Chapter 2
Special Orthogonal Group SO(N)

1 Introduction

Since the exactly solvable higher-dimensional quantum systems with certain central potentials are usually related to the real orthogonal group O(N) defined by orthogonal $n \times n$ matrices, we shall give a brief review of some basic properties of group O(N) based on the monographs and textbooks [136–140]. Before proceeding to do so, we first outline the development in order to make the reader recognize its importance in physics.

We often apply groups throughout mathematics and the sciences to capture the internal symmetry of other structures in the form of automorphism groups. It is well-known that the internal symmetry of the structure is usually related to an invariant mathematical property, and a set of transformations that preserve this kind of property together with the operation of composition of transformations form a group named a symmetry group.

It should be noted that Galois theory is the historical origin of the group concept. He used groups to describe the symmetries of the equations satisfied by the solutions of a polynomial equation. The solvable groups are thus named due to their prominent role in this theory.

The concept of the Lie group named for mathematician Sophus Lie plays a very important role in the study of differential equations and manifolds; they combine analysis and group theory and are therefore the proper objects for describing symmetries of analytical structures.

An understanding of group theory is of importance in physics. For example, groups describe the symmetries which the physical laws seem to obey. On the other hand, physicists are very interested in group representations, especially of the Lie groups, since these representations often point the way to the possible physical theories and they play an essential role in the algebraic method for solving quantum mechanics problems.

As a common knowledge, the study of the groups is always related to the corresponding algebraic method. Up to now, the algebraic method has become the subject of interest in various fields of physics. The elegant algebraic method was first

S.-H. Dong, *Wave Equations in Higher Dimensions*,
DOI 10.1007/978-94-007-1917-0_2,

introduced in the context of the new matrix mechanics around 1925. Since the introduction of the angular momentum in quantum mechanics, which was intimately connected with the representations of the rotation group SO(3) associated with the rotational invariance of central potentials, its importance was soon recognized and the necessary formalism was developed principally by a number of pioneering scientists including Weyl, Racah, Wigner and others [136, 141–144]. Until now, the algebraic method to treat the angular momentum theory can be found in almost all textbooks of quantum mechanics.

On the other hand, it often runs parallel to the differential equation approach due to the great scientist Schrödinger. Pauli employed algebraic method to deal with the hydrogen atom in 1926 [145] and Schrödinger also solved the same problem almost at the same time [146], but their fates were quiet different. This is because the standard differential equation approach was more accessible to the physicists than the algebraic method. As a result, the algebraic approach to determine the energy levels of the hydrogen atom was largely forgotten and the algebraic techniques went into abeyance for several decades. Until the middle of 1950s, the algebraic techniques revived with the development of theories for the elementary particles since the explicit forms of the Hamiltonian for those elementary particle systems are unknown and the physicists have to make certain assumptions on their internal symmetries. Among various attempts to solve this difficult problem, the particle physicists examined some non-compact Lie algebras and hoped that they would provide a clue to the classification of the elementary particles. Unfortunately, this hope did not materialize. Nevertheless, it is found that the Lie algebras of the compact Lie groups enable such a classification for the elementary particles [147] and the non-compact groups are relevant for the dynamic groups in atomic physics [148] and the non-classical properties of quantum optical systems involving coherent and squeezed states as well as the beam splitting and linear directional coupling devices [149–153].

It is worth pointing out that one of the reasons why the algebraic techniques were accepted very slowly and the original group theoretical and algebraic methods proposed by Pauli [145] were neglected is undoubtedly related to the abstract character and inherent complexity of group theory. Even though the proper understanding of group theory requires an intimate knowledge of the standard theory of finite groups and of the topology and manifold theory, the basic concepts of group theory are quite simple, specially when we present them in the context of physical applications. Basically, we attempt to introduce them as simple as possible so that the common reader can master the basic ideas and essence of group theory. The detailed information on group theory can be found in the textbooks [138–140, 154].

On the other hand, during the development of algebraic method, Racah algebra techniques played an important role in physics since it enables us to treat the integration over the angular coordinates of a complex many-particle system analytically and leads to the formulas expressed in terms of the generalized CGCs, Wigner n-j symbols, tensor spherical harmonics and/or rotation matrices. With the development of algebraic method in the late 1950s and early 1960s, the algebraic method proposed by Pauli was systematized and simplified greatly by using the concepts of the Lie algebras. Up to now, the algebraic method has been widely applied to various fields of physics such as nuclear physics [155], field theory and particle physics [156], atomic and molecular physics [157–160], quantum chemistry [161], solid state physics [162], quantum optics [149, 151, 163–168] and others.

2 Abstract Groups

We now give some basic definitions about the abstract groups[1] based on textbooks by Weyl, Wybourne, Miller, Ma and others [136, 137, 139, 140, 169].

Definition A **group** G is a set of elements $\{e, f, g, h, k, \ldots\}$ together with a binary operation. This binary operation named a group multiplication is subject to the following four requirements:

- **Closure**: if $f, g \in G$, then $fg \in G$ too,
- **Identity element**: there exists an identity element e in G (a unit) such that $ef = fe = f$ for any $f \in G$,
- **Inverses**: for every $f \in G$ there exists an inverse element $f^{-1} \in G$ such that $ff^{-1} = f^{-1}f = e$,
- **Associative law**: the identity $f(hk) = (fh)k$ is satisfied for all elements $f, h, k \in G$.

Subgroup: a subgroup of G is a subset $\mathcal{S} \in G$, which is itself a group under the group multiplication defined in G, i.e., $f, h \in \mathcal{S} \to f\ h \in \mathcal{S}$.

Homomorphism: a homomorphism of groups G and $\mathcal{H}$ is a mapping from a group G into a group $\mathcal{H}$, which transforms products into products, i.e., $G \to \mathcal{H}$.

Isomorphism: an isomorphism is a homomorphism which is one-to-one and "onto" [169]. From the viewpoint of the abstract group theory, isomorphic groups can be identified. In particular, isomorphic groups have identical multiplication tables.

Representation: a representation of a group G is a homomorphism of the group into the group of invertible operators on a certain (most often complex) Hilbert space V (called representation space). If the representation is to be finite-dimensional, it is sufficient to consider homomorphisms $G \to GL(n)$. The $GL(n)$ represents a general linear group of non-singular matrices of dimension n. Usually, the image of the group in this homomorphism is called a representation as well.

Irreducible representation: an irreducible representation is a representation whose representation space contains no proper subspace invariant under the operators of the representation.

Commutation relation: since a Lie algebra has an underlying vector space structure we may choose a basis set $\{L_i\}$ $(i = 1, 2, 3, \ldots, N)$ for the Lie algebra. In

[1] There exist two kinds of different meanings of the terminology "abstract group" during the first half of the 20th century. The first meaning was that of a group defined by four axioms given above, but the second one was that of a group defined by generators and commutation relations.

general, the Lie algebra can be completely defined by specifying the commutators of these basis elements:

$$[L_i, L_j] = \sum_k c_{ijk} L_k, \quad i, j, k = 1, 2, 3, \ldots, N, \tag{2.1}$$

in which the coefficients c_{ijk} and the elements L_i are the structure constants and the generators of the Lie algebra, respectively. It is worth noting that the set of operators, which commute with all elements of the Lie algebra, are called Casimir operators.

We shall constraint ourselves in the following parts to study some basic properties of the compact group SO(N) alongside the well-known compact so(n) Lie algebra of the generalized angular momentum theory since it shall be helpful in successive Chapters. We suggest the reader refer to the textbooks on group theory [136–140, 144, 154, 169] or Appendices A–C for more information.

3 Orthogonal Group SO(N)

For every positive integer N, the orthogonal group O(N) is the group of $N \times N$ orthogonal matrices A satisfying

$$AA^T = \mathbf{1}, \qquad A^* = A. \tag{2.2}$$

Because the determinant of an orthogonal matrix is either 1 or -1, and so the orthogonal group has two components. The component containing the identity **1** is the special orthogonal group SO(N). An N-dimensional real matrix contains N^2 real parameters. The column matrices of a real orthogonal matrix are normal and orthogonal to each other. There exist N real matrix constraints for the normalization and $N(N-1)/2$ real constraints for the orthogonality. Thus, the number of independent real parameters for characterizing the elements of the groups SO(N) is equal to $N^2 - [N + N(N-1)/2] = N(N-1)/2$. The group space is a doubly-connected closed region so that the SO(N) is a compact Lie group with rank $N(N-1)/2$.

4 Tensor Representations of the Orthogonal Group SO(N)

In this section we are going to study the reduction of a tensor space of the SO(N) and calculation of the orthonormal irreducible basis tensors [139, 140].

4.1 Tensors of the Orthogonal Group SO(N)

We begin by studying the tensors of the SO(N). For a given rank n of the SO(N), we know that there are N^n components with a following transform,

$$T_{c_1 \cdots c_n} \xrightarrow{R} O_R T_{c_1 \cdots c_n} = \sum_{d_1 \cdots d_n} R_{c_1 d_1} \cdots R_{c_n d_n} T_{d_1 \cdots d_n}, \quad R \in \mathrm{SO}(N). \tag{2.3}$$

It is noted that only one nonvanishing component of a basis tensor is equal to 1, i.e.,

$$(\boldsymbol{\theta}_{a_1\cdots a_n})_{b_1\cdots b_n} = \delta_{a_1b_1}\cdots\delta_{a_nb_n} = (\boldsymbol{\theta}_{a_1})_{b_1}\cdots(\boldsymbol{\theta}_{a_n})_{b_n}, \tag{2.4}$$

$$O_R(\boldsymbol{\theta}_{a_1\cdots a_n}) = \sum_{c_1\cdots c_n}(\boldsymbol{\theta}_{c_1\cdots c_n})R_{c_1a_1}\cdots R_{c_na_n}, \tag{2.5}$$

from which one may expand any tensor in such a way

$$\mathbf{T}_{b_1\cdots b_n} = \sum_{a_1\cdots a_n} T_{a_1\cdots a_n}(\boldsymbol{\theta}_{a_1\cdots a_n})_{b_1\cdots b_n}. \tag{2.6}$$

The tensor space is an invariant linear space both in the SO(N) and in the permutation group S_n. Since the SO(N) transformation commutes with the permutation so that one can reduce the tensor space in the orthogonal groups SO(N) by the projection of the Young operators, which are conveniently used to deal with the permutation group S_n.

