Chapter 2
Eigenmode reciprocity in k-space

2.1 Reciprocity in physical space and in k-space

2.1.1 Overview

Until the mid 1960's the problem of reciprocity in electromagnetics had been developing in two separate, and seemingly unrelated directions. As early as 1896 Lorentz [91] had demonstrated that if two independent current distributions, $\mathbf{J_1}(\mathbf{r})$ and $\mathbf{J_2}(\mathbf{r})$, generated electromagnetic fields, $\mathbf{E_1}(\mathbf{r}),\ \mathbf{H_1}(\mathbf{r})$ and $\mathbf{E_2}(\mathbf{r}),\ \mathbf{H_2}(\mathbf{r})$ respectively, in free space, then

$$\int \mathbf{E}_1(\mathbf{r}) \cdot \mathbf{J}_2(\mathbf{r}) d^3 r = \int \mathbf{E}_2(\mathbf{r}) \cdot \mathbf{J}_1(\mathbf{r}) d^3 r \tag{2.1}$$

and this was recognized as an expression of the 'interchangeability' of transmitting and receiving antennas. This, or an equivalent formulation,

$$\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = 0 \tag{2.2}$$

became to be known as the Lorentz reciprocity theorem, and will be discussed in some detail in Chap. 4. The theorem was used, inter alia, to deduce the properties of transmitting antennas if their properties as receiving antennas were known. Eckersley [51], for instance, used the theorem to deduce the radiation pattern of a transmitting antenna as modified by an imperfectly conducting ground below it, by solving the simpler problem of its response as a receiving antenna.

Sommerfeld [110] and Dällenbach [47] pointed out that the theorem would hold in anisotropic media provided that the electric permittivity $\boldsymbol{\varepsilon}$, the magnetic permeability $\boldsymbol{\mu}$ and the conductivity $\boldsymbol{\sigma}$ were symmetric tensors. Rumsey [106] and Cohen [40] noted that a modified form of Lorentz reciprocity would hold also for non-symmetric tensors provided that the second (reciprocal) system of currents and

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fields were taken in a 'transposed medium', characterized by the transposed tensors $\boldsymbol{\varepsilon}^T$, $\boldsymbol{\mu}^T$ and $\boldsymbol{\sigma}^T$. Harrington and Villeneuve [63] applied the theorem to gyrotropic media, such as magnetoplasmas or ferrites, in which the 'transposed medium' is just the original medium with the direction of the external magnetic field reversed. Kong and Cheng [84] and Kerns [81] extended the result to bianisotropic media (see Sec. 2.2.2) and introduced the concept of a 'complementary' or 'adjoint' medium, which generalizes the earlier concept of the transposed medium.

A parallel, and seemingly unrelated line of development treated what we shall call 'reciprocity in (transverse-) **k**-space', which in its early form dealt with the symmetry properties of the scattering matrices in a plane-stratified ionospheric magnetoplasma. Budden [29] and Barron and Budden [21] found that the 2×2 reflection matrix for plane-wave incidence on a plane-stratified magnetoplasma was the transpose of the reflection matrix for another symmetrically disposed direction of incidence, which we shall subsequently call the 'conjugate direction'. (Because of Snell's law, the component $\mathbf{k}_t$ of the propagation vector in the stratification plane—the 'transverse' component—is the same for the incoming plane wave and for the outgoing, scattered waves.) Pitteway and Jespersen [100] and Heading [66] found similar results relating the transmission coefficients for upgoing waves incident on the ionosphere in a given direction, and downgoing waves incident in a symmetrically disposed, conjugate direction. These results were later generalized by Suchy and Altman [12, 13, 118, 119] who showed that the 4×4 scattering matrices could be expressed in terms of suitably defined eigenmode amplitudes within the gyrotropic medium, and not only in terms of linearly polarized base modes in free space outside of the scattering medium. This result was further extended by Altman et al. [10] to include bianisotropic media, and it was shown that a wide range of 'adjoint' or 'complementary' reciprocal media could be generated by means of orthogonal transformations (rotation, reflection or inversion) of the transposed medium.

2.1.2 *From physical space to k-space*

The two lines of development just described converged from both directions. A passive antenna is a scattering object, and any dielectric scattering object will re-radiate by virtue of the currents induced by the external fields incident on it. Lorentz reciprocity will apply to such scattering objects (see, for instance, Rumsey [106]). Harrington and Villeneuve [63] showed that if a scattering object, characterized by constitutive tensors, $\boldsymbol{\varepsilon}$, $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$, be considered as a generalized N terminal-pair network, with **V** and **I** representing column matrices of 'terminal' voltages and currents at the surface of the scatterer, one may define a scattering matrix S through the relation

$$\mathbf{V} =: \mathsf{S}\,\mathbf{I} \tag{2.3}$$

They showed that if the medium of the object had transposed constitutive tensors $\boldsymbol{\varepsilon}^T$, $\boldsymbol{\mu}^T$ and $\boldsymbol{\sigma}^T$, the scattering matrix would be transposed to S^T.

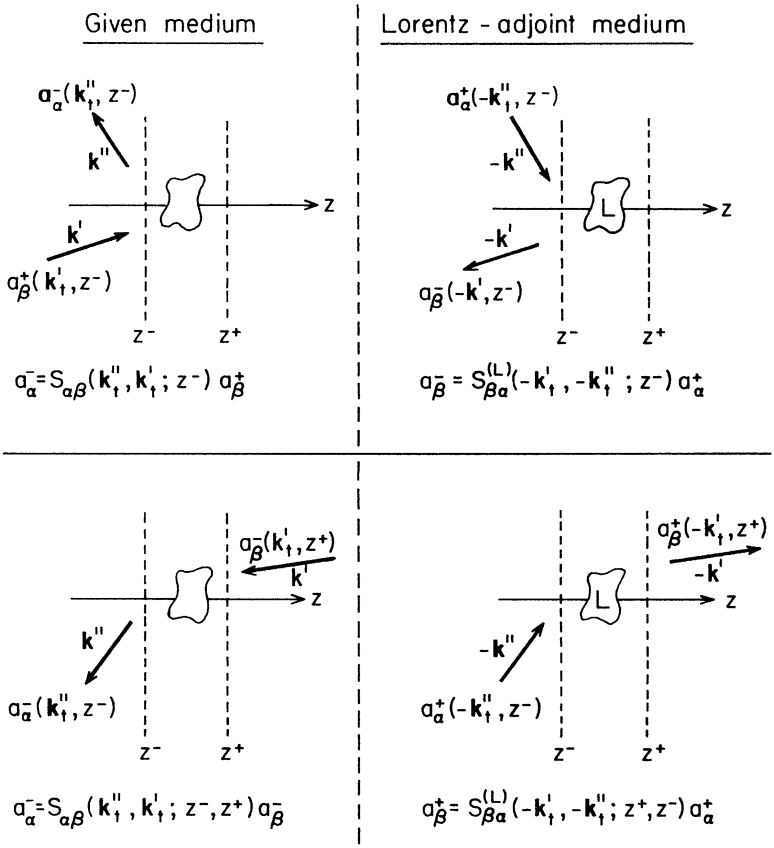


Fig. 2.1 Scattering relations illustrated schematically for object with given or Lorentz-adjoint medium. In all cases $S_{\alpha\beta}\left(\mathbf{k}_t'', \mathbf{k}_t'\right) = S_{\beta\alpha}^{(L)}\left(-\mathbf{k}_t', -\mathbf{k}_t''\right)$

It was the work of Kerns [81], however, that bridged the gap from reciprocity in real (physical) space to reciprocity in k-space. Let us suppose, with Kerns, that a scattering object in free space, Fig. 2.1, is contained between two imaginary planes, z^- and z^+. We consider an incoming wave, with an electric wave field $\mathbf{E}^{in}(z^-)$ or $\mathbf{E}^{in}(z^+)$ incident on the object from the left or right respectively. We shall adapt Kerns' notation to that used by us. The transverse component (transverse to the z-axis) of the electric field, $\mathbf{E}_t^{in}$, may be Fourier analysed in the $z = z^-$ or $z = z^+$ planes. Any Fourier component having a transverse wave vector

$$\mathbf{k}_t \equiv (k_x,\ k_y) \quad \text{with} \quad |\mathbf{k}_t| = \left(k_0{}^2 - k_z{}^2\right)^{1/2}, \quad k_0 := \omega(\varepsilon_0\mu_0)^{1/2}$$

may be decomposed into two 'modes', in which the electric fields, $\mathbf{E}_{t1}$ and $\mathbf{E}_{t2}$, are respectively parallel and perpendicular to the plane of incidence. The basis vectors along these fields will be

$$\hat{\boldsymbol{\epsilon}}_{||} := \mathbf{k}_t/|\mathbf{k}_t| \equiv \hat{\boldsymbol{\epsilon}}_{\pm 1} \qquad \text{and} \qquad \hat{\boldsymbol{\epsilon}}_{\perp} := \hat{\mathbf{z}} \times \hat{\boldsymbol{\epsilon}}_{\pm} \equiv \hat{\boldsymbol{\epsilon}}_{\pm 2}$$

Fourier analysis of $\mathbf{E}_t^{in}(z^{\mp})$ yields the spectral amplitude densities, $\underset{\sim}{A}_{\alpha}^{\pm}(\mathbf{k}_t, z^{\mp})$, in transverse-$\mathbf{k}$ space, with $\alpha = 1, 2$ or $\alpha = -1, -2$ for positive- or negative-going waves respectively:

$$\mathbf{E}_t^{in}(z^{\mp}) = \frac{1}{2\pi} \int \int \underset{\sim}{A}_{\alpha}^{\pm}(\mathbf{k}_t, z^{\mp}) \hat{\boldsymbol{\epsilon}}_{\alpha} \, \exp[-i(k_x x + k_y y)] dk_z dk_y \tag{2.4}$$

integrated over the entire transverse-$\mathbf{k}$ plane, with assumed summation over the characteristic polarizations $\alpha = 1, 2$ for $z = z^{-}$, or $\alpha = -1, -2$ for $z = z^{+}$. Phase factors $\exp(\mp i k_z z^{\mp})$ have been included in the spectral amplitudes $\underset{\sim}{A}_{\alpha}^{\pm}$. Underlying tildes ($\sim$) are used in this section to denote quantities that represent densities in transverse-$\mathbf{k}$ space.

The outgoing scattered wave fields $\mathbf{E}_t^{out}(z^{\pm})$ may similarly be Fourier analyzed to yield outgoing amplitude densities, $\underset{\sim}{A}_{\alpha}^{\pm}(\mathbf{k}_t, z^{\pm})$. It is convenient to define normalized amplitude densities, $\underset{\sim}{a}_{\alpha}^{\pm}$:

$$\underset{\sim}{a}_{\alpha}^{\pm} := {\eta_{\alpha}}^{1/2} {\underset{\sim}{A}_{\alpha}}^{1/2}, \quad \alpha = \pm 1, \pm 2$$

where

$$\eta_{\pm 1} := \omega \varepsilon_0 / |k_z| = \frac{k}{|k_z|} \sqrt{\frac{\varepsilon_0}{\mu_0}} \quad \text{and} \quad \eta_{\pm 2} := |k_z| / \omega \mu_0 = \frac{|k_z|}{k} \sqrt{\frac{\varepsilon_0}{\mu_0}}$$

are the characteristic wave admittances [81, eqs. (1.2–5) and (1.2–6)]. Then $\left|\underset{\sim}{a}_{\alpha}^{\pm}\right|^2$ will represent the spectral densities (in transverse-k space) of the z-component of the time-averaged energy fluxes across the surfaces $z = z^{+}$ or $z = z^{-}$:

$$\left\langle P_{z^{\mp},\alpha} \right\rangle = -\frac{1}{2} \int \int \left|\underset{\sim}{a}_{a}^{\pm}(\mathbf{k}_t, \, z^{\mp})\right|^2 dk_x dk_y, \quad \alpha = \pm 1, \pm 2 \quad \text{for} \quad z = z^{\mp} \tag{2.5}$$

for incoming waves, and

$$\left\langle P_{z^{\mp},\alpha} \right\rangle = +\frac{1}{2} \int \int \left|\underset{\sim}{a}_{a}^{\pm}(\mathbf{k}_t, \, z^{\pm})\right|^2 dk_x dk_y, \quad \alpha = \pm 1, \pm 2 \quad \text{for} \quad z = z^{\pm} \tag{2.6}$$

for outgoing waves, in which, for simplicity, we have ignored the contributions from evanescent modes [81, eqs. (1.4–2) and (1.4–3)].

Outgoing and incoming modal amplitude densities will be related by elements of a scattering-density matrix $\underset{\sim}{\mathbf{S}}$. Symbolically, we may write [81, eq. (1.3–1)]

$$\underset{\sim}{a}_{\alpha}^{out}\left(\mathbf{k}_t''\right) = \int \int \underset{\sim}{S}_{\alpha\beta}\left(\mathbf{k}_t'', \mathbf{k}_t'\right) \underset{\sim}{a}_{\beta}^{in}\left(\mathbf{k}_t'\right) \, dk_x' \, dk_y' \tag{2.7}$$

which, for back- and forward-scattered waves respectively, becomes

$$\underset{\sim}{a}_{\alpha}^{\mp}\left(\mathbf{k}_t'',\ z^{\mp}\right) = \int\int \underset{\sim}{S}_{\alpha\beta}\left(\mathbf{k}_t'', \mathbf{k}_t';\ z^{\mp}\right) \underset{\sim}{a}_{\beta}^{\pm}\left(\mathbf{k}_t', z^{\mp}\right)\ dk_x'\, dk_y'$$

and

$$\underset{\sim}{a}_{\alpha}^{\mp}\left(\mathbf{k}_t'',\ z^{\pm}\right) = \int\int \underset{\sim}{S}_{\alpha\beta}\left(\mathbf{k}_t'', \mathbf{k}_t';\ z^{\pm}, z^{\mp}\right) \underset{\sim}{a}_{\beta}^{\pm}\left(\mathbf{k}_t', z^{\mp}\right)\ dk_x'\, dk_y'$$

Now let the medium of the scattering object be replaced by a '(Lorentz-) adjoint' medium. (Just what is meant by this is explained in Sec. 3.4. In the case of a magnetoplasma, it means the given medium in which the external magnetic field has been reversed in direction). Suppose also that all outgoing wave vectors $\mathbf{k}''$ are reversed in direction ($\mathbf{k}'' \to -\mathbf{k}''$) so that they become *incoming* wave fields. Kerns' (Lorentz-) adjoint scattering theorem [81, eq. (1.5–5)] states that in this case the outgoing wave fields will be just the incoming wave fields in the original problem with their wave vectors reversed ($\mathbf{k}' \to -\mathbf{k}'$). In the case of back-scattering this means that

$$\underset{\sim}{S}_{\alpha\beta}\left(\mathbf{k}_t'', \mathbf{k}_t';\ z^{\mp}\right) = \underset{\sim}{S}_{\beta\alpha}^{(L)}\left(-\mathbf{k}_t', -\mathbf{k}_t''; z^{\mp}\right) \tag{2.8}$$

and for forward scattering

$$\underset{\sim}{S}_{\alpha\beta}\left(\mathbf{k}_t'', \mathbf{k}_t';\ z^{\pm}, z^{\mp}\right) = \underset{\sim}{S}_{\beta\alpha}^{(L)}\left(-\mathbf{k}_t', -\mathbf{k}_t''; z^{\mp}, z^{\pm}\right) \tag{2.9}$$

where $\underset{\sim}{\mathbf{S}}^{(L)}$ is the scattering-density matrix for the Lorentz-adjoint medium. These relations are illustrated schematically in Fig. 2.1.

