2
Preliminaries

In this chapter, we bring together the notations and acronyms used in this book as well as various definitions and facts related to matrices, linear spaces, linear operators, norms of deterministic as well as stochastic signals, norms of linear time- or shift-invariant systems, saturation functions, internal (Lyapunov) stability, and external stability.

2.1 A list of symbols

Throughout this book, we shall adopt the following conventions and notations:

$\mathbb{R}$	Set of real numbers
$\mathbb{R}^+$	Set of nonnegative real numbers
$\mathbb{Z}^+$	Set of nonnegative integers
$\mathbb{C}$	Entire complex plane
$\mathbb{C}^-$	Open left-half complex plane
$\mathbb{C}^0$	Imaginary axis
$\mathbb{C}^+$	Open right-half complex plane
$\mathbb{C}^{-0}$	Closed left-half complex plane
$\mathbb{C}^{+0}$	Closed right-half complex plane
$\mathbb{C}^{\ominus}$	Set of complex numbers inside the unit circle
$\mathbb{C}^{\bigcirc}$	Unit circle
$\mathbb{C}^{\oplus}$	Set of complex numbers outside the unit circle
$\mathbb{C}^{\otimes}$	Set of complex numbers inside and on the unit circle
$\mathbb{C}^{\odot}$	Set of complex numbers outside and on the unit circle
$\mathcal{B}(r)$	The set $\{ x \in \mathbb{R}^n \mid \|x\| < r \}$
$\mathcal{B}(x_0, r)$	The set $\{ x \in \mathbb{R}^n \mid \|x - x_0\| < r \}$

A. Saberi et al., *Internal and External Stabilization of Linear Systems with Constraints*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4787-2_2,

I	An identity matrix
I_k	Identity matrix of dimension $k \times k$
A'	Transpose of A
A^*	Complex conjugate transpose of A
$\lambda(A)$	Set of eigenvalues of A
$\sigma_{\max}(A)$	Maximum singular value of A
$\sigma_{\min}(A)$	Minimum singular value of A
$\rho(A)$	Spectral radius of A
trace A	Trace of A
ker A	The null space of A
im A	The range space of A
$\langle A \mid \operatorname{im} B \rangle$	The controllability subspace of the pair (A, B)
$\langle \ker C \mid A \rangle$	The unobservable subspace of the pair (A, C)
$\mathcal{V}^{\perp}$	Orthogonal complement of a subspace $\mathcal{V}$ in $\mathbb{R}^n$
$\boldsymbol{E}[\cdot]$	The expectation of a stochastic vector
$\mathbb{R}[s]$	Ring of polynomials with real coefficients
$\mathbb{R}^{n \times m}[s]$	Set of all $n \times m$ matrices with coefficients in $\mathbb{R}[s]$
$\mathbb{R}(s)$	Field of rational functions with real coefficients
$\mathbb{R}^{n \times m}(s)$	Set of all $n \times m$ matrices with coefficients in $\mathbb{R}(s)$

For any set $\mathcal{C} \subset \mathbb{R}^n$, $\operatorname{int}\mathcal{C}$ denotes the interior of set $\mathcal{C}$, $\partial\mathcal{C}$ the boundary of set $\mathcal{C}$, and $\overline{\mathcal{C}}$ the closure of set $\mathcal{C}$. For a dynamical system

$$\rho x = f(x, u),$$

the ρ denotes the time-derivative

$$\rho x = \tfrac{\mathrm{d}}{\mathrm{d}t} x$$

for continuous-time systems while it denotes the shift operator

$$(\rho x)(k) = x(k+1)$$

for discrete-time systems.

2.2 Matrices, linear spaces, and linear operators

In this section, we recall certain fundamental facts and properties of matrices, linear spaces, and linear operators that are relevant to this book. We have done so for the ease of readers and to establish the related notations used throughout the book.

We say a matrix A is injective or surjective if A is of full column rank or full row rank, respectively. By $\mathrm{rank}_{\mathcal{K}}$, we denote the rank of a matrix over the field $\mathcal{K}$. We shall write *rank* only for the case when $\mathcal{K} = \mathbb{R}$ or $\mathcal{K} = \mathbb{C}$. Moreover, we use the term *normal rank* or *normrank* for $\mathrm{rank}_{\mathcal{K}}$ whenever $\mathcal{K} = \mathbb{R}(s)$. We note that if $A \in \mathbb{C}^{m\times n}$, we have that $\mathrm{im}\, A = \ker(A^*)^{\perp}$.

We recall next the classical concept of the Jordan form of a general matrix A and the concept of the multiplicity structure of an eigenvalue of a matrix A. Given any matrix A of dimension $n\times n$, we can always find a non-singular transformation matrix X (see [40]) such that

$$X^{-1}AX = J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_k \end{pmatrix}, \tag{2.1}$$

where $J_i, i = 1, \ldots, k$ are some $n_i \times n_i$ Jordan blocks,

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_i \end{pmatrix}. \tag{2.2}$$

We note that

$$\sum_{i=1}^{k} n_i = n.$$

Then, the geometric multiplicity ν_λ of an eigenvalue $\lambda \in \lambda(A)$ is the number of Jordan blocks associated with λ in (2.1) as well as the number of linearly independent eigenvectors associated with λ. On the other hand, the algebraic multiplicity ρ_λ is the total number of repetitions of λ in $\lambda(A)$; equivalently, the algebraic multiplicity is equal to the sum of the number of rows of all Jordan blocks associated with λ.

We introduce next what is known as the multiplicity structure of an eigenvalue. For any given $\lambda \in \lambda(A)$, let there be ν_λ Jordan blocks of A associated with λ. Let

$$n_{\lambda,1} \geqslant n_{\lambda,2} \geqslant \cdots , n_{\lambda,\nu_\lambda}$$

be the dimensions of the corresponding Jordan blocks ordered in size. Then, λ is an eigenvalue of A with multiplicity structure S^*_λ,

$$S^*_\lambda = \{n_{\lambda,1}, n_{\lambda,2}, \ldots , n_{\lambda,\nu_\lambda}\}. \tag{2.3}$$

If $n_{\lambda,1} = n_{\lambda,2} = \cdots = n_{\lambda,\nu_\lambda} = 1$, then λ is called a semi-simple eigenvalue of A. Moreover, we call an eigenvalue a simple eigenvalue if $\nu_\lambda = 1$ and $n_{\lambda,1} = 1$ or equivalently if it has an algebraic multiplicity equal to 1.

The invariant factor $\Psi_i(s)$ of a matrix A is the monic polynomial of lowest degree such that for each eigenvalue λ with $\nu_\lambda \geqslant i$, $\Psi_i(s)$ has $n_{\lambda,i}$ zeros in λ. We note that algebraic multiplicity ρ_λ satisfies

$$\rho_\lambda = n_{\lambda,1} + n_{\lambda,2} + \cdots + n_{\lambda,\nu_\lambda}.$$

We recall next the following classic concepts of generalized eigenvectors and the eigenvector chain associated with an eigenvalue of a matrix. A vector x is said to be a generalized eigenvector of grade k associated with an eigenvalue λ of a matrix A if and only if

$$(A - \lambda I)^k x = 0 \quad \text{and} \quad (A - \lambda I)^{k-1} x \neq 0.$$

A generalized eigenvector of grade one (i.e., $k = 1$) is a standard eigenvector associated with an eigenvalue of a matrix. Let vector x be a generalized eigenvector of grade k associated with an eigenvalue λ of a matrix A. Let

$$\begin{aligned}
x_k &= x \\
x_{k-1} &= (A - \lambda I)V &&= (A - \lambda I)x_k \\
x_{k-2} &= (A - \lambda I)^2 V &&= (A - \lambda I)x_{k-1} \\
&\vdots &&\vdots \\
x_1 &= (A - \lambda I)^{k-1} V &&= (A - \lambda I)x_2.
\end{aligned}$$

Such a set of vectors $\{x_1, x_2, \cdots, x_k\}$ is called a chain of generalized eigenvectors of length k associated with an eigenvalue λ.

For an eigenvalue λ with the multiplicity structure S_λ^* as given in (2.3), there are ν_λ chains of generalized eigenvectors with lengths $n_{\lambda,1}, n_{\lambda,2}, \cdots, n_{\lambda,\nu_\lambda}$. The total number of generalized eigenvectors in these chains equals the algebraic multiplicity ρ_λ. Moreover, these ρ_λ generalized eigenvectors are linearly independent.

