2 Concrete Matrix Groups

In this chapter, we mainly study the general linear group $\mathrm{GL}_n(\mathbb{K})$ of invertible $n \times n$-matrices with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and introduce some of its subgroups. In particular, we discuss some of the connections between matrix groups and also introduce certain symmetry groups of geometric structures like bilinear or sesquilinear forms. In Section 2.3, we introduce also groups of matrices with entries in the quaternions $\mathbb{H}$.

2.1 The General Linear Group

We start with some notation. We write $\mathrm{GL}_n(\mathbb{K})$ for the group of invertible matrices in $M_n(\mathbb{K})$ and note that

$$\mathrm{GL}_n(\mathbb{K}) = \big\{g \in M_n(\mathbb{K})\colon \big(\exists h \in M_n(\mathbb{K})\big)\; hg = gh = \mathbf{1}\big\}.$$

Since the invertibility of a matrix can be tested with its determinant,

$$\mathrm{GL}_n(\mathbb{K}) = \big\{g \in M_n(\mathbb{K})\colon \det g \neq 0\big\}.$$

This group is called the *general linear group*.

On the vector space $\mathbb{K}^n$, we consider the *euclidian norm*

$$\|x\| := \sqrt{|x_1|^2 + \cdots + |x_n|^2}, \quad x \in \mathbb{K}^n,$$

and on $M_n(\mathbb{K})$ the corresponding *operator norm*

$$\|A\| := \sup\big\{\|Ax\|\colon x \in \mathbb{K}^n, \|x\| \leq 1\big\}$$

which turns $M_n(\mathbb{K})$ into a Banach space. On every subset $S \subseteq M_n(\mathbb{K})$, we shall always consider the subspace topology inherited from $M_n(\mathbb{K})$ (otherwise we shall say so). In this sense, $\mathrm{GL}_n(\mathbb{K})$ and all its subgroups carry a natural topology.

Lemma 2.1.1. *The group* $\mathrm{GL}_n(\mathbb{K})$ *has the following properties:*

(i) $\mathrm{GL}_n(\mathbb{K})$ *is open in* $M_n(\mathbb{K})$.

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(ii) *The multiplication map* $m\colon \mathrm{GL}_n(\mathbb{K}) \times \mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K})$ *and the inversion map* $\eta\colon \mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K})$ *are smooth and in particular continuous.*

Proof. (i) Since the determinant function

$$\det\colon M_n(\mathbb{K}) \to \mathbb{K}, \quad \det(a_{ij}) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$$

is continuous and $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$ is open in $\mathbb{K}$, the set $\mathrm{GL}_n(\mathbb{K}) = \det^{-1}(\mathbb{K}^\times)$ is open in $M_n(\mathbb{K})$.

(ii) For $g \in \mathrm{GL}_n(\mathbb{K})$, we define $b_{ij}(g) := \det(g_{mk})_{m \neq j, k \neq i}$. According to Cramer's Rule, the inverse of g is given by

$$\left(g^{-1}\right)_{ij} = \frac{(-1)^{i+j}}{\det g} b_{ij}(g).$$

The smoothness of the inversion therefore follows from the smoothness of the determinant (which is a polynomial) and the polynomial functions b_{ij} defined on $M_n(\mathbb{K})$.

For the smoothness of the multiplication map, it suffices to observe that

$$(ab)_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

is the (ik)-entry in the product matrix. Since all these entries are quadratic polynomials in the entries of a and b, the product is a smooth map. □

Definition 2.1.2. A *topological group* G is a Hausdorff space G, endowed with a group structure, such that the multiplication map $m_G\colon G \times G \to G$ and the inversion map $\eta\colon G \to G$ are continuous, when $G \times G$ is endowed with the product topology.

Lemma 2.1.1(ii) says in particular that $\mathrm{GL}_n(\mathbb{K})$ is a topological group. It is clear that the continuity of group multiplication and inversion is inherited by every subgroup $G \subseteq \mathrm{GL}_n(\mathbb{K})$, so that every subgroup G of $\mathrm{GL}_n(\mathbb{K})$ also is a topological group.

We write a matrix $A = (a_{ij})_{i,j=1,\ldots,n}$ also as (a_{ij}) and define

$$A^\top := (a_{ji}), \quad \overline{A} := (\overline{a_{ij}}), \quad \text{and} \quad A^* := \overline{A}^\top = (\overline{a_{ji}}).$$

Note that $A^* = A^\top$ is equivalent to $\overline{A} = A$, which means that all entries of A are real. Now we can define the most important classes of matrix groups.

Definition 2.1.3. We introduce the following notation for some important subgroups of $\mathrm{GL}_n(\mathbb{K})$:

(1) The *special linear group*: $\mathrm{SL}_n(\mathbb{K}) := \{g \in \mathrm{GL}_n(\mathbb{K})\colon \det g = 1\}$.
(2) The *orthogonal group*: $\mathrm{O}_n(\mathbb{K}) := \{g \in \mathrm{GL}_n(\mathbb{K})\colon g^\top = g^{-1}\}$.
(3) The *special orthogonal group*: $\mathrm{SO}_n(\mathbb{K}) := \mathrm{SL}_n(\mathbb{K}) \cap \mathrm{O}_n(\mathbb{K})$.
(4) The *unitary group*: $\mathrm{U}_n(\mathbb{K}) := \{g \in \mathrm{GL}_n(\mathbb{K})\colon g^* = g^{-1}\}$. Note that $\mathrm{U}_n(\mathbb{R}) = \mathrm{O}_n(\mathbb{R})$, but $\mathrm{O}_n(\mathbb{C}) \neq \mathrm{U}_n(\mathbb{C})$.
(5) The *special unitary group*: $\mathrm{SU}_n(\mathbb{K}) := \mathrm{SL}_n(\mathbb{K}) \cap \mathrm{U}_n(\mathbb{K})$.

One easily verifies that these are indeed subgroups. One simply has to use that $(ab)^\top = b^\top a^\top$, $\overline{ab} = \overline{a}\overline{b}$ and that

$$\det\colon \mathrm{GL}_n(\mathbb{K}) \to (\mathbb{K}^\times, \cdot)$$

is a group homomorphism.

We write $\mathrm{Herm}_n(\mathbb{K}) := \{A \in M_n(\mathbb{K})\colon A^* = A\}$ for the set of *hermitian matrices*. For $\mathbb{K} = \mathbb{C}$, this is not a complex vector subspace of $M_n(\mathbb{K})$, but it is always a real subspace. A matrix $A \in \mathrm{Herm}_n(\mathbb{K})$ is called *positive definite* if for each $0 \neq z \in \mathbb{K}^n$ we have $\langle Az, z\rangle > 0$, where

$$\langle z, w\rangle := \sum_{j=1}^n z_j \overline{w_j}$$

is the natural scalar product on $\mathbb{K}^n$. We write $\mathrm{Pd}_n(\mathbb{K}) \subseteq \mathrm{Herm}_n(\mathbb{K})$ for the subset of positive definite matrices.

Lemma 2.1.4. *The groups*

$$\mathrm{U}_n(\mathbb{C}), \quad \mathrm{SU}_n(\mathbb{C}), \quad \mathrm{O}_n(\mathbb{R}), \quad \textit{and} \quad \mathrm{SO}_n(\mathbb{R})$$

are compact.

Proof. Since all these groups are subsets of $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$, by the Heine–Borel Theorem we only have to show that they are closed and bounded.

Bounded: In view of

$$\mathrm{SO}_n(\mathbb{R}) \subseteq \mathrm{O}_n(\mathbb{R}) \subseteq \mathrm{U}_n(\mathbb{C}) \quad \text{and} \quad \mathrm{SU}_n(\mathbb{C}) \subseteq \mathrm{U}_n(\mathbb{C}),$$

it suffices to see that $\mathrm{U}_n(\mathbb{C})$ is bounded. Let $g_1, \ldots, g_n$ denote the rows of the matrix $g \in M_n(\mathbb{C})$. Then $g^* = g^{-1}$ is equivalent to $gg^* = \mathbf{1}$, which means that $g_1, \ldots, g_n$ form an orthonormal basis for $\mathbb{C}^n$ with respect to the scalar product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w_j}$ which induces the norm $\|z\| = \sqrt{\langle z, z\rangle}$. Therefore, $g \in \mathrm{U}_n(\mathbb{C})$ implies $\|g_j\| = 1$ for each j, so that $\mathrm{U}_n(\mathbb{C})$ is bounded.

Closed: The functions

$$f, h\colon M_n(\mathbb{K}) \to M_n(\mathbb{K}), \quad f(A) := AA^* - \mathbf{1} \quad \text{and} \quad h(A) := AA^\top - \mathbf{1}$$

are continuous. Therefore, the groups

$$\mathrm{U}_n(\mathbb{K}) := f^{-1}(\mathbf{0}) \quad \text{and} \quad \mathrm{O}_n(\mathbb{K}) := h^{-1}(\mathbf{0})$$

are closed. Likewise $\mathrm{SL}_n(\mathbb{K}) = \det^{-1}(\mathbf{1})$ is closed, and therefore the groups $\mathrm{SU}_n(\mathbb{C})$ and $\mathrm{SO}_n(\mathbb{R})$ are also closed because they are intersections of closed subsets. $\square$

2.1.1 The Polar Decomposition

Proposition 2.1.5 (Polar decomposition). *The multiplication map*

$$m\colon \mathrm{U}_n(\mathbb{K}) \times \mathrm{Pd}_n(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K}), \quad (u,p) \mapsto up$$

is a homeomorphism. In particular, each invertible matrix g can be written in a unique way as a product $g = up$ of a unitary matrix u and a positive definite matrix p.

Proof. We know from linear algebra that for each hermitian matrix A there exists an orthonormal basis $v_1, \ldots, v_n$ for $\mathbb{K}^n$ consisting of eigenvectors of A, and that all the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ are real (see [La93, Thm. XV.6.4]). From that it is obvious that A is positive definite if and only if $\lambda_j > 0$ holds for each j. For a positive definite matrix A, this has two important consequences:

(1) A is invertible, and A^{-1} satisfies $A^{-1}v_j = \lambda_j^{-1} v_j$.

(2) There exists a unique positive definite matrix B with $B^2 = A$ which will be denoted $\sqrt{A}$: We define B with respect to the basis $(v_1, \ldots, v_n)$ by $Bv_j = \sqrt{\lambda_j} v_j$. Then $B^2 = A$ is obvious and since all λ_j are real and the v_j are orthonormal, B is positive definite because

$$\left\langle B\left(\sum_i \mu_i v_i\right), \sum_j \mu_j v_j \right\rangle = \sum_{i,j} \mu_i \overline{\mu_j} \langle Bv_i, v_j\rangle = \sum_{j=1}^n |\mu_j|^2 \sqrt{\lambda_j} > 0$$

for $\sum_j \mu_j v_j \neq 0$ and real coefficients μ_j. It remains to verify the uniqueness. So assume that C is positive definite with $C^2 = A$. Then $CA = C^3 = AC$ implies that C preserves all eigenspaces of A, so that we find an orthonormal basis $w_1, \ldots, w_n$ consisting of simultaneous eigenvectors of C and A (cf. Exercise 2.1.1). If $Cw_j = \alpha_j w_j$, we have $Aw_j = \alpha_j^2 w_j$, which implies that C acts on the λ-eigenspace of A by multiplication with $\sqrt{\lambda}$, which shows that $C = B$.