Suppose now that the incoming wave field in (2.7) is that of a single plane wave with a transverse propagation vector $\mathbf{k}_t$. Then $\underset{\sim}{a}_{\beta}^{in}$ becomes a Dirac delta function (aside from a multiplying factor) in transverse-k space,

$$\underset{\sim}{a}_{\beta}^{in}\left(\mathbf{k}_t'\right) = a_{\beta}^{in}(\mathbf{k}_t)\delta\left(\mathbf{k}_t' - \mathbf{k}_t\right) = a_{\beta}^{\pm}(z^{\mp})\delta\left(\mathbf{k}_t' - \mathbf{k}_t\right),\ \ a_{\beta}^{in}(\mathbf{k}_t) = \iint \underset{\sim}{a}_{\beta}^{in}\left(\mathbf{k}_t'\right) dk_x'\, dk_y'$$

and (2.7) becomes

$$\underset{\sim}{a}_{\alpha}^{out}\left(\mathbf{k}_t''\right) = \underset{\sim}{S}_{\alpha,\beta}\left(\mathbf{k}_t'', \mathbf{k}_t\right) a_{\beta}^{in}(\mathbf{k}_t) \tag{2.10}$$

Finally we let the scattering object be a plane-stratified slab, situated between the planes $z = z^-$ and $z = z^+$, i.e. all constitutive parameters of the medium are functions of the z-coordinate only. Because of Snell's law, the scattering-density matrix $\underset{\sim}{S}_{\alpha\beta}$ becomes a delta function in $\mathbf{k}_t$-space for plane-wave incidence, and the amplitude densities of both the incident and scattered waves will also be delta functions in $\mathbf{k}_t$-space:

$$\begin{aligned} \underset{\sim}{S}_{\alpha\beta}\left(\mathbf{k}_t'', \mathbf{k}_t\right) &= S_{\alpha\beta}(\mathbf{k}_t)\delta\left(\mathbf{k}_t'' - \mathbf{k}_t\right) \\ \underset{\sim}{a}_{\alpha}^{out}\left(\mathbf{k}_t''\right) &= a_{\alpha}^{out}(\mathbf{k}_t)\delta\left(\mathbf{k}_t'' - \mathbf{k}_t\right) = a_{\alpha}^{\mp}(z^{\mp})\delta\left(\mathbf{k}_t'' - \mathbf{k}_t\right) \end{aligned} \tag{2.11}$$

If $\underset{\sim}{S}_{\alpha\beta}$ and $\underset{\sim}{a}_{\alpha}^{out}\left(\mathbf{k}_t''\right)$ are now substituted into (2.10), and the equation then integrated over all $\mathbf{k}_t''$, we obtain

$$a_{\alpha}^{out}(\mathbf{k}_t) = S_{\alpha\beta}(\mathbf{k}_t)a_{\beta}^{in}(\mathbf{k}_t) \tag{2.12}$$

The scattering relation (2.7) then reduces to a straightforward matrix relation, with each incident mode, $\alpha = \pm 1, \pm 2$, generating two reflected (back-scattered) and two transmitted (forward-scattered) modes. With

$$\boldsymbol{a}_+ := \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \qquad \boldsymbol{a}_- := \begin{bmatrix} a_{-1} \\ a_{-2} \end{bmatrix}$$

we have

$$\begin{aligned} \boldsymbol{a}_{out} := \begin{bmatrix} \boldsymbol{a}_-(z^-) \\ \boldsymbol{a}_+(z^+) \end{bmatrix} &= \begin{bmatrix} \mathsf{R}_+(\mathbf{k}_t; z^-) & \mathsf{T}_-(\mathbf{k}_t; z^-, z^+) \\ \mathsf{T}_+(k_t; z^+, z^-) & \mathsf{R}_-(\mathbf{k}_t; z^+) \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_+(z^-) \\ \boldsymbol{a}_-(z^+) \end{bmatrix} \\ &= \mathsf{S}(\mathbf{k}_t)\boldsymbol{a}_{in} \end{aligned} \tag{2.13}$$

defining thereby the 4-element modal-amplitude column matrices, $\boldsymbol{a}_{in}$ and $\boldsymbol{a}_{out}$, and the 2×2 reflection and transmission matrices, $\mathsf{R}_\pm$ and $\mathsf{T}_\pm$, which constitute the scattering matrix S. (The signed subscripts indicate the direction of the incident mode with respect to the z-axis.)

Kerns' theorem, (2.8) and (2.9), reduces in this case to

$$\mathsf{S}(\mathbf{k}_t) = \tilde{\mathsf{S}}^{(L)}(-\mathbf{k}_t) \tag{2.14}$$

$$\mathsf{R}_\pm(\mathbf{k}_t; z^\mp) = \tilde{\mathsf{R}}_\pm^{(L)}(-\mathbf{k}_t; z^\mp), \qquad \mathsf{T}_\pm(\mathbf{k}_t; z^+, z^-) = \tilde{\mathsf{T}}_\mp^{(L)}(-\mathbf{k}_t; z^-, z^+)$$

with, typically

$$R_{12}^+(\mathbf{k}_t; z^-) = R_{21}^{(L)+}(-\mathbf{k}_t; z^-), \qquad T_{12}^+\left(\mathbf{k}_t; z^+, z^-\right) = T_{21}^{(L)-}(-\mathbf{k_t}; \mathbf{z}^-, \mathbf{z}^+)$$

where $T_{\alpha\beta}^\pm$ and $R_{\alpha\beta}^\pm$, with $\alpha, \beta = 1$ or 2, denote elements of the 2×2 matrices $\mathsf{T}_\pm$ and $\mathsf{R}_\pm$.

This restricted form of Kerns' scattering theorem will be discussed in Sec. 3.4 in the general context of scattering theorems in plane-stratified media. In Sec. 7.3 Kerns' scattering theorem will be generalized to the case in which an anisotropic scattering object is immersed in a homogeneous or plane-stratified anisotropic medium.

2.1.3 Reciprocity in transverse-k space: a review of the earlier scattering theorems

The interest in the ionosphere, until the mid-fifties, lay primarily in its ability to reflect radio waves. Vertical ionospheric sounding had been employed since the mid-thirties to determine ionospheric structure and maximum usable frequencies for radio communication between fixed ground stations. Point-to-point long-wave and very-long-wave radio links had been tested experimentally to determine diurnal and seasonal variations, as well as the directional dependence of the ionospheric reflection coefficients. The first heroic efforts in the early fifties, especially by Budden and his coworkers [30, 31], to produce full-wave computer programs to solve the differential equations governing the propagation of radio waves in a plane-stratified magnetoplasma, were aimed at producing, as their primary output, a set of reflection coefficients for arbitrary directions of incidence. Various equalities were then discovered in the numerically computed reflection coefficients for certain symmetrically disposed directions of incidence. The analytical proof of these 'reciprocity theorems' was found only later, after the theorems were already known from the computer output [21, 29].

In 1953 Storey [114] showed both experimentally and theoretically that very-low-frequency (whistler) waves, guided by the earth's magnetic field, could penetrate through the ionospheric $X = 1$ level (where waves of similar polarization but higher frequency would normally have been reflected — cf. Sec. 1.2) to reach a magnetically conjugate point in the opposite hemisphere. (The frequency dispersion of these waves—the higher frequencies arriving before the lower—generated a whistling sound of falling frequency when the audio-frequency electromagnetic waves were received by an antenna connected to an audio amplifying system. Hence the name 'whistler'.) Storey's findings were one of the motivating factors in developing computer programs, such as that due to Pitteway [98], to calculate very-low-frequency transmission coefficients for propagation through the ionosphere for plane wave incidence from both below and above the ionosphere. Equalities between the transmission coefficients of downgoing whistler waves and upgoing 'penetrating modes' were again found in the computer output, for certain symmetrically disposed planes of incidence, and the analytical proof then followed [100].

Heading [66] undertook a systematic analysis of reciprocity (scattering) relations in plane-stratified magnetoplasmas, by considering certain general symmetry properties of Maxwell's second-order differential equations in such media. Equalities were again found between elements of the reflection and transmission matrices for certain pairs of symmetrically related directions of plane-wave incidence. The scattering matrix elements were defined, as in Budden's treatment [29], in terms of linearly polarized base modes in the free space bounding the medium.

At this stage there was still no obvious connection between the results of Kerns previously discussed, as applied to plane-stratified media, and those of Barron and Budden, Pitteway and Jespersen, and Heading. Kerns' scattering theorem involved an 'adjoint medium', which in the case of a magnetoplasma meant a magnetic-field reversed medium, with wave vectors reversed in direction too. The work of Budden and others, on the other hand, compared scattering matrix elements in the same medium, but with different directions of incidence.

The thin-layer scattering-matrix numerical technique developed by Altman and Cory [3, 4] (see Sec. 1.5.3) led, fortuitously, to a generalized form of the scattering theorem in plane-stratified media. In this method the elements of the reflection and transmission matrices were just the quantities which were recursively summed in the numerical procedure in which thin elementary layers were added stepwise to the plane-stratified slab. The scattering matrix elements related amplitudes of eigenmodes emerging from slabs of varying thicknesses, imbedded in the given statified medium, to the amplitudes of eigenmodes incident on the slab. The 'amplitude' of an eigenmode was taken initially to be the (square root of the) z-component, normal to the stratification, of the time-averaged Poynting flux of the eigenmode. The computed output yielded the elements of the scattering matrix $\mathsf{S}(\mathbf{k}_t, \phi)$, (2.13), for given values of the transverse wave vector $\mathbf{k}_t$, and for given azimuthal angles, ϕ, between the plane of incidence and the magnetic meridian plane (the plane containing the external magnetic field, $\mathbf{b}$, and the normal to the stratification, $\hat{\mathbf{z}}$). The scattering matrix $\mathsf{S}(\mathbf{k}_t, \phi)$, when the plane of incidence was at an azimuthal angle ϕ, was found to be the transpose of that for a *conjugate* orientation in which the azimuthal angle was $(\pi - \phi)$,

$$\mathsf{S}(\mathbf{k}_t, \phi) = \tilde{\mathsf{S}}(\mathbf{k}_t, \pi - \phi) \tag{2.15}$$

as long as the medium was lossless. The exact equality broke down as soon as collisional losses were introduced. On the basis of a procedure due to Budden and Jull (1964) in their treatment of reciprocity of magnetoionic rays [35], the complex conjugate wave fields, $\mathbf{E}^*$ and $\mathbf{H}^*$, appearing in the expression for the mean Poynting flux (see Sec. 2.3.1), were replaced by the computed adjoint wave fields, $\overline{\mathbf{E}}$ and $\overline{\mathbf{H}}$. This meant that the complex-conjugate transverse wave polarizations, ρ^*, appearing in the expression for the mean Poynting flux (see eq. 2.65 in Sec. 2.3.1) were replaced by the adjoint wave polarizations, $\bar{\rho} = -\rho$ (2.66). (The equality $\rho^* = \bar{\rho} = -\rho$ holds only for loss-free media). The eigenmode scattering theorem (2.15), reported by Altman in 1971 [2], was found to be exact, but the analytic proof was only found much later by Suchy and Altman [12, 119].

This chapter deals with some of the properties of the adjoint Maxwell equations, and their use in the derivation of the eigenmode scattering theorem. In Chap. 3 we consider the generalization of the theorem to base modes which are not eigenmodes of the medium, and discuss some of the earlier reciprocity (scattering) theorems in the light of the generalized theorem.

2.1.4 From transverse-k space back to physical space

The link back from reciprocity in transverse-k space to reciprocity in physical space was found by Schatzberg and Altman [8, 108]. Their procedure was to Fourier-analyse currents and fields in transverse-$\mathbf{k}$ space, and then to set up the angular spectrum of plane-wave eigenmodes associated with an element of current, $\mathbf{J}_1(\mathbf{k}_t, z')dz'$, flowing in an elementary layer of thickness dz' in the medium. With the aid of the scattering matrix $\mathsf{S}(\mathbf{k}_t; z, z')$ a dyadic Green's function, $\mathsf{G}(\mathbf{k}_t; z, z')$, was determined, so that the overall field $\mathbf{e}_1(\mathbf{k}_t, z)$ at a level z in the medium was given by

$$\mathbf{e}_1(\mathbf{k}_t, z) = \int \mathbf{G}(\mathbf{k}_t; z, z')\mathbf{J}_1(\mathbf{k}_t, z')dz' \tag{2.16}$$

In a similar fashion a second, independent current distribution, $\mathbf{J}_2(\mathbf{k}_t, z')$, generated a field $\mathbf{e}_2(\mathbf{k}_t, z)$.

A mirroring (reflection) transformation of the currents and fields with respect to the magnetic meridian $(\mathbf{b}, \hat{\mathbf{z}})$ plane, yielded the 'conjugate' currents and fields, $\mathbf{J}^c\left(\mathbf{k}_t^c, z'\right)$ and $\mathbf{e}^c\left(\mathbf{k}_t^c, z\right)$. Here, $\mathbf{k}_t^c$ has been formed by reversing the sign of $\mathbf{k}_t$ (this will later be seen to be an expression of time reversal, which is inherent in the reciprocity process), and then the y-component, normal to the $(\mathbf{b}, \hat{\mathbf{z}})$ plane, is again sign-reversed by the reflection transformation to give

$$\mathbf{k}_t = k_0(s_x, s_y), \qquad \mathbf{k}_t^c = k_0(-s_x, s_y) \tag{2.17}$$

With the aid of the scattering theorem (2.15), which may be written in the form

$$\mathsf{S}(s_x, s_y) = \mathsf{S}^T(-s_x, s_y) \tag{2.18}$$

a simple relation was found between the given and conjugate dyadic Green's functions, $\mathsf{G}(\mathbf{k}_t; z, z')$ and $\mathsf{G}^c(\mathbf{k}_t^c; z, z')$, in $\mathbf{k}_t$ space. An inverse Fourier transfomation in $\mathbf{k}_t$-space led finally to a Lorentz-type reciprocity relation in real space [8, 108]

$$\int \mathbf{e}_1(\mathbf{r}) \cdot \mathbf{J}_2(\mathbf{r})d^3r = \int \mathbf{e}_2^c(\mathbf{r}) \cdot \mathbf{J}_1^c(\mathbf{r})d^3r \tag{2.19}$$

Eq. (2.19) is derived in Chap. 5. It will be noted that this result does not contain any feature that would indicate that its validity is restricted to plane-stratified media. In fact it is shown in Chap. 6 that in any medium that has 'conjugation symmetry', the reciprocity relation (2.19) will apply. A medium will be said to possess 'conjugation symmetry' if, after being 'time reversed', it can be mapped into itself by means of an orthogonal transformation. A 'time-reversed' magnetoplasma, for example, is one in which the external magnetic field has been reversed.