If M is a subspace of $\mathbb{C}^n$, then we define the *orthogonal projection* P_M of $\mathbb{C}^n$ onto M by $P_M u = u$ if $u \in M$ and $P_M u = 0$ if $u \in M^\perp$. We note that $I - P_M = P_{M^\perp}$.

A matrix $U \in \mathbb{C}^{n\times n}$ is said to be a unitary matrix if $U^* = U^{-1}$. For a matrix $A \in \mathbb{C}^{m\times n}$, the generalized inverse of A (or Moore–Penrose inverse of A) is defined to be a unique matrix $A^\dagger$ in $\mathbb{C}^{n\times m}$ such that:

(a) $AA^\dagger$ is an orthogonal projection onto im A.

(b) $A^\dagger A$ is an orthogonal projection onto $(\ker A)^\perp = \text{im}\, A^*$.

Another equivalent definition for a generalized inverse of $A \in \mathbb{C}^{m\times n}$ is a unique matrix $A^\dagger$ in $\mathbb{C}^{n\times m}$ such that:

(a) $AA^\dagger A = A$.

(b) $A^\dagger AA^\dagger = A^\dagger$.

(c) $AA^\dagger$ is a symmetric matrix.

(d) $A^\dagger A$ is a symmetric matrix.

Some basic properties of the generalized inverse of $A \in \mathbb{C}^{m\times n}$ are listed as follows:

- $(A^\dagger)^\dagger = A$.
- $(A^\dagger)^* = (A^*)^\dagger$.
- If $\lambda \in \mathbb{C}$, $(\lambda A)^\dagger = \lambda^\dagger A^\dagger$, where $\lambda^\dagger = \frac{1}{\lambda}$ if $\lambda \neq 0$ and $\lambda^\dagger = 0$ if $\lambda = 0$.
- $A^* = A^*AA^\dagger = A^\dagger AA^*$.
- $(A^*A)^\dagger = A^\dagger(A^*)^\dagger$.
- $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$.
- $(UAV)^\dagger = V^*A^\dagger U^*$, where U and V are unitary matrices.
- $\operatorname{im} A = \operatorname{im} AA^\dagger = \operatorname{im} AA^*$.
- $\operatorname{im} A^\dagger = \operatorname{im} A^* = \operatorname{im} A^\dagger A = \operatorname{im} A^*A$.
- $\operatorname{im}(I - AA^\dagger) = \ker AA^\dagger = \ker A^* = \ker A^\dagger = (\operatorname{im} A)^\perp$.
- $\operatorname{im}(I - A^\dagger A) = \ker A^\dagger A = \ker A = (\operatorname{im} A^*)^\perp$.
- If $B \in \mathbb{C}^{n\times p}$, then $(AB)^\dagger = (P_{\operatorname{im} A^*} B)^\dagger (AP_{\operatorname{im} B})^\dagger$.
- If $A^*ABB^* = BB^*A^*A$, then $(AB)^\dagger = B^\dagger A^\dagger$.
- If $A = BC$, where $B \in \mathbb{C}^{m\times r}$ and $C \in \mathbb{C}^{r\times n}$, while $r = \operatorname{rank} A$, then $A^\dagger = C^*(CC^*)^{-1}(B^*B)^{-1}B^*$.

The following necessary and sufficient conditions for a partitioned matrix to be *positive semi-definite* and *positive definite* are useful. Consider an arbitrarily partitioned Hermitian matrix Q:

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{pmatrix}.$$

Then Q is *positive semi-definite* if and only if

$$\begin{cases} Q_{22} \geqslant 0 \\ Q_{12} = Q_{12}Q_{22}^\dagger Q_{22} \\ Q_{11} \geqslant Q_{12}Q_{22}^\dagger Q_{12}^*, \end{cases}$$

or, equivalently, Q is *positive semi-definite* if and only if

$$\begin{cases} Q_{11} \geqslant 0 \\ Q_{12} = Q_{11}Q_{11}^{\dagger}Q_{12} \\ Q_{22} \geqslant Q_{12}^{*}Q_{11}^{\dagger}Q_{12}. \end{cases}$$

Similarly, Q is *positive definite* if and only if

$$\begin{cases} Q_{22} > 0 \\ Q_{11} > Q_{12}Q_{22}^{-1}Q_{12}^{*} \end{cases}$$

or, equivalently,

$$\begin{cases} Q_{11} > 0 \\ Q_{22} > Q_{12}^{*}Q_{11}^{-1}Q_{12}. \end{cases}$$

Let us next discuss the addition of subspaces and the associated notations. Suppose $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are some subspaces of $\mathbb{R}^n$ or $\mathbb{C}^n$. Then,

$$\mathcal{X} + \mathcal{Y} = \{x + y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}.$$

If $\mathcal{Z} = \mathcal{X} + \mathcal{Y}$ and $\mathcal{X} \cap \mathcal{Y} = \{0\}$, then $\mathcal{Z}$ is called the direct sum of $\mathcal{X}$ and $\mathcal{Y}$, and, in this case, $\mathcal{Z}$ is written as $\mathcal{X} \oplus \mathcal{Y}$. Consider a subspace $\mathcal{X}$ in $\mathbb{R}^n$. Then, the orthogonal complement $\mathcal{X}^{\perp}$ of the subspace $\mathcal{X}$ is defined as

$$\mathcal{X}^{\perp} = \{u \in \mathbb{R}^n \mid \langle u\,,\, v\rangle\ = 0 \text{ for every } v \in \mathcal{X}\}.$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be two nontrivial subspaces of $\mathbb{R}^n$. If the inner product of x and y is zero for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, then the two subspaces $\mathcal{X}$ and $\mathcal{Y}$ are said to be orthogonal, and this is denoted by $\mathcal{X} \perp \mathcal{Y}$.

Next, for a matrix $M \in \mathbb{R}^{m\times n}$, the linear transformation $M\mathcal{X}$ is defined as

$$M\mathcal{X} := \{Mx \mid x \in \mathcal{X}\}.$$

Also, for a matrix $N \in \mathbb{R}^{n\times m}$,

$$N^{-1}\mathcal{X} := \{z \in \mathbb{R}^m \mid Nz \in \mathcal{X}\}.$$

The following relations will be useful in algebraic manipulations regarding subspaces:

$$\begin{aligned}
\mathcal{X} \cap (\mathcal{Y} + \mathcal{Z}) &\supseteq (\mathcal{X} \cap \mathcal{Y}) + (\mathcal{X} \cap \mathcal{Z}) \\
\mathcal{X} + (\mathcal{Y} \cap \mathcal{Z}) &\subseteq (\mathcal{X} + \mathcal{Y}) \cap (\mathcal{X} + \mathcal{Z}) \\
(\mathcal{X}^{\perp})^{\perp} &= \mathcal{X} \\
(\mathcal{X} + \mathcal{Y})^{\perp} &= \mathcal{X}^{\perp} \cap \mathcal{Y}^{\perp} \\
(\mathcal{X} \cap \mathcal{Y})^{\perp} &= \mathcal{X}^{\perp} + \mathcal{Y}^{\perp} \\
M(\mathcal{X} \cap \mathcal{Y}) &\subseteq M\mathcal{X} \cap M\mathcal{Y} \\
M(\mathcal{X} + \mathcal{Y}) &= M\mathcal{X} + M\mathcal{Y} \\
N^{-1}(\mathcal{X} \cap \mathcal{Y}) &= N^{-1}\mathcal{X} \cap N^{-1}\mathcal{Y} \\
N^{-1}(\mathcal{X} + \mathcal{Y}) &\supseteq N^{-1}\mathcal{X} + N^{-1}\mathcal{Y}.
\end{aligned} \tag{2.4}$$

Also, let $\mathcal{V}$ be a subspace of dimension m. Then we have

$$M\mathcal{X} \subseteq \mathcal{V} \iff M'\mathcal{V}^{\perp} \subseteq \mathcal{X}^{\perp} \tag{2.5}$$

$$(M^{-1}\mathcal{V})^{\perp} = M'\mathcal{V}^{\perp}. \tag{2.6}$$

Let $A = \mathbb{R}^{n\times n}$. Then $\mathcal{T}$, a subspace of $\mathbb{R}^n$, is an A-invariant subspace if

$$A\mathcal{T} \subseteq \mathcal{T}.$$

The following properties of an A-invariant subspace are useful:

(a) A subspace $\mathcal{T}$ with T a matrix such that $\mathcal{T} = \operatorname{im} T$ is A-invariant if and only if a matrix X exists such that

$$AT = TX.$$

(b) Let $\mathcal{T}$ be an A-invariant subspace. Then a similarity transformation L exists such that

$$\tilde{A} := L^{-1}AL = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} \quad \text{and} \quad \mathcal{T} = \operatorname{im} L^{-1}\begin{pmatrix} I \\ 0 \end{pmatrix}$$

with $\tilde{A}_{11} \in \mathbb{R}^{h\times h}$, where $h := \dim \mathcal{T}$.

