Chapter 2

Linear and quasi-linear systems with piecewise constant argument

In this chapter we start investigation with the most simple linear systems of differential equations with piecewise constant argument. Then the analysis will be extended to quasi-linear systems. Existence-uniqueness of solutions, the linear space of solutions, fundamental matrix, stability problems are under discussion.
We consider the following two equations

$$z'(t) = A_0(t)z(t) + A_1(t)z(\gamma(t)), \tag{2.1}$$

and

$$z'(t) = A_0(t)z(t) + A_1(t)z(\gamma(t)) + f(t, z(t), z(\gamma(t))), \tag{2.2}$$

where $z \in \mathbb{R}^n$, $t \in \mathbb{R}$. The argument-function γ was introduced in the last chapter. However, in this chapter, it is considered with $J = \mathbb{R}$ and $\mathscr{A} = \mathbb{Z}$.
The equation (2.1) is a linear homogeneous system with argument-function $\gamma(t)$, and equation (2.2) is a quasilinear system.
The following assumptions will be needed throughout this chapter:

(C1) $A_0, A_1 \in C(\mathbb{R})$ are $n \times n$ real valued matrices;
(C2) $f(t,x,y) \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ is an $n \times 1$ real valued function;
(C3) $f(t,x,y)$ satisfies the condition

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \leqslant L(\|x_1 - x_2\| + \|y_1 - y_2\|), \tag{2.3}$$

for some positive constant L, and satisfies the condition

$$f(t,0,0) = 0, \quad t \in \mathbb{R}; \tag{2.4}$$

(C4) matrices A_0, A_1 are uniformly bounded on $\mathbb{R}$;
(C5) $\inf_{\mathbb{R}} \|A_1(t)\| > 0$;
(C6) there exists a number $\overline{\theta} > 0$ such that $\theta_{i+1} - \theta_i \leqslant \overline{\theta}, i \in \mathbb{Z}$;

(C7) there exists a number $\theta > 0$ such that $\theta_{i+1} - \theta_i \geqslant \theta, i \in \mathbb{Z}$;

(C8) there exists a positive real number p such that

$$\lim_{t\to\infty} \frac{i(t_0,t)}{t-t_0} = p$$

uniformly with respect to $t_0 \in \mathbb{R}$, where $i(t_0,t)$ denotes the number of points θ_i in the interval (t_0,t).

Condition (C7) implies immediately that $|\theta_i| \to \infty$ as $|i| \to \infty$. One can easily see that equations (2.1) and (2.2) have the form of functional differential equations

$$z'(t) = A_0(t)z(t) + A_1(t)z(\zeta_i), \tag{2.5}$$

and

$$z'(t) = A_0(t)z(t) + A_1(t)z(\zeta_i) + f(t,z(t),z(\zeta_i)), \tag{2.6}$$

respectively, if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$.
That is, these systems have the structure of a continuous dynamical system within the intervals $[\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$.
It is useful to specify the Definition 1.1 in the following way.

Definition 2.1. A continuous function $z(t)$ is a solution of (2.1), (2.2) on $\mathbb{R}$ if:

(i) the derivative $z'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception of the points θ_i, $i \in \mathbb{Z}$, where the one-sided derivatives exist;

(ii) the equation is satisfied for $z(t)$ on each interval (θ_i, θ_{i+1}), $i \in \mathbb{Z}$, and it holds for the right derivative of $z(t)$ at the points θ_i, $i \in \mathbb{Z}$.

Remark 2.1. One can easily see that the last definition is appropriate for differential equations with β type arguments, as they also are γ functions.

2.1 Linear homogeneous systems

Let $\mathscr{I}$ be an $n \times n$ identity matrix. Denote by $X(t,s), X(s,s) = \mathscr{I}, t, s \in \mathbb{R}$, the fundamental matrix of solutions of the system

$$x'(t) = A_0(t)x(t) \tag{2.7}$$

which is associated with systems (2.1) and (2.2). We introduce the following matrix-function

Notation 2.1.

$$M_i(t) = X(t,\zeta_i) + \int_{\zeta_i}^{t} X(t,s)A_1(s)ds, \;\; i \in \mathbb{Z}.$$

This matrix is very useful in what follows. From now on we make the assumption:

(C9) For every fixed $i \in \mathbb{Z}$, $\det[M_i(t)] \neq 0$, $\forall t \in [\theta_i, \theta_{i+1}]$.

Remark 2.2. One can easily see that the last condition is equivalent to the following one: $\det[\mathscr{I} + \int_{\zeta_i}^{t} X(\zeta_i,s)A_1(s)ds] \neq 0$, for all $t \in [\theta_i, \theta_{i+1}]$, $i \in \mathbb{Z}$.

Theorem 2.1. *If condition* (C1) *is fulfilled, then for every* $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$ *there exists a unique solution* $z(t) = z(t,t_0,z_0), z(t_0) = z_0$, *of* (2.5) *in the sense of Definition* 2.1 *if and only if condition* (C9) *is valid.*

Proof. *Sufficiency.* Fix a $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$. Without loss of generality assume that $\theta_i \leqslant \zeta_i < t_0 \leqslant \theta_{i+1}$, for a fixed $i \in \mathbb{Z}$. We consider only the construction of the solution for decreasing t, since forward continuation can be investigated in a similar manner.
Condition (C9) implies that the equation

$$v_i = X(\zeta_i, t_0)z(t_0) + \int_{t_0}^{\zeta_i} X(\zeta_i, s)A_1(s)v_i ds$$

can be solved for v_i uniquely. Indeed, since we have

$$\left[\mathscr{I} + \int_{\zeta_i}^{t_0} X(\zeta_i, s)A_1(s)ds\right] v_i = X(\zeta_i, t_0)z(t_0),$$

multiplying both sides of the last expression by $X(t_0, \zeta_i)$, we obtain

$$v_i = M_i^{-1}(t_0)z(t_0).$$

Define $z(t) : [\theta_i, t_0] \to \mathbb{R}^n$ as the unique solution of the system

