

Chapter 2

Riemann Surfaces and the $L^2 \bar{\partial}$ -Method for Scalar-Valued Forms

In this chapter, we consider some elementary properties of Riemann surfaces, as well as a fundamental technique called the $L^2 \bar{\partial}$ -method, Radó's theorem on second countability of Riemann surfaces, and analogues of the Mittag-Leffler theorem and the Runge approximation theorem for open Riemann surfaces. Viewing holomorphic functions as solutions of the homogeneous Cauchy–Riemann equation $\partial f / \partial \bar{z} = 0$ in \mathbb{C} allows one to very efficiently obtain their basic properties (see Chap. 1). The intrinsic form of the homogeneous Cauchy–Riemann equation on a Riemann surface is given by $\bar{\partial} f = 0$ (see Sect. 2.5). In order to obtain holomorphic functions (and holomorphic 1-forms) on a Riemann surface (even on an open subset of \mathbb{C}), it is useful to consider the *inhomogeneous* Cauchy–Riemann equation $\bar{\partial} \alpha = \beta$. One well-known approach to solving this differential equation (as well as differential equations in many other contexts) is to consider weak solutions in L^2 . This is the approach taken in this book. In order to do so, we must develop suitable versions of an L^2 space of differential forms (see Sect. 2.6) and an (intrinsic) distributional $\bar{\partial}$ operator (see Sect. 2.7). The relatively simple approaches to the above appearing in this book are, in part, special to Riemann surfaces; but they do contain important elements of the higher-dimensional versions (see, for example, [Hö] or [De3] for the higher-dimensional versions).

In this chapter, we consider only scalar-valued differential forms. In Chap. 3, we will consider the analogue for forms with values in a holomorphic line bundle. The solution in line bundles is more efficient in some ways, and it also generalizes more readily to higher-dimensional complex manifolds.

In Sects. 2.1–2.9, we consider the definition and basic properties of a Riemann surface, the L^2 spaces of differential forms, and the fundamental theorem regarding the solution of the (inhomogeneous) Cauchy–Riemann equation for scalar-valued differential forms. In the remaining sections, we apply the above to obtain some important facts, namely, the existence of meromorphic 1-forms and functions, Radó's theorem on second countability of Riemann surfaces, the Mittag-Leffler theorem, and the Runge approximation theorem (see [R] for a historical perspective).

2.1 Definitions and Examples

In this section, we consider the definition of a Riemann surface and some examples.

Definition 2.1.1 Let X be a Hausdorff space.

- (a) A homeomorphism $\Phi : U \rightarrow U'$ of an open set $U \subset X$ onto an open set $U' \subset \mathbb{C}$ is called a *local complex chart of dimension 1* (or a *1-dimensional local complex chart*) in X . We also denote this local complex chart by (U, Φ, U') .
- (b) Two 1-dimensional local complex charts (U_1, Φ_1, U'_1) and (U_2, Φ_2, U'_2) in X are *holomorphically compatible* if the *coordinate transformations*

$$\Phi_1 \circ \Phi_2^{-1} : \Phi_2(U_1 \cap U_2) \rightarrow \Phi_1(U_1 \cap U_2)$$

and

$$(\Phi_1 \circ \Phi_2^{-1})^{-1} = \Phi_2 \circ \Phi_1^{-1} : \Phi_1(U_1 \cap U_2) \rightarrow \Phi_2(U_1 \cap U_2)$$

are holomorphic; that is, each is a biholomorphism (see Fig. 2.1).

- (c) A family of holomorphically compatible 1-dimensional local complex charts $\mathcal{A} = \{(U_i, \Phi_i, U'_i)\}_{i \in I}$ that covers X (i.e., that satisfies $X = \bigcup_{i \in I} U_i$) is called a *holomorphic atlas of dimension 1* on X .
- (d) Two 1-dimensional holomorphic atlases \mathcal{A}_1 and \mathcal{A}_2 on X are *holomorphically equivalent* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is a holomorphic atlas (this is an equivalence relation).
- (e) An equivalence class \mathcal{R} of holomorphic atlases on X is called a *holomorphic structure* (or a *complex analytic structure*) of dimension 1 on X , and the pair (X, \mathcal{R}) (which is usually denoted simply by X) is called a *complex manifold of dimension 1* (or a *complex 1-manifold* or a *complex analytic manifold of dimension 1*). If, in addition, X is connected, then the pair (X, \mathcal{R}) (which again is usually denoted simply by X) is called a *Riemann surface*.
- (f) A 1-dimensional local complex chart (U, Φ, U') in a holomorphic atlas in the holomorphic structure on a complex 1-manifold X is called a *local holomorphic chart in X* . Setting $z = \Phi$, we call z a *local holomorphic coordinate*, and for each point $p \in U$, we call (U, z) a *local holomorphic coordinate neighborhood of p* .

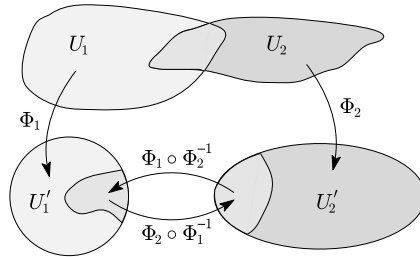


Fig. 2.1 Holomorphic coordinate transformations

Remarks 1. Higher-dimensional complex manifolds are also considered in this book, but only in some of the exercises (see, for example, Exercise 2.2.6). For this reason, all local complex charts, all local holomorphic charts, and all holomorphic atlases should be assumed to be of dimension 1 unless otherwise indicated.

2. A holomorphic structure on X determines a unique underlying real 2-dimensional C^∞ (in fact, real analytic) structure, with C^∞ atlas consisting of local C^∞ coordinates given by $(x, y) = (\operatorname{Re} z, \operatorname{Im} z)$ for any local holomorphic coordinate z (see Chap. 9). A map from (to) an open subset of X from (respectively, to) an open subset of a manifold or complex 1-manifold is said to be of class C^k if the map is of class C^k with respect to the underlying C^k structures.

3. In many treatments, manifolds are assumed to be second countable (often without comment). In fact, according to a theorem of Radó that will be proved in Sect. 2.11, every Riemann surface is automatically second countable.

4. A local holomorphic chart (U, Φ, V) and a local holomorphic coordinate neighborhood $(U, z = \Phi)$ are really the same object, but one generally uses the former terminology when emphasizing the mapping properties, and the latter when emphasizing the coordinate properties.

5. We will call a local holomorphic coordinate neighborhood $(U, z = \Phi)$ in which $\Phi(U)$ is a disk a *holomorphic coordinate disk* (or simply a *coordinate disk*). We will use the analogous terminology in similar contexts (for example, a *holomorphic coordinate annulus* will refer to a local holomorphic coordinate neighborhood with image an annulus). Similarly, given a set $S \subset U$ and a property of $\Phi(U)$, we will often say that S has this property *in* (or *with respect to*) the local coordinate neighborhood. For example, if $\Phi(S)$ is a closed rectangle in \mathbb{R}^2 , then we will say that S is a *closed rectangle in* (U, z) . We will also use terminology analogous to the above in the context of topological and C^∞ manifolds.

6. It turns out that the theory of compact Riemann surfaces and that of noncompact Riemann surfaces differ. A noncompact Riemann surface is also called an *open* Riemann surface.

7. For convenience, most of the main theorems, as well as some of the elementary facts, in this book are stated as applying to Riemann surfaces. However, analogues, with the appropriate modifications, also hold for complex 1-manifolds (see Example 2.1.4 below).

We now consider some examples. The verifications of some details are left to the reader.

Example 2.1.2 (The complex plane) The *complex plane* \mathbb{C} together with the holomorphic structure determined by the holomorphic atlas $\{(\mathbb{C}, z \mapsto z, \mathbb{C})\}$ is a Riemann surface.

Example 2.1.3 An open set Ω in a complex 1-manifold X has a natural induced holomorphic structure in which each triple of the form $(U \cap \Omega, \Phi|_{U \cap \Omega}, \Phi(U \cap \Omega))$, where (U, Φ, U') is a local holomorphic chart in X , is a local holomorphic chart in Ω (see Exercise 2.1.1). Unless otherwise indicated, we take this to be the holomorphic structure on such a given open subset Ω .

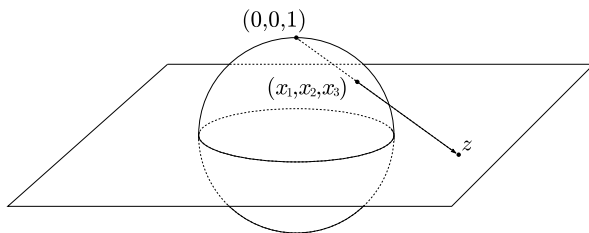


Fig. 2.2 The Riemann sphere

Example 2.1.4 Let $X \equiv \bigsqcup_{\alpha \in A} X_\alpha$ be a disjoint union of complex 1-manifolds $\{X_\alpha\}_{\alpha \in A}$, and let $\iota_\alpha: X_\alpha \hookrightarrow X$ be the inclusion map for each index $\alpha \in A$. Then X has a unique (natural) holomorphic structure in which each triple of the form $(\iota_\alpha(U), \Phi \circ \iota_\alpha^{-1}, U')$, where (U, Φ, U') is a local holomorphic chart in X_α for some $\alpha \in A$, is a local holomorphic chart in X (see Exercise 2.1.2). Unless otherwise indicated, we take this to be the holomorphic structure on such a given disjoint union. In particular, every complex 1-manifold is equal to a disjoint union of Riemann surfaces, that is, of its connected components. Consequently, most statements about Riemann surfaces may be easily modified to give an analogous statement about complex 1-manifolds.

Example 2.1.5 (The Riemann sphere) The *Riemann sphere* (or *extended complex plane*) is the one-point compactification $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ of \mathbb{C} (see Definition 9.1.11) together with the holomorphic structure determined by the holomorphic atlas

$$\{(\mathbb{C}, z \mapsto z, \mathbb{C}), (\mathbb{C}^* \cup \{\infty\}, z \mapsto 1/z, \mathbb{C})\}$$

(where $1/\infty = 0$). The Riemann sphere is diffeomorphic to the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ (see Examples 9.2.3) under the *stereographic projection* $\mathbb{S}^2 \rightarrow \mathbb{P}^1$ (see Fig. 2.2 and Exercise 2.1.3), i.e., the mapping $(x_1, x_2, x_3) \mapsto \frac{x_1}{1-x_3} + i \frac{x_2}{1-x_3}$ ($(0, 0, 1) \mapsto \infty$).

Example 2.1.6 (Complex tori) A *lattice* in \mathbb{C} is a subgroup of the form $\Gamma = \mathbb{Z}\xi_1 + \mathbb{Z}\xi_2$, where $\xi_1, \xi_2 \in \mathbb{C}$ are complex numbers that are linearly independent over \mathbb{R} . We may associate to Γ an equivalence relation \sim given by

$$z \sim w \iff z - w \in \Gamma$$

(in other words, the equivalence class of each element $z \in \mathbb{C}$ is $z + \Gamma$). Let us denote the corresponding quotient space and quotient mapping by $\Upsilon: \mathbb{C} \rightarrow X$. That is, X is the set of equivalence classes for \sim , Υ is the mapping $z \mapsto z + \Gamma$, and X is given the quotient topology (i.e., $U \subset X$ is open if and only if its inverse image $\Upsilon^{-1}(U)$ is open).

Observe that Υ is an open mapping. For if $U \subset \mathbb{C}$ is open, then $\Upsilon^{-1}(\Upsilon(U))$ is the union of the open sets $\{U + \xi\}_{\xi \in \Gamma}$. For each point $z = u_1\xi_1 + u_2\xi_2 \in \mathbb{C}$ with

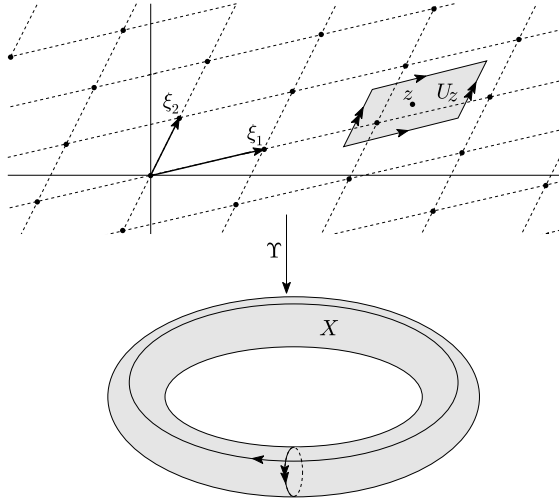


Fig. 2.3 A complex torus

$u_1, u_2 \in \mathbb{R}$, the set

$$U_z \equiv \left\{ z + t_1 \xi_1 + t_2 \xi_2 \mid -\frac{1}{2} < t_1, t_2 < \frac{1}{2} \right\}$$

is the image of the open square $(u_1 - 1/2, u_1 + 1/2) \times (u_2 - 1/2, u_2 + 1/2) \subset \mathbb{R}^2$ under the real linear isomorphism $\mathbb{R}^2 \rightarrow \mathbb{C}$ given by $(t_1, t_2) \mapsto t_1 \xi_1 + t_2 \xi_2$, and hence U_z is a relatively compact neighborhood of z in \mathbb{C} (see Fig. 2.3). It is also easy to check that

$$U_z \cap (U_z + (\Gamma \setminus \{0\})) = \emptyset \quad \text{and} \quad \Upsilon(\overline{U}_z) = X.$$

In particular, the openness of Υ , together with the first equality, implies that Υ maps U_z homeomorphically onto $\Upsilon(U_z)$; and the second equality implies that X is compact. Furthermore, X is Hausdorff. For it is clear that each of the sets $\Upsilon(U_z)$ is Hausdorff, while if $w \in \partial U_z$, then the neighborhoods

$$V \equiv \left\{ z + t_1 \xi_1 + t_2 \xi_2 \mid -\frac{1}{4} < t_1, t_2 < \frac{1}{4} \right\}$$

and

$$W \equiv \left\{ w + t_1 \xi_1 + t_2 \xi_2 \mid -\frac{1}{4} < t_1, t_2 < \frac{1}{4} \right\}$$

of z and w , respectively, have disjoint images in X .

If $U' \subset \mathbb{C}$ is an open set with $U' \cap (U' + \xi) = \emptyset$ for each $\xi \in \Gamma \setminus \{0\}$, then Υ maps U' homeomorphically onto an open set $U \subset X$. Thus we get a local complex chart $(U, (\Upsilon|_{U'})^{-1}, U')$ in X . The collection of such local complex charts determines a holomorphic structure on X . For if $(V, (\Upsilon|_{V'})^{-1}, V')$ is another such

local complex chart in X and $\Psi \equiv (\Upsilon|_{V'})^{-1} \circ \Upsilon|_{(\Upsilon|_{U'})^{-1}(U \cap V)}$ is the associated coordinate transformation, then $z \mapsto \Psi(z) - z$ is a continuous function on $U' \cap (V' + \Gamma) = (\Upsilon|_{U'})^{-1}(U \cap V)$ with values in the discrete set Γ . Hence this function must be locally constant, and it follows that Ψ is holomorphic. Thus X is a compact Riemann surface.

The Riemann surface X is called a *complex torus* because topologically, X is the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. In fact, we have a commutative diagram of C^∞ maps

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\alpha} & \mathbb{C} \\ \Upsilon_0 \downarrow & & \downarrow \Upsilon \\ \mathbb{T}^2 & \xrightarrow{\beta} & X \end{array}$$

where α is the real linear isomorphism (and therefore diffeomorphism) given by $(t_1, t_2) \mapsto t_1 \xi_1 + t_2 \xi_2$, Υ_0 is the C^∞ mapping onto the real torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ given by $(t_1, t_2) \mapsto (e^{2\pi i t_1}, e^{2\pi i t_2})$, and the induced mapping β is a well-defined diffeomorphism. For it is easy to verify that β is a well-defined bijection. Moreover, given a point $t = (t_1, t_2) \in \mathbb{R}^2$, Υ_0 maps the neighborhood $V \equiv (t_1 - 1/2, t_1 + 1/2) \times (t_2 - 1/2, t_2 + 1/2)$ diffeomorphically onto a neighborhood of $\Upsilon_0(t)$ (with inverse given by the product of $1/2\pi$ and a C^∞ argument function in each coordinate). Since locally, β is a composition of the C^∞ map Υ , the C^∞ map α , and a local C^∞ inverse of Υ_0 , β must be a C^∞ map. Similarly, β^{-1} is also a C^∞ map, and therefore β is a diffeomorphism.

Exercises for Sect. 2.1

- 2.1.1 Verify that in any open subset of a complex 1-manifold, the local complex charts described in Example 2.1.3 form a holomorphic atlas.
- 2.1.2 Verify that in any disjoint union of complex 1-manifolds, the local complex charts described in Example 2.1.4 form a holomorphic atlas.
- 2.1.3 Verify that the Riemann sphere (Example 2.1.5) is a Riemann surface, and verify that the stereographic projection (see Fig. 2.2) is a diffeomorphism of the unit sphere \mathbb{S}^2 onto \mathbb{P}^1 .

2.2 Holomorphic Functions and Mappings

Given a holomorphic structure, one gets the corresponding holomorphic functions and mappings.

Definition 2.2.1 Let X and Y be complex 1-manifolds.

- (a) A *holomorphic function* on an open set $\Omega \subset X$ is a complex-valued function f on Ω such that $f \circ \Phi^{-1} \in \mathcal{O}(\Phi(\Omega \cap U))$ for every local holomorphic chart (U, Φ, U') in X . We denote the set of holomorphic functions on Ω by $\mathcal{O}(\Omega)$.
- (b) A continuous mapping $\Psi: X \rightarrow Y$ is *holomorphic* if the function $\Phi \circ \Psi$ belongs to $\mathcal{O}(\Psi^{-1}(U))$ for every local holomorphic chart (U, Φ, U') in Y .

- (c) A bijective holomorphic mapping $\Psi: X \rightarrow Y$ with holomorphic inverse is called a *biholomorphism* (or a *biholomorphic mapping*). If such a biholomorphism exists, then we say that X and Y are *biholomorphically equivalent* (or simply *biholomorphic*). For $Y = X$, we also call Ψ an *automorphism* of X .
- (d) A holomorphic mapping $\Psi: X \rightarrow Y$ is a *local biholomorphism* (or a *locally biholomorphic mapping*) if Ψ maps a neighborhood of each point in X biholomorphically onto an open subset of Y .

Remarks 1. A holomorphic function on an open subset Ω of a complex 1-manifold X is precisely a holomorphic mapping of Ω into \mathbb{C} .

2. A function f on an open subset Ω of a complex 1-manifold X is holomorphic if and only if for each point $p \in \Omega$, there exists a local holomorphic chart (U, Φ, U') in X such that $p \in U$ and $f \circ \Phi^{-1} \in \mathcal{O}(\Phi(\Omega \cap U))$. Similarly, a continuous mapping of complex 1-manifolds $\Psi: X \rightarrow Y$ is holomorphic if and only if for each point $p \in X$, there exists a local holomorphic chart (U, Φ, U') in Y such that $\Psi(p) \in U$ and $\Phi \circ \Psi \in \mathcal{O}(\Psi^{-1}(U))$ (see Exercise 2.2.1). Furthermore, a holomorphic mapping of complex 1-manifolds is of class \mathcal{C}^∞ with respect to the underlying \mathcal{C}^∞ structures (in fact, the mapping is real analytic with respect to the underlying real analytic structures).

3. Biholomorphic equivalence is an equivalence relation (see Exercise 2.2.2), and we usually identify any two biholomorphic complex 1-manifolds.

4. Any sum or product of holomorphic functions is holomorphic, and any composition of holomorphic mappings is holomorphic (see Exercise 2.2.3). In particular, the set of holomorphic functions on an open set is an algebra.

5. In Example 2.1.6, the quotient mapping $\Upsilon: \mathbb{C} \rightarrow X$ of \mathbb{C} to the complex torus X is locally biholomorphic, because the local holomorphic charts are given by local inverses of Υ . Although all complex tori are diffeomorphic, they are not all biholomorphic (see Exercise 5.9.1).

Many of the elementary theorems for holomorphic functions on domains in the plane immediately give analogues for holomorphic mappings on Riemann surfaces. For example, we have the following:

Theorem 2.2.2 *Let X and Y be Riemann surfaces.*

- (a) Identity theorem. *If $\Phi: X \rightarrow Y$ is a nonconstant holomorphic mapping, then the fiber $\Phi^{-1}(p)$ over each point $p \in Y$ is discrete in X (i.e., $\Phi^{-1}(p)$ has no limit points in X). Moreover, if $\Psi: X \rightarrow Y$ is a holomorphic map that is not identically equal to Φ , then $\{x \in X \mid \Phi(x) = \Psi(x)\}$ is discrete in X .*
- (b) Riemann's extension theorem. *A continuous mapping $\Phi: X \rightarrow Y$ that is holomorphic on the complement of a discrete subset of X is holomorphic.*
- (c) Maximum principle. *If $f \in \mathcal{O}(X)$ and $|f|$ attains a local maximum at some point $p \in X$, then f is constant. In particular, if X is compact, then every holomorphic function on X is constant.*
- (d) Open mapping theorem. *If $\Phi: X \rightarrow Y$ is a nonconstant holomorphic mapping, then the image $\Phi(X)$ is open.*

Proof The proof of the first part of (a) is provided here, and the proofs of the second part of (a) and of (b)–(d) are left to the reader (see Exercise 2.2.4). Suppose $\Phi: X \rightarrow Y$ is a holomorphic mapping for which the fiber $F \equiv \Phi^{-1}(p)$ over some point $p \in Y$ has a limit point $q \in X$. We may fix local holomorphic charts (U, Ψ, U') in X and (V, Λ, V') in Y such that $q \in U$, U is connected, and $\Phi(U) \subset V$. It follows that for the holomorphic function $f \equiv \Lambda \circ \Phi \circ \Psi^{-1}: U' \rightarrow \mathbb{C}$, the fiber $f^{-1}(\Lambda(p)) = \Psi(F \cap U)$ has a limit point $\Psi(q)$ in U' , and hence $f \equiv \Lambda(p)$ on U' by the identity theorem in the plane (see Corollary 1.3.3). Thus $\Phi|_U = \Lambda^{-1} \circ f \circ \Psi \equiv p$, and therefore $U \subset F$. Hence the interior $\overset{\circ}{F}$ of F is nonempty, and by the above argument, $\partial(\overset{\circ}{F}) = \emptyset$. It follows that $F = X$, and hence that Φ is constant. \square

Lemma 2.2.3 (Local representation of holomorphic mappings) *Let p be a point in a Riemann surface X .*

- (a) *If f is a nonconstant holomorphic function on X , then there exist a positive integer m and a local holomorphic coordinate neighborhood (U, z) of p such that $z(p) = 0$ and $f = f(p) + z^m$ on U . Moreover, $m = m(f, p)$ is unique, and for any local holomorphic coordinate neighborhood (W, w) of p , the function $(w - w(p))^{-m}(f - f(p)) \in \mathcal{O}(W \setminus \{p\})$ extends to a holomorphic function on W that does not vanish at p .*
- (b) *If $\Psi: X \rightarrow Y$ is a nonconstant holomorphic mapping of X to a Riemann surface Y , then for some $m \in \mathbb{Z}_{>0}$ and for every local holomorphic coordinate neighborhood (V, ζ) of $\Psi(p)$ in Y , there is a local holomorphic coordinate neighborhood (U, z) of p in X such that $z(p) = 0$ and $\zeta(\Psi) = \zeta(\Psi(p)) + z^m$ on $U \cap \Psi^{-1}(V)$. Moreover, the integer $m = m(\Psi, p)$ is unique, and for any local holomorphic coordinate neighborhood (W, w) of p , the holomorphic function $(w - w(p))^{-m}(\zeta(\Psi) - \zeta(\Psi(p)))$ on $W \cap \Psi^{-1}(V) \setminus \{p\}$ extends to a holomorphic function on $W \cap \Psi^{-1}(V)$ that does not vanish at p .*

Proof For the proof of (a), we may fix a local holomorphic chart $(U_0, \Phi = w, V_0)$ with $p \in U_0$. Corollary 1.3.3 then provides a unique integer $m \in \mathbb{Z}_{>0}$ and a unique function $g \in \mathcal{O}(U_0)$ such that $g(p) \neq 0$ and $f - f(p) = (w - w(p))^m \cdot g$ on U_0 . Choosing U_0 to be sufficiently small, we also get a holomorphic branch L of the logarithmic function on a neighborhood of $g(U_0)$ (that is, $e^{L(\zeta)} = \zeta$), and hence we have the holomorphic m th root function $e^{L/m}$. The holomorphic function $z \equiv (w - w(p)) \cdot e^{L(g)/m}$ then satisfies $z(p) = 0$ and $f = f(p) + z^m$ on U_0 . Moreover,

$$(z \circ \Phi^{-1})'(w(p)) = e^{L(g(p))/m} \neq 0,$$

so the holomorphic inverse function theorem (Theorem 1.5.1) implies that the function z maps some neighborhood U of p in U_0 biholomorphically onto a neighborhood of 0 in \mathbb{C} . Finally, we have uniqueness of m . For if $f - f(p) = \zeta^n$ on Q for some $n \in \mathbb{Z}_{>0}$ and some local holomorphic chart $(Q, \Psi = \zeta, Q')$ with $p \in Q$, then by Corollary 1.3.3, since $(z \circ \Psi^{-1})'(0) \neq 0$, there is a holomorphic function h on

a neighborhood of 0 in \mathbb{C} with $h(0) \neq 0$ and $z(\Psi^{-1}(\xi)) = \xi h(\xi)$ for each $\xi \in Q'$ near 0. Hence, for each ξ near 0,

$$\xi^n = f(\Psi^{-1}(\xi)) - f(p) = [z(\Psi^{-1}(\xi))]^m = \xi^m (h(\xi))^m.$$

The uniqueness part of Corollary 1.3.3 implies that $n = m$ (and $h^m \equiv 1$). Thus (a) is proved.

The proof of (b) is left to the reader (see Exercise 2.2.5). \square

Definition 2.2.4 Let Ω be an open subset of a complex 1-manifold X , and let $p \in \Omega$.

- (a) If f is a holomorphic function on Ω and m is a positive integer such that $f = z^m g$ on a neighborhood p for some local holomorphic coordinate z with $z(p) = 0$ and some nonvanishing holomorphic function g (i.e., $f(p) = 0$ and m is the integer provided by part (a) of Lemma 2.2.3), then we say that f has a *zero of order m at p* . For $m = 1$, we also say that f has a *simple zero at p* .
- (b) If $\Psi: \Omega \rightarrow Y$ is a holomorphic mapping into a complex 1-manifold Y and m is a positive integer such that $\zeta(\Psi) - \zeta(\Psi(p))$ has a zero of order m at p for some local holomorphic coordinate ζ on a neighborhood of $\Psi(p)$ (i.e., m is the integer provided by part (b) of Lemma 2.2.3), then we say that Ψ has *multiplicity m at p* , and we write $\text{mult}_p \Psi = m$.

Definition 2.2.5 A *meromorphic function* on an open subset Ω of a complex 1-manifold X is a function $f: \Omega \setminus P \rightarrow \mathbb{C}$ on the complement $\Omega \setminus P$ of a discrete subset P of Ω such that for each point $p \in P$, $|f(x)| \rightarrow \infty$ as $x \rightarrow p$. Each point $p \in P$ is called a *pole* of f . We denote the set of meromorphic functions on Ω by $\mathcal{M}(\Omega)$.

Proposition 2.2.6 For any holomorphic function f on the complement $X \setminus P$ of a discrete subset P of a complex 1-manifold X , the following are equivalent:

- (i) The function f is a meromorphic function on X with pole set P .
- (ii) There exists a holomorphic mapping $h: X \rightarrow \mathbb{P}^1$ such that $h = f$ on $X \setminus P$ and $P = h^{-1}(\infty)$.
- (iii) For each point $p \in P$ and for every local holomorphic coordinate neighborhood (U, z) of p in X , there exist a nonvanishing holomorphic function g on a neighborhood V of p in U and a positive integer v_p such that $f = (z - z(p))^{-v_p} g$ on $V \setminus \{p\}$.
- (iv) For each point $p \in P$, there exist a local holomorphic coordinate neighborhood (U, z) of p in X , a nonvanishing holomorphic function g on a neighborhood V of p in U , and a positive integer v_p such that $f = (z - z(p))^{-v_p} g$ on $V \setminus \{p\}$.
- (v) For each point $p \in P$, there exist a local holomorphic coordinate neighborhood (U, z) of p in X and a positive integer v_p such that $z(p) = 0$ and $f = z^{-v_p}$ on $U \setminus \{p\}$.

Moreover, in the above, for each point $p \in P$, the integer v_p in (iii)–(v) is equal to the multiplicity at p of the holomorphic mapping h in (ii).

Proof Clearly, any one of the conditions (ii)–(v) implies (i). Conversely, if f is meromorphic with pole set P , then f extends to a unique continuous mapping $h: \Omega \rightarrow \mathbb{P}^1$ with $h^{-1}(\infty) = P$, and Riemann's extension theorem (Theorem 2.2.2) implies that h is a holomorphic mapping; that is, (ii) holds. If $p \in P$ and $v_p = \text{mult}_p h$, then by the local representation of holomorphic mappings (Lemma 2.2.3), for every local holomorphic coordinate neighborhood (U, z) of p , there is a non-vanishing holomorphic function u on a neighborhood V of p in U such that $h^{-1}(\infty) \cap V = \{p\}$ and $u = (z - z(p))^{-v_p} \cdot (1/h)$ on $V \setminus \{p\}$. Setting $g \equiv 1/u$, we get (iii), and (iv) follows. Moreover, we may choose (U, z) so that $z(p) = 0$ and $1/h = z^{v_p}$ on U , so we get (v) as well. \square

Definition 2.2.7 We identify any meromorphic function f on an open subset Ω of a complex 1-manifold X with the associated holomorphic mapping $\Omega \rightarrow \mathbb{P}^1$. If $p \in f^{-1}(\infty)$ is a pole of f at which this mapping has multiplicity m , then we say that f has a *pole of order m at p* . We say that f has a *zero of order m at $q \in \Omega$* if $q \in f^{-1}(0)$ and the holomorphic function $f|_{\Omega \setminus f^{-1}(\infty)}$ has a zero of order m at q (note that this is consistent with the above identification). For any point $p \in \Omega$, the *order of f at p* is given by

$$\text{ord}_p f \equiv \begin{cases} 0 & \text{if } p \notin f^{-1}(\{0, \infty\}), \\ m & \text{if } f \text{ has a zero of order } m \text{ at } p, \\ -m & \text{if } f \text{ has a pole of order } m \text{ at } p, \\ \infty & \text{if } f \equiv 0 \text{ on a neighborhood of } p. \end{cases}$$

We say that f has a *simple zero (simple pole) at p* if $\text{ord}_p f = 1$ (respectively, $\text{ord}_p f = -1$).

Remarks 1. If f and g are meromorphic functions on a connected neighborhood of a point p in a complex 1-manifold X and neither f nor g is identically zero, then in terms of some local holomorphic coordinate z on a neighborhood of p , we have $f = z^m f_0$ and $g = z^n g_0$ for some pair of integers m and n and some pair of nonvanishing holomorphic functions f_0 and g_0 on a neighborhood of p . It follows that each of the functions $f + g$, fg , and f/g is holomorphic on a neighborhood of p or has a removable singularity or a pole at p . Consequently, any sum, product, or quotient (provided the denominator does not vanish identically on any open set) of meromorphic functions is a meromorphic function, provided we holomorphically extend the resulting function over any removable singularities. In other words, for any connected open subset Ω of a complex 1-manifold, $\mathcal{M}(\Omega)$ is a field.

2. Suppose X is a complex 1-manifold, (U, z) is a local holomorphic coordinate neighborhood of a point $p \in X$, $f \in \mathcal{O}(X \setminus \{p\})$, and $f = \sum_{n=-\infty}^{\infty} c_n (z - z(p))^n$ is the corresponding Laurent series representation of f about p provided by Theorem 1.3.6. Then f has a pole of order $m \in \mathbb{Z}_{>0}$ at p if and only if $c_n = 0$ for all $n < -m$ and $c_{-m} \neq 0$.