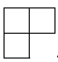
Note that there are several important characteristics for the tensors of the SO(N) group:

- The real and imaginary parts of a tensor of the SO(N) transform independently in Eq. (2.3). As a result, we need only study their real tensors.
- There is no any difference between a covariant tensor and a contra-variant tensor for the SO(N) transformations. The contraction of a tensor can be achieved between any two indices. Therefore, before projecting a Young operator, the tensor space must be decomposed into a series of traceless tensor subspaces, which remain invariant in the SO(N).
- Denote by T the traceless tensor space of rank n. After projecting a Young operator, $\mathrm{T}^{[\lambda]}_{\mu} = \mathrm{y}^{[\lambda]}_{\mu}\mathrm{T}$ is a traceless tensor subspace with a given permutation symmetry. $\mathrm{T}^{[\lambda]}_{\mu}$ will become a null space if the summation of the numbers of boxes in the first two columns of the Young pattern[2] $[\lambda]$ is larger than the dimension N.
- If the row number m of the Young pattern $[\lambda]$ is larger than $N/2$, then the basis tensor $\mathrm{y}^{[\lambda]}_{\mu}\theta_{b_1\cdots b_m c\cdots}$ can be changed to a dual basis tensor by a totally antisymmetric tensor $\epsilon_{a_1\cdots a_N}$,

$$^*[\mathrm{y}^{[\lambda]}\theta]_{a_1\cdots a_{N-m}c\cdots} = \frac{1}{m!}\sum_{a_{N-m+1}\cdots a_N}\epsilon_{a_1\cdots a_{N-m}a_{N-m+1}\cdots a_N}\, \mathrm{y}^{[\lambda]}\theta_{a_N\cdots a_{N-m+1}c\cdots}, \tag{2.7}$$

whose inverse transformation is given by

[2] A Young pattern $[\lambda]$ has n boxes lined up on the top and on the left, where the jth row contains λ_j boxes. For instance, the Young pattern [2, 1] is



It should be noted that the number of boxes in the upper row is not less than in the lower row, and the number of boxes in the left column is not less than that in the right column. We suggest the reader refer to the permutation group S_n in Appendix A for more information.

$$\frac{1}{(N-m)!}\sum_{a_{m+1}\cdots a_N} \epsilon_{b_1\cdots b_m a_{m+1}\cdots a_N}{}^*[\mathrm{y}^{[\lambda]}\theta]_{a_N\cdots a_{m+1}c\cdots}$$
$$= \frac{1}{m!(N-m)!}\sum_{a_1\cdots a_N} \epsilon_{b_1\cdots b_m a_{m+1}\cdots a_N}\epsilon_{a_N\cdots a_{m+1}a_m\cdots a_1}\mathrm{y}^{[\lambda]}\theta_{a_1\cdots a_m c}$$
$$= (-1)^{N(N-1)/2}\mathrm{y}^{[\lambda]}\theta_{b_1\cdots b_m c\cdots}. \tag{2.8}$$

After some algebraic manipulations, it is found that the correspondence between two sets of basis tensors is one-to-one and the difference between them is only in the arranged order. Thus, a traceless tensor subspace $\mathrm{T}^{[\lambda]}_{\mu}$ is equivalent to a traceless tensor subspace $\mathrm{T}^{[\lambda']}_{\nu}$, where the row number of the Young pattern $[\lambda']$ is $(N-m) < N/2$,

$$[\lambda'] \simeq [\lambda], \qquad \lambda'_i = \begin{cases} \lambda_i, & i \le (N-m), \\ 0, & i > (N-m), \end{cases} \tag{2.9}$$

where $m \in (N/2, N]$.

- If $N = 2l$, i.e., the row number l of $[\lambda]$ is equal to $N/2$, then the Young pattern $[\lambda]$ is the same as its dual Young pattern, called the self-dual Young pattern. To remove the phase factor $(-1)^{N(N-1)/2} = (-1)^l$ appearing in Eq. (2.8), we introduce a factor $(-i)^l$ in Eq. (2.7),

$$^*[\mathrm{y}^{[\lambda]}\theta]_{a_1\cdots a_l c\cdots} = \frac{(-i)^l}{l!}\sum_{a_{l+1}\cdots a_{2l}} \epsilon_{a_1\cdots a_l a_{l+1}\cdots a_{2l}}\mathrm{y}^{[\lambda]}\theta_{a_{2l}\cdots a_{l+1}c\cdots}, \tag{2.10}$$

$$\mathrm{y}^{[\lambda]}\theta_{a_1\cdots a_l c\cdots} = \frac{(-i)^l}{l!}\sum_{a_{l+1}\cdots a_{2l}} \epsilon_{a_1\cdots a_l a_{l+1}\cdots a_{2l}}{}^*[\mathrm{y}^{[\lambda]}\theta]_{a_{2l}\cdots a_{l+1}c\cdots}. \tag{2.11}$$

Define

$$\psi^{\pm}_{a_1\cdots a_l c\cdots} = \frac{1}{2}\left\{\mathrm{y}^{[\lambda]}\theta_{a_1\cdots a_l c\cdots} \pm {}^*[\mathrm{y}^{[\lambda]}\theta]_{a_1\cdots a_l c\cdots}\right\}. \tag{2.12}$$

We observe that $\psi^{+}_{a_1\cdots a_l c\cdots}$ keeps invariant in the dual transformation so that we call it self-dual basis tensor. On the contrary, we call $\psi^{-}_{a_1\cdots a_l c\cdots}$ the anti-self-dual basis tensor because it changes its sign in dual transformation. For example, for even $N = 2l$ we may construct the self-dual and anti-self-dual basis tensors as follows:

$$\psi^{\pm}_{1\cdots l} = \frac{1}{2}\left\{\mathrm{y}^{[1^l]}\theta_{1\cdots l} \pm (-i)^l\mathrm{y}^{[1^l]}\theta_{(2l)\cdots(l+1)}\right\}. \tag{2.13}$$

Therefore, when $l = N/2$ the representation space $\mathrm{T}^{[\lambda]}_{\mu}$ can be divided to the self-dual and the anti-self-dual tensor subspaces with the same dimension. Notice that the combinations by the Young operators and the dual transformations (2.7) and (2.13) are all real except that the dual transformation (2.13) with $N = 4l + 2$ is complex.

In conclusion, the traceless tensor subspace $\mathrm{T}^{[\lambda]}_{\mu}$ corresponds to a representation $[\lambda]$ of the SO(N), where the row number l of Young pattern $[\lambda]$ is less than $N/2$. When $l = N/2$ the traceless tensor subspace $\mathrm{T}^{[\lambda]}_{\mu}$ can be decomposed into the self-dual tensor subspace $\mathrm{T}^{[+\lambda]}_{\mu}$ and anti-self-dual tensor subspace $\mathrm{T}^{[-\lambda]}_{\mu}$ corresponding to the representation $[\pm\lambda]$, respectively. All irreducible representations $[\lambda]$ and $[\pm\lambda]$ are real except for $[\pm\lambda]$ with $N = 4l + 2$.

As far as the orthonormal irreducible basis tensors of the SO(N), we are going to address two problems. The first is how to decompose the standard tensor Young tableaux into a sum of the traceless basis tensors. The second is how to combine the basis tensors such that they are the common eigenfunctions of H_j and orthonormal to each other. The advantage of the method based on the standard tensor Young tableaux is that the basis tensors are known explicitly and the multiplicity of any weight is equivalent to the number of the standard tensor Young tableaux with the weight.

For group SO(N), the key issue for finding the orthonormal irreducible basis is to find the common eigenstates of H_i and the highest weight state in an irreducible representation. For odd and even N, i.e., the groups SO($2l + 1$) and SO($2l$), the generators T_{ab} of the self-representation satisfy

$$\begin{aligned} &[T_{ab}]_{cd} = -i(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}), \\ &[T_{ab}, T_{cd}] = -i(\delta_{bc}T_{ad} + \delta_{ad}T_{bc} - \delta_{bd}T_{ac} - \delta_{ac}T_{bd}). \end{aligned} \tag{2.14}$$

The bases H_i in the Cartan subalgebra can be written as

$$H_i = T_{(2i-1)(2i)}, \quad i \in [1, N/2]. \tag{2.15}$$

As what follows, we are going to study the irreducible basis tensors of the SO($2l + 1$) and SO($2l$), respectively.

4.2 Irreducible Basis Tensors of the SO($2l + 1$)

It is known that the Lie algebra of the SO($2l + 1$) is B_l. The simple roots of the SO($2l + 1$) are given by [139, 140]

$$r_\nu = e_\nu - e_{\nu+1}, \quad \nu \in [1, l-1], \qquad r_l = e_l, \tag{2.16}$$

where r_ν are the longer roots with $d_\nu = 1$ and r_l is the shorter root with $d_l = 1/2$. Based on the definition of the Chevalley bases, which include $3l$ bases E_ν, F_ν, and H_ν for the generators,

$$\frac{E_{\mathbf{r}_\nu}}{\sqrt{d_\nu}} \to E_\nu, \qquad \frac{E_{-\mathbf{r}_\nu}}{\sqrt{d_\nu}} \to F_\nu, \qquad \frac{1}{d_\nu}\sum_{i=1}^{l}(\mathbf{r}_\nu)_i H_i \equiv \frac{1}{d_\nu}\mathbf{r}_\nu \cdot \mathbf{H} \to H_\nu, \tag{2.17}$$

one is able to calculate the Chevalley bases of the SO($2l+1$) in the self-representation as follows:

$$\begin{aligned}
H_\nu &= T_{(2\nu-1)(2\nu)} - T_{(2\nu+1)(2\nu+2)},\\
E_\nu &= \frac{1}{2}[T_{(2\nu)(2\nu+1)} - iT_{(2\nu-1)(2\nu+1)} - iT_{(2\nu)(2\nu+2)} - T_{(2\nu-1)(2\nu+2)}],\\
F_\nu &= \frac{1}{2}[T_{(2\nu)(2\nu+1)} + iT_{(2\nu-1)(2\nu+1)} + iT_{(2\nu)(2\nu+2)} - T_{(2\nu-1)(2\nu+2)}],\\
H_l &= 2T_{(2l-1)(2l)},\\
E_l &= T_{(2l)(2l+1)} - iT_{(2l-1)(2l+1)},\\
F_l &= T_{(2l)(2l+1)} + iT_{(2l-1)(2l+1)}.
\end{aligned} \tag{2.18}$$