The proofs of the above relations are simple and can be found in standard books on vector spaces.

Consider a matrix $A \in \mathbb{R}^{n\times n}$ and an A-invariant subspace $\mathcal{T} \subseteq \mathbb{R}^{n\times n}$. Then the *restriction* of A to $\mathcal{T}$ is the linear map $A_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ defined by

$$A_{\mathcal{T}}x = Ax \quad \text{for all } x \in \mathcal{T}.$$

The restriction of A to $\mathcal{T}$ is also often denoted by $A \mid \mathcal{T}$.

Next, we would like to recall some elementary concepts regarding modal subspaces. We first develop some notations used in continuous-time systems. Consider a matrix $A \in \mathbb{R}^{n\times n}$. Let $\alpha(s)$ denote the characteristic polynomial of A and factor it as $\alpha(s) = \alpha_-(s)\cdot\alpha_+(s)$, where $\alpha_-(s)$ has all its roots in the open left-half complex plane $\mathbb{C}^-$ and $\alpha_+(s)$ has all its roots in the closed right-half complex plane $\mathbb{C}^{+0}$. Then the stable and unstable modal subspaces of $\mathbb{R}^n$ related to A are

$$\mathcal{X}_-(A) = \ker \alpha_-(A),$$
$$\mathcal{X}_+(A) = \ker \alpha_+(A).$$

It is easy to show that $\mathcal{X}_-(A)$ is spanned by the real and the imaginary part of the generalized eigenvectors of A corresponding to the eigenvalues in $\mathbb{C}^-$. Similarly, $\mathcal{X}_+(A)$ is spanned by the real and imaginary parts of the generalized eigenvectors of A corresponding to the eigenvalues in $\mathbb{C}^{+0}$. These two modal subspaces are complementary; that is, they are independent and their sum is $\mathbb{R}^n$; thus,

$$\mathbb{R}^n = \mathcal{X}_-(A) \oplus \mathcal{X}_+(A).$$

Standard numerical linear algebra can be used to compute the bases for modal subspaces. For example, one can transform A via orthogonal transformation T to a real Schur form

$$T'AT = \begin{pmatrix} A_- & \star \\ 0 & A_+ \end{pmatrix}, \tag{2.7}$$

where the eigenvalues of A_- and A_+ are, respectively, located in $\mathbb{C}^-$ and $\mathbb{C}^{+0}$ and $\star$ denotes some matrix that is not necessarily zero. If we partition T in conformity with the partitioning on the right-hand side of (2.7),

$$T = \begin{pmatrix} T_1 & T_2 \end{pmatrix},$$

then it is obvious that the columns of T_1 form a basis for $\mathcal{X}_-(A)$. That is,

$$\mathcal{X}_-(A) = \operatorname{im} T_1.$$

Analogously, we develop some notations used in discrete-time systems. Consider a matrix $A \in \mathbb{R}^{n \times n}$. Let $\alpha(z)$ denote the characteristic polynomial of A and factor it as $\alpha(z) = \alpha_\ominus(z) \cdot \alpha_\odot(z)$, where $\alpha_\ominus(z)$ has all its roots within the unit circle $\mathbb{C}^\ominus$ in the complex plane and $\alpha_\odot(z)$ has all its roots on or outside the unit circle $\mathbb{C}^\odot$. Then the stable and unstable modal subspaces of $\mathbb{R}^n$ related to A are

$$\mathcal{X}_\ominus(A) = \ker \alpha_\ominus(A),$$
$$\mathcal{X}_\odot(A) = \ker \alpha_\odot(A).$$

It is easy to show that $\mathcal{X}_\ominus(A)$ is spanned by the generalized real eigenvectors of A corresponding to the eigenvalues in $\mathbb{C}^\ominus$. Similarly, $\mathcal{X}_\odot(A)$ is spanned by the generalized real eigenvectors of A corresponding to the eigenvalues in $\mathbb{C}^\odot$. These two modal subspaces are complementary; that is, they are independent and their sum is $\mathbb{R}^n$; thus,

$$\mathbb{R}^n = \mathcal{X}_\ominus(A) \oplus \mathcal{X}_\odot(A).$$

Again, as in the continuous-time case, standard numerical linear algebra can be used to compute the bases for modal subspaces.

2.3 Norms of deterministic signals

Many measures are used to describe the size of a signal. The measures of size are called norms. In this section, we recall some of the common norms for persistent or transient continuous-time (discrete-time) vector signals. We consider continuous-time vector signals $y : \mathbb{R}^+ \to \mathbb{R}^n$ and discrete-time vector signals $y : \mathbb{Z}^+ \to \mathbb{R}^n$.

Definition 2.1 *The* $\boldsymbol{L_p}$ ***space****, with* $p \in [1, \infty)$*, consists of all vector-valued continuous-time signals from* $\mathbb{R}^+$ *to* $\mathbb{R}^n$ *for which*

$$\int_0^\infty \sum_{i=1}^n |y_i(t)|^p dt$$

is well defined[1] *and finite. The space* L_∞ *consists of all vector-valued continuous-time signals for which*

$$\operatorname*{ess\,sup}_{t \in \mathbb{R}^+} \max_{1 \le i \le n} |y_i(t)|$$

is finite.

The $\boldsymbol{\ell_p}$ ***space****, with* $p \in [1, \infty)$*, consists of all vector-valued discrete-time signals from* $\mathbb{Z}^+$ *to* $\mathbb{R}^n$ *for which*

$$\sum_{k=0}^\infty \sum_{i=1}^n |y_i(k)|^p$$

is finite, and the space ℓ_∞ *consists of all vector-valued discrete-time signals for which*

$$\sup_{k \in \mathbb{Z}^+} \max_{1 \le i \le n} |y_i(k)|$$

is finite.

Remark 2.2 *We will sometimes use* $L_p[t_0, \infty)$ *to refer to vector-valued signals from* $[t_0, \infty)$ *to* $\mathbb{R}^n$ *for which*

$$\int_{t_0}^\infty \sum_{i=1}^n |y_i(t)|^p dt$$

is well defined when $p \in [1, \infty)$ *or*

$$\operatorname*{ess\,sup}_{t \in [t_0, \infty)} \max_{1 \le i \le n} |y_i(t)|$$

is finite in case $p = \infty$*.*

---

[1] This integral needs to be well defined in the sense of Lebesgue. A reader who has no prior acquaintance with the Lebesgue theory of measure and integration can simply think of all functions encountered here as piecewise-continuous functions and of all integrals as Riemann integrals. This would lead to no conceptual difficulties and no loss of insight except that occasionally some results from Lebesgue theory would have to be accepted on faith.

Similarly, we will sometimes use $\ell_p[k_0, \infty)$ *to refer to vector-valued signals from* $\{ k \in \mathbb{Z}^+ \mid k > k_0 \}$ *to* $\mathbb{R}^n$ *for which*

$$\sum_{k=k_0}^{\infty} \sum_{i=1}^{n} |y_i(k)|^p$$

is finite when $p \in [1, \infty)$ *or*

$$\sup_{k \in \mathbb{Z}^+, k \geq k_0} \max_{1 \leq i \leq n} |y_i(k)|$$

is finite when $p = \infty$.

However, we would like to note that L_p *and* ℓ_p *will always refer to functions from* $\mathbb{R}^+$ *or* $\mathbb{Z}^+$ *to* $\mathbb{R}^n$*, respectively.*

The spaces defined above are actually normed linear vector spaces if we define the appropriate norms.