From (1) we derive that the image of the map m is contained in $\mathrm{GL}_n(\mathbb{K})$.

m **is surjective:** Let $g \in \mathrm{GL}_n(\mathbb{K})$. For $0 \neq v \in \mathbb{K}^n$ we then have

$$0 < \langle gv, gv\rangle = \langle g^*gv, v\rangle,$$

showing that g^*g is positive definite. Let $p := \sqrt{g^*g}$ and define $u := gp^{-1}$. Then

$$uu^* = gp^{-1}p^{-1}g^* = gp^{-2}g^* = g(g^*g)^{-1}g^* = gg^{-1}(g^*)^{-1}g^* = \mathbf{1}$$

implies that $u \in \mathrm{U}_n(\mathbb{K})$, and it is clear that $m(u,p) = g$.

m **is injective:** If $m(u,p) = m(w,q) = g$, then $g = up = wq$ implies that

$$p^2 = p^*p = (up)^*up = g^*g = (wq)^*wq = q^2,$$

so that p and q are positive definite square roots of the same positive definite matrix g^*g, hence coincide by (2) above. Now $p = q$, and therefore $u = gp^{-1} = gq^{-1} = w$.

It remains to show that m is a homeomorphism. Its continuity is obvious, so that it remains to prove the continuity of the inverse map m^{-1}. Let $g_j = u_j p_j \to g = up$. We have to show that $u_j \to u$ and $p_j \to p$. Since $\mathrm{U}_n(\mathbb{K})$ is compact, the sequence (u_j) has a subsequence (u_{j_k}) converging to some $w \in \mathrm{U}_n(\mathbb{K})$ by the Bolzano–Weierstraß Theorem. Then $p_{j_k} = u_{j_k}^{-1} g_{j_k} \to w^{-1}g =: q \in \mathrm{Herm}_n(\mathbb{K})$ and $g = wq$. For each $v \in \mathbb{K}^n$, we then have

$$0 \leq \langle p_{j_k} v, v\rangle \to \langle qv, v\rangle,$$

showing that all eigenvalues of q are ≥ 0. Moreover, $q = w^{-1}g$ is invertible, and therefore q is positive definite. Now $m(u,p) = m(w,q)$ yields $u = w$ and $p = q$. Since each convergent subsequence of (u_j) converges to u, the sequence itself converges to u (Exercise 2.1.9), and therefore $p_j = u_j^{-1} g_j \to u^{-1}g = p$. $\square$

We shall see later that the set $\mathrm{Pd}_n(\mathbb{K})$ is homeomorphic to a vector space (Proposition 3.3.5), so that, topologically, the group $\mathrm{GL}_n(\mathbb{K})$ is a product of the compact group $\mathrm{U}_n(\mathbb{K})$ and a vector space. Therefore, the “interesting” part of the topology of $\mathrm{GL}_n(\mathbb{K})$ is contained in the compact group $\mathrm{U}_n(\mathbb{K})$.

Remark 2.1.6 (Normal forms of unitary and orthogonal matrices). We recall some facts from linear algebra:

(a) For each $u \in \mathrm{U}_n(\mathbb{C})$, there exists an orthonormal basis $v_1, \ldots, v_n$ consisting of eigenvectors of g (see [La93, Thm. XV.6.7]). This means that the unitary matrix s whose columns are the vectors $v_1, \ldots, v_n$ satisfies

$$s^{-1}us = \mathrm{diag}(\lambda_1, \ldots, \lambda_n),$$

where $uv_j = \lambda_j v$ and $|\lambda_j| = 1$.

The proof of this normal form is based on the existence of an eigenvector v_1 of u, which in turn follows from the existence of a zero of the characteristic polynomial. Since u is unitary, it preserves the hyperplane $v_1^\perp$ of dimension $n-1$. Now one uses induction to obtain an orthonormal basis $v_2, \ldots, v_n$ consisting of eigenvectors.

(b) For elements of $\mathrm{O}_n(\mathbb{R})$, the situation is more complicated because real matrices do not always have real eigenvectors.

Let $A \in M_n(\mathbb{R})$ and consider it as an element of $M_n(\mathbb{C})$. We assume that A does not have a real eigenvector. Then there exists an eigenvector $z \in \mathbb{C}^n$ corresponding to some eigenvalue $\lambda \in \mathbb{C}$. We write $z = x + iy$ and $\lambda = a + ib$. Then

$$Az = Ax + iAy = \lambda z = (ax - by) + i(ay + bx).$$

Comparing real and imaginary part yields

$$Ax = ax - by \quad \text{and} \quad Ay = ay + bx.$$

Therefore, the two-dimensional subspace generated by x and y in $\mathbb{R}^n$ is invariant under A.

This can be applied to $g \in \mathrm{O}_n(\mathbb{R})$ as follows. The argument above implies that there exists an invariant subspace $W_1 \subseteq \mathbb{R}^n$ with $\dim W_1 \in \{1, 2\}$. Then

$$W_1^\perp := \big\{v \in \mathbb{R}^n : \langle v, W_1 \rangle = \{0\}\big\}$$

is a subspace of dimension $n - \dim W_1$ which is also invariant (Exercise 2.1.14), and we apply induction to see that $\mathbb{R}^n$ is a direct sum of g-invariant subspaces $W_1, \ldots, W_k$ of dimension ≤ 2. Therefore, the matrix g is conjugate by an orthogonal matrix s to a block matrix of the form

$$d = \mathrm{diag}(d_1, \ldots, d_k),$$

where d_j is the matrix of the restriction of the linear map corresponding to g to W_j.

To understand the structure of the d_j, we have to take a closer look at the case $n \leq 2$. For $n = 1$ the group $\mathrm{O}_1(\mathbb{R}) = \{\pm 1\}$ consists of two elements, and for $n = 2$ an element $r \in \mathrm{O}_2(\mathbb{R})$ can be written as

$$r = \begin{pmatrix} a & \mp b \\ b & \pm a \end{pmatrix} \quad \text{with} \quad \det r = \pm\big(a^2 + b^2\big) = \pm 1,$$

because the second column contains a unit vector orthogonal to the first one. With $a = \cos\alpha$ and $b = \sin\alpha$ we get

$$r = \begin{pmatrix} \cos\alpha & \mp\sin\alpha \\ \sin\alpha & \pm\cos\alpha \end{pmatrix}.$$

For $\det r = -1$, we obtain

$$r^2 = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \mathbf{1},$$

but this implies that r is an orthogonal reflection with the two eigenvalues ± 1 (Exercise 2.1.13), hence has two orthogonal eigenvectors.

In view of the preceding discussion, we may therefore assume that the first m of the matrices d_j are of the rotation form

$$d_j = \begin{pmatrix} \cos\alpha_j & -\sin\alpha_j \\ \sin\alpha_j & \cos\alpha_j \end{pmatrix},$$

that $d_{m+1}, \ldots, d_\ell$ are -1, and that $d_{\ell+1}, \ldots, d_n$ are 1:

$$\begin{pmatrix}
\cos\alpha_1 & -\sin\alpha_1 &&&&&&&&& \\
\sin\alpha_1 & \cos\alpha_1 &&&&&&&&& \\
&& \ddots &&&&&&&& \\
&&& \cos\alpha_m & -\sin\alpha_m &&&&&& \\
&&& \sin\alpha_m & \cos\alpha_m &&&&&& \\
&&&&& -1 &&&&& \\
&&&&&& \ddots &&&& \\
&&&&&&& -1 &&& \\
&&&&&&&& 1 && \\
&&&&&&&&& \ddots & \\
&&&&&&&&&& 1
\end{pmatrix}.$$

For $n = 3$, we obtain in particular the normal form

$$d = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

From this normal form we immediately read off that $\det d = 1$ is equivalent to d describing a rotation around an axis consisting of fixed points (the axis is $\mathbb{R}e_3$ for the normal form matrix).

Proposition 2.1.7. (a) *The group* $\mathrm{U}_n(\mathbb{C})$ *is arcwise connected.*

(b) *The group* $\mathrm{O}_n(\mathbb{R})$ *has the two arc components*

$$\mathrm{SO}_n(\mathbb{R}) \quad \text{and} \quad \mathrm{O}_n(\mathbb{R})_- := \big\{g \in \mathrm{O}_n(\mathbb{R}) \colon \det g = -1\big\}.$$

Proof. (a) First we consider $\mathrm{U}_n(\mathbb{C})$. To see that this group is arcwise connected, let $u \in \mathrm{U}_n(\mathbb{C})$. Then there exists an orthonormal basis $v_1, \ldots, v_n$ of eigenvectors of u (Remark 2.1.6(a)). Let $\lambda_1, \ldots, \lambda_n$ denote the corresponding eigenvalues. Then the unitarity of u implies that $|\lambda_j| = 1$, and we therefore find $\theta_j \in \mathbb{R}$ with $\lambda_j = e^{\theta_j i}$. Now we define a continuous curve

$$\gamma \colon [0,1] \to \mathrm{U}_n(\mathbb{C}), \quad \gamma(t)v_j := e^{t\theta_j i} v_j, \ j = 1, \ldots, n.$$

We then have $\gamma(0) = \mathbf{1}$ and $\gamma(1) = u$. Moreover, each $\gamma(t)$ is unitary because the basis $(v_1, \ldots, v_n)$ is orthonormal.

(b) For $g \in \mathrm{O}_n(\mathbb{R})$, we have $gg^\top = \mathbf{1}$, and therefore $1 = \det(gg^\top) = (\det g)^2$. This shows that

$$\mathrm{O}_n(\mathbb{R}) = \mathrm{SO}_n(\mathbb{R}) \,\dot\cup\, \mathrm{O}_n(\mathbb{R})_-$$

and both sets are closed in $\mathrm{O}_n(\mathbb{R})$ because det is continuous. Therefore, $\mathrm{O}_n(\mathbb{R})$ is not connected, and hence not arcwise connected. Suppose we knew that $\mathrm{SO}_n(\mathbb{R})$ is arcwise connected and $x, y \in \mathrm{O}_n(\mathbb{R})_-$. Then $\mathbf{1}, x^{-1}y \in \mathrm{SO}_n(\mathbb{R})$ can be connected by an arc $\gamma \colon [0,1] \to \mathrm{SO}_n(\mathbb{R})$, and then $t \mapsto x\gamma(t)$ defines an arc $[0,1] \to \mathrm{O}_n(\mathbb{R})_-$ connecting x to y. So it remains to show that $\mathrm{SO}_n(\mathbb{R})$ is arcwise connected.