$$z'(t) = A_0(t)z(t) + A_1(t)v_i \tag{2.8}$$

with the initial condition $z(t_0) = z_0$. One can easily see that $z(\zeta_i) = v_i$.
Consider, now, the interval $[\theta_{i-1}, \theta_i]$. Again, by condition (C9), the equation

$$v_{i-1} = X(\zeta_{i-1}, \theta_i)\psi(\theta_i) + \int_{\theta_i}^{\zeta_{i-1}} X(\zeta_{i-1}, s)A_1(s)v_{i-1} ds$$

is uniquely solvable with respect to v_{i-1}. Let $z(t)$ be equal to the solution of the equation

$$z'(t) = A_0(t)z(t) + A_1(t)v_{i-1}, \tag{2.9}$$

on $[\theta_{i-1}, \theta_i]$ with the initial data $(\theta_i, \psi(\theta_i))$. Obviously, the solution exists and is unique, and $z(\zeta_{i-1}) = v_{i-1}$.

Assume that we have defined the solution $z(t)$ on the interval $[\theta_j, t_0], j < i-1$. Then, the equation

$$v_{j-1} = X(\zeta_{j-1}, \theta_j)\psi(\theta_j) + \int_{\theta_j}^{\zeta_{j-1}} X(\zeta_{j-1}, s)A_1(s)v_{j-1}ds$$

is uniquely solvable with respect to v_{j-1}. We assume that $z(t)$ is a solution of the equation

$$z'(t) = A_0(t)z(t) + A_1(t)v_{j-1}, \tag{2.10}$$

on $[\theta_{j-1}, \theta_j]$, with the initial data $(\theta_j, z(\theta_j))$. Consequently, the function $z(t)$ could be continued up to $-\infty$ by induction. One can easily see that $z(t)$ is the unique solution of (2.5) on $(-\infty, t_0]$ by construction.

Necessity. Assume that condition (C9) is not true for some fixed $i \in \mathbb{Z}$ and $\xi \in [\theta_i, \theta_{i+1}]$. That is, $\det M_i(\xi) = 0$. Definitely, $\xi \neq \zeta_i$. If $z(t) = z(t, \xi, z(\xi))$ is a solution, then $z(\xi) = M_i(\xi)z(\zeta_i)$, and $z(\zeta_i)$ could not be defined uniquely. This fact proves the necessity of (C9). The theorem is proved. □

The last theorem is of major importance for this chapter. It arranges the correspondence between the points $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$ and the solutions of (2.1) in the sense of Definition 2.1, and there exists no solution of the equation out of the correspondence. Using this assertion, we can say that the definition of the initial value problem for differential equations with piecewise constant arguments is similar to the problem for ordinary differential equations. Particularly, the dimension of the space of all solutions is n. Hence, the investigation of problems considered in this chapter does not need to be supported by the results of the theory of functional differential equations [133, 150].

The following useful, in some particular cases, assertion is implied by the proof of the last theorem.

Theorem 2.2. *Assume that condition* (C1) *is fulfilled, and a number* $t_0 \in \mathbb{R}$, $\theta_i \leqslant t_0 < \theta_{i+1}$, *is fixed. For every* $z_0 \in \mathbb{R}^n$ *there exists a unique solution* $z(t) = z(t, t_0, z_0)$ *of* (2.1) *in the sense of Definition* 2.1 *such that* $z(t_0) = z_0$ *if and only if* $\det[M_i(t_0)] \neq 0$ *and* $\det[M_j(t)] \neq 0$ *for* $t = \theta_j,\ \theta_{j+1},\ j \in \mathbb{Z}$.

System (2.1) is a differential equation with a deviated argument. That is why it is reasonable to suppose that the initial "interval" must consist of more than one point. The following arguments show that in our case we need only one initial moment. Indeed, assume that (t_0, z_0) is fixed, and $\theta_i \leqslant t_0 < \theta_{i+1}$ for a fixed $i \in \mathbb{Z}$. We suppose that $t_0 \neq \zeta_i$. The solution satisfies, on the interval $[\theta_i, \theta_{i+1}]$, the functional differential equation

$$z'(t) = A_0(t)z + A_1(t)z(\zeta_i). \tag{2.11}$$

Formally, we need the pair of initial points (t_0, z_0) and $(\zeta_i, z(\zeta_i))$ to proceed with the solution. Indeed, since $z_0 = M_i(t_0)z(\zeta_i)$ and the matrix $M_i(t_0)$ is nonsingular, we can say that the initial condition $z(t_0) = z_0$ is sufficient to define the solution.

Theorem 2.1 implies that the set of all solutions of (2.1) is an n-dimensional linear space. Hence, for a fixed $t_0 \in \mathbb{R}$ there exists a fundamental matrix of solutions of (2.1), $Z(t) = Z(t,t_0), Z(t_0,t_0) = I$ such that

$$\frac{dZ}{dt} = A_0(t)Z(t) + A_1(t)Z(\gamma(t)).$$

Let us construct $Z(t)$. Without loss of generality assume that $\theta_i < t_0 < \zeta_i$ for a fixed $i \in \mathbb{Z}$, and define the matrix only for increasing t, as the construction is similar for decreasing t.

We first note that $Z(\zeta_i) = M_i^{-1}(t_0)I = M_i^{-1}(t_0)$. Hence, on the interval $[\theta_i, \theta_{i+1}]$, we have $Z(t,t_0) = M_i(t)M_i^{-1}(t_0)$. Therefore

$$Z(\zeta_{i+1}) = M_{i+1}^{-1}(\theta_{i+1})Z(\theta_{i+1}) = M_{i+1}^{-1}(\theta_{i+1})M_i(\theta_{i+1})M_i^{-1}(t_0),$$

and hence for $t \in [\theta_{i+1}, \theta_{i+2}]$, we have

$$Z(t,t_0) = M_{i+1}(t)Z(\zeta_{i+1}) = M_{i+1}(t)M_{i+1}^{-1}(\theta_{i+1})M_i(\theta_{i+1})M_i^{-1}(t_0).$$

One can continue by induction to obtain

$$Z(t) = M_l(t)\left[\prod_{k=l}^{i+1} M_k^{-1}(\theta_k)M_{k-1}(\theta_k)\right]M_i^{-1}(t_0), \tag{2.12}$$

if $t \in [\theta_l, \theta_{l+1}]$, for arbitrary $l > i$.