Definition 2.3 *For a vector-valued continuous-time signal* $y \in L_p$ *with* $p \in [1, \infty)$*, the* $\boldsymbol{L_p}$ *norm is defined as*

$$\|y\|_p := \left(\int_0^{\infty} \sum_{i=1}^{n} |y_i(t)|^p dt \right)^{\frac{1}{p}},$$

For a vector-valued continuous-time signal $y \in L_\infty$*, the* $\boldsymbol{L_\infty}$ ***norm*** *is defined as*

$$\|y\|_\infty := \operatorname*{ess\,sup}_{t \in \mathbb{R}^+} \max_{1 \leq i \leq n} |y_i(t)|.$$

Analogously, for a vector-valued discrete-time signal $y \in \ell_p$ *with* $p \in [1, \infty)$*, we define the* $\boldsymbol{\ell_p}$ ***norm*** *as*

$$\|y\|_p := \left(\sum_{k=0}^{\infty} \sum_{i=1}^{n} |y_i(k)|^p \right)^{\frac{1}{p}}.$$

Finally, for a vector-valued discrete-time signal $y \in \ell_\infty$*, the* $\boldsymbol{\ell_\infty}$ ***norm*** *is defined as*

$$\|y\|_\infty := \sup_{k \in \mathbb{Z}^+} \max_{1 \leq i \leq n} |y_i(k)|.$$

The following lemmas are useful in concluding attractivity in dealing with L_p stability to be defined shortly in a later section. These lemmas imply that if both a continuous-time signal and its derivative are in L_p for some $p \in [1, \infty)$, then it vanishes as time tends to infinity, and moreover, it is in L_∞.

Lemma 2.4 *If* $\phi : [0, \infty) \to \mathbb{R}$ *is absolutely continuous,* $\phi(t) \in L_{p_1}$ *for some* $p_1 \in [1, \infty)$*, and its derivative* $\dot{\phi}(t) \in L_{p_2}$ *for some* $p_2 \in [1, \infty)$*, then* $\lim_{t\to\infty} \phi(t) = 0$.

Proof : Let $\alpha = p_1 \left(1 - \frac{1}{p_2}\right) \geqslant 0$. Then, $\frac{\alpha}{p_1} + \frac{1}{p_2} = 1$. By Hölder's inequality,

$$\begin{aligned}
\frac{1}{\alpha+1}\left|\phi^{\alpha+1}(\tau) - \phi^{\alpha+1}(s)\right| &= \left|\int_s^\tau \phi^\alpha(t)\dot{\phi}(t)\mathrm{d}t\right| \\
&\leqslant \left\{\int_s^\tau |\phi^\alpha(t)|^{\frac{p_1}{\alpha}}\mathrm{d}t\right\}^{\frac{\alpha}{p_1}} \left\{\int_s^\tau |\dot{\phi}(t)|^{p_2}\mathrm{d}t\right\}^{\frac{1}{p_2}} \\
&= \left\{\int_s^\tau |\phi(t)|^{p_1}\mathrm{d}t\right\}^{\frac{\alpha}{p_1}} \left\{\int_s^\tau |\dot{\phi}(t)|^{p_2}\mathrm{d}t\right\}^{\frac{1}{p_2}}.
\end{aligned} \tag{2.8}$$

Since $\phi \in L_{p_1}$ and $\dot{\phi}(t) \in L_{p_2}$, it is clear that $\{\phi^{\alpha+1}(t_k)\}_{k=1}^\infty$ is a Cauchy sequence for any sequence $t_k \to \infty$. Hence, we can assume that $\phi^{\alpha+1}(t_k) \to c$ as $k \to \infty$. That is, for all $\varepsilon > 0$, there exists a $K > 0$ such that

$$\left|\phi^{\alpha+1}(t_k) - c\right| < \frac{\varepsilon}{2}, \qquad \forall\, k \geqslant K. \tag{2.9}$$

Also, by choosing t_K sufficiently large, we see from (2.8) that

$$\left|\phi^{\alpha+1}(t) - \phi^{\alpha+1}(t_K)\right| < \frac{\varepsilon}{2}, \qquad \forall\, t > t_K. \tag{2.10}$$

Combining (2.9) and (2.10), we get

$$\left|\phi^{\alpha+1}(t) - c\right| \leqslant \left|\phi^{\alpha+1}(t) - \phi^{\alpha+1}(t_K)\right| + \left|\phi^{\alpha+1}(t_K) - c\right| < \varepsilon, \qquad \forall\, t > t_K.$$

Hence, $\lim_{t\to\infty} \phi^{\alpha+1}(t) = c$ or $\lim_{t\to\infty} \phi(t) = (c)^{\frac{1}{\alpha+1}}$. Since $\phi \in L_{p_1}$, it is obvious that $c = 0$. ■

Lemma 2.5 *For* $\phi(t) : [0, \infty) \to \mathbb{R}$*, if* $\phi(t) \in L_{p_1}$ *for some* $p_1 \in [1, \infty)$ *and its derivative* $\dot{\phi}(t) \in L_{p_2}$ *for some* $p_2 \in [1, \infty)$*, then* $\phi \in L_\infty$.

Proof : Let $\alpha = p_1(1 - \frac{1}{p_2})$. Then $\frac{\alpha}{p_1} + \frac{1}{p_2} = 1$. We have

$$\frac{1}{\alpha+1}|\phi^{\alpha+1}(t) - \phi^{\alpha+1}(0)| \leqslant \left|\int_0^t \phi^\alpha(s)\dot{\phi}(s)\mathrm{d}s\right| \tag{2.11}$$

$$\leqslant \left\{\int_0^t |\phi^\alpha(s)|^{\frac{p_1}{\alpha}}\right\}^{\frac{\alpha}{p_1}} \left\{\int_0^t |\dot{\phi}(s)|^{p_2}\right\}^{\frac{1}{p_1}}. \tag{2.12}$$

This implies that

$$|\phi^{\alpha+1}(t)| \leqslant |\phi^{\alpha+1}(0)| + (\alpha+1)\|\phi\|_{p_1}^{\alpha}\|\dot{\phi}\|_{p_2}.$$

This completes the proof. ■

The square of the L_2 or ℓ_2 norm of a signal y is commonly termed as the total energy in the signal y. In many areas of engineering, the energy or square of the L_2 (ℓ_2) norm is used as a measure of the size of a transient signal y that decays to zero as time progresses toward infinity. By Parseval's theorem, $\|y\|_2$ can also be computed in the frequency domain as follows: for the continuous-time case,

$$\|y\|_2 = \Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)^* Y(j\omega)\mathrm{d}\omega\Big)^{1/2},$$

where Y is the Fourier transform of y; similarly, for the discrete-time case,

$$\|y\|_2 = \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})^* Y(e^{j\omega})\mathrm{d}\omega\Big)^{1/2},$$

where Y is the z-transform of y.

Definition 2.6 *A continuous-time signal y for which the following limit is well defined and finite:*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T y(t)'y(t)dt$$

is called an ***RMS (root mean square) or power signal****. The* ***RMS value*** *of such a continuous-time signal y is defined as*

$$\|y\|_{\mathrm{RMS}} = \Big(\lim_{T\to\infty}\frac{1}{T}\int_0^T y(t)'y(t)dt\Big)^{1/2}. \tag{2.13}$$

Similarly, a discrete-time signal y for which the following limit is well defined and finite:

$$\lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T} y(k)' y(k)$$

is called an ***RMS or power signal****. The* ***RMS value*** *of such a discrete-time signal y is defined as*

$$\|y\|_{\text{RMS}} = \Big(\lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T} y(k)' y(k) \Big)^{1/2}. \tag{2.14}$$

Remark 2.7 *Note that sometimes, the RMS is defined by*

$$\|y\|_{\text{RMS}} = \Big(\limsup_{T\to\infty} \frac{1}{T} \int_{0}^{T} y(t)' y(t) dt \Big)^{1/2}$$

and

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T} y(k)' y(k),$$

respectively, for continuous- and discrete-time systems. This has the advantage that the class of signals for which the RMS is well defined and finite becomes a linear vectorspace. This additional structure can sometimes be convenient. Using limsup instead of the standard limit makes some of the derivations a bit more involved, but generally speaking, all properties carry over to this more general case.

The square of the RMS norm[2] of y is commonly termed as the average power of the signal y. Often, in engineering, the RMS norm or average power is used for signals y which are persistent. We note that the RMS norm is a steady-state measure of a signal and is not affected by any transients.