Let $g \in \mathrm{SO}_n(\mathbb{R})$. In the normal form of g discussed in Remark 2.1.6, the determinant of each two-dimensional block is 1, so that the determinant is the product of all -1-eigenvalues. Hence their number is even, and we can write each consecutive pair as a block

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix}.$$

This shows that with respect to some orthonormal basis for $\mathbb{R}^n$ the linear map defined by g has a matrix of the form

$$g = \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & & & & & & \\ \sin\alpha_1 & \cos\alpha_1 & & & & & & \\ & & \ddots & & & & & \\ & & & \cos\alpha_m & -\sin\alpha_m & & & \\ & & & \sin\alpha_m & \cos\alpha_m & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}.$$

Now we obtain an arc $\gamma\colon [0,1] \to \mathrm{SO}_n(\mathbb{R})$ with $\gamma(0) = \mathbf{1}$ and $\gamma(1) = g$ by

$$\gamma(t) := \begin{pmatrix} \cos t\alpha_1 & -\sin t\alpha_1 & & & & & & \\ \sin t\alpha_1 & \cos t\alpha_1 & & & & & & \\ & & \ddots & & & & & \\ & & & \cos t\alpha_m & -\sin t\alpha_m & & & \\ & & & \sin t\alpha_m & \cos t\alpha_m & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}.$$

$\square$

Corollary 2.1.8. *The group* $\mathrm{GL}_n(\mathbb{C})$ *is arcwise connected and the group* $\mathrm{GL}_n(\mathbb{R})$ *has two arc-components given by*

$$\mathrm{GL}_n(\mathbb{R})_\pm := \big\{g \in \mathrm{GL}_n(\mathbb{R})\colon\ \pm \det g > 0\big\}.$$

Proof. If $X = A \times B$ is a product space, then the arc-components of X are the sets of the form $C \times D$, where $C \subseteq A$ and $D \subseteq B$ are arc-components (easy Exercise!). The polar decomposition of $\mathrm{GL}_n(\mathbb{K})$ yields a homeomorphism

$$\mathrm{GL}_n(\mathbb{K}) \cong \mathrm{U}_n(\mathbb{K}) \times \mathrm{Pd}_n(\mathbb{K}).$$

Since $\mathrm{Pd}_n(\mathbb{K})$ is an open convex set, it is arcwise connected (Exercise 2.1.6). Therefore, the arc-components of $\mathrm{GL}_n(\mathbb{K})$ are in one-to-one correspondence with those of $\mathrm{U}_n(\mathbb{K})$ which have been determined in Proposition 2.1.7. $\square$

2.1.2 Normal Subgroups of $\mathbf{GL}_n(\mathbb{K})$

We shall frequently need some basic concepts from group theory which we recall in the following definition.

Definition 2.1.9. Let G be a group with identity element e.

(a) A subgroup $N \subseteq G$ is called *normal* if $gN = Ng$ holds for all $g \in G$. We write this as $N \trianglelefteq G$. The normality implies that the quotient set G/N (the set of all cosets of the subgroup N) inherits a natural group structure by

$$gN \cdot hN := ghN$$

for which eN is the identity element and the quotient map $q\colon G \to G/N$ is a surjective group homomorphism with kernel $N = \ker q = q^{-1}(eN)$.

On the other hand, all kernels of group homomorphisms are normal subgroups, so that the normal subgroups are precisely those which are kernels of group homomorphisms.

It is clear that G and $\{e\}$ are normal subgroups. We call G *simple* if $G \neq \{e\}$ and these are the only normal subgroups.

(b) The subgroup $Z(G) := \{g \in G\colon (\forall x \in G)\, gx = xg\}$ is called the *center of* G. It obviously is a normal subgroup of G. For $x \in G$, the subgroup

$$Z_G(x) := \{g \in G\colon gx = xg\}$$

is called its *centralizer*. Note that $Z(G) = \bigcap_{x \in G} Z_G(x)$.

(c) If $G_1, \ldots, G_n$ are groups, then the product set $G := G_1 \times \cdots \times G_n$ has a natural group structure given by

$$(g_1, \ldots, g_n)(g_1', \ldots, g_n') := (g_1 g_1', \ldots, g_n g_n').$$

The group G is called the *direct product* of the groups G_j, $j = 1, \ldots, n$. We identify G_j with a subgroup of G. Then all subgroups G_j are normal subgroups and $G = G_1 \cdots G_n$.

In the following, we write $\mathbb{R}_+^\times :=]0, \infty[$.

Proposition 2.1.10. (a) $Z(\mathrm{GL}_n(\mathbb{K})) = \mathbb{K}^\times \mathbf{1}$.

(b) *The multiplication map*

$$\varphi\colon (\mathbb{R}_+^\times, \cdot) \times \mathrm{SL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})_+, \quad (\lambda, g) \mapsto \lambda g$$

is a homeomorphism and a group isomorphism, i.e., an isomorphism of topological groups.

Proof. (a) It is clear that $\mathbb{K}^\times \mathbf{1}$ is contained in the center of $\mathrm{GL}_n(\mathbb{K})$. To see that each matrix $g \in Z(\mathrm{GL}_n(\mathbb{K}))$ is a multiple of $\mathbf{1}$, we consider the elementary matrix $E_{ij} := (\delta_{ij})$ with the only nonzero entry 1 in position (i, j). For $i \neq j$, we then have $E_{ij}^2 = 0$, so that $(\mathbf{1} + E_{ij})(\mathbf{1} - E_{ij}) = \mathbf{1}$, which implies that $T_{ij} := \mathbf{1} + E_{ij} \in \mathrm{GL}_n(\mathbb{K})$. From the relation $gT_{ij} = T_{ij}g$ we immediately get $gE_{ij} = E_{ij}g$ for $i \neq j$, so that for $k, \ell \in \{1, \ldots, n\}$ we get

$$g_{ki}\delta_{j\ell} = (gE_{ij})_{k\ell} = (E_{ij}g)_{k\ell} = \delta_{ik}g_{j\ell}.$$

For $k = i$ and $\ell = j$, we obtain $g_{ii} = g_{jj}$; and for $k = j = \ell$, we get $g_{ji} = 0$. Therefore, $g = \lambda\mathbf{1}$ for some $\lambda \in \mathbb{K}$.

(b) It is obvious that φ is a group homomorphism and that φ is continuous. Moreover, the map

$$\psi\colon \mathrm{GL}_n(\mathbb{R})_+ \to \mathbb{R}^\times_+ \times \mathrm{SL}_n(\mathbb{R}), \quad g \mapsto \big((\det g)^{\frac{1}{n}}, (\det g)^{-\frac{1}{n}}g\big)$$

is continuous and satisfies $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$. Hence φ is a homeomorphism. $\square$

Remark 2.1.11. The subgroups

$$Z\big(\mathrm{GL}_n(\mathbb{K})\big) \quad \text{and} \quad \mathrm{SL}_n(\mathbb{K})$$

are normal subgroups of $\mathrm{GL}_n(\mathbb{K})$. Moreover, for $\mathrm{GL}_n(\mathbb{R})$ the subgroup $\mathrm{GL}_n(\mathbb{R})_+$ is a proper normal subgroup and the same holds for $\mathbb{R}^\times_+\mathbf{1}$. One can show that these examples exhaust all normal arcwise connected subgroups of $\mathrm{GL}_n(\mathbb{K})$.

2.1.3 Exercises for Section 2.1

Exercise 2.1.1. Let V be a $\mathbb{K}$-vector space and $A \in \mathrm{End}(V)$. We write $V_\lambda(A) := \ker(A - \lambda\mathbf{1})$ for the *eigenspace of* A corresponding to the eigenvalue λ and $V^\lambda(A) := \bigcup_{n\in\mathbb{N}} \ker(A - \lambda\mathbf{1})^n$ for the *generalized eigenspace of* A corresponding to λ.

(a) If $A, B \in \mathrm{End}(V)$ commute, then

$$BV^\lambda(A) \subseteq V^\lambda(A) \quad \text{and} \quad BV_\lambda(A) \subseteq V_\lambda(A)$$

holds for each $\lambda \in \mathbb{K}$.

(b) If $A \in \mathrm{End}(V)$ is diagonalizable and $W \subseteq V$ is an A-invariant subspace, then $A|_W \in \mathrm{End}(W)$ is diagonalizable.

(c) If $A, B \in \mathrm{End}(V)$ commute and both are diagonalizable, then they are simultaneously diagonalizable, i.e., there exists a basis for V which consists of eigenvectors of A and B.

(d) If $\dim V < \infty$ and $\mathcal{A} \subseteq \mathrm{End}(V)$ is a commuting set of diagonalizable endomorphisms, then $\mathcal{A}$ can be simultaneously diagonalized, i.e., V is a direct sum of simultaneous eigenspaces of $\mathcal{A}$.

(e) For any function $\lambda\colon \mathcal{A} \to V$, we write $V_\lambda(\mathcal{A}) = \bigcap_{a\in\mathcal{A}} V_{\lambda(a)}(a)$ for the corresponding simultaneous eigenspace. Show that the sum $\sum_\lambda V_\lambda(\mathcal{A})$ is direct.

(f) If $\mathcal{A} \subseteq \mathrm{End}(V)$ is a finite commuting set of diagonalizable endomorphisms, then $\mathcal{A}$ can be simultaneously diagonalized.

(g) Find a commuting set of diagonalizable endomorphisms of a vector space V which cannot be diagonalized simultaneously.

Exercise 2.1.2. Let G be a topological group. Let G_0 be the *identity component*, i.e., the connected component of the identity in G. Show that G_0 is a closed normal subgroup of G.

Exercise 2.1.3. $\mathrm{SO}_n(\mathbb{K})$ is a closed normal subgroup of $\mathrm{O}_n(\mathbb{K})$ of index 2 and, for every $g \in \mathrm{O}_n(\mathbb{K})$ with $\det(g) = -1$,

$$\mathrm{O}_n(\mathbb{K}) = \mathrm{SO}_n(\mathbb{K}) \cup g\,\mathrm{SO}_n(\mathbb{K})$$

is a disjoint decomposition.

Exercise 2.1.4. For each subset $M \subseteq M_n(\mathbb{K})$, the *centralizer*

$$Z_{\mathrm{GL}_n(\mathbb{K})}(M) := \big\{g \in \mathrm{GL}_n(\mathbb{K}) : (\forall m \in M) gm = mg\big\}$$

is a closed subgroup of $\mathrm{GL}_n(\mathbb{K})$.

Exercise 2.1.5. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by the map $z = x + iy \mapsto (x, y)$ and write $I(x, y) := (-y, x)$ for the real linear endomorphism of $\mathbb{R}^{2n}$ corresponding to multiplication with i. Then

$$\mathrm{GL}_n(\mathbb{C}) \cong Z_{\mathrm{GL}_{2n}(\mathbb{R})}\big(\{I\}\big)$$

yields a realization of $\mathrm{GL}_n(\mathbb{C})$ as a closed subgroup of $\mathrm{GL}_{2n}(\mathbb{R})$.

Exercise 2.1.6. A subset C of a real vector space V is called a *convex cone* if C is convex and $\lambda C \subseteq C$ for each $\lambda > 0$.

Show that $\mathrm{Pd}_n(\mathbb{K})$ is an open convex cone in $\mathrm{Herm}_n(\mathbb{K})$.

Exercise 2.1.7. Show that

$$\gamma : (\mathbb{R}, +) \to \mathrm{GL}_2(\mathbb{R}), \quad t \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

is a continuous group homomorphism with $\gamma(\pi) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\mathrm{im}\,\gamma = \mathrm{SO}_2(\mathbb{R})$.