Similarly, if $\theta_j \leqslant t \leqslant \theta_{j+1} < \cdots < \theta_i \leqslant t_0 \leqslant \theta_{i+1}$, then

$$Z(t) = M_j(t)\left[\prod_{k=j}^{i-1} M_k^{-1}(\theta_{k+1})M_{k+1}(\theta_{k+1})\right]M_i^{-1}(t_0). \tag{2.13}$$

One can easily see that

$$Z(t,s) = Z(t)Z^{-1}(s), \quad t,s \in \mathbb{R}, \tag{2.14}$$

and a solution $z(t)$ of (2.5) with $z(t_0) = z_0$ for $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$, is given by

$$z(t) = Z(t,t_0)z_0, \quad t \in \mathbb{R}. \tag{2.15}$$

Differential inequalities

Let $I \subseteq \mathbb{R}$ be an open interval, a scalar function $f(t,x,y)$ is defined on $\Omega = [t_0, t_0+T] \times I \times I$, $0 < T$. Function f is continuous in t except possibly moments θ_k, $k \in \mathbb{Z}$, where it admits discontinuities of the first kind, and is locally Lipschitzian in the second argument.

Lemma 2.1. *Assume that a continuous function $u(t)$ satisfies*

$$u'(t) \leqslant f(t, u(t), u(\gamma(t))), \tag{2.16}$$

for $t_0 \leqslant t \leqslant t_0 + T$, and a continuous function $v(t)$ is a solution of the equation

$$v'(t) = f(t, v(t), u(\gamma(t))), \ \ t_0 \leqslant t \leqslant t_0 + T, \tag{2.17}$$

except possibly moments $\theta_k, \ k \in \mathbb{Z}$. Then $u(t) \leqslant v(t)$, if $u(t_0) \leqslant v(t_0)$.

Proof. Assume that $\theta_i \leqslant t_0 \leqslant \theta_{i+1}$. One can easily verify the assertion, by using Theorem 4.1, [153] on the differential inequality for ordinary differential equations, on intervals $[t_0, \theta_{i+1}]$ and $[\theta_k, \theta_{k+1}]$, $k > 1$, consecutively. □

Example 2.1. Consider a non-negative function $u(t) : \mathbb{R} \to \mathbb{R}$, which is continuous, continuously differentiable, except, possibly, points $\theta_k, k \in \mathbb{Z}$, where the right-side derivative exists. Assume that $u(t)$ satisfies the differential inequality

$$u'(t) \leqslant a(t)u(t) + b(t)u(\gamma(t)), \ \ t_0 \leqslant t, \tag{2.18}$$

where a, b are non-negative piecewise continuous functions with discontinuities of the first kind at θ_k, $k \in \mathbb{Z}$. The moment $t_0 \in [\theta_i, \theta_{i+1}]$, $t_0 \leqslant \zeta_i$, is fixed. We assume that

$$\begin{aligned} &\int_{t_0}^{\zeta_i} \mathrm{e}^{\int_s^{\zeta_i} a(r)dr} b(s)ds < 1, \\ &\int_{\theta_k}^{\zeta_k} \mathrm{e}^{\int_s^{\zeta_k} a(r)dr} b(s)ds < 1, \ \ k > i. \end{aligned} \tag{2.19}$$

One can easily verify that these last inequalities relate to conditions of Theorem 2.2.
Moreover, consider the linear equation

$$v'(t) = a(t)v(t) + b(t)u(\gamma(t)), \ \ t_0 \leqslant t, \tag{2.20}$$

and its solution $v(t), v(t_0) = v_0$.
By the last lemma $u(t)$ satisfies $u(t) \leqslant v(t), \ \ t \geqslant t_0$, if $u(t_0) \leqslant v_0$.
If $t \in [t_0, \theta_{i+1}]$, we have that u is not greater than the solution

$$v(t) = \mathrm{e}^{\int_{t_0}^t a(s)ds} v_0 + \int_{t_0}^t \mathrm{e}^{\int_s^t a(r)dr} b(s)u(\zeta_i)ds,$$

of the equation

$$v'(t) = a(t)v(t) + b(t)u(\zeta_i). \tag{2.21}$$

One can also verify that

$$u(\zeta_i) \leqslant \frac{\mathrm{e}^{\int_{t_0}^{\zeta_i} a(s)ds} v_0}{1 - \int_{t_0}^{\zeta_i} \mathrm{e}^{\int_s^{\zeta_i} a(r)dr} b(s)ds},$$

and

$$u(t) \leqslant (e^{\int_{t_0}^{t} a(s)ds} + \frac{e^{\int_{t_0}^{\zeta_i} a(s)ds}}{1 - \int_{t_0}^{\zeta_i} e^{\int_s^{\zeta_i} a(r)dr} b(s)ds} \int_{t_0}^{t} e^{\int_s^t a(r)dr} b(s)ds) v_0.$$