Remark 2.8 *It is obvious that an L_2 (ℓ_2) signal has a zero RMS value. Also, an L_1 signal does not necessarily have a finite or well-defined RMS value, whereas, in contrast, an ℓ_1 signal always has a zero RMS value. Finally, for an L_∞ (ℓ_∞) signal, the RMS value need not be well defined. However, if we use the generalized definition from Remark 2.7, then an L_∞ signal which is locally square Lebesgue integrable has a well-defined and finite RMS value which is less than its L_∞ norm.*

[2] We would like to remark that the RMS norm is a pseudo-norm because the RMS norm of any energy or transient signal is zero.

There exist some relationships among L_p spaces for different values of p. The L_p space can be visualized by a Venn diagram as clarified in the following remark.

Remark 2.9 *Consider a square in the Euclidean plane with vertices* $(0, 1)$, $(1, 0)$, $(0, -1)$, $(-1, 0)$*; that is, the vertices of square satisfy* $|x| + |y| = 1$. *Then, we can interpret the* L_∞ *space in a Venn diagram as the* $[-1, 1]$ *interval on the vertical axis and the* L_1 *space as the* $[-1, 1]$ *interval on the horizontal axis. Also, any* L_p *space can be interpreted as the rectangle whose vertices are given by* $(\pm 1/p, 0)$ *and* $(0, \pm 1 \mp 1/p)$*; that is,* $(1/p, 0), (-1/p, 0), (0, 1 - 1/p), (0, -1 + 1/p)$. *Figure 2.1 illustrates this.*

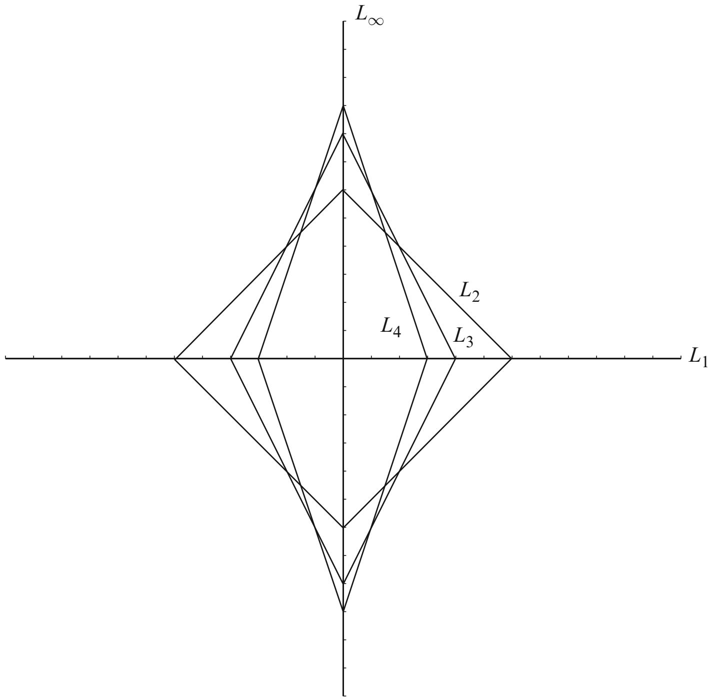

Figure 2.1: Venn diagram of L_p spaces

Note that L_p spaces for different values of p do not contain one another. Only the intersection of L_1 and L_∞ is contained in all L_p spaces. That is, the intersection of all L_p spaces is a single point in the Venn diagram which is equal to the intersection of L_1 and L_∞ spaces. In other words, we have

$$L_1 \cap L_\infty = \bigcap_{p=1}^{\infty} L_p.$$

Regarding the relationship among all ℓ_p spaces, we have for $1 < p < q < \infty$ that

$$\ell_1 \subset \ell_p \subset \ell_q \subset \ell_\infty.$$

To see this, note that $y \in \ell_p$ implies that $\|y(k)\| \leqslant \|y\|_{\ell_p}$ for any $k \geq 0$. Hence, for $p < q < \infty$,

$$\sum_{k=0}^{\infty} \|y(k)\|^q \leq \sum_{k=0}^{\infty} \|y(k)\|^p \|y\|_{\ell_p}^{q-p} = \|y\|_{\ell_p}^{q-p} \|y\|_{\ell_p}^{p} = \|y\|_{\ell_p}^{q} < \infty.$$

Obviously, unlike in the case of L_p space, each ℓ_p is a strict subset of ℓ_q whenever $p < q$.

2.4 Norms of stochastic signals

For a vector signal that is modeled as a wide-sense stationary or an asymptotically wide-sense stationary vector stochastic process (random sequence), the common measure of size is the RMS norm. We recall below the needed definition.

Definition 2.10 *For a wide-sense stationary vector stochastic process y with a bounded variance, we define the* ***stochastic RMS norm*** *as*

$$\|y\|_{\text{RMS}} = \Big(\boldsymbol{E}\,[y(t)'y(t)]\Big)^{1/2}. \tag{2.15}$$

Analogously, for a wide-sense stationary vector random sequence y with a bounded variance, we define the ***stochastic RMS norm*** *as*

$$\|y\|_{\text{RMS}} = \Big(\boldsymbol{E}\,[y(k)'y(k)]\Big)^{1/2}. \tag{2.16}$$

Here $\boldsymbol{E}\,[.]$ denotes the expectation. For stochastic processes (random sequences) that are only asymptotically wide-sense stationary as time goes to infinity [i.e., for asymptotically wide-sense stationary processes (random sequences)], (2.15) *and* (2.16) *need to be rewritten as*

$$\|y\|_{\text{RMS}} = \Big(\lim_{t\to\infty} \boldsymbol{E}\,[y(t)'y(t)]\Big)^{1/2} \tag{2.17}$$

and

$$\|y\|_{\text{RMS}} = \Big(\lim_{k\to\infty} \boldsymbol{E}\,[y(k)'y(k)]\Big)^{1/2}, \tag{2.18}$$

respectively.

Note that in (2.15) and (2.16), the result is independent of t or k because the stochastic process (random sequence) is wide-sense stationary.

We note that if y is an ergodic stochastic process (random sequence), then the deterministic RMS norm of any realization of the stochastic process (random sequence) y is equal to the stochastic RMS norm of y with probability one.

Also, we note that the RMS value of a wide-sense stationary process y can be expressed in terms of its autocorrelation matrix $R_y(\tau)$,

$$R_y(\tau) := \boldsymbol{E}[y(t)y'(t+\tau)],$$

or its power spectral density (PSD) $S_y(\omega)$,

$$S_y(\omega) := \int_{-\infty}^{\infty} R_y(\tau)e^{-j\omega\tau}\mathrm{d}\tau\,.$$

That is,

$$\|y\|_{\mathrm{RMS}} = \Big(\mathrm{trace}[R_y(0)]\Big)^{1/2} = \Big(\frac{1}{2\pi}\,\mathrm{trace}\Big[\int_{-\infty}^{\infty} S_y(\omega)\mathrm{d}\omega\Big]\Big)^{1/2}.$$

Similarly, for a wide-sense stationary random sequence y, let the autocorrelation matrix be

$$R_y(n) := \boldsymbol{E}[y(k)y'(k+n)]$$

and the power spectral density (PSD) be

$$S_y(\omega) := \sum_{n=-\infty}^{\infty} R_y(n)e^{-j\omega n}, \quad -\pi \le \omega \le \pi.$$

Then,

$$\|y\|_{\mathrm{RMS}} = \Big(\mathrm{trace}[R_y(0)]\Big)^{1/2} = \Big(\frac{1}{2\pi}\,\mathrm{trace}\Big[\int_{-\pi}^{\pi} S_y(\omega)\mathrm{d}\omega\Big]\Big)^{1/2}.$$

2.5 Norms of linear time- or shift-invariant systems

We recalled above the definitions of norms of signals. A notion related to the size of a signal is the gain of a transfer function of a linear time- or shift-invariant system. As in the case of a signal, once again, various norms are used to measure the size of a transfer function. In this section, we recall the definitions of certain such norms. Also, we recall methods of computing them.

Two well-known classic norms of linear time- or shift-invariant systems are the H_2 norm (which is the RMS value of the response of a system to white noise input of unit PSD) and the H_∞ norm (which is the RMS gain of the system). The definitions of these norms are recalled below.

Definition 2.11 *Consider a continuous-time system Σ having a $q \times \ell$ stable transfer function G. Then* ***the H_2 norm of the continuous-time system*** *Σ or, equivalently, of the transfer matrix G is defined as*

$$\|G\|_2 = \left(\frac{1}{2\pi} \operatorname{trace} \left[\int_{-\infty}^{\infty} G(j\omega)G^*(j\omega)d\omega \right]\right)^{1/2}. \tag{2.19}$$

We assign ∞ as the H_2 norm of an unstable continuous-time system.