Exercise 2.1.8. Show that the group $\mathrm{O}_n(\mathbb{C})$ is homeomorphic to the topological product of the subgroup

$$\mathrm{O}_n(\mathbb{R}) \cong \mathrm{U}_n(\mathbb{C}) \cap \mathrm{O}_n(\mathbb{C}) \quad \text{and the set} \quad \mathrm{Pd}_n(\mathbb{C}) \cap \mathrm{O}_n(\mathbb{C}).$$

Exercise 2.1.9. Let (X, d) be a compact metric space and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. Show that $\lim_{n\to\infty} x_n = x$ is equivalent to the condition that each convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converges to x.

Exercise 2.1.10. If $A \in \mathrm{Herm}_n(\mathbb{K})$ satisfies $\langle Av, v\rangle = 0$ for each $v \in \mathbb{K}^n$, then $A = 0$.

Exercise 2.1.11. Show that for a complex matrix $A \in M_n(\mathbb{C})$ the following are equivalent:

(1) $A^* = A$.
(2) $\langle Av, v\rangle \in \mathbb{R}$ for each $v \in \mathbb{C}^n$.

Exercise 2.1.12. (a) Show that a matrix $A \in M_n(\mathbb{K})$ is hermitian if and only if there exists an orthonormal basis $v_1, \ldots, v_n$ for $\mathbb{K}^n$ and real numbers $\lambda_1, \ldots, \lambda_n$ with $Av_j = \lambda_j v_j$.

(b) Show that a complex matrix $A \in M_n(\mathbb{C})$ is unitary if and only if there exists an orthonormal basis $v_1, \ldots, v_n$ for $\mathbb{K}^n$ and $\lambda_j \in \mathbb{C}$ with $|\lambda_j| = 1$ and $Av_j = \lambda_j v_j$.

(c) Show that a complex matrix $A \in M_n(\mathbb{C})$ is *normal*, i.e., satisfies $A^*A = AA^*$, if and only if there exists an orthonormal basis $v_1, \ldots, v_n$ for $\mathbb{K}^n$ and $\lambda_j \in \mathbb{C}$ with $Av_j = \lambda_j v_j$.

Exercise 2.1.13. (a) Let V be a vector space and $\mathbf{1} \neq A \in \mathrm{End}(V)$ with $A^2 = \mathbf{1}$ (A is called an *involution*). Show that

$$V = \ker(A - \mathbf{1}) \oplus \ker(A + \mathbf{1}).$$

(b) Let V be a vector space and $A \in \mathrm{End}(V)$ with $A^3 = A$. Show that

$$V = \ker(A - \mathbf{1}) \oplus \ker(A + \mathbf{1}) \oplus \ker A.$$

(c) Let V be a vector space and $A \in \mathrm{End}(V)$ an endomorphism for which there exists a polynomial p of degree n with n different zeros $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ and $p(A) = 0$. Show that A is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$.

Exercise 2.1.14. Let $\beta\colon V \times V \to \mathbb{K}$ be a bilinear map and $g : V \to V$ with $\beta(gv, gw) = \beta(v, w)$ be a β-isometry. For a subspace $E \subseteq V$, we write

$$E^\perp := \big\{v \in V\colon (\forall w \in E)\ \beta(v, w) = 0\big\}$$

for its *orthogonal space*. Show that $g(E) = E$ implies that $g(E^\top) = E^\top$.

Exercise 2.1.15 (Iwasawa decomposition of $\mathrm{GL}_n(\mathbb{R})$). Let

$$T_n^+(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$$

denote the subgroup of upper-triangular matrices with positive diagonal entries. Show that the multiplication map

$$\mu\colon\ \mathrm{O}_n(\mathbb{R}) \times T_n^+(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}), \quad (a, b) \mapsto ab$$

is a homeomorphism.

Exercise 2.1.16. Let $\mathbb{K}$ be a field and $n \in \mathbb{N}$. Show that

$$Z\big(M_n(\mathbb{K})\big) := \big\{z \in M_n(\mathbb{K})\colon \big(\forall x \in M_n(\mathbb{K})\big) zx = xz\big\} = \mathbb{K}\mathbf{1}.$$

2.2 Groups and Geometry

In Definition 2.1.3, we have defined certain matrix groups by concrete conditions on the matrices. Often it is better to think of matrices as linear maps described with respect to a basis. To do that, we have to adopt a more abstract point of view. Similarly, one can study symmetry groups of bilinear forms on a vector space V without fixing a certain basis a priori. Actually, it is much more convenient to choose a basis for which the structure of the bilinear form is as simple as possible.

2.2.1 Isometry Groups

Definition 2.2.1 (Groups and bilinear forms). (a) (The abstract general linear group) Let V be a $\mathbb{K}$-vector space. We write $\mathrm{GL}(V)$ for the group of linear automorphisms of V. This is the group of invertible elements in the ring $\mathrm{End}(V)$ of all linear endomorphisms of V.

If V is an n-dimensional $\mathbb{K}$-vector space and $v_1, \ldots, v_n$ is a basis for V, then the map

$$\Phi\colon M_n(\mathbb{K}) \to \mathrm{End}(V), \quad \Phi(A)v_k := \sum_{j=1}^{n} a_{jk} v_j$$

is a linear isomorphism which describes the passage between linear maps and matrices. In view of $\Phi(\mathbf{1}) = \mathrm{id}_V$ and $\Phi(AB) = \Phi(A)\Phi(B)$, we obtain a group isomorphism

$$\Phi|_{\mathrm{GL}_n(\mathbb{K})}\colon\ \mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}(V).$$

(b) Let V be an n-dimensional vector space with basis $v_1, \ldots, v_n$ and $\beta\colon V \times V \to \mathbb{K}$ a bilinear map. Then $B = (b_{jk}) := (\beta(v_j, v_k))_{j,k=1,\ldots,n}$ is an $(n \times n)$-matrix, but this matrix should NOT be interpreted as the matrix of a linear map. It is the matrix of a bilinear map to $\mathbb{K}$, which is something different. It describes β in the sense that

$$\beta\Big(\sum_j x_j v_j, \sum_k y_k v_k\Big) = \sum_{j,k=1}^{n} x_j b_{jk} y_k = x^\top B y,$$

where $x^\top B y$ with column vectors $x, y \in \mathbb{K}^n$ is viewed as a matrix product whose result is a (1×1)-matrix, i.e., an element of $\mathbb{K}$.

We write

$$\mathrm{Aut}(V, \beta) := \big\{g \in \mathrm{GL}(V)\colon (\forall v, w \in V)\, \beta(gv, gw) = \beta(v, w)\big\}$$

for the *isometry group of the bilinear form* β. Then it is easy to see that

$$\Phi^{-1}\big(\mathrm{Aut}(V, \beta)\big) = \big\{g \in \mathrm{GL}_n(\mathbb{K})\colon g^\top B g = B\big\}.$$

If β is symmetric, we also write $\mathrm{O}(V,\beta) := \mathrm{Aut}(V,\beta)$; and if β is skew-symmetric, we write $\mathrm{Sp}(V,\beta) := \mathrm{Aut}(V,\beta)$.

If $v_1,\ldots,v_n$ is an orthonormal basis for β, i.e., $B = \mathbf{1}$, then

$$\Phi^{-1}\big(\mathrm{Aut}(V,\beta)\big) = \mathrm{O}_n(\mathbb{K})$$

is the orthogonal group defined in Section 2.1. Note that orthonormal bases can only exist for symmetric bilinear forms (Why?).

For $V = \mathbb{K}^{2n}$ and the block (2×2)-matrix

$$B := \begin{pmatrix} \mathbf{0} & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0} \end{pmatrix},$$

we see that $B^\top = -B$, and the group

$$\mathrm{Sp}_{2n}(\mathbb{K}) := \big\{g \in \mathrm{GL}_{2n}(\mathbb{K})\colon g^\top B g = B\big\}$$

is called the *symplectic group*. The corresponding skew-symmetric bilinear form on $\mathbb{K}^{2n}$ is given by

$$\beta(x,y) = x^\top B y = \sum_{i=1}^n x_i y_{n+i} - x_{n+i} y_i.$$

(c) A symmetric bilinear form β on V is called *nondegenerate* if $\beta(v,V) = \{0\}$ implies $v = 0$. For $\mathbb{K} = \mathbb{C}$, every nondegenerate symmetric bilinear form β possesses an orthonormal basis (this builds on the existence of square roots of nonzero complex numbers; see Exercise 2.2.1), so that for every such form β we get

$$\mathrm{O}(V,\beta) \cong \mathrm{O}_n(\mathbb{C}).$$

For $\mathbb{K} = \mathbb{R}$, the situation is more complicated, since negative real numbers do not have a square root in $\mathbb{R}$. There might not be an orthonormal basis, but if β is nondegenerate, there always exists an orthogonal basis $v_1,\ldots,v_n$ and $p \in \{1,\ldots,n\}$ such that $\beta(v_j,v_j) = 1$ for $j = 1,\ldots,p$ and $\beta(v_j,v_j) = -1$ for $j = p+1,\ldots,n$. Let $q := n-p$ and $I_{p,q}$ denote the corresponding matrix

$$I_{p,q} = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} \in M_{p+q}(\mathbb{R}).$$

Then $\mathrm{O}(V,\beta)$ is isomorphic to the group

$$\mathrm{O}_{p,q}(\mathbb{R}) := \big\{g \in \mathrm{GL}_n(\mathbb{R})\colon g^\top I_{p,q} g = I_{p,q}\big\},$$

where $\mathrm{O}_{n,0}(\mathbb{R}) = \mathrm{O}_n(\mathbb{R})$.

(d) Let V be an n-dimensional complex vector space and $\beta\colon V\times V \to \mathbb{C}$ a sesquilinear form, i.e., β is linear in the first and antilinear in the second argument. Then we also choose a basis $v_1,\ldots,v_n$ in V and define $B = (b_{jk}) := (\beta(v_j,v_k))_{j,k=1,\ldots,n}$, but now we obtain

$$\beta\Big(\sum_j x_j v_j, \sum_k y_k v_k\Big) = \sum_{j,k=1}^n x_j b_{jk} \overline{y_k} = x^\top B \overline{y}.$$

We write

$$\mathrm{U}(V,\beta) := \big\{g \in \mathrm{GL}(V) \colon (\forall v, w \in V)\, \beta(gv, gw) = \beta(v,w)\big\}$$

for the corresponding *unitary group* and find

$$\Phi^{-1}\big(\mathrm{U}(V,\beta)\big) = \big\{g \in \mathrm{GL}_n(\mathbb{C}) \colon g^\top B \overline{g} = B\big\}.$$

If $v_1, \ldots, v_n$ is an orthonormal basis for β, i.e., $B = \mathbf{1}$, then

$$\Phi^{-1}\big(\mathrm{U}(V,\beta)\big) = \mathrm{U}_n(\mathbb{C}) = \big\{g \in \mathrm{GL}_n(\mathbb{C}) \colon g^* = g^{-1}\big\}$$

is the unitary group over $\mathbb{C}$. We call β *hermitian* if it is sesquilinear and satisfies $\beta(y,x) = \overline{\beta(x,y)}$. In this case, one has to face the same problems as for symmetric forms on real vector spaces, but there always exists an orthogonal basis $v_1, \ldots, v_n$ and $p \in \{1, \ldots, n\}$ with $\beta(v_j, v_j) = 1$ for $j = 1, \ldots, p$ and $\beta(v_j, v_j) = -1$ for $j = p+1, \ldots, n$. With $q := n - p$ and

$$I_{p,q} := \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} \in M_n(\mathbb{C}),$$

we then define the *indefinite unitary groups* by

$$\mathrm{U}_{p,q}(\mathbb{C}) := \big\{g \in \mathrm{GL}_n(\mathbb{C}) \colon g^\top I_{p,q} \overline{g} = I_{p,q}\big\}.$$

Since $I_{p,q}$ has real entries,

$$\mathrm{U}_{p,q}(\mathbb{C}) = \big\{g \in \mathrm{GL}_n(\mathbb{C}) \colon g^* I_{p,q} g = I_{p,q}\big\},$$

where $\mathrm{U}_{n,0}(\mathbb{C}) = \mathrm{U}_n(\mathbb{C})$.

