i} \cong A_0 \tau^{-1/2} + A_1 + A_2 \tau^{1/2} + A_3 \tau + \dots + A_n \tau^{(n-1)/2}. \quad (2.76)$$

From physical consideration it is clear that when  $\tau = 1$  ( $t \rightarrow \infty$ ), the values of the function  $\bar{i}$  must be equal to unity; hence the sum of all coefficients  $A_n$  in the series must also be equal to unity. The substitution of (2.76) into the differential Eq. 2.74 and some additional operations allow us to find the coefficients  $A_n$  of the Eq. 2.76 whatever the number of terms in the series  $n$ . Thus, in the case of a three-term relationship, passing to dimensional variables, we obtain:

$$i(\bar{i}) \cong k_s \left\{ \frac{\sqrt{2}}{2} \left( \frac{\bar{i}}{1+\bar{i}} \right)^{-1/2} + \frac{2}{3} - \frac{\sqrt{2}}{6} \left( \frac{\bar{i}}{1+\bar{i}} \right)^{1/2} + \frac{1-\sqrt{2}}{3} \left( \frac{\bar{i}}{1+\bar{i}} \right) \right\}. \quad (2.77)$$

The Eq. 2.77 can be integrated, allowing one to obtain the expression for the specific accumulated infiltration volume:

$$I(\bar{i}) \cong k_s \chi \left\{ \left( 1 - \frac{\sqrt{2}}{3} \right) \bar{i} + \left( \frac{\sqrt{2}}{3} \right) (\bar{i} + \bar{i}^2)^{1/2} + \left( \frac{\sqrt{2}-1}{3} \right) \ln(1+\bar{i}) + \left( \frac{\sqrt{2}}{3} \right) \ln \left( 1 + 2\bar{i} + 2(\bar{i} + \bar{i}^2)^{1/2} \right) \right\}. \quad (2.78)$$

Plots in Fig. 2.5 reflect the major features in the behavior of dimensionless functions  $\bar{i}(\bar{i})$  and  $\bar{I}(\bar{i})$ : an abrupt drop in the specific discharge  $i$  over time at the initial stages of infiltration and its asymptotic approximation to the limiting value  $i = k(\theta_s) \equiv k_s$  during long stages (approximately, when  $t > 5S^2/k_s^2$ ); conversely,  $\bar{I}$  is an increasing function, which becomes nearly linear at  $\bar{i} > 3 - 5$ .

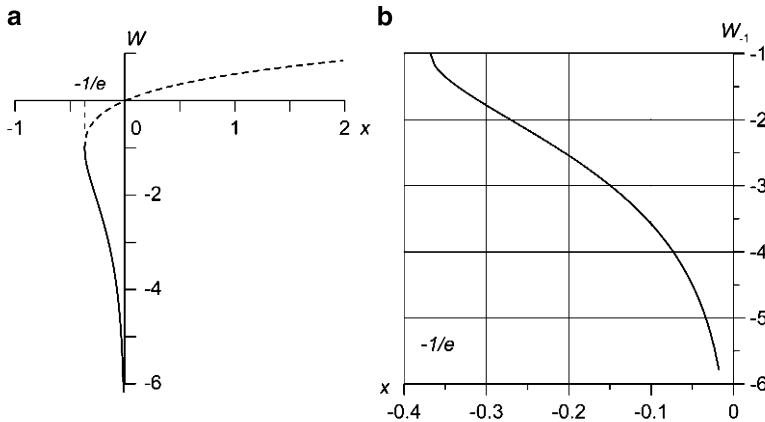
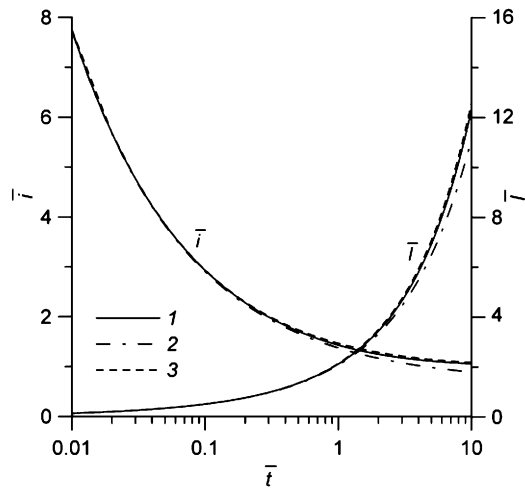
An attractive and productive idea was to transform Green–Ampt equations with the use of Lambert  $W$ -function (Barry et al. 1993, 2005; Parlange et al. 2002), defined as

$$We^W = x, \quad x \geq -e^{-1}. \quad (2.79)$$

The graphical representation of function  $W = W(x)$  (Fig. 2.6a) has two branches:  $W_{-1}(x) \leq -1$  and  $W_0(x) \geq -1$  (the domain  $W_0(x) \geq 0$  is not considered in this class of problems). In the solution of a real problem, it is advisable to transpose the independent variable ( $x$ ) into the same part of the equation, representing the latter as



**Fig. 2.5** Calculation of the basic infiltration functions:  $\bar{i}(\bar{t}) = i/k_s$  (dimensionless instantaneous infiltration capacity) and  $\bar{I}(\bar{t}) = I/\chi k_s$  (dimensionless accumulated infiltration) from analytical solutions: 1 – Eqs. 2.77, 2.78; 2 – Eqs. 2.87, 2.88; 3 – Eqs. 2.82, 2.84.  $\bar{t} = t/\chi$ ,  $\chi = S^2/2k_s^2$ ,  $A = 2k_s/3$



**Fig. 2.6** Lambert  $W$ -function. (a) Two real branches [ $W_0(x)$ (dashed line) and  $W_{-1}(x)$  (full line)]; (b) larger scale plot of  $W_{-1}(x)$