Note that θ_a are not the common eigenvectors of H_ν. By generalizing the spherical harmonic basis vectors for the SO(3) group, we may define the spherical harmonic basis vectors for the self-representation of the SO($2l+1$) as follows:

$$\phi_\beta = \begin{cases} (-1)^{l-\beta+1}\sqrt{\frac{1}{2}}(\theta_{2\beta-1} + i\theta_{2\beta}), & \beta \in [1, l],\\ \theta_{2l+1}, & \beta = l+1,\\ \sqrt{\frac{1}{2}}(\theta_{4l-2\beta+3} - i\theta_{4l-2\beta+4}), & \beta \in [l+2, 2l+1], \end{cases} \tag{2.19}$$

which are orthonormal and complete. In the spherical harmonic basis vectors ϕ_β, the nonvanishing matrix entries in the Chevalley bases are given by

$$\begin{aligned}
&H_\nu\phi_\nu = \phi_\nu, && H_\nu\phi_{\nu+1} = -\phi_{\nu+1},\\
&H_\nu\phi_{2l-\nu+1} = \phi_{2l-\nu+1}, && H_\nu\phi_{2l-\nu+2} = -\phi_{2l-\nu+2},\\
&H_l\phi_l = 2\phi_l, && H_l\phi_{l+2} = -2\phi_{l+2},\\
&E_\nu\phi_{\nu+1} = \phi_\nu, && E_\nu\phi_{2l-\nu+2} = \phi_{2l-\nu+1},\\
&E_l\phi_{l+1} = \sqrt{2}\phi_l, && E_l\phi_{l+2} = \sqrt{2}\phi_{l+1},\\
&F_\nu\phi_\nu = \phi_{\nu+1}, && F_\nu\phi_{2l-\nu+1} = \phi_{2l-\nu+2},\\
&F_l\phi_l = \sqrt{2}\phi_{l+1}, && F_l\phi_{l+1} = \sqrt{2}\phi_{l+2},
\end{aligned} \tag{2.20}$$

where $\nu \in [1, l-1]$. That is to say, the diagonal matrices of H_ν and H_l in the spherical harmonic basis vectors ϕ_β are expressed as follows:

$$\begin{aligned}
H_\nu &= \text{diag}\{\underbrace{0,\ldots,0}_{\nu-1}, 1, -1, \underbrace{0,\ldots,0}_{2l-2\nu-1}, 1, -1, \underbrace{0,\ldots,0}_{\nu-1}\},\\
H_l &= \text{diag}\{\underbrace{0,\ldots,0}_{l-1}, 2, 0, -2, \underbrace{0,\ldots,0}_{l-1}\}.
\end{aligned} \tag{2.21}$$

The spherical harmonic basis tensor $\phi_{\beta_1\cdots\beta_n}$ of rank n for the SO($2l+1$) becomes the direct product of n spherical harmonic basis vectors $\phi_{\beta_1}\cdots\phi_{\beta_n}$. The standard tensor Young tableaux $\mathcal{Y}_\nu^{[\lambda]}\phi_{\beta_1\cdots\beta_n}$ are the common eigenstates of the H_ν, but generally neither orthonormal nor traceless. The eigenvalue of H_ν in the standard tensor

Young tableaux $\mathcal{Y}_\nu^{[\lambda]}\phi_{\beta_1\cdots\beta_n}$ is equal to the number of the digits ν and $(2l-\nu+1)$ in the tableau, minus the number of $(\nu+1)$ and $(2l-\nu+2)$. The eigenvalue of H_l in the standard tensor Young tableau is equal to the number of l in the tableau, minus the number of $(l+2)$, and then multiplied with a factor 2. The action F_ν on the standard tensor Young tableau is equal to the sum of all possible tensor Young tableaux, each of which can be obtained from the original one through replacing one filled digit ν by the digit $(\nu+1)$, or through replacing one filled digit $(2l-\nu+1)$ by the digit $(2l-\nu+2)$. But the action of the F_l on the standard tensor Young tableau is equal to the sum, multiplied with a factor $\sqrt{2}$, of all possible tensor Young tableaux, each of which can be obtained from the original one through replacing one filled digit l by $(l+1)$ or through replacing one filled $(l+1)$ by $(l+2)$. However, the actions of E_ν and E_l on the standard tensor Young tableau are opposite to those of F_ν and F_l. Even though the obtained tensor Young tableaux may be not standard, they can be transformed into the sum of the standard tensor Young tableaux by symmetry.

Two standard tensor Young tableaux with different sets of filled digits are orthogonal to each other. For a given irreducible representation $[\lambda]$ of the SO$(2l+1)$, where the row number of Young pattern $[\lambda]$ is not larger than l, the highest weight state corresponds to the standard tensor Young tableau, in which each box in the βth row is filled with the digit β because each raising operator E_ν annihilates it. The highest weight $\mathbf{M}=\sum_\nu \omega_\nu M_\nu$ can be calculated from (2.20) as follows:

$$M_\nu=\lambda_\nu-\lambda_{\nu+1}, \quad \nu\in[1,l), \qquad M_l=2\lambda_l. \tag{2.22}$$

The tensor representation $[\lambda]$ of the SO$(2l+1)$ with even M_l is a single-valued representation, while the representation with odd M_l becomes a double-valued (spinor) representation.

The standard tensor Young tableaux $\mathcal{Y}_\nu^{[\lambda]}\phi_{\beta_1\cdots\beta_n}$ are generally not traceless, but the standard tensor Young tableau with the highest weight is traceless because it only contains ϕ_β with $\beta<l+1$ as shown in Eq. (2.19). For example, the tensor basis $\theta_1\theta_1$ is not traceless, but $\phi_1\phi_1$ is traceless. Since the highest weight is simple, the highest weight state is orthogonal to any other standard tensor Young tableau in the irreducible representation. Therefore, one is able to obtain the remaining orthonormal and traceless basis tensors in the representation $[\lambda]$ of the SO$(2l+1)$ from the highest weight state by the lowering operators F_ν based on the method of the block weight diagram.

4.3 Irreducible Basis Tensors of the SO($2l$)

The Lie algebra of the SO$(2l)$ is D_l and its simple roots are given by

$$r_\nu=e_\nu-e_{\nu+1}, \quad \nu\in[1,l-1], \qquad r_l=e_{l-1}+e_l. \tag{2.23}$$

The lengths of all simple roots are same, $d_\nu = 1$. Similarly, based on the definition of the Chevalley bases (2.17), we find that its Chevalley bases in the self-representation are same as those of the SO($2l+1$) except for $\nu = l$,

$$\begin{aligned}
H_l &= T_{(2l-3)(2l-2)} + T_{(2l-1)(2l)}, \\
E_l &= \frac{1}{2}[T_{(2l-2)(2l-1)} - iT_{(2l-3)(2l-1)} + iT_{(2l-2)(2l)} + T_{(2l-3)(2l)}], \\
F_l &= \frac{1}{2}[T_{(2l-2)(2l-1)} + iT_{(2l-3)(2l-1)} - iT_{(2l-2)(2l)} + T_{(2l-3)(2l)}].
\end{aligned} \tag{2.24}$$

Likewise, θ_a are not the common eigenvectors of the H_ν. By generalizing the spherical harmonic basis vectors for the SO(4) group, we define the spherical harmonic basis vectors for the self-representation of the SO($2l$) as follows:

$$\phi_\beta = \begin{cases} (-1)^{l-\beta}\sqrt{\frac{1}{2}}(\theta_{2\beta-1} + i\theta_{2\beta}), & \beta \in [1, l], \\ \sqrt{\frac{1}{2}}(\theta_{4l-2\beta+1} - i\theta_{4l-2\beta+2}), & \beta \in [l+1, 2l], \end{cases} \tag{2.25}$$

which are orthonormal and complete. In these basis vectors, the nonvanishing matrix entries of the Chevalley bases are given by

$$\begin{aligned}
&H_\nu\phi_\nu = \phi_\nu, && H_\nu\phi_{\nu+1} = -\phi_{\nu+1}, \\
&H_\nu\phi_{2l-\nu} = \phi_{2l-\nu}, && H_\nu\phi_{2l-\nu+1} = -\phi_{2l-\nu+1}, \\
&H_l\phi_{l-1} = \phi_{l-1}, && H_l\phi_l = \phi_l, \\
&H_l\phi_{l+1} = -\phi_{l+1}, && H_l\phi_{l+2} = -\phi_{l+2}, \\
&E_\nu\phi_{\nu+1} = \phi_\nu, && E_\nu\phi_{2l-\nu+1} = \phi_{2l-\nu}, \\
&E_l\phi_{l+1} = \phi_{l-1}, && E_l\phi_{l+2} = \phi_l, \\
&F_\nu\phi_\nu = \phi_{\nu+1}, && F_\nu\phi_{2l-\nu} = \phi_{2l-\nu+1}, \\
&F_l\phi_{l-1} = \phi_{l+1}, && F_l\phi_l = \phi_{l+2},
\end{aligned} \tag{2.26}$$

where $\nu \in [1, l-1]$. As a result, the diagonal matrices of the H_ν and H_l in the spherical harmonic basis vectors ϕ_β are calculated as:

$$\begin{aligned}
H_\nu &= \text{diag}\{\underbrace{0, \ldots, 0}_{\nu-1}, 1, -1, \underbrace{0, \ldots, 0}_{2l-2\nu-2}, 1, -1, \underbrace{0, \ldots, 0}_{\nu-1}\}, \\
H_l &= \text{diag}\{\underbrace{0, \ldots, 0}_{l-2}, 1, 1, -1, -1, \underbrace{0, \ldots, 0}_{l-2}\}.
\end{aligned} \tag{2.27}$$

The spherical harmonic basis tensor $\phi_{\beta_1\cdots\beta_n}$ of rank n for the SO($2l$) is the direct product of n spherical harmonic basis vectors $\phi_{\beta_1}\cdots\phi_{\beta_n}$. The standard tensor Young tableaux $\mathrm{y}_\nu^{[\lambda]}\phi_{\beta_1\cdots\beta_n}$ are the common eigenstates of the H_ν, but in general neither orthonormal nor traceless. The eigenvalue of H_ν in the standard tensor Young tableaux $\mathrm{y}_\nu^{[\lambda]}\phi_{\beta_1\cdots\beta_n}$ is equal to the number of the digits ν and $(2l-\nu)$ in the tableau, minus the number of $(\nu+1)$ and $(2l-\nu+1)$. The eigenvalue of H_l in the standard tensor Young tableau is equal to the number of the digits $(l-1)$ and l in the tableau, minus the number of $(l+1)$ and $(l+2)$. The eigenvalues form the weight $\mathbf{m}$ of standard tensor Young tableau. The action of F_ν on the standard tensor Young tableau is equal to the sum of all possible tensor Young tableaux, each of which can be obtained from the original one through replacing one filled digit $(2l-\nu)$ by the digit $(2l-\nu+1)$. The action of F_l on the standard tensor Young tableau is equal to the sum of all possible tensor Young tableaux, each of which is obtained from the original one through replacing one filled digit $(l-1)$ by the digit $(l+1)$ or through replacing one filled l by the digit $(l+2)$. However, the actions of E_ν and E_l are opposite to those of F_ν and F_l. The obtained tensor Young tableaux may be not standard, but they can be transformed into the sum of the standard tensor Young tableaux by symmetry.

