II

Algebra

This chapter provides background material; it has two sections. The first briefly introduces universal algebra. The second surveys the many ways that products of algebras may be captured both externally and internally.

1. Universal Algebra

To set the stage for general algebraic systems, we quickly review the definitions, examples, and theorems that are needed to explain and exploit their decompositions by sheaves. Some of the topics covered are the isomorphism theorems, examples of lattices and shells, and the new notion of sesquimorphism. If the reader desires amplification, there are a number of good texts available; closest to our notation are those by Stanley Burris and H. P. Sankapannavar [**BurSa81**], George Grätzer [**Grät79**], and Ralph N. McKenzie, George F. McNulty and Walter F. Taylor [**McMcT87**]. Others are those by George M. Bergman [**Berg98**], Paul M. Cohn [**Cohn81**] and Richard S. Pierce [**Pier68**]. Ross Willard outlines some later work [**Will94**].

1.1. Definition. An **algebra**,

$$\boldsymbol{A} = \langle A; \omega_1, \omega_2, \ldots, \omega_i, \ldots \rangle$$

is a nonempty set A, called the **carrier**, together with a sequence of multiplace functions $\omega_1, \omega_2, \ldots$ from powers of A to A, called **operations**:

$$\omega_i \colon A^{n_i} \to A.$$

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This sequence may be finite, infinite and even uncountable, but each operation ω_i in it must be **finitary**, that is, have a finite number n_i of arguments. Call their sequence,

$$\boldsymbol{n} = \langle n_1, n_2, \ldots, n_i, \ldots \rangle,$$

the **type** of the algebra. To evaluate an operation ω_i, write as usual

$$\omega_i(a_1, a_2, \ldots, a_{n_i}) \qquad (a_1, a_2, \ldots, a_{n_i} \in A).$$

Implicitly included in the operations of an algebra are the **projections**:

$$\pi_i^n(a_1, a_2, \ldots, a_n) = a_i,$$

which allow us to permute and identify variables. An algebra with only one element will be called **trivial**.

In practice, the type is fixed and various algebras of this type are considered simultaneously, in which case we consider ω_i to be an **operation-symbol** and write its evaluation in a particular algebra $\boldsymbol{A}$ as $\omega_i^{\boldsymbol{A}}$. However, we shall be casual about including the superscript. Familiar examples illustrating this are the operation-symbols $+$ and $\times$ acting on the integers, rational numbers or real numbers, as desired.

Nullary operations ω_i with no arguments, where $n_i = 0$, are curious but useful and necessary; each can take on only one value:

$$\omega_i(\,) = c_i.$$

Call such an ω_i a **constant** and write it simply as an element c_i of the algebra. Most algebras occurring in practice do have constants, often designated 0 and 1.

To avoid subscripts, we often write ω as the generic operation of n arguments, and the algebra simply as $\langle A; \ldots, \omega, \ldots \rangle$. To save space, we often write the sequence $a_1, a_2, \ldots, a_n$ of arguments as a vector $\vec{a}$, and the evaluation as $\omega\vec{a} = \omega(a_1, a_2, \ldots, a_n)$.

These concepts are illustrated by many examples: rings, semilattices, lattices, and Boolean algebras. The set of integers,

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\},$$

is the carrier of the unital ring of integers,

$$\boldsymbol{\mathbb{Z}} = \langle \mathbb{Z}; +, \times, 0, 1 \rangle,$$

which is an algebra of type $\langle 2, 2, 0, 0 \rangle$. Of the same type is the ring $\boldsymbol{\mathbb{Z}}_n$ of the integers modulo n:

$$\boldsymbol{\mathbb{Z}}_n = \langle \mathbb{Z}_n; +, \times, 0, 1 \rangle,$$

where $\mathbb{Z}_n = \langle 0, 1, 2, \ldots, n-1 \rangle$ with the operations performed modulo n. There may be divisors of zero; for example, $3 \times 4 = 0$ in $\boldsymbol{\mathbb{Z}}_{12}$.

An algebra may have only one binary operation. Such are **semilattices**, $\boldsymbol{S} = \langle S; \wedge \rangle$, algebras satisfying the idempotent, commutative and associative laws for all a, b and c in S:

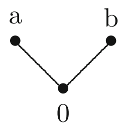
$$\begin{aligned} a \wedge a &= a, \\ a \wedge b &= b \wedge a, \text{ and} \\ a \wedge (b \wedge c) &= (a \wedge b) \wedge c. \end{aligned}$$

To each semilattice is associated a partial order: $a \leq b$ iff $a \wedge b = a$. The identities imply that each pair of elements have a greatest lower bound in the partial order associated with it.

An example is $\boldsymbol{SL}_3$, a three-element semilattice $\langle \{0, a, b\}; \wedge \rangle$ of type $\langle 2 \rangle$, whose binary operation is given by the table:

$\wedge$	0	a	b
0	0	0	0
a	0	a	0
b	0	0	b

It is pictured by its associated partial order presented as a Hasse diagram:



This algebra will serve as a counterexample for many conjectures.

A subset T of a semilattice $\boldsymbol{S}$ has an **infimum** if there is a largest element less than all elements of S, notated $\bigwedge T$. The **supremum** is defined dually, notated $\bigvee T$. If the infimum of each subset of a semilattice exists, then it is said to be **complete**.

A **lattice** $\langle L; \vee, \wedge \rangle$ has two binary operations, $\vee$ and $\wedge$, each a semilattice operation, such that the absorptive laws also hold for all a and b in L:

$$\begin{aligned} a \wedge (b \vee a) &= a \text{ and} \\ a \vee (b \wedge a) &= a. \end{aligned}$$

The identities for a lattice reflect the property that each pair of elements must have both a least upper bound and a greatest lower bound in the partial orders associated with each semilattice, and that they are dually equal:

$$a \leq_{\wedge} b \text{ iff } b \leq_{\vee} a.$$

Traditionally, the order associated with a lattice is that for $\wedge$.

In what follows, we state only the basic notions of lattice theory that are needed in this book. For more detail see the classical text of Birkhoff [**Birk67**], the introductory text of Davey and Priestley [**DavPr02**], or the advanced text of Grätzer [**Grät98**].

By **completeness** of a lattice we mean that both semilattice operations are complete. But completeness of one operation in a lattice insures completeness of the other.

A **distributive** lattice $\boldsymbol{L}$ is one satisfying the law:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad (a, b, c \in L).$$

The other distributive law follows from it and the other lattice identities:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \qquad (a, b, c \in L).$$

A lattice $\boldsymbol{L}$ is **modular** if:

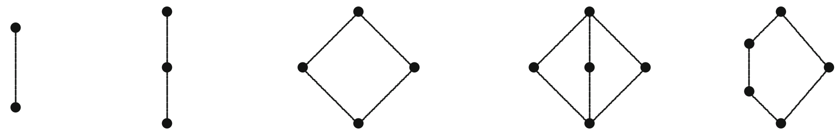
$$a \le b \Rightarrow a \vee (b \wedge c) = b \wedge (a \vee c) \qquad (a, b, c \in L).$$

Any distributive lattice is modular.

When there are two elements, 0 and 1, below and above everything else, we say that the lattice is **bounded**. A bounded lattice may be written:

$$\boldsymbol{L} = \langle L; \vee, \wedge, 0, 1 \rangle \text{ with type } \langle 2, 2, 0, 0 \rangle.$$

Of necessity, finite lattices are always bounded, although the bounds may not be part of the type. These are most easily pictured by Hasse diagrams.



$\boldsymbol{C}_2$ $\boldsymbol{C}_3$ $\boldsymbol{M}_2$ $\boldsymbol{M}_3$ $\boldsymbol{N}_5$

Note that among these examples the first three satisfy the distributive laws, which do not generally hold in lattices. The fourth $\boldsymbol{M}_3$ is not distributive; however, it is modular. The fifth $\boldsymbol{N}_5$ satisfies neither. The last three have divisors of zero. The integers $\mathbb{Z}$ have operations that turn them into an unbounded lattice:

$$m \vee n = \max(m, n) \text{ and } m \wedge n = \min(m, n).$$

In a bounded lattice, an element a may have a **complement** a':

$$a \vee a' = 1, \quad a \wedge a' = 0.$$

Complements do not always exist, and even when they do, there may be more than one. In the chain $\boldsymbol{C}_3$, the middle element has no complement. In the example of $\boldsymbol{M}_3$, complements of any middle element are not unique. However, in $\boldsymbol{M}_2$, the complements of all elements exist and are unique.

A bounded distributive lattice in which all elements have a complement is called **Boolean**. The distributive law guarantees that complements are unique. De Morgan's laws follow:

$$(a \wedge b)' = a' \vee b'; \quad (a \vee b)' = a' \wedge b' \qquad (a, b \in L).$$

Examples are $\boldsymbol{C}_2$ and $\boldsymbol{M}_2$. Putting complementation within bounded lattices into the type gives us **Boolean algebras**:

$$\langle L; \vee, \wedge, ', 0, 1\rangle \quad \text{of type} \quad \langle 2, 2, 1, 0, 0\rangle.$$

There is more about them in III.4.

Each bounded lattice has within it a Boolean sublattice called the **center**, which consists of its complemented neutral elements. As element a of a lattice $\boldsymbol{L}$ is called **neutral** if any set of three elements in L containing a generates a distributive sublattice.

Other examples of algebras are groups, rings, modules, fields, and vector spaces, with which we assume the reader is familiar. In Chap. VII, shells of various kinds generalize many of these examples.

There are several ways to relate algebras to one another; we first talk about subalgebras and homomorphisms. To be compared they must have the same type. For a **subalgebra** the new carrier is a nonempty subset of the old – closure of the new carrier to the old operations is the defining characteristic. An example is the ring of integers, which is a subalgebra of the ring of all rational numbers, which in turn is a subalgebra of the ring of all real numbers, etc. One typically writes $\boldsymbol{A} \subseteq \boldsymbol{B}$ for the subalgebra relationship.

A **homomorphism** is a function φ from the carrier of one algebra $\boldsymbol{A}$ to another $\boldsymbol{B}$ of the same type such that any operation ω of the type is preserved in going from $\boldsymbol{A}$ to $\boldsymbol{B}$:

$$\varphi\big(\omega^{\boldsymbol{A}}(a_1, \dots, a_n)\big) = \omega^{\boldsymbol{B}}\big(\varphi(a_1), \dots, \varphi(a_n)\big) \qquad (a_1, \dots, a_n \in A).$$

Abbreviate this as $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$. Reducing integers modulo 12 is an example of a homomorphism from $\mathbb{Z}$ onto $\mathbb{Z}_{12}$. Any homomorphism $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ that maps A onto B is called **surjective**. If it is one-to-one, it is called **injective**. The examples of lattices given earlier have many homomorphisms, some surjective and others injective. When a homomorphism is both injective and surjective, that is bijective, it is called an **isomorphism**, and notated $\boldsymbol{A} \cong \boldsymbol{B}$. When a homomorphism goes from an algebra back to itself, we have an **endomorphism**. An endomorphism that is also an isomorphism is an **automorphism**.

Here is a useful notion that combines subalgebras with homomorphisms. One algebra $\boldsymbol{B}$ is a **retract** of another $\boldsymbol{A}$ whenever there are two homomorphisms $\mu\colon \boldsymbol{A} \to \boldsymbol{B}$ and $\iota\colon \boldsymbol{B} \to \boldsymbol{A}$ such that their composition $\mu \circ \iota$ is the identity function on $\boldsymbol{B}$. Of necessity, μ is surjective and ι is injective. For example, $\mathbb{Z}_4$ is a retract of $\mathbb{Z}_{12}$.

The external effect of a homomorphism, $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$, can be captured internally within $\boldsymbol{A}$ itself by the concept of a congruence. A **congruence** of an algebra $\boldsymbol{A}$ is an equivalence relation θ on its carrier A such that for any operation ω of $\boldsymbol{A}$:

$$\text{if } a_1 \mathrel{\theta} b_1, \dots, a_n \mathrel{\theta} b_n, \text{ then } \omega(a_1, \dots, a_n) \mathrel{\theta} \omega(b_1, \dots, b_n).$$

An example is the congruence on $\mathbb{Z}$ of the integers modulo 12, another is the equivalence relation corresponding to the partition $\{\{a\}, \{0, b\}\}$ of the semilattice $\boldsymbol{SL}_3$ given earlier. In analogy with number theory, one sometimes writes $a \equiv b \ (\theta)$ for $a \ \theta \ b$. For the set of all congruences of an algebra $\boldsymbol{A}$ write $\operatorname{Con} \boldsymbol{A}$. Also write $\theta(B)$ for the smallest congruence of $\boldsymbol{A}$ in which all elements of a subset B are related. An algebra is called **simple** if it has exactly two congruences; of necessity, these will have to be the largest and smallest congruences. A simple algebra is never trivial. Examples of simple algebras are $\mathbb{Z}_p$ for p a prime and the two-element Boolean algebra $\boldsymbol{B}_2$.