$y = xe^x$ , and consider the inverse function of  $y$ ; mathematically, this procedure can be formulated as

$$y = xe^x \leftrightarrow x = W(y). \quad (2.80)$$

Let us represent the Green–Ampt formula (2.69) in the form

$$\bar{I} = \bar{t} + \ln(1 + \bar{I}), \quad (2.81)$$

which can be also written in the equivalent form:

$$-\exp[-(1 + \bar{t})] = -(1 + \bar{I}) \exp[-(1 + \bar{I})]; \quad (2.81a)$$

here  $\bar{I} = I/(S^2/2k_s)$ ,  $\bar{t} = t/(S^2/2k_s^2)$ ,  $I = \Delta\theta z_f$ ,  $S = \sqrt{2k_s\Delta\theta(h_0 + h_f)}$ . In accordance with Eq. 2.80,  $x = -(1 + \bar{I})$ , and  $y = -\exp[-(1 + \bar{I})]$ . From here it follows

$$\bar{I} = -1 - W_{-1}[-\exp(-1 - \bar{t})], \quad (2.82)$$

i.e., we come to the explicit solution of the Green and Ampt problem for dimensionless cumulative infiltration.

For calculations by (2.82), the plot of the function  $W_{-1}(x)$  in Fig. 2.6b can be used. Another approach is to use the approximation for the function  $W_{-1}(x)$  obtained by a special method (Barry et al. 2005):

$$W_{-1}(x) \approx -1 - \frac{\sigma^2}{2} - \frac{\sigma}{1 + \sigma/6}, \quad (2.83)$$

$\sigma = \sqrt{-2 - 2\ln(-x)}$ . This approximation is valid in the range  $-1/e \leq x \leq 0$  and has a maximum relative error of 0.47%.

The flux  $\bar{i}$  can be calculated directly from (2.82) by the differentiation of (2.83) with respect to  $\bar{t}$ . The flux  $\bar{i}$  can also be calculated using (Barry et al. 1993, 2005):

$$\bar{i} = \frac{W_{-1}[-\exp(-1 - \bar{t})]}{1 + W_{-1}[-\exp(-1 - \bar{t})]}. \quad (2.84)$$

As can be seen from the plot in Fig. 2.5, the solutions of (2.84) and (2.82) almost coincide with the solutions (2.77) and (2.78).

### 2.1.3.7 Some Generalized Approximations

Analysis of the solutions given above allows us to obtain expressions for studying the limiting regimes of the infiltration penetration of water into initially incompletely saturated shallow sediments.

Thus, at small  $t$ , when the moistening front lies near the surface [ $z_f/(h_0 + h_f) \ll 1$ ], the solutions (2.77), (2.78), and (2.82), (2.84) have the asymptotic representations for infiltration capacity and accumulated infiltration:

$$i = \frac{\sqrt{2k_s(h_0 + h_f)\Delta\theta}}{2} \frac{1}{\sqrt{t}}, \quad I = \sqrt{2k_s(h_0 + h_f)\Delta\theta} \sqrt{t}. \quad (2.85)$$

At large time ( $t$ ), functions  $i$  and  $I$  asymptotically tend to the limits

$$i = k_s, \quad I = k_s t. \quad (2.86)$$

The analysis given above clarifies the structure of the generalized expressions (Philip 1957b, 1987)

$$I(t) = S\sqrt{t} + At, \quad (2.87)$$

$$i(t) = \frac{dI}{dt} = \frac{1}{2} \frac{S}{\sqrt{t}} + A, \quad (2.88)$$

which are often used for the approximated description of infiltration process. Now the expression

$$S = \sqrt{S_0^2 + S^2(\theta_s, \theta^0)}, \quad (2.89)$$

$$S_0^2 = 2k_s h_0 \Delta\theta, \quad S_0^2(\theta_s, \theta^0) = 2k_s h_f \Delta\theta, \quad \Delta\theta = \theta_s - \theta^0, \quad (2.89a)$$

can be associated with the notion of sorptivity (see Sect. 2.1.3.3) for the case of infiltration from the surface covered by a water layer;  $A = (1/3 \div 1)k_s$  is the gravity factor. Note, however, that the solutions (2.87) and (2.88) do not directly follow from the model considered here.

More strict expressions for  $S$ , which take into account the features of the retention function can be found in the works (Parlange et al. 1985, 1992; Broadbridge 1990).

Analysis of (2.88) shows that in course of time (tentatively, at  $t > t_g = (3S/2k_s)^2$ ) the effect of the first term, which accounts for the role of capillary-adsorption forces, becomes negligible and the rate of imbibition is determined by hydraulic conductivity of the porous material, i.e., by the second term of the Eq. 2.88, which accounts for the gravity movement of moisture. Most authors recommend using the values of  $A$  between  $k_s/3$  and  $2k_s/3$  for not very long periods.

This range also most often contains the estimates of parameter  $A$  by the formula (Haverkamp et al. 1994; Lassabatère et al. 2006):

$$A = k_s \left\{ \frac{(2 - \beta)}{3} \left[ 1 - \left( \frac{\theta^0}{\theta_s} \right)^n \right] + \left( \frac{\theta^0}{\theta_s} \right)^n \right\}, \quad (2.90)$$

where  $n$  is the exponent in the Averjanov formula (2.10) for the moisture transfer coefficient; the empirical coefficient  $\beta$  for most lithologic members at their moisture content  $\theta^0 < 0.25\theta_s$  is 0.6.

The plots in Fig. 2.5 show that the approximate solutions (2.87) and (2.88) can be used in practical calculations.

The obtained relationships allow us, by analogy with the Green–Ampt problem, to write the generalized expressions determining the position of the moisture transfer front ( $z_f = I/\Delta\theta$ ):

$$z_f(t) = \xi_f \sqrt{t} + \frac{A}{\Delta\theta} t \text{ at } t \leq t_g, \quad (2.91)$$

$$z_f(t) = \frac{k_s}{\Delta\theta} (t - t_g) + z_f(t_g) \text{ at } t > t_g, \quad (2.92)$$

where parameter  $\xi_f = S/\Delta\theta$  accounts for the contribution of capillary forces. It can be seen that the infiltration time is divided into two periods: at times less than  $t_g$ , the moisture front propagation velocity is governed by both the gravity and capillary mechanisms, while the gravity moisture transport dominates for time greater than  $t_g$ .