Two standard tensor Young tableaux with different weights are orthogonal to each other. For a given irreducible representation $[\lambda]$ or $[+\lambda]$ of the SO($2l$), where the row number of Young pattern $[\lambda]$ is not larger than l, the highest weight state corresponds to the standard tensor Young tableau where each box in the βth row is filled with the digit β because every raising operator E_ν annihilates it. In the standard tensor Young tableau with the highest weight of the representation $[-\lambda]$, the box in the βth row is filled with the digit β, but the box in the lth row with the digit $(l+1)$. The highest weight $\mathbf{M}=\sum_\nu \omega_\nu M_\nu$ is calculated from (2.20) as

$$\begin{aligned}
&M_\nu=\lambda_\nu-\lambda_{\nu+1}, && && \nu\in[1,l-1),\\
&M_{l-1}=M_l=\lambda_{l-1}, && && \lambda_l=0,\\
&M_{l-1}=\lambda_{l-1}-\lambda_l, && M_l=\lambda_{l-1}+\lambda_l, && \text{for } [+\lambda],\\
&M_{l-1}=\lambda_{l-1}+\lambda_l, && M_l=\lambda_{l-1}-\lambda_l, && \text{for } [-\lambda].
\end{aligned}\tag{2.28}$$

The tensor representation $[\lambda]$ of the SO($2l$) with even $(M_{l-1}+M_l)$ is a single-valued representation. However, the representation with odd $(M_{l-1}+M_l)$ is a double-valued (spinor) representation.

The standard tensor Young tableaux are generally not traceless, but the standard tensor Young tableau with the highest weight is traceless because it only contains ϕ_β with $\beta<l+2$. Furthermore, l and $l+1$ do not appear in the tableau simultaneously as illustrated in Eq. (2.25). Since the highest weight is simple, the highest weight state is orthogonal to any other standard tensor Young tableau in the irreducible representation. Hence, we can obtain the remaining orthonormal and traceless basis tensors in the irreducible representation of the SO($2l$) from the highest weight state by the lowering operators F_ν in light of the method of the block weight diagram. The multiplicity of a weight in the representation can be easily obtained by counting the number of the traceless tensor Young tableaux with this weight.

4.4 Dimensions of Irreducible Tensor Representations

The dimension $d_{[\lambda]}$ of the representation $[\lambda]$ of the SO(N) can be calculated by hook rule [139, 140]. The dimension is expressed as a quotient, where the numerator and the denominator are denoted by the symbols $Y_T^{[\lambda]}$ and $Y_h^{[\lambda]}$, respectively:

$$\begin{aligned} d_{[\pm\lambda]}[\mathrm{SO}(2l)] &= \frac{Y_T^{[\lambda]}}{2Y_h^{[\lambda]}}, \qquad \text{when } \lambda_l \neq 0, \\ d_{[\lambda]}[\mathrm{SO}(N)] &= \frac{Y_T^{[\lambda]}}{2Y_h^{[\lambda]}}, \qquad \text{others.} \end{aligned} \tag{2.29}$$

The first formula in Eq. (2.29) corresponds to the case where the row number of the Young pattern $[\lambda]$ is equal to $N/2$. The hook path (i, j) in the Young pattern $[\lambda]$ is defined as a path which enters the Young pattern at the rightmost of the ith row, goes leftward in the i row, turns downward at the j column, goes downward in the j column, and leaves from the Young pattern at the bottom of the j column. The inverse hook path denoted by $\overline{(i, j)}$ is the same path as the hook path (i, j), but with opposite direction. The number of boxes contained in the path (i, j), as well as in its inverse, is the hook number h_{ij}. The $Y_h^{[\lambda]}$ represents a tableau of the Young pattern $[\lambda]$ where the box in the jth column of the ith row is filled with the hook number H_{ij}. However, the $Y_T^{[\lambda]}$ is a tableau of the Young pattern $[\lambda]$ where each box is filled with the sum of the digits which are respectively filled in the same box of each tableau $Y_{T_b}^{[\lambda]}$ in the series. The notation $Y_T^{[\lambda]}$ means the product of the filled digits in it, so does the notation $Y_h^{[\lambda]}$. Here, the tableaux $Y_{T_b}^{[\lambda]}$ can be obtained by the following rules:

- $Y_{T_0}^{[\lambda]}$ is a tableau of the Young pattern $[\lambda]$, where the box in the jth column of the ith row is filled with the digit $(N + j - i)$.
- Let $[\lambda^{(1)}] = [\lambda]$. Starting with $[\lambda^{(1)}]$, define recursively the Young pattern $[\lambda^{(b)}]$ by removing the first row and the first column of the Young pattern $[\lambda^{(b-1)}]$ until $[\lambda^{(b)}]$ contains less two columns.
- If $[\lambda^{(b)}]$ contains more than one column, define $Y_{T_b}^{[\lambda]}$ as a tableau of the Young pattern $[\lambda]$ where the boxes in the first $(b - 1)$ row and in the first $(b - 1)$ column are filled with 0, and the remaining part of the Young pattern is $[\lambda^{(b)}]$. Let $[\lambda^{(b)}]$ have r rows. Fill the first r boxes along the hook path $(1, 1)$ of the Young pattern $[\lambda^{(b)}]$, starting with the box on the rightmost, with the digits $(\lambda_1^{(b)} - 1), (\lambda_2^{(b)} - 1), \ldots, (\lambda_r^{(b)} - 1)$, box by box, and fill the first $(\lambda_i^{(b)} - 1)$ boxes in each inverse hook path $\overline{(i, 1)}$ of the Young pattern $[\lambda^{(b)}]$, $i \in [1, r]$ with "-1". The remaining boxes are filled with 0. If several "-1" are filled in the same box, the digits are summed. The sum of all filled digits in the pattern $Y_{T_b}^{[\lambda]}$ with $b > 0$ is equal to 0.

4.5 Adjoint Representation of the SO(N)

We are going to study the adjoint representation of the SO(N) by replacing the tensors. The $N(N-1)/2$ generators T_{ab} in the self-representation of the SO(N) construct the complete bases of N-dimensional antisymmetric matrices. Denote T_{cd} by T_A for convenience, $A \in [1, N(N-1)/2]$. Then we have

$$\mathrm{Tr}(T_A T_B) = 2\delta_{AB}. \tag{2.30}$$

Based on the adjoint representation $D^{\mathrm{ad}}(G)$ satisfying

$$D(R) I_B D(R)^{-1} = \sum_D I_D D^{\mathrm{ad}}_{DB}(R), \quad R \in \mathrm{SO}(N), \tag{2.31}$$

where R is an infinitesimal element, we have

$$R T_A R^{-1} = \sum_{B=1}^{N(N-1)/2} T_B D^{\mathrm{ad}}_{BA}(R). \tag{2.32}$$

The antisymmetric tensor T_{ab} of rank 2 of the SO(N) satisfies a similar relation in the SO(N) transformation R

$$(O_R T)_{cd} = \sum_{ij} R_{ci} T_{ij} (R^{-1})_{jd} = (R T R^{-1})_{cd}, \tag{2.33}$$

where T_{ab} like an antisymmetric matrix can be expanded by $(T_A)_{ab}$ as follows:

$$T_{cd} = \sum_{A=1}^{N(N-1)/2} (T_A)_{cd} F_A, \quad F_A = \frac{1}{2} \sum_{cd} (T_A)_{dc} T_{cd}, \tag{2.34}$$

where the coefficient F_A is a tensor that transforms in the SO(N) transformation R as follows:

$$\begin{aligned}
(O_R T)_{cd} &= (R T R^{-1})_{cd} \\
&= \sum_A (R T_A R^{-1})_{cd} F_A \\
&= \sum_B (T_B)_{cd} \left\{ \sum_A D^{\mathrm{ad}}_{BA}(R) F_A \right\}, \\
(O_R T)_{cd} &= \sum_B (T_B)_{cd} O_R F_B.
\end{aligned} \tag{2.35}$$

Thus, in terms of the adjoint representation of the SO(N) we can transform F_A in such a way

$$(O_R F)_B = \sum_A D^{\mathrm{ad}}_{BA}(R) F_A. \tag{2.36}$$

The adjoint representation of the SO(N) is equal to the antisymmetric tensor representation [1, 1] of rank 2. The adjoint representation of the SO(N) for $N = 3$ or $N > 4$ is irreducible. Except for $N = 2, 4$, the SO(N) is a simple Lie group.

4.6 Tensor Representations of the Groups $\mathbf{O}(N)$

It is known that the group O(N) is a mixed Lie group with two disjoint regions corresponding to $\det R = \pm 1$. Its invariant subgroup SO(N) has a connected group space corresponding to $\det R = 1$. The set of elements related to the $\det R = -1$ is the coset of SO(N). The property of the O(N) can be characterized completely by the SO(N) and a representative element in the coset [139, 140].

For odd $N = 2l + 1$, we may choose $\varepsilon = -1$ as the representative element in the coset since ε is self-inverse and commutes with every element in O($2l+1$). Thus, the representation matrix $D(\varepsilon)$ in the irreducible representation of O($2l+1$) is a constant matrix

$$D(\varepsilon) = c\mathbf{1}, \qquad D(\varepsilon)^2 = \mathbf{1}, \quad c = \pm 1. \tag{2.37}$$

Denote by R the element in SO($2l+1$) and by $R' = \varepsilon R$ the element in the coset. From each irreducible representation $D^{[\lambda]}(\mathrm{SO}(2l+1))$ one obtains two induced irreducible representations $D^{[\lambda]\pm}(\mathrm{O}(2l+1))$,

$$D^{[\lambda]\pm}(R) = D^{[\lambda]}(R), \qquad D^{[\lambda]\pm}(\varepsilon R) = \pm D^{[\lambda]}(R). \tag{2.38}$$

Two representations $D^{[\lambda]\pm}(\mathrm{O}(2l+1))$ are inequivalent because of different characters of the ε in two representations.

For even $N = 2l$, $\varepsilon = -1$ belongs to SO($2l$). We may choose the representative element in the coset to be a diagonal matrix σ, in which the diagonal entries are 1 except for $\sigma_{NN} = -1$. Even though $\sigma^2 = \mathbf{1}$, σ does not commute with some elements in O($2l$). Any tensor Young tableau $\mathcal{Y}_\nu^{[\lambda]}\theta_{\beta_1\cdots\beta_n}$ is an eigentensor of the σ with the eigenvalue 1 or -1 depending on whether the number of filled digits N in the tableau is even or odd. In the spherical harmonic basis tensors, σ interchanges the filled digits l and $l+1$ in the tensor Young tableau $\mathcal{Y}_\nu^{[\lambda]}\phi_{\beta_1\cdots\beta_n}$. Therefore, the representation matrix $D^{[\lambda]}(\sigma)$ is known.