Out of each congruence θ of an algebra $\boldsymbol{A}$, a **quotient algebra** $\boldsymbol{A}/\theta$ of the same type is constructed as follows. First, designate the congruence class $\{b \in A \mid b \ \theta \ a\}$ of θ modulo an element a of A as a/θ. The carrier of $\boldsymbol{A}/\theta$ is the set A/θ of congruence classes a/θ of θ as a runs over A. The operations ω/θ on A/θ are defined by

$$\frac{\omega}{\theta}\left(\frac{a_1}{\theta}, \ldots, \frac{a_n}{\theta}\right) = \frac{\omega(a_1, \ldots, a_n)}{\theta} \qquad (a_1, \ \ldots, a_n \in A).$$

In algebras with a group operation, one can recover the whole congruence θ from only one congruence class o/θ. Rings also do not need congruences θ since every equivalence class is a coset of the ideal $0/\theta$. Thus, it suffices to work with normal subgroups or more generally ideals.

Unfortunately, as the abundance of congruences in most lattices makes clear, there is no longer such a handy one-to-one correspondence between ideals and congruences. To see this consider these two congruences on the three-element chain $\boldsymbol{C}_3$.

The upper two elements in the first lattice are related whereas in the second they are not. The bottom singleton is insufficient to determine the congruence. So, in lattice theory and in most other algebras without a group operation, the broader concept of congruence is essential, replacing normal subgroups and ideals.

Throughout we will implicitly use the three isomorphism theorems originally formulated for modules and rings by Emmy Noether [**Noet26**, p. 40], see also [**BurSa81**, Sect. II.6]. We label them the 'homomorphism', 'cancellation', and 'Noether' theorems. The first makes precise the connection between external homomorphisms and internal congruences, and captures the one-to-one correspondence between them: each homomorphism determines a unique congruence, and each congruence determines a homomorphism that is unique up to a commutative diagram.

1.2. Theorem (Homomorphism). (a) *For any congruence θ of an algebra $\boldsymbol{A}$ there is a surjective homomorphism onto the quotient algebra, $\varphi\colon \boldsymbol{A} \to \boldsymbol{A}/\theta$, given by $\varphi(a) = a/\theta$.*

(b) *For any surjective homomorphism $\psi\colon \boldsymbol{A} \to \boldsymbol{B}$, there is a congruence θ of $\boldsymbol{A}$ such that $\boldsymbol{A}/\theta \underset{\iota}{\cong} \boldsymbol{B}$. This congruence is defined by*

$$a_1\, \theta\, a_2 \textit{ iff } \psi(a_1) = \psi(a_2) \quad (a_1, a_2 \in A)$$

and the isomorphism ι by $\iota(a/\theta) = \psi(a)$. This congruence will be called the **kernel** *of the homomorphism: $\theta = \ker\psi$.*

(c) *Starting with a congruence, passing to the surjective homomorphism, and then to its kernel yields back the original congruence.*

(d) *Starting with a surjective homomorphism $\psi\colon \boldsymbol{A} \to \boldsymbol{B}$, passing to its kernel, $\theta = \ker\psi$, and then to its quotient homomorphism φ yields an algebra isomorphic to $\boldsymbol{B}$ and a composition of functions equal to the original ψ, that is, this diagram commutes.*

$$\begin{tikzcd} \boldsymbol{A} \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & & \boldsymbol{B} \\ & \dfrac{\boldsymbol{A}}{\ker\psi} \arrow[ur, "\iota"'] & \end{tikzcd}$$

The set $\mathrm{Con}\,\boldsymbol{A}$ of congruences of an algebra $\boldsymbol{A}$ has some structure, namely the partial order of inclusion, which turns it into a lattice $\mathbf{Con}\,\boldsymbol{A}$. The intersection $\theta \cap \eta$ of any two congruences θ and η is again a congruence; also for each pair of congruences there is smallest congruence, $\theta \vee \eta$, that includes both. This join is the union of all compositions of θ and η; these few suffice for the union: $\theta \circ \eta$, $\theta \circ \eta \circ \theta$, $\theta \circ \eta \circ \theta \circ \eta$, etc. These two operations create a lattice. There are always the special congruences: the smallest $0_{\mathbf{Con}\,\boldsymbol{A}}$, which is the identity relation, often called the **trivial** congruence, and the largest $1_{\mathbf{Con}\,\boldsymbol{A}}$, the **improper** congruence, the other congruences being called **proper**.

Thus, $\mathbf{Con}\,\boldsymbol{A}$ is a bounded lattice $\langle \mathrm{Con}\,\boldsymbol{A}; \vee, \wedge, 0_{\mathbf{Con}\,\boldsymbol{A}}, 1_{\mathbf{Con}\,\boldsymbol{A}} \rangle$ for any algebra $\boldsymbol{A}$. An example is $\mathbb{Z}_4$, where $\mathbf{Con}\,\mathbb{Z}_4$ is isomorphic to $\boldsymbol{C}_3$. Also, $\mathbf{Con}\,\boldsymbol{A}$ is a complete lattice.

1.3. Theorem (Cancellation). *For any algebra $\boldsymbol{A}$ and any congruence θ, the sublattice of all congruences η between and including θ and $1_{\mathbf{Con}\,\boldsymbol{A}}$ is isomorphic to the congruence lattice $\mathbf{Con}(\boldsymbol{A}/\theta)$ of the quotient algebra. The isomorphism sends η to η/θ where η/θ is the congruence on $\boldsymbol{A}/\theta$ defined by*

$$\frac{a}{\theta}\,\frac{\eta}{\theta}\,\frac{b}{\theta} \quad \textit{iff} \quad a\,\eta\,b.$$

Moreover, $(\boldsymbol{A}/\theta)(\eta/\theta) \cong \boldsymbol{A}/\eta$.

As an illustration in the ring $\mathbb{Z}$ of integers, in the interval within $\mathbf{Con}\,\mathbb{Z}$ from mod 4 to mod 1, there are three congruences: mod 4, mod 2 and mod 1. By the cancellation theorem, this three-element lattice is isomorphic to $\mathbf{Con}\,\mathbb{Z}_4$, which is $\boldsymbol{C}_3$, as we already know.

The subalgebras $\boldsymbol{B}$ and congruences θ of an algebra interact in the last isomorphism theorem. For that, extend $\boldsymbol{B}$ to a larger subalgebra, the union of all those congruence classes of θ with at least one member in B:

$$\theta B = \{c \mid c\;\theta\; b \text{ for some } b \text{ in } B\}.$$

1.4. THEOREM (Noether). *Let $\boldsymbol{B}$ be a subalgebra of $\boldsymbol{A}$ and θ a congruence of $\boldsymbol{A}$. Then*

$$\frac{\theta B}{\theta|(\theta B)} \cong \frac{\boldsymbol{B}}{\theta|B},$$

where $\theta|B$ is the restriction of θ to $\boldsymbol{B}$.

Classically, homomorphisms of one algebra onto another are captured by congruences, as we have just seen. But congruences are relations. Are there simpler structures that will accomplish the same?

There are several: sesquimorphisms, transversals, and ideals. The first work universally for all algebras; these are projections that replace endomorphisms in earlier investigations. Precursors of sesquimorphisms occur in [**GouGr67**] and [**HobMc88**, def. 2.1], the last as idempotent polynomials restricted to their ranges. The last two, transversals and ideals, which are subsets of an algebra, are always definable, but have nice one-to-one correspondences with congruences only for certain algebras.

We define these concepts, give examples, and relate them to each other. Whenever possible, we would like an axiomatic definition for each concept that is independent of the others. Our immediate goal is to establish the isomorphism theorems for sesquimorphisms. Their definition and some of their properties were formulated in [**Knoe07a**].

1.5. DEFINITION. *A* **sesquimorphism**[1] *is a function μ from the carrier of an algebra $\boldsymbol{A}$ to itself such that*

(i) $\mu\big(\mu(a)\big) = \mu(a) \quad (a \in A)$

(ii) $\mu\big(\omega(\mu(a_1), \dots, \mu(a_n))\big) = \mu\big(\omega(a_1, \dots, a_n)\big) \qquad (a_1, \dots, a_2 \in A)$

for all operations ω of $\boldsymbol{A}$.

Although appearing a bit strange, sesquimorphisms do occur in the literature, but are rarely given a name. Here are three examples, with three more given in the next section on products. For any e in a distributive lattice, Birkhoff [1967, Sect. III. 9] displays the functions,

$$\mu\colon \boldsymbol{L} \to \boldsymbol{L} : x \mapsto e \vee x \quad \text{and} \quad \nu\colon \boldsymbol{L} \to \boldsymbol{L} : x \mapsto e \wedge x.$$

[1] Here is the reason for this name. The prefix ‘sesqui’ means a ratio of 3:2. When written with a sequence $\vec{a}$ of elements of A, formula (ii) becomes $\mu\big(\omega(\mu\circ\vec{a})\big) = \mu\big(\omega(\vec{a})\big)$, with three occurrences of μ as opposed to two in the formula for a homomorphism, $\omega(\mu \circ \vec{a}) = \mu(\omega(\vec{a}))$.

These are sesquimorphisms, in fact, idempotent endomorphisms. However, in a Boolean algebra these two functions are no longer endomorphisms, but they are still sesquimorphisms.

Other examples of sesquimorphisms that are not endomorphisms are found in the ring of integers $\mathbb{Z}$. For a fixed integer m define

$$\begin{aligned}\mu_m(n) &= n \bmod m\\ &= \text{that } k \text{ such that } 0 \le k < m \text{ and } k \equiv n \pmod{m}.\end{aligned}$$

The range $\mu_m\mathbb{Z}$ is then just the finite set $\{0, 1, \ldots, m-1\}$, with the usual addition and multiplication reduced modulo m.

While any idempotent endomorphism is a sesquimorphism, an arbitrary sesquimorphism is not necessarily an endomorphism, as just shown. However, in the context of products, Proposition 2.14 will show that for many classical algebras, factor sesquimorphisms are endomorphisms.

Each sesquimorphism μ engenders a congruence θ_μ:

$$a\ \theta_\mu\ b \text{ if } \mu(a) = \mu(b),$$

its **induced congruence**. Unequal sesquimorphisms, $\mu \neq \nu$, may induce equal congruences, $\theta_\mu = \theta_\nu$.

For a precise one-to-one correspondence between sesquimorphisms and congruences, we need also the concept of a transversal. The goal is to capture a homomorphism $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ internally within $\boldsymbol{A}$ with a sesquimorphism that picks an element from each congruence class of $\ker\varphi$. A **transversal** of a congruence θ is a subset T of A such that each congruence class of θ has exactly one representative in T. Overall, call T a **transversal** of an algebra $\boldsymbol{A}$ if it is the transversal of some congruence of $\boldsymbol{A}$. In the integers, the congruence $\bmod\, m$ has the transversal $\{0, 1, \ldots, m-1\}$, which is the range of the sesquimorphism $\mu_m(n) = n \bmod m$. Generally, for a sesquimorphism μ, its range, $\mu A = \{\mu a \mid a \in A\}$, is a transversal of θ_μ. Conversely, a congruence θ and a transversal T of it produce a sesquimorphism μ:

$$\mu(a) = \text{ that unique } b \text{ such that } a\ \theta\ b \text{ and } b \in T. \tag{1.1}$$

These observations may be formalized and proven.

1.6. Proposition. *The sesquimorphisms of an algebra $\boldsymbol{A}$ are in one-to-one correspondence with pairs consisting of a congruence and one of its transversals.*

Proof. The only difficult part might be to show that for any transversal the function μ defined by (1.1) is indeed a sesquimorphism. To that end, let $a_1, \ldots, a_n$ be any sequence of arguments for an operation ω of $\boldsymbol{A}$. Then, $a_1\ \theta\ \mu(a_1), \ldots, a_n\ \theta\ \mu(a_n)$, and hence

$$\omega(a_1, \ldots, a_n)\ \theta\ \omega(\mu(a_1), \ldots, \mu(a_2)).$$

By the uniqueness of transversal elements in congruence classes,

$$\mu\omega(a_1, a_2, \ldots) = \mu\omega(\mu a_1, \mu a_2, \ldots). \qquad \square$$

1.7. PROBLEM. Which algebras have congruences that can be represented by transversals that are subalgebras? Vector spaces are an example.

1.8. DEFINITION. We may go further within an algebra $\boldsymbol{A}$ and turn the range μA of each sesquimorphism μ of it into an algebra. To do this, **relativize** each operation ω of $\boldsymbol{A}$ to μA:

$$\omega^{\mu A}(a_1, a_2, \ldots) = \mu\big(\omega^{\boldsymbol{A}}(a_1, a_2, \ldots)\big) \qquad (a_1, a_2, \ldots \in \mu A).$$

Thus, μA becomes an algebra $\mu\boldsymbol{A}$ of the same type as $\boldsymbol{A}$. Sometimes $\mu\boldsymbol{A}$ is called the **relativization** of $\boldsymbol{A}$ by μ.

1.9. EXERCISE. Consider the four-element Boolean algebra $\boldsymbol{B}$ on the set $\{0, a, b, 1\}$ and the unique congruence θ with transversal $\{a, 1\}$. Find the corresponding sesquimorphism, and write out the operation tables for the relativized Boolean operations on the set $\{a, 1\}$; this is a again a Boolean algebra!

That each external homomorphism casts a shadow as an internal sesquimorphism gives rise to a new version of the Homomorphism Theorem.