Wang et al. (2003) developed an algebraic model for the description of soil water infiltration based on Parlange (1971b) approximation of Richards equation and on soil retention curve (2.6) and hydraulic conductivity Eq. 2.11 given by Brooks and Corey (1964). When the initial soil water content is small and can be approximated

as  $\theta_r = \theta^0$  and the boundary condition is  $\theta(z = 0, t) = \theta_s$ , integrating (2.22) for soil water provides the following four equations (Wang et al. 2003, 2009)

$$\theta - \theta_r = \begin{cases} \left(1 - \frac{z}{z'_f}\right)^\alpha \Delta\theta & z \leq z'_f, \\ 0 & z > z'_f, \end{cases} \quad (2.93)$$

$$i = k_s \left(1 + \frac{1}{\beta z'_f}\right), I = \frac{\Delta\theta}{1 + \alpha} z'_f, t = \frac{\Delta\theta}{(1 + \alpha)k_s} \left(z'_f - \frac{\ln(\beta z'_f + 1)}{\beta}\right), \quad (2.94)$$

where  $z'_f$  is the wetting front distance;  $\alpha = \lambda/m$  is the comprehensive shape coefficient of the soil water content distribution, which determines the amount of water in a soil profile for a fixed wetting front distance;  $\beta = m/a$  ( $a$  is a constant) is a soil suction allocation coefficient;  $i$  is the infiltration rate;  $I$  is the cumulative infiltration;  $\Delta\theta = \theta_s - \theta_r$ .

It is easy to show that the Wang et al. (2003) algebraic model transforms into the Green–Ampt model if one assumes the following relationships to hold between model characteristics

$$z_f = \frac{z'_f}{1 + \alpha}, h_f = \frac{1}{\beta(1 + \alpha)}. \quad (2.95)$$

Formulas (2.77), (2.78), (2.87), (2.88), (2.94) and similar relationships (Ravi and Williams 1998; Williams et al. 1998) are intended for adapting the theoretical models to the regularities observed in atmospheric water infiltration from the land surface (Chahinian et al. 2005). Since infiltration is a component of the general water balance equation for hydrological basins, the relationships presented above give formulations for runoff-infiltration analysis (Sect. 10.7).

### 2.1.3.8 Traveling Wave Solution of the Richards Equation

We consider vertical infiltration into initially wet porous medium domain,  $\theta(z, 0) = \theta^0$ . Pore water moves from  $z = -\infty$  to  $z = +\infty$ . Suppose that the nonlinearity of the process ensures the balance between the gravity and diffusion components of moisture flux, resulting in that a stationary (in moving coordinates) moisture wave ( $\theta^0 < \theta < \theta_0$ ) forms during long-time infiltration (theoretically, at  $t \rightarrow \infty$ ), which moves with nearly constant velocity  $u^w$ , where  $\theta = \theta_0$  at  $z = -\infty$  and  $\theta = \theta^0$  at  $z = +\infty$ .

Multiplying the original Eq. 2.22 by  $\partial z / \partial \theta$  with a change  $z = -z$  (in this case, the  $z$ -axis will be directed vertically downward) we obtain (Youngs 1957)

$$-\frac{\partial z}{\partial t} = \frac{\partial}{\partial \theta} \left( D(\theta) \frac{\partial \theta}{\partial z} \right) - \frac{\partial k(\theta)}{\partial \theta}. \quad (2.96)$$

In this representation of moisture transfer equation, the function  $z = z(\theta, t)$  is the space coordinate of the point where the moisture content in moment  $t$  is  $\theta$ .

The assumption that the moisture front moves in the vertical direction with a constant velocity allows us to write the identity

$$\frac{\partial z}{\partial t} = \frac{k(\theta_0) - k(\theta^0)}{\theta_0 - \theta^0} = u^w = \text{const}, \quad (2.97)$$

where  $k(\theta_0)$  and  $k(\theta^0)$  are the values of unsaturated hydraulic conductivity at  $\theta = \theta_0$  and  $\theta = \theta^0$ . Now the Eq. 2.96 becomes

$$-\frac{d}{d\theta} \left( D(\theta) \frac{d\theta}{dz} \right) + \frac{dk(\theta)}{d\theta} = \frac{k(\theta_0) - k(\theta^0)}{\theta_0 - \theta^0}. \quad (2.98)$$

Integrating (2.98) from  $\theta$  to  $\theta_0$  yields

$$D(\theta) \frac{d\theta}{dz} + (k(\theta_0) - k(\theta)) = (k(\theta_0) - k(\theta^0)) \frac{\theta_0 - \theta}{\theta_0 - \theta^0}. \quad (2.99)$$

Eq. 2.99 can be solved by separation of variables, yielding

$$(\theta_0 - \theta^0) \int_{\theta^0}^{\theta} \frac{D(\theta)}{k(\theta)(\theta_0 - \theta^0) - k(\theta^0)(\theta_0 - \theta) - k(\theta_0)(\theta - \theta^0)} d\theta = z(\theta) - z_f, \quad (2.100)$$

where  $z_f = z(\theta^0) = u^w t$  is the position of the leading edge of the moisture wave. The solution (2.100) of the moisture transfer equation can also be obtained by direct transformation of the initial Eq. 2.23 with the substitution (2.46) (Philip 1957c; Ross and Parlange 1994; Witelski 2005).

Formally speaking, we have come to a solution of the traveling-wave type (2.46). This solution has a self-similar front moving at a velocity of  $u^w$  without changing its shape. Theoretically, the wave front length is infinite, stretching from  $z = -\infty$  where  $\theta = \theta_0$  to  $z = +\infty$  where  $\theta = \theta^0$  at any given time. To make this solution of practical value, Zlotnik et al. (2007) suggested to consider  $(\theta_0)^* = \theta_0 - \Delta_0$  and  $(\theta^0)^* = \theta^0 - \Delta^0$  ( $\Delta_0$  and  $\Delta^0$  are small positive values) to avoid integrand singularities at both integration limits  $\theta = \theta_0$  and  $\theta = \theta^0$ . Zlotnik et al. (2007) showed also that the solution (2.100) allows generating different exact solutions for infiltration problems for arbitrary finite time and can be a good benchmark for the verification of the accuracy of solution methods for transient flow conditions.