2.2 The adjoint wave fields

2.2.1 The need for an auxiliary set of equations adjoint to Maxwell's equations

The field equations of physics may generally be written as a system of first-order partial differential equations or, on elimination of some of the field variables, as higher-order equations. The Maxwell field, containing both electric and magnetic components, may be described by six first-order differential equations with six possible source terms, components of the electric and equivalent magnetic currents. The coefficients of the field components in these equations will be determined by the constitutive relations of the medium considered, as expressed by the constitutive tensor which relates the field vectors **D** and **B** to **E** and **H**. Examples of such tensors and their characteristic symmetries are discussed in Sec. 4.1.

To reveal the basic symmetries of the fields it is useful to make use of an auxiliary or *adjoint* set of equations, which will be satisfied by *adjoint field variables*. These hypothetical adjoint fields will then exhibit a *reciprocity relation* with respect to the fields in the original problem. The adjoint fields will in general be non-physical, insofar as they satisfy the non-physical adjoint equations, but frequently they can be related in a simple and direct way to the physical fields in another *conjugate* problem, derived from the original by some sort of mapping transfomation (such as reflection). The reciprocity relation between the given and adjoint problems then leads to a reciprocity relation between fields (or between currents and fields, if sources are present) in the two physical configurations of the given and conjugate problems.

In the case of plane-stratified media both the given Maxwell field and the adjoint field can in principle be decomposed into characteristic wave fields or eigenmodes. It will be shown that the given and adjoint eigenmodes are biorthogonal, a property which provides a simple procedure for decomposing a wave field into its constituent eigenmodes, and for determining their amplitudes in a manner suitable for application in a general scattering theorem which will be derived in Sec. 2.5.

2.2.2 *Maxwell's equations in anisotropic, plane-stratified media*

The electric and magnetic wave fields in an anisotropic or bianisotropic medium will be related in general by a 6×6 constitutive tensor K:

$$\begin{bmatrix} \mathbf{D} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon} & \boldsymbol{\xi} \\ \boldsymbol{\eta} & \boldsymbol{\mu} \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} \equiv \mathsf{K}\mathbf{e} \tag{2.20}$$

We note that the fundamental fields, defined by the Lorentz force on an electric charge q

$$\mathbf{F} = q(\mathbf{E} + \boldsymbol{v} \times \mathbf{B})$$

are **E** and **B**, whereas **D** and **H** are derived fields which contain the additional contributions of electric polarization and magnetization currents. Nevertheless it is convenient, for the sake of symmetry of Maxwell's equations, to represent the constitutive tensor K in this form. $\boldsymbol{\varepsilon}$ is the 3×3 electric permittivity tensor, and $\boldsymbol{\mu}$ is the magnetic permeability tensor which, for media having no magnetic activity (such as plasmas, with or without an ambient magnetic field) is just the scalar permeability μ_0 of free space, $\boldsymbol{\mu} = \boldsymbol{\mu}_0 \mathsf{I}^{(3)}$, where $\mathsf{I}^{(n)}$ represents the $n \times n$ unit matrix. The 3×3 coupling matrices $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ in (2.20) are usually zero except for a small class of *bianisotropic* media in which a magnetic field produces electric polarization, and an electric field magnetizes the medium. (Moving media are bianisotropic since the electric and magnetic fields are coupled by the Lorentz transformation, and so too are the magneto-electric or so-called Tellegen media [124] in which the elementary electric dipoles also have magnetic moment. Bianisotropic media have been discussed by Post [101], Kong and Cheng [37, 82, 83], van Bladel [127] and others, and are considered in some detail in Sec. 4.1.).

With time-harmonic $\exp(i\omega t)$ variation of all field quantities, Maxwell's equations take the form

$$[i\omega \mathsf{K} + \mathsf{D}]\mathbf{e}(\mathbf{r}) = -\mathbf{j}(\mathbf{r}) \tag{2.21}$$

with D, the differential operator, given by

$$\mathsf{D} := \begin{bmatrix} 0 & -\nabla \times \mathsf{I}^{(3)} \\ \nabla \times \mathsf{I}^{(3)} & 0 \end{bmatrix} = \mathsf{D}^T \tag{2.22}$$

The generalized wave-field and current vectors, $\mathbf{e}$ and $\mathbf{j}$, are given by

$$\mathbf{e} := \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}, \qquad \mathbf{j} := \begin{bmatrix} \mathbf{J}_\mathrm{e} \\ \mathbf{J}_\mathrm{m} \end{bmatrix} \tag{2.23}$$

where $\mathbf{J}_\mathrm{e}$ and $\mathbf{J}_\mathrm{m}$ are the electric and equivalent magnetic current densities.

If we split the differential operator D into three cartesian differential operators, (2.21) becomes

$$\left[i\omega \mathsf{K} + \mathsf{U}_x \frac{\partial}{\partial x} + \mathsf{U}_y \frac{\partial}{\partial y} + \mathsf{U}_z \frac{\partial}{\partial z}\right]\mathbf{e}(\mathbf{r}) = -\mathbf{j}(\mathbf{r}) \tag{2.24}$$

where

$$\mathsf{U}_x := \begin{bmatrix} 0 & -\hat{\mathbf{x}} \times \mathsf{I}^{(3)} \\ \hat{\mathbf{x}} \times \mathsf{I}^{(3)} & 0 \end{bmatrix} = \mathsf{U}_x{}^T, \ \mathsf{U}_y := \begin{bmatrix} 0 & -\hat{\mathbf{y}} \times \mathsf{I}^{(3)} \\ \hat{\mathbf{y}} \times \mathsf{I}^{(3)} & 0 \end{bmatrix} = \mathsf{U}_y{}^T$$

$$\mathsf{U}_z := \begin{bmatrix} 0 & -\hat{\mathbf{z}} \times \mathsf{I}^{(3)} \\ \hat{\mathbf{z}} \times \mathsf{I}^{(3)} & 0 \end{bmatrix} = \mathsf{U}_z{}^T = \left[\begin{array}{ccc:ccc} & & & 0 & 1 & 0 \\ & 0 & & -1 & 0 & 0 \\ & & & 0 & 0 & 0 \\ \hdashline 0 & -1 & 0 & & & \\ 1 & 0 & 0 & & 0 & \\ 0 & 0 & 0 & & & \end{array}\right] \tag{2.25}$$

(Note that U_x, U_y, U_z and D are all symmetric). Now assume the medium to be plane stratified with the z-axis normal to the stratification. We denote the projection of $\mathbf{k}$ on the stratification plane by $\mathbf{k}_t$,

$$\mathbf{k}_t := (k_x, k_y) = k_0(s_x, s_y) \tag{2.26}$$

where s_z and s_y are propagation constants (Snell's law). Fourier-transforming $\mathbf{e}(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$ in (2.24) in the transverse (stratification) plane, we have typically

$$\mathbf{e}(\mathbf{r}) = \frac{k_0^2}{4\pi^2} \int \int \mathbf{e}(\mathbf{k}_t, z) \exp[-ik_0(s_x x + s_y y)] ds_x ds_y \tag{2.27}$$

$$\mathbf{e}(\mathbf{k}_t, z) = \int \int \mathbf{e}(\mathbf{r}) \exp[ik_0(s_x x + s_y y)] dx\, dy \tag{2.28}$$

Substitution in (2.24), (with K independent of x and y), yields

$$ik_0 \left[c\mathsf{K}(z) - s_x \mathsf{U}_x - s_y \mathsf{U}_y - \frac{i}{k_0} \mathsf{U}_z \frac{d}{dz} \right] \mathbf{e}(\mathbf{k}_t, z) = -\mathbf{j}(\mathbf{k}_t, z)$$

or, more concisely

$$ik_0 \left[\mathsf{C} - \frac{i}{k_0} \mathsf{U}_z \frac{d}{dz} \right] \mathbf{e}(\mathbf{k}_t, z) \equiv \mathsf{L}\mathbf{e}(\mathbf{k}_t, z) = -\mathbf{j}(\mathbf{k}_t, z) \tag{2.29}$$

where

$$\mathsf{C} := [c\mathsf{K} - s_x \mathsf{U}_x - s_y \mathsf{U}_y]$$

and L is the Maxwell operator:

$$\mathsf{L} := ik_0 \left[\mathsf{C} - \frac{i}{k_0} \mathsf{U}_z \frac{d}{dz} \right]$$

2.2.3 Eigenmodes in the plane-stratified medium

In order to find the eigenmodes of the plane-stratified medium we set the source term in (2.29) to zero

$$\mathsf{L}\mathbf{e} := ik_0 \left[\mathsf{C} - \frac{i}{k_0} \mathsf{U}_z \frac{d}{dz} \right] \mathbf{e}(\mathbf{k}_t, z) = 0 \tag{2.30}$$

and assume local plane-wave solutions

$$\mathbf{e}_\alpha(\mathbf{k}_t, z) = \mathbf{e}_\alpha(\mathbf{k}_t) \exp(-ik_0\, q_\alpha z) \tag{2.31}$$

to obtain the eigenmode equation

$$[\mathsf{C} - q_\alpha \mathsf{U}_z] \mathbf{e}_\alpha(\mathbf{k}_t, z) = 0 \tag{2.32}$$

Since there are two null rows and columns in U_z, the equation

$$\det[\mathsf{C} - q_\alpha \mathsf{U}_z] = 0 \tag{2.33}$$

gives a quartic equation in q_α (the Booker quartic (1.109), discussed in Sec. 1.3), yielding two positive- and two negative-going waves with respect to the z-axis, corresponding to $\alpha = \pm 1, \pm 2$. The eigenvectors $\mathbf{e}_\alpha$ give the characteristic wave polarizations (the ratios of the various wave-field components) corresponding to each eigenvalue q_α.

2.2.4 The Lagrange identity and the bilinear concomitant

We now construct the equation adjoint to (2.30) by changing the sign of the differential operator *d/dz* and replacing C by its transpose C^T (i.e. replacing K by K^T):

$$\bar{\mathsf{L}}\bar{\mathbf{e}} := i k_0 \left[\mathsf{C}^T + \frac{i}{k_0} \mathsf{U}_z \frac{d}{dz} \right] \bar{\mathbf{e}}(\mathbf{k}_t, z) = 0 \tag{2.34}$$

where $\bar{\mathsf{L}}$ is the adjoint Maxwell operator, and $\bar{\mathbf{e}}(\mathbf{k}_t, z)$ now denotes an *adjoint wave field*, satisfying the adjoint Maxwell equations.

We should note at this point that in the case of a cold magnetoplasma permeated by an external magnetic field $\mathbf{b}$, the constitutive tensor $\mathsf{K} \equiv \mathsf{K}(\mathbf{b})$ has the general form

$$\mathsf{K} = \begin{bmatrix} \boldsymbol{\varepsilon} & \mathbf{0} \\ \mathbf{0} & \mu_0 \mathsf{I}^{(3)} \end{bmatrix} =: \begin{bmatrix} \varepsilon_0(\mathsf{I}^{(3)} + \boldsymbol{\chi}) & \mathbf{0} \\ \mathbf{0} & \mu_0 \mathsf{I}^{(3)} \end{bmatrix} \tag{2.35}$$

defining, for later use, the susceptibility matrix $\boldsymbol{\chi}$; $\boldsymbol{\varepsilon} \equiv \boldsymbol{\varepsilon}(\mathbf{b})$(1.38) is given by

$$\frac{\boldsymbol{\varepsilon}}{\varepsilon_0} = S(\mathsf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}^T) - iD\bar{\mathbf{b}} \times \mathsf{I} + P\hat{\mathbf{b}}\hat{\mathbf{b}}^T, \qquad \hat{\mathbf{b}} := \mathbf{b}/|\mathbf{b}|$$

S, D and P being parameters of the medium, cf. (1.37), (1.40) and (1.39); $\boldsymbol{\varepsilon}(\hat{\mathbf{b}})$ is clearly gyrotropic, i.e. $\boldsymbol{\varepsilon}(-\mathbf{b}) = \boldsymbol{\varepsilon}^T(\mathbf{b})$, by virtue of the antisymmetric term $\hat{\mathbf{b}} \times \mathsf{I}$, and so too are $\mathsf{K}(\mathbf{b})$ (2.35) and $\mathsf{C}(\mathbf{b})$ (2.29),

$$\mathsf{K}(-\mathbf{b}) = \mathsf{K}^T(\mathbf{b}), \qquad \mathsf{C}(-\mathbf{b}) = \mathsf{C}^T(\mathbf{b}) \tag{2.36}$$

The given and adjoint operators L and $\bar{\mathsf{L}}$ will obey the *Lagrange identity*

$$\bar{\mathbf{e}}^T \mathsf{L}\mathbf{e} - \mathbf{e}^T \bar{\mathsf{L}}\bar{\mathbf{e}} = \nabla \cdot \mathbf{P} \tag{2.37}$$

where the vector $\mathbf{P}$ is called the *bilinear concomitant* [94, Sec. 7.5]. In our case, (2.34), the differential operator is just the z-component of ∇, and remembering that $\bar{\mathbf{e}}^T \mathsf{C}\mathbf{e}$ is a scalar which is equal to its transpose, and therefore eliminated on subtraction in (2.37), we find

$$\left[\bar{\mathbf{e}}^T \mathsf{U}_z \frac{d\mathbf{e}}{dz} + \mathbf{e}^T \mathsf{U}_z \frac{d\bar{\mathbf{e}}}{dz}\right] = \frac{d}{dz}\left[\bar{\mathbf{e}}^T \mathsf{U}_z \mathbf{e}\right]$$

which, with $\mathsf{L}\mathbf{e} = \bar{\mathsf{L}}\bar{\mathbf{e}} = 0$, (2.30) and (2.34), gives

$$\frac{d}{dz}\left[\bar{\mathbf{e}}^T \mathsf{U}_z \mathbf{e}\right] = 0 \tag{2.38}$$

Hence the (z-component of the) bilinear concomitant vector is a constant

$$\bar{\mathbf{e}}^T \mathsf{U}_z \mathbf{e} = P_z = \text{const} \tag{2.39}$$

an important result that we shall require later.