Definition 2.2.2. (a) Let V be a vector space. We consider the *affine group* $\mathrm{Aff}(V)$ of all maps $V \to V$ of the form

$$\varphi_{v,g}(x) = gx + v, \quad g \in \mathrm{GL}(V), v \in V.$$

We write elements $\varphi_{v,g}$ of $\mathrm{Aff}(V)$ simply as pairs (v,g). Then the composition in $\mathrm{Aff}(V)$ is given by

$$(v,g)(w,h) = (v + gw, gh),$$

$(0, \mathbf{1})$ is the identity, and inversion is given by

$$(v,g)^{-1} = \big(-g^{-1}v, g^{-1}\big).$$

For $V = \mathbb{K}^n$, we put $\mathrm{Aff}_n(\mathbb{K}) := \mathrm{Aff}(\mathbb{K}^n)$. Then the map

$$\Phi\colon \operatorname{Aff}_n(\mathbb{K}) \to \mathrm{GL}_{n+1}(\mathbb{K}), \quad \Phi(v,g) = \begin{pmatrix} [g] & v \\ 0 & 1 \end{pmatrix}$$

is an injective group homomorphism, where $[g]$ denotes the matrix of the linear map with respect to the canonical basis for $\mathbb{K}^n$.

(b) (The euclidian isometry group) Let $V = \mathbb{R}^n$ and consider the euclidian metric $d(x,y) := \|x-y\|_2$ on $\mathbb{R}^n$. We define

$$\mathrm{Iso}_n(\mathbb{R}) := \big\{g \in \operatorname{Aff}\big(\mathbb{R}^n\big)\colon (\forall x,y \in V)\ d(gv,gw) = d(v,w)\big\}.$$

This is the group of *affine isometries* of the euclidian n-space. Actually one can show that every isometry of a normed space $(V, \|\cdot\|)$ is an affine map (Exercise 2.2.5). This implies that

$$\mathrm{Iso}_n(\mathbb{R}) = \big\{g\colon \mathbb{R}^n \to \mathbb{R}^n\colon \big(\forall x,y \in \mathbb{R}^n\big)\ d(gv,gw) = d(v,w)\big\}.$$

2.2.2 Semidirect Products

We have seen in Definition 2.1.9 how to form direct products of groups. If $G = G_1 \times G_2$ is a direct product of the groups G_1 and G_2, then we identify G_1 and G_2 with the corresponding subgroups of $G_1 \times G_2$, i.e., we identify $g_1 \in G_1$ with (g_1, e) and $g_2 \in G_2$ with (e, g_2). Then G_1 and G_2 are normal subgroups of G and the product map

$$m\colon G_1 \times G_2 \to G, \quad (g_1,g_2) \mapsto g_1g_2 = (g_1,g_2)$$

is a group isomorphism, i.e., each element $g \in G$ has a unique decomposition $g = g_1g_2$ with $g_1 \in G_1$ and $g_2 \in G_2$.

The affine group $\operatorname{Aff}(V)$ has a structure which is similar. The translation group $V \cong \{(v,\mathbf{1})\colon v \in V\}$ and the linear group $\mathrm{GL}(V) \cong \{0\} \times \mathrm{GL}(V)$ are subgroups, and each element (v,g) has a unique representation as a product $(v,\mathbf{1})(0,g)$, but in this case $\mathrm{GL}(V)$ is not a normal subgroup, whereas V is normal. The following lemma introduces a concept that is important to understand the structure of groups which have similar decompositions.

In the following, we write $\operatorname{Aut}(G)$ for the set of automorphisms of the group G and note that this set is a group under composition of maps. In particular, the inverse of a group automorphism is an automorphism.

Lemma 2.2.3. (a) *Let N and H be groups, write $\operatorname{Aut}(N)$ for the group of all automorphisms of N, and suppose that $\delta\colon H \to \operatorname{Aut}(N)$ is a group homomorphism. Then we define a multiplication on $N \times H$ by*

$$(n,h)(n',h') := \big(n\delta(h)(n'), hh'\big). \tag{2.1}$$

This multiplication turns $N \times H$ into a group denoted by $N \rtimes_\delta H$, where $N \cong N \times \{e\}$ is a normal subgroup, $H \cong \{e\} \times H$ is a subgroup, and each element $g \in N \rtimes_\delta H$ has a unique representation as $g = nh$, $n \in N$, $h \in H$.

(b) *If, conversely, G is a group, $N \trianglelefteq G$ a normal subgroup and $H \subseteq G$ a subgroup with the property that the multiplication map $m\colon N \times H \to G$ is bijective, i.e., $NH = G$ and $N \cap H = \{e\}$, then*

$$\delta\colon H \to \mathrm{Aut}(N), \quad \delta(h)(n) := hnh^{-1} \tag{2.2}$$

is a group homomorphism, and the map

$$m\colon N \rtimes_\delta H \to G, \quad (n,h) \mapsto nh$$

is a group isomorphism.

Proof. (a) We have to verify the associativity of the multiplication and the existence of an inverse. The associativity follows from

$$\begin{aligned}
&\big((n,h)(n',h')\big)(n'',h'')\\
&= \big(n\delta(h)(n'), hh'\big)(n'',h'') = \big(n\delta(h)(n')\delta(hh')(n''), hh'h''\big)\\
&= \big(n\delta(h)(n')\delta(h)\big(\delta(h')(n'')\big), hh'h''\big) = \big(n\delta(h)\big(n'\delta(h')(n'')\big), hh'h''\big)\\
&= (n,h)\big(n'\delta(h')(n''), h'h''\big) = (n,h)\big((n',h')(n'',h'')\big).
\end{aligned}$$

With (2.1) we immediately get the formula for the inverse

$$(n,h)^{-1} = \big(\delta\big(h^{-1}\big)\big(n^{-1}\big), h^{-1}\big). \tag{2.3}$$

(b) Since

$$\delta(h_1h_2)(n) = h_1h_2n(h_1h_2)^{-1} = h_1\big(h_2nh_2^{-1}\big)h_1^{-1} = \delta(h_1)\delta(h_2)(n),$$

the map $\delta\colon H \to \mathrm{Aut}(N)$ is a group homomorphism. Moreover, the multiplication map m satisfies

$$m(n,h)m(n',h') = nhn'h' = (nhn'h^{-1})hh' = m\big((n,h)(n',h')\big),$$

hence is a group homomorphism. It is bijective by assumption. ☐

Definition 2.2.4. The group $N \rtimes_\delta H$ constructed in Lemma 2.2.3 from the data (N,H,δ) is called the *semidirect product* of N and H with respect to δ. If it is clear from the context what δ is, then we simply write $N \rtimes H$ instead of $N \rtimes_\delta H$.

If δ is trivial, i.e., $\delta(h) = \mathrm{id}_N$ for each $h \in H$, then $N \rtimes_\delta H \cong N \times H$ is a direct product. In this sense, semidirect products generalize direct products. Below we shall see several concrete examples of groups which can most naturally be described as semidirect products of known groups.

One major point in studying semidirect products is that for any normal subgroup $N \trianglelefteq G$, we think of the groups N and G/N as building blocks of the group G. For each semidirect product $G = N \rtimes H$, we have $G/N \cong H$, so that the two building blocks N and $G/N \cong H$ are the same, although the groups might be quite different, for instance, $\mathrm{Aff}(V)$ and $V \times \mathrm{GL}(V)$ are very different groups: In the latter group, $N = V \times \{\mathbf{1}\}$ is a central subgroup, and in the former group it is not. On the other hand, there are situations where G cannot be built from N and $H := G/N$ as a semidirect product. This works if and only if there exists a group homomorphism $\sigma \colon G/N \to G$ with $\sigma(gN) \in gN$ for each $g \in G$. An example where such a homomorphism does not exist is

$$G = C_4 := \{z \in \mathbb{C}^\times : z^4 = 1\} \quad \text{and} \quad N := C_2 := \{z \in \mathbb{C}^\times : z^2 = 1\} \trianglelefteq G.$$

In this case, $G \ncong N \rtimes H$ for any group H because then $H \cong G/N \cong C_2$, so that the fact that G is abelian would lead to $G \cong C_2 \times C_2$, contradicting the existence of elements of order 4 in G.

Example 2.2.5. (a) We know already the following examples of semidirect products from Definition 2.2.2: The affine group $\mathrm{Aff}(V)$ of a vector space is isomorphic to the semidirect product

$$\mathrm{Aff}(V) \cong V \rtimes_\delta \mathrm{GL}(V), \quad \delta(g)(v) = gv.$$

Similarly, we have

$$\mathrm{Aff}_n(\mathbb{R}) \cong \mathbb{R}^n \rtimes_\delta \mathrm{GL}_n(\mathbb{R}), \quad \delta(g)(v) = gv.$$

We furthermore have the subgroup $\mathrm{Iso}_n(\mathbb{R})$ which, in view of

$$\mathrm{O}_n(\mathbb{R}) = \{g \in \mathrm{GL}_n(\mathbb{R}) \colon (\forall x \in \mathbb{R}^n) \|gx\| = \|x\|\}$$

(cf. Exercise 2.2.6), satisfies

$$\mathrm{Iso}_n(\mathbb{R}) \cong \mathbb{R}^n \rtimes \mathrm{O}_n(\mathbb{R}).$$

The group of *euclidian motions of* $\mathbb{R}^n$ is the subgroup

$$\mathrm{Mot}_n(\mathbb{R}) := \mathbb{R}^n \rtimes \mathrm{SO}_n(\mathbb{R})$$

of those isometries preserving orientation.

(b) For each group G, we can form the semidirect product group

$$G \rtimes_\delta \mathrm{Aut}(G), \quad \delta(\varphi)(g) = \varphi(g).$$

Example 2.2.6 (The concrete Galilei[1] group). We consider the vector space

$$M := \mathbb{R}^4 \cong \mathbb{R}^3 \times \mathbb{R}$$

[1] Galileo Galilei (1564–1642), was an Italian mathematician and philosopher. He held professorships in Pisa and Padua, later he worked at the court in Florence. The Galilei group is the symmetry group of nonrelativistic kinematics in three dimensions.

as the space of pairs (q,t) describing *events* in a four-dimensional (nonrelativistic) *spacetime*. Here q stands for the spatial coordinate of the event and t for the (absolute) time of the event. The set M is called *Galilei spacetime*. There are three types of symmetries of this spacetime:

(1) The special Galilei transformations:

$$G_v \colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}, \quad (q,t) \mapsto (q+vt,t) = \begin{pmatrix} \mathbf{1} & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ t \end{pmatrix}$$

describing movements with constant velocity v.