The solution (2.100) can be represented in a closed form given the analytical functions  $D(\theta)$  and  $k(\theta)$ . Thus, for the Averjanov–Gardner moisture transfer model (2.10, 2.15) in the case of infiltration into relatively dry shallow sediments ( $\theta^0 = \theta_r$ ) at  $k(\theta_r) = 0$  we obtain:

$$\frac{\beta}{\alpha} \int_0^{\tilde{\theta}} \frac{\tilde{\theta}^{n-1}}{\tilde{\theta}^n - \tilde{\theta}} d\tilde{\theta} = z(\tilde{\theta}) - u^w t, \quad (2.101)$$

where

$$\tilde{\theta} = \frac{\theta - \theta_r}{\theta_0 - \theta_r}, \quad \beta = \frac{\theta_s - \theta_r}{\theta_0 - \theta_r}, \quad u^w = \frac{k(\theta_0)}{\theta_0 - \theta_r} = \frac{k_s}{\beta^{n-1} \Delta\theta}, \quad \Delta\theta = \theta_s - \theta_r. \quad (2.101a)$$

Integrating (2.101) we obtain the solution of the problem in the explicit form:

$$\tilde{\theta}(\eta) = \left(1 - e^{(n-1)\alpha\eta/\beta}\right)^{\frac{1}{n-1}}. \quad (2.102)$$

If the soil is not initially dry and has a nonuniform initial distribution of water content then the motion of the wetting front will change due to the interaction of the infiltration flow with the pre-existing soil condition (Witelski 2005). Approximate solutions for this situation were derived by Witelski (2005) who suggested a simple procedure for scaling the wetting front advancing into a soil layer with heterogeneous initial water content profile.

## 2.2 On Models Coupling Water Infiltration and Solute Transport

Unsaturated flow forms as the result of infiltration of precipitation (during rains or snow melting); infiltration from a surface water body underlain by a low-permeability bed (or clayey bottom sediments); or a short-time waterlogging of land surface as the result of accidental spills of liquids (solutions) with different composition. In all cases, the infiltration water can deliver into the subsurface hydrosphere some pollutants whose fate can be predicted by solving migration problems of a specific class.

The mathematical formalization of the transport of chemical components dissolved in infiltration water requires the joint solution of Richards equations describing moisture transfer in the aeration zone and the continuity equation for the mass flux of the matter. Thus, the unsteady process in three-dimensional space following Eq. 2.24 can be described by the mass transfer equation

$$\frac{\partial \theta C}{\partial t} - \nabla \cdot (\mathbf{D} \cdot \nabla C - \mathbf{q} C) = 0. \quad (2.103)$$

However, most hydrogeological problems arising from area-distributed pollution of infiltration water are solved in one-dimensional vertical formulation, which will be the focus of the analysis below (Sects. 2.2.1 and 2.2.2). As was the case with the analysis of mass transfer in aquifers, we will successively consider two calculation schemes (models): a purely advective scheme, when the dispersion of solutes in the porous medium is neglected, and an advection–dispersion one, taking into account the effect of longitudinal dispersion because of fluctuations in flow velocities in pores.

The actual heterogeneity of the near-surface sediments largely limits the use of simplified one-dimensional models, requiring the description of macrodispersion effects, which goes beyond the scope of this work.

### 2.2.1 Advection: A Characteristic Solution

As the first step, we consider a system of moisture and mass transfer equations without regard for dispersion effects, which has the following form:

$$\frac{\partial \theta}{\partial t} + \frac{\partial q}{\partial z} + W_f = 0, \quad (2.104)$$

$$\frac{\partial}{\partial t}(\theta C) + \frac{\partial}{\partial z}(qC) + W_s = 0; \quad (2.105)$$

here  $W_f$  and  $W_s$  are functions accounting for the effect of distributed (three-dimensional) sources–sinks of moisture and solute.

Differentiating the first and second terms of Eq. 2.105 and regrouping the partial derivatives of functions  $\theta$ ,  $q$  and  $C$ , we obtain

$$\theta \frac{\partial C}{\partial t} + q \frac{\partial C}{\partial z} + C \left( \frac{\partial \theta}{\partial t} + \frac{\partial q}{\partial z} \right) + W_s = 0. \quad (2.106)$$

The comparison of (2.106) and (2.104) shows that the expression in the parentheses in (2.106) corresponds to  $-W_f$ . Thus, the system of equations can be transformed into the single equation:

$$\theta \frac{\partial C}{\partial t} + q \frac{\partial C}{\partial z} = CW_f - W_s; \quad (2.107)$$

here, in the general case,  $\theta = \theta(z, t)$  and  $q = q(z, t)$ , the  $z$ -axis is directed vertically downward.

Let us restrict our consideration to the migration process in a homogeneous medium, assuming  $W_f = 0$  and  $W_s = 0$ , to obtain

$$\theta \frac{\partial C}{\partial t} + q \frac{\partial C}{\partial z} = 0. \quad (2.108)$$

The partial differential Eq. 2.108 corresponds to the system of first-order ordinary differential equations

$$\frac{dt}{\theta} = \frac{dz}{q} = \frac{dC}{0}, \quad (2.109)$$

which are referred to as characteristic. From here, in particular, we obtain the equality

$$\theta(z, t)dz - q(z, t)dt = 0. \quad (2.110)$$

The left part of (2.110) is the total differential of some function  $\eta(z, t)$ , i.e.

$$q(z, t)dt - \theta(z, t)dz = d\eta(z, t). \quad (2.111)$$

Since

$$d\eta = \frac{\partial \eta}{\partial t} dt + \frac{\partial \eta}{\partial z} dz, \quad (2.112)$$

this is equivalent to

$$\theta = \frac{\partial \eta}{\partial z}, \quad q = -\frac{\partial \eta}{\partial t}. \quad (2.113)$$

From here it follows that the function  $\eta(z, t)$ , relating the parameters of the initial Eq. 2.108, reflects the dynamics of the frontal moisture transfer in the aeration zone. To determine it, we write the equation in total differentials as:

$$d\eta = 0. \quad (2.114)$$

The general solution of the Eq. 2.114 has the form

$$\eta(z, t) = c, \quad (2.115)$$

i.e., the function  $\eta$  is the integral of (2.110),  $c$  is a constant.