Exercises for Sect. 2.2

- 2.2.1 Prove that a function f on an open subset Ω of a complex 1-manifold X is holomorphic if and only if for each point $p \in \Omega$, *there exists* a local holomorphic chart (U, Φ, U') in X such that $p \in U$ and $f \circ \Phi^{-1} \in \mathcal{O}(\Phi(\Omega \cap U))$. Prove also that a continuous mapping of complex 1-manifolds $\Psi: X \rightarrow Y$ is holomorphic if and only if for each point $p \in X$, *there exists* a local holomorphic chart (U, Φ, U') in Y such that $\Psi(p) \in U$ and $\Phi \circ \Psi \in \mathcal{O}(\Psi^{-1}(U))$.
- 2.2.2 Prove that biholomorphic equivalence of complex 1-manifolds is an equivalence relation.
- 2.2.3 Prove that any sum or product of holomorphic functions on a complex 1-manifold is holomorphic, and any composition of holomorphic mappings of complex 1-manifolds is holomorphic.
- 2.2.4 Prove the second statement in part (a) of Theorem 2.2.2 (i.e., if $\Phi, \Psi: X \rightarrow Y$ are distinct holomorphic mappings of Riemann surfaces, then $\{x \in X \mid \Phi(x) = \Psi(x)\}$ is discrete in X). Also prove parts (b)–(d) of Theorem 2.2.2.
- 2.2.5 Prove part (b) Lemma 2.2.3.
- 2.2.6 *Complex manifolds.* A *holomorphic atlas of dimension n* on a Hausdorff space X is an atlas in which each element is a homeomorphism $\Phi: U \rightarrow U'$ of an open set $U \subset X$ onto an open set $U' \subset \mathbb{C}^n$ and the coordinate transformations are holomorphic mappings (see Exercise 1.6.4 for the definition). Two holomorphic atlases of dimension n are *holomorphically equivalent* if their union is a holomorphic atlas. A *complex manifold of dimension n* is a Hausdorff space X together with an equivalence class of n -dimensional holomorphic atlases (i.e., an n -dimensional *holomorphic structure*). The elements of any atlas in the holomorphic structure are called *local holomorphic charts*. A continuous mapping of complex manifolds is *holomorphic* if its representation in local holomorphic charts is holomorphic. Prove that the Cartesian product of a pair of Riemann surfaces has a natural 2-dimensional holomorphic structure.
- 2.2.7 Prove that no two of the Riemann surfaces \mathbb{C} , \mathbb{P}^1 , and $\Delta \equiv \Delta(0; 1)$ are biholomorphic.
- 2.2.8 Prove the following:
- (a) *Liouville's theorem.* Every bounded entire function is constant.
Hint. Apply Theorem 1.2.10 near ∞ in $\mathbb{P}^1 \supset \mathbb{C}$.
 - (b) *The fundamental theorem of algebra (Gauss).* Every nonconstant complex polynomial has a zero.
Hint. Given a nonvanishing complex polynomial function g , consider the holomorphic function $1/g$.

2.3 Holomorphic Attachment

One may produce infinitely many examples of Riemann surfaces by holomorphic attachment. In fact, one goal of this book is a proof (appearing in Chap. 5) that *every* Riemann surface may be obtained by holomorphic attachment of tubes to a domain

in the Riemann sphere \mathbb{P}^1 . Holomorphic attachment is not required until Chap. 5, so a reader who prefers to skip this section and move on to other topics in this chapter may do so without fear of missing a required concept. It will be convenient to have holomorphic attachment formulated as follows:

Proposition 2.3.1 (Holomorphic attachment) *Let X and X' be complex 1-manifolds, and let $\Psi: G' \rightarrow G$ be a biholomorphism of an open set $G' \subset X'$ onto an open set $G \subset X$ such that $\Psi^{-1}(K \cap G)$ is closed in X' for every compact set $K \subset X$ (i.e., $\Psi(x) \rightarrow \infty$ in the one-point compactification of X as $x \rightarrow p \in \partial G$) and $\Psi(K' \cap G')$ is closed in X for every compact set $K' \subset X'$ (i.e., $\Psi^{-1}(x) \rightarrow \infty$ in the one-point compactification of X' as $x \rightarrow p \in \partial G$). Let $\iota: X \hookrightarrow X \sqcup X'$ and $\iota': X' \hookrightarrow X \sqcup X'$ be the inclusion mappings of X and X' into the disjoint union $X \sqcup X'$, let \sim be the equivalence relation in the disjoint union $X \sqcup X'$ determined by $\iota'(x) \sim \iota(\Psi(x))$ for every point $x \in G'$, and let*

$$\Pi: X \sqcup X' \rightarrow Y = X \cup_{\Psi} X' \equiv (X \sqcup X')/\sim$$

be the associated quotient space. Then there is a unique 1-dimensional holomorphic structure on Y with respect to which $\Pi \circ \iota$ and $\Pi \circ \iota'$ are biholomorphisms of X and X' , respectively, onto open subsets of Y (equivalently, there is a unique holomorphic structure on Y with respect to which Π is a local biholomorphism).

The proof is left to the reader (see Exercise 2.3.1).

Definition 2.3.2 For $X \supset G$, $X' \supset G'$, and $\Psi: G' \rightarrow G$ as in Proposition 2.3.1, the complex 1-manifold $X \cup_{\Psi} X'$ is called the *holomorphic attachment* of X and X' along Ψ .

Remarks 1. It is customary and convenient to identify X and X' with their (disjoint) images in $X \sqcup X'$. However, although it is customary to leave out any explicit mention of the inclusion maps ι and ι' , we will often mention the inclusion maps when considering specific mappings of the above spaces. This will allow us to avoid any danger of confusion (for example, there is danger of confusion whenever $X = X'$).

2. The natural identification of $X \sqcup X'$ with $X' \sqcup X$ gives a natural identification of $X \cup_{\Psi} X'$ with $X' \cup_{\Psi^{-1}} X$.

In applications, usually one first removes a set and then holomorphically attaches a new set of a desired type. For example, several key arguments in Chap. 5 will require that we replace sets with caps (i.e., disks) or tubes (i.e., annuli). For now, we consider the following:

Example 2.3.3 (Holomorphic attachment of a tube) Let Y be a complex 1-manifold; let $R_0, R_1 > 1$; let $\{(D_{\nu}, \Phi_{\nu}, \Delta(0; R_{\nu}))\}_{\nu \in [0,1]}$ be disjoint local holomorphic charts in Y ; let $A'_{\nu} \equiv \Phi_{\nu}^{-1}(\Delta(0; 1, R_{\nu})) \subset D_{\nu}$ for $\nu = 0, 1$; let $T \equiv \Delta(0; 1/R_0, R_1)$; and

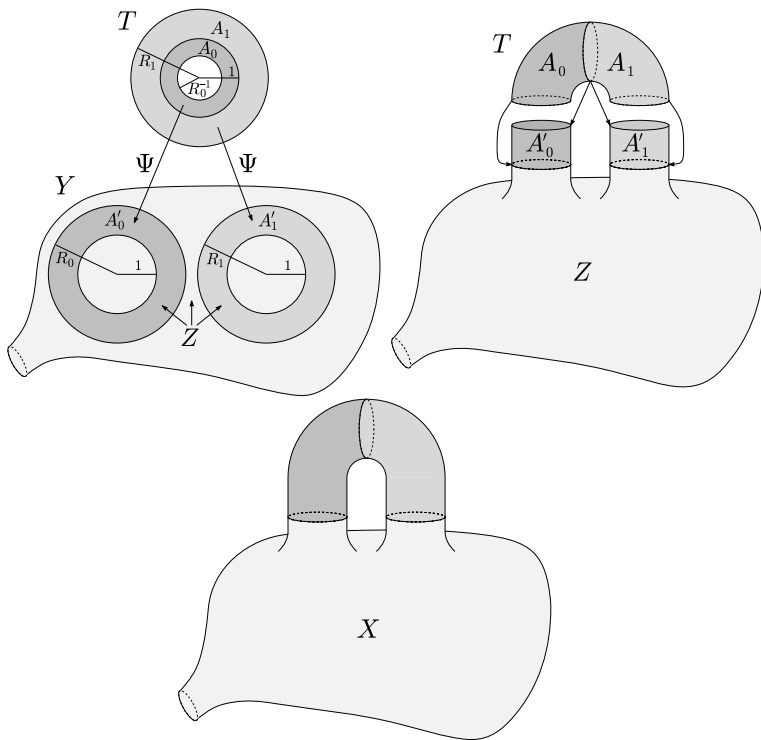


Fig. 2.4 Holomorphic attachment of a tube

let $A_0 \equiv \Delta(0; 1/R_0, 1) \subset T$ and $A_1 \equiv \Delta(0; 1, R_1) \subset T$. Thus we get a biholomorphism $\Psi: A_0 \cup A_1 \rightarrow A'_0 \cup A'_1$ given by

$$\Psi(z) = \begin{cases} \Phi_0^{-1}(1/z) \in A'_0 & \text{if } z \in A_0, \\ \Phi_1^{-1}(z) \in A'_1 & \text{if } z \in A_1. \end{cases}$$

For $Z \equiv Y \setminus [\Phi_0^{-1}(\overline{\Delta(0; 1)}) \cup \Phi_1^{-1}(\overline{\Delta(0; 1)})] \supset A'_0 \cup A'_1$, $\Psi^{-1}(K' \cap (A'_0 \cap A'_1))$ is closed in T for every compact set $K' \subset Z$, and $\Psi(K \cap (A_0 \cup A_1))$ is closed in Z for every compact set $K \subset T$. We may therefore form the holomorphic attachment $X \equiv Z \cup_{\Psi} T = Z \sqcup T / \sim$, where for $p \in A'_0$ and $z \in A_0$, the images p_0 of p and z_0 of z in $Z \sqcup T$ under the inclusions $A'_0, A_0 \hookrightarrow Z \sqcup T$ satisfy $p_0 \sim z_0$ if and only if $z \cdot \Phi_0(p) = 1$, and for $p \in A'_1$ and $z \in A_1$, the respective images p_0 and z_0 satisfy $p_0 \sim z_0$ if and only if $z = \Phi_1(p)$ (see Fig. 2.4). In other words, X is a complex 1-manifold obtained by removing the unit disks in each of the coordinate disks D_0 and D_1 and gluing in a tube (i.e., an annulus) T (equivalently, the boundaries of the unit disks are glued together). We call X the *complex 1-manifold obtained by holomorphic attachment of the tube T to Y at the coordinate disks $\{(D_v, \Phi_v, \Delta(0; R_v))\}_{v \in \{0, 1\}}$ (or simply at $\{D_v\}_{v \in \{0, 1\}}$)*. This is a slight abuse of language, since actually we first removed the set $\Phi_0^{-1}(\overline{\Delta(0; 1)}) \cup \Phi_1^{-1}(\overline{\Delta(0; 1)})$ before performing the attachment.

Observe that if Y is connected, then X is connected; and if Y is compact, then X is compact. If X is compact, then Y is compact. However, X may be connected even if Y is not connected (see Exercise 2.3.2).

Fixing R_v^* with $1 < R_v^* \leq R_v$ for $v = 0, 1$, we may form the holomorphic attachment $X^* = Z \sqcup T^*/\sim$ of the tube $T^* \equiv \Delta(0; 1/R_0^*, R_1^*) \subset T$ to Y at the coordinate disks $\{(D_v^* \equiv \Phi_v^{-1}(\Delta(0; R_v^*)), \Phi_v^* \equiv \Phi_v \upharpoonright_{D_v^*}, \Delta(0; R_v^*))\}_{v \in 0,1}$. The natural inclusion

$Z \sqcup T^* \subset Z \sqcup T$ then descends to a natural biholomorphism $X^* \xrightarrow{\cong} X$ (see Exercise 2.3.4), so we may identify X^* with X . In particular, we may always choose $R_0, R_1 \in (1, \infty)$ to be equal and arbitrarily close to 1.

By holomorphically attaching tubes, one obtains examples of complex 1-manifolds with complicated topology. In fact, one of the main goals of Chap. 5 will be to show that *every* Riemann surface may be obtained by holomorphic attachment of tubes at elements of a locally finite family of disjoint coordinate disks in a domain in \mathbb{P}^1 (see Theorem 5.13.1 and Theorem 5.14.1).

Exercises for Sect. 2.3

- 2.3.1 Prove Proposition 2.3.1.
- 2.3.2 Let X be a complex 1-manifold obtained by holomorphically attaching a tube to a complex 1-manifold Y as in Example 2.3.3. Prove that X is connected if Y is connected. Give an example that shows that X may be connected even if Y is not.
- 2.3.3 Write out a specific example of holomorphic attachment of a tube to the Riemann sphere (observe that the resulting Riemann surface appears to have the topological type of a torus).
- 2.3.4 In the notation of Example 2.3.3, verify that the natural inclusion of $Z \sqcup T^*$ into $Z \sqcup T$ descends to a biholomorphism of X^* onto X .
- 2.3.5 Let X and X' be complex 1-manifolds, let $\Psi: G' \rightarrow G$ be a biholomorphism of an open set $G' \subset X'$ onto an open set $G \subset X$, let $\iota: X \hookrightarrow X \sqcup X'$ and $\iota': X' \hookrightarrow X \sqcup X'$ be the inclusion mappings of X and X' into the disjoint union $X \sqcup X'$, let \sim be the equivalence relation in the disjoint union $X \sqcup X'$ determined by $\iota'(x) \sim \iota(\Psi(x))$ for every point $x \in G'$, and let $Y \equiv (X \sqcup X')/\sim$ be the associated quotient space. Prove that if there exists a compact set $K \subset X$ for which $\Psi^{-1}(K \cap G)$ is *not* closed in X' , then Y is *not* Hausdorff.

2.4 Holomorphic Tangent Bundle

A complex 1-manifold X has an underlying real 2-dimensional C^∞ structure, and therefore a tangent bundle and a complexified tangent bundle

$$\Pi_{TX}: TX \rightarrow X \quad \text{and} \quad \Pi_{(TX)_\mathbb{C}}: (TX)_\mathbb{C} \rightarrow X,$$

respectively (see Sect. 9.4). We recall that for $p \in X$, a tangent vector $v \in (T_p X)_\mathbb{C}$ is a linear derivation on the germs of C^∞ functions at p ; that is, for every pair

of \mathcal{C}^∞ functions f, g on a neighborhood of p , we have $v(fg) = v(f) \cdot g(p) + f(p) \cdot v(g) \in \mathbb{C}$. Given a local holomorphic coordinate neighborhood $(U, z = x + iy)$, the vector fields $\partial/\partial x$ and $\partial/\partial y$ form a basis for the tangent space at each point in U . Furthermore, the corresponding complex vector fields $\partial/\partial z$ and $\partial/\partial \bar{z}$ as in Chap. 1 form a different complex basis for the complexified tangent space at each point, and their spans yield a natural decomposition into a sum of 1-dimensional vector spaces. Moreover, a \mathcal{C}^1 function g on U is holomorphic if and only if $\partial g/\partial \bar{z} = \overline{\partial \bar{g}/\partial z} = 0$. Similarly, the differentials dz and $d\bar{z}$ form a dual basis that yields a decomposition of the complexified cotangent space at each point. The precise definitions and verifications appear below.

Definition 2.4.1 Let X be a complex 1-manifold.

- (a) For each point $p \in X$, a tangent vector $v \in (T_p X)_{\mathbb{C}}$ is of *type* $(1, 0)$ (of *type* $(0, 1)$) if $d\bar{f}(v) = v(\bar{f}) = 0$ (respectively, $df(v) = v(f) = 0$) for every holomorphic function f on a neighborhood of p . The subspace $(T_p X)^{1,0}_{\mathbb{C}}$ of $(T_p X)_{\mathbb{C}}$ formed by the tangent vectors at p of type $(1, 0)$ is called the *holomorphic tangent space* (or $(1, 0)$ -*tangent space*) at p . The subspace $(T_p X)^{0,1}_{\mathbb{C}} \subset (T_p X)_{\mathbb{C}}$ formed by the tangent vectors at p of type $(0, 1)$ is called the $(0, 1)$ -*tangent space* at p . The spaces

$$\Pi_{(TX)^{1,0}} = \Pi_{(TX)_{\mathbb{C}}} \upharpoonright_{(TX)^{1,0}} : (TX)^{1,0} \equiv \bigcup_{p \in X} (T_p X)^{1,0} \rightarrow X$$

and

$$\Pi_{(TX)^{0,1}} = \Pi_{(TX)_{\mathbb{C}}} \upharpoonright_{(TX)^{0,1}} : (TX)^{0,1} \equiv \bigcup_{p \in X} (T_p X)^{0,1} \rightarrow X$$

are called the *holomorphic tangent bundle* (or $(1, 0)$ -*tangent bundle*) and $(0, 1)$ -*tangent bundle*, respectively.

- (b) For each $p \in X$, an element $\alpha \in (T_p^* X)_{\mathbb{C}}$ is of *type* $(1, 0)$ (of *type* $(0, 1)$) if $\alpha(v) = 0$ for every tangent vector $v \in (T_p X)^{0,1}_{\mathbb{C}}$ (respectively, $v \in (T_p X)^{1,0}_{\mathbb{C}}$). The subspace $(T_p^* X)^{1,0}_{\mathbb{C}} \subset (T_p^* X)_{\mathbb{C}}$ formed by the elements of type $(1, 0)$ is called the *holomorphic cotangent space* (or $(1, 0)$ -*cotangent space*) at p . The subspace $(T_p^* X)^{0,1}_{\mathbb{C}} \subset (T_p^* X)_{\mathbb{C}}$ formed by the elements of type $(0, 1)$ is called the $(0, 1)$ -*cotangent space* at p . The spaces

$$\Pi_{(T^*X)^{1,0}} = \Pi_{(T^*X)_{\mathbb{C}}} \upharpoonright_{(T^*X)^{1,0}} : (T^*X)^{1,0} \equiv \bigcup_{p \in X} (T_p^* X)^{1,0} \rightarrow X$$

and

$$\Pi_{(T^*X)^{0,1}} = \Pi_{(T^*X)_{\mathbb{C}}} \upharpoonright_{(T^*X)^{0,1}} : (T^*X)^{0,1} \equiv \bigcup_{p \in X} (T_p^* X)^{0,1} \rightarrow X$$

are called the *holomorphic cotangent bundle* (or $(1, 0)$ -*cotangent bundle*) and $(0, 1)$ -*cotangent bundle*, respectively. The holomorphic cotangent bundle

$(T^*X)^{1,0}$ is also called the *canonical line bundle* (or *canonical bundle*) and is also denoted by $\Pi_{K_X}: K_X \rightarrow X$.

- (c) Given a local holomorphic coordinate neighborhood $(U, \Phi = z = x + iy)$ in X (with $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$), we call the vector fields

$$\frac{\partial}{\partial z} \equiv \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} \equiv \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) = \overline{\partial/\partial z}$$

the *partial derivative operators with respect to z and \bar{z}* , respectively. In other words, denoting the standard complex coordinate on \mathbb{C} by w , we have, for any suitable function f ,

$$\frac{\partial f}{\partial z} = \frac{\partial(f \circ \Phi^{-1})}{\partial w} \circ \Phi \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = \frac{\partial(f \circ \Phi^{-1})}{\partial \bar{w}} \circ \Phi.$$

Proposition 2.4.2 *Let X be a complex 1-manifold.*

- (a) *For each point $p \in X$, we have direct sum decompositions*

$$(T_p X)_{\mathbb{C}} = (T_p X)^{1,0} \oplus (T_p X)^{0,1} \quad \text{and} \quad (T_p^* X)_{\mathbb{C}} = (T_p^* X)^{1,0} \oplus (T_p^* X)^{0,1};$$

and we have isomorphisms

$$(T_p^* X)^{1,0} \xrightarrow{\cong} ((T_p X)^{1,0})^* \quad \text{and} \quad (T_p^* X)^{0,1} \xrightarrow{\cong} ((T_p X)^{0,1})^*$$

given by $\alpha \mapsto \alpha|_{(T_p X)^{1,0}}$ and $\alpha \mapsto \alpha|_{(T_p X)^{0,1}}$, respectively.

- (b) *Conjugation gives $\overline{(TX)^{1,0}} = (TX)^{0,1}$ and $\overline{(T^*X)^{1,0}} = (T^*X)^{0,1}$.*
(c) *For every local holomorphic coordinate neighborhood $(U, z = x + iy)$ of a point $p \in X$, we have*

$$\begin{aligned} (T_p X)^{1,0} &= \mathbb{C} \cdot (\partial/\partial z)_p, & (T_p X)^{0,1} &= \mathbb{C} \cdot (\partial/\partial \bar{z})_p, \\ (T_p^* X)^{1,0} &= \mathbb{C} \cdot (dz)_p, & (T_p^* X)^{0,1} &= \mathbb{C} \cdot (d\bar{z})_p, \end{aligned}$$

and

$$\begin{aligned} dz((\partial/\partial z)_p) &= 1, & d\bar{z}((\partial/\partial \bar{z})_p) &= 1, \\ d\bar{z}((\partial/\partial \bar{z})_p) &= 0, & d\bar{z}((\partial/\partial z)_p) &= 0. \end{aligned}$$

Moreover, for every complex number $\zeta = a + ib$ with $a, b \in \mathbb{R}$, we have

$$a \left(\frac{\partial}{\partial x} \right)_p + b \left(\frac{\partial}{\partial y} \right)_p = \zeta \left(\frac{\partial}{\partial z} \right)_p + \bar{\zeta} \left(\frac{\partial}{\partial \bar{z}} \right)_p$$

and

$$a(dx)_p + b(dy)_p = \frac{\bar{\zeta}}{2}(dz)_p + \frac{\zeta}{2}(d\bar{z})_p.$$

(d) A \mathcal{C}^1 function f on an open set $\Omega \subset X$ is holomorphic if and only if for each point $p \in \Omega$, $\partial f / \partial \bar{z} \equiv 0$ on $U \cap \Omega$ for some (equivalently, for every) local holomorphic coordinate neighborhood (U, z) of p . If $\Psi: X \rightarrow Y$ is a \mathcal{C}^1 mapping of X into a complex 1-manifold Y and $(r, s) \in \{(1, 0), (0, 1)\}$, then the following are equivalent:

- (i) Ψ is holomorphic.
- (ii) $\Psi_*(TX)^{r,s} \subset (TY)^{r,s}$.
- (iii) $\Psi^*(T^*Y)^{r,s} \subset (T^*X)^{r,s}$.

Proof Let $(U, \Phi = z, U')$ be a local holomorphic chart in X and let $p \in U$. A function $f \in \mathcal{C}^1(U)$ is holomorphic if and only if $f \circ \Phi^{-1} \in \mathcal{O}(U')$. Let w denote the standard complex coordinate on \mathbb{C} . Since

$$\frac{\partial f}{\partial \bar{z}} \circ \Phi^{-1} = \frac{\partial(f \circ \Phi^{-1})}{\partial \bar{w}},$$

we see that f is holomorphic if and only if $\overline{\partial \bar{f} / \partial \bar{z}} = \partial f / \partial \bar{z} \equiv 0$, and hence that $(\partial / \partial z)_p \in (T_p X)^{1,0} \setminus \{0\}$ and $(\partial / \partial \bar{z})_p \in (T_p X)^{0,1} \setminus \{0\}$. Moreover,

$$\left(\frac{\partial}{\partial x} \right)_p = \left(\frac{\partial}{\partial z} \right)_p + \left(\frac{\partial}{\partial \bar{z}} \right)_p \quad \text{and} \quad \left(\frac{\partial}{\partial y} \right)_p = i \left(\frac{\partial}{\partial z} \right)_p - i \left(\frac{\partial}{\partial \bar{z}} \right)_p.$$

Therefore, since $\dim_{\mathbb{C}}(T_p X)_{\mathbb{C}} = 2$, we have the direct sum decomposition

$$(T_p X)_{\mathbb{C}} = \mathbb{C} \cdot \left(\frac{\partial}{\partial z} \right)_p \oplus \mathbb{C} \cdot \left(\frac{\partial}{\partial \bar{z}} \right)_p = (T_p X)^{1,0} \oplus (T_p X)^{0,1}.$$

We also have

$$dz \left(\frac{\partial}{\partial z} \right) = (dx + i dy) \left(\frac{1}{2} \frac{\partial}{\partial x} - \frac{i}{2} \frac{\partial}{\partial y} \right) = \frac{1}{2} - \frac{i}{2} \cdot 0 + i \cdot \frac{1}{2} \cdot 0 - i \cdot \frac{i}{2} \cdot 1 = 1.$$

A similar computation gives $dz(\partial / \partial \bar{z}) = 0$, and it follows that

$$d\bar{z}(\partial / \partial \bar{z}) = \overline{dz(\partial / \partial z)} = 1 \quad \text{and} \quad d\bar{z}(\partial / \partial z) = \overline{dz(\partial / \partial \bar{z})} = 0.$$

Thus $(dz)_p \in (T_p^* X)^{1,0}$ and $(d\bar{z})_p \in (T_p^* X)^{0,1}$ form the basis of $(T_p^* X)_{\mathbb{C}}$ which is dual to the basis $(\partial / \partial z)_p, (\partial / \partial \bar{z})_p$. Parts (a)–(c) now follow easily.

If Ψ is a \mathcal{C}^1 mapping of X into a complex 1-manifold Y and (V, ξ) is a local holomorphic coordinate neighborhood of $\Psi(p)$ in Y , then

$$\begin{aligned} \frac{\partial(\xi \circ \Psi)}{\partial \bar{z}} &= d(\xi \circ \Psi)(\partial / \partial \bar{z}) = d\xi(\Psi_*(\partial / \partial \bar{z})) = (\Psi^* d\xi)(\partial / \partial \bar{z}) \\ &= \overline{d(\bar{\xi} \circ \Psi)(\partial / \partial z)} = \overline{d\bar{\xi}(\Psi_*(\partial / \partial z))} = \overline{(\Psi^* d\bar{\xi})(\partial / \partial z)}. \end{aligned}$$

Part (c) now gives the equivalence of (i)–(iii) in (d). □

Remarks 1. It follows from the above proposition that if $(U, z = x + iy)$ is a local holomorphic coordinate neighborhood in a complex 1-manifold X , $p \in U$, $v \in (T_p X)_{\mathbb{C}}$, $\alpha \in (T_p^* X)_{\mathbb{C}}$, and f is a C^1 function on a neighborhood of p , then

$$\begin{aligned} v &= dx(v) \left(\frac{\partial}{\partial x} \right)_p + dy(v) \left(\frac{\partial}{\partial y} \right)_p = dz(v) \left(\frac{\partial}{\partial z} \right)_p + d\bar{z}(v) \left(\frac{\partial}{\partial \bar{z}} \right)_p, \\ \alpha &= \alpha((\partial/\partial x)_p)(dx)_p + \alpha((\partial/\partial y)_p)(dy)_p \\ &= \alpha((\partial/\partial z)_p)(dz)_p + \alpha((\partial/\partial \bar{z})_p)(d\bar{z})_p, \\ (df)_p &= \frac{\partial f}{\partial z}(p)(dz)_p + \frac{\partial f}{\partial \bar{z}}(p)(d\bar{z})_p. \end{aligned}$$

In particular, f is holomorphic if and only if $df = (\partial f/\partial z) \cdot dz$. If $\Psi: X \rightarrow Y$ is a holomorphic mapping into a Riemann surface Y , (V, w) is a local holomorphic coordinate neighborhood of $\Psi(p)$ in Y , and $g = w \circ \Psi$, then for all $a, b \in \mathbb{C}$,

$$(\Psi_*)_p \left[a \left(\frac{\partial}{\partial z} \right)_p + b \left(\frac{\partial}{\partial \bar{z}} \right)_p \right] = a \frac{\partial g}{\partial z}(p) \left(\frac{\partial}{\partial w} \right)_{\Psi(p)} + b \overline{\frac{\partial g}{\partial z}(p)} \left(\frac{\partial}{\partial \bar{w}} \right)_{\Psi(p)}.$$

Consequently, if any one of the following linear mappings is nontrivial (i.e., not the zero mapping), then all of the mappings are isomorphisms:

$$\begin{aligned} (\Psi_*)_p: (T_p X)_{\mathbb{C}} &\rightarrow (T_{\Phi(p)} Y)_{\mathbb{C}}, & (\Psi_*)_p: T_p X &\rightarrow T_{\Phi(p)} Y, \\ (\Psi_*)_p: (T_p X)^{1,0} &\rightarrow (T_{\Phi(p)} Y)^{1,0}, & (\Psi_*)_p: (T_p X)^{0,1} &\rightarrow (T_{\Phi(p)} Y)^{0,1}, \\ (dg)_p: (T_p X)^{1,0} &\rightarrow \mathbb{C}. \end{aligned}$$

2. We express the decomposition of the complexified tangent and cotangent bundles by writing $(TX)_{\mathbb{C}} = (TX)^{1,0} \oplus (TX)^{0,1}$ and $(T^*X)_{\mathbb{C}} = (T^*X)^{1,0} \oplus (T^*X)^{0,1}$, and for each pair $(r, s) \in \{(1, 0), (0, 1)\}$, we let

$$\mathcal{P}^{r,s}: (TX)_{\mathbb{C}} \rightarrow (TX)^{r,s} \quad \text{and} \quad \mathcal{P}^{r,s}: (T^*X)_{\mathbb{C}} \rightarrow (T^*X)^{r,s}$$

denote the mappings for which the restrictions

$$\mathcal{P}^{r,s}|_{(T_p X)_{\mathbb{C}}}: (T_p X)_{\mathbb{C}} = (T_p X)^{1,0} \oplus (T_p X)^{0,1} \rightarrow (T_p X)^{r,s}$$

and

$$\mathcal{P}^{r,s}|_{(T_p^* X)_{\mathbb{C}}}: (T_p^* X)_{\mathbb{C}} = (T_p^* X)^{1,0} \oplus (T_p^* X)^{0,1} \rightarrow (T_p^* X)^{r,s}$$

are the corresponding vector space projections for each point $p \in X$. For any element ξ of $(T_p X)_{\mathbb{C}}$ or $(T_p^* X)_{\mathbb{C}}$, we call $\mathcal{P}^{r,s} \xi$ the (r, s) part of ξ .

3. The holomorphic tangent bundle $(TX)^{1,0}$ and cotangent bundle $(T^*X)^{1,0}$ are examples of holomorphic line bundles (see Example 3.1.4). The $(0, 1)$ tangent bundle $(TX)^{0,1}$ and cotangent bundle $(T^*X)^{0,1}$ are examples of C^∞ line bundles.

4. A reader familiar with vector bundles will recognize the decomposition $(TX)_{\mathbb{C}} = (TX)^{1,0} \oplus (TX)^{0,1}$ as a decomposition of the C^∞ vector bundle $(TX)_{\mathbb{C}}$ into a sum of C^∞ subbundles (as is the decomposition of $(T^*X)_{\mathbb{C}}$).

5. For $(r, s) \in \{(1, 0), (0, 1)\}$ and for any open subset Ω of a complex 1-manifold X with inclusion mapping $\iota: \Omega \hookrightarrow X$, we identify $(T\Omega)^{r,s}$ and $(T^*\Omega)^{r,s}$ with the sets $\Pi_{(TX)^{r,s}}^{-1}(\Omega) \subset (TX)^{r,s}$ and $\Pi_{(T^*X)^{r,s}}^{-1}(\Omega) \subset (T^*X)^{r,s}$, respectively, under the bijections $\iota_*: (T\Omega)^{r,s} \rightarrow \Pi_{(TX)^{r,s}}^{-1}(\Omega)$ and $\iota^*: \Pi_{(T^*X)^{r,s}}^{-1}(\Omega) \rightarrow (T^*\Omega)^{r,s}$, respectively.

Guided by Definition 9.4.5, we make the following definition:

Definition 2.4.3 Let X be a complex 1-manifold.

- (a) The *coefficient functions* (or simply the *coefficients*) of a vector field v on a set $S \subset X$ with respect to (or in) a local holomorphic coordinate neighborhood (U, z) are the functions $dz(v) = v(z)$ and $d\bar{z}(v) = v(\bar{z})$ on $S \cap U$.
- (b) We call a vector field v of type $(1, 0)$ on an open set $\Omega \subset X$ (that is, v_p is of type $(1, 0)$ for each point p) a *holomorphic vector field* if the coefficient function $f = dz(v) = v(z): \Omega \cap U \rightarrow \mathbb{C}$ in every local holomorphic coordinate neighborhood (U, z) is holomorphic (we have $d\bar{z}(v) = v(\bar{z}) = 0$, since v is of type $(1, 0)$); that is, $v = f \cdot (\partial/\partial z)$ on $\Omega \cap U$ for some function $f \in \mathcal{O}(\Omega \cap U)$.

Remark A vector field on a set $S \subset X$ is of class \mathcal{C}^k if and only if its coefficient functions with respect to every local holomorphic coordinate neighborhood (or equivalently, for each point $p \in S$, with respect to *some* local holomorphic coordinate neighborhood of p) are of class \mathcal{C}^k (see Exercise 2.4.1).

We close this section with the observation that the holomorphic inverse function theorem for domains in the plane (Theorem 1.5.1) gives the following analogue for Riemann surfaces:

Theorem 2.4.4 (Holomorphic inverse function theorem for Riemann surfaces) *Let $\Phi: X \rightarrow Y$ be a holomorphic mapping of Riemann surfaces.*

- (a) *If $p \in X$ and $(\Phi_*)_p \neq 0$, then Φ maps some neighborhood of p biholomorphically onto a neighborhood of $q \equiv \Phi(p)$. In particular, if f is a holomorphic function on a neighborhood of a point $p \in X$ and $(df)_p \neq 0$, then $(U, f|_U)$ is a local holomorphic coordinate neighborhood for some neighborhood U of p .*
- (b) *If Φ is one-to-one, then Φ maps X biholomorphically onto an open subset $\Phi(X)$ of Y .*

The proof is left to the reader (see Exercise 2.4.2).