Consider the initial moment $t = \theta_{i+1}$, and interval $[\theta_{i+1}, \theta_{i+2}]$. We have that $u(\theta_{i+1}) \leqslant v(\theta_{i+1})$. Consequently, $u(t) \leqslant v(t)$ on the interval. Recursively we can obtain that the following inequality is correct

$$u(t) \leqslant (e^{\int_{\theta_l}^{t} a(s)ds} + \frac{e^{\int_{\theta_l}^{\zeta_l} a(s)ds}}{1 - \int_{\theta_l}^{\zeta_l} e^{\int_s^{\zeta_l} a(r)dr} b(s)ds} \int_{\theta_l}^{t} e^{\int_s^t a(r)dr} b(s)ds) \times$$
$$\prod_{k=l}^{i+1} (e^{\int_{\theta_k}^{\theta_{k+1}} a(s)ds} + \frac{e^{\int_{\theta_k}^{\zeta_k} a(s)ds}}{1 - \int_{\theta_k}^{\zeta_k} e^{\int_s^{\zeta_k} a(r)dr} b(s)ds} \int_{\theta_k}^{\theta_{k+1}} e^{\int_s^{\theta_{k+1}} a(r)dr} b(s)ds) \times$$
$$(e^{\int_{t_0}^{\theta_{i+1}} a(s)ds} + \frac{e^{\int_{t_0}^{\zeta_i} a(s)ds}}{1 - \int_{t_0}^{\zeta_i} e^{\int_s^{\zeta_i} a(r)dr} b(s)ds} \int_{t_0}^{\theta_{i+1}} e^{\int_s^{\theta_{i+1}} a(r)dr} b(s)ds) v_0, \qquad (2.22)$$

if $t \in [\theta_l, \theta_{l+1}]$.

The results of this example will be useful in Chapter 4.

2.2 Quasi-linear systems

Let us consider system (2.2). One can easily see that (C4)–(C7) imply the existence of positive constants m, M and $\overline{M}$ such that $m \leqslant \|Z(t,s)\| \leqslant M$, $\|X(t,s)\| \leqslant \overline{M}$ for t, $s \in [\theta_i, \theta_{i+1}]$, $i \in \mathbb{Z}$.

From now on we make the assumption

(C10) $2\overline{M}L(1+M)\overline{\theta} < 1.$

Then, we can see that $\overline{M}(1+M)L\overline{\theta}e^{\overline{M}L(1+M)\overline{\theta}} < 1$, and the expression

Notation 2.2. $\kappa(L) = \frac{Me^{\overline{M}L(1+M)\overline{\theta}}}{1-\overline{M}(1+M)L\overline{\theta}e^{\overline{M}L(1+M)\overline{\theta}}}$

can be introduced. The following assumption is also needed:

(C11) $2\overline{M}L\overline{\theta}\kappa(L)(1+M) < m.$

The following Lemma is the most important auxiliary result of this chapter.

Lemma 2.2. *Suppose that conditions* (C1)–(C7), (C9)–(C11) *hold, and fix* $i \in \mathbb{Z}$. *Then, for every* $(\xi, z_0) \in [\theta_i, \theta_{i+1}] \times \mathbb{R}^n$, *there exists a unique solution* $z(t) = z(t, \xi, z_0)$ *of* (2.6) *on* $[\theta_i, \theta_{i+1}]$.

Proof. *Existence*. Fix $i \in \mathbb{Z}$. We assume without loss of generality that $\theta_i \leqslant \zeta_i < \xi \leqslant \theta_{i+1}$. Set $\|z(t)\|_0 = \max\limits_{[\theta_i,\theta_{i+1}]} \|z(t)\|$, take $z_0(t) = Z(t,\xi)z_0$ and define a sequence $\{z_k(t)\}$ by

$$\begin{aligned} z_{k+1}(t) = Z(t,\xi)\left[z_0 + \int_{\xi}^{\zeta_i} X(\zeta_i,s)f(s,z_k(s),z_k(\zeta_i))ds\right] \\ + \int_{\zeta_i}^{t} X(t,s)f(s,z_k(s),z_k(\zeta_i))ds, \quad k \geqslant 0. \end{aligned}$$

The last expression implies that

$$\|z_{k+1}(t) - z_k(t)\|_0 \leqslant [2\overline{M}L(1+M)\overline{\theta}]^{k+1}M\|z_0\|.$$

Thus, there exists a unique solution $z(t) = z(t,\xi,z_0)$ of the equation

$$z(t) = Z(t,\xi)\left[z_0 + \int_{\xi}^{\zeta_i} X(\zeta_i,s)f(s,z(s),z(\zeta_i))ds\right] + \int_{\zeta_i}^{t} X(t,s)f(s,z(s),z(\zeta_i))ds, \quad (2.23)$$

which is a solution of (2.6) on $[\theta_i,\theta_{i+1}]$ as well. This proves the existence.

Uniqueness. Denote by $z_j(t) = z(t,\xi,z_0^j)$, $z_j(\xi) = z_0^j$, $j = 1,2$, the solutions of (2.6), where $\theta_i \leqslant \xi \leqslant \theta_{i+1}$. Without loss of generality, we assume that $\xi \leqslant \zeta_i$. It is sufficient to check that for every $t \in [\theta_i,\theta_{i+1}]$, $z_0^1 \neq z_0^2$ implies $z_1(t) \neq z_2(t)$. We have that

$$\begin{aligned} z_1(t) - z_2(t) = Z(t,\xi)(z_0^1 - z_0^2) + \int_{\xi}^{\zeta_i} X(\zeta_i,s)[f(s,z_1(s),z_1(\zeta_i)) - f(s,z_2(s),z_2(\zeta_i))]ds \\ + \int_{\zeta_i}^{t} X(t,s)[f(s,z_1(s),z_1(\zeta_i)) - f(s,z_2(s),z_2(\zeta_i))]ds. \end{aligned}$$

Hence,

$$\begin{aligned} \|z_1(t) - z_2(t)\| \leqslant M\|z_0^1 - z_0^2\| + \overline{M}L\overline{\theta}(1+M)\|z_1(\zeta_i) - z_2(\zeta_i)\| \\ + \overline{M}L(1+M)\left|\int_{\xi}^{t} \|z_1(s) - z_2(s)\|ds\right|. \end{aligned}$$

