Preface

My involvement in the line of research leading to this book began in 1963 when I was a graduate student working under the direction of Alfred Foster, and was first learning about representing algebras as subdirect products. In particular, for a starter I learned that Stone's representation theorem was valid not just for Boolean algebras but for any class of algebras satisfying the identities of a primal algebra. Foster perceived in these algebras a Boolean part whose representation theory could be levered into representing many other kinds of algebras.

The broad motivation was to break up a complicated algebra into simpler pieces; if the pieces could be understood, then hopefully so could the whole algebra. The obvious decomposition to try first is a direct product. The advantage of direct products is the simplicity of their construction. The overwhelming disadvantage is that most algebras are indecomposable in this sense, and even when decomposable there may be no ultimate refinement. Subdirect products overcome both of these liabilities, as first demonstrated by Garrett Birkhoff.

The main drawback to subdirect products is that, while factors may be commonplace and well understood, the transfer of an argument from the components to the whole algebra may fail because one may not know in sufficient detail how the components fit together to form the original algebra. Thus one grafts topological spaces onto subdirect products to form significantly superior sheaves. Elements of the subdirect product become continuous functions, and are easier to recognize. Boolean spaces are often used since they arise naturally in representing Boolean algebras and have been the key to many other representation theorems. However, the topological spaces of algebra are intuitively quite different from the more traditional topological spaces such as manifolds with a local Euclidean topology. They may be totally disconnected or not even Hausdorff.

The question we address then is, how far can one go in representing arbitrary algebras by sheaves over general topologies, and in particular Boolean spaces? The overall structure of a given algebra should come from a systematic synthesis of the components, that is, the stalks of the sheaf. Many questions about any algebra in such a class should be answerable by analyzing locally what is happening in the components, rather than working globally with formulas over the whole algebra.

My first exposure to sheaves over Boolean spaces was in a seminar run by Joseph Kist in the spring of 1972, in which he presented the seminal paper of Stephen Comer. Here I learned of the rich and productive world of ring spaces as expounded by Richard Pierce in his memoir.

It was in this seminar that I discovered factor elements, which generalize central idempotents in rings, and how they correspond to factor congruences. Later, factor bands, ideals, and sesquimorphisms were added. The goal was to extend the classical representation of regular commutative rings as subdirect products of fields.

Although general tools are developed, applicable to all algebras, the best efforts come from settling on those that I dub 'shells', which assert the existence of a zero and a one for a multiplicative operation and perhaps an addition that otherwise need not satisfy any of the usual identities such as commutativity and associativity. In this context, one can generalize well beyond ring theory a number of classical results on biregularity, strong regularity, and lack of nilpotents.

This monograph adapts the intuitive idea of a metric space to universal algebra, leading to the useful device of a complex. Then a sheaf is constructed directly from a complex.

The core of this book does not look at all congruences of an algebra, but at only some of them comprising a Boolean subsemilattice of congruences, and more typically, at others splitting the algebra into a product of complementary factors. Thus there are no restrictions on the whole lattice of congruences, but only on parts of it. This is one of the themes of this monograph.

Over the course of time, terms and notations tend to grow like Topsy. In synthesizing disparate fields and even extending them, inconsistencies across them pose a dilemma for an author. Should he completely streamline the terminology, thereby shutting out the casual reader who is merely browsing but already knows something of the traditional notation? Or, should he leave every term as it has originally arisen, thereby making it difficult for the serious reader to correlate similar ideas? I have taken a middle course, respecting most terms and notations already in the literature, but occasionally changing some to better reflect the overall picture. For example, congruences that *permute* elsewhere *commute* here since other internal factor objects, such as idempotent endomorphisms, always commute when creating a product. But I left unchanged directly *indecomposable* and subdirectly *irreducible*, although one ought to have a common root word for the many kinds of algebraic atoms. The definitions of the rather general algebras, *shells* and *half-shells*, have broadened over time as weaker and weaker conditions were observed to create sheaves that would accomplish most of the same ends. *Nullity* is used for an element annihilating a binary operation as a zero does in ring theory. And *unity* is the term used where others might use 'unit element' or 'identity;' it even means 'object' in categories. Likewise, the adjective *unital* adds a unity to a ring or shell.

Many exercises and problems have been included. The distinction between them is as follows. On the one hand, the exercises come from notes I wrote to myself while trying to understand the relationships between new concepts. There was no attempt to create other exercises that might fill out the book; thus the density of exercises varies from section to section. The reader may enjoy more healthy exercise by filling in wherever a proof trails off with a phrase such as 'straightforward to prove', 'trivial', or 'left to the reader'. This is especially so in the categorical sections establishing adjointness and equivalence.

On the other hand, the problems are open questions that I have not resolved because I did not take the time. Thus, such problems may range from the trivial to the significant, perhaps to promising research to pursue. I have not attempted to distinguish these possibilities.

As for prerequisites, a reader should have a nodding acquaintance with universal algebra, logic, categories, topology, and Boolean algebra. By recalling useful facts about these topics, prerequisites have been kept to a minimum. All concepts beyond these are defined. However, as the goal is new theorems, and the ideas already in the literature are lightly illustrated here, the prospective reader will be well motivated if he is familiar with some of the classical results that are being generalized.

I am thankful to the participants who asked penetrating questions in algebra seminars at New Mexico State University, Tennessee Technological University, the University of Tennessee and Vanderbilt University; some of these led to additional insights and examples. Fruitful conversations with Joseph Kist have cleared up a number of murky points. Mai Gehrke pointed out non sequiturs, and shortened several long-winded proofs. Isadore Fleischer corrected several of the early chapters. Paul Cohn offered suggestions on the history of the subject, and Ross Willard pointed out a significant

extension of the concept of a shell. Diego Vaggione quickly dispatched several of the original open problems. All of these, including three anonymous reviewers, deserve warm handshakes for their many comments and thought-provoking suggestions. As for remaining faux pas that I should have caught, may the sympathetic reader forgive me for any difficulties they might cause.

Albuquerque, New Mexico Arthur Knoebel

http://www.springer.com/978-0-8176-4218-1

Sheaves of Algebras over Boolean Spaces
Knoebel, A.
2012, XII, 331 p. 63 illus., Hardcover
ISBN: 978-0-8176-4218-1
A product of Birkhäuser Basel