Corollary 2.4.5 *Let f be a holomorphic function on a Riemann surface X , let r be a positive regular value (see Definition 9.4.6) of the function $\rho \equiv |f|$ ($|f|$ is of class \mathcal{C}^∞ on the complement of its zero set), and let $\Omega \equiv \{p \in X \mid |f(p)| < r\}$. Then every point $p \in \partial\Omega$ admits a local holomorphic coordinate neighborhood $(U, z = x + iy)$ in which*

$$\Omega \cap U = \{q \in U \mid x(q) < 0\}.$$

Remark In particular, Ω is a C^∞ open set (see Definition 9.7.14).

Proof We have $2\rho d\rho = d\rho^2 = \bar{f} df + f d\bar{f}$ on $X \setminus f^{-1}(0)$, and hence, given a point $p \in \partial\Omega$, we have $(df)_p \neq 0$. Therefore, by Theorem 2.4.4, there exists a local holomorphic chart of the form $(U, f|_U, U')$ with $p \in U$. Moreover, we may choose U and U' so small that there exists a holomorphic branch g of the logarithmic function on U' (see Example 1.6.2). Since g is a biholomorphism (with holomorphic inverse function $\zeta \mapsto e^\zeta$), we get a local holomorphic coordinate neighborhood

$$(U, z = x + iy \equiv -\log r + g \circ f)$$

in which $x = \operatorname{Re} z = \log |f/r| |_U$. In particular, $\Omega \cap U = \{q \in U \mid x(q) < 0\}$. \square

Exercises for Sect. 2.4

2.4.1 Let X be a Riemann surface.

- (a) For a vector field v on X and for $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, prove that the following are equivalent (cf. Exercise 9.4.2):
 - (i) The vector field v is of class \mathcal{C}^k .
 - (ii) The coefficients of v in every local holomorphic coordinate neighborhood are of class \mathcal{C}^k .
 - (iii) For every point in X , there exists a local holomorphic coordinate neighborhood with respect to which the coefficients of v are of class \mathcal{C}^k .
 - (iv) The $(1, 0)$ and $(0, 1)$ parts of v are of class \mathcal{C}^k .
- (b) For a vector field v of type $(1, 0)$ on X , prove that the following are equivalent:
 - (i) The vector field v is holomorphic.
 - (ii) The coefficient $dz(v)$ of v in some local holomorphic coordinate neighborhood (U, z) of each point in X is holomorphic.
 - (iii) For every holomorphic function f on an open set $U \subset X$, the function $df(v): p \mapsto df(v_p) = v_p(f)$ is holomorphic.
- (c) Prove that any sum of holomorphic vector fields, and any product of a holomorphic function and a holomorphic vector field, are holomorphic (in particular, the set of holomorphic vector fields on X is a complex vector space).

2.4.2 Prove Theorem 2.4.4.

2.4.3 Let X be a Riemann surface.

- (a) Prove that there is a unique 2-dimensional holomorphic structure on $(TX)^{1,0}$ such that for each local holomorphic chart $(U, \Phi = z, U')$ in X , the triple

$$(\Pi_{(TX)^{1,0}}^{-1}(U), (\Phi \circ \Pi_{(TX)^{1,0}}, dz), U' \times \mathbb{C})$$

is a local holomorphic chart in $(TX)^{1,0}$ (see Exercise 2.2.6 for the definition of a complex manifold, and cf. Exercise 9.4.3).

- (b) Prove that a vector field v of type $(1, 0)$ on X is holomorphic if and only if the mapping $v: X \rightarrow (TX)^{1,0}$ is holomorphic as a mapping of complex manifolds (cf. Exercise 9.4.5).
- (c) Prove that there is a unique 2-dimensional holomorphic structure on $(T^*X)^{1,0}$ such that for each local holomorphic chart $(U, \Phi = z, U')$ in X , the triple

$$(\Pi_{(T^*X)^{1,0}}^{-1}(U), \Psi, U' \times \mathbb{C}),$$

where the map $\Psi: \Pi_{(T^*X)^{1,0}}^{-1}(U) \rightarrow U' \times \mathbb{C}$ is given by $\alpha \mapsto (z(p), \alpha(\frac{\partial}{\partial z}))$ for each point $p \in U$ and each element $\alpha \in (T_p^*X)^{1,0}$, is a local holomorphic chart in $(T^*X)^{1,0}$.

- (d) Find (natural) \mathcal{C}^∞ structures in $(TX)^{0,1}$ and $(T^*X)^{0,1}$.

2.5 Differential Forms on a Riemann Surface

We recall that a differential form α of degree r on a subset S of a \mathcal{C}^∞ manifold M is a mapping of S into $\Lambda^r(T^*M)_\mathbb{C}$ with $\alpha_p \in \Lambda^r(T_p^*M)_\mathbb{C}$ for each point $p \in S$ (see Sect. 9.5). On a Riemann surface, the decomposition of the complexified tangent space into $(1, 0)$ and $(0, 1)$ parts induces a decomposition of differential forms.

Definition 2.5.1 Let X be a complex 1-manifold, and let $(r, s) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$.

- (a) We set

$$\Lambda^{r,s}T^*X \equiv \begin{cases} \Lambda^0(T^*X)_\mathbb{C} = X \times \mathbb{C} & \text{if } (r, s) = (0, 0), \\ (T^*X)^{r,s} & \text{if } (r, s) = (1, 0) \text{ or } (0, 1), \\ \Lambda^2(T^*X)_\mathbb{C} & \text{if } (r, s) = (1, 1), \\ X \times \{0\} & \text{if } r \geq 2 \text{ or } s \geq 2, \end{cases}$$

we let $\Pi_{\Lambda^{r,s}T^*X}: \Lambda^{r,s}T^*X \rightarrow X$ be the corresponding projection, and we let $\Lambda^{r,s}T_p^*X = \Pi_{\Lambda^{r,s}T^*X}^{-1}(p)$ for each point $p \in X$.

- (b) Elements of $\Lambda^{r,s}T^*X$ are said to be of *type* (r, s) . A differential form α on a set $S \subset X$ is of *type* (r, s) if $\alpha_p \in \Lambda^{r,s}T_p^*X$ for each point $p \in S$. We also call α a (*differential*) (r, s) -*form*.
- (c) For each open set $\Omega \subset X$, $\mathcal{E}^{r,s}(\Omega)$ denotes the vector space of \mathcal{C}^∞ differential forms of type (r, s) . The vector space of \mathcal{C}^∞ (r, s) -forms with compact support in Ω is denoted by $\mathcal{D}^{r,s}(\Omega)$.
- (d) Let α be a differential form of degree q on a set $S \subset X$. The *coefficient function(s)* (or simply the *coefficient(s)*) of α with respect to (or in) a local holomorphic coordinate neighborhood (U, z) is (are) the function(s) on $S \cap U$ given

by

$$\begin{cases} \alpha & \text{if } q = 0, \\ a \equiv \alpha(\partial/\partial z), \quad b \equiv \alpha(\partial/\partial \bar{z}) \quad (\text{i.e., } \alpha = a dz + b d\bar{z}) & \text{if } q = 1, \\ a \equiv \alpha(\partial/\partial z, \partial/\partial \bar{z}) = \frac{\alpha}{dz \wedge d\bar{z}} \quad (\text{i.e., } \alpha = a dz \wedge d\bar{z}) & \text{if } q = 2, \\ 0 & \text{if } q > 2. \end{cases}$$

Remarks 1. For each point $p \in X$, we have $\alpha \wedge \beta = \bar{\alpha} \wedge \bar{\beta} = 0$ for all $\alpha, \beta \in \Lambda^{1,0} T_p^* X$. Observe also that $\xi \wedge \zeta$ is of type $(r+t, s+u)$ if $\xi \in \Lambda^{r,s} T_p^* X$ and $\zeta \in \Lambda^{t,u} T_p^* X$.

2. On any local holomorphic coordinate neighborhood $(U, z = x + iy)$, we have $(i/2) dz \wedge d\bar{z} = dx \wedge dy$.

3. For all $r, s \in \mathbb{Z}_{\geq 0}$, conjugation gives $\overline{\Lambda^{r,s} T^* X} = \Lambda^{s,r} T^* X$; that is, $\bar{\alpha} \in \Lambda^{s,r} T^* X$ for each element $\alpha \in \Lambda^{r,s} T^* X$ (see Exercise 2.5.1).

4. Clearly, a differential form of type (r, s) is of degree $r + s$.

5. Exercises 9.5.4 and 2.4.3 provide a natural C^∞ structure on $\Lambda^{r,s} T^* X$.

Guided by Definition 9.5.2 and Definition 2.4.3, we make the following definition:

Definition 2.5.2 Let Ξ be an open subset of a complex 1-manifold X .

- (a) A *holomorphic 1-form* on Ξ is a differential form θ of type $(1, 0)$ on Ξ such that for every local holomorphic coordinate neighborhood (U, z) in X , the coefficient function $\theta(\partial/\partial z) = \theta/dz$ is holomorphic on $\Xi \cap U$; that is, for some function $f \in \mathcal{O}(\Xi \cap U)$, we have $\theta = f dz$ on $\Xi \cap U$. The vector space of holomorphic 1-forms on Ξ is denoted by $\Omega_X(\Xi)$ or $\Omega(\Xi)$.
- (b) A *meromorphic 1-form* on Ξ is a differential form θ of type $(1, 0)$ defined on the complement in Ξ of a discrete subset P such that for every local holomorphic coordinate neighborhood (U, z) in X , the coefficient function θ/dz is meromorphic on $\Xi \cap U$ with pole set $P \cap U$; that is, for some function $f \in \mathcal{M}(\Xi \cap U)$, we have $\theta = f dz$ on $\Xi \cap U \setminus f^{-1}(\infty) = \Xi \cap U \setminus P$ (we normally say simply $\theta = f dz$ on $\Xi \cap U$).
- (c) A holomorphic (meromorphic) function on Ξ is also called a *holomorphic* (respectively, *meromorphic*) 0-form.
- (d) A meromorphic 1-form θ on Ξ has a *zero* (a *pole*) of order ν at a point $p \in \Xi$ if for every local holomorphic coordinate neighborhood (U, z) of p , the meromorphic function θ/dz has a zero (respectively, a pole) of order ν at p . If $\nu = 1$, then we also say that θ has a *simple zero* (respectively, a *simple pole*) at p .

Remarks 1. A holomorphic 1-form is of class C^∞ (see Proposition 2.5.3 below).

2. For a nontrivial (i.e., not everywhere zero) meromorphic 1-form on a Riemann surface, the set of zeros (i.e., the set of points at which the 1-form is equal to 0) is discrete (see Exercise 2.5.2).

The proof of the following characterization of continuous, C^k , holomorphic, and meromorphic differential forms is left to the reader (see Exercise 2.5.3):

Proposition 2.5.3 (Cf. Proposition 9.5.3 and Definition 9.7.12) *Let α be a differential form of degree r defined at points of a subset S of a Riemann surface X , let $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, and let $d \in [1, \infty]$. Then:*

(a) *The following are equivalent:*

- (i) *The differential form α is continuous (of class C^k , measurable, in L_{loc}^d).*
- (ii) *The coefficients of α in every local holomorphic coordinate neighborhood are continuous (respectively, of class C^k , measurable, in L_{loc}^d).*
- (iii) *For every point in S , there exists a local holomorphic coordinate neighborhood with respect to which the coefficients of α are continuous (respectively, of class C^k , measurable, in L_{loc}^d).*

Moreover, for $r = 1$, α is continuous (of class C^k , measurable, in L_{loc}^d) if and only if the $(1, 0)$ and $(0, 1)$ parts of α are continuous (respectively, of class C^k , measurable, in L_{loc}^d).

- (b) *For S open and α of type $(1, 0)$, α is holomorphic if and only if for every point in S , there exists a local holomorphic coordinate neighborhood (U, z) with respect to which the coefficient $\alpha(\partial/\partial z) = \alpha/dz$ is holomorphic.*
- (c) *Suppose α is of type $(1, 0)$ and $S = \Xi \setminus P$ for some discrete set P in an open set $\Xi \subset X$. Then α is a meromorphic 1-form on Ξ with pole set P if and only if for each point $p \in \Xi$, there exists a local holomorphic coordinate neighborhood (U, z) of p such that the coefficient $\alpha(\partial/\partial z) = \alpha/dz$ is a meromorphic function on $\Xi \cap U$ with pole set $\Xi \cap U \cap P$. Moreover, if α is a meromorphic 1-form on Ξ , then α has a zero (a pole) of order v at a point $p \in \Xi$ if and only if there exists a local holomorphic coordinate neighborhood (U, z) of p such that the coefficient α/dz has a zero (respectively, a pole) of order v at p .*

The decomposition of $\Lambda^1(T^*X)_{\mathbb{C}}$ yields a decomposition of the exterior derivative operator d (see Definition 9.5.5):

Definition 2.5.4 Let X be a complex 1-manifold, let α be a differential form of degree r on an open subset Ω of X , and let $\mathcal{P}^{p,q} : \Lambda^1(T^*X)_{\mathbb{C}} \rightarrow \Lambda^{p,q}T^*X$ be the associated projection for each pair $(p, q) \in \{(1, 0), (0, 1)\}$.

(a) If α is of class C^1 , then we define

$$\partial\alpha = \begin{cases} \mathcal{P}^{1,0}(d\alpha) & \text{if } r = 0, \\ d(\mathcal{P}^{0,1}\alpha) & \text{if } r = 1, \\ 0 & \text{if } r \geq 2, \end{cases} \quad \text{and} \quad \bar{\partial}\alpha = \begin{cases} \mathcal{P}^{0,1}(d\alpha) & \text{if } r = 0, \\ d(\mathcal{P}^{1,0}\alpha) & \text{if } r = 1, \\ 0 & \text{if } r \geq 2. \end{cases}$$

(b) If α is of class C^1 and we have $\bar{\partial}\alpha = 0$ ($\partial\alpha = 0$), then we say that α is $\bar{\partial}$ -closed (respectively, ∂ -closed).

- (c) If $\alpha = \bar{\partial}\beta$ ($\alpha = \partial\beta$) for some C^1 differential form β of degree $r - 1$ on Ω (in particular, $r > 0$), then we say that α is $\bar{\partial}$ -exact (respectively, ∂ -exact). If β may be chosen to be of class C^k with $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then we also say that α is C^k $\bar{\partial}$ -exact (respectively, C^k ∂ -exact). If for each point in Ω , there exists a C^1 differential form β_0 of degree $r - 1$ on a neighborhood U_0 with $\bar{\partial}\beta_0 = \alpha|_{U_0}$ ($\partial\beta_0 = \alpha|_{U_0}$), then we say that α is *locally* $\bar{\partial}$ -exact (respectively, *locally* ∂ -exact). If for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, each of the local forms β_0 may be chosen to be of class C^k , then we also say that α is *locally* C^k $\bar{\partial}$ -exact (respectively, *locally* C^k ∂ -exact). It is also convenient to consider the trivial 0-form $\alpha \equiv 0$ to be C^∞ $\bar{\partial}$ -exact and C^∞ ∂ -exact, and to write $0 = \partial 0 = \bar{\partial} 0$.

Remark If α is a $\bar{\partial}$ -exact (∂ -exact) r -form on a Riemann surface X , then we may choose a C^1 differential form β with $\bar{\partial}\beta = \alpha$ (respectively, $\partial\beta = \alpha$). For $r = 1$, β is of type $(0, 0)$ and α is of type $(0, 1)$ (respectively, α is of type $(1, 0)$). For $r = 2$, α is of type $(1, 1)$ and we have $\bar{\partial}\mathcal{P}^{1,0}\beta = \bar{\partial}\beta = \alpha$ (respectively, $\partial\mathcal{P}^{0,1}\beta = \partial\beta = \alpha$). Thus, in either case ($r = 1$ or 2), α is of type $(p, q + 1)$ (respectively, of type $(p + 1, q)$) with $p + q + 1 = r$, and we may choose β to be of type (p, q) . Moreover, according to part (c) of Proposition 2.5.5 below, if α is of class C^k for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then *any* such β of type (p, q) is also of class C^k .

Proposition 2.5.5 *For any C^1 differential form α of degree r on an open subset Ω of a Riemann surface X , we have the following:*

- (a) *The exterior derivative satisfies $d\alpha = \partial\alpha + \bar{\partial}\alpha$, and if α is of class C^2 , then $d^2\alpha = \bar{\partial}^2\alpha = \partial^2\alpha = \partial\bar{\partial}\alpha + \bar{\partial}\partial\alpha = 0$; in other words,*

$$d = \partial + \bar{\partial} \text{ on } C^1 \text{ forms}$$

and

$$d^2 = \partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0 \text{ on } C^2 \text{ forms.}$$

We also have $\overline{\partial\alpha} = \bar{\partial}\bar{\alpha}$ and $\overline{\bar{\partial}\alpha} = \partial\bar{\alpha}$. Consequently, α is $\bar{\partial}$ -closed (∂ -closed) if and only if $\bar{\alpha}$ is ∂ -closed (respectively, $\bar{\partial}$ -closed); and if $\alpha = \bar{\partial}\beta$ ($\alpha = \partial\beta$) for some differential form β of class C^2 , then α is $\bar{\partial}$ -closed (respectively, ∂ -closed).

- (b) *If α is of type (p, q) , then $\partial\alpha$ is of type $(p + 1, q)$ and $\bar{\partial}\alpha$ is of type $(p, q + 1)$. In particular, $\partial\alpha = 0$ if $p \geq 1$ and $\bar{\partial}\alpha = 0$ if $q \geq 1$.*
- (c) *If α is $\bar{\partial}$ -exact (∂ -exact) and of class C^k for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is $\bar{\partial}$ -closed (respectively, ∂ -closed) and of type $(p, q + 1)$ (respectively, of type $(p + 1, q)$) for some pair of integers (p, q) . Moreover, there exists a C^1 differential form β of type (p, q) such that $\alpha = \bar{\partial}\beta$ (respectively, $\alpha = \partial\beta$), and any such form β is actually of class C^k .*
- (d) *Let (U, z) be a local holomorphic coordinate neighborhood in Ω . If $r = 0$, then on U ,*

$$d\alpha = \frac{\partial\alpha}{\partial z} dz + \frac{\partial\alpha}{\partial \bar{z}} d\bar{z}, \quad \partial\alpha = \frac{\partial\alpha}{\partial z} dz, \quad \text{and} \quad \bar{\partial}\alpha = \frac{\partial\alpha}{\partial \bar{z}} d\bar{z}.$$

If $r = 1$ and $\alpha = a dz + b d\bar{z}$ on U , then on U , we have

$$d\alpha = \left(\frac{\partial b}{\partial z} - \frac{\partial a}{\partial \bar{z}} \right) dz \wedge d\bar{z},$$

and

$$\partial\alpha = \frac{\partial b}{\partial z} dz \wedge d\bar{z} \quad \text{and} \quad \bar{\partial}\alpha = \frac{\partial a}{\partial \bar{z}} d\bar{z} \wedge dz = -\frac{\partial a}{\partial \bar{z}} dz \wedge d\bar{z}.$$

(e) For any C^1 differential form β on Ω , we have

$$\partial(\alpha \wedge \beta) = (\partial\alpha) \wedge \beta + (-1)^r \alpha \wedge \partial\beta$$

and

$$\bar{\partial}(\alpha \wedge \beta) = (\bar{\partial}\alpha) \wedge \beta + (-1)^r \alpha \wedge \bar{\partial}\beta.$$

- (f) If α is of type $(0, 0)$ or $(1, 0)$, then α is a holomorphic r -form if and only if $\bar{\partial}\alpha = 0$. If α is a holomorphic 0-form (i.e., a holomorphic function), then $\partial\alpha$ is a holomorphic 1-form.
- (g) If $\Phi: Y \rightarrow X$ is a holomorphic mapping of a Riemann surface Y into X , then $\partial\Phi^*\alpha = \Phi^*\partial\alpha$ and $\bar{\partial}\Phi^*\alpha = \Phi^*\bar{\partial}\alpha$ on $\Phi^{-1}(\Omega)$. In particular, if α is a holomorphic r -form, then $\Phi^*\alpha$ is also a holomorphic r -form. If α is a meromorphic r -form and $\Phi(Y)$ is not contained in the pole set of α , then $\Phi^*\alpha$ is a meromorphic r -form.

Proof Let (U, z) be a local holomorphic coordinate neighborhood in Ω . If $r = 0$, then, by (the remarks following) Proposition 2.4.2,

$$d\alpha = \frac{\partial\alpha}{\partial z} dz + \frac{\partial\alpha}{\partial \bar{z}} d\bar{z};$$

and since the first summand on the right-hand side is of type $(1, 0)$ and the second is of type $(0, 1)$, we get the expressions in (d) for $\partial\alpha$ and $\bar{\partial}\alpha$. If $r = 1$ and $\alpha = a dz + b d\bar{z}$ on U , then since $d^2 = 0$ (on class C^2 forms), $dz \wedge dz = d\bar{z} \wedge d\bar{z} = 0$, and $dz \wedge d\bar{z} = -d\bar{z} \wedge dz$, we have

$$d\alpha = d(\mathcal{P}^{0,1}\alpha + \mathcal{P}^{1,0}\alpha) = d(\mathcal{P}^{0,1}\alpha) + d(\mathcal{P}^{1,0}\alpha) = \partial\alpha + \bar{\partial}\alpha,$$

$$\begin{aligned} \partial\alpha &= d(\mathcal{P}^{0,1}\alpha) = d(b d\bar{z}) = db \wedge d\bar{z} \\ &= \left(\frac{\partial b}{\partial z} dz + \frac{\partial b}{\partial \bar{z}} d\bar{z} \right) \wedge d\bar{z} = \frac{\partial b}{\partial z} dz \wedge d\bar{z}, \end{aligned}$$

and

$$\begin{aligned} \bar{\partial}\alpha &= d(\mathcal{P}^{1,0}\alpha) = d(a dz) = da \wedge dz \\ &= \left(\frac{\partial a}{\partial z} dz + \frac{\partial a}{\partial \bar{z}} d\bar{z} \right) \wedge dz = -\frac{\partial a}{\partial \bar{z}} d\bar{z} \wedge dz. \end{aligned}$$

In particular, we get (d). The verifications of the remaining claims are left to the reader (see Exercise 2.5.5). \square

We have the following analogue of the Poincaré lemma (Lemma 9.5.7) for the operators ∂ and $\bar{\partial}$:

Lemma 2.5.6 (Dolbeault lemma) *Let p be a point in a Riemann surface X . Then there exists a neighborhood D of p in X such that for every $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, all $r, s \in \mathbb{Z}_{\geq 0}$, and every C^k differential form β of type $(r, s+1)$ (respectively, $(r+1, s)$) on a neighborhood of \bar{D} , there is a C^k differential form α of type (r, s) on D with $\bar{\partial}\alpha = \beta$ (respectively, $\partial\alpha = \beta$) on D . Consequently, every C^k differential form of type $(r, s+1)$ (of type $(r+1, s)$) is locally C^k $\bar{\partial}$ -exact (respectively, locally C^k ∂ -exact).*

Proof We may fix a local holomorphic coordinate neighborhood (U, z) of p in X and a neighborhood $D \Subset U$ such that $z(D)$ is a disk in \mathbb{C} . Suppose $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, $r, s \in \mathbb{Z}_{\geq 0}$, and β is a C^k differential form of type $(r, s+1)$ on a neighborhood of \bar{D} , which we may assume to be U . Clearly, $\beta \equiv 0 = \bar{\partial}0$ if $r > 1$ or $s > 0$, so we may also assume that $r = 0$ or 1 and that $s = 0$. Thus we have either $\beta = b d\bar{z}$ or $\beta = b dz \wedge d\bar{z}$ on U for some function $b \in C^k(U)$. By Lemma 1.2.2, there exists a function $a \in C^k(D)$ with $\partial a / \partial \bar{z} = b$ on D . Thus

$$\bar{\partial}a = b d\bar{z} \quad \text{and} \quad \bar{\partial}(-a dz) = b dz \wedge d\bar{z},$$

and it follows that $\beta = \bar{\partial}\alpha$ on D , where $\alpha = a$ if $r = 0$ and $\alpha = -a dz$ if $r = 1$.

If instead, we take β to be of type $(r+1, s)$, then $\bar{\beta}$ is of type $(s, r+1)$. Hence, by the above, there is a C^k differential form α of type (s, r) such that $\bar{\partial}\alpha = \bar{\beta}$ on D . Thus the C^k form $\bar{\alpha}$ is of type (r, s) and $\partial\bar{\alpha} = \bar{\partial}\alpha = \bar{\beta} = \beta$ on D . \square

We close this section with some remarks concerning integration on a Riemann surface. We first observe that line integrals (Definition 9.7.18) may be expressed in terms of local holomorphic coordinates. For example, if (U, z) is a local holomorphic coordinate neighborhood in a complex 1-manifold X , $\alpha = f dz + g d\bar{z}$ is a continuous differential form of degree 1 on U , and $\gamma: [a, b] \rightarrow U$ is a piecewise C^1 path, then

$$\int_{\gamma} \alpha = \int_a^b \left(f(\gamma(t)) \frac{d}{dt} [z(\gamma(t))] + g(\gamma(t)) \frac{d}{dt} [\overline{z(\gamma(t))}] \right) dt.$$

For integration of 2-forms, observe that there is a natural orientation (see Sect. 9.7) in a complex 1-manifold X determined by the local holomorphic charts. For if $(U, \Phi = z = x + iy \leftrightarrow (x, y), U')$ and $(V, \Psi = w = u + iv \leftrightarrow (u, v), V')$ are two local holomorphic charts, then on $U \cap V$,

$$\mathcal{J}_{\Psi \circ \Phi^{-1}} \circ \Phi = \frac{du \wedge dv}{dx \wedge dy} = \frac{dw \wedge d\bar{w}}{dz \wedge d\bar{z}} = \left| \frac{\partial w}{\partial z} \right|^2 > 0.$$

In particular, $(i/2) dz \wedge d\bar{z} = dx \wedge dy$ is a positive $\mathcal{C}^\infty (1, 1)$ -form (i.e., a positive \mathcal{C}^∞ 2-form) on U . A positive $\mathcal{C}^\infty (1, 1)$ -form ω on X is also called a *Kähler form*. By part (g) of Proposition 9.7.9, for any nonnegative measurable $(1, 1)$ -form α defined on a measurable set $S \subset X$, we have

$$\int_S \alpha = \sup \sum_{j=1}^m \int_{S_j} \alpha,$$

where the supremum is taken over all choices of disjoint measurable subsets S_1, \dots, S_m of S each of which is contained in a local holomorphic coordinate neighborhood.

Remark By definition, a positive $\mathcal{C}^\infty (1, 1)$ -form on a complex manifold of arbitrary dimension is called a *Kähler form* if it is d -closed. This condition holds automatically in complex dimension 1, since every 2-form on a \mathcal{C}^∞ manifold of real dimension 2 is closed.

We make the following observation:

Proposition 2.5.7 (Cf. Exercise 2.5.8 and Exercise 6.7.1) *For a nontrivial meromorphic function f (i.e., f is not everywhere zero) on a compact Riemann surface X , counting multiplicities, the number of zeros is equal to the number of poles. More precisely, if Z is the set of zeros, P is the set of poles, μ_p is the order of the zero at each point $p \in Z$, and ν_p is the order of the pole at each point $p \in P$, then*

$$\sum_{p \in Z \cup P} \text{ord}_p f = \sum_{p \in Z} \mu_p - \sum_{p \in P} \nu_p = 0.$$

In particular, if f has exactly one (simple) pole, then f maps X biholomorphically onto \mathbb{P}^1 .

Proof For $r > 0$ sufficiently small, the local representation of meromorphic functions provided by Proposition 2.2.6 allows us to fix disjoint local holomorphic charts $\{(U_p, \Phi_p = z_p, \Delta(0; 2r))\}_{p \in Z \cup P}$ in X such that for each point $p \in Z \cup P$, we have $z_p(p) = 0$ and on U_p ,

$$f = \begin{cases} z_p^{\mu_p} & \text{if } p \in Z, \\ z_p^{-\nu_p} & \text{if } p \in P. \end{cases}$$

Thus the meromorphic 1-form $\theta \equiv df/f$ satisfies, on U_p ,

$$\theta = \begin{cases} \mu_p \cdot z_p^{-1} \cdot dz_p & \text{if } p \in Z, \\ -\nu_p \cdot z_p^{-1} \cdot dz_p & \text{if } p \in P. \end{cases}$$

Stokes' theorem now gives

$$\begin{aligned} \sum_{p \in Z} 2\pi i \mu_p - \sum_{p \in P} 2\pi i \nu_p &= \sum_{p \in Z \cup P} \int_{\partial \Phi_p^{-1}(\Delta(0;r))} \theta \\ &= - \int_{X \setminus \bigcup_{p \in Z \cup P} \Phi_p^{-1}(\overline{\Delta(0;r)})} d\theta = 0. \end{aligned}$$

In particular, if f is holomorphic except for a simple pole at $p \in X$ (i.e., $P = \{p\}$ and $\nu_p = 1$), then $f^{-1}(\infty) = \{p\}$, and for each point $q \in X \setminus \{p\}$, the meromorphic function $f - f(q)$ is nonvanishing except for a simple zero at q . Thus $f: X \rightarrow \mathbb{P}^1$ is injective, and by the holomorphic inverse function theorem (Theorem 2.4.4), f maps X biholomorphically onto an open subset of \mathbb{P}^1 . On the other hand, $f(X)$ is compact, hence closed, so we must have $f(X) = \mathbb{P}^1$. \square

Exercises for Sect. 2.5

- 2.5.1 Verify that for all $r, s \in \mathbb{Z}_{\geq 0}$, conjugation gives $\overline{\Lambda^{r,s} T^* X} = \Lambda^{s,r} T^* X$ on a Riemann surface X .
- 2.5.2 Let θ be a nontrivial meromorphic 1-form on a Riemann surface X . Verify that $\{x \in X \mid \theta_x = 0\}$ is discrete.
- 2.5.3 Prove Proposition 2.5.3.
- 2.5.4 Prove that \mathbb{P}^1 does not admit any nontrivial holomorphic 1-forms.
- 2.5.5 Prove parts (a)–(c) and (e)–(g) of Proposition 2.5.5.
- 2.5.6 Let α be a differential form of type $(1, 0)$ on a Riemann surface X . Prove that α is a holomorphic 1-form if and only if α is holomorphic as a mapping of X into the 2-dimensional complex manifold $(T^*X)^{1,0}$ (see Exercise 2.4.3 for a description of the natural holomorphic structure on $(T^*X)^{1,0}$).
- 2.5.7 Let X be a complex torus (Example 2.1.6). Prove that the vector space of holomorphic 1-forms on X is 1-dimensional.
- 2.5.8 Let Ω be a (nonempty) smooth relatively compact domain in a Riemann surface X , and for any given nontrivial meromorphic function h on X that does not have any zeros or poles in $\partial\Omega$, let μ_h denote the number of zeros of h in Ω , and ν_h the number of poles of h in Ω , counting multiplicities.
- (a) Prove the following version of the *argument principle* (cf. Proposition 2.5.7 and Exercises 5.1.6, 5.1.7, 5.9.5, 6.7.1, and 6.7.6): If f is a nontrivial meromorphic function on X that does not have any zeros or poles in $\partial\Omega$, then

$$\mu_f - \nu_f = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{df}{f}.$$

- (b) Prove the following version of *Rouché's theorem*: If f and g are nontrivial meromorphic functions on X that do not have any zeros or poles in $\partial\Omega$, and $|g| < |f|$ on $\partial\Omega$, then $\mu_f - \nu_f = \mu_{f+g} - \nu_{f+g}$.

Hint. Using part (a), show that the integer-valued function $t \mapsto \mu_{f+tg} - \nu_{f+tg}$ is continuous on the interval $[0, 1]$.

2.5.9 If X is a complex 1-manifold, $p \in X$, θ is a holomorphic 1-form on $V \setminus \{p\}$ for some neighborhood V of p in X , $(U, \Phi = z, \Delta(z_0; R))$ is a local holomorphic coordinate neighborhood of p with $\Phi(p) = z_0$ and $U \subset V$, $f = \theta/dz$ (i.e., $\theta = f dz$ on $U \setminus \{p\}$), and $f = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n$ is the corresponding Laurent series representation of f , then the coefficient c_{-1} is called the *residue* of θ at p and is denoted by $\text{res}_p \theta$. Equivalently, if $r \in (0, R)$ and $D \equiv \Phi^{-1}(\Delta(0; r))$, then

$$\text{res}_p \theta = \frac{1}{2\pi i} \int_{\partial D} \theta.$$

- (a) Prove that the residue of a holomorphic 1-form at an isolated singularity is well defined (that is, prove that the above definition is independent of the choice of the local holomorphic coordinate).
- (b) Prove the following version of the *residue theorem* (cf. Exercises 5.1.6, 5.1.7, 5.9.5, 6.7.1, and 6.7.6): If Ω is a (nonempty) smooth relatively compact domain in a Riemann surface X , S is a finite subset of Ω , and θ is a holomorphic 1-form on $X \setminus S$, then

$$\frac{1}{2\pi i} \int_{\partial \Omega} \theta = \sum_{p \in S} \text{res}_p \theta.$$

In particular, if X is a compact Riemann surface, S is a finite subset of X , and θ is a holomorphic 1-form on $X \setminus S$, then $\sum_{p \in S} \text{res}_p \theta = 0$.