The first equations (2.113) is satisfied by the function

$$\eta(z, t) = \int_{z_0}^z \theta(z, t)dz + \varphi(t), \quad (2.116)$$

where  $\varphi(t)$  is an arbitrary function of  $t$ ;  $z_0$  is an arbitrary point on the coordinate axis; in this problem it corresponds to the position of the front in the initial moment ( $t_0$ ). We choose the function  $\varphi(t)$  to satisfy the second equality (2.113), i.e.,

$$\frac{\partial \eta}{\partial t} = \frac{\partial}{\partial t} \int_{z_0}^z \theta(z, t)dz + \varphi'(t) = -q(z, t), \quad (2.117)$$

or

$$\int_{z_0}^z \frac{\partial \theta(z, t)}{\partial t} dz + \varphi'(t) = -q(z, t). \quad (2.118)$$

By using the original equality (2.108), we rewrite Eq 2.118. as

$$-\int_{z_0}^z \frac{\partial q(z, t)}{\partial z} dz + \varphi'(t) = -q(z, t). \quad (2.119)$$



Integration yields

$$-q(z, t) + q(z_0, t) + \varphi'(t) = -q(z, t), \quad (2.120)$$

whence

$$\varphi'(t) = -q(z_0, t), \quad (2.121)$$

and hence

$$\varphi(t) = -\int_{t_0}^t q(z_0, t) dt + c', \quad (2.122)$$

where  $c'$  is an arbitrary constant, which, with appropriately chosen initial conditions  $z_0, t_0$ , can be taken to be zero.

Thus, the integral function  $\eta(z, t)$  (2.113) can be written as:

$$\eta(z, t) = \int_{z_0}^z \theta(z, t) dz - \int_{t_0}^t q(z_0, t) dt. \quad (2.123)$$

With  $q(z, t)$  taken as the initial function for calculating the general integral (the second formula in 2.113), we come to another integral formula equivalent to (2.123):

$$\eta(z, t) = \int_{z_0}^z \theta(z, t_0) dz - \int_{t_0}^t q(z, t) dt. \quad (2.124)$$

The equalities (2.123)–(2.124) with the additional condition

$$dC(\eta) = 0, \text{ or } C = \text{const along the characteristic } \eta(\theta, t), \quad (2.125)$$

following from (2.109), can be regarded as the generalized characteristic solution of the problem (2.108). This solution implies that the solute concentration does not change along the characteristic  $\eta(z, t)$ .

Analysis of (2.123) allows us to come to useful physical conclusions. Physically,  $\eta(z, t)$  could be thought of as a moving spatial coordinate in relation to a solute front that was located at the point  $z = z_0$  at time  $t = t_0$  (Wilson and Gelhar 1981; Nachabe et al. 1995). The location of the solute front  $z = z_f(t)$  can be determined by setting  $\eta = 0$ . The condition  $\eta > 0$  corresponds to points located ahead of the front, while  $\eta < 0$  corresponds to points behind the front. The first term on the right-hand side of the characteristic (2.123) represents the cumulative change in water content,  $U(z, t)$ , between  $z$  and  $z_0$ ,

$$U(z, t) = \int_{z_0}^z \theta(z, t) dz. \quad (2.126)$$

The second term is the cumulative volumetric water infiltrated at the section  $z = z_0$  between time  $t$  and  $t_0$ :

$$V(t) = \int_{t_0}^t q(z_0, t) dt. \quad (2.127)$$

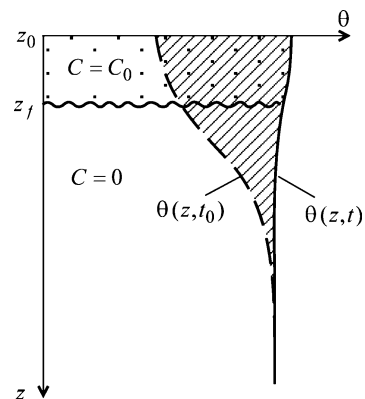
Therefore, the characteristic solution can be written as (Wilson and Gelhar 1981):

$$\eta(z, t) = U(z, t) - V(t), \quad (2.128)$$

and the position of the concentration front is determined by the equality

$$U(z, t) = V(t). \quad (2.129)$$

Let us give some graphic illustration to the obtained result (Fig. 2.7). Suppose that a volume  $V(t)$  of water containing some indicator solute in concentration  $C_0$  entered the aeration zone within time interval  $t - t_0$ , resulting in a change in the moisture content profile as shown in Fig. 2.7 (the shaded area). The position of the concentration front  $z_f$  (the wavy line in Fig. 2.7) is determined by (2.128) at  $\eta = 0$ . Since  $U(z_f) = V$ , the area of the domain  $z_0 < z < z_f$  (shown by dots in Fig. 2.7), where  $C = C_0$ , must be equal to the shaded area. From here it follows that the vertical displacement velocity of the solute front in the incomplete-saturation zone is always less than the velocity of the moisture front. Physically, this phenomenon can be explained by the fact that during the motion of the solute, the transport process involves all water contained in the pores,  $\theta(z, t)$ , while the propagation dynamics of the moisture front is controlled only by an increase in moisture content, which is equal to the difference between the current value of the volumetric moisture content of the subsurface material,  $\theta(z, t)$ , and the initial moisture content,  $\theta(z, t_0)$ . The velocity of the concentration front is determined by the dynamics of moisture distribution, however the concentration behind the transport front,  $C_0$ , in the absence of dispersion, remains unchanged.