We note that the H_2 norm is induced by an inner product; that is, we have

$$\|G\|_2 = \langle G, G\rangle^{1/2}$$

with the inner product defined by

$$\langle G_1, G_2\rangle = \frac{1}{2\pi} \operatorname{trace} \left[\int_{-\infty}^{\infty} G_1(j\omega)G_2^*(j\omega)\mathrm{d}\omega \right].$$

By Parseval's theorem, the H_2 norm of the transfer matrix G can equivalently be defined as

$$\|G\|_2 = \left(\operatorname{trace} \left[\int_{0}^{\infty} g(t)g'(t)\mathrm{d}t \right]\right)^{1/2}, \tag{2.20}$$

where $g(t)$ is the inverse Laplace transform of the transfer matrix or the unit impulse (Dirac distribution) response of the associated linear system. Thus, $\|G\|_2 = \|g\|_2$. It is also known that $\|G\|_2$ can be expressed in terms of the singular values of the matrix G at each frequency,

$$\|G\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i=1}^{\min\{q,\ell\}} \sigma_i^2(G(j\omega))\mathrm{d}\omega\right)^{1/2}. \tag{2.21}$$

where $\sigma_i(G(j\omega))$ is the ith singular value of $G(j\omega)$.

Remark 2.12 ***(Stochastic interpretation of the H_2 norm of a continuous-time system)*** *Let us consider a continuous-time system with a stable transfer function G. Let the input w to the system be a wide-sense stationary stochastic process. Let z be the corresponding output. It is well known that*

$$S_z(\omega) = G(j\omega)S_w(\omega)G^*(-j\omega), \tag{2.22}$$

where S_w and S_z are the PSDs of $w(t)$ and $z(t)$, respectively. Then, the H_2 norm of $G(s)$ can be interpreted as the *RMS* value of the output z when the given system is driven by zero mean white noise with unit PSD. *Note that formally, white noise with unit PSD does not exist, but the above can be formalized using Brownian motion and stochastic differential equations.*

Remark 2.13 *Note that the H_2 norm of a stable continuous-time system or transfer function $G(s)$ is finite if and only if it is strictly proper.*

Definition 2.14 *Consider a discrete-time system Σ having a $q \times \ell$ stable transfer function G. Then the* ***H_2 norm of the discrete-time system*** *Σ or, equivalently, of the transfer matrix G is defined as*

$$\|G\|_2 = \left(\frac{1}{2\pi}\,\mathrm{trace}\left[\int_{-\pi}^{\pi} G(e^{j\omega})G^*(e^{j\omega})d\omega\right]\right)^{1/2}. \tag{2.23}$$

We assign ∞ to the H_2 norm of an unstable discrete-time system.

Again, we note that the H_2 norm is induced by an inner product; that is, we have

$$\|G\|_2 = \langle G, G\rangle^{1/2}$$

with the inner product defined by

$$\langle G_1, G_2\rangle = \left(\frac{1}{2\pi}\,\mathrm{trace}\left[\int_{-\pi}^{\pi} G_1(e^{j\omega})G_2^*(e^{j\omega})\mathrm{d}\omega\right]\right)^{1/2}.$$

Again, by Parseval's theorem, $\|G\|_2$ can equivalently be defined as

$$\|G\|_2 = \left(\mathrm{trace}\left[\sum_{k=0}^{\infty} g(k)g'(k)\right]\right)^{1/2} \tag{2.24}$$

where g is the inverse z-transform of the transfer matrix which is equal to the unit impulse response of the associated linear system. Thus, $\|G\|_2 = \|g\|_2$.

Remark 2.15 ***(Stochastic interpretation of the H_2 norm of a discrete-time system)*** *Let us consider a discrete-time system with a stable transfer function $G(z)$. Let the input $w(k)$ to the system be a wide-sense stationary random sequence. Let $z(k)$ be the corresponding output. Then, it is well known that*

$$S_z(\omega) = G(e^{j\omega})S_w(\omega)G^*(e^{-j\omega}), \quad -\pi \le \omega \le \pi, \tag{2.25}$$

where S_w and S_z are the PSDs of w and z, respectively. Then, once again, the H_2 norm of G can be interpreted as the RMS value of the output z when the given system is driven by a zero mean white noise random sequence having unit variance.

State-space method for computing the H_2 norm: We present here briefly some results on the computation of the H_2 norm of a transfer function matrix when its realization is given in a state-space form (for details, see [15]). Consider the transfer function G of a continuous-time system with realization (A, B, C, D) where A is Hurwitz stable. Let W^c_{obs} denote the observability grammian of the pair (A, C) and W^c_{con} the controllability grammian of (A, B). Note that D needs to be zero for a finite H_2 norm. We note that W^c_{obs} and W^c_{con} are the unique solutions of continuous-time Lyapunov equations:

$$\begin{aligned} A'W^c_{\text{obs}} + W^c_{\text{obs}}A + C'C &= 0,\\ AW^c_{\text{con}} + W^c_{\text{con}}A' + BB' &= 0. \end{aligned}$$

The H_2 norm of $G(s)$ can now be computed by

$$\begin{aligned} \|G\|_2 &= \left(\text{trace}\, B'W^c_{\text{obs}}B\right)^{1/2}\\ &= \left(\text{trace}\, CW^c_{\text{con}}C'\right)^{1/2}. \end{aligned}$$

The computation of the H_2 norm of a transfer function G of a discrete-time system with realization (A, B, C, D) where A is Schur stable, can be given along the same lines. That is, let W^d_{obs} and W^d_{con} be the unique solutions of discrete-time Lyapunov equations:

$$\begin{aligned} A'W^d_{\text{obs}}A - W^d_{\text{obs}} + C'C &= 0,\\ AW^d_{\text{con}}A' - W^d_{\text{con}} + BB' &= 0. \end{aligned}$$

The H_2 norm of G can now be computed by

$$\begin{aligned} \|G(z)\|_2 &= \left(\text{trace}[B'W^d_{\text{obs}}B + D'D]\right)^{1/2}\\ &= \left(\text{trace}[CW^d_{\text{con}}C' + DD']\right)^{1/2}. \end{aligned}$$

Definition 2.16 *Consider a continuous-time system having a $q \times \ell$ stable transfer function G. Then* ***the H_∞ norm*** *of G is defined as*

$$\|G\|_\infty := \sup_{\omega} \sigma_{\max}[G(j\omega)]. \tag{2.26}$$

Similarly, consider a discrete-time system having a $q \times \ell$ stable transfer function G. Then the H_∞ norm of G is defined as

$$\|G\|_\infty := \sup_{-\pi \leq \omega \leq \pi} \sigma_{\max}[G(e^{j\omega})]. \tag{2.27}$$

For a continuous-time system having a stable transfer function G, let w and z be energy signals that are, respectively, the input and the corresponding output of the given system. Similarly, for a discrete-time system having a stable transfer function G, let w and z be energy signals that are, respectively, the input and the corresponding output. Then it is easy to see that $\|G\|_\infty$ has the following interpretation for both continuous-time and discrete-time systems (where $\|\cdot\|_2$ denotes the L_2 and ℓ_2 norm, respectively):

$$\|G\|_\infty = \sup_{w \neq 0} \frac{\|z\|_2}{\|w\|_2}.$$

Also, when the input and the corresponding output (i.e., w and z) are power signals, the H_∞ norm of G turns out to coincide with its RMS gain, namely,

$$\|G\|_\infty = \|G\|_{\text{RMS gain}} = \sup_{\substack{w \\ \|w\|_{\text{RMS}} \neq 0}} \frac{\|z\|_{\text{RMS}}}{\|w\|_{\text{RMS}}}.$$

We have the following remarks.