2.6 L^2 Scalar-Valued Differential Forms on a Riemann Surface

Throughout this section, X denotes a complex 1-manifold. The goal of this section is the development of a suitable L^2 space of differential forms. Since the objects that we integrate on oriented surfaces are the 2-forms (see Sect. 9.7) and we wish to pair forms of the same degree and integrate in order to get an inner product, it is natural to consider a pointwise pairing that gives a 2-form. Let us first consider integration on \mathbb{C} . We have the natural unit-length $(1, 0)$ -form $\theta = (1/\sqrt{2}) dz$ and the natural positive $(1, 1)$ -form $\omega = \sqrt{-1} \theta \wedge \bar{\theta} = (\sqrt{-1}/2) dz \wedge d\bar{z}$. In the notation of Sect. 9.7, the Lebesgue measure λ in \mathbb{C} is equal to the measure λ_ω associated to ω (see Definition 9.7.10). If $\alpha = a\theta$ and $\beta = b\bar{\theta}$ are differential forms of type $(1, 0)$ with L^2 coefficients, then

$$\langle \alpha, \beta \rangle_{L^2(\mathbb{C})} = \int_{\mathbb{C}} a \bar{b} d\lambda = \int_{\mathbb{C}} a \bar{b} \omega = \int_{\mathbb{C}} a \bar{b} i \theta \wedge \bar{\theta} = \int_{\mathbb{C}} i \alpha \wedge \bar{\beta}.$$

If α and β are L^2 differential forms of type $(0, 0)$ (i.e., L^2 functions), then

$$\langle \alpha, \beta \rangle_{L^2(\mathbb{C})} = \int_{\mathbb{C}} \alpha \bar{\beta} d\lambda = \int_{\mathbb{C}} \alpha \bar{\beta} \omega.$$

Finally, if $\alpha = a\omega$ and $\beta = b\omega$ are differential forms of type $(1, 1)$ with L^2 coefficients, then

$$\langle a, b \rangle_{L^2(\mathbb{C})} = \int_{\mathbb{C}} a \bar{b} d\lambda = \int_{\mathbb{C}} a \bar{b} \omega = \int_{\mathbb{C}} \frac{\alpha}{\omega} \cdot \frac{\bar{\beta}}{\omega} \cdot \omega.$$

The right-hand sides in the above point to a reasonable definition for the L^2 inner product of differential forms on the complex 1-manifold X . Observe that the expression for the inner product of a pair of forms of type $(0, 0)$ and $(1, 1)$ involves the Kähler form ω , although that for the inner product of a pair of $(1, 0)$ -forms does *not*. Thus, on X , for forms of type $(0, 0)$ and $(1, 1)$, we must define the inner product with respect to some choice of a positive $(1, 1)$ -form. For functions, it turns out to be useful to weaken the requirement on the $(1, 1)$ -form by allowing it to be only nonnegative. After all, it is not even a priori clear that X admits a global Kähler form; although, as will be shown in Sect. 2.11, it turns out that every Riemann surface is second countable and therefore that X *does* admit a Kähler form (see Corollary 2.11.3). As will be seen in Sect. 2.9, it is also useful to include a *weight function* $e^{-\varphi}$ (in an abuse of language, we also call φ a *weight function*). For example, a Riemann surface need not admit any L^2 holomorphic 1-forms, but given a holomorphic 1-form, one may construct a C^∞ function φ that grows so large at infinity that the holomorphic 1-form becomes L^2 with respect to the weight function $e^{-\varphi}$. Finally, observe that $i\alpha \wedge \bar{\alpha} > 0$ for every nonzero $\alpha \in \Lambda^{1,0}T^*X$. Based on these considerations, we make the following definition:

Definition 2.6.1 Let S be a measurable subset of X , let φ be a measurable real-valued function that is defined on S , and let α and β be measurable differential forms of type (p, q) that are defined on S .

(a) For $(p, q) = (1, 0)$, we define

$$\|\alpha\|_{L^2_{1,0}(S, \varphi)} \equiv \left[\int_S \sqrt{-1} \alpha \wedge \bar{\alpha} \cdot e^{-\varphi} \right]^{1/2} \in [0, \infty].$$

If $\sqrt{-1} \alpha \wedge \bar{\beta} \cdot e^{-\varphi}$ is integrable on S , then we define

$$\langle \alpha, \beta \rangle_{L^2_{1,0}(S, \varphi)} \equiv \int_S \sqrt{-1} \alpha \wedge \bar{\beta} \cdot e^{-\varphi} \in \mathbb{C}.$$

(b) For $(p, q) = (0, 1)$, we define

$$\|\alpha\|_{L^2_{0,1}(S, \varphi)} \equiv \left[- \int_S \sqrt{-1} \alpha \wedge \bar{\alpha} \cdot e^{-\varphi} \right]^{1/2} \in [0, \infty].$$

If $-\sqrt{-1} \alpha \wedge \bar{\beta} \cdot e^{-\varphi}$ is integrable on S , then we define

$$\langle \alpha, \beta \rangle_{L^2_{0,1}(S, \varphi)} \equiv - \int_S \sqrt{-1} \alpha \wedge \bar{\beta} \cdot e^{-\varphi} \in \mathbb{C}.$$

- (c) If $(p, q) = (0, 0)$ and ω is a *nonnegative* measurable form of type $(1, 1)$ defined on S , then we define

$$\|\alpha\|_{L^2_{0,0}(S,\omega,\varphi)} \equiv \left[\int_S |\alpha|^2 e^{-\varphi} \omega \right]^{1/2} \in [0, \infty].$$

If $\alpha \bar{\beta} \cdot e^{-\varphi} \omega$ is integrable on S , then we define

$$\langle \alpha, \beta \rangle_{L^2_{0,0}(S,\omega,\varphi)} \equiv \int_S \alpha \bar{\beta} \cdot e^{-\varphi} \omega \in \mathbb{C}.$$

- (d) If $(p, q) = (1, 1)$ and ω is a *positive* measurable form of type $(1, 1)$ defined on S , then we define

$$\|\alpha\|_{L^2_{1,1}(S,\omega,\varphi)} \equiv \left[\int_S \left| \frac{\alpha}{\omega} \right|^2 e^{-\varphi} \omega \right]^{1/2} = \|\alpha/\omega\|_{L^2_{0,0}(S,\omega,\varphi)} \in [0, \infty].$$

If the form

$$\frac{\alpha}{\omega} \cdot \frac{\bar{\beta}}{\omega} \cdot e^{-\varphi} \omega$$

is integrable on S , then we define

$$\langle \alpha, \beta \rangle_{L^2_{1,1}(S,\omega,\varphi)} \equiv \int_S \frac{\alpha}{\omega} \cdot \frac{\bar{\beta}}{\omega} \cdot e^{-\varphi} \omega = \left\langle \frac{\alpha}{\omega}, \frac{\beta}{\omega} \right\rangle_{L^2_{0,0}(S,\omega,\varphi)} \in \mathbb{C}.$$

- (e) When there is no danger of confusion (for example, when the choice of (p, q) , S , ω , or φ is understood from the context), we will suppress parts of the notation. For example, we will often write $\|\alpha\|_{L^2_{p,q}(S,\omega,\varphi)}$ simply as $\|\alpha\|_{S,\omega,\varphi}$, $\|\alpha\|_{S,\varphi}$, $\|\alpha\|_{S,\omega}$, $\|\alpha\|_{\omega,\varphi}$, $\|\alpha\|_{\varphi}$, $\|\alpha\|_{\omega}$, $\|\alpha\|_{L^2(S,\omega,\varphi)}$, $\|\alpha\|_{L^2(S,\varphi)}$, $\|\alpha\|_{L^2(\omega,\varphi)}$, $\|\alpha\|_{L^2(\varphi)}$, $\|\alpha\|_{L^2(\omega)}$, or $\|\alpha\|$. We will also use the analogous simplified notation for $\langle \alpha, \beta \rangle_{L^2_{p,q}(S,\omega,\varphi)}$, for $\|\alpha\|_{L^2_{p,q}(S,\varphi)}$, and for $\langle \alpha, \beta \rangle_{L^2_{p,q}(S,\varphi)}$. Moreover, when no mention of a weight function appears in the context, then the weight function will be assumed to be $\varphi \equiv 0$ and we will write $\|\alpha\|_{L^2_{p,q}(S,\omega,0)}$ simply as $\|\alpha\|_{L^2_{p,q}(S,\omega)}$ and so on.
- (f) Let γ and η be measurable differential 1-forms defined on S with (r, s) parts $\gamma^{r,s}$ and $\eta^{r,s}$, respectively, for each $(r, s) \in \{(1, 0), (0, 1)\}$. Then we set

$$\|\gamma\|_{L^2_1(S,\varphi)}^2 \equiv \|\gamma^{1,0}\|_{\varphi}^2 + \|\gamma^{0,1}\|_{\varphi}^2.$$

We also set $\langle \gamma, \eta \rangle_{L^2_1(S,\varphi)} \equiv \langle \gamma^{1,0}, \eta^{1,0} \rangle_{\varphi} + \langle \gamma^{0,1}, \eta^{0,1} \rangle_{\varphi}$, provided each of the summands on the right-hand side is defined. We also use the simplified notation analogous to that appearing in (e).

Definition 2.6.2 Let S be a measurable subset of X , and let φ be a measurable real-valued function that is defined on S .

- (a) For $(p, q) = (1, 0)$ or $(0, 1)$, $L_{p,q}^2(S, \varphi)$ consists of all equivalence classes of measurable differential forms α of type (p, q) on S with $\|\alpha\|_{L^2(S, \varphi)} < \infty$, where we identify any two elements that are equal almost everywhere. The set $L_1^2(S, \varphi)$ consists of all equivalence classes of measurable 1-forms α with $\|\alpha\|_{L^2(S, \varphi)} < \infty$, where again, we identify any two elements that are equal almost everywhere.
- (b) For ω a *nonnegative* measurable differential form of type $(1, 1)$ that is defined on S , $L_{0,0}^2(S, \omega, \varphi)$ consists of all equivalence classes of measurable functions α on S with $\|\alpha\|_{L^2(S, \omega, \varphi)} < \infty$, where we identify any two elements that are equal almost everywhere.
- (c) For ω a *positive* measurable differential form of type $(1, 1)$ that is defined on S , $L_{1,1}^2(S, \omega, \varphi)$ consists of all equivalence classes of measurable differential forms α of type $(1, 1)$ on S with $\|\alpha\|_{L^2(S, \omega, \varphi)} < \infty$, where we identify any two elements that are equal almost everywhere.
- (d) When no mention of a weight function appears in the context and there is no danger of confusion, then the weight function will be assumed to be $\varphi \equiv 0$ and we will write $L_{p,q}^2(S, 0)$ simply as $L_{p,q}^2(S)$, and $L_{p,q}^2(S, \omega, 0)$ simply as $L_{p,q}^2(S, \omega)$.

Remark For our purposes, we will need only weight functions φ and nonnegative (or positive) $(1, 1)$ -forms ω that are defined and of class C^∞ on an open subset of X .

Proposition 2.6.3 *Let S be a measurable subset of X , and let φ be a continuous real-valued function that is defined on S .*

- (a) *The pair $(L_1^2(S, \varphi), \langle \cdot, \cdot \rangle_{L^2(S, \varphi)})$, and the pair $(L_{p,q}^2(S, \varphi), \langle \cdot, \cdot \rangle_{L^2(S, \varphi)})$ for $(p, q) = (1, 0)$ or $(0, 1)$, are Hilbert spaces (where the inner product of any two equivalence classes is given by the pairing of any representatives). Moreover, we have the Hilbert space orthogonal decomposition*

$$L_1^2(S, \varphi) = L_{1,0}^2(S, \varphi) \oplus L_{0,1}^2(S, \varphi).$$

- (b) *For ω a continuous positive differential form of type $(1, 1)$ that is defined on S and for $(p, q) = (0, 0)$ or $(1, 1)$, $(L_{p,q}^2(S, \omega, \varphi), \langle \cdot, \cdot \rangle_{L^2(S, \omega, \varphi)})$ is a Hilbert space (where the inner product of any two equivalence classes is given by the pairing of any representatives).*

Moreover, in each of the above spaces, any sequence converging to an element α admits a subsequence that converges to α pointwise almost everywhere in S .

Proof Let $p, q \in \{0, 1\}$. We set

$$L_{p,q}^2 \equiv \begin{cases} L_{p,q}^2(S, \varphi) \text{ as in (a)} & \text{if } (p, q) = (1, 0) \text{ or } (0, 1), \\ L_{p,q}^2(S, \omega, \varphi) \text{ as in (b)} & \text{if } (p, q) = (0, 0) \text{ or } (1, 1); \end{cases}$$

and for each point $r \in S$ and each pair of elements $\eta, \theta \in \Lambda^{p,q} T_r^* X$, we set

$$H(\eta, \theta) \equiv \begin{cases} \eta \bar{\theta} \cdot e^{-\varphi(r)} \cdot \omega_r & \text{if } (p, q) = (0, 0), \\ i\eta \wedge \bar{\theta} \cdot e^{-\varphi(r)} & \text{if } (p, q) = (1, 0), \\ -i\eta \wedge \bar{\theta} \cdot e^{-\varphi(r)} & \text{if } (p, q) = (0, 1), \\ \frac{\eta}{\omega_r} \cdot \frac{\bar{\theta}}{\omega_r} \cdot e^{-\varphi(r)} \omega_r & \text{if } (p, q) = (1, 1). \end{cases}$$

We first show that $(L_{p,q}^2, \langle \cdot, \cdot \rangle)$ is an inner product space. For this, observe that if $\alpha, \beta \in L_{p,q}^2$, then the $(1, 1)$ -form $H(\alpha, \beta): r \mapsto H(\alpha_r, \beta_r)$ is measurable. Suppose $(U, \Phi = z = x + iy, U')$ is a local holomorphic chart and $\omega_\Phi \equiv (i/2) dz \wedge d\bar{z} = dx \wedge dy$. Then the map

$$h: (\eta, \theta) \mapsto h(\eta, \theta) \equiv \frac{H(\eta, \theta)}{(\omega_\Phi)_r} \in \mathbb{C} \quad \forall \eta, \theta \in \Lambda^{p,q} T_r^* X$$

is a Hermitian inner product in $\Lambda^{p,q} T_r^* X$ for each point $r \in S \cap U$. We also have the corresponding pointwise norm $\eta \mapsto |\eta|_h = [h(\eta, \eta)]^{1/2}$. Hence, for each measurable set $R \subset S \cap U$ and each complex number ζ with $|\zeta| = 1$, we have (by the Schwarz inequality)

$$\begin{aligned} \int_R [\operatorname{Re}(\zeta H(\alpha, \beta))]^+ &= \int_R [\operatorname{Re}(\zeta h(\alpha, \beta))]^+ d\lambda_{\omega_\Phi} \leq \int_R |\alpha|_h |\beta|_h d\lambda_{\omega_\Phi} \\ &\leq \left[\int_R |\alpha|_h^2 d\lambda_{\omega_\Phi} \right]^{1/2} \left[\int_R |\beta|_h^2 d\lambda_{\omega_\Phi} \right]^{1/2} \\ &= \left[\int_R H(\alpha, \alpha) \right]^{1/2} \cdot \left[\int_R H(\beta, \beta) \right]^{1/2} \end{aligned}$$

(where λ_{ω_Φ} is the positive measure associated to ω_Φ as in Definition 9.7.10). Thus, if S_1, \dots, S_m are disjoint measurable subsets of S each of which lies in some local holomorphic coordinate neighborhood, then, by the Schwarz inequality for sums,

$$\begin{aligned} \sum_{j=1}^m \int_{S_j} [\operatorname{Re}(\zeta H(\alpha, \beta))]^+ &\leq \sum_{j=1}^m \left[\int_{S_j} H(\alpha, \alpha) \right]^{1/2} \cdot \left[\int_{S_j} H(\beta, \beta) \right]^{1/2} \\ &\leq \left[\sum_{j=1}^m \int_{S_j} H(\alpha, \alpha) \right]^{1/2} \cdot \left[\sum_{j=1}^m \int_{S_j} H(\beta, \beta) \right]^{1/2} \\ &\leq \left[\int_S H(\alpha, \alpha) \right]^{1/2} \cdot \left[\int_S H(\beta, \beta) \right]^{1/2} = \|\alpha\| \cdot \|\beta\|. \end{aligned}$$

Passing to the supremum, we get

$$\int_S [\operatorname{Re}(\zeta H(\alpha, \beta))]^+ \leq \|\alpha\| \cdot \|\beta\| < \infty.$$

Taking $\zeta = \pm 1, \pm i$, we see that $H(\alpha, \beta)$ is integrable and therefore that $\langle \alpha, \beta \rangle = \int_S H(\alpha, \beta)$ is defined. Furthermore, choosing $\zeta \in \mathbb{C}$ so that $|\zeta| = 1$ and

$$\zeta \cdot \langle \alpha, \beta \rangle = |\langle \alpha, \beta \rangle|,$$

we get

$$\begin{aligned} |\langle \alpha, \beta \rangle| &= \int_S \zeta H(\alpha, \beta) = \int_S [\operatorname{Re}(\zeta H(\alpha, \beta))]^+ - \int_S [\operatorname{Re}(\zeta H(\alpha, \beta))]^- \\ &\leq \int_S [\operatorname{Re}(\zeta H(\alpha, \beta))]^+ \leq \|\alpha\| \cdot \|\beta\|. \end{aligned}$$

For each constant $s \in \mathbb{C}$, we have

$$H(s\alpha + \beta, s\alpha + \beta) = |s|^2 H(\alpha, \alpha) + 2 \operatorname{Re}(s H(\alpha, \beta)) + H(\beta, \beta),$$

so $H(s\alpha + \beta, s\alpha + \beta)$ is integrable. If $\alpha = \alpha'$ a.e. (almost everywhere) in S , then clearly, $H(\alpha', \alpha') = H(\alpha, \alpha)$ and $H(\alpha', \beta) = H(\alpha, \beta)$ a.e. Thus $L^2_{p,q}$ is a well-defined vector space, and as is now easy to verify, $\langle \cdot, \cdot \rangle$ is a well-defined Hermitian inner product.

It remains to show that $(L^2_{p,q}, \langle \cdot, \cdot \rangle)$ is complete with respect to the norm $\|\alpha\| = \langle \alpha, \alpha \rangle^{1/2}$. We again apply (appropriately modified) standard arguments (cf., for example, [Rud1]). We must show that a given Cauchy sequence $\{\alpha_v\}$ in $L^2_{p,q}$ converges. For this, it suffices to show that some subsequence converges. Hence, after replacing the sequence with a suitable subsequence, we may assume without loss of generality that $\sum \|\alpha_{v+1} - \alpha_v\| < 1$. For a local holomorphic chart $(U, \Phi = z, U')$, for $\omega_\Phi = dx \wedge dy = (i/2) dz \wedge d\bar{z}$ and for $h(\cdot, \cdot) = H(\cdot, \cdot)/\omega_\Phi$, as before, let us set

$$\varphi_N \equiv \sum_{v=1}^N |\alpha_{v+1} - \alpha_v|_h \quad \forall N \in \mathbb{Z}_{>0} \quad \text{and} \quad \varphi \equiv \sum_{v=1}^{\infty} |\alpha_{v+1} - \alpha_v|_h.$$

For each $N \in \mathbb{Z}_{>0}$, we have

$$\|\varphi_N\|_{L^2(S \cap U, \lambda_{\omega_\Phi})} \leq \sum_{v=1}^N \|\alpha_{v+1} - \alpha_v\|_{L^2(S \cap U, \lambda_{\omega_\Phi})} \leq \sum_{v=1}^N \|\alpha_{v+1} - \alpha_v\| < 1.$$

Fatou's lemma now implies that $\|\varphi\|_{L^2(S \cap U, \lambda_{\omega_\Phi})} \leq 1$. In particular, $\varphi < \infty$ almost everywhere in $S \cap U$. It follows that the series

$$\alpha_1 + \sum_{v=1}^{\infty} (\alpha_{v+1} - \alpha_v)$$

converges pointwise almost everywhere in S to a measurable differential form α of type (p, q) , and hence $\alpha_v \rightarrow \alpha$ pointwise a.e.

It remains to show that $\alpha \in L^2_{p,q}$ and that $\|\alpha - \alpha_v\| \rightarrow 0$. Given $\epsilon > 0$, we may choose $N \in \mathbb{Z}_{>0}$ so that $\|\alpha_\mu - \alpha_v\| < \epsilon$ for all $\mu, v > N$. For any $v > N$, Fatou's lemma (Theorem 9.7.11) gives

$$\|\alpha - \alpha_v\|^2 = \int_S H(\alpha - \alpha_v, \alpha - \alpha_v) \leq \liminf_{\mu \rightarrow \infty} \int_S H(\alpha_\mu - \alpha_v, \alpha_\mu - \alpha_v) \leq \epsilon^2.$$

Thus $\alpha = (\alpha - \alpha_v) + \alpha_v \in L^2_{p,q}$ and $\|\alpha - \alpha_v\| \leq \epsilon$. Therefore $\alpha_v \rightarrow \alpha$ in $L^2_{p,q}$ (and pointwise almost everywhere in S) and $L^2_{p,q}$ is a Hilbert space. The proposition (including the orthogonal decomposition in (a)) now follows. \square

We have the following useful version of Theorem 1.2.4:

Theorem 2.6.4 *Let φ be a continuous real-valued function and let ω be a continuous positive differential form of type $(1, 1)$ on X . Then, for every compact set $K \subset X$, there is a constant $C = C(X, K, \omega, \varphi) > 0$ such that*

$$\max_K |f| \leq C \|f\|_{L^2(X, \omega, \varphi)} \quad \forall f \in \mathcal{O}(X).$$

Consequently, $\mathcal{O}(X) \cap L^2_{0,0}(X, \omega, \varphi)$ is a closed subspace of $L^2_{0,0}(X, \omega, \varphi)$.

Proof Given a point $p \in K$, we may fix a relatively compact local holomorphic coordinate neighborhood (U, z) on which the function $(i/2)(dz \wedge d\bar{z})/\omega$ is bounded. Therefore, for some constant $R > 0$, we have, for every holomorphic function f on U ,

$$\int_U |f|^2 \frac{i}{2} dz \wedge d\bar{z} = \int_U |f|^2 e^\varphi \frac{(i/2) dz \wedge d\bar{z}}{\omega} \cdot e^{-\varphi} \cdot \omega \leq R \|f\|_{L^2(U, \omega, \varphi)}^2.$$

Fixing a relatively compact neighborhood V_p of p in U and applying Theorem 1.2.4, we get a constant $C_p > 0$ such that

$$\sup_{V_p} |f| \leq C_p \|f\|_{L^2(U, \omega, \varphi)} \leq C_p \|f\|_{L^2(X, \omega, \varphi)} \quad \forall f \in \mathcal{O}(X).$$

Covering K by finitely many such neighborhoods V_p and taking C to be the maximum of the associated constants C_p , we get the claim. The proof that $\mathcal{O}(X) \cap L^2_{0,0}(X, \omega, \varphi)$ is a closed subspace is left to the reader (see Exercise 2.6.3). \square

Exercises for Sect. 2.6

2.6.1 Show that there are no nontrivial L^2 holomorphic 1-forms on \mathbb{C} (cf. Exercise 2.5.4).

2.6.2 For the function $\varphi: z \mapsto |z|^2$ on \mathbb{C} , show that there exists a nontrivial holomorphic 1-form in $L^2_{1,0}(\mathbb{C}, \varphi)$.

2.6.3 Let φ be a continuous real-valued function on a Riemann surface X . Prove that the vector space of holomorphic 1-forms in $L^2_{1,0}(X, \varphi)$ is a closed subspace of $L^2_{1,0}(X, \varphi)$. Also prove that if ω is a continuous positive $(1, 1)$ -form on X , then the vector space of holomorphic functions in $L^2_{0,0}(X, \omega, \varphi)$ is a closed subspace (i.e., prove the second part of Theorem 2.6.4).

2.7 The Distributional $\bar{\partial}$ Operator on Scalar-Valued Forms

Throughout this section, X again denotes a complex 1-manifold. We now develop a suitable distributional version of the $\bar{\partial}$ operator. In particular, we are led to consider a modified version of the exterior derivative d called the *canonical* (or *Chern*) *connection*.

For locally integrable functions f and g on a local holomorphic coordinate neighborhood (U, z) , we have $(\partial f / \partial \bar{z})_{\text{distr}} = g$ (see Definition 9.8.2 and Proposition 9.8.3) if and only if for each function $u \in \mathcal{D}(U)$, we have

$$\int_U f \overline{\left(-\frac{\partial u}{\partial z}\right)} \frac{i}{2} dz \wedge d\bar{z} = \int_U g \bar{u} \frac{i}{2} dz \wedge d\bar{z}$$

(equivalently, $\int_U f \overline{(-\partial u / \partial \bar{z})} dz \wedge d\bar{z} = \int_U g \bar{u} dz \wedge d\bar{z}$). One natural definition for the distributional $\bar{\partial}$ operator is the following:

Definition 2.7.1 Let α and β be locally integrable differential forms on an open set $\Omega \subset X$. We write $\bar{\partial}_{\text{distr}} \alpha = \beta$ if for every local holomorphic coordinate neighborhood (U, z) , one of the following holds:

- (i) On $U \cap \Omega$, α is a 0-form, $\beta = b d\bar{z}$, and $(\partial \alpha / \partial \bar{z})_{\text{distr}} = b$;
- (ii) On $U \cap \Omega$, $\alpha = a_1 dz + a_2 d\bar{z}$, $\beta = b dz \wedge d\bar{z}$, and $(\partial a_1 / \partial \bar{z})_{\text{distr}} = -b$;
- (iii) The form α is of degree > 1 and $\beta \equiv 0$.

Remark Given a locally integrable differential form α , $\bar{\partial}_{\text{distr}} \alpha$ need not exist as a form (see the remarks following Definition 7.4.2). For example, let u be the characteristic function of the unit disk $\Delta \equiv \Delta(0; 1)$. Then $\bar{\partial} u \equiv 0$ on the complement $\mathbb{C} \setminus \partial \Delta$ of the measure-zero set $\partial \Delta$. Thus, if $\bar{\partial}_{\text{distr}} u$ were to exist on \mathbb{C} , then it would be the zero form, and hence by the regularity theorem (Theorem 1.2.8), u would be holomorphic, which it is not. It also follows that for the measurable differential form $\alpha = u dz$, $\bar{\partial}_{\text{distr}} \alpha$ does not exist as a form.

It is often more convenient to work with an intrinsic form of an operator, so we look for an intrinsic description of $\bar{\partial}_{\text{distr}}$. Distributional differential operators in \mathbb{R}^n are defined via integration against test functions (as in Sect. 7.4.2). On a surface, the objects that we integrate on open sets (or, more generally, measurable sets) are 2-forms, and for a differential form, its wedge product with a form of complementary degree is a 2-form. Thus, in the present context, it is natural to

integrate against *test forms* of complementary degree. For example, suppose α and β are C^∞ differential forms of type $(1, 0)$ and $(1, 1)$, respectively, and $\bar{\partial}\alpha = \beta$. Since we integrate C^∞ forms of type $(1, 1)$ (i.e., 2-forms) on open sets in X , it is natural to integrate the wedge product of α and the conjugate of a $(1, 0)$ -form, and the scalar product (i.e., the wedge product) of β and a function (i.e., a $(0, 0)$ -form). Given a function $f \in \mathcal{D}(X)$, we have

$$\begin{aligned}\beta \cdot \bar{f} &= (\bar{\partial}\alpha) \cdot \bar{f} = (d\alpha) \cdot \bar{f} = d(\alpha \cdot \bar{f}) + \alpha \wedge d\bar{f} \\ &= d(\alpha \cdot \bar{f}) + \alpha \wedge \partial\bar{f} + \alpha \wedge \bar{\partial}\bar{f} \\ &= d(\alpha \cdot \bar{f}) + \alpha \wedge \overline{\partial f}.\end{aligned}$$

Integrating and applying Stokes' theorem, we get

$$\int_X \beta \cdot \bar{f} = \int_X \alpha \wedge \overline{\partial f}.$$

It turns out to be useful to insert a weight function φ (more precisely, $e^{-\varphi}$), where φ is a real-valued C^∞ (usually) function on X . Doing so allows one to obtain estimates on L^2 norms that lead to solutions of the Cauchy–Riemann equation as well as certain bounds on the L^2 norms of the solutions (see Sect. 2.9). Moreover, a Hermitian metric in a holomorphic line bundle (see Chap. 3) is locally represented by such weight functions, and this point of view yields results that generalize readily to that context. For α , β , and f as above, we have

$$\begin{aligned}\beta \cdot \bar{f} \cdot e^{-\varphi} &= (\bar{\partial}\alpha) \cdot \bar{f} \cdot e^{-\varphi} = (d\alpha) \cdot (\bar{f}e^{-\varphi}) = d(\alpha \cdot e^{-\varphi}\bar{f}) + \alpha \wedge d(e^{-\varphi}\bar{f}) \\ &= d(\alpha \cdot e^{-\varphi}\bar{f}) + \alpha \wedge \bar{\partial}(e^{-\varphi}\bar{f}) \\ &= d(\alpha \cdot e^{-\varphi}\bar{f}) + \alpha \wedge \overline{e^\varphi \partial(e^{-\varphi}f)} \cdot e^{-\varphi}.\end{aligned}$$

Thus

$$\int_X \beta \cdot \bar{f} \cdot e^{-\varphi} = \int_X \alpha \wedge \overline{e^\varphi \partial(e^{-\varphi}f)} \cdot e^{-\varphi}.$$

The above suggests a natural intrinsic form of the definition of $\bar{\partial}_{\text{distr}}$ on forms of type $(1, 0)$. A similar computation applies to forms of type $(0, 0)$. Based on the above, we make the following definition:

Definition 2.7.2 Given a real-valued function $\varphi \in C^\infty(X)$, the associated *canonical connection* (or the *Chern connection*) is the operator $D = D_\varphi = D' + D''$, where for each C^1 differential form α on an open subset of X , we define

$$D'\alpha = D'_\varphi\alpha \equiv e^\varphi \partial(e^{-\varphi}\alpha) = \partial\alpha + e^\varphi(\partial e^{-\varphi}) \wedge \alpha = \partial\alpha - \partial\varphi \wedge \alpha$$

and

$$D''\alpha = D''_\varphi\alpha \equiv \bar{\partial}\alpha.$$

We call D' and D'' , respectively, the $(1, 0)$ part and $(0, 1)$ part of the connection.

Remarks 1. Observe that $D'' = \bar{\partial}$ does not depend on the choice of φ .

2. We have $D = d + e^\varphi (\partial e^{-\varphi}) \wedge (\cdot) = d - \partial \varphi \wedge (\cdot)$.

3. If α is of type (p, q) , then $D'\alpha$ is of type $(p+1, q)$ (in particular, $D'\alpha = 0$ if $p \geq 1$) and $D''\alpha$ is of type $(p, q+1)$ ($D''\alpha = 0$ if $q \geq 1$).

4. If α and β are \mathcal{C}^1 differential forms and α is of degree p , then (see Exercise 2.7.1)

$$D(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge D\beta = (D\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta.$$

5. D_φ , D'_φ , and D''_φ are defined in the same way as above for $\varphi \in \mathcal{C}^1(X)$.

We now get the following equivalent form for the definition of $\bar{\partial}_{\text{distr}}$ (which we also denote by D''_{distr}):

Proposition 2.7.3 *Let φ be a real-valued \mathcal{C}^∞ function on X , and let α and β be locally integrable differential forms on an open set $\Omega \subset X$.*

(a) *If α is of type $(1, 0)$ and β is of type $(1, 1)$, then $\bar{\partial}_{\text{distr}}\alpha = \beta$ if and only if*

$$\int_{\Omega} \alpha \wedge \overline{D'f} e^{-\varphi} = \int_{\Omega} \beta \cdot \bar{f} \cdot e^{-\varphi} \quad \forall f \in \mathcal{D}(\Omega).$$

(b) *If α is of type $(0, 0)$ and β is of type $(0, 1)$, then $\bar{\partial}_{\text{distr}}\alpha = \beta$ if and only if*

$$\int_{\Omega} \alpha \cdot \overline{(-D'\gamma)} \cdot e^{-\varphi} = \int_{\Omega} \beta \wedge \bar{\gamma} \cdot e^{-\varphi} \quad \forall \gamma \in \mathcal{D}^{0,1}(\Omega).$$

(c) *If α is of type (p, q) with $p \geq 2$ or $q \geq 1$, then $\bar{\partial}_{\text{distr}}\alpha = 0$.*

Remark It will follow from the proof that if the conditions in Definition 2.7.1 hold for the forms α and β in *some* local holomorphic coordinate neighborhood of every point, then they hold in *every* local holomorphic coordinate neighborhood.