1.10. THEOREM (Internal Homomorphism). (a) *Any sesquimorphism μ of an algebra $\boldsymbol{A}$ is a surjective homomorphism:*

$$\mu\colon \boldsymbol{A} \to \mu\boldsymbol{A},$$

where $\mu\boldsymbol{A}$ is its transversal.

(b) *Any surjective homomorphism $\psi\colon \boldsymbol{A} \to \boldsymbol{B}$ is realized internally by a sesquimorphism $\mu\colon \boldsymbol{A} \to \mu\boldsymbol{A}$ that makes this diagram commute:*

$$\begin{tikzcd}
\boldsymbol{A} \arrow[rr, "\psi"] \arrow[dr, "\mu"'] & & \boldsymbol{B} \\
& \mu\boldsymbol{A} \arrow[ur, leftrightarrow, "isom."'] &
\end{tikzcd}$$

Here, μ is any sesquimorphism associated with the kernel of ψ, and the isomorphism from $\mu\boldsymbol{A}$ to $\boldsymbol{B}$ is $\psi|(\mu\boldsymbol{A})$. Thus $\ker\psi = \ker\mu$.

To see what this has to do with the traditional Homomorphism Theorem phrased in terms of congruences, realize that

$$\frac{\boldsymbol{A}}{\ker\psi} \cong \mu\boldsymbol{A}.$$

The Cancellation Theorem, when phrased in terms of sesquimorphisms, is simpler than when phrased in terms of congruences. To state it, sesquimorphisms need to be partially ordered, using their composition.

1.11. DEFINITION. For sesquimorphisms μ and ν in an algebra $\boldsymbol{A}$,

$$\mu \le \nu \quad \text{if} \quad \mu \circ \nu = \mu = \nu \circ \mu.$$

Consequently, if $\mu \leq \nu$, then

(a) $\mu \circ \nu$ and $\nu \circ \mu$ are sesquimorphisms,
(b) $\nu(a) = \nu(b)$ implies $\mu(a) = \mu(b)$ $(a, b \in A)$,
(c) $\mu(A) \subseteq \nu(A)$,
(d) $\theta_\mu \supseteq \theta_\nu$.

The reversal of ordering in (d) will fit in with the Boolean algebras of factor objects to be defined in Sect. VI.2, and in particular Theorem VI.3.2. For now, accept that we are ordering sesquimorphisms by the size of their transversals.

1.12. Theorem (Internal Cancellation). *If μ and ν are sesquimorphims of an algebra $\boldsymbol{A}$ such that $\mu \leq \nu$ and $\bar{\mu} = \mu|(\nu A)$, then $\bar{\mu}$ is a sesquimorphism of $\nu \boldsymbol{A}$ and $\mu = \bar{\mu} \circ \nu$.*

Proof. Since $\nu(\nu(a)) = \nu(a)$ and $\mu \leq \nu$, then $\bar{\mu}(\nu(a)) = \mu(a)$. □

To connect with the traditional Cancellation Theorem, usually couched in terms of two congruences, $\eta \subseteq \theta$, let ν be a sesquimorphism for η. Choose a transversal T for θ such that $T \subseteq \nu A$, thereby determining a sesquimorphism μ such that $T = \mu A$. Then $\mu \leq \nu$. By the Internal Homomorphism Theorem,

$$\frac{\boldsymbol{A}}{\eta} \cong \nu \boldsymbol{A}, \quad \frac{\boldsymbol{A}/\eta}{\theta/\eta} \cong \bar{\mu}\nu \boldsymbol{A}, \quad \frac{\boldsymbol{A}}{\theta} \cong \mu \boldsymbol{A}.$$

The conclusion of the Internal Cancellation Theorem yields the traditional Cancellation Theorem.

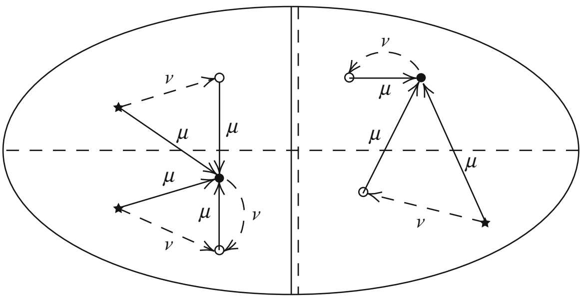


Figure 1. The action of sesquimorphisms.

Figure 1 illustrates the general interaction of two sesquimorphisms, μ and ν, of an algebra $\boldsymbol{A}$, with corresponding congruences, $\theta \supseteq \eta$. Their congruence classes are outlined in solid and dashed lines, respectively – two classes for θ and four for η. The symbol $\star$ represents arbitrary elements of A, the symbol $\circ$ their images under ν, and $\bullet$ their images under μ. Thus, there is one $\circ$ for each congruence class of ν, and one $\bullet$ for each class of μ.

As ν and μ are idempotent, $\circ$ and $\bullet$ go to themselves under the respective sesquimorphisms. If also $\mu \le \nu$, as in Theorem 1.12, then $\circ$ and $\bullet$ would become one and the same in the first and third quadrants of this figure.

1.13. PROBLEM. Given an algebra $\boldsymbol{A}$, can one choose for each congruence θ a sesquimorphism μ_θ representing it ($\theta_{\mu_\theta} = \theta$) so that altogether they are compatible:

$$\theta \subseteq \eta \quad \text{if, and only if,} \quad \mu_\theta \ge \mu_\eta \qquad (\theta \in \operatorname{Con} \boldsymbol{A})?$$

The Noether Theorem becomes even simpler when viewed internally.

1.14. THEOREM (Internal Noether). *Let $\boldsymbol{B}$ be a subalgebra of $\boldsymbol{A}$. Suppose μ is a sesquimorphism of $\boldsymbol{A}$ such that $b \in B$ implies $\mu(b) \in B$. Define $\boldsymbol{C} = \mu^{-1}\boldsymbol{B}$. Then $\boldsymbol{C}$ is a subalgebra of $\boldsymbol{A}$, and*

$$\mu\boldsymbol{C} = \mu\boldsymbol{B}.$$

PROOF. To check closure of $\boldsymbol{C}$ to an n-ary operation ω of $\boldsymbol{A}$, apply it to elements $c_1, \ldots, c_n$ of $\boldsymbol{C}$. There must be elements b_i of $\boldsymbol{B}$ such that $\mu c_i = b_i$. Thus,

$$\mu\omega(c_1, \ldots, c_n) = \mu\omega(\mu b_1, \ldots, \mu b_n) = \mu\omega(b_1, \ldots, b_n) \in B,$$

since μ is a sesquimorphism. Hence, $\omega\vec{c} \in \boldsymbol{C}$.
The equality of the subalgebras is straightforward to verify. □

Again, we see simple parallels with congruences. If θ is the congruence determined by μ, then $\boldsymbol{C}$ is the subalgebra of $\boldsymbol{A}$ that is the union of all those congruence classes of θ with at least one member in B, that is, $\boldsymbol{C} = \theta\boldsymbol{B}$, as defined in the external Noether theorem. Applying μ to both sides of this equation yields the quotient algebras of Theorem 1.4.

1.15. PROBLEM. (a) When can sesquimorphisms be chosen to be term-operations or polynomials?
(b) When can sesquimorphisms be chosen to be endomorphisms?

The last internal concept for capturing homomorphisms is that of ‘ideal’. It depends on fixing one element of the carrier of an algebra as an ‘origin’ o. For a congruence θ, its **ideal** or **o-class** is the equivalence class o/θ. This notion captures several common concepts in algebra: in a group an ideal is a normal subgroup, providing the origin is the unity 1; in a ring or Boolean algebra, an ideal is the usual notion, providing that $o = 0$; and in a Boolean algebra, if $o = 1$, then we obtain ‘filters’. In these cases, any other element can also serve as the origin, in the sense that its ideals uniquely determine the congruences from which they come. Ivan Chajda, Günther Eigenthaler and Helmut Länger [**ChaEL03**, p. 64] name this property: an algebra is **regular** if any congruence θ of $\boldsymbol{A}$ is determined by o/θ for any o in A, that is, for any congruences θ and η, and any element o of A, if $o/\theta = o/\eta$, then $\theta = \eta$. However, an ideal generally does not uniquely determine a congruence, as shown earlier.

Classically, the concept of an ideal as an internal determiner of homomorphisms is defined in terms of equations, as for normal subgroups. In [**ChaEL03**, p. 137], ideals are defined in this way with what are called 'ideal terms'. Their notion agrees with ours for the specific algebras mentioned in the last paragraph, but otherwise they may disagree. We do not pursue how generally equations may be used to capture the concept of an ideal since our more special interest is in complementary factor ideals, where these will be characterized in shells by somewhat different sentences at the end of Sect. VII.3.

Ideals are subalgebras when there is an element o of the algebra such that for any operation, $\omega(o, o, \ldots, o) = o$, that is, $\{o\}$ is a subalgebra. An ideal of an algebra is **maximal** if any larger ideal is the whole algebra.

This section closes with a miscellanea of useful concepts and conventions. Sometimes we need to forget about some of the operations of an algebra; this is called a **reduct**. For example, the group $\langle \mathbb{Z}; +, 0\rangle$ of integers is a reduct of the ring $\langle \mathbb{Z}; +, \times, 0, 1\rangle$. The opposite of a reduct is an **expanse**, where more operations are adjoined to an algebra.

Other times we want an operation to act on subsets of the carrier rather than on elements. This is done by mimicking the complex multiplication often found in group and ring theory:

$$\omega(A_1, A_2, \ldots, A_n) = \{\omega(a_1, a_2, \ldots a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}\,.$$

Occasionally, as in categories and sheaves, we need to relax the totality of the operations. A **partial algebra** is an algebra in which some or all of the operations are not defined everywhere. An example is the notion of a field in which division by 0 is not allowed. We might specify the field as a partial algebra $\langle F; +, -, \times, /, 0, 1\rangle$. A composition of these operations is said to exist for particular arguments if each stage of evaluation exists. Thus, in a field, $0 + 1/(1 + 1)$ exists, but $0 + 1/(1 - 1)$ does not. An equation in which each side is a composition of partial operations is said to be **satisfied** for particular arguments if, when one side exists, so does the other, and they are equal.

The definition of homomorphism for partial algebras also needs a proviso. A **homomorphism** of partial algebras, $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$, must satisfy for each n-ary operation ω:

$$\varphi(\omega(a_1, a_2, \ldots, a_n)\big) = \omega\big(\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_n)\big)$$

whenever $\omega(a_1, a_2, \ldots, a_n)$ exists, and thus also $\omega\big(\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_n)\big)$ must exist.

The symbols 0 and 1 are used throughout this book in many contexts, most loosely connected. In a bounded partial order, $\boldsymbol{P} = \langle P; \leq\rangle$, the bottom element is designated $0_{\boldsymbol{P}}$ and the top $1_{\boldsymbol{P}}$. Thus, in the congruence lattice $\mathbf{Con}\,\boldsymbol{A}$ of an algebra $\boldsymbol{A}$, the identity relation, which is the smallest, is $0_{\mathbf{Con}\,\boldsymbol{A}}$, and the universal relation, the largest, is $1_{\mathbf{Con}\,\boldsymbol{A}}$. This goes along with notations in bounded lattices. Further 1_S represents the identity function on a set S, in other words, 1_S is the unity of the semigroup of functions on S. When obvious, these subscripts may be dropped to improve clarity. Also, this notation may be used for identity morphisms in a category; we will see in Sect. IV.3 that the identity morphism $1_{\mathcal{A}}$ of a sheaf $\mathcal{A}$ has a more complicated structure than a simple identity function has on a set. Chapter VII will use 0 and 1 in shells analogously to their use in unital rings, monoids and bounded lattices.

Here are other common terms and notations. The **power set** $\mathcal{P}(A)$ of a set A is the set of all its subsets. The cardinality of a set A is notated $|A|$.

The **composition** of two binary relations η and θ is given by:

$$\eta \circ \theta = \{\langle a, b\rangle \mid a\ \eta\ x \text{ and } x\ \theta\ b \text{ for some } x\}.$$

If $\eta \circ \theta = \theta \circ \eta$, then η and θ are said to **commute** or **permute**.[2] For example, in an algebra with a group operation, any two congruences commute. Generally, the composition of two congruences is not again a congruence; but when they commute it is.

Composition specializes to functions α and β, when they are considered as sets of ordered pairs, so that upon evaluation:

$$(\alpha \circ \beta)(a) = \alpha(\beta(a)).$$

The next exercise is about commuting sesquimorphisms.

1.16. EXERCISE. Find proofs and counterexamples.

(a) A composition of sesquimorphism is not necessarily a sesquimorphism.
(b) If two sesquimorphisms commute, then their composition is a sesquimorphism.
(c) Even if two sesquimorphisms commute, their congruences may not.
(d) Even if two congruences commute, some corresponding sesquimorphisms may not.

The anomalies of Exercise 1.16 will vanish in the next section when complementary sesquimorphisms create products.