From the Gronwall-Bellman Lemma, it follows that

$$\|z_1(t) - z_2(t)\| \leqslant [M\|z_0^1 - z_0^2\| + \overline{M}L\overline{\theta}(1+M)\|z_1(\zeta_i) - z_2(\zeta_i)\|]\mathrm{e}^{\overline{M}L(1+M)\overline{\theta}}.$$

In particular,

$$\|z_1(\zeta_i) - z_2(\zeta_i)\| \leqslant [M\|z_0^1 - z_0^2\| + \overline{M}L\overline{\theta}(1+M)\|z_1(\zeta_i) - z_2(\zeta_i)\|]\mathrm{e}^{\overline{M}L(1+M)\overline{\theta}},$$

which gives us

$$\|z_1(\zeta_i) - z_2(\zeta_i)\| \leqslant \kappa(L)\|z_0^1 - z_0^2\|,$$

and hence

$$\|z_1(t) - z_2(t)\| \leqslant \kappa(L)\|z_0^1 - z_0^2\|. \quad (2.24)$$

Assume on the contrary that there exists $t \in [\theta_i, \theta_{i+1}]$ such that $z_1(t) = z_2(t)$. Then

$$Z(t,\xi)(z_0^1 - z_0^2) = Z(t,\xi)\int_{\xi}^{\zeta_i} X(\zeta_i,s)[f(s,z_1(s),z_1(\zeta_i)) - f(s,z_2(s),z_2(\zeta_i))]ds + \int_{\zeta_i}^{t} X(t,s)[f(s,z_1(s),z_1(\zeta_i)) - f(s,z_2(s),z_2(\zeta_i))]ds. \tag{2.25}$$

We have that

$$\|Z(t,\xi)(z_0^2 - z_0^1)\| \geqslant m\|z_0^2 - z_0^1\|, \tag{2.26}$$

and (2.24) implies that

$$\begin{aligned}&\|Z(t,\xi)\int_{\xi}^{\zeta_i} X(\zeta_i,s)[f(s,z_1(s),z_1(\zeta_i)) - f(s,z_2(s),z_2(\zeta_i))]ds\\ &+ \int_{\zeta_i}^{t} X(t,s)[f(s,z_1(s),z_1(\zeta_i)) - f(s,z_2(s),z_2(\zeta_i))]ds\|\\ &\leqslant 2\overline{M}L\overline{\theta}\kappa(L)(1+M)\|z_0^2 - z_0^1\|.\end{aligned} \tag{2.27}$$

Finally, we can see that (C11), (2.26) and (2.27) contradict (2.25). The lemma is proved. □

Remark 2.3. Inequality (2.24) implies the continuous dependence of solutions of (2.2) on initial value.

Theorem 2.3. *Suppose that conditions* (C1)–(C7), (C9)–(C11) *are fulfilled. Then for every* $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$ *there exists a unique solution* $z(t) = z(t,t_0,z_0)$ *of* (2.2) *in the sense of Definition* 2.1 *such that* $z(t_0) = z_0$.

Proof. We prove the theorem only for decreasing t, but one can easily see that the proof is similar for increasing t.

Let us assume without loss of generality that $\theta_i \leqslant \zeta_i < t_0 \leqslant \theta_{i+1}$ for some $i \in \mathbb{Z}$.

Using Lemma 2.2 for $\xi = t_0$ one can check that solution $z(t) = z(t,t_0,z_0)$ of (2.2) exists on $[\zeta_i, t_0]$ as a solution of (2.6) and is unique. Then conditions (C1)–(C3) imply that $z(t)$ can be continued to $t = \theta_i$, as it is a solution of the system of ordinary differential equations $z' = A_0(t)z(t) + A_1(t)z(\zeta_i) + f(t,z(t),z(\zeta_i))$ on $[\theta_i, \theta_{i+1})$.

Next, using the lemma again, we can continue $z(t)$ from $t = \theta_i$ to $t = \zeta_{i-1}$, and then to $t = \theta_{i-1}$. Since $\theta_i \to -\infty$ as $i \to -\infty$, the induction completes the proof. □

Lemma 2.3. *Suppose that conditions* (C1)–(C7), (C9)–(C11) *hold. Then, the solution*

$z(t) = z(t, t_0, z_0)$, $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$, of (2.2) *is a solution, on $\mathbb{R}$, of the integral equation*

$$\begin{aligned} z(t) = Z(t,t_0)\left[z_0 + \int_{t_0}^{\zeta_i} X(t_0,s)f(s,z(s),z(\gamma(s)))\,ds\right] \\ + \sum_{k=i}^{k=j-1} Z(t,\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s)f(s,z(s),z(\gamma(s)))\,ds \\ + \int_{\zeta_j}^{t} X(t,s)f(s,z(s),z(\gamma(s)))\,ds, \end{aligned} \tag{2.28}$$

where $\theta_i \leqslant t_0 \leqslant \theta_{i+1}$ and $\theta_j \leqslant t \leqslant \theta_{j+1}$, $i < j$.

Proof. We shall prove the Lemma only for $\theta_i < t_0 < \theta_{i+1} < t \leqslant \theta_{i+2}$. All other cases can be proved analogously. Consider at first $t \in [\theta_i, \theta_{i+1}]$. The solution uniquely satisfies the integral equation

$$z(t) = X(t,\zeta_i)z(\zeta_i) + \int_{\zeta_i}^{t} X(t,s)A_1(s)z(\zeta_i)ds + \int_{\zeta_i}^{t} X(t,s)f(s,z(s),z(\gamma(s)))ds.$$

Using the last expression one can easily see that

$$z(\zeta_i) = M_i^{-1}(t_0)\left[z_0 - \int_{\zeta_i}^{t_0} X(t_0,s)f(s,z(s),z(\gamma(s)))ds\right].$$