**Fig. 2.7** Relative rates of movement of moisture and solute during transient infiltration (Wilson and Gelhar 1981). Explanations see in the text (Reproduced according to terms of use of licensed materials of American Geophysical Union, AGU)

From this viewpoint it is clear that, when the initial moisture distribution is not uniform, the thickness of the concentration parcel that forms during polluted water input into the aeration zone within a limited time interval will vary in the process of water infiltration from the surface. Thus, when moisture content increases with depth, for the solute balance to hold, the thickness of the layer of originally polluted water must decrease. The picture will be inverse if the moisture content decreases with depth. Nachabe et al. (1995) used a relationship (2.123) (assuming  $t_0 = 0$  and  $z_0 = 0$ ) to determine the coordinate  $\eta$  of the solute front exactly using the parametric flow solution for  $\theta(z, t)$ -function obtained earlier by Broadbridge and White (1988).

To determine the position of the concentration front in the case of constant infiltration rate  $q$  through the aeration zone with steady-state moisture profile  $\theta(z)$ , one can use the first solution of the initial characteristic Eq. 2.108 in the form (2.123):

$$qt = \int_0^z \theta(z) dz, \quad (2.130)$$

with the assumption that  $t_0 = 0$ ,  $z_0 = 0$ . The propagation velocity of the concentration front is (Vanderborght et al. 2000a):

$$u(z) = \frac{qz}{\int_0^z \theta(z) dz}. \quad (2.130a)$$

From general physical considerations it is clear that the moistening front running appreciably ahead of the concentration front provides conditions for solute migration (with the infiltration rate  $q = \text{const}$ ) under quasisteady conditions in terms of the moisture regime. If the moisture function behind the moistening front is steady, and the values of  $\theta$  vary over depth only slightly, the formula (2.130) yields

$$u = \frac{q}{\bar{\theta}}, \quad (2.131)$$

where, in the problems on the regional scale,  $\bar{\theta}$  is identified with the value of mean rock moisture content that has formed under infiltration recharge of groundwater.

In some cases, the assumption that the moisture motion is steady allows us to find the distribution  $\theta(z)$  in the explicit form thus obtaining an expression describing the displacement of the concentration in a closed analytical form.

## 2.2.2 Dispersion During Adsorption of Water by Soil

### 2.2.2.1 A General Solution

As well as in Sect. 2.1.3.3, we consider here a limiting case of moisture imbibition by soil (2.132), when the gravity effect may be neglected

$$\frac{\partial \bar{\theta}}{\partial t} = \frac{\partial}{\partial z} \bar{q}(\bar{\theta}), \quad \bar{q}(\bar{\theta}) = -D(\bar{\theta}) \frac{\partial \bar{\theta}}{\partial z}, \quad (2.132)$$

which is true in the first moments of infiltration; the normalized specific flux,  $\bar{q}$ , is determined by  $\bar{q}(\bar{\theta}) = q/\Delta\theta$ ,  $\bar{\theta} = (\theta - \theta^0)/\Delta\theta$ ,  $\Delta\theta = \theta_s - \theta^0$ . In this case, Eq. 2.103 becomes

$$\left(\bar{\theta} + \frac{\theta^0}{\Delta\theta}\right) \frac{\partial \bar{C}}{\partial t} - D(\theta) \frac{\partial \bar{\theta}}{\partial z} \frac{\partial \bar{C}}{\partial z} - \frac{\partial}{\partial z} \left[ D_s \left( \bar{\theta} + \frac{\theta^0}{\Delta\theta} \right) \frac{\partial \bar{C}}{\partial z} \right] = 0. \quad (2.133)$$

One may assume that both  $\bar{\theta}$  and  $\bar{C}$  profiles preserve similarity in terms of the Boltzmann variable  $\lambda = z/\sqrt{t}$ . With  $\lambda = z/\sqrt{t}$  (Smiles et al. 1978) (2.132) and (2.133) become respectively (Smiles et al. 1978; Gandola et al. 2001)

$$\frac{\lambda}{2} \frac{d\bar{\theta}}{d\lambda} + \frac{d}{d\lambda} \left( D(\bar{\theta}) \frac{d\bar{\theta}}{d\lambda} \right) = 0, \quad (2.134)$$

$$\frac{d}{d\lambda} \left( D'_s(\bar{\theta}) \frac{d\bar{C}}{d\lambda} \right) + \frac{G(\lambda)}{2} \frac{d\bar{C}}{d\lambda} = 0; \quad (2.135)$$

here  $\bar{C} = (C - C^0)/(C_s - C^0)$ ;  $D_s$  is the coefficient of dispersion,  $D'_s = \theta D_s$ ,

$$G(\lambda) = \left( \bar{\theta} + \frac{\theta^0}{\Delta\theta} \right) \lambda + 2D(\bar{\theta}) \frac{d\bar{\theta}}{d\lambda} = \left( \bar{\theta} + \frac{\theta^0}{\Delta\theta} \right) \lambda - \int_0^{\bar{\theta}} \lambda du. \quad (2.136)$$

A solution to (2.134) and (2.135) is sought subject to the boundary condition:  $\bar{\theta} = \bar{C} = 1$  at  $\lambda = 0$ , and  $\bar{\theta} = \bar{C} = 0$  as  $\lambda \rightarrow \infty$ .