If we were to consider an arbitrary source-free medium, i.e. not necessarily plane-stratified, governed by (2.21) with $\mathbf{j}(\mathbf{r}) = 0$, we would write the formally adjoint equation as before [46, p. 234–236] by replacing K by its transpose K^T, and the differential operator D (2.22) by its negative transpose $-\mathsf{D}^T$. If therefore the Maxwell system, (2.21) or (2.24) with $\mathbf{j}(\mathbf{r}) = 0$, is given by

$$\mathsf{L}\mathbf{e} := [i\omega\mathsf{K} + \mathsf{D}]\mathbf{e}(\mathbf{r}) = \left[i\omega\mathsf{K} + \mathsf{U}_x\frac{\partial}{\partial x} + \mathsf{U}_y\frac{\partial}{\partial y} + \mathsf{U}_z\frac{\partial}{\partial z}\right]\mathbf{e}(\mathbf{r}) = 0 \tag{2.40}$$

the formally adjoint equation, with $\mathsf{D}^T = \mathsf{D}$, (2.22), will be

$$\bar{\mathsf{L}}\bar{\mathbf{e}} := [i\omega\mathsf{K}^T - \mathsf{D}^T]\bar{\mathbf{e}}(\mathbf{r}) = \left[i\omega\mathsf{K}^T - \mathsf{U}_x\frac{\partial}{\partial x} - \mathsf{U}_y\frac{\partial}{\partial y} - \mathsf{U}_z\frac{\partial}{\partial z}\right]\bar{\mathbf{e}}(\mathbf{r}) = 0 \tag{2.41}$$

Application of the Lagrange identity, (2.37), then yields the result

$$\nabla \cdot \mathbf{P} = 0, \qquad \mathbf{P} = \mathbf{E} \times \overline{\mathbf{H}} + \overline{\mathbf{E}} \times \mathbf{H} \tag{2.42}$$

and the expression $\bar{\mathbf{e}}^T \mathsf{U}_z \mathbf{e}$ appearing in (2.39) is seen to be the z-component of the Poynting-like product in (2.42). This bilinear concomitant vector $\mathbf{P}$ was introduced by Budden and Jull [35] in their study of reciprocity of ray paths in magnetoionic media, and a variant of it was used by Pitteway and Jespersen [100] to derive their reciprocity theorem discussed in Sec. 3.2.4.

It should be remarked that the above prescription ($\mathsf{K} \to \mathsf{K}^T,\ \mathsf{D} \to -\mathsf{D}^T$) for forming the adjoint system is not unique, and any other prescription that will satisfy a Lagrange identity like (2.37) is equally valid. The particular form chosen by us, yields a bilinear concomitant vector $\mathbf{P}$ (2.42) which reduces to the time-averaged Poynting vector in loss-free media (see Sec. 2.3.1), and is particularly useful in the applications discussed in this and the next chapters. Other prescriptions may be formulated by certain orthogonal transformations of the adjoint Maxwell system.

They have been used by Kong and Cheng [84] and by Kerns [81], and are useful in analysing Lorentz-type reciprocity when the waves which are compared travel in opposite directions (i.e. when the wave vectors are reversed in **k**-space, or the roles of receiving and transmitting antennas are interchanged in real space). Such transformed adjoint systems are introduced in Sec. 3.4, and discussed in some detail in Chaps. 4 and 6.

2.2.5 Biorthogonality of the given and adjoint eigenmodes

We now derive another important result that links the given and adjoint eigenvectors. Assuming local plane-wave solutions to the adjoint equation (2.34) of the form

$$\bar{\mathbf{e}}_\beta(\mathbf{k}_t, z) = \bar{\mathbf{e}}_\beta(\mathbf{k}_t) \exp(i k_0\, \bar{q}_\beta\, z) \tag{2.43}$$

we obtain the adjoint eigenmode equation

$$[\mathbf{C}^T - \bar{q}_\beta \mathbf{U}_z]\;\; \bar{\mathbf{e}}_\beta(\mathbf{k}_t, z) = 0 \tag{2.44}$$

The eigenvalues are determined by

$$\det\;\; [\mathbf{C}^T - \bar{q}_\beta \mathbf{U}_z] = 0 \tag{2.45}$$

which is seen to give the same quartic equation in q as (2.33). Hence the given and adjoint eigenvalues are identical

$$\bar{q}_\beta = q_\beta \tag{2.46}$$

This implies that the given and adjoint modal refractive indices are also equal

$$\bar{n}_\beta(s_x, s_y, \bar{q}_\beta) = n_\beta(s_x, s_y, q_\beta) \tag{2.47}$$

Note however that q_β and $\bar{q}_\beta$ appear in the plane-wave representations (2.31) and (2.43) with opposite signs, but since both representations have the same $\exp(i\omega t)$ time dependence, this means that the given and adjoint waves propagate in opposite directions with respect to the z-axis (but of course in the same transverse direction, since $\bar{\mathbf{k}}_t = \mathbf{k}_t$).

Again applying the Lagrange identity (2.37) to the eigenmode equations (2.32) and (2.44), and remembering that $\bar{q}_\beta = q_\beta$, we find that

$$(q_\beta - q_\alpha)\bar{\mathbf{e}}_\beta^T \mathbf{U}_z\, \mathbf{e}_\alpha = 0 \tag{2.48}$$

which gives the well known biorthogonality relation [18, 53, 105] between the given and adjoint eigenmodes

$$\bar{\mathbf{e}}_\beta^T \mathbf{U}_z\, \mathbf{e}_\alpha = \text{const}\, \delta_{\alpha\beta} = \delta_{\alpha\beta}\;\, P_{z,\alpha} \tag{2.49}$$

with the aid of (2.39). If the eigenmodes are suitably normalized this relation may be written

$$\hat{\tilde{\mathbf{e}}}_\beta^T \, \mathbf{U}_z \, \hat{\mathbf{e}}_\alpha = \delta_{\alpha\beta} \, \mathrm{sgn}(\alpha), \qquad \alpha, \beta = \pm 1, \pm 2 \tag{2.50}$$

To discuss the nature of the normalization, it will be necessary to define the *amplitude* of an eigenmode, and this will be crucial to the scattering theorems which will be derived later, in which ingoing and outgoing eigenmode amplitudes will be related.

2.3 The amplitude of an eigenmode

2.3.1 Amplitude in a loss-free medium

In discussing the propagation of a characteristic (eigen-) mode in a plane-stratified medium in which there are no collisional losses, it is useful to define the modal amplitude as the square root of the z-component (normal to the stratification) of the time-averaged Poynting vector (see, for instance, [100, 126]). If the medium varies slowly, so that there are no losses due to reflection or to mode coupling, it will be shown that the amplitude is conserved, i.e. it will remain constant even though the parameters of the medium vary in the direction normal to the stratification.

The time-averaged Poynting vector $\langle \mathbf{S} \rangle$ is given by

$$\langle \mathbf{S} \rangle = \mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H} \tag{2.51}$$

aside from a factor 1/4 which we have absorbed into $\langle \mathbf{S} \rangle$, and its z-component is given by

$$\langle S_z \rangle = \tilde{\mathbf{e}}^* \mathbf{U}_z \, \mathbf{e} \tag{2.52}$$

Now the complex-conjugate wave field $\mathbf{e}^*$ obeys an equation given by the complex conjugate of (2.30)

$$\left[\mathbf{C}^* + \frac{i}{k_0} \mathbf{U}_z \frac{d}{dz} \right] \mathbf{e}^*(\mathbf{k}_t, z) = \left[\mathbf{C}^T + \frac{i}{k_0} \mathbf{U}_z \frac{d}{dz} \right] \mathbf{e}^*(\mathbf{k}_t, z) = 0 \tag{2.53}$$

since the dielectric tensor $\boldsymbol{\varepsilon}$ in $\mathbf{C}$ is hermitian. [The hermiticity of $\boldsymbol{\varepsilon}$, or of $\mathbf{K}$, can be shown to stem from the requirement of energy conservation (see for instance [1, p. 9] or [113, p. 65]) and conversely, as in the present discussion, will be shown to lead to energy conservation].

Similarly, the complex-conjugate eigenwave field for a progressive plane wave (q_α real) becomes, from (2.32)

$$[\mathbf{C}^* - q_\alpha \mathbf{U}_z] \mathbf{e}^* = [\mathbf{C}^T - q_\alpha \mathbf{U}_z] \mathbf{e}^* = 0 \tag{2.54}$$

Thus the complex-conjugate wave fields in loss-free media obey the adjoint Maxwell equations, and eqs. (2.38) and (2.39) will apply here too, with

$$\tilde{\mathbf{e}}^* \mathsf{U}_z \mathbf{e} = P_z = \text{const} \tag{2.55}$$

Comparison with (2.52) gives

$$\langle S_z \rangle = P_z = \text{const} \tag{2.56}$$

so that in loss-free media the (z-component of the) bilinear concomitant is seen to be just the (z- component of the) mean Poynting vector, as already noted by Budden and Jull [35] and others [12,118,119], which expresses conservation of mean energy flux. Analogy with (2.49) also gives the biorthogonality of the given and complex-conjugate eigenmodes in loss-free media

$$\tilde{\mathbf{e}}^*_\alpha \mathsf{U}_z \mathbf{e}_\beta = \delta_{\alpha\beta}\, P_{z,\alpha} \tag{2.57}$$

We could now define a modal amplitude (or at least its modulus) by equating its square to the modal energy flux

$$\tilde{\mathbf{e}}^*_\alpha \mathsf{U}_z \mathbf{e}_\beta = \text{sgn}(\alpha)\delta_{\alpha\beta}\, |\boldsymbol{a}_\alpha|^2 \tag{2.58}$$

and then define normalized modal wave fields, $\hat{\mathbf{e}}_\alpha$ or $\hat{\mathbf{e}}^\star_\alpha$, by dividing the given fields by the modulus of the amplitudes:

$$\mathbf{e}_\alpha = |a_\alpha|\, \hat{\mathbf{e}}_\alpha \qquad\qquad \mathbf{e}^*_\alpha = |a_\alpha|\, \hat{\mathbf{e}}^*_\alpha$$

thereby letting the normalized wave fields carry the phase information of the given fields. Such a procedure is manifestly unsatisfactory, in that a normalized wave field would not be uniquely defined at a given level, and it is preferable to let the complex amplitude carry the phase information by letting it have the same phase as one of the components of $\mathbf{e}_\alpha$, say $\mathbf{e}_x$ or $\mathbf{e}_\xi$ (depending on the coordinate system in which the components of $\mathbf{e}_\alpha$ are expressed). We then have

$$\mathbf{e}_\alpha = a_\alpha \hat{\mathbf{e}}_\alpha \qquad\qquad \mathbf{e}^*_\alpha = a^*_\alpha\, \hat{\mathbf{e}}^*_\alpha \tag{2.59}$$

In either case a normalized modal wave field is that which generates unit energy flux normal to the stratification:

$$\left(\hat{\mathbf{e}}^*_\alpha\right)^T \mathsf{U}_z \hat{\mathbf{e}}_\beta = \text{sgn}(\alpha)\delta_{\alpha\beta} \tag{2.60}$$

Now an arbitrary wave field $\mathbf{e}(\mathbf{k}_t)$ can be expressed as a linear superposition of the eigenmodes of the medium

$$\mathbf{e}(\mathbf{k}_t) = \sum_{\alpha=\pm1,\pm2} a_\alpha \hat{\mathbf{e}}_\alpha(\mathbf{k}_t) \tag{2.61}$$

where, by virtue of (2.60),

$$a_\alpha = \left(\hat{\mathbf{e}}_\alpha^*\right)^T \mathbf{U}_z\, \mathbf{e}\, \mathrm{sgn}(\alpha), \qquad a_\alpha^* = (\hat{\mathbf{e}}_\alpha)^T \mathbf{U}_z \mathbf{e}^* \mathrm{sgn}(\alpha) \tag{2.62}$$

and hence, with $\mathbf{e} = \mathbf{e}(\mathbf{k}_t)$,

$$\begin{aligned} \langle S_z \rangle &= \tilde{\mathbf{e}}^* \mathbf{U}_z \mathbf{e} = a_1^* a_1 + a_2^* a_2 - a_{-1}^* a_{-1} - a_{-2}^* a_{-2} \\ &= \sum_\alpha |a_\alpha|^2\, \mathrm{sgn}(\alpha) \end{aligned} \tag{2.63}$$

We have thus expressed the energy flux normal to the stratification of an arbitrary wave field as the algebraic sum of the energy fluxes of the component eigenmodes.

Results analogous to those derived in this section (modal orthogonality in loss-free media and separation of overall energy flux into contributions of the component eigenmodes) have been given by Marcuse [92, Sec. 8.5] in his discussion of optical fibres and dielectric waveguides having cylindrical symmetry. There the form of the modes is dictated by the geometry of the problem (i.e. by the boundary conditions) and by a radiation condition at infinity, but the formalism is somewhat similar. In Sec. 2.6 we discuss the problem of curved stratified media in some detail.

Suppose we wish to determine the z-component of the energy flux associated with an eigenmode in a loss-free magnetoplasma. One method (not necessarily the simplest) would be to determine the eigenmode components (the wave polarizations) in the (ξ, η, ζ) coordinate system (1.77), in which the ζ-axis is along the wave-normal direction and the ξ-axis is in the plane spanned by the wave normal and the external magnetic field, cf. (1.82)–(1.85) in Sec. 1.2,

$$\begin{aligned} \mathbf{E} &:= (E_\xi,\ E_\eta,\ E_\zeta) = (1,\ \rho,\ \sigma)\, E_\xi \\ \mathbf{H} &:= (H_\xi,\ H_\eta,\ H_\zeta) = Y_0(-\rho,\ 1,\ 0) n E_\xi \end{aligned} \tag{2.64}$$

where ρ is purely imaginary and σ purely real, as may be seen from (1.81) in Sec. 1.2 with S, P, D and n^2 all real. $Y_0 \equiv 1/Z_0 := (\varepsilon_0/\mu_0)^{1/2}$ is the free-space admittance. The mean Poynting vector becomes

$$\begin{aligned} \langle S \rangle &= Y_0 \{-(\sigma + \sigma^*),\ -(\rho^* \sigma + \rho \sigma^*),\ 2(1 + \rho\rho^*)\} n E_\xi^* E_\xi \\ &= 2Y_0 \{-\sigma,\ 0,\ 1 - \rho^2\} n E_\xi^* E_\xi \end{aligned} \tag{2.65}$$

and the z-component of $\langle \mathbf{S} \rangle$, as well as the components of $\mathbf{e}_\alpha$ if required, are then determined by a coordinate transformation from the (ξ, η, ζ) to the (x, y, z) system (1.80).

2.3.2 *Amplitude of an eigenmode in the general case*

Normalization of the wave fields

For lossy media the constitutive tensors are no longer hermitian, and the orthogonality of eigenmodes with respect to the complex-conjugate modes is thereby lost, together with the manifest advantage of being able to express modal amplitudes via the complex-conjugate wave fields.