Denote by R the element in the SO($2l$) and by $R' = \sigma R$ the element in the coset. From each irreducible representation $D^{[\lambda]}(\mathrm{SO}(2l))$, where the row number of $[\lambda]$ is less than l, we obtain two induced irreducible representations $D^{[\lambda]\pm}(\mathrm{O}(2l))$,

$$D^{[\lambda]\pm}(R) = D^{[\lambda]}(R), \qquad D^{[\lambda]\pm}(\sigma R) = \pm D^{[\lambda]}(\sigma)D^{[\lambda]}(R). \tag{2.39}$$

Likewise, two representations $D^{[\lambda]\pm}(\mathrm{O}(2l))$ are inequivalent due to the different characters of the σ in two representations.

When $l = N/2$ there are two inequivalent irreducible representations $D^{[(\pm)\lambda]}$ of the SO($2l$). Their basis tensors are given in Eq. (2.12). Since two terms in Eq. (2.12) contain different numbers of the subscripts N, then σ changes the tensor Young tableau in $[\pm\lambda]$ to that in $[\mp\lambda]$, i.e., the representation spaces of both $D^{[\pm\lambda]}(\mathrm{SO}(2l))$ correspond to an irreducible representation $D^{[\lambda]}$ of the O($2l$),

$$D^{[\lambda]}(R) = D^{[+\lambda]}(R) \oplus D^{[-\lambda]}(R), \qquad D^{[\lambda]}(\sigma R) = D^{[\lambda]}(\sigma)D^{[\lambda]}(R), \tag{2.40}$$

where the representation matrix $D^{[\lambda]}(\sigma)$ is calculated by interchanging the filled digits l and $(l+1)$ in the tensor Young tableau $\mathcal{Y}_\nu^{[\lambda]}\phi_{\beta_1\cdots\beta_n}$. Two representations with different signs of $D^{[\lambda]}(\sigma)$ are equivalent since they might be related by a similarity transformation

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.41}$$

5 Γ Matrix Groups

Dirac generalized the Pauli matrices to four γ matrices, which satisfy the anticommutation relations. In terms of the γ matrices, Dirac established the Dirac equation to describe the relativistic particle with spin $1/2$. In the language of group theory, Dirac found the spinor representation of the Lorentz group. In this section we first generalize the γ matrices and find that the set of products of the γ matrices forms the matrix group Γ.

5.1 Fundamental Property of Γ Matrix Groups

First, let us review the property of the Γ matrix groups [88–90]. We define N matrices γ_a, which satisfy the following anticommutation relations

$$\{\gamma_a, \gamma_b\} = \gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab}\mathbf{1}, \quad a, b \in [1, N]. \tag{2.42}$$

That is, $\gamma_a^2 = \mathbf{1}$ and $\gamma_a\gamma_b = -\gamma_b\gamma_a$ for $a \neq b$. The set of all products of the γ_a matrices, in the multiplication rule of matrices, forms a group, denoted by Γ_N. In a product of γ_a matrices, two γ_b with the same subscript can be moved together and eliminated by Eq. (2.42) so that Γ_N is a finite matrix group.

We choose a faithful irreducible unitary representation of the Γ_N as its self-representation. It is known from Eq. (2.42) that γ_a is unitary and hermitian,

$$\gamma_a^\dagger = \gamma_a^{-1} = \gamma_a, \tag{2.43}$$

whose eigenvalue is 1 or -1.

Let

$$\gamma_\xi^{(N)} = \gamma_1\gamma_2\cdots\gamma_N, \qquad \left(\gamma_\xi^{(N)}\right)^2 = (-1)^{N(N-1)/2}\mathbf{1}. \tag{2.44}$$

For odd N, since $\gamma_\xi^{(N)}$ commutes with every γ_a matrix, then it is a constant matrix according to the Schur theorem (see Appendix B):

$$\gamma_\xi^{(N)} = \begin{cases} \pm\mathbf{1}, & \text{for } N = 4l+1, \\ \pm i\mathbf{1}, & \text{for } N = 4l-1. \end{cases} \tag{2.45}$$

Two groups with different $\gamma_\xi^{(4l+1)}$ are isomorphic through a one-to-one correspondence, say

$$\gamma_a \leftrightarrow \gamma_a', \quad a \in [1, 4l], \qquad \gamma_{4l+1} \leftrightarrow -\gamma_{4l+1}'. \tag{2.46}$$

On the other hand, for a given $\gamma_\xi^{(4l+1)}$, the $\gamma_{4l+1}^{(4l+1)}$ can be expressed as a product of other γ_a matrices. As a result, all elements both in Γ_{4l} and in Γ_{4l+1} can be expressed as the products of matrices γ_a, $a \in [1, 4l]$ so that they are isomorphic. In addition, since $\gamma_\xi^{(4l-1)}$ is equal to either $i\mathbf{1}$ or $-i\mathbf{1}$, Γ_{4l-1} is isomorphic onto a group composed of the Γ_{4l-2} and $i\Gamma_{4l-2}$,

$$\Gamma_{4l+1} \approx \Gamma_{4l}, \qquad \Gamma_{4l-1} \approx \{\Gamma_{4l-2}, i\Gamma_{4l-2}\}. \tag{2.47}$$

5.2 Case $N = 2l$

- Let us calculate the order $g^{(2l)}$ of the Γ_{2l}. Obviously, if $R \in \Gamma_{2l}$, then $-R \in \Gamma_{2l}$, too. If we choose one element in each pair of elements $\pm R$, then we obtain a set Γ'_{2l} containing $g^{(2l)}/2$ elements. Denote by $\mathcal{S}_n$ a product of n different γ_a. Since the number of different $\mathcal{S}_n$ contained in the set Γ'_{2l} is equal to the combinatorics of n among $2l$, then we have

$$g^{(2l)} = 2\sum_{n=0}^{2l} \binom{2l}{n} = 2(1+1)^{2l} = 2^{2l+1}. \tag{2.48}$$

- For any element $\mathcal{S}_n \in \Gamma_{2l}$ except for ± 1, we may find a matrix γ_a which is anticommutable with $\mathcal{S}_n$. In fact, when n is even and γ appears in the product $\mathcal{S}_n$, one has $\gamma_a \mathcal{S}_n = -\mathcal{S}_n \gamma_a$. However, when n is odd there exists at least one γ_a which does not appear in the product $\mathcal{S}_n$ so that $\gamma_a \mathcal{S}_n = -\mathcal{S}_n \gamma_a$. Therefore, we find that

$$\operatorname{Tr} \mathcal{S}_n = \operatorname{Tr}(\gamma_a^2 \mathcal{S}_n) = -\operatorname{Tr}(\gamma_a \mathcal{S}_n \gamma_a) = -\operatorname{Tr} \mathcal{S}_n = 0. \tag{2.49}$$

  That is to say, the character of the element $\mathcal{S}$ in the self-representation of the Γ_{2l} is

$$\xi(\mathcal{S}) = \begin{cases} \pm d^{(2l)}, & \text{when } \mathcal{S} = \pm\mathbf{1}, \\ 0, & \text{when } \mathcal{S} \neq \pm\mathbf{1}, \end{cases} \tag{2.50}$$

  where $d^{(2l)}$ is the dimension of the γ_a. Since the self-representation of the Γ_{2l} is irreducible, we have

$$2\big(d^{(2l)}\big)^2 = \sum_{\mathcal{S} \in \Gamma_{2l}} |\xi(\mathcal{S})|^2 = g^{(2l)} = 2^{2l+1}, \qquad d^{(2l)} = 2^l. \tag{2.51}$$

  Based on Eqs. (2.43) and (2.50), we have $\det \gamma_a = 1$ for $l > 1$.
- Since $\gamma_\xi^{(2l)}$ is anticommutable with every γ_a, one may define $\gamma_f^{(2l)}$ by multiplying $\gamma_\xi^{(2l)}$ with a factor such that $\gamma_f^{(2l)}$ satisfies Eq. (2.42), i.e.,

$$\gamma_f^{(2l)} = (-i)^l \gamma_\xi^{(2l)} = (-i)^l \gamma_1 \gamma_2 \cdots \gamma_{2l}, \qquad \big(\gamma_f^{(2l)}\big)^2 = \mathbf{1}. \tag{2.52}$$

  Actually, $\gamma_f^{(2l)}$ can also be defined as the matrix γ_{2l+1} in Γ_{2l+1}.

- The matrices in the set Γ'_{2l} are linearly independent. Otherwise, there exists a linear relation $\sum_{\mathcal{S}} C(\mathcal{S})\mathcal{S} = 0$, $\mathcal{S} \in \Gamma'_{2l}$. By multiplying it with $R^{-1}/d^{(2l)}$ and taking the trace, one obtains any coefficient $C(R) = 0$. Thus, the set Γ'_{2l} contains 2^{2l} linear independent matrices of dimension $d^{(2l)} = 2^l$ so that they form a complete set of basis matrices. Any matrix M of dimension $d^{(2l)}$ can be expanded by $\mathcal{S} \in \Gamma'_{2l}$ as follows:

$$M = \sum_{\mathcal{S}\in\Gamma'_{2l}} C(\mathcal{S})\mathcal{S}, \qquad C(\mathcal{S}) = \frac{1}{d^{(2l)}} \mathrm{Tr}(\mathcal{S}^{-1}M). \tag{2.53}$$

- According to Eq. (2.42), the $\pm\mathcal{S}$ form a class, while $\mathbf{1}$ and $-\mathbf{1}$ form two classes, respectively. The Γ_{2l} group contains $(2^{2l} + 1)$ classes. Their representation is one-dimensional. Arbitrary chosen n matrices γ_a correspond to 1 and the remaining matrices γ_b correspond to -1. The number of the one-dimensional nonequivalent representations is calculated as

$$\sum_{n=0}^{2l} \binom{2l}{n} = 2^{2l}. \tag{2.54}$$

The remaining irreducible representation of the Γ_{2l} must be $d^{(2l)}$-dimensional, which is faithful. The γ_a matrices in the representation are called the irreducible γ_a matrices, which may be written as:

$$\begin{aligned}
\gamma_{2n-1} &= \underbrace{\mathbf{1} \times \cdots \times \mathbf{1}}_{n-1} \times \sigma_1 \times \underbrace{\sigma_3 \times \cdots \times \sigma_3}_{l-n}, \\
\gamma_{2n} &= \underbrace{\mathbf{1} \times \cdots \times \mathbf{1}}_{n-1} \times \sigma_2 \times \underbrace{\sigma_3 \times \cdots \times \sigma_3}_{l-n}, \\
\gamma_f^{(2l)} &= \underbrace{\sigma_3 \times \cdots \times \sigma_3}_{l}.
\end{aligned} \tag{2.55}$$

Since $\gamma_f^{(2l)}$ is diagonal, the forms of Eq. (2.55) are called the reduced spinor representations. Remember that the eigenvalues ± 1 are arranged in the diagonal line of the $\gamma_f^{(2l)}$ in mixed way.

- Let us mention an equivalent theorem for the γ_a matrices.