Equation (2.135) may be written as the linear equation

$$\frac{d}{d\lambda} \left( D'_s(\lambda) \frac{d\bar{C}}{d\lambda} \right) + \frac{G(\lambda)}{2} \frac{d\bar{C}}{d\lambda} = 0. \quad (2.137)$$

The substitution  $V = D'_s(\lambda) d\bar{C}/d\lambda$  results in the equation

$$\frac{dV}{d\lambda} + \frac{G(\lambda)}{2D'_s(\lambda)} V = 0. \quad (2.138)$$

Integration of (2.138) yields

$$V = V_0 \exp \left[ -\frac{1}{2} \int_0^{\lambda} G(\lambda) \frac{d\lambda}{D'_s(\lambda)} \right]. \quad (2.139)$$

It then follows (Smiles et al. 1978)

$$\bar{C} = 1 - \frac{M(\lambda)}{M(\infty)}, \quad (2.140)$$

in which

$$M(\lambda) = \int_0^\lambda \left\{ \frac{1}{D'_s(\lambda)} \exp \left[ -\frac{1}{2} \int_0^\lambda \frac{d\lambda}{D'_s(\lambda)} \right] \right\} d\lambda. \quad (2.140a)$$

If  $D'_s(\theta)$  is known, the calculation of  $D'_s(\lambda)$  is straightforward since  $\theta(\lambda)$  is unique.

### 2.2.2.2 Derivation of a Particular Solution

Suppose that the moisture profile in the porous medium can be described by the equation:

$$\bar{\theta} = (1 - \bar{\lambda})^{1/N}, \quad (2.141)$$

where  $\bar{\theta} = (\theta - \theta_r)/\Delta\theta$ ,  $\Delta\theta = \theta_s - \theta_r$ , meaning that the moisture transfer during infiltration into the shallow sediment is considered for the initial moisture content of  $\theta^0 = \theta_r$ ;  $\bar{\lambda} = \lambda/\lambda_f$ ;  $\lambda = z/\sqrt{t}$ ;  $\lambda_f$  is a variable characterizing the position of the moistening front  $z_f$  at fixed  $t$  ( $\lambda_f = z_f/\sqrt{t}$ );  $N > 1$ . A moisture profile described by a power function similar to (2.141) was considered in works [Gandola et al. \(2001\)](#) and [Wang et al. \(2003, 2009\)](#). In this case, the expression for the diffusivity coefficient (2.38) becomes

$$D(\bar{\theta}) = \frac{\lambda_f^2}{2} N \left( \bar{\theta}^N - \frac{1}{N+1} \bar{\theta}^{2N} \right) \approx \frac{\lambda_f^2}{2} N \bar{\theta}^N. \quad (2.142)$$

Comparing this expression with the first formula (2.15), following from the model of Averjanov and Gardner (2.5), (2.10), we obtain

$$\lambda_f^2 = \frac{2k_s}{\Delta\theta\alpha(n-1)}, N = n-1, \Delta\theta = \theta_s - \theta_r. \quad (2.143)$$

With  $h_f = 1/\alpha(n-1)$ ,  $h_0 = 0$  in the expression (2.89) for sorptivity, we obtain

$$\lambda_f = \frac{S(\theta_s, \theta_r)}{\Delta\theta}, \quad (2.143a)$$

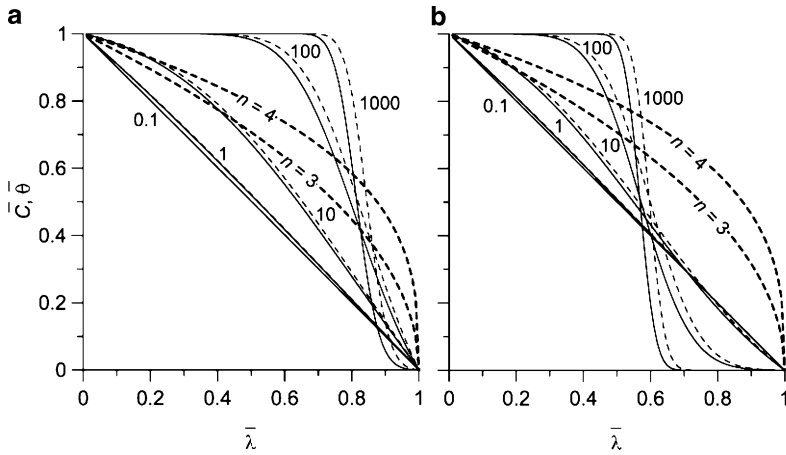
i.e., we have the exact equality  $z_f = S(\theta_s, \theta_r)\sqrt{t}/\Delta\theta$ , which is in agreement with (2.91) at  $t/t_g \ll 1$ .

In the simplest case of  $D'_s = D_s^0 = \text{const}$ , the initial Eq. 2.135 becomes

$$\frac{d^2\bar{C}}{d\bar{\lambda}^2} - \frac{\text{Pe}}{2} g(\bar{\lambda}) \frac{d\bar{C}}{d\bar{\lambda}} = 0, \quad (2.144)$$

where  $g(\bar{\lambda}) = (1 - \bar{\lambda})^m - \bar{\theta}^0$ ,  $\bar{\theta}^0 = \theta_r/\Delta\theta$ ,  $m = n/(n-1)$ ,  $\text{Pe} = \lambda_f^2/D_s^0$  (an analogue of the Peclet number). The solution of this equation in the segment  $0 \leq \bar{\lambda} \leq 1$  has the form:

$$\bar{C} = 1 - \frac{M(\bar{\lambda})}{M(1)}, \quad (2.145)$$



**Fig. 2.8** Plots of functions  $\bar{C}(\bar{\lambda})$  (full lines for  $n = 3$  and thin dashed lines for  $n = 4$ ) and  $\bar{\theta}(\bar{\lambda})$  (heavy dashed lines). (a)  $\bar{\theta}^0 = 0.1$ , (b)  $\bar{\theta}^0 = 0.5$ . Numbers at curves are the values of parameter  $Pe$

where

$$M(\bar{\lambda}) = \int_0^{\bar{\lambda}} \exp \left[ -\frac{Pe}{2} \left( \frac{((1 - \bar{\lambda})^{m+1} - 1)}{(m+1)} + \frac{\bar{\theta}^0 \bar{\lambda}^2}{2} \right) \right] d\bar{\lambda}. \quad (2.146)$$

Figure 2.8 gives plots of the solution (2.145), drawn for characteristic values of parameters. Analysis of these plots leads to several useful conclusions regarding the physics of the process: (1) the front of solute lags behind of the moisture front, and this lag is the longer, the greater the value of parameter  $\bar{\theta}^0$ , characterizing the system capacity; (2) an increase in parameter  $Pe$  (a decrease in the dispersion coefficient) causes the compression of concentration fronts (a decrease in the dispersion zone); (3) the shape of the concentration profile features low sensitivity to  $n$ .