The *adjoint* wave fields, however, retain their biorthogonality with respect to the given fields (2.49), and we may use this property in the definition of modal amplitudes which will be valid for lossy media too. The constant bilinear concomitant $P_z = \bar{\mathbf{e}}^T \mathbf{U}_z \mathbf{e}$ (2.39) evidently no longer represents the z-component of the mean Poynting vector if absorption is present, since the energy flux would attenuate in the direction of propagation of the wave. The point is that the amplitude of an eigenmode, a_α, is no longer equal in magnitude to the amplitude, $\bar{a}_\alpha$, of the adjoint eigenmode since the constancy of the Poynting cross product (2.49) implies that as $\mathbf{e}_\alpha$ attenuates, $\bar{\mathbf{e}}_\alpha$ will grow correspondingly.

To obtain the adjoint eigenmode components it is convenient to express field quantities in the (ξ, η, ζ) system, as in (2.64). In a magnetoplasma, as pointed out in Sec. 2.2.4, the adjoint medium is obtained by reversing the direction of the external magnetic field $\mathbf{b}$, so that the transverse wave polarization $\rho := E_\eta / E_\xi$ (1.84) in this system is reversed in sign, while the longitudinal polarization $\sigma := E_\zeta / E_\xi$ (1.85) is unchanged, cf. (1.78),

$$\bar{\rho}_\alpha = -\rho_\alpha, \qquad \bar{\sigma}_\alpha = \sigma_\alpha \tag{2.66}$$

(the corresponding relations for the complex-conjugate polarizations

$$\rho^*_\alpha = -\rho_\alpha, \qquad \sigma^*_\alpha = \sigma_\alpha$$

are valid only in loss-free media). Hence, if the modal field

$$\mathbf{e}_\alpha \equiv (E_\xi,\ E_\eta,\ E_\zeta;\ H_\xi,\ H_\eta,\ H_\zeta)_\alpha$$

has the form

$$\mathbf{E}_\alpha = (1,\ \rho_\alpha,\ \sigma_\alpha)\, E_{\alpha\xi}, \qquad \mathbf{H}_\alpha = Y_0(-\rho_\alpha,\ 1,\ 0) n_\alpha\, E_{\alpha\xi} \tag{2.67}$$

the adjoint field will be

$$\bar{\mathbf{E}}_\alpha = (1,\ -\rho_\alpha,\ \sigma_\alpha)\, \bar{E}_{\alpha\xi}, \qquad \bar{\mathbf{H}}_\alpha = Y_0(\rho_\alpha,\ 1,\ 0) n_\alpha\, \bar{E}_{\alpha\xi} \tag{2.68}$$

and we may form the bilinear concomitant vector **P** from the Poynting-like product

$$\begin{aligned}\mathbf{P}_\alpha &= \mathbf{E}_\alpha \times \bar{\mathbf{H}}_\alpha + \bar{\mathbf{E}}_\alpha \times \mathbf{H}_\alpha \\ &= 2Y_0 \left(-\sigma_\alpha,\ 0,\ 1-{\rho_\alpha}^2\right) n_\alpha \bar{E}_{\alpha\xi}\, E_{\alpha\xi}\end{aligned} \tag{2.69}$$

which is formally identical to (2.65), except that both σ and ρ^2 may now be complex. The z-component may now be obtained by a coordinate transformation (1.80) to the (x, y, z) system

$$P_{z,\alpha} = \bar{\mathbf{e}}_\alpha^T \mathbf{U}_z \mathbf{e}_\alpha = 2Y_0 \left(-\sigma_\alpha,\ 0,\ 1-{\rho_\alpha}^2\right)_z n_\alpha \bar{E}_{\alpha\xi}\, E_{\alpha\xi} \tag{2.70}$$

This is a convenient representation to use for normalizing eigenmodes. If we choose

$$\hat{E}_{\alpha\xi} = \hat{\bar{E}}_{\alpha\xi} = \{2Y_0 n_\alpha \mathrm{sgn}(\alpha)(-\alpha_\alpha,\ 0,\ 1-{\rho_\alpha}^2)_z\}^{-\frac{1}{2}} \tag{2.71}$$

we can define the normalized eigenfields through (2.67) and (2.68)

$$\begin{aligned}\hat{\mathbf{e}}_\alpha &= (1,\ \rho_\alpha,\ \sigma_\alpha;\ -Y_0 n_\alpha \rho_\alpha,\ Y_0 n_\alpha,\ 0)\hat{E}_{\alpha\xi} \\ \hat{\bar{\mathbf{e}}}_\alpha &= (1,\ -\rho_\alpha,\ \sigma_\alpha;\ Y_0 n_\alpha \rho_\alpha,\ Y_0 n_\alpha,\ 0)\hat{\bar{E}}_{\alpha\xi}\end{aligned} \tag{2.72}$$

which yield immediately the required biorthogonality normalization

$$\left(\hat{\bar{\mathbf{e}}}_\alpha\right)^T \mathbf{U}_z \hat{\mathbf{e}}_\beta = \delta_{\alpha\beta}\mathrm{sgn}(\alpha) \tag{2.73}$$

Eigenmode amplitudes

We now relate a modal wave field to a normalized field via the modal amplitude, as in the previous section,

$$\mathbf{e}_\alpha = a_\alpha \hat{\mathbf{e}}_\alpha, \qquad \bar{\mathbf{e}}_\alpha = \bar{a}_\alpha \hat{\bar{\mathbf{e}}}_\alpha \tag{2.74}$$

so that (2.49) becomes

$$\bar{\mathbf{e}}_\alpha^T \mathbf{U}_z \mathbf{e}_\beta = \mathrm{sgn}(\alpha)\delta_{\alpha\beta}\bar{a}_\alpha a_\alpha \tag{2.75}$$

Now an arbitrary wave field $\mathbf{e}(\mathbf{k}_t)$, as well as its adjoint, can be expressed in terms of the eigenmodes

$$\mathbf{e}(\mathbf{k}_t) = \sum_{\alpha=\pm1,\pm2} a_\alpha \hat{\mathbf{e}}_\alpha(\mathbf{k}_t), \qquad \bar{\mathbf{e}}(\mathbf{k}_t) = \sum_{\alpha=\pm1,\pm2} \bar{a}_\alpha \hat{\bar{\mathbf{e}}}_\alpha(\mathbf{k}_t) \tag{2.76}$$

whence

$$a_\alpha = \left(\hat{\bar{\mathbf{e}}}_\alpha\right)^T \mathbf{U}_z \, \mathbf{e} \, \mathrm{sgn}(\alpha), \qquad \bar{a}_\alpha = \hat{\mathbf{e}}_\alpha^T \mathbf{U}_z \, \bar{\mathbf{e}} \, \mathrm{sgn}(\alpha) \tag{2.77}$$

by virtue of (2.73) and (2.76).

Finally we have the generalized Poynting flux density (2.39)

$$\begin{aligned} P_z &= [\mathbf{E} \times \bar{\mathbf{H}} + \bar{\mathbf{E}} \times \mathbf{H}]_z = \bar{\mathbf{e}}^T \mathbf{U}_z \mathbf{e} \\ &= \bar{a}_1 a_1 + \bar{a}_2 a_2 - \bar{a}_{-1} a_{-1} - \bar{a}_{-2} a_{-2} \\ &= \sum_\alpha \bar{a}_\alpha \, a_\alpha \, \mathrm{sgn}(\alpha) = \mathrm{const} \end{aligned} \tag{2.78}$$

expressed as the sum of the generalized flux densities of the eigenmodes, by analogy with (2.63).

We remark in conclusion that the procedure adopted here for determining an eigenmode amplitude in an absorbing medium may seem somewhat cumbersome, but it is straightforward and easily incorporated into a computer program for calculating wave fields in plane-stratified media. In most cases of practical interest the aim of such calculations is to determine fields or scattering coefficients outside the absorbing regions, where the squares of the modal amplitudes reduce simply to the z-components (normal to the stratification) of the Poynting flux of each mode. For our purposes, however, the important result is that modal amplitudes can *in principle* be defined in absorbing (and hence in all) media which, in conjunction with modal biorthogonality, permits the decomposition of generalized energy flux into the sum of the contributions of each of the eigenmodes.

2.4 The conjugate wave fields

2.4.1 The physical content of the conjugate problem

In our review in Sec. 2.1 we noted that earlier scattering (reciprocity) theorems for plane-stratified magnetoplasmas related the ingoing and outgoing amplitudes of waves incident from two different directions — the given and *conjugate* directions. If the transverse components (i.e. in the plane of the statification) of the incident wave vector are $\mathbf{k}_t = k_0(s_x, s_y)$, those of the conjugate wave vector are defined to be $\mathbf{k}_t^c = k_0(-s_x, s_y)$ see Fig. 2.2. To characterize the relation between the given and conjugate wave vectors geometrically, Barron and Budden [21], Pitteway and Jespersen [100] and others, when discussing incidence on the earth's ionosphere from below, have pointed out that the planes of incidence in the two cases are 'symmetrically disposed about the vertical East–West plane, at right angles to the magnetic meridian plane', i.e. if the plane of incidence in the one case is at an azimuthal angle ϕ with respect to the meridian plane, the conjugate plane of incidence is at an angle $(\pi - \phi)$.

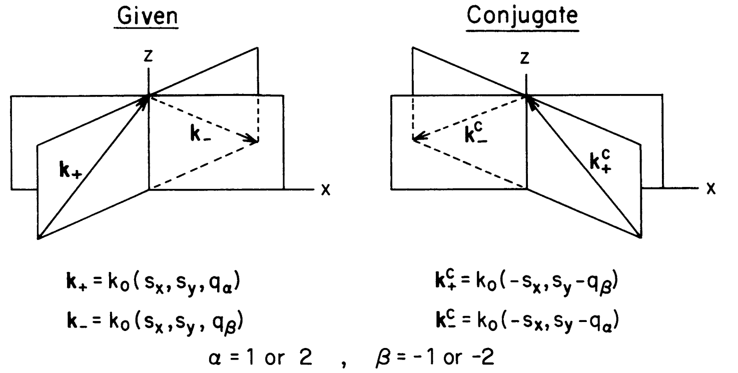


Fig. 2.2 Given and conjugate eigenmodes. The z-axis is normal to the stratification, and the external magnetic field lies in the $(x,\ z)$ plane (the magnetic meridian plane)

This characterization, although perfectly true, concealed the physical nature of the symmetry. With the hindsight provided by a number of later papers, [9, 11, 108], we note that the given and conjugate planes of incidence are reflections with respect to the *magnetic meridian plane*. But this is only part of the story.

If we take the original (given) problem and perform a reflection mapping, $\mathcal{R}$, with respect to some arbitrary plane, then all proper (polar) vectors, such as the position vector **r**, the electric field $\mathbf{E}(\mathbf{r})$ or the electric current density $\mathbf{J}_\mathrm{e}(\mathbf{r})$, undergo 'geometric mirroring', in the sense that physical arrows would be imaged by a mirror. All axial (pseudo-) vectors, on the other hand, such as the wave field $\mathbf{H}(\mathbf{r})$, the external magnetic field **b** or the equivalent magnetic current density $\mathbf{J}_\mathrm{m}(\mathbf{r})$ undergo mirroring too, but in addition, are *reversed in direction*. Such mappings will be considered in detail in Chap. 6. It is well known (and will be demonstrated in Sec. 6.2) that Maxwell's equations are invariant under such orthogonal mappings.

We now perform a time-reversal transformation, $\mathcal{T}$, on the reflected problem. The operation $\mathcal{T}$ can be visualized by imagining the original process to have been recorded on a movie film, and then observed when the film is run backwards. Maxwell's equations are invariant under time reversal, as will be demonstrated in Chap. 7, and it will be shown in particular that quantities such as **H**, **B**, $\mathbf{J}_\mathrm{e}$ and **S** (the Poynting vector) are odd (i.e. change sign) under time reversal, whereas **E, D** and $\mathbf{J}_\mathrm{m}$ are even. For our purposes this means that the combined action of $\mathcal{R}$ (with respect to the magnetic meridian plane) and $\mathcal{T}$ leaves the original external magnetic field **b** unchanged, i.e.

$$\mathcal{T}\mathcal{R}\mathbf{b} = \mathcal{R}\mathcal{T}\mathbf{b} = \mathbf{b}$$

and the mapped eigenmodes (i.e. reflected and time-reversed) will remain eigenmodes of the (unchanged) mapped medium.

Absorption losses in the medium require special attention. These will be expressed in the constitutive tensor K through an imaginary term $i\nu$, where ν is the effective collision frequency. Time reversal, as will be shown in Sec. 7.2, has the effect of changing the sign of the collision term, or to be more precise, converts the constitutive tensor into its complex conjugate, thereby changing the sign not only of ν, but of $\mathbf{b}$ which appears also as an imaginary term, $i\mathbf{b}$, in gyrotropic media. The effect of time reversal will then be to transform the eigenvalue q_α in the plane-wave representation, $\exp(-ik_0\, q_\alpha\, z)$ into its negative complex conjugate,

$$\mathcal{T} q_\alpha = -q_\alpha^*$$

so that a damped plane wave, propagating in the positive z-direction, would be transformed into a growing plane wave propagating in the negative z-direction. However, insofar as we wish to describe physical processes in a physical absorbing medium after applying our reflection-time-reversal transformation, we shall not transform the collision frequency. Under this *restricted time reversal* the wave eigenvectors $\mathbf{k}_\alpha$ will reverse their directions

$$\mathbf{k}_\alpha(s_x, s_y, q_\alpha) \rightarrow \mathbf{k}_{-\alpha}(-s_x, -s_y, -q_\alpha)$$

(but not $q_\alpha \rightarrow -q_\alpha^*$, which would yield growing waves), and the signs of the magnetic wave-field components will also be changed, leading to a reversal in direction of the Poynting vector.

This, then, is the rationale of the mathematical procedure (in itself quite rigorous) which will now be used to generate the 'conjugate eigenmodes' by a reflection-time-reversal transformation.

2.4.2 *The conjugating transformation*

The restricted time-reversal procedure

We start off by exhibiting explicitly the components, s_x and s_y, of $\mathbf{k}_t$ (2.26) in the eigenmode equation (2.32), as well as the dependence of K, and consequently of $\mathbf{e}_\alpha$, on the external magnetic field $\mathbf{b}$,

$$[c\mathcal{K}(\mathbf{b}) - s_x\mathsf{U}_x - s_y\mathsf{U}_y - q_\alpha\mathsf{U}_z]\;\; \mathbf{e}_\alpha(\mathbf{b}; s_x,\ s_y) = 0 \tag{2.79}$$

We reverse the direction of $\mathbf{b}$, so that the adjoint eigenmode equation, (2.44) in conjunction with (2.36), is satisfied by the field $\bar{\mathbf{e}}_\alpha(s_x, s_y)$, adjoint to $\mathbf{e}_\alpha(\mathbf{b};\ s_x, s_y)$

$$\mathsf{L}(-\mathbf{b})\bar{\mathbf{e}}_\alpha \equiv \bar{\mathsf{L}}\ \bar{\mathbf{e}}_\alpha := ik_0[c\mathsf{K}(-\mathbf{b}) - s_x\mathsf{U}_x - s_y\mathsf{U}_y - q_\alpha\mathsf{U}_z]\ \bar{\mathbf{e}}_\alpha(s_x,\ s_y) = 0 \tag{2.80}$$

where we have used the result (2.46), $\bar{q}_\alpha = q_\alpha$, and it will be remembered that $\bar{\mathbf{e}}_\alpha$ has the plane-wave ansatz $\exp(ik_0 q_\alpha z)$ (2.43).

