

Preface

A Riemann surface X is a connected 1-dimensional complex manifold, that is, a connected Hausdorff space that is locally homeomorphic to open subsets of \mathbb{C} with complex analytic coordinate transformations (this also makes X a real 2-dimensional smooth manifold). Although a connected second countable real 1-dimensional smooth manifold is simply a line or a circle (up to diffeomorphism), the real 2-dimensional character of Riemann surfaces makes for a much more interesting topological characterization and a still more interesting complex analytic characterization.

A noncompact (i.e., an open) Riemann surface satisfies analogues of the classical theorems of complex analysis, for example the Mittag-Leffler theorem, the Weierstrass theorem, and the Runge approximation theorem (this development began only in the 1940s with the work of, for example, Behnke and Stein). On the other hand, by the maximum principle, a compact Riemann surface admits only constant holomorphic functions. However, compact Riemann surfaces do admit a great many meromorphic functions. This property leads to powerful theorems, in particular, the crucial Riemann–Roch theorem.

The theory of Riemann surfaces occupies a unique position in modern mathematics, lying at the intersection of analysis, algebra, geometry, and topology. Most earlier books on this subject have tended to focus on its algebraic-geometric and number-theoretic aspects, rather than its analytic aspects. This book takes the point of view that Riemann surface theory lies at the root of much of modern analysis, and it exploits this happy circumstance by using it as a way to introduce some fundamental ideas of analysis, as well as to illustrate some of the interactions of analysis with geometry and topology. The analytic methods applied in this book to the study of one complex variable are also useful in the study of several complex variables. Moreover, they contain the essence of techniques used generally in the study of partial differential equations and in the application of analytic tools to problems in geometry. Thus a careful reader will be rewarded not only with a good command of the classical theory of Riemann surfaces, but also with an introduction to these important modern techniques. While much of the book is intended for students at the second-year graduate level, Chaps. 1 and 2 and Sect. 5.2 (along with the required

background material) could serve as the basis for the complex analysis component of a year-long first-year graduate-level course on real and complex analysis. A successful student in such a course would be well prepared for further study in analysis and geometry.

The analytic approach in this book is based on the solution of the inhomogeneous Cauchy–Riemann equation with L^2 estimates (or the L^2 $\bar{\partial}$ -method) in a holomorphic line bundle with positive curvature. This powerful technique from several complex variables (see, for example, Andreotti–Vesentini [AnV], Hörmander [Hö], Skoda [Sk1], [Sk2], [Sk3], [Sk4], [Sk5], and Demailly [De1]) takes an especially nice form on a Riemann surface. Moreover, the 1-variable version serves as a gentle introduction to, and demonstration of, this important technique. For example, one may sometimes, with care, check signs in the formulas for higher dimensions by considering the dimension-one case. On the other hand, the higher-dimensional analogues of the main theorems considered in this book do require additional hypotheses.

Two central features of this book are a simple construction of a strictly subharmonic exhaustion function (which is a modified version of the construction in [De2]) and a simple construction of a positive-curvature Hermitian metric in the holomorphic line bundle associated to a nontrivial effective divisor. The simplicity of these constructions and the power of the L^2 $\bar{\partial}$ -method make this approach to Riemann surfaces very efficient. The recent book of Varolin [V] also uses L^2 methods. However, although there is some overlap in the choice of topics, the proofs themselves are quite different. For a different treatment of this material that uses the solution of the inhomogeneous Cauchy–Riemann equation, but not L^2 methods, the reader may refer to, for example, [GueNs].

This book also contains (in Chaps. 5 and 6) proofs of some fundamental facts concerning the holomorphic, smooth, and topological structure of a Riemann surface, such as the Koebe uniformization theorem, the biholomorphic classification of Riemann surfaces, the embedding theorems, the integrability of almost complex structures, Schönflies’ theorem (and the Jordan curve theorem), and the existence of a smooth structure on a second countable surface. The approach in this book to the above facts differs from the usual approaches (see, for example, [Wey], [Sp], or [AhS]) in that it mostly relies on the L^2 $\bar{\partial}$ -method (in place of harmonic functions and forms) and on explicit holomorphic attachment of disks and annuli (in place of triangulations). The above facts, along with the facts concerning compact Riemann surfaces considered in Chap. 4, constitute some of the background required for the study of Teichmüller theory (as considered in, for example, [Hu]).

Riemann first introduced Riemann surfaces partly as a way of understanding multiple-valued holomorphic functions (see, for example, [Wey] or [Sp]), and Riemann surface theory has since grown into a vast area of study. Weyl’s book [Wey] was the first book on the subject, and some elements of the point of view in [Wey] (and in the similar book [Sp]) are present in this book. For different approaches to the study of open Riemann surfaces, the reader may refer to, for example, [AhS] or [For]. For further study of compact Riemann surfaces (the main focus of most books on Riemann surfaces), the reader may refer to, for example, [FarK], [For], or [Ns4].

Much of the background material required for study of this book is provided in detail in Part III (Chaps. 7–11). It is strongly recommended that, rather than first studying all of Part III, the reader instead focus on Parts I and II, and consult the appropriate sections in Part III only as needed (tables providing the interdependence of the sections appear after table of contents). In fact, the background material has been placed in the last part of the book instead of the first in order to make such consultation more convenient. The main prerequisite for the book is some knowledge of point-set topology (as in, for example, [Mu]) and elementary measure theory (as in, for example, Chap. 1 of [Rud1]), although parts of these subjects are reviewed in Sects. 7.1 and 9.1. On the other hand, smooth manifolds (as in Chap. 9 and in, for example, [Mat], [Ns3], and [Wa]) and Hilbert spaces (as in Chap. 7 and in, for example, [Fol] and [Rud1]) are essential objects in this book, so it would be to the reader's advantage to have had some previous experience with these objects. It may surprise the reader to learn that previous experience with complex analysis in the plane (as in, for example, [Ns5]), while helpful, is not necessary. In fact, as indicated earlier, Chaps. 1 and 2 and Sect. 5.2 (along with the required background material and the corresponding exercises) together actually provide the material for a fairly complete course on complex analysis on domains in the plane along with the analogous study of complex analysis on Riemann surfaces (other suggested course outlines appear after the tables listing the interdependence of the sections).

Exercises are included for most of the sections, and some of the theory is developed in the exercises.

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