Proof of Proposition 2.7.3 Suppose $\alpha = a dz$ and $\beta = b dz \wedge d\bar{z}$ in some local holomorphic coordinate neighborhood (U, z) with $U \subset \Omega$. Then, for every function $f \in \mathcal{D}(U)$ (which we may view as a \mathcal{C}^∞ function with compact support in Ω), we have

$$\int_{\Omega} \beta \bar{f} e^{-\varphi} = \int_U b \overline{(e^{-\varphi} f)} dz \wedge d\bar{z}$$

and

$$\int_{\Omega} \alpha \wedge \overline{D'f} \cdot e^{-\varphi} = \int_U \alpha \wedge \overline{\partial(e^{-\varphi} f)} = \int_U a \cdot \overline{\frac{\partial}{\partial z}(e^{-\varphi} f)} dz \wedge d\bar{z}.$$

Given $u \in \mathcal{D}(U)$, setting $f = e^\varphi u$, we see that if the above left-hand sides are always equal, then we have $\bar{\partial}_{\text{distr}}\alpha = \beta$. Conversely, if $\bar{\partial}_{\text{distr}}\alpha = \beta$ and $f \in \mathcal{D}(\Omega)$, then, choosing finitely many \mathcal{C}^∞ functions $\{\eta_\nu\}_{\nu=1}^n$ on Ω such that $\sum \eta_\nu \equiv 1$ on $\text{supp } f$

and such that for each v , the support of η_v is contained in some local holomorphic coordinate neighborhood U_v , the above gives

$$\begin{aligned} \int_{\Omega} \alpha \wedge \overline{D'f} \cdot e^{-\varphi} &= \sum_v \int_{\Omega} \alpha \wedge \overline{D'(\eta_v \cdot f)} \cdot e^{-\varphi} = \sum_v \int_{U_v} \alpha \wedge \overline{D'(\eta_v \cdot f)} \cdot e^{-\varphi} \\ &= \sum_v \int_{U_v} \beta \overline{\eta_v f} e^{-\varphi} = \sum_v \int_{\Omega} \beta \overline{\eta_v f} e^{-\varphi} = \int_{\Omega} \beta \bar{f} \cdot e^{-\varphi}. \end{aligned}$$

Thus part (a) is proved.

Part (c) is obvious and the proof of part (b) is left to the reader (see Exercise 2.7.2). \square

The regularity theorem (Theorem 1.2.8) gives the following in this context:

Theorem 2.7.4 *If $p \in \{0, 1\}$, α is a locally integrable form of type $(p, 0)$ on X , and $\bar{\partial}_{\text{distr}} \alpha = \beta$ for a form β of class C^k for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class C^k . In particular, if $\bar{\partial}_{\text{distr}} \alpha = 0$, then α is a holomorphic p -form.*

Exercises for Sect. 2.7

2.7.1 Let φ be a C^∞ real-valued function on a Riemann surface X . Show that if α and β are C^1 differential forms on X and α is of degree p , then

$$D_\varphi(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge D_\varphi \beta = (D_\varphi \alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta.$$

2.7.2 Prove part (b) of Proposition 2.7.3.

2.7.3 Let X be a Riemann surface, let φ be a real-valued C^∞ function on X , let $\{\theta_v\}$ be a sequence of holomorphic 1-forms in $L^2_{1,0}(X, \varphi)$, and let $\theta \in L^2_{1,0}(X, \varphi)$. Assume that for each element $\alpha \in L^2_{1,0}(X, \varphi)$, we have

$$\lim_{v \rightarrow \infty} \langle \theta_v, \alpha \rangle_{L^2_{1,0}(X, \varphi)} = \langle \theta, \alpha \rangle_{L^2_{1,0}(X, \varphi)}$$

(that is, $\{\theta_v\}$ converges *weakly* to θ in $L^2_{1,0}(X, \varphi)$). Prove that θ is a holomorphic 1-form.

2.8 Curvature and the Fundamental Estimate for Scalar-Valued Forms

Throughout this section, X again denotes a complex 1-manifold. Suppose φ is a real-valued C^∞ function on X . We recall that for any C^∞ differential form α on an open subset of X , we have $D_\varphi \alpha = D\alpha = D'\alpha + D''\alpha$, where

$$D'\alpha = e^\varphi \partial[e^{-\varphi} \alpha] = \partial\alpha - (\partial\varphi) \wedge \alpha \quad \text{and} \quad D''\alpha = \bar{\partial}\alpha.$$

Consequently, $(D')^2 = (D'')^2 = 0$ and $D^2 = D'D'' + D''D'$. Since $\partial\bar{\partial} + \bar{\partial}\partial = 0$, we have

$$\begin{aligned} D^2\alpha &= \partial\bar{\partial}\alpha - (\partial\varphi) \wedge \bar{\partial}\alpha + \bar{\partial}\partial\alpha - \bar{\partial}[(\partial\varphi) \wedge \alpha] \\ &= -(\partial\varphi) \wedge \bar{\partial}\alpha - (\bar{\partial}\partial\varphi) \wedge \alpha + (\partial\varphi) \wedge \bar{\partial}\alpha = \Theta_\varphi \wedge \alpha, \end{aligned}$$

where $\Theta_\varphi \equiv \partial\bar{\partial}\varphi$ is a differential form of type $(1, 1)$ (of course, $\theta_\varphi \wedge \alpha = 0$ if $\deg \alpha > 0$).

Definition 2.8.1 The *curvature* (or *curvature form* or *Levi form*) associated to a real-valued C^∞ function φ on X is the differential form $\Theta = \Theta_\varphi \equiv \partial\bar{\partial}\varphi$. In other words, Θ is defined by

$$D^2 = D'D'' + D''D' = \Theta \wedge (\cdot).$$

The function φ is called *subharmonic* (*strictly subharmonic*, *harmonic*, *superharmonic*, *strictly superharmonic*) if the real $(1, 1)$ -form $i\Theta_\varphi$ satisfies $i\Theta_\varphi \geq 0$ (respectively, $i\Theta_\varphi > 0$, $i\Theta_\varphi = 0$, $i\Theta_\varphi \leq 0$, $i\Theta_\varphi < 0$). In a slight abuse of language, we also say that φ has *nonnegative* (respectively, *positive*, *zero*, *nonpositive*, *negative*) *curvature*.

Remarks 1. The above terminology, with the same definitions, is also applied to C^2 functions. There is also a natural and useful notion of a *continuous* subharmonic function (see, for example, [Ns5]), but continuous subharmonic functions are not considered in this book.

2. For any local holomorphic coordinate neighborhood $(U, z = x + iy)$ in X and for any real-valued C^∞ function φ on U , we have

$$i\Theta_\varphi = \frac{\partial^2\varphi}{\partial z\partial\bar{z}} i dz \wedge d\bar{z} = 2 \frac{\partial^2\varphi}{\partial z\partial\bar{z}} dx \wedge dy = \frac{1}{2} \left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} \right) dx \wedge dy.$$

Thus φ is subharmonic (strictly subharmonic, harmonic) on U if and only if

$$\frac{\partial^2\varphi}{\partial z\partial\bar{z}} \geq 0 \quad (\text{respectively, } > 0, \quad = 0).$$

3. It is easy to see that a strictly subharmonic function cannot attain a local maximum (see Exercise 2.8.2). In fact, one can show that subharmonic functions satisfy a strong maximum principle (see, for example, [Ns5]). In particular, every subharmonic function on a compact Riemann surface is constant.

Proposition 2.8.2 (Fundamental estimate for scalar-valued forms) *Let φ be a real-valued C^∞ function on X , and let $D' = D'_\varphi$, $\Theta = \Theta_\varphi$, and $D'' = \bar{\partial}$. Then, for all functions $u, v \in \mathcal{D}(X)$, we have*

$$\langle D'u, D'v \rangle_{L^2(X, \varphi)} = \langle D''u, D''v \rangle_{L^2(X, \varphi)} + \int_X i\Theta u \bar{v} e^{-\varphi}.$$

In particular, $\|D'u\|_{L^2(X, \varphi)}^2 = \|D''u\|_{L^2(X, \varphi)}^2 + \int_X i\Theta |u|^2 e^{-\varphi} \geq \int_X i\Theta |u|^2 e^{-\varphi}$.

Remark If $i\Theta \geq 0$, then we get

$$\langle D'u, D'v \rangle_{L^2(X, \varphi)} = \langle D''u, D''v \rangle_{L^2(X, \varphi)} + \langle u, v \rangle_{L^2(X, i\Theta, \varphi)}$$

$$\text{and } \|D'u\|_{L^2(X, \varphi)}^2 = \|D''u\|_{L^2(X, \varphi)}^2 + \|u\|_{L^2(X, i\Theta, \varphi)}^2 \geq \|u\|_{L^2(X, i\Theta, \varphi)}^2.$$

Proof of Proposition 2.8.2 For each pair of functions $u, v \in \mathcal{D}(X)$, Proposition 2.7.3 gives

$$\begin{aligned} \langle D'u, D'v \rangle_{L^2(X, \varphi)} &= \int_X i D'u \wedge \overline{D'v} e^{-\varphi} = \int_X i (D'' D'u) \cdot \bar{v} e^{-\varphi} \\ &= - \int_X i (D' D''u) \cdot \bar{v} e^{-\varphi} + \int_X i \Theta u \bar{v} e^{-\varphi} \\ &= -i \left[\int_X v \overline{D' D''u} \cdot e^{-\varphi} \right] + \int_X i \Theta u \bar{v} e^{-\varphi} \\ &= i \left[\int_X (D''v) \wedge \overline{D''u} \cdot e^{-\varphi} \right] + \int_X i \Theta u \bar{v} e^{-\varphi} \\ &= -i \int_X (D''u) \wedge \overline{D''v} \cdot e^{-\varphi} + \int_X i \Theta u \bar{v} e^{-\varphi} \\ &= \langle D''u, D''v \rangle_{L^2(X, \varphi)} + \int_X i \Theta u \bar{v} e^{-\varphi}. \quad \square \end{aligned}$$

Exercises for Sect. 2.8

2.8.1 Show that the function $z \mapsto |z|^2$ on \mathbb{C} is strictly subharmonic.

2.8.2 Show that if φ is a \mathcal{C}^2 strictly subharmonic function (i.e., $i\partial\bar{\partial}\varphi > 0$) on a Riemann surface X , then φ cannot attain a local maximum.

2.9 The L^2 $\bar{\partial}$ -Method for Scalar-Valued Forms of Type (1, 0)

The following theorem (in various guises) is the main tool in this book:

Theorem 2.9.1 *Let X be a Riemann surface, let φ be a real-valued \mathcal{C}^∞ function on X with $i\Theta = i\Theta_\varphi = i\partial\bar{\partial}\varphi \geq 0$, and let $Z = \{x \in X \mid \Theta_x = 0\}$. Then, for every measurable (1, 1)-form β on X with $\beta|_{X \setminus Z} \in L^2_{1,1}(X \setminus Z, i\Theta, \varphi)$ and $\beta = 0$ a.e. in Z (in particular, β is in L^2_{loc} on X), there exists a form $\alpha \in L^2_{1,0}(X, \varphi)$ such that*

$$D''_{\text{distr}} \alpha = \bar{\partial}_{\text{distr}} \alpha = \beta \quad \text{and} \quad \|\alpha\|_{L^2(X, \varphi)} \leq \|\beta\|_{L^2(X \setminus Z, i\Theta, \varphi)}.$$

In particular, if β is of class \mathcal{C}^k for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class \mathcal{C}^k and $\bar{\partial}\alpha = \beta$.

Remark The form β in the above theorem is in L^2_{loc} because β vanishes on Z , while for any compact set K in any local holomorphic coordinate neighborhood (U, z) , we have

$$\|\cdot\|_{L^2_{1,1}(K \setminus Z, (i/2) dz \wedge d\bar{z})} \leq A \|\cdot\|_{L^2_{1,1}(K \setminus Z, i\Theta, \varphi)},$$

where

$$A \equiv \left(\max_K \frac{i\Theta}{(i/2) dz \wedge d\bar{z}} e^\varphi \right)^{1/2} < \infty.$$

Proof of Theorem 2.9.1 Let $N \equiv \|\beta\|_{L^2(X \setminus Z, i\Theta, \varphi)} = \|\beta/(i\Theta)\|_{L^2(X \setminus Z, i\Theta, \varphi)}$. For each function $f \in \mathcal{D}(X)$, the Schwarz inequality and the fundamental estimate (Proposition 2.8.2) give

$$\begin{aligned} \left| \int_X f \bar{\beta} e^{-\varphi} \right| &= \left| \int_{X \setminus Z} f \cdot \frac{\bar{\beta}}{i\Theta} \cdot e^{-\varphi} \cdot i\Theta \right| = |\langle f, \beta/(i\Theta) \rangle_{L^2(X \setminus Z, i\Theta, \varphi)}| \\ &\leq \|f\|_{L^2(X \setminus Z, i\Theta, \varphi)} \cdot \|\beta/(i\Theta)\|_{L^2(X \setminus Z, i\Theta, \varphi)} \\ &\leq N \cdot \|f\|_{L^2(X, i\Theta, \varphi)} \leq N \cdot \|D'f\|_{L^2(X, \varphi)}. \end{aligned}$$

It follows that the mapping $\Upsilon : [D'f] \mapsto -i \int_X f \bar{\beta} e^{-\varphi}$ is a bounded complex linear functional on the subspace $D'[\mathcal{D}(X)]$ of $L^2_{1,0}(X, \varphi)$. For by the above inequality, Υ is well defined, and for each $f \in \mathcal{D}(X)$, we have $|\Upsilon[D'f]| \leq N \cdot \|D'f\|_{L^2(X, \varphi)}$. In particular, $\|\Upsilon\| \leq N$. By the Hahn–Banach theorem (Theorem 7.5.11), there exists a bounded linear functional $\hat{\Upsilon}$ on $L^2_{1,0}(X, \varphi)$ such that $\hat{\Upsilon}|_{D'[\mathcal{D}(X)]} = \Upsilon$ and $\|\hat{\Upsilon}\| = \|\Upsilon\|$. Therefore, by Theorem 7.5.10, there exists a (unique) element $\alpha \in L^2_{1,0}(X, \varphi)$ such that $\|\alpha\|_{L^2(X, \varphi)} = \|\hat{\Upsilon}\| \leq N$ and $\hat{\Upsilon}(\cdot) = \langle \cdot, \alpha \rangle_{L^2(X, \varphi)}$. Moreover, for each $f \in \mathcal{D}(X)$, we have

$$\int_X i\alpha \wedge \overline{D'f} e^{-\varphi} = \overline{\Upsilon(D'f)} = \overline{\int_X (-i)f \bar{\beta} e^{-\varphi}} = \int_X i\bar{\beta} f e^{-\varphi}.$$

Therefore, by Proposition 2.7.3, $D''_{\text{distr}}\alpha = \beta$, as required. Finally, the regularity statement at the end follows from Theorem 2.7.4. \square

It is often more convenient to apply Theorem 2.9.1 in one of the following forms, the proofs of which are left to the reader (see Exercises 2.9.1 and 2.9.2):

Corollary 2.9.2 *Suppose that X is a Riemann surface, ω is a Kähler form on X , φ is a real-valued C^∞ function on X , ρ is a nonnegative measurable function on X with $i\Theta_\varphi \geq \rho\omega$, and $Z = \{x \in X \mid \rho(x) = 0\}$. Then, for every measurable $(1, 1)$ -form β on X with $\beta = 0$ a.e. in Z and*

$$\beta|_{X \setminus Z} \in L^2_{1,1}(X \setminus Z, \rho\omega, \varphi) = L^2_{1,1}(X \setminus Z, \omega, \varphi + \log \rho)$$

(in particular, β is in L^2_{loc} on X), there exists a form $\alpha \in L^2_{1,0}(X, \varphi)$ such that

$$D''_{\text{distr}}\alpha = \bar{\partial}_{\text{distr}}\alpha = \beta \quad \text{and} \quad \|\alpha\|_{L^2(X, \varphi)} \leq \|\beta\|_{L^2(X \setminus Z, \rho\omega, \varphi)}.$$

In particular, if β is of class C^k for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class C^k and $\bar{\partial}\alpha = \beta$.

Remark The main point is that $\|\beta\|_{L^2(X \setminus Z, i\Theta_\varphi, \varphi)} \leq \|\beta\|_{L^2(X \setminus Z, \rho\omega, \varphi)}$ if $i\Theta_\varphi \geq \rho\omega$.

Corollary 2.9.3 Suppose that X is a Riemann surface, ω is Kähler form on X , φ is a real-valued C^∞ function on X , and C is a positive constant with $i\Theta_\varphi \geq C^2\omega$. Then, for every form $\beta \in L^2_{1,1}(X, \omega, \varphi)$, there exists a form $\alpha \in L^2_{1,0}(X, \varphi)$ such that

$$D''_{\text{distr}}\alpha = \bar{\partial}_{\text{distr}}\alpha = \beta \quad \text{and} \quad \|\alpha\|_{L^2(X, \varphi)} \leq C^{-1}\|\beta\|_{L^2(X, \omega, \varphi)}.$$

In particular, if β is of class C^k for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class C^k and $\bar{\partial}\alpha = \beta$.

Remark In Sect. 2.11, we will show that every Riemann surface admits a Kähler form.

Exercises for Sect. 2.9

2.9.1 Prove Corollary 2.9.2.

2.9.2 Prove Corollary 2.9.3.

2.9.3 Let X be a Riemann surface, let φ be a real-valued C^∞ function on X with $i\Theta = i\Theta_\varphi \geq 0$, and let $Z = \{x \in X \mid \Theta_x = 0\}$. Prove that for every C^∞ $(1, 1)$ -form β on X with $\beta = 0$ a.e. in Z and $\beta|_{X \setminus Z} \in L^2_{1,1}(X \setminus Z, i\Theta, \varphi)$, there exists a C^∞ form $\alpha \in L^2_{0,1}(X, \varphi)$ such that

$$\partial\alpha = \beta \quad \text{and} \quad \|\alpha\|_{L^2_{0,1}(X, \varphi)} \leq \|\beta\|_{L^2_{1,1}(X \setminus Z, i\Theta, \varphi)}.$$

2.9.4 Let X be a Riemann surface, let φ be a real-valued C^∞ function on X , let ω be a nonnegative C^∞ $(1, 1)$ -form on X , let $Z = \{x \in X \mid \omega_x = 0\}$, and let β be a C^∞ $(1, 1)$ -form on X such that $\beta = 0$ a.e. in Z and $\beta|_{X \setminus Z} \in L^2_{1,1}(X \setminus Z, \omega, \varphi)$. Prove that $\beta \equiv 0$ on Z (hence in the C^∞ case of Theorem 2.9.1 and in Exercise 2.9.3, there is no loss of generality if we assume that $\beta \equiv 0$ on Z).

2.10 Existence of Meromorphic 1-Forms and Meromorphic Functions

The power of Theorem 2.9.1 is demonstrated by the following important application, which will play a crucial role in the proof of such central facts as Radó's theorem on second countability (see Sect. 2.11) and the Riemann mapping theorem (see Chap. 5):

Theorem 2.10.1 *For every point in a Riemann surface, there exists a meromorphic 1-form that is holomorphic except for a pole of arbitrary prescribed order ≥ 2 at the point. In fact, for each integer $m \geq 2$, there exists a universal constant $C_m > 0$ such that if X is any Riemann surface, p is any point in X , and $(D, \Phi = z, \Delta(0; 1))$ is any local holomorphic chart with $p \in D$ and $\Phi(p) = z(p) = 0$, then there exists a meromorphic 1-form θ on X with the following properties:*

- (i) *The meromorphic 1-form θ is holomorphic on $X \setminus \{p\}$ and has a pole of order m at p ;*
- (ii) *We have $\|\theta\|_{L^2(X \setminus D)} \leq C_m$;*
- (iii) *The meromorphic 1-form $\theta - z^{-m} dz$ on D has at worst a simple pole at p ; and*
- (iv) *We have $\|z\theta - z^{-m+1} dz\|_{L^2(D)} \leq C_m$.*

Remark The value for the constant C_m that we will obtain is far from optimal.

Lemma 2.10.2 *For any choice of constants $a, b, c \in \mathbb{R}$ with $a < b < c$, there exists a C^∞ function $\chi: \mathbb{R} \rightarrow \mathbb{R}$ such that*

- (i) *For each $t \in \mathbb{R}$, $\chi'(t) \geq 0$ and $\chi''(t) \geq 0$;*
- (ii) *For each $t \leq a$, $\chi(t) = 0$; and*
- (iii) *For each $t \geq c$, $\chi(t) = t - b$.*

Proof For $s > 0$, the function

$$\rho_s(t) \equiv \begin{cases} \frac{s}{s + \exp(\frac{1}{t-a} + \frac{1}{t-c})} = \frac{s \exp(\frac{1}{a-t})}{s \exp(\frac{1}{a-t}) + \exp(\frac{1}{t-c})} & \text{if } a < t < c, \\ 0 & \text{if } t \leq a, \\ 1 & \text{if } c \leq t, \end{cases}$$

is of class C^∞ , $\rho_s \geq 0$, and $\rho'_s \geq 0$. Moreover, the function $\mu: s \mapsto \int_a^c \rho_s(t) dt$ is continuous on $(0, \infty)$, $\mu(s) \rightarrow 0$ as $s \rightarrow 0^+$, and $\mu(s) \rightarrow c - a > c - b$ as $s \rightarrow \infty$ (for example, by the dominated convergence theorem). Therefore, by the intermediate value theorem, there is a number $s_0 > 0$ such that $\mu(s_0) = c - b$. The function $t \mapsto \chi(t) \equiv \int_a^t \rho_{s_0}(u) du$ then has the required properties. \square

Lemma 2.10.3 *Let $r > 0$ be a constant and let ψ be a C^∞ strictly subharmonic function on $\Delta(0; r)$. Then there is a constant $b_0 = b_0(r, \psi) > 0$ such that for every constant $b > b_0$, there exist a constant $R = R(b, r, \psi) \in (0, r)$ and a nonnegative C^∞ subharmonic function φ on \mathbb{C}^* with $\varphi \equiv 0$ on a neighborhood of $\mathbb{C} \setminus \Delta(0; r)$ and*

$$\varphi(z) = \psi(z) - \log |z|^2 - b \quad \forall z \in \Delta^*(0; R).$$

Proof Setting $\rho(z) \equiv \psi(z) - \log |z|^2$ for all $z \in \Delta^*(0; r)$, we may fix positive constants R_0 and R_1 with $0 < R_0 < R_1 < r$ and a positive constant $b_0 > \sup_{\Delta(0; R_0, R_1)} \rho$. Given a constant $b > b_0$, we may fix constants a and c with

$c > b > a > b_0 > 0$, and applying Lemma 2.10.2, we get a C^∞ function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\chi' \geq 0$ and $\chi'' \geq 0$ on \mathbb{R} , $\chi(t) = 0$ for $t \leq a$, and $\chi(t) = t - b$ for $t \geq c$. For $R \in (0, r)$ sufficiently small, we have $\rho > c$ on $\Delta^*(0; R)$. This constant R and the function φ on \mathbb{C}^* given by

$$\varphi \equiv \begin{cases} \chi(\rho) & \text{on } \Delta^*(0; R_1), \\ 0 & \text{on } \mathbb{C} \setminus \Delta(0; R_1), \end{cases}$$

then have the required properties. For the choice of χ guarantees that φ is nonnegative and of class C^∞ , φ vanishes on $\mathbb{C} \setminus \Delta(0; R_0)$, and $\varphi = \rho - b$ on $\Delta^*(0; R)$. Furthermore, on $\Delta^*(0; R_1)$, we have $i\Theta_\rho = i\Theta_\psi > 0$ and hence

$$i\Theta_\varphi = \chi'(\rho) \cdot i\Theta_\psi + \chi''(\rho) \cdot i\partial\rho \wedge \overline{\partial\rho} \geq 0. \quad \square$$

Proof of Theorem 2.10.1 The idea of the proof (an idea that is applied in various guises throughout this book) is to first produce a C^∞ solution, that is, a C^∞ 1-form τ that has all of the required properties except that it is not meromorphic (away from p). Forming a suitable L^2 solution α of the equation $\bar{\partial}\alpha = \bar{\partial}\tau$, one gets a meromorphic 1-form $\theta = \tau - \alpha$ with the required properties.

Letting ζ denote the (standard) coordinate function on \mathbb{C} , setting $\psi_0 \equiv |\zeta|^2$, and applying Lemma 2.10.3, we get positive constants b , R_0 , and R_1 and a nonnegative C^∞ subharmonic function φ_0 on \mathbb{C}^* such that $R_0 < R_1 < 1$, $\varphi_0 \equiv 0$ on $\mathbb{C} \setminus \Delta(0; R_1)$, and

$$\varphi_0 = |\zeta|^2 - \log |\zeta|^2 - b \quad \text{on } \Delta^*(0; R_0).$$

In particular, $i\Theta_{\varphi_0} = i\Theta_{\psi_0} = i d\zeta \wedge d\bar{\zeta} > 0$ on $\Delta^*(0; R_0)$. We may also fix a constant $R_2 \in (0, R_0)$ and a function $\eta_0 \in \mathcal{D}(\Delta(0; R_0))$ such that $\eta_0 \equiv 1$ on $\Delta(0; R_2)$.

Given an integer $m \geq 2$, we have the meromorphic 1-form $\gamma_m \equiv \zeta^{-m} d\zeta$ on \mathbb{C} and the C^∞ form $\tau_m \equiv \eta_0 \gamma_m$ of type $(1, 0)$ on \mathbb{C}^* . In particular, $\tau_m \equiv 0$ on a neighborhood of $\mathbb{C} \setminus \Delta(0; R_0)$ and τ_m is holomorphic on $\Delta^*(0; R_2)$. Hence $\beta_m \equiv \bar{\partial}\tau_m = \bar{\partial}\eta_0 \wedge \gamma_m$ is a C^∞ 2-form that vanishes on $\mathbb{C}^* \setminus \Delta(0; R_2, R_0)$. Observe that

$$A_m \equiv \|\beta_m\|_{L^2(\Delta^*(0; R_0), i\Theta_{\varphi_0}, \varphi_0)} = \|\beta_m\|_{L^2(\Delta(0; R_2, R_0), i d\zeta \wedge d\bar{\zeta}, \varphi_0)} < \infty$$

and

$$B_m \equiv \|\zeta \tau_m - \gamma_m\|_{L^2(\Delta^*(0; 1))} = \|(\eta_0 - 1)d\zeta / \zeta^{m-1}\|_{L^2(\Delta(0; R_2, 1))} < \infty.$$

By construction, the function $\varphi_0 + \log |\zeta|^2$ is bounded on $\Delta^*(0; 1)$, so we may define a positive constant C_m by

$$C_m \equiv A_m + B_m + \sup_{\Delta^*(0; 1)} e^{(\varphi_0 + \log |\zeta|^2)/2} \cdot A_m < \infty.$$

Suppose now that p is a point in a Riemann surface X ($D, \Phi = z, \Delta(0; 1)$) is a local holomorphic chart with $p \in D$ and $\Phi(p) = z(p) = 0$, and $Y \equiv X \setminus \{p\}$. The

form τ on Y that is equal to $\Phi^*\tau_m$ on $D \setminus \{p\}$ and 0 elsewhere is a \mathcal{C}^∞ differential form of type $(1, 0)$ that vanishes on $X \setminus \Phi^{-1}(\Delta(0; R_0))$ and that is holomorphic on $\Phi^{-1}(\Delta^*(0; R_2))$. The function φ on Y given by $\varphi = \varphi_0(z)$ on $D \setminus \{p\}$ and 0 elsewhere is nonnegative, subharmonic, and of class \mathcal{C}^∞ . Moreover, φ vanishes on $X \setminus \Phi^{-1}(\Delta(0; R_1))$ and $i\Theta_\varphi = i dz \wedge d\bar{z} > 0$ on $\Phi^{-1}(\Delta^*(0; R_0))$. The \mathcal{C}^∞ $(1, 1)$ -form $\beta \equiv \bar{\partial}\tau$ on Y is equal to $\Phi^*\beta_m$ on $D \setminus \{p\}$, vanishes on

$$Y \setminus \Phi^{-1}(\Delta(0; R_2, R_0)) \supset Z \equiv \{q \in Y \mid i(\Theta_\varphi)_q = 0\},$$

and satisfies

$$\|\beta\|_{L^2(Y \setminus Z, i\Theta_\varphi, \varphi)} = \|\beta_m\|_{L^2(\Delta^*(0; R_0), i\Theta_{\varphi_0}, \varphi_0)} = A_m < \infty.$$

Therefore, by Theorem 2.9.1, there exists a \mathcal{C}^∞ $(1, 0)$ -form α on Y such that

$$\bar{\partial}\alpha = \beta \quad \text{and} \quad \|\alpha\|_{L^2(Y, \varphi)} \leq A_m.$$

Thus \mathcal{C}^∞ $(1, 0)$ -form $\theta \equiv \tau - \alpha$ on Y satisfies $\bar{\partial}\theta = 0$ and is therefore holomorphic on Y . Since τ and therefore α are holomorphic on $\Phi^{-1}(\Delta^*(0; R_2))$, we have $\alpha|_{\Phi^{-1}(\Delta^*(0; R_2))} = f(z)dz$ for some function $f \in \mathcal{O}(\Delta^*(0; R_2))$. In particular,

$$\begin{aligned} \int_{\Delta^*(0; R_2)} |\zeta f(\zeta)|^2 \frac{i}{2} d\zeta \wedge d\bar{\zeta} &= \int_{\Phi^{-1}(\Delta^*(0; R_2))} \frac{i}{2} \alpha \wedge \bar{\alpha} \cdot e^{-\varphi} \cdot e^{|z|^2 - b} \\ &\leq \frac{1}{2} e^{R_2^2 - b} \cdot \|\alpha\|_{L^2(Y, \varphi)}^2 < \infty. \end{aligned}$$

Hence, by Riemann's extension theorem (Theorem 1.2.10), the function $\zeta \mapsto \zeta f(\zeta)$ extends to a (unique) holomorphic function on $\Delta(0; R_2)$; that is, f extends to a meromorphic function with at worst a simple pole at 0. Therefore, since $\eta_0 \equiv 1$ on $\Delta(0; R_2)$ and γ_m is a meromorphic 1-form with a pole of order $m > 1$ at 0, we see that $\theta = \tau - \alpha$ determines a meromorphic 1-form on X that is holomorphic on Y and that has a pole of order m at p .

For the bounds, we observe that

$$\|\theta\|_{L^2(X \setminus D)} = \|\alpha\|_{L^2(X \setminus D)} = \|\alpha\|_{L^2(X \setminus D, \varphi)} \leq A_m \leq C_m$$

and that

$$\begin{aligned} \|z\theta - z^{-m+1}dz\|_{L^2(D)} &\leq \|z\tau - z^{-m+1}dz\|_{L^2(D)} + \|z\alpha\|_{L^2(D)} \\ &\leq B_m + \sup_{\Delta^*(0; 1)} e^{(\varphi_0 + \log|\zeta|^2)/2} \cdot \|\alpha\|_{L^2(D, \varphi)} \\ &\leq B_m + \sup_{\Delta^*(0; 1)} e^{(\varphi_0 + \log|\zeta|^2)/2} \cdot \|\alpha\|_{L^2(Y, \varphi)} \\ &\leq B_m + \sup_{\Delta^*(0; 1)} e^{(\varphi_0 + \log|\zeta|^2)/2} \cdot A_m \leq C_m. \end{aligned}$$

□

Corollary 2.10.4 *Every Riemann surface X admits a nonconstant meromorphic function.*

Proof We may choose two distinct points p and q in X . Applying Theorem 2.10.1, we get two meromorphic 1-forms η and θ that are holomorphic except for poles of order 2 at p and q , respectively. The quotient $f = \eta/\theta$ then determines a nonconstant meromorphic function on X . \square

Exercises for Sect. 2.10

- 2.10.1 Show that there exists a constant $A > 0$ such that for each integer $m \geq 2$, the constant $C_m \equiv A^m$ has the properties described in Theorem 2.10.1.
- 2.10.2 Prove the following generalization of Theorem 2.10.1. Let X_0 be a Riemann surface, let θ_0 be a meromorphic 1-form on X_0 that is holomorphic except for a pole of order $m \geq 2$ at some point $p_0 \in X_0$, let Ω_0 be a relatively compact neighborhood of p_0 in X_0 , and let f_0 be a holomorphic function on X_0 that vanishes at p_0 . Then there exists a constant $C = C(X_0, \theta_0, \Omega_0, f_0) > 0$ such that if X is any Riemann surface, p is a point in X , and $\Phi: \Omega \rightarrow \Omega_0$ is a biholomorphic mapping of a neighborhood Ω of p onto Ω_0 with $\Phi(p) = p_0$, then there exists a meromorphic 1-form θ on X with the following properties:
- (i) The meromorphic 1-form θ is holomorphic $X \setminus \{p\}$ and has a pole of order m at p ;
 - (ii) We have $\|\theta\|_{L^2(X \setminus \Omega)} \leq C$;
 - (iii) The meromorphic 1-form $\theta - \Phi^*\theta_0$ on Ω has at worst a simple pole at p ; and
 - (iv) We have $\|f_0(\Phi) \cdot \theta - f_0(\Phi) \cdot \Phi^*\theta_0\|_{L^2(\Omega)} \leq C$.