(2) Rotations:

$$\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}, \quad (q,t) \mapsto (Aq,t), \quad A \in \mathrm{SO}_3(\mathbb{R}),$$

(3) Space translations

$$T_v \colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}, \quad (q,t) \mapsto (q+v,t),$$

and time translations

$$T_\beta \colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}, \quad (q,t) \mapsto (q,t+\beta).$$

All these maps are affine maps on $\mathbb{R}^4$. The subgroup $\varGamma \subseteq \mathrm{Aff}_4(\mathbb{R})$ generated by the maps in (1), (2) and (3) is called the *proper* (*orthochrone*) *Galilei group*. The *full Galilei group* $\varGamma_{\mathrm{ext}}$ is obtained if we add the time reversion $T(q,t) := (q,-t)$ and the space reflection $S(q,t) := (-q,t)$. Both are not contained in $\varGamma$.

Roughly stated, *Galilei's relativity principle* states that *the basic physical laws of closed systems are invariant under transformations of the proper Galilei group* (see [Sch95], Section II.2, for more information on this perspective). It means that $\varGamma$ is the natural symmetry group of nonrelativistic mechanics.

To describe the structure of the group $\varGamma$, we first observe that by (3) it contains the subgroup $\varGamma_t \cong (\mathbb{R}^4,+)$ of all spacetime translations. The maps under (1) and (2) are linear maps on $\mathbb{R}^4$. They generate the group

$$\varGamma_\ell := \big\{(v,A) \colon A \in \mathrm{SO}_3(\mathbb{R}), v \in \mathbb{R}^3\big\},$$

where we write (v,A) for the affine map given by $(q,t) \mapsto (Aq+vt,t)$. The composition of two such maps is given by

$$(v,A).\big((v',A').(q,t)\big) = \big(A(A'q+v't)+vt,t\big) = \big(AA'q+(Av'+v)t,t\big),$$

so that the product in $\varGamma_\ell$ is

$$(v,A)(v',A') = (v+Av',AA').$$

We conclude that

$$\Gamma_\ell \cong \mathbb{R}^3 \rtimes \mathrm{SO}_3(\mathbb{R})$$

is isomorphic to the group $\mathrm{Mot}_3(\mathbb{R})$ of motions of euclidian space. We thus obtain the description

$$\Gamma \cong \mathbb{R}^4 \rtimes \big(\mathbb{R}^3 \rtimes \mathrm{SO}_3(\mathbb{R})\big) \cong \mathbb{R}^4 \rtimes \mathrm{Mot}_3(\mathbb{R}),$$

where $\mathrm{Mot}_3(\mathbb{R})$ acts on $\mathbb{R}^4$ by $(v, A).(q, t) := (Aq + vt, t)$, which corresponds to the natural embedding $\mathrm{Aff}_3(\mathbb{R}) \to \mathrm{GL}_4(\mathbb{R})$ discussed in Example 2.2.2.

For the extended Galilei group, one easily obtains

$$\Gamma_{\mathrm{ext}} \cong \Gamma \rtimes \{S, T, ST, \mathbf{1}\} \cong \Gamma \rtimes (C_2 \times C_2)$$

because the group $\{S, T, ST, \mathbf{1}\}$ generated by S and T is a four element group intersecting the normal subgroup Γ trivially. Therefore, the description as a semidirect product follows from the second part of Lemma 2.2.3.

Example 2.2.7 (The concrete Poincaré group). In the preceding example, we have viewed four-dimensional spacetime as a product of space $\mathbb{R}^3$ with time $\mathbb{R}$. This picture changes if one wants to incorporate special relativity. Here the underlying spacetime is *Minkowski space*, which is $M = \mathbb{R}^4$, endowed with the *Lorentz form*

$$\beta(x, y) := x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4.$$

The group

$$L := \mathrm{O}_{3,1}(\mathbb{R}) \cong \mathrm{O}\big(\mathbb{R}^4, \beta\big)$$

is called the *Lorentz group*. This is the symmetry group of relativistic (classical) mechanics.

The Lorentz group has several important subgroups:

$$L_+ := \mathrm{SO}_{3,1}(\mathbb{R}) := L \cap \mathrm{SL}_4(\mathbb{R}) \quad \text{and} \quad L^\uparrow := \{g \in L \colon g_{44} \geq 1\}.$$

The condition $g_{44} \geq 1$ comes from the observation that for $e_4 = (0, 0, 0, 1)^\top$ we have

$$-1 = \beta(e_4, e_4) = \beta(ge_4, ge_4) = g_{14}^2 + g_{24}^2 + g_{34}^2 - g_{44}^2,$$

so that $g_{44}^2 \geq 1$. Therefore, either $g_{44} \geq 1$ or $g_{44} \leq -1$. To understand geometrically why $L^\uparrow$ is a subgroup, we consider the quadratic form

$$q(x) := \beta(x, x) = x_1^2 + x_2^2 + x_3^2 - x_4^2$$

on $\mathbb{R}^4$. Since q is invariant under L, the action of the group L on $\mathbb{R}^4$ preserves the double cone

$$C := \big\{x \in \mathbb{R}^4 \colon q(x) \leq 0\big\} = \big\{x \in \mathbb{R}^4 \colon |x_4| \geq \big\|(x_1, x_2, x_3)\big\|\big\}.$$

Let

$$C_\pm := \{x \in C\colon\ \pm x_4 \geq 0\} = \big\{x \in \mathbb{R}^4\colon\ \pm x_4 \geq \big\|(x_1, x_2, x_3)\big\|\big\}.$$

Then $C = C_+ \cup C_-$ with $C_+ \cap C_- = \{0\}$ and the sets $C_\pm$ are both convex cones, as follows easily from the convexity of the norm function on $\mathbb{R}^3$ (Exercise). Each element $g \in L$ preserves the set $C \setminus \{0\}$ which has the two arc-components $C_\pm \setminus \{0\}$. The continuity of the map $g\colon C \setminus \{0\} \to C \setminus \{0\}$ now implies that we have two possibilities. Either $gC_+ = C_+$ or $gC_+ = C_-$. In the first case, $g_{44} \geq 1$ and in the latter case $g_{44} \leq -1$.

In the physics literature, one sometimes finds $\mathrm{SO}_{3,1}(\mathbb{R})$ as the notation for $L_+^\uparrow := L_+ \cap L^\uparrow$, which is inconsistent with the standard notation for matrix groups.

The (*proper*) *Poincaré group* is the corresponding affine group

$$P := \mathbb{R}^4 \rtimes L_+^\uparrow.$$

This group is the identity component of the *inhomogeneous Lorentz group* $\mathbb{R}^4 \rtimes L$. Some people use the name Poincaré group only for the universal covering group $\widetilde{P}$ of P which is isomorphic to $\mathbb{R}^4 \rtimes \mathrm{SL}_2(\mathbb{C})$, as we shall see below in Example 9.5.16(3).

The topological structure of the Poincaré- and Lorentz group will become more transparent when we have refined information on the polar decomposition obtained from the exponential function (Example 4.3.4). Then we shall see that the Lorentz group L has four arc-components

$$L_+^\uparrow, \quad L_+^\downarrow, \quad L_-^\uparrow, \quad \text{and} \quad L_-^\downarrow,$$

where

$$L_\pm := \{g \in L\colon\ \det g = \pm 1\}, \quad L^\downarrow := \{g \in L\colon g_{44} \leq -1\}$$

and

$$L_\pm^\uparrow := L_\pm \cap L^\uparrow, \quad L_\pm^\downarrow := L_\pm \cap L^\downarrow.$$

2.2.3 Exercises for Section 2.2

Exercise 2.2.1. (a) Let β be a symmetric bilinear form on a finite-dimensional complex vector space V. Show that there exists an orthogonal basis $v_1, \ldots, v_n$ with $\beta(v_j, v_j) = 1$ for $j = 1, \ldots, p$ and $\beta(v_j, v_j) = 0$ for $j > p$.

(b) Show that each invertible symmetric matrix $B \in \mathrm{GL}_n(\mathbb{C})$ can be written as $B = AA^\top$ for some $A \in \mathrm{GL}_n(\mathbb{C})$.

Exercise 2.2.2. Let β be a symmetric bilinear form on a finite-dimensional real vector space V. Show that there exists an orthogonal basis $v_1, \ldots, v_{p+q}$ with $\beta(v_j, v_j) = 1$ for $j = 1, \ldots, p$, $\beta(v_j, v_j) = -1$ for $j = p+1, \ldots, p+q$, and $\beta(v_j, v_j) = 0$ for $j > p + q$.

Exercise 2.2.3. Let β be a skew-symmetric bilinear form on a finite-dimensional vector space V which is nondegenerate in the sense that $\beta(v, V) = \{0\}$ implies $v = 0$. Show that there exists a basis $v_1, \ldots, v_n, w_1, \ldots, w_n$ of V with

$$\beta(v_i, w_j) = \delta_{ij} \quad \text{and} \quad \beta(v_i, v_j) = \beta(w_i, w_j) = 0.$$

Exercise 2.2.4 (Metric characterization of midpoints). Let $(X, \|\cdot\|)$ be a normed space and $x, y \in X$ distinct points. Let

$$M_0 := \big\{z \in X : \|z - x\| = \|z - y\| = \tfrac{1}{2}\|x - y\|\big\} \quad \text{and} \quad m := \frac{x + y}{2}.$$

For a subset $A \subseteq X$, we define its *diameter*

$$\delta(A) := \sup\big\{\|a - b\| : a, b \in A\big\}.$$

Show that:

(1) If X is a pre-Hilbert space (i.e., a vector space with a hermitian scalar product), then $M_0 = \{m\}$ is a one-element set.
(2) $\|z - m\| \leq \frac{1}{2}\delta(M_0) \leq \frac{1}{2}\|x - y\|$ for $z \in M_0$.
(3) For $n \in \mathbb{N}$, we define inductively:

$$M_n := \big\{p \in M_{n-1} : (\forall z \in M_{n-1})\ \|z - p\| \leq \tfrac{1}{2}\delta(M_{n-1})\big\}.$$

Then, for each $n \in \mathbb{N}$:
(a) M_n is a convex set.
(b) M_n is invariant under the point reflection $s_m(a) := 2m - a$ in m.
(c) $m \in M_n$.
(d) $\delta(M_n) \leq \frac{1}{2}\delta(M_{n-1})$.

(4) $\bigcap_{n\in\mathbb{N}} M_n = \{m\}$.

Exercise 2.2.5 (Isometries of normed spaces are affine maps). Let $(X, \|\cdot\|)$ be a normed space endowed with the metric $d(x, y) := \|x - y\|$. Show that each isometry $\varphi\colon (X, d) \to (X, d)$ is an affine map by using the following steps:

(1) It suffices to assume that $\varphi(0) = 0$ and to show that this implies that φ is a linear map.
(2) $\varphi(\frac{x+y}{2}) = \frac{1}{2}(\varphi(x) + \varphi(y))$ for $x, y \in X$.
(3) φ is continuous.
(4) $\varphi(\lambda x) = \lambda\varphi(x)$ for $\lambda \in 2^{\mathbb{Z}} \subseteq \mathbb{R}$.
(5) $\varphi(x + y) = \varphi(x) + \varphi(y)$ for $x, y \in X$.
(6) $\varphi(\lambda x) = \lambda\varphi(x)$ for $\lambda \in \mathbb{R}$.