Theorem 2.1 *Two sets of $d^{(2l)}$-dimensional matrices γ_a and $\bar{\gamma}_a$ satisfying the anticommutation relation* (2.42), *where $N = 2l$, are equivalent*

$$\bar{\gamma}_a = X^{-1}\gamma_a X, \quad a \in [1, 2l]. \tag{2.56}$$

The similarity transformation matrix X is determined up to a constant factor. If the determinant of the matrix X is constrained to be 1, there are $d^{(2l)}$ choices for the factor:

$$\exp[-i2n\pi/d^{(2l)}], \quad n \in [0, d^{(2l)}). \tag{2.57}$$

5.3 Case $N = 2l + 1$

Since $\gamma_f^{(2l)}$ and $(2l)$ matrices γ_a in Γ_{2l}, $a \in [1, 2l]$, satisfy the antisymmetric relation (2.42), then they can be defined to be the $(2l + 1)$ matrices γ_a in Γ_{2l+1}. In this definition, $\gamma_\xi^{(2l+1)}$ in Γ_{2l+1} is chosen as

$$\gamma_{2l+1} = \gamma_f^{(2l)}, \qquad \gamma_\xi^{(2l+1)} = \gamma_1 \cdots \gamma_{2l+1} = i^l \mathbf{1}. \tag{2.58}$$

Obviously, the dimension $d^{(2l+1)}$ of the matrices in Γ_{2l+1} is the same as $d^{(2l)}$ in Γ_{2l},

$$d^{(2l+1)} = d^{(2l)} = 2^l. \tag{2.59}$$

For odd N, the equivalent theorem must be modified because the multiplication rule of elements in Γ_{2l+1} includes Eq. (2.45). A similarity transformation cannot change the sign of $\gamma_\xi^{(2l+1)}$, i.e., the equivalent condition for two sets of γ_a and $\bar{\gamma}_a$ has to include a new condition $\gamma_\xi = \bar{\gamma}_\xi$, in addition to those given in Theorem 2.1.

If we take $\bar{\gamma}_a = -(\gamma_a)^T$, then we have

$$\begin{aligned}
\bar{\gamma}_\xi^{(2l+1)} &= \bar{\gamma}_1 \cdots \bar{\gamma}_{2l+1} = -\{\gamma_{2l+1} \cdots \gamma_1\}^T \\
&= (-1)^{l+1} \{\gamma_\xi^{(2l+1)}\}^T \\
&= (-1)^{l+1} \gamma_\xi^{(2l+1)}.
\end{aligned} \tag{2.60}$$

6 Spinor Representations of the SO(N)

6.1 Covering Groups of the SO(N)

Based on a set of N irreducible unitary matrices γ_a satisfying the anticommutation relation (2.42), we define

$$\bar{\gamma}_a = \sum_{i=1}^{N} R_{ai} \gamma_i, \qquad R \in \mathrm{SO}(N). \tag{2.61}$$

Since R is a real orthogonal matrix, then $\bar{\gamma}_a$ satisfy

$$\begin{aligned}
\bar{\gamma}_a \bar{\gamma}_b + \bar{\gamma}_b \bar{\gamma}_a &= \sum_{ij} R_{ai} R_{bj} \{\gamma_i \gamma_j + \gamma_j \gamma_i\} \\
&= 2 \sum_i R_{ai} R_{bi} \mathbf{1} \\
&= 2\delta_{ab} \mathbf{1}.
\end{aligned} \tag{2.62}$$

Due to Eq. (2.42) and $\sum_a R_{1a} R_{2a} = 0$, we have

$$\sum_{c_1 c_2} R_{1c_1} R_{2c_2} \gamma_{c_1} \gamma_{c_2} = \frac{1}{2} \sum_{c_1 \neq c_2} R_{1c_1} R_{2c_2} (\gamma_{c_1} \gamma_{c_2} - \gamma_{c_2} \gamma_{c_1}), \tag{2.63}$$

$$\begin{aligned} \bar{\gamma}_1 \bar{\gamma}_2 \cdots \bar{\gamma}_N &= \sum_{c_1 \cdots c_N} R_{1\,c_1} \cdots R_{N\,c_N} \gamma_{c_1} \cdots \gamma_{c_N} \\ &= \sum_{c_1 \cdots c_N} R_{1\,c_1} \cdots R_{N\,c_N} \epsilon_{c_1 \cdots c_N} \gamma_1 \gamma_2 \cdots \gamma_N \\ &= (\det R) \gamma_1 \gamma_2 \cdots \gamma_N = \gamma_1 \gamma_2 \cdots \gamma_N. \end{aligned} \tag{2.64}$$

From Theorem 2.1, we know that γ_a and $\bar{\gamma}_a$ are related by a unitary similarity transformation $D(R)$ with determinant 1,

$$D(R)^{-1} \gamma_a D(R) = \sum_{i=1}^{N} R_{ai} \gamma_i, \quad \det D(R) = 1, \tag{2.65}$$

where $D(R)$ is determined up to a constant $\exp[-i2n\pi/d^{(N)}], n \in [0, d^{(N)})$. In terms of the definition of the group, the set of $D(R)$ defined in Eq. (2.65) and operated in the multiplication rule of matrices, forms a Lie group G'_N. There exists a $d^{(N)}$-to-one correspondence between the elements in G'_N and those in SO(N), and the correspondence keeps invariant in the multiplication of elements. Therefore, the G'_N is homomorphic to SO(N). Because the group space of the SO(N) is doubly-connected, its covering group is homomorphic to it by a two-to-one correspondence. As a result, the group space of the G'_N must fall into several disjoint pieces, where the piece containing the identity element E forms an invariant subgroup G_N of the G'_N. The G_N is a connected Lie group and becomes the covering group of the SO(N) . Since the group space of G_N is connected, based on the property of the infinitesimal elements, a discontinuous condition can be found to pick up G_N from the G'_N.

Let R be an infinitesimal element. We may expand R and $D(R)$ with respect to the infinitesimal parameters $\omega_{\alpha\beta}$ as follows

$$\begin{aligned} R_{ab} &= \delta_{ab} - i \sum_{\alpha<\beta} \omega_{\alpha\beta} (T_{\alpha\beta})_{ab} = \delta_{ab} - \omega_{ab}, \\ D(R) &= \mathbf{1} - i \sum_{\alpha<\beta} \omega_{\alpha\beta} S_{\alpha\beta}, \end{aligned} \tag{2.66}$$

where $T_{\alpha\beta}$ are the generators in the self-representation of the SO(N) as given in Eq. (2.14). The $S_{\alpha\beta}$ are the generators in G_N. From Eq. (2.65) one has

$$[\gamma_c, S_{\alpha\beta}] = \sum_d (T_{\alpha\beta})_{cd} \gamma_d = -i \{\delta_{\alpha c} \gamma_\beta - \delta_{\beta c} \gamma_\alpha\}, \tag{2.67}$$

from which we obtain

$$S_{\alpha\beta} = \frac{1}{4i} (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha). \tag{2.68}$$

It is easy to prove that $S_{\alpha\beta}$ is hermitian since $D(R)$ is unitary.

Define[3]

$$C = \begin{cases} B^{(N)}, & \text{when } N = 4l+1, \\ C^{(N)}, & \text{when } N \neq 4l+1. \end{cases} \tag{2.69}$$

Based on this, we have

$$\begin{aligned} C^{-1} S_{\alpha\beta} C &= -(S_{\alpha\beta})^T = -S^*_{\alpha\beta}, \\ C^{-1} D(R) C &= \{D(R^{-1})\}^T = D(R)^*. \end{aligned} \tag{2.70}$$

This discontinuous condition restricts the factor in $D(R)$ such that there is a two-to-one correspondence between $\pm D(R)$ in G_N and R in SO(N) through relations (2.65) and (2.70). That is to say, the G_N is the covering group of SO(N),

$$\mathrm{SO}(N) \sim G_N, \tag{2.71}$$

where the G_N is the fundamental spinor representation denoted by $D^{[s]}(\mathrm{SO}(N))$. Therefore, the $S_{\alpha\beta}$ represent the spinor angular momentum operators [88–90]. The irreducible tensor representation $[\lambda]$ is a single-valued representation of the SO(N), but a non-faithful representation of G_N because its faithful representation is a double-valued representation of the SO(N).

Since the products $\mathcal{S}_n$ span a complete set of the $d^{(N)}$-dimensional matrices, this can be decided by checking the commutation relations of the $\mathcal{S}_n$ with the generators $S_{\alpha\beta}$ whether there exists a non-constant matrix commutable with all $S_{\alpha\beta}$. It is found that only $\gamma_\xi^{(N)}$ is commutable with all $S_{\alpha\beta}$. The $\gamma_\xi^{(2l+1)}$ is a constant matrix so that the fundamental spinor representation $D^{[s]}(\mathrm{SO}(2l+1))$ is irreducible and self-conjugate.

On the contrary, since $\gamma_\xi^{(2l)}$ is not a constant matrix so that the fundamental spinor representation $D^{[s]}(\mathrm{SO}(2l))$ is reducible. By a similarity transformation X, the $\gamma_f^{(2l)}$ can be transferred to $\sigma_3 \times \mathbf{1}$ and $D^{[s]}(\mathrm{SO}(2l))$ is reduced to the direct sum of two irreducible representations

$$X^{-1} D^{[s]}(R) X = \begin{pmatrix} D^{[+s]}(R) & 0 \\ 0 & D^{[-s]}(R) \end{pmatrix}. \tag{2.72}$$

Two representations $D^{[\pm s]}(\mathrm{SO}(2l))$ are proved inequivalent by leading to an absurdity. In fact, if $Z^{-1} D^{[-s]}(R) Z = D^{[+s]}(R)$ and $Y = \mathbf{1} \oplus Z$, then all generators $(XY)^{-1} S_{\alpha\beta} XY$ are commutable with $\sigma_1 \times \mathbf{1}$, but their product is not commutable with it,

$$2^l (XY)^{-1} (S_{12} S_{34} \cdots S_{(2l-1)(2l)}) XY = Y^{-1} [X^{-1} \gamma_f^{(2l)} X] Y = \sigma_3 \times \mathbf{1}, \tag{2.73}$$

which results in a contradiction.

[3] B^N is the strong space-time reflection matrix and C^N is the charge conjugation matrix, which are usually used in particle physics.