2.11 Radó's Theorem on Second Countability

The goal of this section is the following important theorem:

Theorem 2.11.1 (Radó) *Every Riemann surface X is second countable; that is, the topology in X admits a countable basis.*

The proof considered here is similar to that in [Sp].

Lemma 2.11.2 *Every nonempty open subset Ω of a Riemann surface X has only countably many connected components.*

Proof Fixing a point $p \in \Omega$ and a connected neighborhood U of p in Ω , and applying Theorem 2.10.1, we get a nontrivial (i.e., not everywhere zero) holomorphic 1-form θ on $X \setminus \{p\}$ with $\|\theta\|_{L^2(X \setminus U)} < \infty$. If $\{\Omega_i\}_{i \in I}$ is the family of (distinct)

connected components of Ω that do *not* meet (i.e., which do not contain) U and $I \neq \emptyset$, then for every finite set $J \subset I$, we have

$$\|\theta\|_{L^2(X \setminus U)}^2 \geq \|\theta\|_{L^2(\bigcup_{i \in J} \Omega_i)}^2 = \sum_{i \in J} \|\theta\|_{L^2(\Omega_i)}^2.$$

Therefore $\infty > \sum_{i \in I} \|\theta\|_{L^2(\Omega_i)}^2$, and hence, since the right-hand side is an unordered sum of (strictly) positive terms (see Example 7.1.3), the index set I must be countable. \square

Proof of Radó's theorem Corollary 2.10.4 provides a nonconstant meromorphic function, that is, a nonconstant holomorphic mapping $f: X \rightarrow \mathbb{P}^1$. Let \mathcal{P} be the countable collection of all sets $D \subset \mathbb{P}^1$, where D is either a disk in \mathbb{C} with rational radius and center in $\mathbb{Q} + i\mathbb{Q}$, or $D = \mathbb{P}^1 \setminus \overline{\Delta(0; r)}$ for some $r \in \mathbb{Q}_{>0}$. Let \mathcal{B} be the collection of all relatively compact open subsets B of X such that B is a connected component of $f^{-1}(D)$ for some set $D \in \mathcal{P}$. It follows that \mathcal{B} is a basis for the topology in X (see Exercise 2.11.1). Moreover, by Lemma 2.11.2, the set $f^{-1}(D)$ has only countably many connected components for each $D \in \mathcal{P}$, and therefore the basis \mathcal{B} must be countable. \square

Radó's theorem and Proposition 9.7.6 together give the following:

Corollary 2.11.3 *Every Riemann surface X admits a Kähler form (i.e., a positive C^∞ differential form of type $(1, 1)$). In fact, given a continuous real $(1, 1)$ -form τ on X , there exists a Kähler form ω with $\omega \geq \tau$ on X .*

Corollary 2.11.4 (Montel's theorem for a Riemann surface) *If $\{f_n\}$ is a sequence of holomorphic functions on a Riemann surface X and $\{f_n\}$ is uniformly bounded on each compact subset of X , then some subsequence of $\{f_n\}$ converges uniformly on compact subsets of X to a holomorphic function on X .*

Proof We may choose a countable open covering $\{U_\nu\}$ of X such that for each ν , U_ν is relatively compact in some local holomorphic coordinate neighborhood. Montel's theorem in the plane (Corollary 1.2.7) and Cantor's diagonal process together yield a subsequence $\{f_{n_k}\}$ such that the sequence $\{f_{n_k}|_{U_\nu}\}$ converges uniformly to a holomorphic function on U_ν for each ν . It follows that $\{f_{n_k}\}$ converges uniformly on compact subsets of X to some holomorphic function on X . \square

Remark A C^∞ surface need not be second countable. Moreover, Radó's theorem is false in higher dimensions; that is, there exist connected 2-dimensional complex manifolds that are *not* second countable, provided, of course, one does not include second countability as part of the definition (see, for example, [Hu]).

Exercises for Sect. 2.11 In Exercises 2.11.1 and 2.11.2 below, as in the proof of Radó's theorem (Theorem 2.11.1), X denotes a Riemann surface; $f: X \rightarrow \mathbb{P}^1$ denotes a nonconstant holomorphic mapping; \mathcal{P} denotes the collection of all sets

$D \subset \mathbb{P}^1$, where D is either a disk in \mathbb{C} with rational radius and center in $\mathbb{Q} + i\mathbb{Q}$, or $D = \mathbb{P}^1 \setminus \Delta(0; r)$ for some $r \in \mathbb{Q}_{>0}$; and \mathcal{B} denotes the collection of all relatively compact open subsets B of X such that B is a connected component of $\Phi^{-1}(D)$ for some set $D \in \mathcal{P}$.

2.11.1 Prove that \mathcal{B} is a basis for the topology in X (this fact was used in the proof of Radó's theorem).

2.11.2 Another way to see that \mathcal{B} is countable is to observe that each element $B \in \mathcal{B}$ meets at most countably many connected components of $f^{-1}(D)$ for each set $D \in \mathcal{P}$. Prove this observation, and using it in place of Lemma 2.11.2, prove Radó's theorem.

Remark This argument is essentially the proof of a special case of the Poincaré–Volterra theorem (see, for example, [Ns5]).

2.12 The $L^2 \bar{\partial}$ -Method for Scalar-Valued Forms of Type $(0, 0)$

It is also useful to consider solutions of the inhomogeneous Cauchy–Riemann equation $\bar{\partial}\alpha = \beta$ in which α is a function and β is a differential form of type $(0, 1)$. In fact, this problem is essentially equivalent to the problem considered in Sect. 2.9. In order to obtain the L^2 solution, we will have to consider the curvature form associated to a Kähler form ω (i.e., a C^∞ positive differential form of type $(1, 1)$) on a complex 1-manifold X . Suppose θ_1 and θ_2 are two nonvanishing holomorphic 1-forms on an open set $U \subset X$ and $G_j \equiv \omega / (i\theta_j \wedge \bar{\theta}_j)$ for $j = 1, 2$. Then the difference

$$(-\log G_1) - (-\log G_2) = \log |\theta_1 / \theta_2|^2$$

is a harmonic function; that is, $\partial\bar{\partial}[-\log G_1] = \partial\bar{\partial}[-\log G_2]$. Thus we may make the following definition:

Definition 2.12.1 For any Kähler form ω on a complex 1-manifold X , the associated *curvature* is the unique differential form $\Theta_{(X, \omega)} = \Theta_\omega$ that satisfies

$$\Theta_\omega \upharpoonright_U = \Theta_{-\log(\omega/(i\theta \wedge \bar{\theta}))} = \partial\bar{\partial}[-\log(\omega/(i\theta \wedge \bar{\theta}))]$$

for every nonvanishing local holomorphic 1-form θ on an open set $U \subset X$.

Remarks 1. Equivalently, if $\omega = G \cdot (i/2) dz \wedge d\bar{z}$ (i.e., $G = -2i\omega(\partial/\partial z, \partial/\partial \bar{z}) = \omega(\partial/\partial x, \partial/\partial y)$) in a local holomorphic coordinate neighborhood $(U, z = x + iy)$, then $\Theta_\omega = \Theta_{-\log G} = \partial\bar{\partial}[-\log G]$ on U .

2. If φ is a real-valued C^∞ function on X , then $\Theta_{e^{-\varphi}\omega} = \Theta_\omega + \Theta_\varphi$.

Example 2.12.2 For the Euclidean Kähler form $\omega \equiv (i/2) dz \wedge d\bar{z}$ on \mathbb{C} , $\Theta_\omega \equiv 0$.

Example 2.12.3 The *chordal Kähler form* on the Riemann sphere \mathbb{P}^1 is the unique Kähler form ω for which $\omega \upharpoonright_{\mathbb{C}} = 2i(1 + |z|^2)^{-2} dz \wedge d\bar{z}$. The corresponding curvature form satisfies $i\Theta_\omega = \omega$ (see Exercise 2.12.1).

The main goal of this section is the following:

Theorem 2.12.4 *Let X be a Riemann surface, let ω be a Kähler form on X , let φ be a real-valued C^∞ function on X with*

$$i\Theta \equiv i\Theta_{e^{-\varphi}\omega} = i\Theta_\omega + i\Theta_\varphi \geq 0,$$

let $\rho \equiv i\Theta/\omega$, and let $Z = \{x \in X \mid \Theta_x = 0\} = \{x \in X \mid \rho(x) = 0\}$. Then, for every measurable $(0, 1)$ -form β on X with $\beta = 0$ a.e. in Z and $\beta|_{X \setminus Z} \in L^2_{0,1}(X \setminus Z, \varphi + \log \rho)$ (in particular, β is in L^2_{loc} on X), there exists a function $\alpha \in L^2_{0,0}(X, \omega, \varphi)$ such that

$$D''_{\text{distr}}\alpha = \bar{\partial}_{\text{distr}}\alpha = \beta \quad \text{and} \quad \|\alpha\|_{L^2(X, \omega, \varphi)} \leq \|\beta\|_{L^2(X \setminus Z, \varphi + \log \rho)}.$$

In particular, if β is of class C^k for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class C^k and $\bar{\partial}\alpha = \beta$.

Remarks 1. In order to apply Theorem 2.12.4, one needs a Kähler form ω . Fortunately, by Corollary 2.11.3, a Kähler form always exists. In fact, this version is actually equivalent to Theorem 2.9.1.

2. Although it is possible to obtain the theorem directly, we will instead prove the theorem by first forming the exterior product of the given form and a meromorphic 1-form (provided by Theorem 2.10.1), and then applying the solution for forms of type $(1, 0)$ (Theorem 2.9.1).

3. The proof that β is in L^2_{loc} in the above (as well as in Corollary 2.12.5 below) is similar to that in the remark following the statement of Theorem 2.9.1.

It is often more convenient to apply Theorem 2.12.4 in one of the following forms, the proofs of which are left to the reader (see Exercises 2.12.2 and 2.12.3):

Corollary 2.12.5 *Suppose that X is a Riemann surface, ω is a Kähler form on X , φ is a real-valued C^∞ function on X , ρ is a nonnegative measurable function on X with $i\Theta_\omega + i\Theta_\varphi \geq \rho\omega$, and $Z = \{x \in X \mid \rho(x) = 0\}$. Then, for every measurable $(0, 1)$ -form β on X with $\beta = 0$ a.e. in Z and $\beta|_{X \setminus Z} \in L^2_{0,1}(X \setminus Z, \varphi + \log \rho)$ (in particular, β is in L^2_{loc} on X), there exists a function $\alpha \in L^2_{0,0}(X, \omega, \varphi)$ such that*

$$D''_{\text{distr}}\alpha = \bar{\partial}_{\text{distr}}\alpha = \beta \quad \text{and} \quad \|\alpha\|_{L^2(X, \omega, \varphi)} \leq \|\beta\|_{L^2(X \setminus Z, \varphi + \log \rho)}.$$

In particular, if β is of class C^k for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class C^k and $\bar{\partial}\alpha = \beta$.

Corollary 2.12.6 *Suppose that X is a Riemann surface, ω is a Kähler form on X , φ is a real-valued C^∞ function on X , and C is a positive constant for which $i\Theta_\omega + i\Theta_\varphi \geq C^2\omega$ on X . Then, for every form $\beta \in L^2_{0,1}(X, \varphi)$, there exists a func-*

tion $\alpha \in L^2_{0,0}(X, \omega, \varphi)$ such that

$$D''_{\text{distr}}\alpha = \bar{\partial}_{\text{distr}}\alpha = \beta \quad \text{and} \quad \|\alpha\|_{L^2(X, \omega, \varphi)} \leq C^{-1} \|\beta\|_{L^2(X, \varphi)}.$$

In particular, if β is of class C^k for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class C^k and $\bar{\partial}\alpha = \beta$.

Proof of Theorem 2.12.4 According to Theorem 2.10.1, we may fix a nontrivial meromorphic 1-form θ on X , and we may let Q be the (discrete) set of zeros and poles of θ . On the Riemann surface $Y \equiv X \setminus Q$, the function $\tau \equiv \omega/(i\theta \wedge \bar{\theta})$ is then positive and of class C^∞ , and we have $\Theta_{-\log \tau} = \Theta_\omega$; that is, $i\Theta_{\varphi - \log \tau} = i\Theta = \rho\omega$. The form $\beta_0 \equiv \theta \wedge \beta$ is a measurable differential form of type $(1, 1)$ that vanishes on $Z \cap Y$ and that satisfies (since the set Q is discrete, and therefore of measure 0)

$$\begin{aligned} & \|\beta_0\|_{L^2(Y \setminus Z, i\Theta_{\varphi - \log \tau}, \varphi - \log \tau)}^2 \\ &= \|\beta_0\|_{L^2(Y \setminus Z, \rho\omega, \varphi - \log \tau)}^2 = \int_{Y \setminus Z} \left| \frac{\theta \wedge \beta}{\rho\omega} \right|^2 e^{-(\varphi - \log \tau)} \rho\omega \\ &= \int_{Y \setminus Z} \frac{\theta \wedge \beta}{\omega} \cdot \frac{\bar{\theta} \wedge \bar{\beta}}{\omega} \cdot \frac{\omega}{i\theta \wedge \bar{\theta}} \cdot e^{-(\varphi + \log \rho)} \omega \\ &= \int_{Y \setminus Z} i\theta \wedge \beta \cdot \frac{i\theta \wedge \bar{\theta}}{i\theta \wedge \bar{\theta}} \cdot \frac{\bar{\beta}}{\theta} \cdot e^{-(\varphi + \log \rho)} \\ &= \int_{X \setminus Z} (-i)\beta \wedge \bar{\beta} \cdot e^{-(\varphi + \log \rho)} = \|\beta\|_{L^2(X \setminus Z, \varphi + \log \rho)}^2 < \infty. \end{aligned}$$

Therefore, β_0 is in L^2_{loc} on Y , and by Theorem 2.9.1, there exists a form $\alpha_0 \in L^2_{1,0}(Y, \varphi - \log \tau)$ such that $D''_{\text{distr}}\alpha_0 = \bar{\partial}_{\text{distr}}\alpha_0 = \beta_0$ in Y and

$$\|\alpha_0\|_{L^2(Y, \varphi - \log \tau)} \leq \|\beta_0\|_{L^2(Y \setminus Z, i\Theta_{\varphi - \log \tau}, \varphi - \log \tau)} = \|\beta\|_{L^2(X \setminus Z, \varphi + \log \rho)}.$$

The measurable function $\alpha \equiv -\alpha_0/\theta$ on X (again, Q is a set of measure 0) then satisfies

$$\begin{aligned} \|\alpha\|_{L^2(X, \omega, \varphi)}^2 &= \int_Y |\alpha|^2 e^{-\varphi} \omega = \int_Y |\alpha_0/\theta|^2 e^{-\varphi} \omega = \int_Y \frac{i\alpha_0 \wedge \bar{\alpha}_0}{i\theta \wedge \bar{\theta}} e^{-\varphi} \omega \\ &= \int_Y i\alpha_0 \wedge \bar{\alpha}_0 \cdot e^{-(\varphi - \log \tau)} \\ &= \|\alpha_0\|_{L^2(Y, \varphi - \log \tau)}^2 \leq \|\beta\|_{L^2(X \setminus Z, \varphi + \log \rho)}^2. \end{aligned}$$

In particular, α is in L^2_{loc} on X , and it remains to show that $\bar{\partial}_{\text{distr}}\alpha = \beta$ on X . For each point $p \in X$, we may choose a local holomorphic coordinate neighborhood $(U, \Phi = z, \Delta(0; 2))$ such that $U \cap Q \subset \{p\}$ and $z(p) = 0$. The meromorphic function $f \equiv \theta/dz$ on U is then nonvanishing and holomorphic on $U \setminus \{p\} \subset Y$, and

the measurable functions $a \equiv \alpha_0/dz$ and $b \equiv \beta/d\bar{z}$ on U are in $L^2_{\text{loc}} \subset L^1_{\text{loc}}$. In particular, $\alpha = -a/f$ and $\beta_0 = fb dz \wedge d\bar{z}$ on U . We may also choose a nonnegative C^∞ function χ such that $\chi \equiv 1$ on $\mathbb{C} \setminus \Delta(0; 2)$ and $\chi \equiv 0$ on $\Delta(0; 1)$. For every C^∞ function η with compact support in $D \equiv \Phi^{-1}(\Delta(0; 1))$, the dominated convergence theorem gives (here, we denote the coordinate on \mathbb{C} by w)

$$\begin{aligned}
& \int_D \alpha \cdot \frac{\partial \eta}{\partial \bar{z}} \cdot \frac{i}{2} dz \wedge d\bar{z} \\
&= \lim_{\epsilon \rightarrow 0^+} \int_D \chi(z/\epsilon) \alpha \cdot \frac{\partial \eta}{\partial \bar{z}} \cdot \frac{i}{2} dz \wedge d\bar{z} \\
&= \lim_{\epsilon \rightarrow 0^+} \int_D \chi(z/\epsilon) \frac{-a}{f} \cdot \frac{\partial \eta}{\partial \bar{z}} \cdot \frac{i}{2} dz \wedge d\bar{z} \\
&= \lim_{\epsilon \rightarrow 0^+} \left[- \int_D a \cdot \frac{\partial}{\partial \bar{z}} \left(\chi(z/\epsilon) \frac{\eta}{f} \right) \cdot \frac{i}{2} dz \wedge d\bar{z} \right. \\
&\quad \left. + \int_D a \frac{1}{\epsilon} \frac{\partial \chi}{\partial \bar{w}}(z/\epsilon) \frac{\eta}{f} \cdot \frac{i}{2} dz \wedge d\bar{z} \right] \\
&= \lim_{\epsilon \rightarrow 0^+} \left[- \int_D b \cdot \chi(z/\epsilon) \eta \cdot \frac{i}{2} dz \wedge d\bar{z} \right. \\
&\quad \left. - \int_D \alpha \frac{1}{\epsilon} \frac{\partial \chi}{\partial \bar{w}}(z/\epsilon) \eta \cdot \frac{i}{2} dz \wedge d\bar{z} \right] \\
&= - \int_D b \eta \cdot \frac{i}{2} dz \wedge d\bar{z} - \lim_{\epsilon \rightarrow 0^+} \left[\int_D \alpha \frac{1}{\epsilon} \frac{\partial \chi}{\partial \bar{w}}(z/\epsilon) \eta \cdot \frac{i}{2} dz \wedge d\bar{z} \right].
\end{aligned}$$

On the other hand, since $\alpha \in L^2_{\text{loc}}(X)$ and $\partial \chi / \partial \bar{w} \equiv 0$ on $\mathbb{C} \setminus \Delta(0; 1, 2)$, we have, for some constant $C > 0$,

$$\left| \int_D \alpha \frac{1}{\epsilon} \frac{\partial \chi}{\partial \bar{w}}(z/\epsilon) \eta \cdot \frac{i}{2} dz \wedge d\bar{z} \right| \leq C \|\alpha \circ \Phi^{-1}\|_{L^2(\Delta(0; \epsilon, 2\epsilon))} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0^+.$$

It follows that $\bar{\partial}_{\text{distr}} \alpha = \beta$. □

Exercises for Sect. 2.12

2.12.1 Verify that (see Example 2.12.3) there is a unique Kähler form ω on \mathbb{P}^1 such that

$$\omega|_{\mathbb{C}} = 2i(1 + |z|^2)^{-2} dz \wedge d\bar{z}.$$

Verify also that $i\Theta_\omega = \omega$.

2.12.2 Prove Corollary 2.12.5.

2.12.3 Prove Corollary 2.12.6.

2.12.4 Let X be a Riemann surface, let ω be a Kähler form on X , let φ be a real-valued C^∞ function on X with $i\Theta \equiv i\Theta_{e^{-\varphi}\omega} = i\Theta_\omega + i\Theta_\varphi \geq 0$, let

$\rho \equiv i\Theta/\omega$, and let $Z = \{x \in X \mid \Theta_x = 0\} = \{x \in X \mid \rho(x) = 0\}$. Prove that for every $\mathcal{C}^\infty(1, 0)$ -form β on X with $\beta = 0$ a.e. in Z and $\beta|_{X \setminus Z} \in L^2_{1,0}(X \setminus Z, \varphi + \log \rho)$, there exists a \mathcal{C}^∞ function $\alpha \in L^2_{0,0}(X, \omega, \varphi)$ such that

$$\partial\alpha = \beta \quad \text{and} \quad \|\alpha\|_{L^2(X, \omega, \varphi)} \leq \|\beta\|_{L^2(X \setminus Z, \varphi + \log \rho)}$$

(cf. Exercises 2.9.3 and 2.9.4).

2.13 Topological Hulls and Chains to Infinity

As suggested by Theorem 2.9.1 and its applications (see also Theorem 3.9.1 and its applications), it is useful to have available a large supply of \mathcal{C}^∞ strictly subharmonic functions on an (open) Riemann surface, i.e., \mathcal{C}^∞ functions φ for which $i\Theta_\varphi > 0$ (see Definition 2.8.1). In Sect. 2.14, it will be shown that every open Riemann surface admits a \mathcal{C}^∞ strictly subharmonic exhaustion function. Recall that a function ρ on a Hausdorff space X is an *exhaustion function* if $\{x \in X \mid \rho(x) < a\} \Subset X$ for each $a \in \mathbb{R}$ (see Definition 9.3.10). The first proof of the existence of a \mathcal{C}^∞ strictly subharmonic exhaustion function on a connected noncompact Riemannian manifold was obtained by Greene and Wu [GreW]. Demailly [De2] provided an elementary proof using a local construction. Demailly's proof may be modified (as well as simplified) to give an exhausting \mathcal{C}^∞ strict subsolution for an arbitrary second-order linear elliptic differential operator with continuous coefficients on a second countable noncompact \mathcal{C}^∞ manifold (for the details see [NR]). The proof of the existence of a \mathcal{C}^∞ strictly subharmonic exhaustion function on an open Riemann surface appearing in this book is adapted from [NR]. In this section, we first consider some of the topological facts required for the proof.

Definition 2.13.1 For a subset A of a Hausdorff space X , the *topological hull* $\mathfrak{h}_X(A)$ of A in X is the union of A with all of the connected components of $X \setminus A$ that are relatively compact in X . An open subset Ω of X is called *topologically Runge* in X if $\mathfrak{h}_X(\Omega) = \Omega$.

Remark Intuitively, one obtains $\mathfrak{h}_X(A)$ by filling in the holes of A (see Fig. 2.5).

The proofs of the following properties are left to the reader (see Exercise 2.13.1):

Lemma 2.13.2 *Let X be a Hausdorff space.*

- (a) *For every set $A \subset X$, we have $\mathfrak{h}_X(\mathfrak{h}_X(A)) = \mathfrak{h}_X(A) \supset A$.*
- (b) *If $A \subset B \subset X$, then $\mathfrak{h}_X(A) \subset \mathfrak{h}_X(B)$.*
- (c) *If X is locally connected and A is a closed subset of X , then $\mathfrak{h}_X(A)$ is closed.*

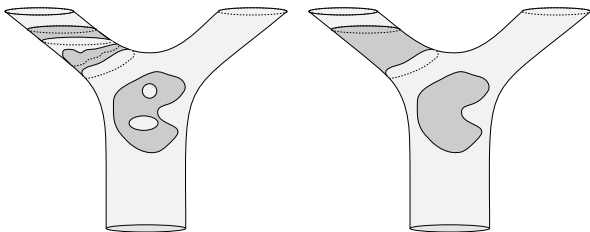


Fig. 2.5 A subset and its topological hull

For a compact subset of a suitable topological space (such as a manifold), the topological hull has a nice form (cf. [Mal] and [Ns5]):

Lemma 2.13.3 *If K is a compact subset of a connected, noncompact, locally connected, locally compact Hausdorff space X , then $\mathfrak{h}_X(K)$ is compact; $X \setminus \mathfrak{h}_X(K)$ has only finitely many components; and for each point $p \in X \setminus \mathfrak{h}_X(K)$, there is a connected noncompact closed subset C of X with $p \in C \subset X \setminus \mathfrak{h}_X(K)$.*

Remark One gets a version in which X has more than one, but only finitely many, connected components by applying the lemma to each connected component (see Exercise 2.13.2).

Proof of Lemma 2.13.3 The lemma is trivial for K empty, so we may assume that $K \neq \emptyset$. Since X is locally connected, $\mathfrak{h}_X(K)$ is a closed set whose complement has no relatively compact components (by Lemma 2.13.2). Since X is locally compact Hausdorff, we may choose a relatively compact neighborhood Ω of K in X . The components of $X \setminus K$ are open and disjoint, so only finitely many meet the compact set $\partial\Omega \subset X \setminus K$. By replacing Ω with the union of Ω and all of the relatively compact components of $X \setminus K$ meeting $\partial\Omega$, we may assume that no relatively compact component of $X \setminus K$ meets $\partial\Omega$. On the other hand, every component E of $X \setminus K$ must satisfy

$$\overline{E} \cap K = \partial E \neq \emptyset.$$

For E is open and closed in $X \setminus K$, so $\partial E \subset K$, while $E \neq X$, so $\partial E = \overline{E} \setminus E \neq \emptyset$ (E cannot be both open and closed in the connected space X). It follows that if E meets $X \setminus \Omega$, then E meets $\partial\Omega$, and hence E is *not* relatively compact in X . Thus

$$X \setminus \Omega \subset E_1 \cup \cdots \cup E_m$$

for finitely many components E_1, \dots, E_m of $X \setminus K$, none of which are relatively compact in X . It follows that $\mathfrak{h}_X(K) \subset \Omega$ and $X \setminus \mathfrak{h}_X(K) = E_1 \cup \cdots \cup E_m$.

Applying the above argument with $K' = \overline{\Omega}$ in place of K , we get a relatively compact neighborhood Ω' of K' in X such that $X \setminus \Omega' \subset X \setminus \mathfrak{h}_X(K') = E'_1 \cup \cdots \cup E'_k$, where E'_1, \dots, E'_k are distinct components of $X \setminus K'$, none of which are relatively compact in X . For each $i = 1, \dots, m$, we get $\overline{E'_j} \subset E_i$ for some

$j \in \{1, \dots, k\}$. Thus, if A_i is the set of points in E_i that lie some connected noncompact closed subset of X that is contained in E_i , then $A_i \neq \emptyset$. Given a point $p \in E_i \cap \overline{A_i}$, we may choose a relatively compact connected neighborhood V of p in E_i . We may then choose a point $q \in A_i \cap V$ and a connected noncompact closed subset B of X with $q \in B \subset E_i$. The set $C \equiv \overline{V} \cup B$ is then a connected noncompact closed subset of X that lies in E_i and that contains V . Thus $V \subset A_i$, and hence the nonempty set A_i is both open and closed in the connected set E_i . Therefore, $A_i = E_i$, and the lemma follows. \square

Lemma 2.13.4 *Let X be a second countable, noncompact, connected, locally connected, locally compact Hausdorff space. Then there is a sequence of compact sets $\{K_v\}_{v=1}^\infty$ such that $X = \bigcup_{v=1}^\infty K_v$, and for each v , $K_v \subset \overset{\circ}{K}_{v+1}$ and $\mathfrak{h}_X(K_v) = K_v$.*

Proof By Lemma 9.3.6, we may fix a sequence of compact sets $\{H_v\}$ with $X = \bigcup_{v=1}^\infty H_v$. Set $K_1 \equiv \mathfrak{h}_X(H_1)$. Given K_v , we may choose a compact set K'_{v+1} with $H_v \cup K_v \subset \overset{\circ}{K}'_{v+1}$, and we may set $K_{v+1} \equiv \mathfrak{h}_X(K'_{v+1})$. This yields the desired sequence. \square

Lemma 2.13.5 *Let X be a connected, locally connected, locally compact Hausdorff space; let \mathcal{B} be a countable collection of connected open subsets that is a basis for the topology in X ; let K be a compact subset of X ; and let U be a connected component of $X \setminus \mathfrak{h}_X(K)$. Then, for each point $p \in U$, there exists a sequence of basis elements $\{B_j\}$ that tends to infinity (i.e., $\{B_j\}$ is a locally finite family in X) such that $p \in B_1$, and for each j , $B_j \subseteq U$ and $B_j \cap B_{j+1} \neq \emptyset$.*

Proof Lemma 2.13.3 provides a connected noncompact closed subset C of X with $p \in C \subset U$, and Lemma 9.3.6 provides a countable locally finite (in X) covering \mathcal{A} of C by basis elements that are relatively compact in U . For each point $q \in C$, there is a finite sequence of elements B_1, \dots, B_k of \mathcal{A} that forms a chain from p to q ; that is, $p \in B_1$, $q \in B_k$, and $B_j \cap B_{j+1} \neq \emptyset$ for $j = 1, \dots, k-1$ (we will call k the length of the chain). For the set E of points q in C for which there is a chain from p to q is clearly nonempty and open in C . On the other hand, E is also closed, because if $q \in \overline{E}$, then $q \in B$ for some set $B \in \mathcal{A}$ and there must be some point $r \in B \cap E$. A chain B_1, \dots, B_k from p to r yields the chain B_1, \dots, B_k, B from p to q . Thus $E = C$. Observe that if $q \in C$ and B_1, \dots, B_k is a chain of minimal length from p to q , then the sets B_1, \dots, B_k are distinct.

Now since C is noncompact and closed, we may choose a locally finite sequence of points $\{q_v\}$ in C ; i.e., $q_v \rightarrow \infty$ in X (for example, we may fix an increasing sequence $\{\Omega_v\}$ of relatively compact open subsets of X with union X and choose $q_v \in C \setminus \Omega_v$ for each v). For each v , we may choose a chain $B_1^{(v)}, \dots, B_{k_v}^{(v)}$ of minimal length from p to q_v . Since the elements of \mathcal{A} are relatively compact in U and \mathcal{A} is locally finite in X , there are only finitely many possible choices for $B_j^{(v)}$ for each j (only finitely many elements of \mathcal{A} are in some chain of length j from p). Moreover, for each fixed $j \in \mathbb{Z}_{>0}$, we have $k_v > j$ for $v \gg 0$, because the set of

points in C joined to p by a chain of length $\leq j$ is relatively compact in C while $q_\nu \rightarrow \infty$. Therefore, after applying a diagonal process and passing to the associated subsequence of $\{q_\nu\}$, we may assume that for each j , there is an element $B_j \in \mathcal{A}$ with $B_j^{(\nu)} = B_j$ for all $\nu \gg 0$. Thus we get an infinite chain of distinct elements $\{B_j\}$ from p to infinity as required in (ii) (local finiteness in X is guaranteed since \mathcal{A} is locally finite and the elements $\{B_j\}$ are distinct). \square

Remark The above lemma actually holds even if the basis elements are not necessarily connected, but it is easier to picture chains of connected sets.

Exercises for Sect. 2.13

2.13.1 Prove Lemma 2.13.2.

2.13.2 Let K be a compact subset of a noncompact, locally connected, locally compact Hausdorff space X . Prove that if X has only finitely many connected components, then $\mathfrak{h}_X(K)$ is compact, $X \setminus \mathfrak{h}_X(K)$ has only finitely many components, and for each point $p \in X \setminus \mathfrak{h}_X(K)$, there is a connected noncompact closed subset C of X with $p \in C \subset X \setminus \mathfrak{h}_X(K)$.

2.13.3 Let K be a compact subset of a connected, noncompact, locally connected, locally compact Hausdorff space X .

(a) Prove that $\mathfrak{h}_X(K)$ is equal to the intersection of all topologically Runge neighborhoods of K .

(b) Prove that for every neighborhood U of $\mathfrak{h}_X(K)$ in X , there exists a topologically Runge open set Ω with $\mathfrak{h}_X(K) \subset \Omega \subset U$.

Hint. Show that one may assume without loss of generality that $U \Subset X$ and that there exists a finite covering of ∂U by connected noncompact closed subsets of X that lie in $X \setminus \mathfrak{h}_X(K)$. Then set Ω equal to the complement of these sets in U .

2.13.4 Prove that if X is a locally connected Hausdorff space and A is a connected subset of X , then $\mathfrak{h}_X(A)$ is connected.

2.13.5 Let Ω be a relatively compact open subset of a connected, noncompact, locally connected, locally compact Hausdorff space X . Prove that $\mathfrak{h}_X(\Omega)$ is a relatively compact open subset of X and that $X \setminus \mathfrak{h}_X(\Omega)$ has only finitely many connected components (cf. Exercise 2.13.6 below).

2.13.6 This exercise will be applied in Exercise 2.16.5 (cf. [Ns5]).

(a) Prove that if Y is a locally compact Hausdorff space, C is a compact connected component of Y , and U is a neighborhood of C in Y , then there exists a set A such that $C \subset A \subset U$ and A is open and closed in Y .

Hint. First work in a relatively compact neighborhood V of C in Y . Let D be the intersection of all subsets of \bar{V} that are open and closed in \bar{V} and that contain C . Show that each neighborhood of D in \bar{V} contains a set $A \supset D$ that is open and closed in \bar{V} . Using this observation, prove that D is connected and conclude that $D = C$. Finally, construct the desired open and closed subset of Y for a given neighborhood U .