Composition may be extended to functions of more than one argument. For an m-ary function α and an n-ary function β the **composition** $\alpha \circ \beta$ is an mn-ary function.

$$\begin{aligned}&(\alpha \circ \beta)(a_1^1, a_2^1, \ldots, a_n^1,\ a_1^2, a_2^2, \ldots, a_n^2,\ \ldots,\ a_1^m, a_2^m, \ldots, a_n^m)\\ &= \alpha\big(\beta(a_1^1, a_2^1, \ldots, a_n^1),\ \beta(a_1^2, a_2^2, \ldots, a_n^2),\ \ldots,\ \beta(a_1^m, a_2^m, \ldots, a_n^m)\big).\end{aligned}$$

It is most convenient to express this by adapting matrix notation:

$$\alpha(M\beta) = \alpha\big(\beta(a_1^1, a_2^1, \ldots, a_n^1),\ \beta(a_1^2, a_2^2, \ldots, a_n^2),\ \ldots,\ \beta(a_1^m, a_2^m, \ldots, a_n^m)\big),$$

[2] The traditional term for relations is 'permute', but the preferred term in this book is 'commute' since other related notions, such as endomorphisms, traditionally commute.

where

$$M = \begin{pmatrix} a_1^1, & a_2^1, & \dots, & a_n^1 \\ a_1^2, & a_2^2, & \dots, & a_n^2 \\ \vdots & \vdots & & \\ a_1^m, & a_2^m, & \dots, & a_n^m \end{pmatrix}.$$

Here, $M\beta$ is the column vector resulting from β operating on each row, and α operates on this column to produce a single element. Similarly, αM is the row vector resulting from α operating on each column, and β operates on this row to yield $(\alpha M)\beta$. If $\alpha(M\beta) = (\alpha M)\beta$, then α and β are said to **commute**; equivalently, if $\beta(M^T\alpha) = (\beta M^T)\alpha$, using the transpose. This is an extension of commuting functions of one variable. For example, when α and β are binary operations of an algebra, they commute iff

$$\alpha\big(\beta(a,b), \beta(c,d)\big) = \beta\big(\alpha(a,c), \alpha(b,d)\big).$$

We extend this notation to the preservation of relations by functions. Let a^i be the ith row of the matrix M and a_j its jth column. For an n-ary function φ and an m-ary relation ρ, both on the set A, we say that φ **preserves** ρ

$$\text{if } \rho(a_1), \rho(a_2), \dots, \rho(a_n), \text{ then } \rho(\varphi(a^1), \varphi(a^2), \dots, \varphi(a^m)),$$

for any m by n matrix M with entries a_j^i in A.

The product $\alpha \times \beta$ of two m-ary relations, α on A and β on B, is an m-ary relation on $A \times B$ given by:

$$\langle\langle a_1, b_1\rangle, \dots, \langle a_m, b_m\rangle\rangle \in \alpha \times \beta \text{ if } \langle a_1, \dots, a_m\rangle \in \alpha \text{ and } \langle b_1, \dots, b_m\rangle \in \alpha.$$

It should be clear how to define products of more than two relations and powers of a single relation.

Since functions are so fundamental, composition of them will often be abbreviated, $\alpha\beta = \alpha \circ \beta$, and their evaluation, $\alpha a = \alpha(a)$. Confusion might result when composition and evaluation are juxtaposed and iterated, but associative laws save the day – at least for functions of one argument:

$$\alpha \circ (\beta \circ \gamma) = \alpha\beta\gamma = (\alpha \circ \beta) \circ \gamma;$$
$$(\alpha \circ \beta)(a) = \alpha\beta a = \alpha(\beta(a)).$$

By $\Phi(\varphi)$(a) where φ is a function and Φ is a function of functions, we mean of course $\big(\Phi(\varphi)\big)$(a). The **domain** of a function, $\varphi\colon A \to B$, is A; and its **range** is denoted: $\operatorname{rng}\varphi = \{\varphi(a) \mid a \in A\}$. If S is a subset of A, then

$$\varphi(S) = \{\varphi(s) \mid s \in S\} \quad \text{and}$$
$$\varphi|S = \{\langle s, \varphi(s)\rangle \mid s \in S\},$$

the restriction of a function.

In analogy with the set former $\{n^2 \mid n \in \mathbb{Z}\}$ – the collection of integers that are squares – we write $\langle n^2 \mid n \in \mathbb{Z}\rangle$ for the function φ that squares integers. We may call this a **family** when the emphasis is on the range of values. This function former may also be spelled out as

$$\varphi : \mathbb{Z} \to \mathbb{Z} : n \mapsto n^2;$$

more generally,

$$\varphi : \boldsymbol{A} \to \boldsymbol{B} : a \mapsto \varepsilon(a),$$

where ε is some expression.

To make formulas easily readable we continue to juxtapose symbols when no confusion arises. For example, in shells the second binary operation-symbol may be omitted: $ab = a \times b$ in rings, and $ab = a \wedge b$ in lattices. These implicit operations have the greatest cohesiveness in groupings; in lattices for example,

$$a(bc \vee de) \vee f = \Big(a \wedge \big((b \wedge c) \vee (d \wedge e)\big)\Big) \vee f.$$

For algebras with a binary operation such as $\times$, we may combine the convention of juxtaposition with complex multiplication:

$$eA = e \times A = \{e \times a \mid a \in A\} = \{ea \mid a \in A\}.$$

The repeated argument o of a function ω is written

$$\omega(\dot{o}) = \omega(o, o, \ldots, o).$$

Here, $\dot{o}$ is o repeated enough times to fill out ω. It is also convenient in long derivations to abbreviate the sequence of arguments $a_1, a_2, \ldots, a_n$ of a function ω as $\vec{a}$. Thus,

$$\omega(\vec{a}) = \omega(a_1, a_2, \ldots, a_n).$$

We continue the convention adopted in model theory of distinguishing between a structure and its carrier: a bold letter for the algebra and an ordinary font for the carrier, and similarly for a topological space. This convention breaks down when both operations and a topology are present on the carrier and the definition of each takes several stages, as in the definition of sheaves in Sect. IV.1.

This convention continues over functions. Thus, when $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, $\varphi(A)$ is a *set* that is the range of φ. But $\varphi(\boldsymbol{A})$ is the image of φ as a *subalgebra* of $\boldsymbol{B}$. A similar comment applies to topological spaces.

Operators, such as Con and Γ, need different conventions. Thus, Con $\boldsymbol{A}$ is the *set* of congruences on an algebra $\boldsymbol{A}$, whereas **Con** $\boldsymbol{A}$ is the *lattice* of congruences. Similarly, $\Gamma(\boldsymbol{\mathcal{A}})$ will be the *set* of global sections of the sheaf on an algebra $\boldsymbol{A}$, whereas $\boldsymbol{\Gamma}(\boldsymbol{\mathcal{A}})$ will be the algebra of such sections.

A few words about emptiness. In general we have excluded empty algebras, the conventional stand, although there are good reasons for including them. Among these are that each variety would then have a free algebra on zero generators; this free algebra would be the initial object of the variety when viewed as a category. Also, there would be no need to distinguish between subalgebras (ordinarily nonempty) and subuniverses (possibly empty). When nullary operations are present, empty algebras do not exist: so in this case, these points are moot. And this is so for most of our applications. To have included empty algebras in this book would have meant adding extra clauses and some ad hoc constructions to the definitions involving specific categories.

2. Products and Factor Objects

Homomorphic images and subalgebras of an algebra are smaller algebras. We turn our attention now to constructions that create larger algebras: direct products, subdirect products, and disjoint unions. Decomposing an algebra into various products of smaller and more manageable pieces is the foundation for constructing the Pierce sheaf. This will be achieved in Chap. VI by refining a family of factor congruences into a subdirect product indexed by a topological space.

This section studies how to factor an arbitrary algebra into a product, both externally and internally. Relating outer direct products to inner direct products is natural and well developed in classical systems. Jónsson and Tarski [**JónTa47**] extended this correspondence to more general algebras (JT-algebras) when they proved uniqueness of direct decompositions of their finite algebras.

We exhibit the many ways in which factorizations may be characterized. The external ways are the outer direct product and the categorical product. There are up to five ways to recreate these outer products internally. Complementary factor congruences are well known, and factor bands less so; there are also complementary factor ideals and elements, the analog of central idempotents in rings. Complementary factor sesquimorphisms have been defined up to now in the literature only for special algebras, and often they are just endomorphisms.

In unital rings and bounded lattices, as well as in their common generalization, unital shells, all external and internal concepts are equivalent. But in general, among all the ways to express factorizations in arbitrary algebras, there is a bijective correspondence between only some of these: outer direct products, categorical products, bands, complementary congruences and complementary sesquimorphisms. Equivalence with the remaining two, elements and ideals, requires something like a weak sum or multiplication in Chap. VII. This section concludes with subdirect products and disjoint unions of algebras.

2.1. Definition. The **outer direct product $\boldsymbol{P}$**, or just **product**, of two algebras $\boldsymbol{A}$ and $\boldsymbol{A}'$ of the same type has as a carrier the Cartesian product $A \times A'$, with the operations defined on it coordinate-wise:

$$\omega^{\boldsymbol{P}}\big(\langle a_1, a_1' \rangle, \ldots, \langle a_n, a_n' \rangle\big) = \langle \omega^{\boldsymbol{A}}(a_1, \ldots, a_n), \omega^{\boldsymbol{A}'}(a_1', \ldots, a_n') \rangle,$$

for any n-ary operation ω of the given type with a_i in A and a'_i in A'. Associated with it are the **projections**: $\pi\colon \boldsymbol{P} \to \boldsymbol{A}\colon \langle a, a'\rangle \mapsto a$ and $\pi'\colon \boldsymbol{P} \to \boldsymbol{A}'\colon \langle a, a'\rangle \mapsto a'$. One writes $\boldsymbol{P} = \boldsymbol{A} \times \boldsymbol{A}'$, and omits the words modifying 'product' when clear. For the same factor repeated n times, we have the power $\boldsymbol{A}^n$.

Carried throughout this discussion will be the example of a product of cyclic rings: $\mathbb{Z}_{12} \cong \mathbb{Z}_3 \times \mathbb{Z}_4$, where the isomorphism is given by projections: $m \mapsto \langle m \bmod 3, m \bmod 4\rangle$.

Power sets, viewed as Boolean algebras, provide more products:

$$\mathcal{P}(A \cup B) \times \mathcal{P}(A \cap B) \cong \mathcal{P}(A) \times \mathcal{P}(B).$$

2.2. DEFINITION. An algebra $\boldsymbol{P}$ is said to be a **categorical product** of algebras $\boldsymbol{A}$ and $\boldsymbol{A}'$ if there are homomorphisms $\pi\colon \boldsymbol{P} \to \boldsymbol{A}$ and $\pi'\colon \boldsymbol{P} \to \boldsymbol{A}'$ such that for any other algebra $\boldsymbol{Q}$ and homomorphisms $\rho\colon \boldsymbol{Q} \to \boldsymbol{A}$ and $\rho'\colon \boldsymbol{Q} \to \boldsymbol{A}'$ there is a unique homomorphism $\chi\colon \boldsymbol{Q} \to \boldsymbol{P}$ for which $\rho = \pi \circ \chi$ and $\rho' = \pi' \circ \chi$; that is, this diagram commutes:

$$\begin{tikzcd}
\boldsymbol{A} & \boldsymbol{P} \arrow[l, "\pi"'] \arrow[r, "\pi'"] & \boldsymbol{A}' \\
& \boldsymbol{Q} \arrow[ul, "\rho"] \arrow[u, dashed, "\chi"] \arrow[ur, "\rho'"'] &
\end{tikzcd}$$

Categorical notions are defined relative to a class of objects and mappings between them. For now, it suffices to consider all algebras of a given type, and all homomorphisms between them. In this case, the outer direct product $\boldsymbol{A} \times \boldsymbol{A}'$ defined earlier will also be a categorical product. Conversely, for any categorical product P as notated above, there is the isomorphism, $\chi : \boldsymbol{P} \cong \boldsymbol{A} \times \boldsymbol{A}'$ (in the defining diagram take $\boldsymbol{Q} = \boldsymbol{A} \times \boldsymbol{A}'$ with ρ and ρ' the Cartesian projections).

One says 'a' product since categorically products are defined only up to isomorphism. For example, in the category of sets, a product of $\{0, 1, 2\}$ and $\{0, 1\}$ may be any six-element set. Note also that the projections are considered an integral part of a categorical product $\langle \boldsymbol{P}, \pi, \pi'\rangle$. Two categorical products on the same algebra, $\langle \boldsymbol{P}; \pi, \pi'\rangle$ of $\boldsymbol{A}$ and $\boldsymbol{A}'$, and $\langle \boldsymbol{P}; \rho, \rho'\rangle$ of $\boldsymbol{B}$ and $\boldsymbol{B}'$, are said to be **isomorphic** if there are isomorphisms making this diagram commute.

$$\begin{tikzcd}
\boldsymbol{A} \arrow[dd, leftrightarrow, "\text{isom.}"'] & & \boldsymbol{A}' \arrow[dd, leftrightarrow, "\text{isom.}"] \\
& \boldsymbol{P} \arrow[ul, "\pi"'] \arrow[ur, "\pi'"] \arrow[dl, "\rho"] \arrow[dr, "\rho'"'] & \\
\boldsymbol{B} & & \boldsymbol{B}'
\end{tikzcd}$$

The internal ways of factoring are five in number. If a unital ring factors, $\boldsymbol{R} = \boldsymbol{S} \times \boldsymbol{T}$, then this product may be captured within $\boldsymbol{R}$ as follows:

(a) By complementary central idempotents, $e = \langle 1, 0\rangle$ and $e' = \langle 0, 1\rangle$

(b) By complementary ideals, $I = eR$ and $I' = e'R$
(c) By complementary endomorphisms onto these ideals, $\pi(r) = er$ and $\pi'(r) = e'r$
(d) By complementary congruences, θ and θ', where $r\ \theta\ s$ iff $er = es$, and $r\ \theta' s$ iff $e'r = e's$
(e) By a band, $\beta(r, s) = er + e's$.