Introduce two project operators $P_\pm$ [139, 140],

$$\begin{aligned}
&P_\pm = \frac{1}{2}\big(\mathbf{1} \pm \gamma_f^{(2l)}\big), && P_\pm D^{[s]}(R) = D^{[s]} P_\pm, \\
&X^{-1} P_+ X = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}, && X^{-1} P_+ D^{[s]}(R) X = \begin{pmatrix} D^{[+s]}(R) & 0 \\ 0 & 0 \end{pmatrix}, \\
&X^{-1} P_- X = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}, && X^{-1} P_- D^{[s]}(R) X = \begin{pmatrix} 0 & 0 \\ 0 & D^{[-s]}(R) \end{pmatrix}.
\end{aligned} \tag{2.74}$$

From the following relation

$$(C^{(2l)})^{-1} \gamma_f^{(2l)} C^{(2l)} = (-i)^l (\gamma_1)^T (\gamma_2)^T \cdots (\gamma_{2l})^T = (-1)^l (\gamma_f^{(2l)})^T, \tag{2.75}$$

where $C^{(2l)}$ and T denote the charge conjugation matrix and the transpose of the matrix, respectively, one has

$$C^{-1} D^{[s]}(R) P_\pm C = \begin{cases} D^{[s]}(R)^* P_\pm, & \text{when } N = 4l, \\ D^{[s]}(R)^* P_\mp, & \text{when } N = 4l + 2. \end{cases} \tag{2.76}$$

Two non-equivalent representations $D^{[\pm s]}(R)$ are conjugate to each other when $N = 4l + 2$, while they are self-conjugate when $N = 4l$. The dimension of the irreducible spinor representations of the SO(N) is calculated as

$$d_{[s]}[\mathrm{SO}(2l+1)] = 2^l, \qquad d_{[\pm s]}[\mathrm{SO}(2l)] = 2^{(l-1)}. \tag{2.77}$$

6.2 *Fundamental Spinors of the* SO(N)

For an SO(N) transformation R, Ψ is called the fundamental spinor of the SO(N) if it transforms through the fundamental spinor representation $D^{[s]}(R)$:

$$(O_R \Psi)_\nu = \sum_\mu D^{[s]}_{\nu\mu}(R) \Psi_\mu, \qquad O_R \Psi = D^{[s]}(R) \Psi, \tag{2.78}$$

where Ψ is a column matrix with $d_{[s]}$ components.

The Chevalley bases $H_\nu(S)$, $E_\nu(S)$ and $F_\nu(S)$ with respect to the spinor angular momentum can be obtained from Eqs. (2.18) and (2.24) through replacing T_{ab} by S_{ab}. In the chosen forms of γ_a given in Eq. (2.55), the Chevalley bases for the SO($2l + 1$) group are given by

$$\begin{aligned}
H_\nu(S) &= \underbrace{\mathbf{1} \times \cdots \times \mathbf{1}}_{\nu-1} \times \frac{1}{2}\{\sigma_3 \times \mathbf{1} - \mathbf{1} \times \sigma_3\} \times \underbrace{\mathbf{1} \times \cdots \times \mathbf{1}}_{l-\nu-1}, \\
H_l(S) &= \underbrace{\mathbf{1} \times \cdots \times \mathbf{1}}_{l-1} \times \sigma_3, \\
E_\nu(S) &= \underbrace{\mathbf{1} \times \cdots \times \mathbf{1}}_{\nu-1} \times \{\sigma_+ \times \sigma_-\} \times \underbrace{\mathbf{1} \times \cdots \times \mathbf{1}}_{l-\nu-1} = F_\nu(S)^T, \\
E_l(S) &= \underbrace{\sigma_3 \times \cdots \times \sigma_3}_{l-1} \times \sigma_+ = F_l(S)^T,
\end{aligned} \tag{2.79}$$

where $\nu \in [1, l)$. The Chevalley bases for the SO($2l$) group are the same as those for the SO($2l+1$) except for $\nu = l$,

$$\begin{aligned} H_l(S) &= \underbrace{\mathbf{1} \times \cdots \times \mathbf{1}}_{l-2} \times \frac{1}{2}\{\sigma_3 \times \mathbf{1} + \mathbf{1} \times \sigma_3\}, \\ E_l(S) &= -\underbrace{\mathbf{1} \times \cdots \times \mathbf{1}}_{l-2} \times \{\sigma_+ \times \sigma_+\} = F_l(S)^T. \end{aligned} \tag{2.80}$$

The basis spinor $\xi[\mathbf{m}]$ of the SO(N) can also be expressed as a direct product of l two-dimensional basis spinors $\xi(\beta)$,

$$\xi[\mathbf{m}] = \xi(\beta_1, \beta_2, \ldots, \beta_l) = \xi(\beta_1)\xi(\beta_2)\ldots\xi(\beta_l), \tag{2.81}$$

$$\xi(+) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \xi(-) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.82}$$

For even N, the fundamental spinor space can be decomposed into two subspaces by the project operators $P_\pm$, $\Psi_\pm = P_\pm \Psi$, corresponding to irreducible spinor representations $D^{[\pm s]}$. The basis spinor in the representation space of $D^{[+s]}$ contains even number of factors $\xi(-)$, and that of $D^{[-s]}$ contains odd number of factors $\xi(-)$. The highest weight states $\xi[\mathbf{M}]$ and their highest weights $\mathbf{M}$ are given by

$$\begin{aligned} &\underbrace{\xi(+)\ldots\xi(+)}_{l-1}\xi(+), && \mathbf{M} = [\underbrace{0, \ldots, 0}_{l-1}, 1], && [s] \text{ of the SO}(2l+1), \\ &\underbrace{\xi(+)\ldots\xi(+)}_{l-1}\xi(+), && \mathbf{M} = [\underbrace{0, \ldots, 0}_{l-2}, 0, 1], && [+s] \text{ of the SO}(2l), \\ &\underbrace{\xi(+)\ldots\xi(+)}_{l-1}\xi(-), && \mathbf{M} = [\underbrace{0, \ldots, 0}_{l-2}, 1, 0], && [-s] \text{ of the SO}(2l). \end{aligned} \tag{2.83}$$

The remaining basis states are calculated by the applications of lowering operators $F_\nu(S)$.

6.3 Direct Products of Spinor Representations

Since the spinor representation is unitary so that we have

$$O_R\Psi^\dagger = \Psi^\dagger D^{[s]}(R)^{-1}, \tag{2.84}$$

$$\begin{aligned} &\Psi^\dagger\Psi = \textstyle\sum_\mu \Psi_\mu^*\Psi_\mu = \sum_{\mu\nu} \Psi_\mu^*\delta_{\mu\nu}\Psi_\nu, \\ &O_R(\Psi^\dagger\Psi) = \Psi^\dagger D^{[s]}(R)^{-1} D^{[s]}(R)\Psi = \Psi^\dagger\Psi, \end{aligned} \tag{2.85}$$

which means that $\Psi^\dagger\Psi$ keeps invariant in the SO(N) transformations and is a scalar of the SO(N). In other words, the products of $\Psi_\mu^\dagger$ and Ψ_ν span an invariant linear space, corresponding to the direct product representation $D^{[s]*} \times D^{[s]}$ of the SO(N). In the reduction of $D^{[s]*} \times D^{[s]}$ there is an identical representation where the CGCs are $\delta_{\mu\nu}$. In general, one has

$$\begin{aligned} O_R(\Psi^\dagger \gamma_{a_1}\cdots\gamma_{a_n}\Psi) &= \Psi^\dagger D^{[s]}(R)^{-1}\gamma_{a_1}\cdots\gamma_{a_n}D^{[s]}(R)\Psi \\ &= \sum_{c_1\ldots c_n} R_{a_1c_1}\cdots R_{a_nc_n}\Psi^\dagger\gamma_{c_1}\cdots\gamma_{c_n}\Psi, \end{aligned} \tag{2.86}$$

where $\Psi^\dagger\gamma_{a_1}\cdots\gamma_{a_n}\Psi$ is an antisymmetric tensor of rank n of the SO(N) corresponding to the Young pattern $[1^n]$ with $n \le N$. Otherwise, the respective γ_a can be moved together and eliminated.

When $N = 2l+1$, the $\gamma_f^{(2l+1)}$ is a constant matrix so that the product of $(N-n)$ matrices γ_a can be changed to a product of n matrices γ_a. Thus, the rank n of the tensor (2.86) is less than $N/2$, and the Clebsch-Gordan series is given by

$$[s]^* \times [s] \simeq [s]\times[s] \simeq [0]\oplus[1]\oplus[1^2]\oplus\cdots\oplus[1^l], \quad \text{for SO}(2l+1). \tag{2.87}$$

The matrix entries of product of γ_a are the CGCs. The highest weight in product space is given by $\mathbf{M} = [0,\ldots,0,2]$ corresponding to representation $[1^l]$.

When $N = 2l$, according to the property of the project operators $P_\pm$,

$$\begin{aligned} &P_+P_- = P_-P_+ = 0, \qquad P_\pm P_\pm = P_\pm, \qquad \gamma_f^{(2l)}P_\pm = \pm P_\pm, \\ &P_\mp\gamma_{b_1}\cdots\gamma_{b_{2n}}P_\pm = 0, \qquad P_\pm\gamma_{b_1}\cdots\gamma_{b_{2n+1}}P_\pm = 0, \end{aligned} \tag{2.88}$$

the product of the $(N-n)$ matrices γ_b can still be changed to a product of n matrices γ_b. If $n = l$, we have

$$\begin{aligned} &\gamma_1\gamma_2\cdots\gamma_l = (-i)^l\gamma_{2l}\gamma_{2l-1}\cdots\gamma_{l+1}\gamma_f^{(2l)}, \\ &\gamma_1\gamma_2\cdots\gamma_l P_\pm = \frac{1}{2}\{\gamma_1\gamma_2\cdots\gamma_l \pm (-i)^l\gamma_{2l}\gamma_{2l-1}\cdots\gamma_{l+1}\}P_\pm. \end{aligned} \tag{2.89}$$

If $N = 4l$, we have

$$\begin{aligned} &[\pm s]^* \times [\pm s] \simeq [\pm s]\times[\pm s] \simeq [0]\oplus[1^2]\oplus[1^4]\oplus\cdots\oplus[(\pm 1)1^{2l}], \\ &[\mp s]^* \times [\pm s] \simeq [\mp s]\times[\pm s] \simeq [1]\oplus[1^3]\oplus[1^5]\oplus\cdots\oplus[1^{2l-1}]. \end{aligned} \tag{2.90}$$

If $N = 4l+2$, one has

$$\begin{aligned} &[\pm s]^* \times [\pm s] \simeq [\mp s]\times[\pm s] \simeq [0]\oplus[1^2]\oplus[1^4]\oplus\cdots\oplus[1^{2l}], \\ &[\mp]^* \times [\pm s] \simeq [\pm s]\times[\pm s] \simeq [1]\oplus[1^3]\oplus[1^5]\oplus\cdots\oplus[(\pm)1^{2l+1}]. \end{aligned} \tag{2.91}$$

The self-dual and anti-self-dual representations occur in the reduction of the direct product $[\pm s]\times[\pm s]$, but not in the reduction of $[+s]\times[-s]$. The highest weights are $\mathbf{M} = [0,\ldots,0,0,2]$ in the product space $[+s]\times[+s]$, $\mathbf{M} = [0,\ldots,0,2,0]$ in $[-s]\times[-s]$, and $\mathbf{M} = [0,\ldots,0,1,1]$ in $[+s]\times[-s]$.