- (b) Let Ω be an open subset of a locally compact Hausdorff space X , and let C be a compact connected component of $X \setminus \Omega$. Prove that for every neighborhood U of C in X , there is a neighborhood V of C in U with $\partial V \subset \Omega$.
- (c) Let Ω be an open subset of a locally compact Hausdorff space X . Prove that $\mathfrak{h}_X(\Omega)$ is open.

2.13.7 Let X be a connected, locally *path* connected, locally compact Hausdorff space; let \mathcal{B} be a countable collection of connected open subsets that is a basis for the topology in X ; let K be a compact subset of X ; and let U be a connected component of $X \setminus \mathfrak{h}_X(K)$. Prove that for each point $p \in U$, there is a proper continuous map $\gamma : [0, \infty) \rightarrow X$ with $\gamma(0) = p$ and $\gamma([0, \infty)) \subset U$ (i.e., a path in U from p to ∞).

In the remaining exercises in this section, and in some exercises in later sections, generalizations in various contexts are obtained by considering a different hull as follows. For a subset A of a Hausdorff space X , the *extended topological hull* $\mathfrak{h}_X^*(A)$ of A in X is the union of A with all of the connected components of $X \setminus A$ that do not contain any connected noncompact closed subsets of X .

- 2.13.8 Prove that Lemma 2.13.2 holds with the extended topological hull in place of the topological hull. Prove also that $\mathfrak{h}_X(A) \subset \mathfrak{h}_X^*(A)$.
- 2.13.9 Give an example of a closed set K in a Hausdorff space X with $\mathfrak{h}_X^*(K) \neq \mathfrak{h}_X(K)$.
- 2.13.10 Prove that if Ω is an open subset of a Hausdorff space X , then $\mathfrak{h}_X^*(\Omega) = \mathfrak{h}_X(\Omega)$.
- 2.13.11 Prove that if K is a compact subset of a connected, noncompact, locally connected, locally compact Hausdorff space X , then $\mathfrak{h}_X^*(K) = \mathfrak{h}_X(K)$.
- 2.13.12 Let X be a second countable connected, locally connected, locally compact Hausdorff space, and let K be a closed subset of X .
 - (a) Prove that if \mathcal{B} is a countable collection of connected open subsets that is a basis for the topology in X and U is a connected component of $X \setminus \mathfrak{h}_X^*(K)$, then for each point $p \in U$, there exists a sequence of basis elements $\{B_j\}$ that tends to infinity (i.e., $\{B_j\}$ is a locally finite family in X) such that $p \in B_1$ and for each j , $B_j \subseteq U$ and $B_j \cap B_{j+1} \neq \emptyset$ (cf. Lemma 2.13.5).
 - (b) Prove that if $D \subset X \setminus K$ is a closed subset of X with no compact connected components, then there exist a countable locally finite (in X) family of disjoint connected noncompact closed sets $\{C_\lambda\}_{\lambda \in \Lambda}$ and a locally finite (in X) family of disjoint connected open sets $\{U_\lambda\}_{\lambda \in \Lambda}$ such that

$$D \subset C \equiv \bigcup_{\lambda \in \Lambda} C_\lambda \quad \text{and} \quad C_\lambda \subset U_\lambda \subset \overline{U}_\lambda \subset X \setminus K \quad \forall \lambda \in \Lambda.$$

Hint. First show that there is a countable locally finite covering \mathcal{A} of D by connected open relatively compact subsets of $X \setminus K$ each of which meets D , that $X \setminus K$ contains the closure $C \equiv \overline{V}$ of the union V

of the elements of \mathcal{A} , that the family of components $\{V_\gamma\}_{\gamma \in \Gamma}$ of V is countable and locally finite, and that $\overline{V_\gamma}$ is noncompact for each $\gamma \in \Gamma$. Choosing suitable neighborhoods $\{U_\lambda\}_{\lambda \in \Lambda}$ of the connected components $\{C_\lambda\}_{\lambda \in \Lambda}$ of C , one gets the claim.

2.13.13 Let K be a subset of a second countable, connected, noncompact, locally connected, locally compact Hausdorff space X .

- (a) Prove that, if K is closed, then $\mathfrak{h}_X^*(K)$ is equal to the intersection of all topologically Runge neighborhoods of K (cf. Exercise 2.13.3). Give an example that shows that this need not be the case if K is not closed.
- (b) Give an example of such a space X and a closed subset K such that $K = \mathfrak{h}_X(K) = \mathfrak{h}_X^*(K)$, but not every neighborhood of K contains a neighborhood of K that is topologically Runge in X .

2.14 Construction of a Subharmonic Exhaustion Function

This section contains the construction, alluded to in the beginning of Sect. 2.13, of a C^∞ strictly subharmonic exhaustion function on an open Riemann surface. The main goal is the following:

Theorem 2.14.1 *Suppose X is an open Riemann surface, K is a compact subset of X satisfying $\mathfrak{h}_X(K) = K$, τ is a continuous real-valued function on X , θ is a continuous real $(1, 1)$ -form on X , and Ω is a neighborhood of K in X . Then there exists a C^∞ exhaustion function φ on X such that*

- (i) *On X , $\varphi \geq 0$ and $i\Theta_\varphi \geq 0$;*
- (ii) *On $X \setminus \Omega$, $\varphi > \tau$ and $i\Theta_\varphi \geq \theta$; and*
- (iii) *We have $\text{supp } \varphi \subset X \setminus K$.*

Remark It suffices to consider the case $\tau \geq 0$ and $\theta \geq 0$, since in general, one may construct φ with the nonnegative function $|\tau| \geq \tau$ and the nonnegative form $\theta^+ + \theta^- \geq \theta$ in place of τ and θ , respectively.

Setting $K = \Omega = \emptyset$ and choosing $\theta > 0$ (as we may by Corollary 2.11.3), we get the following:

Corollary 2.14.2 *Every open Riemann surface admits a C^∞ strictly subharmonic exhaustion function.*

After fixing a Kähler form, one may work with a convenient 0-form in place of the curvature form:

Definition 2.14.3 The Laplace operator associated to a Kähler form ω on a complex 1-manifold X is the second-order linear differential operator Δ_ω with C^∞ co-

efficients given by

$$\Delta_\omega \varphi \equiv \frac{i \partial \bar{\partial} \varphi}{\omega}$$

for every C^2 function φ . In particular, for φ real-valued, we have $\Delta_\omega \varphi = i \Theta_\varphi / \omega$.

Remarks 1. Writing $\omega = G \cdot (i/2) dz \wedge d\bar{z} = G dx \wedge dy$ with respect to a local holomorphic coordinate $z = x + iy$, we get

$$\Delta_\omega = \frac{2}{G} \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{2G} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

For example, if $\omega = (i/2) dz \wedge d\bar{z} = dx \wedge dy$ is the standard Euclidean volume form on \mathbb{C} , then

$$\Delta_\omega = 2 \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),$$

which is $1/2$ the standard Laplace operator.

2. Clearly, a real-valued C^∞ (or C^2) function φ on an open set $\Omega \subset X$ is subharmonic (strictly subharmonic) if and only if $\Delta_\omega \varphi \geq 0$ (respectively, $\Delta_\omega \varphi > 0$) on Ω .

For the rest of this section, X will denote an *open* Riemann surface. According to Radó's theorem, X is second countable, and applying Corollary 2.11.3, we get a Kähler form ω on X . Instead of proving Theorem 2.14.1 directly, we will prove the following equivalent version (in Exercise 2.14.1, the reader is asked to verify that the two versions are equivalent):

Theorem 2.14.4 (Cf. [GreW], [De2], and [NR]) *Suppose K is a compact subset of X satisfying $\mathfrak{h}_X(K) = K$, ρ is a continuous real-valued function on X , and Ω is a neighborhood of K in X . Then there exists a C^∞ exhaustion function φ on X such that*

- (i) *On X , $\varphi \geq 0$ and $\Delta_\omega \varphi \geq 0$;*
- (ii) *On $X \setminus \Omega$, $\varphi > \rho$ and $\Delta_\omega \varphi > \rho$; and*
- (iii) *We have $\text{supp } \varphi \subset X \setminus K$.*

The main step in the proof of Theorem 2.14.4 is the following:

Proposition 2.14.5 *Let K be a compact subset of X satisfying $\mathfrak{h}_X(K) = K$. Then, for each point $p \in X \setminus K$, there is a nonnegative C^∞ function α on X such that $\Delta_\omega \alpha \geq 0$ on X , $\text{supp } \alpha \subset X \setminus K$, $\alpha(p) > 0$, and $\Delta_\omega \alpha(p) > 0$.*

Remark According to the maximum principle for subharmonic functions (see, for example, [Ns5]), a nonconstant subharmonic function on a domain cannot attain a maximum value (in particular, any subharmonic function on a compact Riemann

surface is constant). Therefore, the converse of the proposition also holds; that is, if such a function α exists for some point $p \in X \setminus K$, then the component of $X \setminus K$ containing p is not relatively compact in X .

Assuming Proposition 2.14.5 for now, we may prove Theorem 2.14.4.

Proof of Theorem 2.14.4 By Proposition 9.3.11, we may assume that ρ is a positive exhaustion function. Let $K_0 = K$. By Lemma 2.13.4, we may choose a sequence of nonempty compact sets $\{K_v\}$ such that $X = \bigcup_{v=1}^{\infty} K_v$ and such that for each $v = 1, 2, 3, \dots$, $K_{v-1} \subset \overset{\circ}{K}_v$ and $\mathfrak{h}_X(K_v) = K_v$.

Given a point $p \in X \setminus \Omega$, there is a unique $v = v(p) > 0$ with $p \in K_v \setminus K_{v-1}$, and Proposition 2.14.5 provides a nonnegative C^∞ function α_p and a relatively compact neighborhood V_p of p in $X \setminus K_{v-1}$ such that $\Delta_\omega \alpha_p \geq 0$ on X , $\text{supp } \alpha_p \subset X \setminus K_{v-1}$, and $\alpha_p > \rho$ and $\Delta_\omega \alpha_p > \rho$ on V_p (one obtains the last two conditions by multiplying by a sufficiently large positive constant). We may choose a sequence of points $\{p_k\}$ in X , and corresponding functions $\{\alpha_{p_k}\}$ and neighborhoods $\{V_{p_k}\}$, so that $\{V_{p_k}\}$ forms a locally finite covering of $X \setminus \Omega$ (for example, we may take $\{p_k\}$ to be an enumeration of the countable set $\bigcup_{v=0}^{\infty} Z_v$, where for each v , Z_v is a finite set of points in $X \setminus [\overset{\circ}{K}_{v-1} \cup \Omega]$ such that $\{V_p\}_{p \in Z_v}$ covers $K_v \setminus [\overset{\circ}{K}_{v-1} \cup \Omega]$). The collection $\{\text{supp } \alpha_{p_k}\}$ is then locally finite in X , since $\text{supp } \alpha_{p_k} \subset X \setminus K_{v-1}$ whenever $p_k \notin K_{v-1}$. Hence the sum $\sum_{k=1}^{\infty} \alpha_{p_k}$ is locally finite and therefore convergent to a C^∞ function φ on X satisfying $\varphi \geq \alpha_{p_k} > \rho$ and $\Delta_\omega \varphi \geq \Delta_\omega \alpha_{p_k} > \rho$ on V_{p_k} for each k . Therefore, since $\{V_{p_k}\}$ covers $X \setminus \Omega$, we get $\varphi > \rho$ and $\Delta_\omega \varphi > \rho$ on $X \setminus \Omega$. In particular, φ is an exhaustion function. Similarly, we also have $\text{supp } \varphi \subset X \setminus K$, and $\varphi \geq 0$ and $\Delta_\omega \varphi \geq 0$ on X . \square

It remains to prove Proposition 2.14.5. The idea is to form a sequence of functions, each of which is subharmonic outside a small set and strictly subharmonic on the bad set of the previous function. Multiplying each function by a sufficiently large positive constant (obtained inductively), one pushes these small bad sets off to infinity.

Lemma 2.14.6 *There exists a nonnegative C^∞ function χ on \mathbb{R} such that $\chi \equiv 0$ on $(-\infty, 0]$ and $\chi, \chi', \chi'' > 0$ on $(0, \infty)$.*

Proof For example, the C^∞ function

$$\chi(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \exp(t - (1/t)) & \text{if } t > 0, \end{cases}$$

has the required properties. The verification is left to the reader (see Exercise 2.14.3). \square

Lemma 2.14.7 *For every $r \in (0, 1)$, there exists a nonnegative C^∞ function α on \mathbb{P}^1 such that*

- (i) We have $\alpha > 0$ on $\Delta(0; 1)$ and $\alpha \equiv 0$ on $\mathbb{P}^1 \setminus \Delta(0; 1)$; and
- (ii) The function α is strictly subharmonic on the annulus $\Delta(0; r, 1)$ (and therefore subharmonic on $\mathbb{P}^1 \setminus \overline{\Delta(0; r)}$).

Proof Letting $\chi: \mathbb{R} \rightarrow [0, \infty)$ be the function provided by Lemma 2.14.6, it is easy to see that the function given by $\alpha(\infty) = 0$ and

$$\alpha(z) \equiv \chi(e^{-|z|^2/r^2} - e^{-1/r^2}) \quad \forall z \in \mathbb{C}$$

has the required properties. Again, the verification is left to the reader (see Exercise 2.14.4). \square

Proposition 2.14.8 *Let $c \in \Delta \equiv \Delta(0; 1)$, and let $\Phi: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the map given by $z \mapsto (z - c)/(1 - \bar{c}z)$ (with $\Phi(1/\bar{c}) = \infty$ and $\Phi(\infty) = -1/\bar{c}$). Then we have the following:*

- (i) Φ is an automorphism of \mathbb{P}^1 (i.e., a biholomorphic mapping of \mathbb{P}^1 onto itself);
- (ii) $\Phi(c) = 0$, $\Phi'(0) = 1 - |c|^2$, and $\Phi'(c) = 1/(1 - |c|^2)$; and
- (iii) $\Phi(\Delta) = \Delta$ and $\Phi(\partial\Delta) = \partial\Delta$.

The proof is left to the reader (see Exercise 2.14.5). It follows from the above that $\Phi|_{\Delta}$ is an automorphism of Δ . In Chapter 5, we will see that in fact, every automorphism of Δ is of the form $b\Phi|_{\Delta}$ for some $b \in \partial\Delta$ and some Φ as above (see Theorem 5.8.2).

Lemma 2.14.9 *For every nonempty relatively compact open subset U of the unit disk $\Delta \equiv \Delta(0; 1)$, there exists a nonnegative \mathcal{C}^∞ function β on \mathbb{P}^1 such that*

- (i) We have $\beta > 0$ on Δ and $\beta \equiv 0$ on $\mathbb{P}^1 \setminus \Delta$; and
- (ii) The function β is strictly subharmonic on the set $\Delta \setminus \overline{U}$ (and therefore subharmonic on $\mathbb{P}^1 \setminus \overline{U}$).

Proof Fixing a point $c \in U$, we get the automorphism $\Phi: z \mapsto (z - c)/(1 - \bar{c}z)$ of \mathbb{P}^1 mapping c to 0 as in Proposition 2.14.8, and for $r \in (0, 1)$ sufficiently small, we have $\Phi^{-1}(\Delta(0; r)) \subset U$. By Lemma 2.14.7, there exists a nonnegative \mathcal{C}^∞ function α on \mathbb{P}^1 such that $\alpha > 0$ on Δ , $\alpha \equiv 0$ on $\mathbb{P}^1 \setminus \Delta$, and α is strictly subharmonic on $\Delta(0; r, 1)$. The function $\beta \equiv \alpha(\Phi)$ then has the required properties. \square

Proof of Proposition 2.14.5 By Lemma 2.13.5, given a point $p \in X \setminus K$, there is a locally finite (in X) sequence of relatively compact open subsets $\{V_m\}$ of $X \setminus K$ such that $p \in V_1$ and such that for each m , we have $V_m \cap V_{m+1} \neq \emptyset$ and there is a biholomorphism of a neighborhood of the closure \overline{V}_m of V_m onto a neighborhood of the closure $\overline{\Delta}$ of the unit disk $\Delta \equiv \Delta(0; 1)$ that maps V_m onto Δ . Hence we may choose a sequence of nonempty open sets $\{W_m\}_{m=0}^\infty$ with disjoint closures such that $p \in W_0 \subseteq V_1$ and such that for each $m \geq 1$, $W_m \subseteq V_m \cap V_{m+1}$ (see Fig. 2.6).

By Lemma 2.14.9, there is a sequence of nonnegative \mathcal{C}^∞ functions $\{\beta_m\}_{m=1}^\infty$ such that for each m , $\beta_m \equiv 0$ on $X \setminus V_m$, $\Delta_\omega \beta_m \geq 0$ on $X \setminus W_m$, and

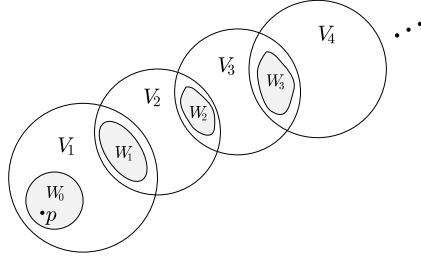


Fig. 2.6 Construction of a subharmonic function that is strictly subharmonic near the point p

$\beta_m > 0$ and $\Delta_\omega \beta_m > 0$ on \overline{W}_{m-1} . We will choose positive constants $\{R_m\}_{m=1}^\infty$ inductively so that for each $m = 1, 2, 3, \dots$,

$$\Delta_\omega \left(\sum_{j=1}^m R_j \beta_j \right) \begin{cases} \geq 0 & \text{on } X \setminus W_m, \\ > 0 & \text{on } \overline{W}_0. \end{cases}$$

Fix $R_1 > 0$. Given $R_1, \dots, R_{m-1} > 0$ with the above property, using the fact that $\Delta_\omega \beta_m > 0$ on \overline{W}_{m-1} , we get, for $R_m \gg 0$,

$$\Delta_\omega \left(\sum_{j=1}^m R_j \beta_j \right) = R_m \Delta_\omega \beta_m + \Delta_\omega \left(\sum_{j=1}^{m-1} R_j \beta_j \right) > 0 \quad \text{on } \overline{W}_{m-1}.$$

On $X \setminus (W_{m-1} \cup W_m)$, we have $\Delta_\omega \beta_m \geq 0$ and hence

$$\Delta_\omega \left(\sum_{j=1}^m R_j \beta_j \right) \geq \Delta_\omega \left(\sum_{j=1}^{m-1} R_j \beta_j \right) \geq 0.$$

On $\overline{W}_0 \subset X \setminus (W_{m-1} \cup W_m)$, the above middle expression, and hence the expression on the left, is positive. Moreover, $\sum_{j=1}^m R_j \beta_j \geq R_1 \beta_1 > 0$ on \overline{W}_0 . Proceeding, we get the sequence $\{R_m\}$. The sum $\sum R_m \beta_m$ is locally finite in X and the sequence of sets $\{W_m\}$ is locally finite in X , so the sum converges to a function α with the required properties. \square

Remark When second countability of a particular open Riemann surface X is evident (for example, when X is a proper nonempty open subset of a compact Riemann surface), one gets Theorem 2.14.4 without any reliance on Radó's theorem, and hence with almost no reliance on Sects. 2.6–2.12.

The existence of strictly subharmonic exhaustion functions allows one to use the full power of the $L^2 \bar{\partial}$ -method. For example, we have the following:

Theorem 2.14.10 *Let $p \in \{0, 1\}$ and let β be a locally square-integrable differential form of type $(p, 1)$ on the open Riemann surface X . Then there exists a locally*

square-integrable differential form α of type $(p, 0)$ with $\bar{\partial}_{\text{distr}}\alpha = \beta$. In particular, if β is of class C^k for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class C^k and $\bar{\partial}\alpha = \beta$.

Proof We may fix a Kähler form ω on X . Applying Theorem 2.14.4, with the compact set given by $K = \emptyset$ and the function ρ chosen (with the help of Lemma 9.7.13) so large that $\beta \in L^2_{0,1}(X, \rho)$ or $L^2_{1,1}(X, \omega, \rho)$ and $\rho > 1 + |i\Theta_\omega/\omega|$, we get a positive C^∞ strictly subharmonic (exhaustion) function φ on X such that

$$\begin{aligned} i\Theta_\omega + i\Theta_\varphi &\geq \omega & \text{and} & & \|\beta\|_{L^2(X, \varphi)} < \infty & \text{if } p = 0, \\ i\Theta_\varphi &\geq \omega & \text{and} & & \|\beta\|_{L^2(X, \omega, \varphi)} < \infty & \text{if } p = 1. \end{aligned}$$

Corollary 2.12.6 and Corollary 2.9.3 now give the desired differential form α . \square

In Sects. 2.15 and 2.16, we will consider other applications to open Riemann surfaces.

Exercises for Sect. 2.14

2.14.1 Prove that Theorem 2.14.1 and Theorem 2.14.4 are equivalent.

2.14.2 Let φ be a real-valued C^∞ function on a Riemann surface X .

(a) Prove that if χ is a real-valued C^∞ function on \mathbb{R} , then

$$\Theta_{\chi(\varphi)} = \chi'(\varphi)\Theta_{\chi(\varphi)} + \chi''(\varphi)\partial\varphi \wedge \bar{\partial}\varphi.$$

From this conclude that if φ is subharmonic and χ' and χ'' are nonnegative, then $\chi(\varphi)$ is subharmonic. Show also that if φ is strictly subharmonic and $\chi' > 0$ and $\chi'' \geq 0$, then $\chi(\varphi)$ is strictly subharmonic.

(b) Prove that if φ is a strictly subharmonic exhaustion function on X , τ is a continuous real-valued function on X , and θ is a continuous real $(1, 1)$ -form on X , then there exists a C^∞ function $\chi: \mathbb{R} \rightarrow \mathbb{R}$ such that $\chi(\varphi) > \tau$ and $i\Theta_{\chi(\varphi)} \geq \theta$.

2.14.3 Verify that the function constructed in the proof of Lemma 2.14.6 has the required properties.

2.14.4 Verify that the function constructed in the proof of Lemma 2.14.7 has the required properties.

2.14.5 Prove Proposition 2.14.8.

2.14.6 Let X be an open Riemann surface.

(a) Prove that if $q \in \{0, 1\}$ and β is a C^∞ differential form of type $(1, q)$ on X , then there exists a C^∞ differential form α of type $(0, q)$ with $\bar{\partial}\alpha = \beta$.

(b) Prove that if ω is a Kähler form on X , then there exists a C^∞ strictly subharmonic function φ on X such that $i\Theta_\varphi = \omega$.

2.14.7 This exercise requires Exercises 2.13.8 and 2.13.12. Let X be an open Riemann surface, let ω be a Kähler form on X , and let K be a closed subset of X satisfying $\mathfrak{h}_X^*(K) = K$.

- (a) Prove that for each connected component U of $X \setminus K$ and each point $p \in X \setminus K$, there is a nonnegative C^∞ function α on X such that $\Delta_\omega \alpha \geq 0$ on X , $\text{supp } \alpha \subset U$, $\alpha(p) > 0$, and $\Delta_\omega \alpha(p) > 0$ (cf. Proposition 2.14.5).
- (b) Suppose that U is a connected component of $X \setminus K$, C is a connected noncompact closed subset of X contained in U , and ρ is a continuous real-valued function on X . Prove that there exists a C^∞ function φ on X such that
- (i) We have $\varphi \geq 0$ and $\Delta_\omega \varphi \geq 0$ on X ;
 - (ii) We have $\varphi > \rho$ and $\Delta_\omega \varphi > \rho$ on C ; and
 - (iii) We have $\text{supp } \varphi \subset U$.

Hint. First show that there exists a connected *locally connected* closed set D in X with $C \subset D \subset U$ (for example, by forming the union of the closures of elements of a suitable locally finite covering of C by coordinate disks each of which meets C). Then show that there is a sequence of compact sets $\{K_\nu\}$ such that $X = \bigcup_\nu K_\nu$ and such that for each ν , $K_\nu \subset \overset{\circ}{K}_{\nu+1}$ and $\mathfrak{h}_D(K_\nu \cap D) = K_\nu \cap D$. Proceed now as in the proof of Theorem 2.14.4.

- (c) Suppose that $D \subset X \setminus K$ is a closed subset of X with no compact connected components and ρ is a real-valued continuous function on X . Prove that there exists a C^∞ function φ on X such that
- (i) We have $\varphi \geq 0$ and $\Delta_\omega \varphi \geq 0$ on X ;
 - (ii) We have $\varphi > \rho$ and $\Delta_\omega \varphi > \rho$ on D ; and
 - (iii) We have $\text{supp } \varphi \subset X \setminus K$.

2.15 The Mittag-Leffler Theorem

One of the main applications of the $L^2 \bar{\partial}$ -method is the construction of a holomorphic or meromorphic function (or section of a holomorphic line bundle) with prescribed values on a given discrete set, or as in the generalization of the classical Mittag-Leffler theorem below, with some prescribed Laurent series terms (see Theorem 1.3.6). This generalization is due to Behnke–Stein [BehS] (see also Flo-rack [Fl]):

Theorem 2.15.1 (Mittag-Leffler theorem) *Let X be an open Riemann surface, let P be a discrete subset of X (i.e., a closed set with no limit points in X), and for each point $p \in P$, let U_p be a neighborhood of p with $U_p \cap P = \{p\}$, let f_p be a holomorphic function on $U_p \setminus \{p\}$, and let m_p be a positive integer. Then there exists a holomorphic function f on $X \setminus P$ such that for each point $p \in P$, $f - f_p$ extends to a holomorphic function on U_p that either vanishes on a neighborhood of p or has a zero of order at least m_p at p (in other words, if z is a local holomorphic coordinate on a neighborhood of p , and $f = \sum_{n \in \mathbb{Z}} a_n(z - z(p))^n$ and $f_p = \sum_{n \in \mathbb{Z}} b_n(z - z(p))^n$ are the corresponding Laurent series expansions centered at p , then $a_{m_p-n} = b_{m_p-n}$ for $n = 1, 2, 3, \dots$).*

Remark It follows that one may actually choose the above function $f \in \mathcal{O}(X \setminus P)$ so that for each point $p \in P$, $f - f_p$ extends to a holomorphic function on U_p with a zero of order equal to m_p at p (see Exercise 2.15.1).

The proofs of Corollaries 2.15.2, 2.15.3, and 2.15.4 below are left to the reader (see Exercises 2.15.2, 2.15.3, and 2.15.4).

Corollary 2.15.2 *Let P be a discrete subset of an open Riemann surface X .*

- (a) *If f_p is a meromorphic function on a neighborhood of p and $m_p \in \mathbb{Z}_{>0}$ for each point $p \in P$, then there exists a function $f \in \mathcal{M}(X)$ such that f is holomorphic on $X \setminus P$ and $\text{ord}_p(f - f_p) \geq m_p$ for every $p \in P$.*
- (b) *If f_p is a holomorphic function on a neighborhood of p and $m_p \in \mathbb{Z}_{>0}$ for each point $p \in P$, then there exists a function $f \in \mathcal{O}(X)$ with $\text{ord}_p(f - f_p) \geq m_p$ for every $p \in P$.*
- (c) *If $\zeta_p \in \mathbb{C}$ for each point $p \in P$, then there exists a function $f \in \mathcal{O}(X)$ with $f(p) = \zeta_p$ for every $p \in P$.*

Corollary 2.15.3 (Behnke–Stein theorem [BehS]) *Every open Riemann surface X is Stein; that is, X has the following properties:*

- (i) (Holomorphic convexity) *If P is any infinite discrete subset of X , then there exists a holomorphic function on X that is unbounded on P ;*
- (ii) (Separation of points) *If $p, q \in X$ and $p \neq q$, then there exists a holomorphic function f on X such that $f(p) \neq f(q)$; and*
- (iii) (Global functions give local coordinates) *For each point $p \in X$, there exists a holomorphic function f on X such that $(df)_p \neq 0$.*

Corollary 2.15.4 *Let X be an open Riemann surface, let P be a discrete subset of X , and for each point $p \in P$, let U_p be a neighborhood of p with $U_p \cap P = \{p\}$, let θ_p be a holomorphic 1-form on $U_p \setminus \{p\}$, and let m_p be a positive integer. Then there exists a holomorphic 1-form θ on $X \setminus P$ such that for each point $p \in P$, $\theta - \theta_p$ extends to a holomorphic 1-form on U_p that either vanishes on a neighborhood of p or has a zero of order at least m_p at p .*

Proof of Theorem 2.15.1 Let $Y = X \setminus P$. We may choose a Kähler form ω on X , and we may choose a locally finite family of disjoint local holomorphic coordinate neighborhoods $\{(V_p, z_p)\}_{p \in P}$ and a family of open sets $\{W_p\}_{p \in P}$ such that for each point $p \in P$, we have $p \in W_p \subseteq V_p \subseteq U_p$ and $z_p(p) = 0$. Let $V = \bigcup_{p \in P} V_p$ and $W = \bigcup_{p \in P} W_p$.

By cutting off, we may construct a real-valued C^∞ function ρ_0 on Y such that $\text{supp } \rho_0 \subset V$ and $\rho_0 = m_p \log |z_p|^2$ on $W_p \setminus \{p\}$ for each $p \in P$. We may also fix a C^∞ function γ on Y such that $\text{supp } \gamma \subset V$ and $\gamma = f_p$ on $W_p \setminus \{p\}$ for each $p \in P$. The form $\beta = \bar{\partial}\gamma$ is then a C^∞ differential form of type $(0, 1)$ on Y with $\text{supp } \beta \subset V \setminus W$. Since $\log |z_p|^2$ is a harmonic function on $V_p \setminus \{p\}$ for each $p \in P$, by applying Theorem 2.14.4 (or Theorem 2.14.1), with the compact set equal to the

empty set and the function ρ chosen (with the help of Lemma 9.7.13) so large that $\beta \in L^2_{0,1}(Y, \rho - |\rho_0|)$ and $\rho > 1 + |i\Theta_\omega/\omega| + |\Delta_\omega\rho_0|$ on Y , we get a positive C^∞ strictly subharmonic (exhaustion) function ρ_1 on X such that $i\Theta_\omega + i\Theta_{\rho_0} + i\Theta_{\rho_1} \geq \omega$ on Y and such that $\|\beta\|_{L^2(Y, \rho_0 + \rho_1)} < \infty$.

Corollary 2.12.6 now provides a C^∞ function α on Y such that

$$\bar{\partial}\alpha = \beta \quad \text{and} \quad \|\alpha\|_{L^2(Y, \omega, \rho_0 + \rho_1)} < \infty.$$

In particular, the C^∞ function $f \equiv \gamma - \alpha$ is holomorphic on Y . Moreover, for each $p \in P$, we have $f - f_p = -\alpha$ on $W_p \setminus \{p\}$. On the other hand, for some positive constant C (depending on p), we have

$$\begin{aligned} \|z_p^{-m_p} \alpha\|_{L^2(W_p \setminus \{p\}, (i/2)dz_p \wedge d\bar{z}_p)} \\ = \|\alpha\|_{L^2(W_p \setminus \{p\}, (i/2)dz_p \wedge d\bar{z}_p, \rho_0)} \leq C \|\alpha\|_{L^2(W_p \setminus \{p\}, \omega, \rho_0 + \rho_1)} \\ \leq C \|\alpha\|_{L^2(Y, \omega, \rho_0 + \rho_1)} < \infty. \end{aligned}$$

Hence the holomorphic function $z_p^{-m_p} \alpha$ on $W_p \setminus \{p\}$ is square-integrable, and therefore, by Riemann's extension theorem (Theorem 1.2.10), this function extends to a holomorphic function on W_p . Thus α extends holomorphically past p with order at least m_p at p , and therefore $f - f_p$ extends to a holomorphic function on U_p with order at least m_p at p . \square

Exercises for Sect. 2.15

- 2.15.1 Prove that in the Mittag-Leffler theorem (Theorem 2.15.1), one may actually choose the function $f \in \mathcal{O}(X \setminus P)$ so that for each point $p \in P$, $f - f_p$ extends to a holomorphic function on U_p with a zero of order *equal to* m_p at p .
- 2.15.2 Prove Corollary 2.15.2.
- 2.15.3 Prove the Behnke–Stein theorem (Corollary 2.15.3).
- 2.15.4 Prove Corollary 2.15.4.
- 2.15.5 Let f be a meromorphic function on an open Riemann surface X . Prove that there exist holomorphic functions g and h on X such that h is not the zero function and $f = g/h$.
- 2.15.6 The goal of this exercise is a generalization of the Mittag-Leffler theorem that, in higher dimensions, is known as the solution of the *additive Cousin problem* (or the *Cousin problem I*). Let X be an open Riemann surface, let P be a discrete subset of X , let $\{m_p\}_{p \in P}$ be a collection of positive integers, let $\{U_i\}_{i \in I}$ be an open covering of X , and for each pair of indices $i, j \in I$, let f_{ij} be a holomorphic function on $U_i \cap U_j$ with $\text{ord}_p f_{ij} \geq m_p$ for each point $p \in P \cap U_i \cap U_j$. Assume that the family $\{f_{ij}\}$ satisfies the (additive) *cocycle relation*

$$f_{ik} = f_{ij} + f_{jk} \text{ on } U_i \cap U_j \cap U_k \quad \forall i, j, k \in I.$$

Prove that there exist functions $\{g_i\}_{i \in I}$ such that

- (i) For each index $i \in I$, $g_i \in \mathcal{O}(U_i)$ and $\text{ord}_p g_i \geq m_p$ for each point $p \in P \cap U_i$; and
 - (ii) For each pair of indices $i, j \in I$, we have $f_{ij} = g_j - g_i$ on $U_i \cap U_j$.
- Prove also that the above implies the standard Mittag-Leffler theorem (Theorem 2.15.1).