In addition one can talk about an inner direct product, $\boldsymbol{R} = e\boldsymbol{R} + e'\boldsymbol{R}$, associated with this outer direct product.

In analogy with analytic geometry, Fig. 2 illustrates these concepts with the product of rings, $\mathbb{Z}_{12} \cong \mathbb{Z}_3 \times \mathbb{Z}_4$. Think of the bottom row, the ideal

$9 \sim \langle 0,1\rangle = e'$	$5 \sim \langle 2,1\rangle$	$1 \sim \langle 1,1\rangle$
$6 \sim \langle 0,2\rangle$	$2 \sim \langle 2,2\rangle$	$10 \sim \langle 1,2\rangle$
$3 \sim \langle 0,3\rangle$	$11 \sim \langle 2,3\rangle$	$7 \sim \langle 1,3\rangle$
$0 \sim \langle 0,0\rangle$	$8 \sim \langle 2,0\rangle$	$4 \sim \langle 1,0\rangle = e$

FIGURE 2. Representing the ring $\mathbb{Z}_{12}$ as $\mathbb{Z}_3 \times \mathbb{Z}_4$

$4\mathbb{Z}_{12}$, as the X-axis – this is isomorphic to $\mathbb{Z}_3$; and the left-hand column, the ideal $3\mathbb{Z}_{12}$, as the Y- or X'-axis – this is isomorphic to $\mathbb{Z}_4$. An X-coordinate x at the bottom points up and an X'-coordinate x' at the left points to the right, giving us an entry in the body that is $\beta(x, x')$. Endomorphisms do the opposite: π projects down to the X-axis and π' projects left to the X'-axis. More generally, $\beta(r, s)$ is that entry that is in same column as r and the same row as s. The corresponding congruences are $\theta = \bmod 3$, and $\theta' = \bmod 4$. As for their congruence classes, the columns are the θ-classes and the rows the θ'-classes.

We now define these internal notions within any general algebra, calling them collectively **factor objects**, starting with bands and ending with elements, prefixing the adjective 'factor' to each to indicate their origin in a product. But to fully capture products by all these notions we will have to wait till additional assumptions on the algebra are added.

For any product of algebras, $\boldsymbol{P} = \boldsymbol{A} \times \boldsymbol{A}'$, there is the homomorphism, $\beta \colon \boldsymbol{P} \times \boldsymbol{P} \to \boldsymbol{P}$, given by $\beta(\langle a, a'\rangle, \langle b, b'\rangle) = \langle a, b'\rangle$. A product given merely as an isomorphism, $\boldsymbol{P} \cong \boldsymbol{A} \times \boldsymbol{A}'$, will carry this binary function back to $\boldsymbol{P}$. In any case, it satisfies these equations for elements in $\boldsymbol{P}$:

$$\beta(p, p) = p, \tag{2.1}$$

$$\beta(\beta(p, q), \beta(s, t)) = \beta(p, t), \tag{2.2}$$

$$\beta(\omega(p_1, \ldots, p_n), \omega(q_1, \ldots, q_n)) = \omega(\beta(p_1, q_1), \ldots, \beta(p_n, q_n)), \tag{2.3}$$

where ω is any n-ary operation of the given type. We call such a binary function β on any algebra a **factor band** ; it is also called a **decomposition operation** (see [**McMcT87**, vol. 1, pp. 162 . . .]). The last equation amounts to the commutativity of β and ω, expressed more compactly: $\beta \circ \omega = \omega \circ \beta$. In the example of the ring $\mathbb{Z}_{12}$, which is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_4$, one has that $\beta(r, s) = 4r + 9s$, the coefficients to be found by the Chinese Remainder Theorem. It follows from its defining identities that a factor band is always associative:

$$\beta(p, \beta(q, r)) = \beta(\beta(p, p), \beta(q, r)) = \beta(p, r) = \ldots = \beta(\beta(p, q), r);$$

hence, the name 'band', as it is applied to any idempotent and associative binary operation. There are the trivial factor bands, $\beta(a, b) = a$ and $\beta(a, b) = b$ for all a and b in A, corresponding to factoring an algebra $\boldsymbol{A}$ as a product of itself with a one-element algebra.

Factor bands may also be obtained directly from the categorical definition. Let $\boldsymbol{P}$ be the outer direct product $\boldsymbol{A} \times \boldsymbol{A}'$ with projections π and π'. As $\boldsymbol{P} \times \boldsymbol{P}$ is also an outer direct product, there are two projections associated with it: $\Pi, \Pi' \colon \boldsymbol{P} \times \boldsymbol{P} \to \boldsymbol{P}$. If we view $\boldsymbol{P}$ as a categorical product, there must be a unique $\beta \colon \boldsymbol{P} \times \boldsymbol{P} \to \boldsymbol{P}$ for which $\pi \circ \beta = \pi \circ \Pi$ and $\pi' \circ \beta = \pi' \circ \Pi'$. One easily checks that β is the previously defined factor band. A commutative diagram illustrates this construction.

$$\begin{tikzcd}
\boldsymbol{P} \arrow[d, "\pi"'] & \boldsymbol{P} \times \boldsymbol{P} \arrow[l, "\Pi"'] \arrow[r, "\Pi'"] \arrow[d, dashed, "\beta"'] & \boldsymbol{P} \arrow[d, "\pi'"] \\
\boldsymbol{A} & \boldsymbol{P} \arrow[l, "\pi"] \arrow[r, "\pi'"'] & \boldsymbol{A}'
\end{tikzcd}$$

2.3. PROPOSITION. *Two categorical products, $\langle \boldsymbol{P} : \pi, \pi' \rangle$ and $\langle \boldsymbol{P} : \rho, \rho' \rangle$ of a common algebra $\boldsymbol{P}$, are isomorphic if, and only if, their respective factor bands β and γ are equal.*

PROOF. From the diagram above the factor bands are defined by

$$\beta(p, q) = \chi\langle \pi p, \pi' q \rangle,$$
$$\gamma(p, q) = \psi\langle \rho p, \rho' q \rangle.$$

In the first formula, the isomorphism χ comes from inserting the outer direct product $\boldsymbol{A} \times \boldsymbol{A}'$ into the definition of categorical product:

$$\begin{tikzcd}
 & \boldsymbol{A} \times \boldsymbol{A}' \arrow[dl] \arrow[dr] \arrow[dd, "\chi"'] & \\
\boldsymbol{A} & & \boldsymbol{A}' \\
 & \boldsymbol{P} \arrow[ul, "\pi"'] \arrow[ur, "\pi'"] &
\end{tikzcd}$$

In the second formula, ψ comes from a similar diagram for $\boldsymbol{B} \times \boldsymbol{B}'$.

The two directions of logical implication are proven by developing the next commutative diagram in two different ways.

$$
\begin{tikzcd}
& \boldsymbol{A} \times \boldsymbol{A}' \arrow[dl] \arrow[dr] \arrow[dd, "\chi"] & \\
\boldsymbol{A} \arrow[dd, "\varphi"'] & & \boldsymbol{A}' \arrow[dd, "\varphi'"] \\
& \boldsymbol{P} \arrow[ul, "\pi"] \arrow[ur, "\pi'"'] \arrow[dl, "\rho"] \arrow[dr, "\rho'"'] & \\
\boldsymbol{B} & & \boldsymbol{B}' \\
& \boldsymbol{B} \times \boldsymbol{B}' \arrow[ul] \arrow[ur] \arrow[uu, "\psi"] &
\end{tikzcd}
$$

$\Rightarrow$. Create this diagram from two earlier diagrams for categorical products and their isomorphism. Clearly, $\chi = \psi \circ \langle \varphi, \varphi' \rangle$. Hence, for all p and q in $\boldsymbol{P}$,

$$\gamma pq = \psi\langle \rho p, \rho' q\rangle = \psi\langle \varphi\pi p, \varphi'\pi' q\rangle = \chi\langle \pi p, \pi' q\rangle = \beta pq.$$

$\Leftarrow$. Create this diagram again by defining $\langle \varphi, \varphi' \rangle$ as $\psi^{-1} \circ \chi$. $\square$

The four remaining inner factor objects come in pairs.

2.4. Definition. Two congruences θ and θ' of an algebra $\boldsymbol{A}$ are **complementary factor** congruences if

$$\theta \cap \theta' = 0_{\mathbf{Con}\,\boldsymbol{A}}, \tag{2.4}$$

$$\theta \circ \theta' = 1_{\mathbf{Con}\,\boldsymbol{A}}. \tag{2.5}$$

Pierce [**Pier68**, p. 88] calls these **decomposition** congruences .

It follows from (2.5), by taking the converse of each side, that also $\theta' \circ \theta = 1$, and hence for their join that $\theta \vee \theta' = 1$. (More generally, for any two congruences θ and η of an algebra $\boldsymbol{A}$, $\theta \vee \eta = \theta \circ \eta$ iff $\theta \circ \eta = \eta \circ \theta$.)

Conditions (2.4) and (2.5) have useful interpretations. Let θ and θ' be arbitrary congruences of an algebra $\boldsymbol{A}$, not necessarily complementary. There is always the **canonical** homomorphism,

$$\varphi\colon \boldsymbol{A} \to \frac{\boldsymbol{A}}{\theta} \times \frac{\boldsymbol{A}}{\theta'} : a \mapsto \left\langle \frac{a}{\theta}, \frac{a}{\theta'} \right\rangle.$$

Now (2.4) holds just when φ is injective; and (2.5) just when φ is surjective.

Another viewpoint considers, for any a and b in A, the possible solutions x to the system of congruences:

$$\begin{cases} x \equiv a \ (\theta), \\ x \equiv b \ (\theta'). \end{cases}$$

Condition (2.5) insures that solutions exist, and (2.4) promises uniqueness.

Up to isomorphism there is a one-to-one correspondence between outer direct products, factor bands and complementary factor congruences, as stated next.

2.5. THEOREM. (a) *An algebra decomposes as a product,* $\boldsymbol{P} \cong \boldsymbol{A} \times \boldsymbol{A}'$, *if, and only if,* $\boldsymbol{P}$ *has a pair of complementary factor congruences* θ *and* θ' *such that*

$$\boldsymbol{A} \cong \frac{\boldsymbol{P}}{\theta} \text{ and } \boldsymbol{A}' \cong \frac{\boldsymbol{P}}{\theta'}.$$

(b) *In any algebra, via its product decompositions in part (a), factor bands* β *correspond one-to-one to pairs* $\langle\theta, \theta'\rangle$ *of complementary factor congruences:*
(1) $\beta(a,b) = c$ *if, and only if,* $a\ \theta\ c$ *and* $c\ \theta'\ b$*;*
(2) $a\ \theta\ b$ *if, and only if,* $\beta(a,b) = b$*;* $a\ \theta'\ b$ *if, and only if,* $\beta(a,b) = a$*.*

PROOF. See [**McMcT87**, Theorem 4.33]. □

Perhaps the meaning of this proposition should be amplified: if a factor band β comes from a pair of complementary factor congruences θ and θ' by (b1) and new factor congruences $\widehat{\theta}$ and $\widehat{\theta}'$ are subsequently defined by (b2), then $\widehat{\theta} = \theta$ and $\widehat{\theta}' = \theta'$. And vice versa, starting with a factor band and going full circle via (b2) on through (b1) gives back the original band. Note that the trivial band β in which $\beta(a,b) = a$ for all a and b corresponds to the trivial congruences 0 and 1.

Observe that $\mathbf{Con}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ depends on whether we are talking about groups or rings. For groups it is isomorphic to the lattice $\boldsymbol{M}_3$, whereas for rings it is isomorphic to $\boldsymbol{M}_2$ (defined in Sect. 1). This seemingly innocuous discrepancy will be significant in later chapters.

2.6. EXERCISE. For any factor band β and its corresponding complementary factor congruences θ and θ' in an algebra $\boldsymbol{A}$, prove that

$$\beta(a,b) = \beta(c,d) \text{ if, and only if, } a\ \theta\ c \text{ and } b\ \theta'\ d \qquad (a,b,c,d \in A)$$

Interpret this as showing that the operation table of any factor band has a characteristic appearance. Namely, it breaks up into rectangular blocks, not necessarily contiguous, with each block containing one element of A and each element of A appearing in one block.

2.7. EXERCISE. Show that any rectangular band β on a set A is Abelian:

$$\beta(a,c) = \beta(a,d) \Rightarrow \beta(b,c) = \beta(b,d),$$
$$\beta(a,c) = \beta(b,c) \Rightarrow \beta(a,d) = \beta(b,d).$$

But find an Abelian binary function that is not a rectangular band. (For the general meaning of 'Abelian', see Definition V.3.10).