We note that (2.80) is also satisfied by the Maxwell field $\mathbf{e}_\alpha(-\mathbf{b}; s_x, s_y)$, i.e. by a physical wave field in the magnetic-field reversed medium, that has exactly the same wave polarization as the adjoint mode, but of course a different z-dependence. This polarization, and specifically the relation between the **E** and **H** fields, prescribes the direction of the Poynting flux, which will be consistent with the direction imposed by the sign of $\Im m(q_\alpha)$ in the Maxwell eigenmode $\mathbf{e}_\alpha(-\mathbf{b}; s_x, s_y)$, but inconsistent with the direction of propagation of the (unphysical) adjoint eigenmode.

Next we apply the *Poynting-vector reversing operator* $\bar{\mathsf{I}}$

$$\bar{\mathsf{I}} \equiv \bar{\mathsf{I}}^{(6)} := \begin{bmatrix} \mathsf{I}^{(3)} & 0 \\ 0 & -\mathsf{I}^{(3)} \end{bmatrix} = \bar{\mathsf{I}}^{-1} = \bar{\mathsf{I}}^{T} \tag{2.81}$$

[the direction of the Poynting vector of the wave field $\bar{\mathsf{I}}\mathbf{e}$ is opposite to that of the field $\mathbf{e}$] to (2.80):

$$\bar{\mathsf{I}}[c\mathsf{K}(-\mathbf{b}) - s_x\mathsf{U}_x - s_y\mathsf{U}_y - q_\alpha\mathsf{U}_z]\ \bar{\mathsf{I}}\bar{\mathsf{I}}\mathbf{e}_\alpha(-\mathbf{b}; s_x, s_y) = 0 \tag{2.82}$$

in which, for clarity, $\bar{\mathbf{e}}_\alpha(s_x, s_y)$ has been replaced by $\mathbf{e}_\alpha(-\mathbf{b}; s_x, s_y)$. Noting that

$$\bar{\mathsf{I}}\mathsf{U}_i\bar{\mathsf{I}} = -\mathsf{U}_i \qquad (i = x, y, z),$$

[see (2.25)], and

$$\bar{\mathsf{I}}\mathsf{K}\bar{\mathsf{I}} = \mathsf{K}$$

when K is of the form given by (2.35), we get

$$[c\mathsf{K}(-\mathbf{b}) + s_x\mathsf{U}_x + s_y\mathsf{U}_y + q_\alpha\mathsf{U}_z]\ \bar{\mathsf{I}}\,\mathbf{e}_\alpha(-\mathbf{b}; s_x, s_y) = 0 \tag{2.83}$$

This completes the (restricted) time-reversal transformation of the Maxwell system (2.79), and we now proceed to reflect the system with respect to the magnetic-meridian plane.

Reflection of wave fields

In general a (polar-) vector field, such as $\mathbf{E}(\mathbf{r})$, will be mapped by reflection with respect to the magnetic meridian plane, $y = 0$, into $\mathbf{E}'(\mathbf{r}') = \mathcal{R}\mathbf{E}(\mathbf{r})$, where

$$\mathbf{E}'(\mathbf{r}') = \mathbf{q}_y\mathbf{E}(\mathbf{r}), \qquad \mathbf{r}' = \mathbf{q}_y\mathbf{r}, \qquad \mathbf{q}_y := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.84}$$

On the other hand an axial-vector field such as $\mathbf{H}(\mathbf{r})$ is, in addition, reversed in sign on reflection, so that the overall reflected electromagnetic field $\mathcal{R}\mathbf{e}(\mathbf{r}) \equiv \mathbf{e}'(\mathbf{r}')$ is given by

$$\mathbf{e}'(\mathbf{r}') = \mathbf{Q}_y\mathbf{e}(\mathbf{r}) \equiv \begin{bmatrix} \mathbf{q}_y & 0 \\ 0 & -\mathbf{q}_y \end{bmatrix} \begin{bmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{bmatrix}; \quad \mathbf{r}' = \mathbf{q}_y\mathbf{r} \tag{2.85}$$

We apply the reflection matrix $\mathbf{Q}_y = \mathbf{Q}_y^{-1}$ to (2.83)

$$\mathbf{Q}_y[c\mathbf{K}(-\mathbf{b}) - s_x\mathbf{U}_x - s_y\mathbf{U}_y - q_\alpha\mathbf{U}_z]\ \mathbf{Q}_y\ \{\mathbf{Q}_y\bar{\mathbf{I}}\mathbf{e}_\alpha(-\mathbf{b}; s_x, s_y)\} = 0 \tag{2.86}$$

and note that

$$\mathbf{Q}_y\mathbf{U}_x\mathbf{Q}_y = \mathbf{U}_x, \qquad \mathbf{Q}_y\mathbf{U}_y\mathbf{Q}_y = -\mathbf{U}_y, \qquad \mathbf{Q}_y\mathbf{U}_z\mathbf{Q}_y = \mathbf{U}_z \tag{2.87}$$

Furthermore, if the magnetic field $\mathbf{b}$ is parallel to the $y = 0$ plane, then $\mathbf{K}$, given by (2.35), with $\boldsymbol{\varepsilon}$ given by (1.45),

$$\mathbf{K} = \begin{bmatrix} \boldsymbol{\varepsilon} & \mathbf{0} \\ \mathbf{0} & \mu_0\mathbf{I}^{(3)} \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \varepsilon_0 \begin{bmatrix} S - C\hat{b}_x^2 & iD\hat{b}_z & -C\hat{b}_x\hat{b}_z \\ -iD\hat{b}_z & S & iD\hat{b}_x \\ -C\hat{b}_x\hat{b}_z & -iD\hat{b}_x & S - C\hat{b}_z^2 \end{bmatrix} \tag{2.88}$$

is magnetic-field reversed by $\mathbf{Q}_y$:

$$\mathbf{Q}_y\mathbf{K}(\mathbf{b})\mathbf{Q}_y = \mathbf{K}(-\mathbf{b}) \tag{2.89}$$

Hence (2.86) becomes

$$\begin{aligned} &[c\mathbf{K}(\mathbf{b}) + s_x\mathbf{U}_x - s_y\mathbf{U}_y + q_\alpha\mathbf{U}_z]\ \mathbf{Q}_y\ \bar{\mathbf{I}}\ \mathbf{e}_\alpha(-\mathbf{b}; s_x, s_y) \\ &\quad = [\mathbf{C}(\mathbf{b}; -s_x, s_y) - q^c_{-\alpha}\mathbf{U}_z]\ \mathbf{e}^c_{-\alpha} = 0 \end{aligned} \tag{2.90}$$

with the notation of (2.29), and we have thereby formally identified the transformed (time-reversed, reflected) wave field as the *conjugate eigenmode*:

$$-q^c_{-\alpha} = q_\alpha = \bar{q}_\alpha, \qquad \mathbf{e}^c_{-\alpha}(\mathbf{b}; -s_x, s_y) = \mathbf{Q}_y\ \bar{\mathbf{I}}\ \mathbf{e}_\alpha(-\mathbf{b}; s_x, s_y) \tag{2.91}$$

with $q_\alpha = \bar{q}_\alpha$ taken from (2.46). In terms of the adjoint eigenmode this gives

$$\mathbf{e}^c_{-\alpha}(-s_x, s_y) = \mathbf{Q}_y\ \bar{\mathbf{I}}\ \mathbf{e}_\alpha(s_x, s_y) \equiv \mathbf{Q}^c_y\bar{\mathbf{e}}_\alpha(s_x, s_y) \tag{2.92}$$

where the diagonal matrix $\mathbf{Q}^c_y$ is given by

$$\mathbf{Q}^c_y \equiv \mathbf{Q}_y\ \bar{\mathbf{I}} = \begin{bmatrix} \mathbf{q}_y & 0 \\ 0 & \mathbf{q}_y \end{bmatrix}; \qquad \mathbf{q}_y := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.93}$$

with

$$\mathbf{Q}^c_y = \tilde{\mathbf{Q}}^c_y = [\mathbf{Q}^c_y]^{-1}$$

Since the adjoint operation is involutary, i.e. $\bar{\bar{\mathbf{e}}} = \mathbf{e}$, (2.92) may be written as

$$\bar{\mathbf{e}}^c_{-\alpha} = \mathbf{Q}^c_y \mathbf{e}_\alpha \tag{2.94}$$

determining the adjoint of a mode in the conjugate system. We note that the *conjugating matrix* $\mathbf{Q}^c_y$ imposes 'geometrical mirroring' on both polar and axial vectors, i.e. it does not reverse the sign of the reflected (axial-vector) wave fields. This leads to a reversal of the direction of the z-component of the Poynting vector, so that upgoing waves are transformed into downgoing.

The conjugate modal amplitudes

We now apply (2.92) and (2.94) to relate the normalized eigenvectors and their adjoints in the given and conjugate problems:

$$\hat{\mathbf{e}}^c_\alpha = \mathbf{Q}^c_y \hat{\bar{\mathbf{e}}}_{-\alpha}, \qquad \hat{\bar{\mathbf{e}}}_\alpha = \mathbf{Q}^c_y \hat{\mathbf{e}}_{-\alpha} \tag{2.95}$$

and use them, with the aid of (2.77), to determine the amplitudes of a conjugate eigenmode a^c_α and its adjoint $\bar{a}^c_\alpha$:

$$\begin{aligned} a^c_\alpha &= (\hat{\bar{\mathbf{e}}}^c_\alpha)^T\, \mathbf{U}_z\, \mathbf{e}^c\, \mathrm{sgn}(\alpha) = [\mathbf{Q}^c_y \hat{\mathbf{e}}_{-\alpha}]^T\, \mathbf{U}_z\, [\mathbf{Q}^c_y \bar{\mathbf{e}}]\, \mathrm{sgn}(\alpha) \\ &= -\hat{\mathbf{e}}^T_\alpha \mathbf{U}_z\, \bar{\mathbf{e}}\, \mathrm{sgn}(\alpha) \end{aligned} \tag{2.96}$$

since

$$\mathbf{Q}^c_y \mathbf{U}_z \mathbf{Q}^c_y = -\mathbf{U}_z$$

and hence, with $-\mathrm{sgn}(\alpha) = \mathrm{sgn}(-\alpha)$, we find

$$a^c_\alpha = \bar{a}_{-\alpha}, \qquad \bar{a}^c_\alpha = a_{-\alpha} \tag{2.97}$$

2.4.3 Resumé

Before proceeding let us retrace some of the relevant steps we have taken till now in this chapter. We considered a solution to Maxwell's equations in a plane-stratified medium, consisting of a set of eigenmodes having a common value of $\mathbf{k}_t$, the projection of the propagation vector $\mathbf{k}$ on the stratification plane, which is transverse to the z-axis. We constructed mathematically a set of adjoint eigenmodes, biorthogonal to the original set, and used the biorthogonality condition to define amplitudes of the given and adjoint eigenmodes at any level, z. Next, we performed a conjugating transformation (reflection and time reversal) of these eigenmodes to obtain a set of conjugate eigenmodes which was shown to be a solution of Maxwell's equations (or the adjoint equations) in the conjugate problem, in which the plane of incidence is a mirror image with respect to the magnetic meridian plane of the original plane of incidence. Finally, a simple relation was found between the eigenmode amplitudes in the given and conjugate problems, which will be required in the next section to derive the scattering theorem.

The reader may well ask why we are using this somewhat elaborate conjugating transformation, when we could have reached the same end result by a more direct transformation which maps $\mathbf{k}_t(s_x, s_y)$ into $\mathbf{k}_t^c(-s_x, s_y)$, as will indeed be demonstrated in Sec. 3.2.5. The reason is that the method described is much more general in its scope than that used in the special case of planar stratification, and will be applied in later chapters to problems possessing quite general spatial symmetries.

2.5 The eigenmode scattering theorem

2.5.1 The scattering matrix

The motivation for most numerical or analytic calculations of wave propagation through a plane-stratified medium, is to derive eventually the reflection, transmission and intermode coupling coefficients for plane-wave incidence from either end. These coefficients are conveniently grouped into the scattering matrix S.

Let $\hat{\mathbf{e}}_\alpha(z), \alpha = \pm1, \pm2$, represent one of the 6-component normalized eigenvectors, defined in Sec. 2.3.2, at a level z, for positive- or negative-going characteristic waves propagating in a plane-stratified medium, with equal prescribed values of the transverse wave vector $\mathbf{k}_t = k_0(s_x, s_y)$; $\hat{\bar{\mathbf{e}}}_\alpha$ is the corresponding normalized adjoint eigenmode. The overall wave fields, $\mathbf{e}(z)$ and $\bar{\mathbf{e}}(z)$, at any level may be decomposed into the respective eigenvectors $\mathbf{e}_\alpha$, or their adjoints $\bar{\mathbf{e}}_\alpha$, as in (2.76) and (2.78)

$$\mathbf{e} = \sum_\alpha a_\alpha \hat{\mathbf{e}}_\alpha, \qquad \bar{\mathbf{e}}_\alpha = \sum_\alpha \bar{a}_\alpha \hat{\bar{\mathbf{e}}}_\alpha \tag{2.98}$$

where

$$a_\alpha = \hat{\bar{\mathbf{e}}}_\alpha^T \mathsf{U}_z \mathbf{e}\ \ \mathrm{sgn}(\alpha) \qquad \bar{a}_\alpha = \hat{\mathbf{e}}_\alpha^T \mathsf{U}_z\, \bar{\mathbf{e}}\, \mathrm{sgn}(\alpha) \tag{2.99}$$

It will be convenient to replace the summation representation in (2.98) by matrix notation:

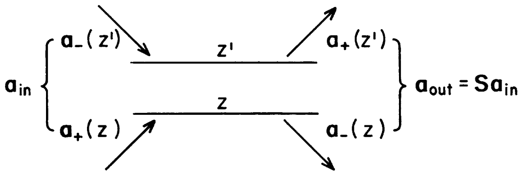
$$\mathbf{e} = \mathbf{E}_+ \boldsymbol{a}_+ + \mathbf{E}_- \boldsymbol{a}_- = \mathbf{E}\,\boldsymbol{a}, \qquad \bar{\mathbf{e}} = \bar{\mathbf{E}}\,\bar{\boldsymbol{a}} \tag{2.100}$$

where

$$\begin{gathered}\mathsf{E}_\pm = [\hat{\mathbf{e}}_{\pm1}\ \hat{\mathbf{e}}_{\pm2}], \quad \mathsf{E} = [\mathsf{E}_+\ \mathsf{E}_-] = [\hat{\mathbf{e}}_1\ \hat{\mathbf{e}}_2\ \hat{\mathbf{e}}_{-1}\ \hat{\mathbf{e}}_{-2}] \\ \boldsymbol{a}_\pm = \begin{bmatrix} a_{\pm1} \\ a_{\pm2} \end{bmatrix}, \quad \boldsymbol{a} = \begin{bmatrix} \boldsymbol{a}_+ \\ \boldsymbol{a}_- \end{bmatrix} = [a_1\ a_2\ a_{-1}\ a_{-2}]^T \end{gathered} \tag{2.101}$$

with adjoint quantities similarly defined.