Hint. Using a \mathcal{C}^∞ partition of unity $\{\lambda_\nu\}$ such that each point in P lies in $\text{supp } \lambda_\nu$ for exactly one index ν and such that for each ν , $\text{supp } \lambda_\nu \subset U_{k_\nu}$ for some index $k_\nu \in I$, one may form a \mathcal{C}^∞ solution of the problem of the form $v_i = \sum_\nu \lambda_\nu \cdot f_{k_\nu i}$. In particular, the forms $\{\bar{\partial} v_i\}$ agree on the overlaps and therefore determine a well-defined $(0, 1)$ -form β on X . Suitable weight functions (as in the proof of Theorem 2.15.1) now give a suitable solution of $\bar{\partial}\alpha = \beta$.

2.16 The Runge Approximation Theorem

According to the Mittag-Leffler theorem (Theorem 2.15.1), on an open Riemann surface X , one may prescribe values for a holomorphic function (to arbitrary order) at the points in a given discrete set. The identity theorem implies that it is not possible to prescribe values on a set that is not discrete. However, for a compact set K with $\mathfrak{h}_X(K) = K$, one can *uniformly approximate* a holomorphic function on a neighborhood of K by a global holomorphic function. In other words, we have the following Riemann surface analogue, due to Behnke and Stein [BehS], of the classical Runge approximation theorem [Run] for domains in the plane:

Theorem 2.16.1 (Runge approximation theorem) *Suppose K is a compact subset of an open Riemann surface X with $\mathfrak{h}_X(K) = K$, f_0 is a holomorphic function on a neighborhood of K in X , and $\epsilon > 0$. Then there exists a holomorphic function f on X such that $|f - f_0| < \epsilon$ on K .*

The converse is also true (see Exercise 2.16.4). Until now, we have not applied, in an essential way, the L^2 estimate in Theorem 2.9.1; but this estimate will play an important role in the proof of this approximation theorem. We will also consider the more general version Theorem 2.16.3, so the reader may wish to skip the proof of Theorem 2.16.1 below and instead, consider only the proof of Theorem 2.16.3.

Proof of Theorem 2.16.1 Multiplying f_0 by a \mathcal{C}^∞ function that has support contained in a small neighborhood of K but that is equal to 1 on some smaller neighborhood, and then extending by 0 to all of X and choosing suitable neighborhoods, we get a \mathcal{C}^∞ function τ on X and open sets Ω_0 and Ω_1 such that $K \subset \Omega_0 \Subset \Omega_1 \Subset X$, $\tau = f_0$ on Ω_0 , and $\text{supp } \tau \subset \Omega_1$.

We may also fix a Kähler form ω on X (by Corollary 2.11.3). Applying Theorem 2.14.1 (with the compact set and its neighborhood given by the empty set, and the nonnegative $(1, 1)$ -form chosen so that its sum with $i\Theta_\omega$ is greater than or equal to ω), we get a positive \mathcal{C}^∞ strictly subharmonic (exhaustion) function ρ_0 on

X such that $i\Theta_\omega + i\Theta_{\rho_0} \geq \omega$ on X . Applying the theorem again (this time with the compact set given by K), we get a nonnegative C^∞ subharmonic (exhaustion) function ρ_1 on X such that $\text{supp } \rho_1 \subset X \setminus K$ and $\rho_1 > 0$ on $X \setminus \Omega_0$. In particular, $r \equiv \inf_{\Omega_1 \setminus \Omega_0} \rho_1 > 0$.

For each positive integer v , let $\varphi_v \equiv \rho_0 + v\rho_1$. The form $\beta \equiv \bar{\partial}\tau$ is a compactly supported C^∞ differential form of type $(0, 1)$ on X . Since we have

$$i\Theta_\omega + i\Theta_{\varphi_v} = i\Theta_\omega + i\Theta_{\rho_0} + v i\Theta_{\rho_1} \geq \omega,$$

Corollary 2.12.6 provides a C^∞ function α on X such that $\bar{\partial}\alpha = \beta$ and

$$\|\alpha\|_{L^2(X, \omega, \varphi_v)}^2 \leq \|\beta\|_{L^2(X, \varphi_v)}^2 = \|\bar{\partial}\tau\|_{L^2(\Omega_1 \setminus \Omega_0, \varphi_v)}^2 \leq e^{-vr} \|\bar{\partial}\tau\|_{L^2(\Omega_1 \setminus \Omega_0, \rho_0)}^2.$$

By construction, we have $\rho_1 \equiv 0$ on some relatively compact neighborhood Ω_2 of K in Ω_0 , and hence

$$\|\alpha\|_{L^2(\Omega_2, \omega, \rho_0)}^2 = \|\alpha\|_{L^2(\Omega_2, \omega, \varphi_v)}^2 \leq \|\alpha\|_{L^2(X, \omega, \varphi_v)}^2.$$

Therefore, by choosing v sufficiently large, we can make $\|\alpha\|_{L^2(\Omega_2, \omega, \rho_0)}$ arbitrarily small. Furthermore, the C^∞ function $f \equiv \tau - \alpha$ on X is actually holomorphic (since $\bar{\partial}f = 0$), and we have $\|f - f_0\|_{L^2(\Omega_2, \omega, \rho_0)} = \|\alpha\|_{L^2(\Omega_2, \omega, \rho_0)}$. Applying Theorem 2.6.4 and choosing $v \gg 0$, we get $|f - f_0| < \epsilon$ on K . \square

We have the following consequence (the converse, which also holds, is considered in Exercise 2.16.5):

Corollary 2.16.2 *Let Ω be a topologically Runge open subset of an open Riemann surface X . Then, for every holomorphic function f_0 on Ω , for every compact set $K \subset \Omega$, and for every $\epsilon > 0$, there exists a holomorphic function f on X such that $|f - f_0| < \epsilon$ on K .*

Proof For every compact set $K \subset \Omega$, we have $\mathfrak{h}_X(K) \subset \mathfrak{h}_X(\Omega) = \Omega$. Theorem 2.16.1 now gives the claim. \square

We also have the following more general version of Theorem 2.16.1:

Theorem 2.16.3 (Runge approximation with poles at prescribed points) *Suppose K is a compact subset of a Riemann surface X , P is a finite subset of $X \setminus K$, $Y = X \setminus P$, $\mathfrak{h}_Y(K) = K$, f_0 is a holomorphic function on a neighborhood of K in X , and $\epsilon > 0$. Then there exists a meromorphic function f on X such that f is holomorphic on $X \setminus P = Y$, f has a pole at each point in P , and $|f - f_0| < \epsilon$ on K .*

For the proof, we will need the following combined version of Lemma 2.10.3 and Theorem 2.14.1:

Lemma 2.16.4 *Suppose X is a Riemann surface, K is a compact subset of X , P is a finite subset of $X \setminus K$, $Y = X \setminus P$, $\mathfrak{h}_Y(K) = K$, $\{(U_p, z_p)\}_{p \in P}$ is a collection of disjoint local holomorphic coordinate neighborhoods in X with $p \in U_p$ and $z_p(p) = 0$ for each $p \in P$, ω is a Kähler form on X , and Ω is a neighborhood of K in X . Then, for every sufficiently large positive constant b , there exist open sets $\{V_p\}_{p \in P}$ with $p \in V_p \subseteq U_p$ for each point $p \in P$ such that for every sufficiently large positive constant R (depending on the above choices), there exists a nonnegative C^∞ subharmonic exhaustion function φ on Y with the following properties:*

- (i) *On $Y \setminus \Omega$, $\varphi > 0$ and $i\Theta_\varphi \geq \omega$;*
- (ii) *We have $\text{supp } \varphi \subset Y \setminus K$; and*
- (iii) *For each $p \in P$, we have $\varphi = R \cdot (|z_p|^2 - \log |z_p|^2 - b)$ on $V_p \setminus \{p\}$.*

Proof If $P = \emptyset$ and X is compact, then we have $K = X$ and the claim is trivial. Thus we may assume without loss of generality that Y is noncompact. By shrinking Ω and the sets $\{(U_p, z_p)\}_{p \in P}$ if necessary, we may also assume without loss of generality that $U_p \subseteq X \setminus \overline{\Omega}$ for each point $p \in P$. We have $\mathfrak{h}_Y(K) = K$, so Theorem 2.14.1 provides a nonnegative C^∞ subharmonic exhaustion function α on Y such that $\text{supp } \alpha \subset Y \setminus K$ and such that $\alpha > 0$ and $i\Theta_\alpha \geq \omega$ on $Y \setminus \Omega$. For $b \gg 0$, Lemma 2.10.3 provides, for each point $p \in P$, a nonnegative C^∞ subharmonic function β_p on $X \setminus \{p\}$ such that $\beta_p \equiv 0$ on $X \setminus U_p$ and $\beta_p = |z_p|^2 - \log |z_p|^2 - b > 0$ on $W_p \setminus \{p\}$ for some relatively compact neighborhood W_p of p in U_p (the finiteness of P allows us to choose a single sufficiently large constant b that works for each of the points $p \in P$). Choosing a relatively compact neighborhood V_p of p in W_p for each $p \in P$, setting $V \equiv \bigcup_{p \in P} V_p \subseteq W \equiv \bigcup_{p \in P} W_p$, and choosing a nonnegative C^∞ function η on X such that $\eta \equiv 1$ on a neighborhood of $X \setminus W$ and $\text{supp } \eta \subset X \setminus V$, we see that if $R \gg 0$, then the function $\varphi = \eta \cdot \alpha + \sum_{p \in P} R \cdot \beta_p$ has the required properties. \square

Proof of Theorem 2.16.3 We have $Y = X \setminus P$ and $\mathfrak{h}_Y(K) = K$, and as in the proof of Lemma 2.16.4, we may assume without loss of generality that Y is noncompact. Multiplying f_0 by a C^∞ function that has support contained in a small neighborhood of K but that is equal to 1 on some smaller neighborhood, and then extending by 0 to all of X and choosing suitable neighborhoods, we get a C^∞ function τ on X and open sets Ω_0 and Ω_1 such that $K \subset \Omega_0 \subseteq \Omega_1 \subseteq X \setminus P$, $\tau = f_0$ on Ω_0 , and $\text{supp } \tau \subset \Omega_1$. We may also choose disjoint local holomorphic coordinate neighborhoods $\{(U_p, z_p)\}_{p \in P}$ in X such that for each $p \in P$, we have $p \in U_p \subseteq X \setminus \overline{\Omega_1}$ and $z_p(p) = 0$. We may also fix a Kähler form ω on X (by Corollary 2.11.3).

By applying Lemma 2.16.4 (with the compact set and its neighborhood given by the empty set, and the Kähler form chosen so that its sum with $i\Theta_\omega$ is greater than or equal to ω), we get a positive constant b_0 , open sets $\{V_p\}_{p \in P}$ with $p \in V_p \subseteq U_p$ for each point $p \in P$, a positive integer k , and a positive C^∞ strictly subharmonic (exhaustion) function ρ_0 on Y such that $i\Theta_\omega + i\Theta_{\rho_0} \geq \omega$ on Y and such that for each point $p \in P$, $\rho_0 = k(|z_p|^2 - \log |z_p|^2 - b_0)$ on $V_p \setminus \{p\}$. Applying Lemma 2.16.4 again (this time with the compact set given by K) and shrinking each of the neigh-

neighborhoods V_p for $p \in P$, we get a positive constant b_1 and a nonnegative C^∞ subharmonic (exhaustion) function ρ_1 on Y such that $\text{supp } \rho_1 \subset Y \setminus K$, $\rho_1 > 0$ on $X \setminus \Omega_0$, and $\rho_1 = |z_p|^2 - \log |z_p|^2 - b_1$ on $V_p \setminus \{p\}$ for each $p \in P$ (here, we apply the lemma to get constants b and R , and we then divide the resulting function by R and set $b_1 \equiv b/R$). In particular, $r \equiv \inf_{\Omega_1 \setminus \Omega_0} \rho_1 > 0$. We may also fix a C^∞ function η with compact support in the neighborhood $\bigcup_{p \in P} V_p$ of P such that $\eta \equiv 1$ on a neighborhood of P .

Now for each positive integer v , let $\varphi_v \equiv \rho_0 + v\rho_1$. Given a positive integer μ and a constant $\delta > 0$, we get a C^∞ function γ on Y by setting $\gamma = \delta\eta/z_p^{(\mu+v+k)}$ on $U_p \setminus \{p\}$ for each $p \in P$ and $\gamma = \tau$ elsewhere. The form $\beta \equiv \bar{\partial}\gamma$ is then a compactly supported C^∞ differential form of type $(0, 1)$ on Y . Since we have

$$i\Theta_\omega + i\Theta_{\varphi_v} = i\Theta_\omega + i\Theta_{\rho_0} + v i\Theta_{\rho_1} \geq \omega,$$

Corollary 2.12.6 provides a C^∞ function α on Y such that $\bar{\partial}\alpha = \beta$ and

$$\begin{aligned} \|\alpha\|_{L^2(Y, \omega, \varphi_v)}^2 &\leq \|\beta\|_{L^2(Y, \varphi_v)}^2 = \|\bar{\partial}\tau\|_{L^2(\Omega_1 \setminus \Omega_0, \varphi_v)}^2 + \sum_{p \in P} \delta^2 \|z_p^{-(\mu+v+k)} \bar{\partial}\eta\|_{L^2(V_p, \varphi_v)}^2 \\ &\leq e^{-v\tau} \|\bar{\partial}\tau\|_{L^2(\Omega_1 \setminus \Omega_0, \rho_0)}^2 + \sum_{p \in P} \delta^2 \|z_p^{-(\mu+v+k)} \bar{\partial}\eta\|_{L^2(V_p, \varphi_v)}^2. \end{aligned}$$

By construction, we have $\rho_1 \equiv 0$ on some relatively compact neighborhood Ω_2 of K in Ω_0 , and hence

$$\|\alpha\|_{L^2(\Omega_2, \omega, \rho_0)}^2 = \|\alpha\|_{L^2(\Omega_2, \omega, \varphi_v)}^2 \leq \|\alpha\|_{L^2(Y, \omega, \varphi_v)}^2.$$

Therefore, by choosing v sufficiently large and then choosing $\delta > 0$ sufficiently small (depending on v), we can make $\|\alpha\|_{L^2(\Omega_2, \omega, \rho_0)}$ arbitrarily small.

The C^∞ function $f \equiv \gamma - \alpha$ on Y is actually holomorphic, since $\bar{\partial}f = 0$. Near each point $p \in P$, the function $z_p^{v+k}\alpha = \delta z_p^{-\mu} - z_p^{v+k}f$ is holomorphic except for an isolated singularity at p , and $|z_p^{v+k}\alpha|^2 = |\alpha|^2 e^{-(v+k)\log|z_p|^2}$ is locally integrable near p by the choice of φ_v . Therefore, by Riemann's extension theorem (Theorem 1.2.10), $z_p^{v+k}\alpha$ extends to a holomorphic function in a neighborhood of p , and hence the function $-\alpha = f - \gamma$, which is equal to $f - \delta z_p^{-(\mu+v+k)}$ near p , is meromorphic on a neighborhood of p with at worst a pole of order $v+k$ at p . It follows that f extends to a meromorphic function on X that is holomorphic on $X \setminus P$ and that has a pole of order $\mu + v + k$ at each point $p \in P$. Finally, since $\|f - f_0\|_{L^2(\Omega_2, \omega, \rho_0)} = \|\alpha\|_{L^2(\Omega_2, \omega, \rho_0)}$, Theorem 2.6.4 implies that for $v \gg 0$ and δ sufficiently small (depending on v), we have $|f - f_0| < \epsilon$ on K . \square

Remarks 1. Note that we may choose the function f in the above proof to have a pole of any order $> v + k$ at each point in P (see Exercise 2.16.3).

2. A version in which the discrete set P may be infinite is considered in Exercise 2.16.6.

We close this section with a consequence of Theorem 2.16.3 that will play an important role in the proof of the Riemann mapping theorem in Chap. 5 (see Sect. 5.4).

Lemma 2.16.5 *Given a compact subset K of an open Riemann surface X , there exist a holomorphic function f on a neighborhood Y of K in X , a positive regular value r of the function $|f|$, and a C^∞ domain Ω such that Ω is a connected component of $\{x \in Y \mid |f(x)| < r\}$ and $K \subset \Omega \Subset Y$.*

Proof Clearly, we may assume that K is nonempty and connected. Thus we may choose a compact set $K_1 \subsetneq X$ with $K \subset K_1$ and $\mathfrak{h}_X(K_1) = K_1$ (in fact, we could replace K with $\mathfrak{h}_X(K)$ and set $K = K_1$, but this would require the fact, considered in Exercise 2.13.4, that the topological hull of a connected set in a manifold is connected, and is not really necessary here). Similarly, we may choose a relatively compact neighborhood Ω_0 of K_1 in X and a compact set K_2 such that $\partial\Omega_0 \subset K_2 \subset X \setminus K_1$ and $\mathfrak{h}_{X \setminus K_1}(K_2) = K_2$ (see Exercise 2.13.2). In particular, the set $X \setminus (K_1 \cup K_2) = (X \setminus K_1) \setminus K_2$ has only finitely many components, and hence we may choose a finite set $P \subset X \setminus (K_1 \cup K_2)$ that meets each of these components. The domain $Y \equiv X \setminus P$ then contains $K_1 \cup K_2$, and furthermore, $\mathfrak{h}_Y(K_1 \cup K_2) = K_1 \cup K_2$. For each component of $Y \setminus (K_1 \cup K_2)$ is of the form $U \setminus P$, where U is a component of $X \setminus (K_1 \cup K_2)$, and by construction, P must meet U . Thus $U \setminus P$ must contain points arbitrarily close to $P = X \setminus Y$, and hence $U \setminus P$ cannot be relatively compact in Y .

Now, by applying Theorem 2.16.3 to a locally constant function on a neighborhood of $K_1 \cup K_2$ that is equal to 0 on K_1 and 3 on K_2 , we get a meromorphic function g on X such that g is holomorphic on $X \setminus P = Y$, g has a pole at each point in P , $|g| < 1$ on $K_1 \subset \Omega_0$, and $|g| > 2$ on $K_2 \supset \partial\Omega_0$. Since g is nonconstant, the set of positive critical values of the function $\rho \equiv |g|_Y$ is countable ($dg = 0$ at any critical point of ρ in $Y \setminus g^{-1}(0) = X \setminus g^{-1}(\{0, \infty\})$ since $2\rho d\rho = d\rho^2 = \bar{g} dg + g d\bar{g}$ on this set). Thus we may fix a regular value $r \in (1, 2)$. The connected component Ω of $\{x \in Y \mid \rho(x) < r\}$ containing the connected compact set K must then be a nonempty C^∞ (by Corollary 2.4.5) domain that is relatively compact in $\Omega_0 \setminus P = \Omega_0 \cap Y$, since $\rho \rightarrow \infty$ at P and $\rho > 2$ on $\partial\Omega_0$. Setting $f \equiv g|_Y$, we get the desired objects. \square

Exercises for Sect. 2.16

2.16.1 Let X be an open Riemann surface. Using the Runge approximation theorem (not the results of Sect. 2.15), prove the following (cf. Corollary 2.15.3):

- (i) *Separation of points.* If $p, q \in X$ and $p \neq q$, then there exists a holomorphic function f on X such that $f(p) \neq f(q)$; and
- (ii) *Global holomorphic functions give local coordinates.* For each point $p \in X$, there exists a holomorphic function f on X such that $(df)_p \neq 0$.

2.16.2 Let P be a finite subset of a compact Riemann surface X , and let U be a neighborhood of P . Prove that there is a meromorphic function f on X

such that f is holomorphic on $X \setminus P$, f has a pole at each point in P , and the set of zeros of f is contained in U .

2.16.3 Verify that in the proof of the Theorem 2.16.3, the constructed function f may be chosen to have a pole of any order $> v + k$ at each point in P .

2.16.4 Let X be an open Riemann surface and let K be a nonempty compact subset of X .

(a) Prove that $\mathfrak{h}_X(K) = \{p \in X \mid |f(p)| \leq \max_K |f| \ \forall f \in \mathcal{O}(X)\}$.

Hint. Show that for each point $p \in X \setminus \mathfrak{h}_X(K)$, we have $\mathfrak{h}_X(\mathfrak{h}_X(K) \cup \{p\}) = \mathfrak{h}_X(K) \cup \{p\}$. Then form a holomorphic approximation to the function that is equal to 0 near $\mathfrak{h}_X(K)$ and 1 near p . Given a point $p \in \mathfrak{h}_X(K) \setminus K$, apply the maximum principle.

(b) Prove that $\mathfrak{h}_X(K) = K$ if and only if for every holomorphic function f_0 on a neighborhood of K in X and for every $\epsilon > 0$, there exists a holomorphic function f on X such that $|f - f_0| < \epsilon$ on K .

Hint. Assuming the approximation condition, suppose U is a connected component of $X \setminus K$ with $U \Subset X$. Fixing a point $p \in U$, the results of Sect. 2.16 (or Sect. 2.15) provide a function $f \in \mathcal{M}(X)$ that is holomorphic except for a pole at p . Show that there is a sequence $\{g_n\}$ in $\mathcal{O}(X)$ that converges uniformly to f on K . Applying the maximum principle to the sequence $\{g_n|_{\bar{U}}\}$ (along with the Cauchy criterion), one gets a continuous function on \bar{U} that is holomorphic on U and that is equal to f on ∂U . This leads to a contradiction.

2.16.5 This exercise requires facts proved in Exercise 2.13.6. Let Ω be a nonempty open subset of an open Riemann surface X . Prove that the following are equivalent:

- (i) Ω is topologically Runge.
- (ii) For every compact set $K \subset \Omega$, we have $\mathfrak{h}_X(K) \subset \Omega$.
- (iii) For every holomorphic function f_0 on Ω , every compact set $K \subset \Omega$, and every $\epsilon > 0$, there exists a holomorphic function f on X such that $|f - f_0| < \epsilon$ on K (that is, Ω is *holomorphically Runge*).

Hint. The proof of (iii) \Rightarrow (i) is similar to that of part (b) of Exercise 2.16.4.

2.16.6 Let X be a Riemann surface, let K be a compact subset of X , let P be a discrete subset of X with $P \subset X \setminus K$, and let $Y = X \setminus P$. Assume that $\mathfrak{h}_Y(K) = K$.

(a) Suppose that $\{(U_p, z_p)\}_{p \in P}$ is a locally finite collection of disjoint local holomorphic coordinate neighborhoods in X with $p \in U_p$ and $z_p(p) = 0$ for each $p \in P$, ω is a Kähler form on X , and Ω is a neighborhood of K in X . Prove that for every collection of sufficiently large positive constants $\{b_p\}_{p \in P}$, there exist open sets $\{V_p\}_{p \in P}$ with $p \in V_p \Subset U_p$ for each point $p \in P$ such that for every collection of sufficiently large positive constants $\{R_p\}_{p \in P}$ (depending on the above choices), there exists a nonnegative \mathcal{C}^∞ subharmonic exhaustion function φ on Y with the following properties (cf. Lemma 2.16.4):

- (i) On $Y \setminus \Omega$, $\varphi > 0$ and $i\Theta_\varphi \geq \omega$;
 - (ii) We have $\text{supp } \varphi \subset Y \setminus K$; and
 - (iii) For each $p \in P$, we have $\varphi = R_p \cdot (|z_p|^2 - \log |z_p|^2 - b_p)$ on $V_p \setminus \{p\}$.
- (b) Suppose that f_0 is a holomorphic function on a neighborhood of K in X and $\epsilon > 0$. Prove that there exists a meromorphic function f on X such that f is holomorphic on $X \setminus P = Y$, f has a pole at each point in P , and $|f - f_0| < \epsilon$ on K (cf. Theorem 2.16.3).
- 2.16.7 Let X be an open Riemann surface, let $K \subset X$ be a compact subset with $\mathfrak{h}_X(K) = K$, let $P \subset X$ be a discrete subset with $P \subset X \setminus K$, let f_0 be a holomorphic function on a neighborhood of K in X , and for each point $p \in P$, let f_p be a holomorphic function on $U_p \setminus \{p\}$ for some neighborhood U_p of p in X with $U_p \cap P = \{p\}$, and let m_p be a positive integer. Prove that for every $\epsilon > 0$, there exists a holomorphic function f on $X \setminus P$ such that $|f - f_0| < \epsilon$ on K and such that for each point $p \in P$, $f - f_p$ extends to a holomorphic function on U_p that either vanishes on a neighborhood of p or has a zero of order at least m_p at p (note that this is a combined version of the Mittag-Leffler theorem and the Runge approximation theorem).
- 2.16.8 Suppose K is compact subset of an open Riemann surface X with $\mathfrak{h}_X(K) = K$ and θ_0 is a holomorphic 1-form on a neighborhood of K in X . Prove that there exists a sequence of holomorphic 1-forms $\{\theta_v\}$ on X such that for every local holomorphic coordinate neighborhood (U, z) in X , $\theta_v/dz \rightarrow \theta_0/dz$ uniformly on compact subsets of $K \cap U$.
- 2.16.9 This exercise requires Exercises 2.13.8, 2.13.10, 2.13.12, and 2.14.7. Let Ω be a topologically Runge open subset of an open Riemann surface X .
- (a) Suppose that ω is a Kähler form on X and φ is a real-valued C^∞ function on X with $i\Theta_\omega + i\Theta_\varphi \geq 0$ on X . Prove that for every holomorphic function f_0 on Ω , every closed set $K \subset \Omega$, and every $\epsilon > 0$, there exists a holomorphic function f on X such that $\|f - f_0\|_{L^2_{0,0}(K, \omega, \varphi)} < \epsilon$.
 - (b) Suppose that φ is a C^∞ subharmonic function on X . Prove that for every holomorphic 1-form θ_0 on Ω , every closed set $K \subset \Omega$, and every $\epsilon > 0$, there exists a holomorphic 1-form θ on X such that $\|\theta - \theta_0\|_{L^2_{1,0}(K, \varphi)} < \epsilon$.
- 2.16.10 The goal of this exercise is a special case of the Mergelyan–Bishop theorem. Let λ denote Lebesgue measure on \mathbb{C} .
- (a) Prove that if ρ_0 is a continuous complex-valued function on a compact set $K \subset \mathbb{C}$ and $\epsilon > 0$, then there exists a C^∞ function ρ on \mathbb{C} such that $|\rho - \rho_0| < \epsilon$ on K .
- Hint.* Patch together local constant approximations using a C^∞ partition of unity.
- (b) Prove that if S is a measurable subset of \mathbb{C} and $z \in \mathbb{C}$, then

$$\int_S \frac{1}{|\zeta - z|} d\lambda(\zeta) \leq 2\pi R + \frac{\lambda(S)}{R} \quad \forall R > 0.$$

Conclude from this that in particular, $\int_S (1/|\zeta - z|) d\lambda(\zeta) \leq \sqrt{8\pi\lambda(S)}$.

- (c) Prove that if $f \in \mathcal{C}^\infty(\mathbb{C})$ and Ω is a smooth relatively compact domain in \mathbb{C} , then

$$\left| f(z) - \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta \right| \leq \sup_{\Omega} \left| \frac{\partial f}{\partial \bar{\zeta}} \right| \cdot \sqrt{8\lambda(\Omega)/\pi}.$$

- (d) *Hartogs–Rosenthal theorem* (see [HarR]). Suppose K is a compact set of measure 0 in \mathbb{C} . Prove that if f_0 is a continuous function on K and $\epsilon > 0$, then there exists a holomorphic function f on a neighborhood of K in \mathbb{C} such that $|f - f_0| < \epsilon$ on K . Prove also that if in addition, $\mathbb{C} \setminus K$ is connected, then there actually exists an entire function f such that $|f - f_0| < \epsilon$ on K .
- (e) Prove that given an open set $\Omega \subset \mathbb{C}$ and a compact set $K \subset \Omega$, there exists a constant $C = C(K, \Omega) > 0$ such that

$$\max_K |f| \leq C \left[\|f\|_{L^2(\Omega)} + \sup_{\Omega} \left| \frac{\partial f}{\partial \bar{\zeta}} \right| \right] \quad \forall f \in \mathcal{C}^\infty(\Omega).$$

Hint. First consider the case in which $K = \overline{\Delta(z_0; R)}$ and $\Omega = \Delta(z_0; 4R)$ for some point $z_0 \in \mathbb{C}$ and some $R > 0$. For this, apply the Cauchy integral formula (Lemma 1.2.1) to the disk $\Delta(z_0; 3R)$ and to the function ηf , where $\eta \in \mathcal{D}(\Delta(z_0; 3R))$ and $\eta \equiv 1$ on $\Delta(z_0; 2R)$. For the general case, cover K by finitely many disks of the form $\Delta(z_0; R)$ with $\Delta(z_0; 4R) \subset \Omega$.

- (f) Let X be a Riemann surface. Prove that given an open set $\Omega \subset X$, a compact set $K \subset \Omega$, a Kähler form ω on X , and a real-valued \mathcal{C}^∞ function φ on X , there exists a constant $C = C(K, \Omega, \omega, \varphi) > 0$ such that

$$\max_K |f| \leq C \left[\|f\|_{L^2(\Omega, \omega, \varphi)} + \sup_{\Omega} \left| \frac{\bar{\partial} f \wedge \overline{\bar{\partial} f}}{\omega} \right|^{1/2} \right] \quad \forall f \in \mathcal{C}^\infty(\Omega).$$

- (g) *Bishop–Kodama localization theorem* (see [Bis] and [Kod]). Let X be an open Riemann surface, let K be a compact subset of X , and let f_0 be a continuous function on K . Assume that each point $p \in K$ admits a neighborhood U in X such that for every $\epsilon > 0$, there exists a holomorphic function f on a neighborhood of the compact set $K' \equiv K \cap \overline{U}$ in X with $|f - f_0| < \epsilon$ on K' . Prove that for every $\epsilon > 0$, there exists a holomorphic function f on a neighborhood of K in X such that $|f - f_0| < \epsilon$ on K .

Hint. By replacing X with a large domain, one may assume that there are a Kähler metric ω and a \mathcal{C}^∞ strictly subharmonic function φ on X as in Corollary 2.12.6. Fix a finite covering of K by relatively compact open subsets of X with the above approximation property, and fix a suitable partition of unity. Given $\delta > 0$, one may form local approximations to within δ on each of these sets. Patching these local

approximations using the partition of unity, one gets a \mathcal{C}^∞ function τ ; and the $(0, 1)$ -form $\bar{\partial}\tau$ is controlled by δ at points in K . Multiplying $\bar{\partial}\tau$ by a \mathcal{C}^∞ (cutoff) function that is equal to 1 on K and that vanishes outside a small neighborhood of K (this function depends on the choice of δ), one gets a $(1, 0)$ -form β that is controlled by δ *everywhere* in X . Corollary 2.12.6 then provides a solution of the equation $\bar{\partial}\alpha = \beta$ along with an L^2 estimate. Guided by part (f), one sees that if $\delta > 0$ is sufficiently small, then the restriction of $\tau - \alpha$ to a small neighborhood of K (which depends on the choice of δ) has the required properties.

- (h) Let X be an open Riemann surface, and let $K \subset X$ be a compact set of measure 0. Prove that for every continuous function f_0 on K and for every $\epsilon > 0$, there exists a holomorphic function f on a neighborhood of K in X such that $|f - f_0| < \epsilon$ on K . Prove also that if in addition, $\mathfrak{h}_X(K) = K$, then there actually exists a function $f \in \mathcal{O}(X)$ such that $|f - f_0| < \epsilon$ on K .

Remarks According to Mergelyan's theorem (see [Me] and [Rud1]), given a compact set $K \subset \mathbb{C}$ with connected complement, a continuous function f_0 on K that is holomorphic on the interior of K , and a constant $\epsilon > 0$, there exists an entire function f with $|f - f_0| < \epsilon$ on K . The Mergelyan–Bishop theorem includes the natural analogue for an open Riemann surface X : Given a compact set $K \subset X$ with $\mathfrak{h}_X(K) = K$, a continuous function f_0 on K that is holomorphic on the interior of K , and a constant $\epsilon > 0$, there exists a holomorphic function f on X with $|f - f_0| < \epsilon$ on K . The proof of the Hartogs–Rosenthal theorem outlined in parts (b)–(d) is due to Hartogs and Rosenthal. The proof of the Bishop–Kodama localization theorem outlined in parts (e)–(g) is due to Jarnicki and Pflug (see [JP] and [Ga]).

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