2.8. PROBLEM. When are factor bands term-operations or polynomials?

As there are outer projections, so there are inner ones, $\mu, \mu' \colon A \to A$ within an algebra $\boldsymbol{A}$, to be called 'factor sesquimorphisms'; but only in special cases are these endomorphisms, such as in rings, groups and lattices. We may create them out of a factor band β of $\boldsymbol{A}$ and an element o:

(2.6) $$\mu(a) = \beta(a, o) \qquad (a \in A),$$

(2.7) $$\mu'(a) = \beta(o, a) \qquad (a \in A).$$

Here, o is a fixed element of A. From the properties of a factor bands, one easily verifies the following properties of μ and μ'.

2.9. Proposition. *For a factor band β of an algebra $\boldsymbol{A}$ with the sesquimorphisms μ and μ' as defined by (2.6) and (2.7):*

(a) $\mu(\mu(a)) = \mu(a)$, *and* $\mu'(\mu'(a)) = \mu'(a)$;
(b) $\mu(\mu'(a)) = \mu'(\mu(a))$;
(c) *for any n-ary operation ω of $\boldsymbol{A}$,*
  (1) $\mu(\omega(\mu(a_1), \ldots, \mu(a_n)) = \mu(\omega(a_1, \ldots, a_n))$ *and*
  (2) $\mu'(\omega(\mu'(a_1), \ldots, \mu'(a_n))) = \mu'(\omega(a_1, \ldots, a_n))$;
(d) *if $\mu(a) = \mu(b)$ and $\mu'(a) = \mu'(b)$, then $a = b$;*
(e) *for all a and b there is an x such that $\mu(x) = \mu(a)$ and $\mu'(x) = \mu'(b)$.*

2.10. Definition. Any two functions, $\mu, \mu' \colon A \to A$, satisfying conditions (a)–(e) of this proposition will be called **complementary factor sesquimorphisms** of an algebra $\boldsymbol{A}$. And any function μ on the carrier of an algebra for which there exists another function μ' that satisfies these properties will be called a **factor sesquimorphism**. (See the previous section for an introduction to sesquimorphisms.)

Here are three examples of factor sesquimorphisms – three other examples of sesquimorphisms were given after Definition 1.5. Jónsson and Tarski algebras (Definition VII.3.17) exhibit factor sesquimorphisms that are idempotent endomorphisms (see [**JónTa47**] and [**McMcT87**, p. 283]). In a categorical setting, [**Hofm72**, p. 323] has factor sesquimorphisms, which are also endomorphisms. In the context of identities, factor sesquimorphisms are to be found in [**Knoe73, Knoe82**]; these are not necessarily endomorphisms.

Note that the conditions (a)–(e) in Proposition 2.9 do not presuppose a fixed element o. But they do yield such an element.

2.11. Proposition. *In an algebra $\boldsymbol{A}$ with complementary factor sesquimorphisms, μ and μ', there is an element o of A such that:*

$$\mu(\mu'(a)) = o = \mu(o) = \mu'(o) \qquad (a \in A).$$

Additionally,

$$\mu(\mu'(a)) = \mu(\mu'(b)) = \mu'(\mu(a)) = o \qquad (a, b \in A).$$

PROOF. To see that $\mu\mu' a = \mu\mu' b$ for any a and b in A, realize by property (e) that there exists an x such that $\mu x = \mu a$ and $\mu' x = \mu' b$, and hence

$$\mu\mu' a = \mu' \mu a = \mu' \mu x = \mu\mu' x = \mu\mu' b.$$

Call this fixed element o. Then $\mu o = \mu\mu\mu' a = \mu\mu' a = o$, and likewise $\mu' o = o$. □

Call this o the **origin** of this particular pair of sesquimorphisms. It is convenient to have an origin common to all the sesquimorphisms chosen to factor an algebra. To that end, in analogy with pointed spaces in topology, define a **pointed algebra** to be a pair $\langle \boldsymbol{A}, o\rangle$ where $\boldsymbol{A}$ is an algebra and o is any element of A, called the **origin**. An origin will also be needed to define factor ideals and elements. Its choice is arbitrary in general but for specific systems such as rings it is best to choose the nullity, and for groups the unity. Choosing the origin of $\mathbb{Z}_{12}$ to be 0 in our running example, we find from β that $\mu(m) = 4m$ and $\mu'(m) = 9m$, so that multiplication effects the action of a sesquimorphism. Factor sesquimorphisms correlate well with other factor objects; collate the next theorem with Theorem 2.5.

2.12. THEOREM. *Let $\langle \boldsymbol{A}, o\rangle$ be a pointed algebra. There is a one-to-one correspondence between factor bands β and pairs $\langle \mu, \mu'\rangle$ of complementary factor sesquimorphisms with origin o, and another one between these $\langle \mu, \mu'\rangle$ and pairs $\langle \theta, \theta'\rangle$ of complementary factor congruences of $\boldsymbol{A}$. This means that, with formulas (a)–(d) below, factor objects on the left may be defined uniquely in terms of those on the right. Going full circle returns us to the original factor objects. For example, going from a factor band to a pair of sequimorphisms by (b) and then returning by (a) gives back the same band:*

(a) $\beta(a,b) = c$ *if, and only if,* $\mu(c) = \mu(a)$ *and* $\mu'(c) = \mu'(b)$*;*
(b) $\mu(a) = \beta(a,o)$ *and* $\mu'(a) = \beta(o,a)$*;*
(c) $a\,\theta\,b$ *iff* $\mu(a) = \mu(b)$ *and* $a\,\theta'b$ *iff* $\mu'(a) = \mu'(b)$*;*
(d) $\mu(a)$ *is that unique* x *such that* $x\,\theta\,a$ *and* $x\,\theta'o$*, and*
$\mu'(a)$ *is that unique* y *such that* $y\,\theta'a$ *and* $y\,\theta\,o$*.*

Also for a factor band β with corresponding sesquimorphisms μ and μ':

(e) $\mu\beta(a,b) = \mu(a)$ *and* $\mu'\beta(a,b) = \mu'(b)$*;*
(f) $\beta(\mu(a), \mu'(a)) = a$*.*

PROOF. It is long but elementary, and so it is not given. Hint: it is easiest to prove (e) and (f) first. □

This theorem is in contrast to the last section where a transversal as a well as a congruence was needed in order to uniquely determine a sesquimorphism. Note that in a pointed algebra $\langle \boldsymbol{A}, o\rangle$ the trivial band β, in which $\beta(a,b) = a$ for all a and b, has the trivial sesquimorphisms, $\mu(a) = a$ and $\mu'(a) = o$.

2.13. Exercise. In the group $(\mathbb{Z}_2)^2$ there are three nontrivial decompositions as a product. Find the corresponding factor bands and factor sesquimorphisms, assuming $\langle 0, 0\rangle$ is the origin.

In JT-algebras [**JónTa47**] (see Definition VII.3.17), factor sesquimorphisms are endomorphisms. Generally, this is not the case; for example, complementation in Boolean algebras complicates the situation. But when the origin is a one-element subalgebra, this is so, and conversely, as shown next.

2.14. Proposition. *For the factor sesquimorphisms of a pointed algebra* $\boldsymbol{A}$ *with origin* o*, the following are equivalent:*

(a) *The singleton* $\{o\}$ *is a subalgebra of* $\boldsymbol{A}$*.*
(b) *All factor sesquimorphisms are endomorphisms of* $\boldsymbol{A}$*.*
(c) *At least one pair of complementary factor sesquimorphisms are endomorphisms of* $\boldsymbol{A}$*.*
(d) *The constant sesquimorphism,* $x \mapsto o$*, is an endomorphism of* $\boldsymbol{A}$*.*

Proof. (a) ⇒ (b). Use (c) and (d) of Proposition 2.9 to show that $\mu\omega\vec{a} = \omega\mu\vec{a}$. One also needs Proposition 2.11.

(b) ⇒ (d). Obvious

(d) ⇒ (c). The complement of the constant sesquimorphism is the identity map, and it is obviously an endomorphism.

(c) ⇒ (a). That $\mu\mu' a = o$ for any a implies, for a generic operation ω with the repeated argument o and endomorphisms μ and μ', that

$$\omega\dot{o} = \omega(\mu'\mu o, \mu'\mu o, \ldots, \mu'\mu o) = \mu'\mu\omega\dot{o} = o. \qquad \square$$

Rings and groups have an obvious origin in the one-element subalgebra 0; hence all factor sesquimorphisms are endomorphisms. Lattices are another good example of this proposition, where any element may be taken as the origin, proving that all factor sesquimorphisms are endomorphisms.

2.15. Exercise. Consider the 4-element Boolean algebra $\boldsymbol{B}$ on the set $\{0, a, b, 1\}$, and write out the operation table for a factor band β corresponding to the nontrivial factorization of $\boldsymbol{B}$. Be perverse by choosing a to be the origin o, and describe the corresponding factor sesquimorphisms. Compare with Exercise 1.9.

2.16. Exercise. Do an exercise similar to the preceding for the ring $\mathbb{Z}_6$ with 2 taken as the origin o; write out the operation tables for the relativized ring on $\{2, 5\}$.

2.17. Exercise. Let β be the factor band for a product $\boldsymbol{A} \times \boldsymbol{B}$, and choose an origin $\langle a_o, b_o\rangle$. For the corresponding sesquimorphisms μ and μ' show that $\mu(\langle a, b\rangle = \langle a, b_o\rangle$ and $\mu(\langle a, b\rangle = \langle a_o, b\rangle$.

2.18. Problem. Which of the axioms for complementary factor sesquimorphisms are independent of the rest?

As a projection, what does a factor sesquimorphism project onto? Answer: its image can be turned into an algebra isomorphic to a factor. To do this, as in Definition 1.8, confine each operation ω of an algebra $\boldsymbol{A}$ to the subset, $\mu(A) = \{\mu(a) \mid a \in A\}$, and evaluate it by restricting it:

$$\omega^{\mu(A)}(a_1, \ldots, a_n) = \mu(\omega^{\boldsymbol{A}}(a_1, \ldots, a_n)) \quad (a_1, \ldots, a_n \in \mu(A));$$

designate the resulting algebra $\mu(\boldsymbol{A})$. Then call the pair, $\mu(\boldsymbol{A})$ and $\mu'(\boldsymbol{A})$, **complementary factor ideals** of the algebra $\boldsymbol{A}$ whenever μ and μ' are complementary factor sesquimorphisms. That these are ideals in the sense of Sect. 1 will surface in the next proposition. An axiomatic definition, independent of sesquimorphisms, will have to wait till there is more structure on $\boldsymbol{A}$, as in Chap. VII. Also $x \in \mu(A)$ iff $\mu(x) = x$, since μ is idempotent.

In our running example where $\mathbb{Z}_{12} \cong \mathbb{Z}_3 \times \mathbb{Z}_4$ and $\beta(r, s) = 4r + 9s$, we have that $\mu(\mathbb{Z}_{12}) = \{0, 4, 8\}$ and $\mu'(\mathbb{Z}_{12}) = \{0, 3, 6, 9\}$. The ring operations of $+$ and $\times$, when confined to these factor ideals, do not change, thanks to the ring identities. For the constants, 0 stays the same, but 1 becomes 4 in $\mu(\mathbb{Z}_{12})$ and 9 in $\mu'(\mathbb{Z}_{12})$. See Fig. 2.

Classically, in the presence of a group operation, factor ideals are known as 'direct summands'. Thus, inner direct products might just as well be called sums, and often are, but we have chosen a term consistent with the previously defined outer direct products of algebras.

The next two results use the commutativity of congruences to connect complementary factor ideals, μA and $\mu' A$, with products and with the transversals defined in the last section, thereby showing that μA and $\mu' A$ are truly ideals in the sense of that section. The proof of the first result is straightforward.

2.19. Theorem. *Let μ and μ' be complementary factor sesquimorphisms corresponding to the factor band β of a pointed algebra $\boldsymbol{A}$ with an origin o. Then, μ and μ' are homomorphisms from $\boldsymbol{A}$ onto $\mu(\boldsymbol{A})$ and $\mu'(\boldsymbol{A})$, respectively; and $\boldsymbol{A} \cong \mu(\boldsymbol{A}) \times \mu'(\boldsymbol{A})$ by the isomorphism $a \mapsto \langle \mu(a), \mu'(a) \rangle$, with inverse β. Further, for the corresponding factor congruences θ and θ',*

$$\frac{\boldsymbol{A}}{\theta} \cong \mu(\boldsymbol{A}) \ \textit{ with } \ \mu(A) = \frac{o}{\theta'}; \ \textit{ and } \ \frac{\boldsymbol{A}}{\theta'} \cong \mu'(\boldsymbol{A}) \ \textit{ with } \ \mu'(A) = \frac{o}{\theta}.$$

2.20. Proposition. *Let $\boldsymbol{A}$ be a pointed algebra with an origin o. Two congruences, θ and η, are complementary factor congruences if, and only if,*

(a) $\theta \circ \eta = \eta \circ \theta$,
(b) $\theta \cap \eta = 0_{\mathbf{Con}\,\boldsymbol{A}}$,
(c) *o/θ is a transversal of η,*
(d) *o/η is a transversal of θ.*

Proof. $\Rightarrow$. Clear.