### 2.2.3 Advection–Dispersion Transport

Microdispersion of the solute in the unsaturated zone can be described by Eq. 2.103, which in the one-dimensional case becomes

$$\frac{\partial}{\partial t}(\theta C) + \frac{\partial}{\partial z}(qC) - \frac{\partial}{\partial z} \left( D' \frac{\partial C}{\partial z} \right) = 0, \quad (2.147)$$

where  $D' = D\theta$  is the longitudinal dispersion coefficient. As was the case with the advection mass transfer, the Eq. 2.147 can be combined with the moisture transport Eq. 2.104. With  $W_f = 0$  and  $W_s = 0$ , we obtain

$$\theta(z, t) \frac{\partial C}{\partial t} + q(z, t) \frac{\partial C}{\partial z} - \frac{\partial}{\partial z} \left( D'(z, t) \frac{\partial C}{\partial z} \right) = 0. \quad (2.148)$$

The majority of approximate solutions of the Eq. 2.148 are based on the assumption that the coefficients  $q$  and  $D$  near the front vary only slightly over the depth but are functions of time. In this case, the Eq. 2.148 can be written as

$$\frac{\partial C}{\partial t} + u(z, t) \frac{\partial C}{\partial z} - D(z, t) \frac{\partial^2 C}{\partial z^2} = 0, \quad (2.149)$$

where  $u(z, t) = q(z, t)/\theta(z, t)$ .

We assume that the Eq. 2.149 can describe solute dispersion near the front, whose coordinate  $\xi(t)$  in any time  $t$  is determined by the characteristic Eq. 2.124, rewritten as

$$\int_0^{\xi(t)} \theta(z, t=0) dz = \int_0^t q[\xi(t), \tau] d\tau, \quad (2.150)$$

i.e.,  $\xi(t)$  is a variable coordinate characterizing the penetration depth of the piston-like concentration front (Vanderborgh et al. 2000b). From Eq. 2.150, we have

$$\frac{d\xi(t)}{dt} = \frac{q[\xi(t)]}{\theta[\xi(t)]}. \quad (2.151)$$

Let us express  $t$  in (2.149) in terms of  $\xi = \xi(t)$ , assuming  $u(z, t) = u(\xi)$  and  $D(z, t) = D(\xi)$ . We obtain

$$\frac{\partial C}{\partial \xi} + \frac{\partial C}{\partial z} - \frac{D(\xi)}{u(\xi)} \frac{\partial^2 C}{\partial z^2} = 0. \quad (2.152)$$

If the linear relationship  $D = \delta_L u$  is valid, then (2.152) becomes the advection–dispersion equation with constant coefficients

$$\frac{\partial C}{\partial \xi} + \frac{\partial C}{\partial z} - \delta_L \frac{\partial^2 C}{\partial z^2} = 0, \quad (2.153)$$

where  $\xi = \xi(t)$  serves as the space coordinate of the piston-like displacement front. An approximate solution of (2.153) can be written as

$$\bar{C} = \frac{1}{2} \operatorname{erfc} \frac{z - \xi(t)}{\sqrt{4\delta_L \xi(t)}}, \quad (2.154)$$

similar to Eq. 1.96.

The time coordinate  $t$  can be replaced by another cumulative coordinate (Vanderborgh et al. 2000b):

$$I(z, t) = \int_0^t q(z, t) dt; \quad (2.155)$$

here  $q(z, t)$  is the velocity (specific rate of infiltration) at depth  $z$  in moment  $t$ .

The Eq. 2.149 for function  $C(I)$  with  $\theta = \theta(z)$  and  $q = q(t)$  (when the moisture content does not depend on the time coordinate, while the velocity does not depend on the space coordinate) becomes

$$\frac{\partial C}{\partial I} + \frac{1}{\theta(z, 0)} \frac{\partial C}{\partial z} - \frac{\delta_L}{\theta(z, 0)} \frac{\partial^2 C}{\partial z^2} = 0. \quad (2.156)$$

With  $\delta_L = 0$ , the solution (2.156) yields the position of the piston-like displacement front

$$I(z, t) = \int_0^z \theta(z, 0) dz, \quad (2.157)$$

where  $\theta(z, 0)$  is moisture distribution in the aeration zone in the initial moment  $t = 0$ . If  $\theta(z, 0) = \theta = \text{const}$ , the solution of the advection–dispersion Eq. 2.156 becomes

$$\bar{C} = \frac{1}{2} \text{erfc} \frac{z\theta - I}{\sqrt{4\delta_L I \theta}}. \quad (2.158)$$

If the moisture content varies over depth,  $\theta = \theta(z)$ , the function  $C(I)$  can be described by expression (2.158) with  $\theta$  replaced by  $\bar{\theta}$ :

$$\bar{\theta} = \int_0^z \theta(z, t) dz. \quad (2.159)$$

Finally, it is worth mentioning that the dispersion of solutes in unsaturated sub-surface material has some specific features. As it was shown by experiments (Padilla et al. 1999), the dispersion coefficient in incompletely saturated porous media is greater than that in the same sediments when completely saturated. The general structure of the dispersion coefficient is  $D = \zeta_L(\theta_m)u^n$ , where the exponent can reach 1.5;  $\theta_m$  is the volumetric moisture content of flow-through pores. This difference is due to the relatively large volume of stagnant pores (zones) as compared with completely saturated media. Solute exchange between the active pores, through which it is transported, and the stagnant pores is slow. However, from the practical viewpoint, such specification is not fundamental, and the analytical solutions given above are well suited for the analysis of the significance of microdispersion effects.

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