Fig. 2.3 Incoming and outgoing eigenmodes related by the scattering matrix $\mathsf{S}(z, z')$



Now consider the wave amplitudes $\boldsymbol{a}(z)$ and $\boldsymbol{a}(z')$ at two levels, z and z', with $z' > z$. In terms of the wave amplitudes $\boldsymbol{a}_\pm$ at z and z', we write in condensed notation

$$\boldsymbol{a}_{in} = \begin{bmatrix} \boldsymbol{a}_+(z) \\ \boldsymbol{a}_-(z') \end{bmatrix}, \qquad \boldsymbol{a}_{out} = \begin{bmatrix} \boldsymbol{a}_-(z) \\ \boldsymbol{a}_+(z') \end{bmatrix} \tag{2.102}$$

(see Fig. 2.3), and define the scattering matrix $\mathsf{S} = \mathsf{S}(s_x, s_y; z, z')$, and its adjoint $\bar{\mathsf{S}} = \bar{\mathsf{S}}(s_x, s_y; z, z')$, by means of

$$\boldsymbol{a}_{out} = \mathsf{S}\,\boldsymbol{a}_{in}, \qquad \bar{\boldsymbol{a}}_{out} = \bar{\mathsf{S}}\,\bar{\boldsymbol{a}}_{in} \tag{2.103}$$

Written out in full, in terms of the 2×2 reflection and transmission matrices, $\mathsf{R}_\pm$ and $\mathsf{T}_\pm$, this becomes

$$\begin{bmatrix} \boldsymbol{a}_-(z) \\ \boldsymbol{a}_+(z') \end{bmatrix} = \begin{bmatrix} \mathsf{R}_+(z) & \mathsf{T}_-(z, z') \\ \mathsf{T}_+(z', z) & \mathsf{R}_-(z') \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_+(z) \\ \boldsymbol{a}_-(z') \end{bmatrix} \tag{2.104}$$

2.5.2 *Derivation of the eigenmode scattering theorem*

Relation between given and adjoint scattering matrices

Our derivation is based on the constancy of the bilinear concomitant vector, (2.39) and (2.78),

$$P_z = \bar{\mathbf{e}}^T \mathsf{U}_z \mathbf{e} = \sum_\alpha \bar{a}_\alpha a_\alpha \operatorname{sgn}(\alpha) = \text{const} \tag{2.105}$$

Applying this result to the modal amplitudes at z' and z, we have

$$\bar{\boldsymbol{a}}_+^T(z')\boldsymbol{a}_+(z') - \bar{\boldsymbol{a}}_-^T(z')\boldsymbol{a}_-(z') = \bar{\boldsymbol{a}}_+^T(z)\boldsymbol{a}_+(z) - \bar{\boldsymbol{a}}_-^T(z)\boldsymbol{a}_-(z)$$

and, regrouping

$$\bar{\boldsymbol{a}}_+^T(z')\;\boldsymbol{a}_+(z') + \bar{\boldsymbol{a}}_-^T(z)\,\boldsymbol{a}_-(z) = \bar{\boldsymbol{a}}_+^T(z)\;\boldsymbol{a}_+(z) + \bar{\boldsymbol{a}}_-^T(z')\;\boldsymbol{a}_-(z') \tag{2.106}$$

so that, with (2.102)

$$\bar{\boldsymbol{a}}_{out}^T \boldsymbol{a}_{out} = \bar{\boldsymbol{a}}_{in}^T \boldsymbol{a}_{in} \tag{2.107}$$

Application of (2.103) yields

$$\bar{\boldsymbol{a}}_{in}^T \bar{\mathbf{S}}^T \,\mathbf{S}\, \boldsymbol{a}_{in} = \bar{\boldsymbol{a}}_{in}^T \boldsymbol{a}_{in}$$

and finally

$$\bar{\mathbf{S}}^T \,\mathbf{S} = \mathbf{I}^{(4)} = \mathbf{S}\, \bar{\mathbf{S}}^T \tag{2.108}$$

since $\bar{\mathbf{S}}^T = \mathbf{S}^{-1}$.

Relation between adjoint, conjugate and given scattering matrices

When the eigenmodes in Fig. 2.3 undergo a conjugating transformation, the incoming and outgoing amplitudes are correspondingly transformed, (2.97), to yield

$$\boldsymbol{a}_{\pm}^c = \begin{bmatrix} \boldsymbol{a}_{\pm 1}^c \\ \boldsymbol{a}_{\pm 2}^c \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{a}}_{\mp 1} \\ \bar{\boldsymbol{a}}_{\mp 2} \end{bmatrix} = \bar{\boldsymbol{a}}_{\mp} \tag{2.109}$$

$$\boldsymbol{a}_{in}^c = \begin{bmatrix} \boldsymbol{a}_{+}^c(z) \\ \boldsymbol{a}_{-}^c(z') \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{a}}_{-}(z) \\ \bar{\boldsymbol{a}}_{+}(z') \end{bmatrix} = \boldsymbol{a}_{out}, \qquad \boldsymbol{a}_{out}^c = \bar{\boldsymbol{a}}_{in} \tag{2.110}$$

and since, by (2.103)

$$\bar{\boldsymbol{a}}_{out} = \bar{\mathbf{S}}\, \bar{\boldsymbol{a}}_{in}$$

this transforms to

$$\boldsymbol{a}_{in}^c = \bar{\mathbf{S}}\, \boldsymbol{a}_{out}^c = (\mathbf{S}^c)^{-1} \boldsymbol{a}_{out}^c \tag{2.111}$$

by definition of $\mathbf{S}^c$. Hence, with $\bar{\mathbf{S}}^{-1} = \mathbf{S}^T$ from (2.108), we get

$$\mathbf{S}^c \equiv \begin{bmatrix} \mathbf{R}_{+}^c & \mathbf{T}_{-}^c \\ \mathbf{T}_{+}^c & \mathbf{R}_{-}^c \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{R}}_{+} & \tilde{\mathbf{T}}_{+} \\ \tilde{\mathbf{T}}_{-} & \tilde{\mathbf{R}}_{-} \end{bmatrix} \equiv \tilde{\mathbf{S}} \tag{2.112}$$

This is the eigenmode scattering theorem [12, 119], expressing ‘reciprocity in k-space’, that we set out to prove. The reciprocity relations

$$\mathbf{R}_{\pm}^c = \tilde{\mathbf{R}}_{\pm}, \qquad \mathbf{T}_{\pm}^c = \tilde{\mathbf{T}}_{\mp} \tag{2.113}$$

are illustrated in Fig. 2.4.

A word as to notation. The elements of the matrix $\mathbf{R}_{\pm}$ (or analogously $\mathbf{T}_{\pm}$) will be written as $R_{11}^{\pm}$, $R_{12}^{\pm}$, $R_{21}^{\pm}$ and $R_{22}^{\pm}$, the $\pm$ sign indicating the direction of incidence with respect to the z-axis. It will sometimes be convenient, however, when the modal species or polarization is specifically characterized, e.g. parallel ($||$) or perpendicular ($\perp$) to the plane of incidence, right- or left-circular (r or ℓ), to adopt and extend Budden’s [32] notation, so that ${}_{||}R_{\perp}^{+}$ represents the conversion

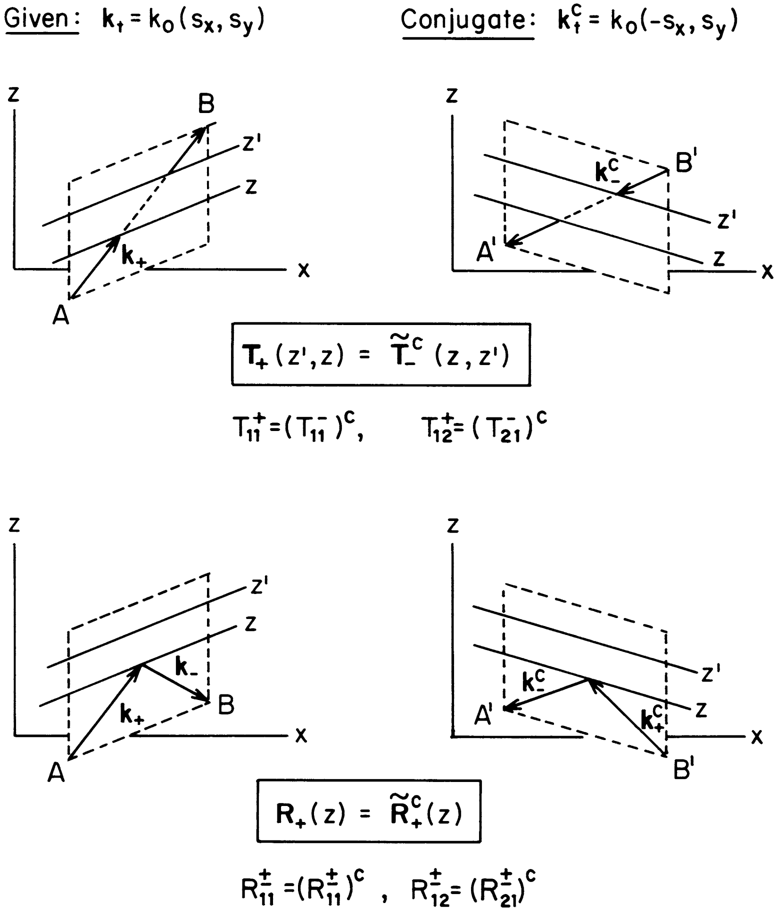


Fig. 2.4 Reciprocity in k-space in a plane-stratified magnetoplasma. The z-axis is normal to the stratification, and the external magnetic field lies in the magnetic-meridian (x, z) plane

coefficient from a positive-going perpendicularly polarized incident mode to a reflected (converted) negative-going parallel-polarized mode. Similarly, $_{\ell}T_r^-$ means negative-going right circular to negative-going left circular.

2.6 Curved stratified media

2.6.1 Curvilinear coordinates

In all previous sections the spatial variation of the media under consideration was taken to be in one cartesian coordinate only, i.e. the media were assumed to be plane stratified. This allowed the full use of Fourier transformations in the two coordinates,

x and y, transverse to the normal z-direction of the stratification. This restriction to plane stratification is usually sufficient when a curved stratified medium can be approximated by a plane stratified one in the region of propagation. But the question remains whether the scattering theorems, $\bar{\mathsf{S}}^{-1} = \mathsf{S}^T$ (2.108) and $\mathsf{S}^c = \mathsf{S}^T$ (2.112), hold also in curved stratified media. This problem has been addressed by Suchy and Altman [120]. Replacing the generalized Poynting flux densities, $P_{z,\alpha}$ (2.70) and P_z (2.78), by the corresponding *Poynting fluxes*, which are the integrals of $P_{z,\alpha}$ and P_z over (parts of) the curved stratification surfaces, the scattering theorems (2.108), and consequently (2.112), can be generalized to curved statified media.

To prove this statement we have to apply the Lagrange identity twice, first to a system of partial differential equations for the two coordinates u and υ in the stratification surfaces, and then to a system of partial differential equations for u, υ and w, where the w-coordinate is directed along the normal to these surfaces, thus generalizing (2.30).

Since we cannot employ the transverse Fourier transforms, (2.27) and (2.28), we decompose the differential operator

$$\nabla := \mathbf{g}^u \frac{\partial}{\partial u} + \mathbf{g}^\upsilon \frac{\partial}{\partial \upsilon} + \mathbf{g}^w \frac{\partial}{\partial w} \tag{2.114}$$

into a tangential part

$$\nabla_t := \mathbf{g}^u \frac{\partial}{\partial u} + \mathbf{g}^\upsilon \frac{\partial}{\partial \upsilon} \tag{2.115}$$

and a normal part

$$\nabla_w := \mathbf{g}^w \frac{\partial}{\partial w} \tag{2.116}$$

with the reciprocal set $\mathbf{g}^u,\ \mathbf{g}^\upsilon,\ \mathbf{g}^w$ of base vectors, obeying

$$\mathbf{g}^i \mathbf{g}_j = \delta_{ij}.$$

The base vectors $\mathbf{g}_u,\ \mathbf{g}_\upsilon,\ \mathbf{g}_w$ span the arc length element

$$d\mathbf{r} = \mathbf{g}_u du + \mathbf{g}_\upsilon d\upsilon + \mathbf{g}_w dw$$

[115, Sec. 1.14]. With the corresponding decomposition of the symmetric differential operator $\mathsf{D} = \mathsf{D}^T$ (2.22), viz

$$\mathsf{D} = \mathsf{D}_t + \mathsf{D}_w \tag{2.117}$$

into a tangential and a normal part

$$\mathsf{D}_t := \begin{bmatrix} \mathbf{0} & -\nabla_t \times \mathsf{I} \\ \nabla_t \times \mathsf{I} & \mathbf{0} \end{bmatrix} = \mathsf{D}_t^T, \qquad \mathsf{D}_w := \begin{bmatrix} \mathbf{0} & -\nabla_w \times \mathsf{I} \\ \nabla_w \times \mathsf{I} & \mathbf{0} \end{bmatrix} = \mathsf{D}_w^T \tag{2.118}$$

with $\mathsf{I} \equiv \mathsf{I}^{(3)}$, Maxwell's equations (2.21) and (2.22) become

$$[i\omega\mathsf{K} + \mathsf{D}_w + \mathsf{D}_t]\mathbf{e} = -\mathbf{j} \tag{2.119}$$

2.6.2 The biorthogonality relation

To establish a set of eigenmodes in the curved stratified medium, we proceed in a manner analogous to that in Sec. 2.2.3, equating the source term $\mathbf{j}$ to zero and keeping the normal coordinate w constant [53, Sec. 8.2a]. Then all six (covariant) components $E_u \dots H_w$ of the generalized wave-field vector $\mathbf{e} := (\mathbf{E}, \mathbf{H})$ (2.23) have the same harmonic factor $\exp(-i\kappa w)$, where κ is the separation constant. Application of $\nabla_w := \mathbf{g}^w\, \partial/\partial w$ (2.116) leads to

$$\mathsf{D}_w\mathbf{e} = -i\kappa\mathsf{U}_w\mathbf{e} \quad \text{with} \quad \mathsf{U}_w := \begin{bmatrix} 0 & -\mathbf{g}^w \times \mathsf{I} \\ \mathbf{g}^w \times \mathsf{I} & 0 \end{bmatrix} = \mathsf{U}_w^T \tag{2.120}$$

and to the eigenvalue equation

$$\mathsf{L}_\alpha\mathbf{e}_\alpha = [i\omega\mathsf{K} - i\kappa_\alpha\mathsf{U}_w + \mathsf{D}_t]\mathbf{e}_\alpha = 0 \tag{2.121}$$

instead of (2.32), which applied to plane-stratified media.