$\Leftarrow$. We need only to prove that $\theta \circ \eta = 1_{\mathbf{Con}\,\boldsymbol{A}}$. Let μ be the sesquimorphism coming from the congruence θ and the transversal o/η, and let λ come from η and o/θ, as in Sect. II.1. For arbitrary a and b in A,

$$a \;\theta\; \mu(a) \;\eta\; o \;\theta\; \lambda(b) \;\eta\; b;$$

thus, $a\ (\theta \vee \eta)\ b$. By commutativity, $\theta \vee \eta = \theta \circ \eta$; hence $\theta \circ \eta = 1_{\mathbf{Con}\,\boldsymbol{A}}$. □

In general, condition (a) of Proposition 2.20 is necessary; the semilattice $\boldsymbol{SL}_3$ on three elements that is not a chain demonstrates this. Of course, (a) is automatically fulfilled in varieties with commuting congruences.

2.21. Exercise. Groups and vector spaces, which have natural origins, are a good place to observe factor bands and other factor objects. In their common generalization, groups with operators (see Sect. 1), factor sesquimorphisms become endomorphisms of a special kind and factor ideals become normal subgroups.

(a) Interpret our language and propositions into that of groups with operators. For example, two normal subgroups $\boldsymbol{M}$ and N of a group $\langle G; \times, 1, \ldots, \omega, \ldots\rangle$ with operators form an inner direct product if, and only if,
   (1) $\boldsymbol{M} \times N = \boldsymbol{G}$,
   (2) $\boldsymbol{M} \cap N = \{1\}$.
(b) Prove that two functions $\mu, \mu' \colon V \to V$ on a vector space $\boldsymbol{V}$ are complementary factor sesquimorphisms if, and only if, μ and μ' are idempotent linear transformations such that for all v in V:
   (1) $\mu(v) + \mu'(v) = v$, and
   (2) $\mu(\mu'(v)) = \mu'(\mu(v))$.
(c) Show that any factor band β of a vector space $\boldsymbol{V}$ is of the form:

$$\beta(v, w) = Mv + Nw \qquad (v, w \in V)$$

for linear transformations M and N such that $M^2 = M$, $N^2 = N$, $M + N = 1$, and $MN = 0 = NM$. And conversely, show that any binary function of this form on a vector space is a factor band.

Although $\mu(\boldsymbol{A})$ and $\mu'(\boldsymbol{A})$ appear to form what might be called an 'inner product', there is no longer, in general, a one-to-one correspondence between pairs of complementary factor ideals and the previous factor objects, even up to isomorphism or a choice of the origin. To illustrate, consider the set $\{0, 1, \ldots, 5\}$ as an algebra $\boldsymbol{A}$ with no operations but with an origin 0. It may be factored in two different ways yielding the same factor ideals, $\{0, 1\}$ and $\{0, 2, 4\}$.

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 0 & 2 & 4 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|}\hline 1 & 5 & 3 \\ \hline 0 & 2 & 4 \\ \hline \end{array}$$

In view of this unfortunate insufficiency of factor ideals to capture inner products uniquely, it seems reasonable to define this concept with sesquimorphisms.

2.22. Definition. An **inner direct product** in an arbitrary algebra is a pair of complementary factor sesquimorphisms. Their images become a pair of algebras, although not subalgebras, whose product is isomorphic to the original algebra.

This is as close as we may get to the classical notion of inner product in groups and rings as a pair of subalgebras satisfying certain conditions. Of interest here are JT-algebras [**JónTa47**], where pairs of complementary factor ideals, now subalgebras, do correspond one-to-one to inner products. By Theorem 2.12, any pair of complementary factor congruences θ and θ' in a pointed algebra $\langle \boldsymbol{A}, o\rangle$ gives an inner direct product in which their ideals o/θ and o/θ' are the images of the corresponding sesquimorphisms. In unital shells and half-shells, to be studied intensively later, their sets of factor congruences are Boolean algebras. More generally, we will show in Sect. VI.2 that, whenever the factor congruences form a Boolean algebra, their factor ideals can give back the congruences from which they came. Inner products relate to outer products as follows.

2.23. Proposition. *Let $\langle \boldsymbol{P}, \pi, \pi'\rangle$ be the outer product of the algebras $\boldsymbol{A}$ and $\boldsymbol{A}'$, that is, $\boldsymbol{P} = \boldsymbol{A} \times \boldsymbol{A}'$. Assume that $\boldsymbol{P}$ is pointed with origin o. Further, assume that β is the induced factor band of $\boldsymbol{P}$. Write μ and μ' for the complementary factor sesquimorphisms created by β via the origin. There are isomorphisms, $\boldsymbol{A} \cong \mu(\boldsymbol{P})$ and $\boldsymbol{A}' \cong \mu(\boldsymbol{P}')$, such that the outer product is related to the inner product by this commutative diagram.*

$$\begin{tikzcd}
\boldsymbol{A} \arrow[dd, leftrightarrow, "\text{isom.}"'] & & \boldsymbol{A}' \arrow[dd, leftrightarrow, "\text{isom.}"] \\
& \boldsymbol{P} \arrow[ul, "\pi"'] \arrow[ur, "\pi'"] \arrow[dl, "\mu"] \arrow[dr, "\mu'"'] & \\
\mu(\boldsymbol{P}) & & \mu'(\boldsymbol{P})
\end{tikzcd}$$

Factor elements are the last kind of factor object to be investigated; these are the counterpart in universal algebra to central idempotents in ring theory. Like factor ideals they apparently need to be defined in terms of factor bands or sesquimorphisms, at least in general.

In order to identify factor elements another fixed element is needed. Call a triple $\langle \boldsymbol{A}, o, t\rangle$ a **doubly pointed algebra** when o and t are any elements of an algebra $\boldsymbol{A}$ – these need not be constant operations; name o the **origin**, as before, and t the **terminus**. Although there are no restrictions on the choice of o and t for now, when we do come to unital rings $\langle R; +, \times, 0, 1\rangle$ and their generalizations, it will be most advantages to assume that o is 0 and t is 1.

There are at least three equivalent ways to define factor elements: through a factor band, through its factor sesquimorphisms, or through the corresponding factor congruences.

2.24. Definition. Call two elements e and e' **complementary factor elements** of a doubly pointed algebra $\langle \boldsymbol{A}, o, t\rangle$ if there is a factor band β of $\boldsymbol{A}$ for which

$$\beta(t,o) = e, \text{ and } \beta(o,t) = e'.$$

With the corresponding sesquimorphisms, $\mu(t) = e$ and $\mu'(t) = e'$. A **factor element** is one of such a complementary pair.[3]

2.25. Proposition. *Let e and e' be complementary factor elements that come from a factor band β in a doubly pointed algebra $\langle \boldsymbol{A}, o, t\rangle$. The factor congruences corresponding to β uniquely define e and e' by the relationships:*

$$o\theta' e \quad \text{and} \quad e\theta t,$$
$$o\theta e' \quad \text{and} \quad e'\theta' t.$$

By choosing the terminus to be 1 and the origin 0 in the running example of $\mathbb{Z}_{12}$ with $\beta(a,b) = 4a+9b$, this definition and proposition are illustrated by computing in several ways its complementary factor elements: $e = 4$ and $e' = 9$. In unital rings, factor elements are central idempotents, which generate the corresponding factor ideals as principal ideals. Significantly for the future unfolding of the theory, $\mu(a) = e \times a$ and $\mu'(a) = e' \times a$.

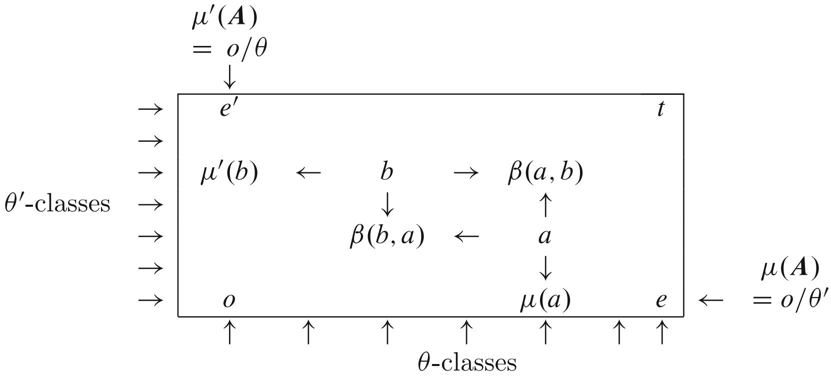


Figure 3. An algebra decomposed as a product

Figure 3 is an abstraction to general algebras of the earlier Fig. 2 for a product of two unital rings. The box encloses the elements of a doubly pointed algebra $\boldsymbol{A}$ with origin o and terminus t, sorted into the congruences classes of two complementary congruences, θ and θ' with the θ-classes being the columns and the θ'-classes the rows. In this picture analogous to the

[3] Swamy and Murti [**SwaMu81a**] discuss factor elements in semigroups, where they call them 'central' elements. Central elements are more generally defined as sequences of elements in [**VagSá04**] and [**SánVa09**]. There the origin and terminus become sequences of unary operations.

Cartesian plane, the algebra $\mu\boldsymbol{A}$ ($= o/\theta'$ qua ideal) is the X-axis and $\mu'\boldsymbol{A}$ ($= o/\theta$) the Y-axis. The sesquimorphisms μ, μ' applied to an element a give its 'co-ordinates' and β recovers a by the formula $\beta\big(\mu(a), \mu'(a)\big) = a$. Two arbitrary elements a and b go to $\beta(a, b)$ and $\beta(b, a)$ as indicated. One should take this figure with a few grains of salt, as there may be some equalities and collapsing among o, e, t and e'.

Although we have defined factor elements, they may be useless in general. An example illustrates how there may be no choice for the origin and the terminus so that each pair of complementary factor elements determines the factor band from which they came. Consider the lattice of all finite subsets of an infinite set, with union and intersection being the lattice operations. Then one can show that, for any choice of the origin and terminus, there are an infinite number of factor bands yielding the same pair of complementary factor elements.

One virtue of unital rings is that each of the five kinds of factor objects uniquely characterize products internally. Especially easy to use are factor elements. This will also be true of two-sided unital shells. But, in broader classes of algebras more complicated set-theoretical structures are needed, such as congruence relations or bands. The less structure needed the better: elements are better than subsets, such as ideals, and subsets are better than relations, etc., as shown in Fig. 4.

	Factor object	as a Structure	
Most	band	binary operation	$\downarrow$
widely	congruence	binary relation	$\downarrow$
applicable	sesquimorphism	unary function	Most
$\uparrow$	ideal	set	desirable
$\uparrow$	element	element	to use

FIGURE 4. Hierarchy of factor objects as structures

2.26. EXERCISE. Can an origin and a terminus be chosen in each vector space so as to create pairs of complementary factor elements via factor bands such that these pairs uniquely determine the factor bands whence they came? Hint: use Exercise 2.21. This was originally a problem, subsequently settled by Diego Vaggione [**Vagg10**].

2.27. PROBLEM. What properties should an algebra or variety possess in order to have complementary factor ideals or elements from which all products may be reconstructed? The shells of Chap. VII are a partial answer. Pedro Sánchez Terraf and Diego Vaggione [**SánVa09**] have a more extensive answer; see Theorem VI.3.11.

Although factor elements play a pivotal role in unital rings, and in shells in subsequent chapters, there is no more to say about them at this point.

Products of more than two factors are possible. Infinite products lead to the important subjects of refinements and subdirect products, discussed only briefly here. As in the case with two factors there are both outer and inner characterizations. Taking to heart remarks made a page ago, we might define inner direct products in terms of sesquimorphisms. However, in anticipation of conventional definitions of refinement to come in Sect. VI.2, we present them equivalently in terms of congruences, leaving their development with sesquimorphisms and infinitary bands to the reader.

2.28. DEFINITION. The **outer direct product** or simply 'direct product' $\prod_{i\in I} \boldsymbol{B}_i$, of the algebras $\boldsymbol{B}_i$ indexed by i in I is an algebra $\boldsymbol{P}$ with carrier $\prod_{i\in I} B_i$ and each n-ary operation ω defined coordinate-wise, as previously done with two factors:

$$\omega^{\boldsymbol{P}}(\langle a_1^i \mid i \in I\rangle, \ldots, \langle a_n^i \mid i \in I\rangle) = \langle \omega^{\boldsymbol{B}_i}(a_1^i, \ldots, a_n^i) \mid i \in I\rangle \qquad (a_j^i \in B_i).$$

If the index set I is empty, then the product has just one element. For two factors (or just a few) we write $\boldsymbol{A} \times \boldsymbol{B}$, etc. If α is a congruence of $\boldsymbol{A}$ and β is a congruence of $\boldsymbol{B}$, their product $\alpha \times \beta$ in $\boldsymbol{A} \times \boldsymbol{B}$ is defined

$$\langle a_1, b_1\rangle\, (\alpha \times \beta)\, \langle a_2, b_2\rangle \text{ if } a_1 \,\alpha\, a_2 \text{ and } b_1 \,\beta\, b_2 \qquad (a_i \in A,\ b_i \in B).$$

A nontrivial algebra is called **directly indecomposable** if it is not isomorphic to a product of two nontrivial algebras.