Exercise 2.2.6. Let $\beta\colon V \times V \to V$ be a symmetric bilinear form on the vector space V and

$$q\colon V \to V, \quad v \mapsto \beta(v, v)$$

the corresponding quadratic form. Then for $\varphi \in \mathrm{End}(V)$ the following are equivalent:

(1) $(\forall v \in V)\ q(\varphi(v)) = q(v)$.
(2) $(\forall v, w \in V)\ \beta(\varphi(v), \varphi(w)) = \beta(v, w)$.

Exercise 2.2.7. We consider $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$, where the elements of $\mathbb{R}^4$ are considered as spacetime events (q, t), $q \in \mathbb{R}^3$, $t \in \mathbb{R}$. On $\mathbb{R}^4$, we have the linear (time) functional

$$\Delta\colon \mathbb{R}^4 \to \mathbb{R}, (x, t) \mapsto t$$

and we endow $\ker \Delta \cong \mathbb{R}^3$ with the euclidian scalar product

$$\beta(x, y) := x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Show that

$$H := \big\{g \in \mathrm{GL}_4(\mathbb{R})\colon g \ker \Delta \subseteq \ker \Delta, g|_{\ker \Delta} \in \mathrm{O}_3(\mathbb{R})\big\} \cong \mathbb{R}^3 \rtimes \big(\mathrm{O}_3(\mathbb{R}) \times \mathbb{R}^\times\big)$$

and

$$G := \{g \in H\colon \Delta \circ g = \Delta\} \cong \mathbb{R}^3 \rtimes \mathrm{O}_3(\mathbb{R}).$$

In this sense, the linear part of the Galilei group (extended by the space reflection S) is isomorphic to the symmetry group of the triple $(\mathbb{R}^4, \beta, \Delta)$, where Δ represents a universal time function and β is the scalar product on $\ker \Delta$. In the relativistic picture (Example 2.2.7), the time function is combined with the scalar product in the Lorentz form.

Exercise 2.2.8. On the four-dimensional real vector space $V := \mathrm{Herm}_2(\mathbb{C})$, we consider the symmetric bilinear form β given by

$$\beta(A, B) := \operatorname{tr}(AB) - \operatorname{tr} A \operatorname{tr} B.$$

Show that:

(1) The corresponding quadratic form is given by $q(A) := \beta(A, A) = -2 \det A$.
(2) Show that $(V, \beta) \cong \mathbb{R}^{3,1}$ by finding a basis $E_1, \ldots, E_4$ of $\mathrm{Herm}_2(\mathbb{C})$ with

$$q(a_1 E_1 + \cdots + a_4 E_4) = a_1^2 + a_2^2 + a_3^2 - a_4^2.$$

(3) For $g \in \mathrm{GL}_2(\mathbb{C})$ and $A \in \mathrm{Herm}_2(\mathbb{C})$, the matrix gAg^* is hermitian and satisfies

$$q(gAg^*) = |\det(g)|^2 q(A).$$

(4) For $g \in \mathrm{SL}_2(\mathbb{C})$, we define a linear map $\rho(g) \in \mathrm{GL}(\mathrm{Herm}_2(\mathbb{C}))$ by $\rho(g)(A) := gAg^*$. Then we obtain a homomorphism

$$\rho\colon\ \mathrm{SL}_2(\mathbb{C}) \to \mathrm{O}(V, \beta) \cong \mathrm{O}_{3,1}(\mathbb{R}).$$

(5) Show that $\ker \rho = \{\pm \mathbf{1}\}$.

Exercise 2.2.9. Let $\beta\colon V \times V \to \mathbb{K}$ be a bilinear form.

(1) Show that there exist a unique symmetric bilinear form β_+ and a unique skew-symmetric bilinear form β_- with $\beta = \beta_+ + \beta_-$.

(2) $\mathrm{Aut}(V, \beta) = \mathrm{O}(V, \beta_+) \cap \mathrm{Sp}(V, \beta_-)$.

Exercise 2.2.10. (a) Let G be a group, $N \subseteq G$ a normal subgroup and $q\colon G \to G/N, g \mapsto gN$ the quotient homomorphism. Show that:

(1) If $G \cong N \rtimes_\delta H$ for a subgroup H, then $H \cong G/N$.

(2) There exists a subgroup $H \subseteq G$ with $G \cong N \rtimes_\delta H$ if and only if there exists a group homomorphism $\sigma\colon G/N \to G$ with $q \circ \sigma = \mathrm{id}_{G/N}$.

(b) Show that

$$\mathrm{GL}_n(\mathbb{K}) \cong \mathrm{SL}_n(\mathbb{K}) \rtimes_\delta \mathbb{K}^\times$$

for a suitable homomorphism $\delta\colon \mathbb{K}^\times \to \mathrm{Aut}(\mathrm{SL}_n(\mathbb{K}))$.

Exercise 2.2.11. Show that $\mathrm{O}_{p,q}(\mathbb{C}) \cong \mathrm{O}_{p+q}(\mathbb{C})$ for $p, q \in \mathbb{N}_0$, $p + q > 0$.

Exercise 2.2.12. Let (V, β) be a euclidian vector space, i.e., a real vector space endowed with a positive definite symmetric bilinear form β. An element $\sigma \in \mathrm{O}(V, \beta)$ is called an *orthogonal reflection* if $\sigma^2 = \mathbf{1}$ and $\ker(\sigma - \mathbf{1})$ is a hyperplane. Show that for any finite-dimensional euclidian vector space (V, β), the orthogonal group $\mathrm{O}(V, \beta)$ is generated by reflections.

Exercise 2.2.13. (i) Show that, if n is odd, each $g \in \mathrm{SO}_n(\mathbb{R})$ has the eigenvalue 1.

(ii) Show that each $g \in \mathrm{O}_n(\mathbb{R})_-$ has the eigenvalue -1.

Exercise 2.2.14. Let V be a $\mathbb{K}$-vector space. An element $\varphi \in \mathrm{GL}(V)$ is called a *transvection* if $\dim_{\mathbb{K}}(\mathrm{im}\,(\varphi - \mathrm{id}_V)) = 1$ and $\mathrm{im}\,(\varphi - \mathrm{id}_V) \subseteq \ker(\varphi - \mathrm{id}_V)$. Show that:

(i) For each transvection φ, there exist a $v_\varphi \in V$ and a $\alpha_\varphi \in V^*$ such that $\varphi(v) = v - \alpha_\varphi(v)v_\varphi$ and $\alpha_\varphi(v_\varphi) = 0$.
(ii) For each transvection φ, there exist a $v_\varphi \in V$ and a $\alpha_\varphi \in V^*$ such that $\varphi(v) = v - \alpha_\varphi(v)v_\varphi$ and $\alpha_\varphi(v_\varphi) = 0$.
(ii) If $\dim V < \infty$, then $\det(\varphi) = 1$ for each transvection φ.
(iii) If $\psi \in \mathrm{GL}(V)$ commutes with all transvections, then every element of V is an eigenvector of ψ, so that $\psi \in \mathbb{K}^\times \mathrm{id}_V$.
(iv) $Z(\mathrm{GL}(V)) = \mathbb{K}^\times \mathbf{1}$.
(v) If $\dim V = n < \infty$, then $Z(\mathrm{SL}(V)) = \Gamma \mathbf{1}$, where $\Gamma := \{z \in \mathbb{K}^\times : z^n = 1\}$.

Exercise 2.2.15. Let V be a finite-dimensional $\mathbb{K}$-vector space for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and β be a skew symmetric bilinear form on V. Show that:

(i) A transvection $\varphi(v) = v - \alpha_\varphi(v)v_\varphi$ preserves β if and only if

$$(\forall v, w \in V): \quad \alpha_\varphi(v)\beta(v_\varphi, w) = \alpha_\varphi(w)\beta(v_\varphi, v).$$

If, in addition, β is nondegenerate, we call φ a *symplectic transvection*.

(ii) If β is nondegenerate and $\psi \in \mathrm{GL}(V)$ commutes with all symplectic transvections, then every vector in V is an eigenvector of ψ.

Exercise 2.2.16. Let V be a finite-dimensional $\mathbb{K}$-vector space for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and β be a non-degenerate symmetric bilinear form on V. An involution $\varphi \in \mathrm{O}(V, \beta)$ is called an *orthogonal reflection* if $\dim_{\mathbb{K}}(\mathrm{im}(\varphi - \mathrm{id}_V)) = 1$. Show that:

(i) For each orthogonal reflection φ, there exists a non-isotropic $v_\varphi \in V$ such that $\varphi(v) = v - 2\frac{\beta(v, v_\varphi)}{\beta(v_\varphi, v_\varphi)}$.

(ii) If $\psi \in \mathrm{GL}(V)$ commutes with all orthogonal reflections, then every non-isotropic vector for β is an eigenvector of ψ, and this implies that $\psi \in \mathbb{K}^\times \mathrm{id}_V$.

(iv) $Z(\mathrm{O}(V, \beta)) = \{\pm \mathbf{1}\}$.

2.3 Quaternionic Matrix Groups

It is an important conceptual step to extend the real number field $\mathbb{R}$ to the field $\mathbb{C}$ of complex numbers. There are numerous motivations for this extension. The most obvious one is that not every algebraic equation with real coefficients has a solution in $\mathbb{R}$, and that $\mathbb{C}$ is *algebraically closed* in the sense that every nonconstant polynomial, even with complex coefficients, has zeros in $\mathbb{C}$. This is the celebrated Fundamental Theorem of Algebra. For analysis, the main point in passing from $\mathbb{R}$ to $\mathbb{C}$ is that the theory of holomorphic functions permits us to understand many functions showing up in real analysis from a more natural viewpoint, which leads to a thorough understanding of singularities and of integrals which can be computed with the calculus of residues.

It is therefore a natural question whether there exists an extension $\mathbb{F}$ of the field $\mathbb{C}$ which would similarly enrich analysis and algebra if we pass from $\mathbb{C}$ to $\mathbb{F}$. It is an important algebraic result that there exists no finite-dimensional field extension of $\mathbb{R}$ other than $\mathbb{C}$ (cf. Exercise 2.3.4). This is most naturally obtained in Galois theory, i.e., the theory of extending fields by adding zeros of polynomials. It is closely related to the fact that every real polynomial is a product of linear factors and factors of degree 2. Fortunately, this does not mean that one has to give up, but that one has to sacrifice one of the axioms of a field to obtain something new.

We call a unital (associative) algebra A a *skew field* or a *division algebra* if every nonzero element $a \in A^\times$ is invertible, i.e., $A = A^\times \cup \{0\}$. Now the question is: Are there any division algebras which are finite-dimensional real vector spaces, apart from $\mathbb{R}$ and $\mathbb{C}$. Here the answer is yes: there is the four-dimensional division algebra $\mathbb{H}$ of *quaternions*, and this is the only finite-dimensional real noncommutative division algebra.