Hence,

$$z(t) = M_i(t)M_i^{-1}(t_0)\left[z_0 - \int_{\zeta_i}^{t_0} X(t_0,s)f(s,z(s),z(\gamma(s)))ds\right] + \int_{\zeta_i}^{t} X(t,s)f(s,z(s),z(\gamma(s)))ds,$$

and

$$\begin{aligned} z(\theta_{i+1}) = M_i(\theta_{i+1})M_i^{-1}(t_0)\left[z_0 - \int_{\zeta_i}^{t_0} X(t_0,s)f(s,z(s),z(\gamma(s)))ds\right] \\ + \int_{\zeta_i}^{\theta_{i+1}} X(\theta_{i+1},s)f(s,z(s),z(\gamma(s)))ds. \end{aligned}$$

Then, for $t \in [\theta_{i+1}, \theta_{i+2}]$,

$$z(t) = X(t,\zeta_{i+1})z(\zeta_{i+1}) + \int_{\zeta_{i+1}}^{t} X(t,s)A_1(s)z(\gamma(s))ds + \int_{\zeta_{i+1}}^{t} X(t,s)f(s,z(s),z(\gamma(s)))ds,$$

and

$$z(\zeta_{i+1}) = M_{i+1}^{-1}(\theta_{i+1})[z(\theta_{i+1}) - \int_{\zeta_{i+1}}^{\theta_{i+1}} X(\theta_{i+1},s)f(s,z(s),z(\gamma(s)))ds].$$

Hence,

$$
\begin{aligned}
z(t) &= M_{i+1}(t)M_{i+1}^{-1}(\theta_{i+1})[z(\theta_{i+1}) - \int_{\zeta_{i+1}}^{\theta_{i+1}} X(\theta_{i+1},s)f(s,z(s),z(\gamma(s)))ds] \\
&\quad + \int_{\zeta_{i+1}}^{t} X(t,s)f(s,z(s),z(\gamma(s)))ds \\
&= M_{i+1}(t)M_{i+1}^{-1}(\theta_{i+1})\{M_i^{-1}(\theta_{i+1})M_i^{-1}(t_0)[z_0 \\
&\quad - \int_{\zeta_i}^{t_0} X(t_0,s)f(s,z(s),z(\gamma(s)))ds] \\
&\quad + \int_{\zeta_i}^{\theta_{i+1}} X(\theta_{i+1},s)f(s,z(s),z(\gamma(s)))ds \\
&\quad - \int_{\zeta_{i+1}}^{\theta_{i+1}} X(\theta_{i+1},s)f(s,z(s),z(\gamma(s)))ds\} \\
&\quad + \int_{\zeta_{i+1}}^{t} X(t,s)f(s,z(s),z(\gamma(s)))ds \\
&= M_{i+1}(t)M_{i+1}^{-1}(\theta_{i+1})M_i^{-1}(\theta_{i+1})M_i^{-1}(t_0)z_0 \\
&\quad + M_{i+1}(t)M_{i+1}^{-1}(\theta_{i+1})M_i^{-1}(\theta_{i+1})M_i^{-1}(t_0)\int_{t_0}^{\zeta_i} X(t,s)f(s,z(s),z(\gamma(s)))ds \\
&\quad + M_{i+1}(t)M_{i+1}^{-1}(\theta_{i+1})\int_{\zeta_i}^{\zeta_{i+1}} X(\theta_{i+1},s)f(s,z(s),z(\gamma(s)))ds \\
&\quad + \int_{\zeta_{i+1}}^{t} X(t,s)f(s,z(s),z(\gamma(s)))ds \\
&= Z(t,t_0)z_0 + Z(t,t_0)\int_{t_0}^{\zeta_i} X(t_0,s)f(s,z(s),z(\gamma(s)))ds \\
&\quad + Z(t,\theta_{i+1})\int_{\zeta_i}^{\zeta_{i+1}} X(\theta_{i+1},s)f(s,z(s),z(\gamma(s)))ds \\
&\quad + \int_{\zeta_{i+1}}^{t} X(t,s)f(s,z(s),z(\gamma(s)))ds.
\end{aligned}
$$

The lemma is proved. □

2.3 Stability

In this section, we assume that conditions (C1)–(C11) are fulfilled, and hence, all solutions of the considered systems are defined on the whole real axis and their integral curves do not intersect each other. For the stability investigation, we consider the systems on $\mathbb{R}_+ = [0,\infty)$. Definitions of Lyapunov stability for the solutions of both discussed systems can be done in the same way as for ordinary differential equations. Let us formulate them.

Definition 2.2. The zero solution of (2.1), (2.2) is stable if to any positive ε and t_0, there corresponds a number $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|x(t, t_0, x_0)\| < \varepsilon$, $t \geqslant t_0 \geqslant 0$, whenever $\|x_0\| < \delta$.

Definition 2.3. The zero solution of (2.1), (2.2) is uniformly stable, if δ in the previous definition is independent of t_0.

Definition 2.4. The zero solution of (2.1), (2.2) is asymptotically stable if it is stable in the sense of Definition 2.2, and there exists a positive number $\kappa(t_0)$ such that if $\psi(t)$ is any solution of (2.1), (2.2) with $\|\psi(t_0)\| < \kappa(t_0)$, then $\|\psi(t)\| \to 0$ as $t \to \infty$.

Definition 2.5. The zero solution of (2.1), (2.2) is uniformly asymptotically stable if it is uniformly stable in the sense of Definition 2.3, and for any $\varepsilon > 0$, we can find $T(\varepsilon) > 0$ such that any solution $\psi(t)$ of (2.1), (2.2) with $\|\psi(t_0)\| < \kappa$, where κ is independent of t_0, satisfies $\|\psi(t)\| < \varepsilon$ for $t \geqslant t_0 + T(\varepsilon)$.

Definition 2.6. The zero solution of (2.1), (2.2) is unstable if there exist numbers $\varepsilon_0 > 0$ and $t_0 \in I$ such that for any $\delta > 0$ there exists a solution $y_\delta(t)$, $\|y_\delta(t_0)\| < \delta$, of the system such that either it is not continuable to ∞ or there exists a moment t_1, $t_1 > t_0$ such that $\|y_\delta(t_1)\| \geqslant \varepsilon_0$.