Remark 2.17 *An important property of the H_∞ norm, for both continuous-time and discrete-time systems, is that it is submultiplicative. That is, for transfer matrices G_1 and G_2, we have*

$$\|G_1 G_2\|_\infty \leqslant \|G_1\|_\infty \|G_2\|_\infty.$$

Remark 2.18 *It is interesting to contrast the H_2 and H_∞ norms. Consider a transfer matrix H. Then the fact that $\|H\|_\infty < \alpha$ for some $\alpha > 0$ implies that*

$$\|Hu\|_{\text{RMS}} \leqslant \alpha \text{ for any input } u \text{ with } \|u\|_{\text{RMS}} \leqslant 1.$$

In contrast, the H_2 norm-bound specification $\|H\|_2 \leqslant \alpha$ implies that

$$\|Hu\|_{\text{RMS}} \leqslant \alpha \text{ when input } u \text{ is a white noise with unit intensity.}$$

State-space method for computing the continuous-time H_∞ norm:

Regarding the H_∞ norm computation for continuous time, there is a simple method to determine whether the inequality specification $\|G\|_\infty < \gamma$ is satisfied. To state this, given $\gamma > 0$, we define the matrix

$$M_\gamma = \begin{pmatrix} A + BR^{-1}D'C & \gamma^{-2}BR^{-1}B' \\ -C'(I + DR^{-1}D')C & -(A + BR^{-1}D'C)', \end{pmatrix}$$

where $R := \gamma^2 I - D'D > 0$. Then, we have

$$\|G\|_\infty < \gamma \Longleftrightarrow M_\gamma \text{ has no imaginary eigenvalues and } \sigma_{\max}(D) < \gamma. \quad (2.28)$$

The above discussion provides a simple bisection algorithm that enables us to compute the H_∞ norm numerically to any degree of numerical accuracy. In the following algorithm, the first three steps represent initialization, whereas the last step represents the bisection principle:

(*i*) Set $i = 0$, and set $\gamma_\ell = \|D\|$.

(*ii*) Choose any $\gamma_0 > \gamma_\ell$.

(*iii*) Use (2.28) to test whether $\|G\|_\infty < \gamma_i$. If so, $\gamma_u = \gamma_i$ and continue with step (d). Otherwise, set $\gamma_{i+1} = 2\gamma_i$ and $i = i + 1$ and continue with step (*iii*).

(*iv*) Set $\gamma = (\gamma_u + \gamma_\ell)/2$. Use (2.28) to test whether $\|G\|_\infty < \gamma$. If so, set $\gamma_u = \gamma$ and otherwise set $\gamma_\ell = \gamma$ and then repeat step (*iv*).

We observe from step (d) that $\|G\|_\infty$ is within the interval $[\gamma_\ell,\ \gamma_u]$. After each iteration, the size of the interval divides itself into half. Hence, one can stop the iterations when the desired level of accuracy is reached.

For more details concerning the computation of the H_∞ norm, we refer to [14, 16].

State-space method for computing the discrete-time H_∞ norm:

Similar to the continuous time, for discrete time, there is a simple method to determine whether the inequality specification $\|G\|_\infty < \gamma$ is satisfied. To state this, given $\gamma > 0$, we define the matrix pencil

$$M_\gamma(z) = \begin{pmatrix} zI - A & -B & 0 \\ C'C & C'D & I - zA' \\ D'C & D'D - \gamma^2 I & -zB' \end{pmatrix}.$$

Then, we have

$$\|G\|_\infty < \gamma \iff M_\gamma(z) \text{ has no zeros on the unit circle and } \sigma_{\max}(D + C(I - A)^{-1}B) < \gamma.$$

The above theorem provides a simple bisection algorithm that enables us to compute the H_∞ norm numerically to any degree of numerical accuracy, which is completely similar to the continuous-time case.

The computation of the H_∞ norm of a transfer matrix of a discrete-time system through a bisection algorithm then follows similarly to the continuous-time case.

2.6 A class of saturation functions

As we said in Chap. 1, one of the most prevalent constraints is the one that arises from actuator saturation. We introduce below the class of saturation functions we consider throughout this book.

Definition 2.19 *The function $\sigma_1 : \mathbb{R}^m \to \mathbb{R}^m$ is called the* ***standard saturation function*** *if*

$$\sigma_1(s) = \begin{pmatrix} \mathrm{sat}_1(s_1) \\ \mathrm{sat}_1(s_2) \\ \vdots \\ \mathrm{sat}_1(s_m) \end{pmatrix},$$

where

$$\mathrm{sat}_1(s) = \mathrm{sgn}(s)\ \min\{|s|, 1\}.$$

For convenience, we also introduce a scaled version of the standard saturation function:

$$\mathrm{sat}_\Delta(s) = \Delta\,\mathrm{sat}\left(\frac{s}{\Delta}\right)$$

and

$$\sigma_\Delta(s) = \begin{pmatrix} \mathrm{sat}_\Delta(s_1) \\ \mathrm{sat}_\Delta(s_2) \\ \vdots \\ \mathrm{sat}_\Delta(s_m) \end{pmatrix}.$$

For ease of notation, we will often use $\sigma(s)$ as abbreviation for $\sigma_\Delta(s)$.

In reality, a saturation occurring in some device will never be equal to the above standard saturation function. Therefore, we introduce a class of saturation functions satisfying some minimum characteristics which are common for all saturation functions.

Definition 2.20 *A function $\widetilde{\sigma} : \mathbb{R}^m \to \mathbb{R}^m$ is called a* ***saturation function*** *if:*

(i) *$\widetilde{\sigma}(u)$ is decentralized, that is,*

$$\widetilde{\sigma}(s) = \begin{pmatrix} \widetilde{\sigma}_1(s_1) \\ \widetilde{\sigma}_2(s_2) \\ \vdots \\ \widetilde{\sigma}_m(s_m) \end{pmatrix}.$$

(ii) *$\widetilde{\sigma}_i$ is globally Lipschitz, that is, for some $\delta > 0$,*

$$|\widetilde{\sigma}_i(s_1) - \widetilde{\sigma}_i(s_2)| \leqslant \delta |s_1 - s_2|.$$

(iii) *$s\widetilde{\sigma}_i(s) > 0$ whenever $s \neq 0$ and $\widetilde{\sigma}_i(0) = 0$.*

(iv) *The two limits*

$$\lim_{s \to 0^+} \frac{\widetilde{\sigma}_i(s)}{s}, \quad \lim_{s \to 0^-} \frac{\widetilde{\sigma}_i(s)}{s}$$

both exist and are strictly positive.

(v) $\liminf\limits_{|s| \to \infty} |\widetilde{\sigma}_i(s)| > 0.$

Remark 2.21 *Note that the above definition for a saturation function does not enforce that a saturation function is bounded. Actually, $\widetilde{\sigma}(s) = s$ satisfies all the properties above. Especially when establishing necessary conditions, it is sometimes useful to require the additional condition:*

(vi) *There exists a $M > 0$ such that $|\widetilde{\sigma}_i(s)| < M$ for all $s \in \mathbb{R}$.*

Remark 2.22 *In some cases, we actually use the following condition:*

(vii) *There exist $\theta > 0$ and $\psi > 0$ such that*

$$|\widetilde{\sigma}(s)| > \min\{\, \theta|s|, \psi \,\}$$

which is a consequence of Condition (iv), which guarantees that $|\sigma(s)|$ is larger than $\theta|s|$ for some positive θ for small s Condition (v), which guarantees that $\sigma(s)$ is bounded away from zero for large s and Condition (iii), which guarantees that $\sigma(s)$ is never equal to zero for $s \neq 0$.

2.7 Internal stability

In this section, we review various notions and definitions all pertaining to internal stability. Most of the definitions and results presented here are classical and can be found in many textbooks such as [58, 142, 189]. The most complete set of results can be found in [44]. We concentrate here only on continuous-time systems; however, all the following definitions can be easily modified for discrete-time systems.

We consider throughout this section a nonlinear ordinary differential equation of the form

$$\dot{x} = f(t, x), \quad x(t_0) = x_0, \tag{2.29}$$

where $x(t) \in \mathbb{R}^n$, $f : [0, \infty) \times \mathbb{R}^n \mapsto \mathbb{R}^n$ and

$$\mathcal{B}(r) = \{x \in \mathbb{R}^n \mid \|x\| < r\}.$$

Unless stated otherwise, we assume that the function f is such that, for all initial conditions in some open neighborhood of the equilibrium, the system (2.29) possesses a unique solution $x(t; t_0, x_0)$ for all $t_0 \geqslant 0$ and $t > t_0$.

The system (2.29) is referred to as a time-varying system. We also consider the case when the function f in (2.29) is not explicitly dependent on time t. In this case, the resulting system (2.29) is referred to as a time-invariant system which can be written as

$$\dot{x} = f(x), \quad x(t_0) = x_0. \tag{2.30}$$

We have the following definitions.