The easiest way to define the quaternions is to take

$$\mathbb{H} := \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \in M_2(\mathbb{C}) \colon a, b \in \mathbb{C} \right\}.$$

Lemma 2.3.1. *$\mathbb{H}$ is a real subalgebra of $M_2(\mathbb{C})$ which is a division algebra.*

Proof. It is clear that $\mathbb{H}$ is a real vector subspace of $M_2(\mathbb{C})$. For the product of elements of $\mathbb{H}$, we obtain

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \begin{pmatrix} c & -\overline{d} \\ d & \overline{c} \end{pmatrix} = \begin{pmatrix} ac - \overline{b}d & -a\overline{d} - \overline{b}\overline{c} \\ bc + \overline{a}d & -b\overline{d} + \overline{ac} \end{pmatrix} \in \mathbb{H}.$$

This implies that $\mathbb{H}$ is a real subalgebra of $M_2(\mathbb{C})$.

We further have

$$\det \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} = |a|^2 + |b|^2, \tag{2.4}$$

so that every nonzero element of $\mathbb{H}$ is invertible in $M_2(\mathbb{C})$, and its inverse

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}^{-1} = \frac{1}{|a|^2 + |b|^2} \begin{pmatrix} \overline{a} & \overline{b} \\ -b & a \end{pmatrix} \tag{2.5}$$

is again contained in $\mathbb{H}$. $\square$

A convenient basis for $\mathbb{H}$ is given by

$$\mathbf{1}, \quad I := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad K := IJ = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Then the multiplication in $\mathbb{H}$ is completely determined by the relations

$$I^2 = J^2 = K^2 = -\mathbf{1} \quad \text{and} \quad IJ = -JI = K.$$

Here $\mathbb{C} \cong \mathbb{R}\mathbf{1} + \mathbb{R}I$ as *real* vector spaces, but $\mathbb{H}$ is not a complex algebra because the multiplication in $\mathbb{H}$ is not a complex bilinear map.

Since $\mathbb{H}$ is a division algebra, its group of units is $\mathbb{H}^\times = \mathbb{H} \setminus \{0\}$, and (2.4) implies that

$$\mathbb{H}^\times = \mathbb{H} \cap \mathrm{GL}_2(\mathbb{C}).$$

On $\mathbb{H}$, we consider the euclidian norm given by

$$|x| := \sqrt{\det x}, \qquad \left| \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \right| = \sqrt{|a|^2 + |b|^2}.$$

From the multiplicativity of the determinant, we immediately derive that

$$|xy| = |x| \cdot |y| \quad \text{for} \quad x, y \in \mathbb{H}. \tag{2.6}$$

It follows in particular that $\mathbb{S} := \{x \in \mathbb{H} \colon |x| = 1\}$ is a subgroup of $\mathbb{H}$. In terms of complex matrices, we have $\mathbb{S} = \mathrm{SU}_2(\mathbb{C})$.

Many results about vector spaces and matrices over fields generalize to matrices over division rings. If the division ring is noncommutative, however, one has to be careful on which side one wants to let the ring act. We want to recover the usual identification of linear maps with matrices acting from the left on column vectors such that the composition of maps corresponds to matrix multiplication. To this end, one has to consider the column vectors with entries in $\mathbb{H}$ as a *right* $\mathbb{H}$-module via componentwise multiplication. See Exercises 2.3.1 and 2.3.2 for the basics of quaternionic linear algebra (a systematic treatment of linear algebra on division rings can be found in [Bou70], Chapter II).

In contrast to bases, linear maps and representing matrices, determinants do not have a straightforward generalization to linear algebra over division rings. Thus we cannot characterize the *quaternionic general linear group* $\mathrm{GL}_n(\mathbb{H})$ of invertible elements in the ring $M_n(\mathbb{H})$ of $n \times n$-matrices with entries in $\mathbb{H}$ via an $\mathbb{H}$-valued determinant.

Proposition 2.3.2. *View $M_n(\mathbb{H})$ as a real subalgebra of $M_{2n}(\mathbb{C})$ writing each entry of $A \in M_n(\mathbb{H})$ as a complex 2×2-matrix. Then*

$$\mathrm{GL}_n(\mathbb{H}) = \left\{A \in M_n(\mathbb{H}) : \det_{\mathbb{C}}(A) \neq 0\right\},$$

where $\det_{\mathbb{C}} \colon M_{2n}(\mathbb{C}) \to \mathbb{C}$ is the ordinary determinant.

Proof. It suffices to show that $M_n(\mathbb{H}) \cap \mathrm{GL}_{2n}(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{H})$. So pick $A \in M_n(\mathbb{H})$ which is invertible in $M_{2n}(\mathbb{C})$. Then the left multiplication λ_A by A on $M_n(\mathbb{H})$ is injective, hence bijective. Thus we have $A^{-1} = \lambda_A^{-1}(\mathbf{1}) \in M_n(\mathbb{H})$. ☐

It follows from Proposition 2.3.2 that $\mathrm{GL}_n(\mathbb{H})$ is a (closed) subgroup of $\mathrm{GL}_{2n}(\mathbb{C})$. Moreover, it allows us to define the *quaternionic special linear group*

$$\mathrm{SL}_n(\mathbb{H}) := \mathrm{GL}_n(\mathbb{H}) \cap \mathrm{SL}_{2n}(\mathbb{C}).$$

Observe that $\mathbb{H}$ as a subset of $M_2(\mathbb{C})$ can be characterized as

$$\mathbb{H} = \left\{A \in M_2(\mathbb{C}) : \overline{A}J = JA\right\},$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the matrix used to build the symplectic group $\mathrm{Sp}_2(\mathbb{K})$ in Definition 2.2.1. Thus $\mathrm{GL}_n(\mathbb{H})$, viewed as a subgroup of $\mathrm{GL}_{2n}(\mathbb{C})$ is given by

$$\mathrm{GL}_n(\mathbb{H}) = \left\{A \in \mathrm{GL}_{2n}(\mathbb{C}) : \overline{A}J_n = J_nA\right\},$$

where J_n is the block diagonal matrix in $M_{2n}(\mathbb{C})$ having J as diagonal entries.

It turns out that inside $\mathrm{GL}_n(\mathbb{H})$ one can define analogs of unitary groups which are closely related to the symplectic groups. We note first that we can write the norm on $\mathbb{H}$ as

$$|x| = \sqrt{x^*x},$$

where $x^* = a\mathbf{1} - bI - cJ - dK$ for $x = a\mathbf{1} + bI + cJ + dK$. We extend this conjugation to matrices with entries in $\mathbb{H}$ setting

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix}^* = \begin{pmatrix} x_{11}^* & x_{21}^* & \cdots & x_{m1}^* \\ x_{12}^* & x_{22}^* & \cdots & x_{m2}^* \\ \vdots & \ddots & \ddots & \vdots \\ x_{1n}^* & x_{2n}^* & \cdots & x_{nm}^* \end{pmatrix}.$$

Note that with respect to the embedding $M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ this involution agrees with the standard involution $A \mapsto A^* = \overline{A}^\top$ on $M_{2n}(\mathbb{C})$. Now

$$\mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}, \quad (v, w) \mapsto v^*w$$

defines a *quaternionic inner product* on $\mathbb{H}^n$ and $v \mapsto |v| := \sqrt{v^*v}$ is a euclidian norm on the real vector space $\mathbb{H}^n \cong \mathbb{R}^{4n}$.

Definition 2.3.3. For $p + q = n \in \mathbb{N}$ view the matrix $I_{p,q}$ from Definition 2.2.1 as an element of $M_n(\mathbb{H})$ and define *quaternionic unitary groups* via

$$\mathrm{U}_{p,q}(\mathbb{H}) := \big\{g \in \mathrm{GL}_n(\mathbb{H}) : g^* I_{p,q} g = I_{p,q}\big\}.$$

If p or q is zero, then we simply write $\mathrm{U}_n(\mathbb{H})$.

Proposition 2.3.4. *Viewed as a subset of* $\mathrm{GL}_{2n}(\mathbb{C})$*, the quaternionic unitary group* $\mathrm{U}_{p,q}(\mathbb{H})$*, is given by*

$$\mathrm{U}_{p,q}(\mathbb{H}) = \mathrm{U}_{2p,2q}(\mathbb{C}) \cap \mathrm{Sp}\big(\mathbb{C}^{2n}, \beta\big),$$

where $\beta\colon \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$ *is the skew-symmetric bilinear form given by the matrix* $J_n^\top I_{2p,2q}$*. The group* $\mathrm{Sp}(\mathbb{C}^{2n}, \beta)$ *is conjugate to* $\mathrm{Sp}_{2n}(\mathbb{C})$ *in* $\mathrm{GL}_{2n}(\mathbb{C})$*. In particular,* $\mathrm{U}_n(\mathbb{H})$ *is isomorphic to a compact subgroup of* $\mathrm{Sp}_{2n}(\mathbb{C})$*.*

Proof. Let $g \in \mathrm{U}_{p,q}(\mathbb{H})$ be viewed as an element of $\mathrm{GL}_{2n}(\mathbb{C})$. Then we have $g^* I_{2p,2q} g = I_{2p,2q}$ and $\overline{g} J_n = J_n g$. Therefore, $J_n^\top g^* = g^\top J_n^\top$ and

$$g^\top J_n^\top I_{2p,2q} g = J_n^\top I_{2p,2q}. \qquad \square$$

2.3.1 Exercises for Section 2.3

For the first two exercises, recall that a right module M over a (noncommutative) ring R is an abelian group M together with a map $M \times R \to M, (m, r) \mapsto mr$ such that $r \mapsto (m \mapsto mr)$ defines a ring homomorphism $R \to \mathrm{End}(M)$.

Exercise 2.3.1. Let V be a right $\mathbb{H}$-module. Show that

(i) V is free, i.e., it admits an $\mathbb{H}$-basis.
(ii) If V is finitely generated as an $\mathbb{H}$-module, then it admits a finite $\mathbb{H}$-basis. In this case, all $\mathbb{H}$-bases have the same number of elements. This number is called the *dimension* of V over $\mathbb{H}$ and denoted by $\dim_{\mathbb{H}}(V)$.

Exercise 2.3.2. Let V and W be two right $\mathbb{H}$-modules with $\mathbb{H}$-bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$. Given an $\mathbb{H}$-linear map $\varphi\colon V \to W$, write

$$\varphi(v_j) = \sum_{k=1}^{n} w_k a_{kj}$$

with $a_{kj} \in \mathbb{H}$. Show that

(i) If $\varphi(v) = w$ with $v = \sum_{r=1}^{m} v_r x_r$ and $W = \sum_{s=1}^{n} w_s y_s$, then

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}.$$

(ii) The map $\varphi \mapsto (a_{kj})$ is a bijection between the set of $\mathbb{H}$-linear maps $\varphi\colon V \to W$ and matrices $A \in M_n(\mathbb{H})$ intertwining the composition of maps with the ordinary matrix multiplication (whenever composition makes sense).

Exercise 2.3.3. Show that the group $\mathrm{U}_n(\mathbb{H})$ is compact and connected.

Exercise 2.3.4. Show that each finite-dimensional complex division algebra is one-dimensional.

2.3.2 Notes on Chapter 2

The material covered in this chapter is standard and only touches the surfaces of what is known about the structure of matrix groups. For much more detailed presentations, see [GW09] or [Gr01].

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