In order to illuminate the inner factoring of congruences, we develop some definitions and propositions.

2.29. DEFINITION. For a family of homomorphisms, $\varphi_i\colon \boldsymbol{A} \to \boldsymbol{B}_i$ with i in I, with common domain, there is the **canonical homomorphism** φ to the outer direct product:

$$\varphi : \boldsymbol{A} \underset{\text{can.}}{\to} \prod_{i\in I} \boldsymbol{B}_i \colon a \mapsto \langle \varphi_i(a) \mid i \in I\rangle.$$

More generally, assume that $\eta \in \operatorname{Con} \boldsymbol{A}$ and $\eta \subseteq \bigcap \Theta$ for a collection of congruences in $\boldsymbol{A}$ with product, $\boldsymbol{P} = \prod_{\theta\in\Theta} \boldsymbol{A}/\theta$. Its **canonical homomorphism**, $\psi : \boldsymbol{A}/\eta \underset{\text{can.}}{\to} \boldsymbol{P}$, maps $a/\eta \mapsto \langle a/\theta \mid \theta \in \Theta\rangle$.

2.30. PROPOSITION. *Assume $\theta, \eta \in \operatorname{Con} \boldsymbol{A}$. Then, θ and η are complementary factor congruences if, and only if, the canonical homomorphism,*

$$\varphi : \boldsymbol{A} \underset{\text{can.}}{\to} \frac{\boldsymbol{A}}{\theta} \times \frac{\boldsymbol{A}}{\eta},$$

is an isomorphism.

2.31. DEFINITION. An **inner direct product** of a congruence η of an algebra $\boldsymbol{A}$ is a nonempty collection Θ of congruences of $\boldsymbol{A}$ such that $\eta = \bigcap \Theta$ and the canonical homomorphism of quotients is an isomorphism:

$$\frac{\boldsymbol{A}}{\eta} \underset{\text{can.}}{\cong} \prod_{\theta\in\Theta} \frac{\boldsymbol{A}}{\theta}.$$

Abbreviate this as $\eta = \prod \Theta$. With only two factors, write $\eta = \theta_1 \sqcap \theta_2$. An **inner direct product** of congruences of an algebra $\boldsymbol{A}$ is an inner direct product of $0_{\mathbf{Con}\,\boldsymbol{A}}$.

For the moment we neglect to have an origin, which would return from complementary θ and θ' the sesquimorphisms of a true inner product in the sense of Definition 2.22. The cancellation isomorphism theorem of Sect. 1 allows us to pass easily between inner products of algebras and their congruences. Also, any congruence θ of an inner direct product of an algebra is a factor congruence, that is, θ and $\bigcap(\Theta{\sim}\{\theta\})$ are complementary factor congruences.

2.32. PROPOSITION. *A congruence η of an algebra $\boldsymbol{A}$ is an inner direct product Θ if, and only if, for any family $\langle a_\theta \mid \theta \in \Theta\rangle$ of elements of A, the family of congruences*

$$x \equiv a_\theta \ \ (\theta) \qquad (\theta \in \Theta)$$

has a unique solution x modulo η; that is, if y is another solution, then $x \equiv y \ \ (\eta)$.

The next proposition will prove to be useful in Sect. VI.2. Atomic and complete Boolean algebras are defined near the end of Sect. III.4.

2.33. PROPOSITION. *Let Θ be an inner direct product of congruences of an algebra $\boldsymbol{A}$, and consider the collection of all intersections of them:*

$$\overline{\Theta} = \left\{\bigcap H \mid H \subseteq \Theta\right\}.$$

(a) *Then, $\overline{\Theta}$ is a complete and atomic Boolean lattice of commuting factor congruences of $\boldsymbol{A}$.*

(b) *If $1_{\mathbf{Con}\,\boldsymbol{A}} \notin \Theta$, then $\overline{\Theta}$ is anti-isomorphic to the set of all subsets of Θ; this is given by the correspondence:*

$$\psi \colon \mathcal{P}\Theta \to \overline{\Theta} : H \mapsto \bigcap H,$$

where the lattice operations are transformed:

$$\psi(Z \cup H) = \psi(Z) \cap \psi(H) \qquad (Z, H \subseteq \Theta), \tag{2.8}$$

$$\psi(Z \cap H) = \psi(Z) \vee \psi(H) \qquad (Z, H \subseteq \Theta), \tag{2.9}$$

$$\psi(\varnothing) = 1_{\mathbf{Con}\,\boldsymbol{A}}, \tag{2.10}$$

$$\psi(\Theta) = 0_{\mathbf{Con}\,\boldsymbol{A}}. \tag{2.11}$$

PROOF. (a) That $\overline{\Theta}$ is a complete, atomic Boolean algebra follows from (b) unless $1 \in \Theta$. But this makes no difference, since 1 is already in $\overline{\Theta}$ as $\cap\varnothing$.

View this as a problem in the corresponding outer direct products. Any partition of Θ, say H and $\Theta{\sim}H$, produces a product:

$$\prod_{\theta\in\Theta} \frac{\boldsymbol{A}}{\theta} \cong \frac{\boldsymbol{A}}{\bigcap H} \times \frac{\boldsymbol{A}}{\bigcap(\Theta{\sim}H)}.$$

Hence, $\bigcap H$ and $\bigcap(\Theta{\sim}H)$ are complementary factor congruences.

To prove that the congruences of $\overline{\Theta}$ commute, assume for some a and b in A that $a\,(\bigcap Z \circ \bigcap H)\,b$, where $Z, H \subseteq \Theta$; one needs to show that $a\,(\bigcap H \circ \bigcap Z)\,b$. Now there exists an x in A such that $a \bigcap Z\; x$ and $x \bigcap H\; b$. So $a\,\zeta\,x$ when $\zeta \in Z$, and $x\,\eta\,b$ when $\eta \in H$. Because Θ is an inner product, there is a solution y to the system of congruences:

$$y \equiv \begin{cases} b\ (\zeta) & (\zeta \in Z), \\ a\ (\eta) & (\eta \in H{\sim}Z). \end{cases}$$

For any η in $H \cap Z$, then $y\,\eta\,b\,\eta\,x\,\eta\,a$. Consequently, $a \bigcap H\; y \bigcap Z\; b$.

(b) To show that ψ is one-to-one, assume that there are two unequal subsets H and Z of Θ; the object is to prove that $\bigcap H \neq \bigcap N$. Without loss of generality, assume that there is an η_0 in H but not in Z. Since $\eta_0 \neq 1_{\mathbf{Con}\,\boldsymbol{A}}$, there are a and b in A not related by η_0. By solvability in an inner product, there is an x in A such that

$$x \equiv \begin{cases} a\ (\zeta) & (\zeta \in Z), \\ b\ (\eta_0). \end{cases}$$

So $x \equiv a\ (\bigcap Z)$ but $x \not\equiv a\ (\bigcap H)$.

The passage of the lattice operations through ψ is clear except for the second one transforming $\cap$ into $\vee$. An argument similar to but simpler than the one for commutation will do the trick. □

2.34. Exercise. Phrase and prove a converse to this proposition.

2.35. Problem. Go beyond factor bands on just two arguments and fashion a theory of products on arbitrary index sets that includes as many as possible of the remaining factor objects.

Throughout this section and the previous one, one sees a general philosophy at work: for any outer concept find a corresponding inner concept. This will reappear in the notion of 'refinement' in Sect. VI.3, where the outer and inner notions diverge.

Decomposing an algebra into a product of other nontrivial algebras is not always possible, even though the algebra may break up in some other way. It might be only the subalgebra of a direct product, as developed next.

2.36. Definition. *An algebra $\boldsymbol{A}$ is an* **outer subdirect product** *of a family $\langle A_i \mid i \in I\rangle$ of algebras all of the same type if $\boldsymbol{A}$ is a subalgebra of the direct product of the family and each projection π_i of $\boldsymbol{A}$ to each factor $\boldsymbol{A}_i$ is surjective. In notation,*

$$\boldsymbol{A} \underset{s.d.}{\subseteq} \prod_{i \in I} \boldsymbol{A}_i .$$

Or more briefly, call this a 'subdirect product' or even just 'subproduct'.

The group $\mathbb{Z}$ is a case in point; it is isomorphic to a subproduct of all groups of prime order:

$$\mathbb{Z} \underset{s.d.}{\subseteq} \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \cdots .$$

In this example, one maps $\mathbb{Z}$ to $\mathbb{Z}_p$ by projecting each integer z to it: $\pi_p(z) = z \bmod p$. Since each factor is fully utilized, these projections are surjective and we have a subproduct. They have an internal characterization.

2.37. Proposition. *An algebra $\boldsymbol{A}$ is an outer subdirect product of a family $\langle \boldsymbol{A}_i \mid i \in I \rangle$ of algebras if, and only if, there are congruences θ_i of $\boldsymbol{A}$ such that*

$$\boldsymbol{A}_i \cong \boldsymbol{A}/\theta_i \quad (i \in I),$$

$$\bigcap\{\theta_i : i \in I\} = 0_{\mathbf{Con}\,\boldsymbol{A}}.$$

This proposition suggests a definition.

2.38. Definition. An **inner subdirect product** in an algebra $\boldsymbol{A}$ is a collection Θ of congruences of $\boldsymbol{A}$ such that their intersection is the identity relation:

$$\bigcap \Theta = 0_{\mathbf{Con}\,\boldsymbol{A}}.$$

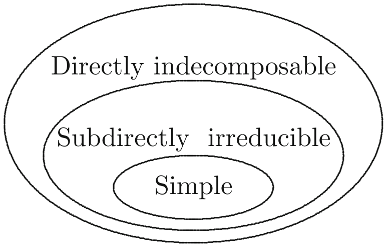
2.39. Definition. A nontrivial algebra $\boldsymbol{A}$ is called **subdirectly irreducible** if, for any inner subdirect product in $\boldsymbol{A}$, at least one of the congruences θ_i is already the identity relation. Likewise, a congruence θ of an algebra $\boldsymbol{A}$ is **subdirectly irreducible** if $\boldsymbol{A}/\theta$ is.

This concept is needed to state a well-known and much used result due to Birkhoff [**Birk44**]; see also [**BurSa81**, Sect. II.8].

2.40. Theorem. *Every nontrivial algebra is a subdirect product of subdirectly irreducible algebras.*

Thus subdirect decompositions have the advantage that they always exist. However, the quotient algebras may be too small for a particular purpose; also it may be impossible to conveniently specify the subalgebra of the product. Direct products on the other hand may be too coarse. For these reasons we seek an intermediate path where we take as initial ingredients a Boolean algebra of congruences, and then take suprema of maximal ideals of these; these suprema are again congruences, but usually not factor congruences. The quotient algebras coming from these suprema have a trivial intersection, thereby yielding a subdirect product, and even better, these stalks, as they will be called, will bind together topologically to form a sheaf in Theorem V.2.1.

A Venn diagram of algebras relates different kinds of factors.



These inclusions are all proper. The three-element semilattice $\boldsymbol{SL}_3$ that is not a chain is directly indecomposable as a product but it is subdirectly reducible. And the group $\mathbb{Z}_4$ is subdirectly irreducible but not simple.

2.41. PROBLEM. In a subdirect product of two algebras, a factor band may be approached as for a product, but it is now a partial function, not defined for all pairs of arguments. Carefully define, if possible, this new concept of a 'subdirect band' so that it corresponds one-to-one to subdirect products of any number of algebras. Likewise, can these subdirect definitions and propositions be rephrased in terms of sesquimorphisms?

There is one last construction needed for making sheaves in Chap. IV.

2.42. DEFINITION. For a family $\{\boldsymbol{A}_x \mid x \in X\}$ of algebras of the same type, consider the disjoint union of their carriers: $\mathcal{A} = \biguplus_{x\in X} A_x$. It is not necessary for the components to be disjoint to start with. To make them disjoint, employ the set construction:

$$\biguplus_{x\in X} A_x = \{\langle x, a\rangle \mid x \in X \text{ and } a \in A_x\}.$$

This **disjoint union** has the natural structure of a partial algebra of the same type, $\boldsymbol{\mathcal{A}} = \biguplus_{x\in X} \boldsymbol{A}_x$: for a generic operation ω with n arguments, when $a_1, \ldots, a_n \in A_x$ for some x in X,

$$\omega^{\boldsymbol{\mathcal{A}}}(\langle x, a_1\rangle, \ldots, \langle x, a_n\rangle) = \langle x, \omega^{\boldsymbol{A}_x}(a_1, \ldots, a_n)\rangle;$$

otherwise, it is undefined.

Nullary operations, that is, constants in the type, pose a problem. By fiat, we postulate that they are undefined in the disjoint union. We also allow the index set X to be empty, whence the disjoint union is also empty; so partial algebras in this case may have an empty carrier. This trivial allowance will be useful in capturing the one-element algebra by a sheaf.

In Chap. VII much more will be said about all the internal factor objects in the context of shells with binary operations and constants. Inner products, as sums of ideals, will capture external products uniquely. Each kind of factor object will be defined independently of the others, and directly in terms of the operations of the shell.

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