Let $Z(t)$ be a fundamental matrix of (2.1). We can prove the following assertions, using representations (2.14) and (2.15) in exactly the same way as theorems for ordinary differential equations [87, 153].

Theorem 2.4. *The zero solution of* (2.1) *is stable if and only if* $Z(t)$ *is bounded on* $t \geqslant 0$.

Theorem 2.5. *The zero solution of* (2.1) *is asymptotically stable if and only if* $Z(t) \to 0$ *as* $t \to \infty$.

Theorem 2.6. *The zero solution of* (2.1) *is uniformly stable if and only if there exists a number* $M > 0$ *such that* $\|Z(t)Z^{-1}(s)\| \leqslant M$, $t \geqslant s \geqslant 0$.

Theorem 2.7. *The zero solution of* (2.1) *is uniformly asymptotically stable if and only if there exist two positive numbers* N *and* ω *such that* $\|Z(t)Z^{-1}(s)\| \leqslant N\mathrm{e}^{-\omega(t-s)}$, $t \geqslant s \geqslant 0$.

On the basis of the last theorems we can formulate the following theorems which provide sufficient conditions for the stability of linear systems.

Theorem 2.8. *Suppose* (C1)–(C6) *hold and* $\|M_k^{-1}(\theta_k)M_{k-1}(\theta_k)\| \leqslant 1$, $k \in \mathbb{N}$. *Then the zero solution of* (2.1) *is stable.*

Theorem 2.9. *Suppose* (C1)–(C6) *hold and there exists a nonnegative number* $\kappa < 1$ *such that* $\|M_k^{-1}(\theta_k)M_{k-1}(\theta_k)\| \leqslant \kappa$, $k \in \mathbb{N}$. *Then the zero solution of* (2.1) *is asymptotically stable.*

Theorem 2.10. *Suppose* (C1)–(C7) *hold. The zero solution of* (2.1) *is uniformly stable if* $\|M_k^{-1}(\theta_k)M_{k-1}(\theta_k)\| \leqslant 1$, $k \in \mathbb{N}$.

Theorem 2.11. *Suppose* (C1)–(C5), (C8) *hold. The zero solution of* (2.1) *is uniformly stable if* $\|M_k^{-1}(\theta_k)M_{k-1}(\theta_k)\| \leqslant 1$, $k \in \mathbb{N}$.

Theorem 2.12. *Suppose* (C1)–(C7) *hold. The zero solution of* (2.1) *is uniformly asymptotically stable if there exists a nonnegative number* $\kappa < 1$ *such that* $\|M_k^{-1}(\theta_k)M_{k-1}(\theta_k)\| \leqslant \kappa$, $k \in \mathbb{N}$.

Theorem 2.13. *Suppose* (C1)–(C5), (C8) *hold. The zero solution of* (2.1) *is uniformly asymptotically stable if there exists a nonnegative number* $\kappa < 1$ *such that* $\|M_k^{-1}(\theta_k)M_{k-1}(\theta_k)\| \leqslant \kappa$, $k \in \mathbb{N}$.

Interesting results comparable to the last theorems can be found in [317].

Example 2.2. Consider the following differential equation with piecewise constant function as argument

$$x'(t) = \alpha x(t) + \beta x(\gamma(t)), \tag{2.29}$$

in which α, β are fixed real constants, the identification function $\gamma(t)$ is defined by the sequences $\theta_i = \kappa i$, $\zeta_i = \theta_i + \kappa_1$, $i \in \mathbb{Z}$, where $\kappa > 0$, $\kappa > \kappa_1 > 0$, are fixed numbers. We will find conditions on the coefficients and the sequences to provide uniform asymptotic stability of the zero solution. One can evaluate that

$$M_i(t) = \mathrm{e}^{\alpha(t-\zeta_i)} + \int_{\zeta_i}^{t} \mathrm{e}^{\alpha(t-s)}\beta ds = \mathrm{e}^{\alpha(t-\zeta_i)} + \frac{\beta}{\alpha}\left(\mathrm{e}^{\alpha(t-\zeta_i)} - 1\right).$$

Then

$$M_i(\theta_i) = \mathrm{e}^{-\alpha\kappa_1} + \frac{\beta}{\alpha}(\mathrm{e}^{-\alpha\kappa_1} - 1).$$

Moreover, if we denote $\kappa_2 = \kappa - \kappa_1$, then

$$M_{i-1}(\theta_i) = \mathrm{e}^{\alpha\kappa_2} + \frac{\beta}{\alpha}(\mathrm{e}^{-\alpha\kappa_1} - 1),$$

and

$$M_i^{-1}(\theta_i)M_{i-1}(\theta_i) = \frac{\mathrm{e}^{\alpha\kappa_2} + \frac{\beta}{\alpha}(\mathrm{e}^{\alpha\kappa_2} - 1)}{\mathrm{e}^{-\alpha\kappa_1} + \frac{\beta}{\alpha}(\mathrm{e}^{-\alpha\kappa_1} - 1)}.$$