The corresponding adjoint eigenvalue equation reads, with $\mathsf{U}_w = \mathsf{U}_w^T$ (2.120) and $\mathsf{D}_t = \mathsf{D}_t^T$ (2.118),

$$\bar{\mathsf{L}}_\alpha\bar{\mathbf{e}}_\alpha = \left[i\omega\mathsf{K}^T - i\bar{\kappa}_\alpha\mathsf{U}_w - \mathsf{D}_t\right]\bar{\mathbf{e}}_\alpha = 0 \tag{2.122}$$

The operators L_α and $\bar{\mathsf{L}}_\alpha$ may now be combined to form the Lagrange identity

$$\bar{\mathbf{e}}_\beta^T\mathsf{L}_\alpha\mathbf{e}_\alpha - \mathbf{e}_\alpha^T\bar{\mathsf{L}}_\beta\bar{\mathbf{e}}_\beta = \mathsf{D}_t : \mathbf{e}_\alpha\bar{\mathbf{e}}_\beta^T - \boldsymbol{i}(\kappa_\alpha - \bar{\kappa}_\beta)\bar{\mathbf{e}}_\beta^T\mathsf{U}_w\,\mathbf{e}_\alpha \tag{2.123}$$

in which, for compactness, we have used the scalar 'double-dot product' introduced by Gibbs [58, Sec. 117]; see also [94, p.57]:

$$\mathsf{D} : \mathbf{e}\bar{\mathbf{e}}^T := \sum_{i,j} D_{ij}(e_j\bar{e}_i) = \sum_{i,j}(\bar{e}_i D_{ij}e_j + e_j\, D_{ji}\bar{e}_i)$$

with $D_{ij} = D_{ji}$ representing any symmetric differential operator. In (2.123), with $\mathsf{D} \to \mathsf{D}_t$ (2.118), $\mathbf{e}_\alpha := (\mathbf{E}_\alpha, \mathbf{H}_\alpha)$ (2.23) and $\bar{\mathbf{e}}_\beta := (\bar{\mathbf{E}}_\beta, \bar{\mathbf{H}}_\beta)$, the first term on the right-hand side is just the tangential divergence of the bilinear concomitant vector $\mathbf{P}_{\alpha\beta}$, viz.

$$
\begin{aligned}
\mathsf{D}_t : \mathbf{e}_\alpha \bar{\mathbf{e}}_\beta^T &= \bar{\mathbf{e}}_\beta^T \mathsf{D}_t \, \mathbf{e}_\alpha + \mathbf{e}_\alpha^T \mathsf{D}_t \, \bar{\mathbf{e}}_\beta \\
&= \bar{\mathbf{E}}_\beta \cdot \nabla_t \times \mathbf{H}_\alpha + \bar{\mathbf{H}}_\beta \cdot \nabla_t \times \mathbf{E}_\alpha + \mathbf{E}_\alpha \cdot \nabla_t \times \bar{\mathbf{H}}_\beta + \mathbf{H}_\alpha \cdot \nabla_t \times \bar{\mathbf{E}}_\beta \\
&= \nabla_t \cdot (\mathbf{E}_\alpha \times \bar{\mathbf{H}}_\beta + \bar{\mathbf{E}}_\beta \times \mathbf{H}_\alpha) =: \nabla_t \cdot \mathbf{P}_{\alpha\beta}
\end{aligned} \tag{2.124}
$$

The second term on the right-hand side of (2.123) contains its (contravariant) normal component:

$$
\bar{\mathbf{e}}_\beta^T \mathsf{U}_w \, \mathbf{e}_\alpha = \mathbf{g}^w \cdot (\mathbf{E}_\alpha \times \bar{\mathbf{H}}_\beta + \bar{\mathbf{E}}_\beta \times \mathbf{H}_\alpha) = P_{\alpha\beta}^w \tag{2.125}
$$

With $\mathsf{L}_\alpha \mathbf{e}_\alpha = 0$ (2.121) and $\bar{\mathsf{L}}_\beta \bar{\mathbf{e}}_\beta = 0$ (2.122), the Lagrange identity (2.123) gives

$$
\nabla_t \cdot \mathbf{P}_{\alpha\beta} = i\,(\kappa_\alpha - \bar{\kappa}_\beta)\, P_{\alpha\beta}^w \tag{2.126}
$$

To derive a biorthogonality relation for the eigenvectors $\mathbf{e}_\alpha$ and the adjoint eigenvectors $\bar{\mathbf{e}}_\beta$, we apply Gauss' divergence theorem in two dimensions

$$
\int \nabla_t \cdot \mathbf{P}_{\alpha\beta} dS = \oint \hat{\boldsymbol{\nu}} \cdot \mathbf{P}_{\alpha\beta} ds \tag{2.127}
$$

to a (finite part of a) stratification surface. (In the integral on the right, the boundary curve on the surface is encompassed in a right-hand sense about the normal $\mathbf{g}^w$. The unit normal vector $\hat{\boldsymbol{\nu}}$ lies on the surface and points in an outward direction with respect to the boundary curve.)

With the boundary conditions [53, eqs. 8.2.4c and 1.1.23b], (see also Secs. 4.3 and 6.3 in this book),

$$
\hat{\boldsymbol{\nu}} \times \mathbf{E}_\alpha = \mathsf{Z}\, \mathbf{H}_\alpha \qquad \hat{\boldsymbol{\nu}} \times \bar{\mathbf{E}}_\beta = -\mathsf{Z}^T \bar{\mathbf{H}}_\beta \tag{2.128}
$$

the contour integral vanishes, and the integrated Lagrange identity (2.126), which becomes a Green's theorem, yields

$$
(\kappa_\alpha - \bar{\kappa}_\beta) \int \mathbf{g}^w \cdot \mathbf{P}_{\alpha\beta} dS = 0 \tag{2.129}
$$

As a further requirement for the derivation of a biorthogonality relation, we exclude modal concomitant vectors $\mathbf{P}_\alpha := \mathbf{E}_\alpha \times \bar{\mathbf{H}}_\alpha + \bar{\mathbf{E}}_\alpha \times \mathbf{H}_\alpha$ (2.124) that lie on a stratification surface, i.e. we require that

$$
P_\alpha^w := \mathbf{g}^w \cdot \mathbf{P}_\alpha \neq 0 \tag{2.130}
$$

For loss-free media with $\bar{\mathbf{e}} = \mathbf{e}^*$ (2.124), and therefore $P_\alpha^w = \langle S_\alpha^w \rangle$ (2.56), this condition excludes surface-wave modes whose (time-averaged) Poynting vectors $\langle \mathbf{S}_\alpha \rangle$ are tangential to the stratification surfaces.

Under the condition (2.130) we can derive from Green's theorem (2.129) first, the identity

$$\bar{\kappa}_\alpha = \kappa_\alpha \tag{2.131}$$

of the adjoint and the given eigenmodes $\bar{\kappa}_\alpha$ and κ_α, and second, the biorthogonality relation

$$\int \bar{\mathbf{e}}_\beta^T \mathsf{U}_w \, \mathbf{e}_\alpha \, dS = \delta_{\alpha\beta} \int P_\alpha^w \, dS \tag{2.132}$$

(Similar reasoning has been employed by Felsen and Marcuvitz [53, p. 53] with the time t in place of the normal coordinate w.)

2.6.3 The generalized Poynting flux

We have obtained the simple harmonic w-dependence $\mathbf{e}_\alpha \sim \exp(-i\kappa_\alpha w)$ of the modal eigenvectors by keeping the normal coordinate w constant in the Maxwell system (2.119). An analogous dependence, $\exp\{-i(k_u u + k_\upsilon \upsilon)\}$, on the surface coordinates u and υ is only possible if all coefficients of the (covariant) components $E_u \ldots H_w$ in the Maxwell system (2.119) do not depend on u and υ. Since

$$\nabla_t \times \mathbf{E} = \frac{1}{\sqrt{g}} \left[\mathbf{g}_u \frac{\partial E_w}{\partial \upsilon} - \mathbf{g}_\upsilon \frac{\partial E_w}{\partial u} + \mathbf{g}_w \left(\frac{\partial E_\upsilon}{\partial u} - \frac{\partial E_u}{\partial \upsilon} \right) \right]$$

and

$$\mathbf{g}^w \times \mathbf{H} = \frac{1}{\sqrt{g}} (\mathbf{g}_\upsilon H_u - \mathbf{g}_u H_\upsilon)$$

with the Jacobian

$$\sqrt{g} := \mathbf{g}_u \times \mathbf{g}_\upsilon \cdot \mathbf{g}_w = (\mathbf{g}^u \times \mathbf{g}^\upsilon \cdot \mathbf{g}^w)^{-1}$$

[115, Secs. 1.14 and 1.15], this requires that the Jacobian be independent of u and υ. The only coordinate systems satisfying this requirement are those with cartesian coordinates in which $g = 1$, and (circular) cylindrical coordinates $\rho,\ \phi,\ z$ in which

$$u = \phi, \quad \upsilon = z, \quad w = \rho, \quad \sqrt{g} = \rho$$

[56, eqs. 19 and 21]. For these two cases the application of $\nabla_t = \mathbf{g}^u \partial/\partial u + \mathbf{g}^\upsilon \partial/\partial \upsilon$ (2.115) leads to

$$\mathsf{D}_t \mathbf{e} = [i\, k_u \mathsf{U}_u + i\, k_\upsilon \mathsf{U}_\upsilon] \mathbf{e} \tag{2.133}$$

with

$$\mathsf{U}_u := \begin{bmatrix} \mathbf{0} & -\mathbf{g}^u \times \mathsf{I} \\ \mathbf{g}^u \times \mathsf{I} & \mathbf{0} \end{bmatrix}, \quad \mathsf{U}_\upsilon := \begin{bmatrix} \mathbf{0} & -\mathbf{g}^\upsilon \times \mathsf{I} \\ \mathbf{g}^\upsilon \times \mathsf{I} & \mathbf{0} \end{bmatrix} \tag{2.134}$$

and to an algebraic eigenvalue problem

$$[i\omega\mathsf{K} - ik_u\mathsf{U}_u - ik_\upsilon\mathsf{U}_\upsilon - i\kappa_\alpha\mathsf{U}_w]\ \mathbf{e}_\alpha = 0 \tag{2.135}$$

With $u = x$, $\upsilon = y$, $w = z$, $k_u = k_0 s_x$, $k_\upsilon = k_0 s_y$, $\kappa_\alpha = k_0 q_\alpha$, we recover the eigenvalue equation (2.32) for plane-stratified media. Now the reasoning in Secs. 2.2.3 to 2.5.2 can be applied without the integration over a (finite) stratification surface as in Sec. 2.6.2, but this holds only for media whose stratification surfaces are either planes or (circular) cylinders.

For media with other stratification surfaces we go back to Maxwell's equations (2.21) without sources

$$\mathsf{L}\mathbf{e} := [i\omega\mathsf{K} + \mathsf{D}]\mathbf{e} = 0 \tag{2.136}$$

The adjoint equation is

$$\bar{\mathsf{L}}\bar{\mathbf{e}} := [i\omega\mathsf{K}^T - \mathsf{D}]\bar{\mathbf{e}} = 0 \tag{2.137}$$

and the Lagrange identity, by analogy with (2.123) and (2.124), reads

$$\bar{\mathbf{e}}^T\,\mathsf{L}\,\mathbf{e} - \mathbf{e}^T\,\bar{\mathsf{L}}\,\bar{\mathbf{e}} = \mathsf{D} : \mathbf{e}\bar{\mathbf{e}}^T = \nabla\cdot\mathbf{P} \tag{2.138}$$

with the bilinear concomitant vector

$$\mathbf{P} := \mathbf{E}\times\bar{\mathbf{H}} + \bar{\mathbf{E}}\times\mathbf{H} \tag{2.139}$$

For $\mathsf{L}\mathbf{e} = 0$ and $\bar{\mathsf{L}}\bar{\mathbf{e}} = 0$, the left-hand side of (2.138) vanishes. To the right-hand side we apply Gauss' divergence theorem (in three dimensions)

$$\int \nabla\cdot\mathbf{P}d^3r = \oint \mathbf{P}\cdot d\mathbf{S} \tag{2.140}$$

with the surface elements $d\mathbf{S}$ directed outwards. The integration volume is bounded by two stratification surfaces, and between them by walls for which the boundary conditions (2.128) hold. The latter leave us with the contributions of the bounding stratification surfaces:

$$\int \bar{\mathbf{e}}^T\,\mathsf{U}_\omega\,\mathbf{e}\,dS \equiv \int P^\omega dS = \text{const} \tag{2.141}$$

where (2.125) has been used. This is the generalization of the result (2.39) for plane stratified media.

2.6.4 Scattering theorems

For the generalization of the scattering theorem (2.108) we introduce modal amplitudes, a_α and $\bar{a}_\alpha$, as in (2.74)

$$\mathbf{e}_\alpha = a_\alpha\hat{\mathbf{e}}_\alpha \qquad \bar{\mathbf{e}}_\alpha = \bar{a}_\alpha\hat{\bar{\mathbf{e}}}_\alpha \tag{2.142}$$

but with $\hat{\mathbf{e}}_\alpha$ and $\hat{\bar{\mathbf{e}}}_\alpha$ now normalized so that, cf. (2.132),

$$\int \hat{\bar{\mathbf{e}}}_\beta^T \, \mathbf{U}_{\boldsymbol{\omega}} \, \hat{\mathbf{e}}_\alpha \, dS = \delta_{\alpha\beta} \, \mathrm{sgn}\,(\alpha) \tag{2.143}$$

Putting this and the decompositions

$$\mathbf{e} = \sum_\alpha a_\alpha \hat{\mathbf{e}}_\alpha \qquad \bar{\mathbf{e}} = \sum_\beta \bar{a}_\beta \hat{\bar{\mathbf{e}}}_\beta \tag{2.144}$$

into (2.141), we obtain

$$\sum_\alpha \bar{a}_\alpha a_\alpha \, \mathrm{sgn}\,(\alpha) = \mathrm{const} \tag{2.145}$$

which is the same result as in (2.78) and (2.105) for plane-stratified media. This was the basis for the derivation of the scattering theorems (2.108) and (2.112) in Secs. 2.5.1 and 2.5.2, which need not be changed for the generalization to curved stratified media.

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