6.4 Spinor Representations of Higher Ranks

In the SO(3) group, $D^{1/2}$ is a fundamental spinor representation. The spinor representations D^j of higher ranks can be obtained by reducing the direct product of the fundamental spinor representation and a tensor representation,

$$D^{1/2}\times D^l \simeq D^{l+1/2}\oplus D^{l-1/2}. \tag{2.92}$$

The spinor representations of higher ranks of the SO(N) can also be obtained in a similar way.

A spinor $\Psi_{a_1\ldots a_n}$ with the tensor indices is called a spin-tensor if it transforms in $R \in \mathrm{SO}(N)$ as follows:

$$(O_R\Psi)_{a_1\cdots a_n} = \sum_{c_1\cdots c_n} R_{a_1c_1}\cdots R_{a_nc_n} D^{[s]}(R)\Psi_{c_1\cdots c_n}. \tag{2.93}$$

The tensor part of the spin-tensor can be decomposed into a direct sum of the traceless tensors with different ranks. Each traceless tensor subspace can be reduced by the projection of the Young operators. Thus, the reduced subspace of the traceless tensor part of the spin-tensor is denoted by a Young pattern $[\lambda]$ or $[\pm\lambda]$ where the row number of $[\lambda]$ is not larger than $N/2$. However, this subspace of the spin-tensor corresponds to the direct product of the fundamental spinor representation $[s]$ and the irreducible tensor representation $[\lambda]$ or $[\pm\lambda]$, and it is still reducible. It is required to find a new restriction to pick up the irreducible subspace like the subspace of $D^{l+1/2}$ in Eq. (2.92) for the SO(3) group. The restriction is from the so-called trace of the second kind of the spin-tensor which keeps invariant in the SO(N) transformations:

$$\Phi_{a_1\cdots a_{i-1}a_{i+1}\cdots a_n} = \sum_{c=1}^{N} \gamma_c \Psi_{a_1\cdots a_{i-1}\,c\,a_{i+1}\cdots a_n}, \tag{2.94}$$

and

$$\begin{aligned}
&(O_R\Phi)_{a_1\cdots a_{i-1}a_{i+1}\cdots a_n} \\
&\quad = \sum_{c_1\cdots c_n c'} R_{a_1c_1}\cdots R_{a_nc_n}\left[\sum_c \gamma_c R_{cc'}\right] D^{[s]}(R)\Psi_{c_1\cdots c_{i-1}c'c_{i+1}\cdots c_n} \\
&\quad = \sum_{c_1\cdots c_n} R_{a_1c_1}\cdots R_{a_nc_n} D^{[s]}(R)\left[\sum_{c'} \gamma_{c'}\Psi_{c_1\cdots c_{i-1}c'c_{i+1}\cdots c_n}\right] \\
&\quad = \sum_{c_1\cdots c_n} R_{a_1c_1}\cdots R_{a_nc_n} D^{[s]}(R)\Psi_{c_1\cdots c_{i-1}c_{i+1}\cdots c_n}.
\end{aligned} \tag{2.95}$$

The irreducible subspace of the SO(N) contained in the spin-tensor space, in addition to the projection of a Young operator, satisfies the usual traceless conditions of tensors and the traceless conditions of the second kind

$$\sum_d \psi_{a\cdots d\cdots d\cdots c} = 0, \qquad \sum_d \gamma_d \psi_{a\cdots d\cdots c} = 0. \tag{2.96}$$

The highest weight $\mathbf{M}$ of the irreducible representation is the highest weight in the direct product space. The irreducible representation is denoted by $[s,\lambda]$ for the SO($2l+1$)

$$\begin{cases} [s]\times[\lambda] \simeq [s,\lambda]\oplus\cdots, \\ \mathbf{M} = [(\lambda_1-\lambda_2),\ldots,(\lambda_{l-1}-\lambda_l),(2\lambda_l+1)], \end{cases} \tag{2.97}$$

and $[\pm s, \lambda]$ for the SO($2l$)

$$\begin{cases} [+s] \times [\lambda] \text{ or } [+s] \times [+\lambda] \simeq [+s, \lambda] \oplus \cdots, \\ \mathbf{M} = [(\lambda_1 - \lambda_2), \ldots, (\lambda_{l-1} - \lambda_l), (\lambda_{l-1} + \lambda_\lambda + 1)], \end{cases}$$
$$\begin{cases} [-s] \times [\lambda] \text{ or } [-s] \times [-\lambda] \simeq [-s, \lambda] \oplus \cdots, \\ \mathbf{M} = [(\lambda_1 - \lambda_2), \ldots, (\lambda_{l-1} + \lambda_l + 1), (\lambda_{l-1} - \lambda_l)], \end{cases}$$
$$\begin{cases} [+s] \times [-\lambda] \simeq [-s, \lambda_1, \lambda_2, \ldots, \lambda_{l-1}, (\lambda_l - 1)] \oplus \cdots, \\ \mathbf{M} = [(\lambda_1 - \lambda_2), \ldots, (\lambda_{l-1} + \lambda_l), (\lambda_{l-1} - \lambda_l + 1)], \end{cases} \tag{2.98}$$
$$\begin{cases} [-s] \times [+\lambda] \simeq [+s, \lambda_1, \lambda_2, \ldots, \lambda_{l-1}, (\lambda_l - 1)] \oplus \cdots, \\ \mathbf{M} = [(\lambda_1 - \lambda_2), \ldots, (\lambda_{l-1} - \lambda_l + 1), (\lambda_{l-1} + \lambda_l)]. \end{cases}$$

These irreducible representations $[s, \lambda]$ of the SO($2l + 1$) and $[\pm s, \lambda]$ of the SO($2l$) are called the spinor representations of higher ranks. It should be noted that the row number of the Young pattern $[\lambda]$ in the spinor representation of higher rank is not larger than l. Otherwise, the space is null.

The remaining representations in the Clebsch-Gordan series (2.97) and (2.98) are calculated by the method of dominant weight diagram. For example, when $[\lambda]$ is a one-row Young diagram, one has

$$\begin{aligned} &\text{SO}(2l+1): \quad [s] \times [\lambda, 0, \ldots, 0] \simeq [s, \lambda, 0, \ldots, 0] \oplus [s, \lambda - 1, 0, \ldots, 0], \\ &\text{SO}(2l): \quad [\pm s] \times [\lambda, 0, \ldots, 0] \simeq [\pm s, \lambda, 0, \ldots, 0] \oplus [\mp s, \lambda - 1, 0, \ldots, 0], \end{aligned} \tag{2.99}$$

where $[\mp s, \lambda - 1, 0, \ldots, 0]$ appears because the factor γ_b in Eq. (2.96) is anticommutable with γ_f in $P_\pm$.

6.5 Dimensions of the Spinor Representations

In a similar way, the dimension of a spinor representation $[s, \lambda]$ of the SO($2l+1$) or $[\pm s, \lambda]$ of the SO($2l$) can be calculated by hook rule. The dimension is expressed as a quotient multiplied with the dimension of the fundamental spinor representation, where the numerator and the denominator are denoted by the symbols $Y_S^{[\lambda]}$ and $Y_h^{[\lambda]}$, respectively:

$$\begin{aligned} d_{[s,\lambda]}[\text{SO}(2l+1)] &= 2^l \frac{Y_S^{[\lambda]}}{Y_h^{[\lambda]}}, \\ d_{[\pm s,\lambda]}[\text{SO}(2l)] &= 2^{l-1} \frac{Y_S^{[\lambda]}}{Y_h^{[\lambda]}}. \end{aligned} \tag{2.100}$$

The concepts of a hook path (i, j) and an inverse hook path $\overline{i, j}$ have been discussed above. The number of boxes contained in the hook path (i, j) is the hook number h_{ij} of the box in the jth column of the ith row. The $Y_h^{[\lambda]}$ is a tableau of the Young pattern $[\lambda]$ where the box in the jth column of the ith row is filled with the hook number h_{ij}. The $Y_S^{[\lambda]}$ is a tableau of the Young pattern $[\lambda]$ where each box is filled with the sum of the digits which are respectively filled in the same box of each tableau $Y_{S_b}^{[\lambda]}$ in the series. The notation $Y_S^{[\lambda]}$ means the product of the filled digits in it, so does the notation $Y_h^{[\lambda]}$. The tableaux $Y_{S_b}^{[\lambda]}$ are defined by the following rules:

- $Y_{S_0}^{[\lambda]}$ is a tableau of the Young pattern $[\lambda]$ where the box in the jth column of the ith row is filled with the digit $(N-1+j-i)$.
- Let $[\lambda^{(1)}]=[\lambda]$. Staring with $[\lambda^{(1)}]$, we define recursively the Young pattern $[\lambda^{(b)}]$ by removing the first row and the first column of the Young pattern $[\lambda^{(b-1)}]$ until $[\lambda^{(b)}]$ contains less two columns.
- If $[\lambda^{(b)}]$ contains more than one column, we define $Y_{S_b}^{[\lambda]}$ as the tableau of the Young pattern $[\lambda]$ where the boxes in the first $(b-1)$ row and column are filled with 0, and the remaining part of the Young pattern is $[\lambda^{(b)}]$. Let $[\lambda^{(b)}]$ have r rows. Fill the first r boxes along the hook path $(1,1)$ of the Young pattern $[\lambda^{(b)}]$, starting with the box on the rightmost, with the digits $(\lambda_1^{(b)}-1)$, $(\lambda_2^{(b)}-1),\ldots,(\lambda_r^{(b)}-1)$, box by box, and fill the first $(\lambda_i^{(b)}-1)$ boxes in each inverse hook path $\overline{(i,1)}$ of the Young pattern $[\lambda^{(b)}]$, $i\in[2,r]$ with "-1". The remaining boxes are filled with 0. If several "-1" are filled in the same box, the digits are summed. The sum of all filled digits in the pattern $Y_{S_b}^{[\lambda]}$ with $b>0$ is equal to 0.

7 Concluding Remarks

In this Chapter we have sketched some basic properties for the Lie group SO(N) since it shall be very helpful in successive several Chapters. The tensor and spinor representations of the SO(N) group, the calculation of the dimensions of irreducible tensor and spinor representations have been addressed. The more information about the properties of the Lie groups and Lie algebras, in particular the SO(N) group as well as the corresponding Lie algebra may refer to textbooks [136, 138–140].

Springer

http://www.springer.com/978-94-007-1916-3

Wave Equations in Higher Dimensions
Dong, S.-H.
2011, XXV, 295 p., Hardcover
ISBN: 978-94-007-1916-3