On the basis of Theorem 2.13 the inequality

$$\left|\frac{e^{\alpha\kappa_2}+\frac{\beta}{\alpha}(e^{\alpha\kappa_2}-1)}{e^{-\alpha\kappa_1}+\frac{\beta}{\alpha}(e^{-\alpha\kappa_1}-1)}\right|<1$$

is necessary and sufficient for the zero solution to be uniformly asymptotically stable.
It is of particular interest to consider the case when $\kappa_1 = 0$ and $\kappa_1 = \kappa$. Consider the first one. Then conditions from the last inequality imply that the zero solution is uniformly asymptotically stable if and only if $\alpha < 0$ and $|\beta| < -\alpha$.
Similarly, if $\kappa_1 = \kappa$, then for the zero solution to be uniformly asymptotically stable it is necessary and sufficient that one of the following two conditions is true: $\beta < -\alpha$ or $\beta > \alpha\frac{1+e^{-\alpha\kappa}}{1-e^{-\alpha\kappa}}$.
Let us obtain an evaluation of the fundamental solution $Z(t,t_0)$ of the equation using (2.12). Set

$$\xi = \left|\frac{e^{\alpha\kappa_2}+\frac{\beta}{\alpha}(e^{\alpha\kappa_2}-1)}{e^{-\alpha\kappa_1}+\frac{\beta}{\alpha}(e^{-\alpha\kappa_1}-1)}\right|,$$

and assume that condition of the uniform asymptotic stability is valid. Then $\xi < 1$ and

$$\min_{\theta_i\leqslant t\leqslant\theta_{i+1}} M_i(t) = e^{-\alpha\kappa_1}+\frac{\beta}{\alpha}(e^{-\alpha\kappa_1}-1), \qquad \max_{\theta_i\leqslant t\leqslant\theta_{i+1}} M_i(t) = e^{\alpha\kappa_2}+\frac{\beta}{\alpha}(e^{\alpha\kappa_2}-1).$$

Hence, for

$$N = \max\left\{e^{\alpha\kappa_2}+\frac{\beta}{\alpha}(e^{\alpha\kappa_2}-1), \left(e^{-\alpha\kappa_1}+\frac{\beta}{\alpha}(e^{-\alpha\kappa_1}-1)\right)^{-1}\right\}e^{-\ln\xi},$$

we have

$$|Z(t,t_0)| \leqslant N e^{\frac{\ln\xi}{\kappa}(t-t_0)}, \quad t \geqslant t_0.$$

If the zero solution of (2.1) is uniformly asymptotically stable, then by Theorem 2.7 there exist two numbers $K \geqslant 1$ and $\omega > 0$ such that $\|Z(t,s)\| \leqslant K e^{-\omega(t-s)}$, $t \geqslant s \geqslant 0$.
Let $\gamma(L) = \frac{M}{1-2\overline{M}L(1+M)\overline{\theta}}$. We may make the following assumption

(C12) $2K e^{2\omega\overline{\theta}}\overline{M}L\max(1,\gamma(L)) < \omega$.

Theorem 2.14. *Suppose* (C1)–(C7), (C9)–(C12) *are fulfilled and the zero solution of* (2.1) *is uniformly asymptotically stable. Then the zero solution of* (2.2) *is uniformly asymptotically stable.*

Proof. If $z(t)=z(t,t_0,z_0)$ is a solution of (2.2), then by (2.23), (2.28) and assuming, without loss of generality, that $t_0<\zeta_i\leqslant\cdots\leqslant\zeta_j<t$, we have that

$$\begin{aligned}
\|z(t)\| \leqslant{}& K\,\mathrm{e}^{-\omega(t-t_0)}\|z_0\|+K\,\mathrm{e}^{-\omega(t-t_0)}\int_{t_0}^{\zeta_i}2\overline{M}L\max(1,\gamma(L))\|z(s)\|\,ds\\
&+\sum_{k=i}^{j}K\,\mathrm{e}^{-\omega(t-\theta_{k+1})}\int_{\zeta_k}^{\zeta_{k+1}}2\overline{M}L\max(1,\gamma(L))\|z(s)\|\,ds\\
&+\int_{\zeta_j}^{t}2\overline{M}L\max(1,\gamma(L))\|z(s)\|\,ds\\
\leqslant{}& K\,\mathrm{e}^{-\omega(t-t_0)}\|z_0\|+\int_{t_0}^{t}2K\,\mathrm{e}^{-\omega(t-s-2\overline{\theta})}\overline{M}L\max(1,\gamma(L))\|z(s)\|\,ds.
\end{aligned}$$

If we set $u(t)=\|z(t)\|K\,\mathrm{e}^{\omega t}$, then the last inequality implies that

$$u(t)\leqslant Ku(t_0)+\int_{t_0}^{t}2K\,\mathrm{e}^{2\omega\overline{\theta}}\overline{M}L\max(1,\gamma(L))u(s)ds.$$

Now, by virtue of the Gronwall-Bellmann Lemma, we obtain

$$\|z(t)\|\leqslant K\,\mathrm{e}^{(-\omega+2K\,\mathrm{e}^{2\omega\overline{\theta}}\overline{M}L\max(1,\gamma(L)))(t-t_0)}\|z_0\|.$$

The last inequality, in conjunction with (C11), proves that the zero solution is uniformly asymptotically stable. The theorem is proved. □

Notes

Since the method of reduction to discrete systems, presumes that an equation has to be solved explicitly or implicitly with respect to the values of the solution at switching moments, the most appropriate equations considered by the method are either linear systems or equations, where solutions present linearly if their argument is non-deviated. These equations have been investigated mostly in literature [2–7], [81–85], [121, 133, 143, 317]. Consequently, basic problems of the theory have been left out of the discussion. Some of these problems are: Existence of solutions with initial moments, which are not moments of switching of constancy; stability of solutions with an arbitrary initial moment; uniform stability, etc.

The method of equivalent integral equations proposed in the papers [8–15] combined with the comparison of values of solutions at switching moments help to construct the fundamental matrix of solutions (2.13) and the Cauchy's formula (2.28). These formulas will play decisive role for the theory, similar they have for ordinary differential equations. We attract the reader's attention to the result that a linear differential equation with piecewise constant arguments has a finite dimensional space of solutions. It may be used as a basis for a new deep analysis of functional differential equations. The main results of this chapter were published in [11] for the first time. Lemma 2.1 and results of Example 2.1 are newly obtained in this chapter. Based on these achievements, the theory of linear systems for differential equations with piecewise constant arguments can be effectively developed further. Using this method, in papers [13,263], asymptotic behavior of solutions of quasilinear systems with piecewise constant argument is analyzed.

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