Definition 2.23 *A state x_e is said to be an **equilibrium state** of the system (2.29) if*

$$f(t, x_e) \equiv 0 \ \textit{for all} \ \ t \geqslant 0.$$

Definition 2.24 *The equilibrium state x_e of (2.29) is said to be an **isolated equilibrium state** if there exists a constant $\alpha > 0$ such that the system (2.29) does not contain any equilibrium other than x_e in the region*

$$\mathcal{B}(x_e, \alpha) := \{x \mid \|x - x_e\| < \alpha\} \subset \mathbb{R}^n.$$

Definition 2.25 *The equilibrium state x_e of (2.29) is said to be **stable** if for any $t_0 \geqslant 0$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x_0 \in \mathbb{R}^n$ with $\|x_0 - x_e\| < \delta$, we have that $\|x(t; t_0, x_0) - x_e\| < \varepsilon$ for all $t \geq t_0$.*

Definition 2.26 *The equilibrium state x_e of* (2.29) *is said to be* ***unstable*** *if it is not stable.*

Definition 2.27 *The equilibrium state x_e of* (2.29) *is said to be* ***uniformly stable*** *if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x_0 \in \mathbb{R}^n$ with $\|x_0 - x_e\| < \delta$, we have $\|x(t; t_0, x_0) - x_e\| < \varepsilon$ for all $t_0 \geqslant 0$ and for all $t \geq t_0$.*

Definition 2.28 *The equilibrium state x_e of* (2.29) *is said to be* ***asymptotically stable*** *if:*

(*i*) *It is stable.*

(*ii*) *For every $t_0 \geqslant 0$, there exists a $\delta > 0$ such that for all $x_0 \in \mathbb{R}^n$ with $\|x_0 - x_e\| < \delta$, we have $\lim_{t\to\infty} \|x(t; t_0, x_0) - x_e\| = 0$.*

Definition 2.29 *For any $t_0 \geqslant 0$, the set of all $x_0 \in \mathbb{R}^n$ such that $x(t; t_0, x_0) \to x_e$ as $t \to \infty$ is called the* ***region of attraction*** *at time t_0 of the equilibrium state x_e. If condition* (*ii*) *of Definition 2.28 is satisfied, then the equilibrium state x_e is said to be* ***attractive****.*

Definition 2.30 *The equilibrium state x_e of* (2.29) *is said to be* ***uniformly asymptotically stable*** *if*

(*i*) *It is uniformly stable.*

(*ii*) *For every $\varepsilon > 0$, there exist $T > 0$ and $\delta_0 > 0$ such that for all initial conditions $x_0 \in \mathbb{R}^n$ with $\|x_0 - x_e\| < \delta_0$, we have $\|x(t; t_0, x_0) - x_e\| < \varepsilon$ for all $t_0 \geqslant 0$ and for all $t \geq t_0 + T$.*

Definition 2.31 *The equilibrium state x_e of* (2.29) *is said to be* ***exponentially stable*** *if there exists an $\alpha > 0$ with the property that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all initial conditions $x_0 \in \mathbb{R}^n$ for which $\|x_0 - x_e\| < \delta$, we have*

$$\|x(t; t_0, x_0) - x_e\| \leq \varepsilon e^{-\alpha(t-t_0)}$$

for all $t_0 \geqslant 0$ and for all $t \geq t_0$.

Definition 2.32 *A solution $x(t; t_0, x_0)$ for some $t_0 \geqslant 0$ and $x_0 \in \mathbb{R}^n$ of* (2.29) *is said to be* ***bounded*** *if there exists a $\beta > 0$ such that $\|x(t; t_0, x_0)\| < \beta$ for all $t \geq t_0$.*

Definition 2.33 *The solutions of* (2.29) *are said to be* ***uniformly bounded*** *if for any $\alpha > 0$, there exists a $\beta > 0$ such that for all initial conditions $x_0 \in \mathbb{R}^n$ for which $\|x_0\| < \alpha$, we have $\|x(t; t_0, x_0)\| < \beta$ for all $t_0 \geqslant 0$ and for all $t \geq t_0$.*

Definition 2.34 *The set of solutions of* (2.29) *is said to be* ***uniformly ultimately bounded*** *if there exists a $B > 0$ such that for any $\alpha > 0$, there exists a $T > 0$ such that for all initial conditions $x_0 \in \mathbb{R}^n$ for which $\|x_0\| < \alpha$, we have $\|x(t; t_0, x_0)\| < B$ for all $t_0 \geqslant 0$ and for all $t \geq t_0 + T$.*

Definition 2.35 *The equilibrium state x_e of* (2.29) *is said to be* ***globally asymptotically stable*** *if it is stable and every solution of* (2.29) *tends to x_e as $t \to \infty$ (i.e., the region of attraction of x_e is all of $\mathbb{R}^n$).*

Definition 2.36 *The equilibrium state x_e of* (2.29) *is said to be* ***uniformly globally asymptotically stable*** *if:*

(*i*) *It is uniformly stable.*

(*ii*) *The solutions of* (2.29) *are uniformly bounded.*

(*iii*) *For all $\alpha > 0$ and $\varepsilon > 0$, there exists a $T > 0$ such that for all initial conditions $x_0 \in \mathbb{R}^n$ satisfying $\|x_0 - x_e\| < \alpha$, we have $\|x(t; t_0, x_0) - x_e\| < \varepsilon$ for all for all $t_0 \geqslant 0$ and $t \geq t_0 + T$.*

Definition 2.37 *The equilibrium state x_e of* (2.29) *is said to be* ***globally exponentially stable*** *if there exists an $\alpha > 0$ with the property that for any $\beta > 0$, there exists a $k > 0$ such that for all initial conditions $x_0 \in \mathbb{R}^n$ for which $\|x_0 - x_e\| < \beta$, we have*

$$\|x(t; t_0, x_0) - x_e\| \leq k e^{-\alpha(t - t_0)}$$

for all $t_0 \geqslant 0$ and for all $t \geq t_0$.

Definition 2.38 *The trajectory $x(t;t_0,x_0)$ is said to be **stable (unstable, uniformly stable, asymptotically stable, uniformly asymptotically stable, exponentially stable)** if the equilibrium state $z_e = 0$ of the system*

$$\dot{z} = f(t, z + x(t;t_0,x_0)) - f(t, x(t;t_0,x_0))$$

is stable (unstable, uniformly stable, asymptotically stable, uniformly asymptotically stable, exponentially stable, respectively).

Definition 2.39 *The system (2.29) is said to be **internally L_p stable at t_0** if the trajectory $x(t;t_0,x_0)$ for any $x_0 \in \mathbb{R}^n$ belongs to $L_p[t_0,\infty)$.*

Definition 2.40 *The system (2.29) is said to be **uniformly internally L_p stable** if it is internally L_p stable at t_0 for all $t_0 \geqslant 0$.*

Remark 2.41 *For autonomous systems, internal L_p stability implies global attractivity of the origin when f is continuous at the origin (see Lemma 2.79).*

It is easy to see that in the case of autonomous system (2.30), all the references to the word "uniform" in the above definitions need not be evoked. That is, if the equilibrium state x_e is stable (asymptotically stable, exponentially stable, globally asymptotically stable), it is always uniformly stable (uniformly asymptotically stable, uniformly exponentially stable, uniformly globally asymptotically stable, respectively). Similarly, if the solution $x(t;t_0,x_0)$ is bounded, it is uniformly bounded as well.

2.7.1 Lyapunov's direct method

The stability properties of the equilibrium state x_e or the solution $x(t;t_0,x_0)$ of (2.29) can be verified by utilizing the well-known direct method of Lyapunov (also called as the second method of Lyapunov). The method seeks to answer various questions of stability by using the form of the function $f(t,x)$ in (2.29) rather than the explicit knowledge of the solutions. We need the following additional definitions before we introduce the method.

Definition 2.42 *A continuous function $\phi : [0,\infty) \mapsto \mathbb{R}^+$ is said to belong to class $\mathcal{K}$ (denoted by $\phi \in \mathcal{K}$) if:*

(*i*) $\phi(0) = 0$.

(*ii*) *ϕ is strictly increasing.*

Definition 2.43 *A continuous function* $\phi : [0, \infty) \mapsto \mathbb{R}^+$ *is said to belong to class* $\mathcal{K}_\infty$ *(denoted by* $\phi \in \mathcal{K}_\infty$*) if:*

(i) $\phi(0) = 0$.

(ii) ϕ *is strictly increasing.*

(iii) $\lim_{\tau \to \infty} \phi(\tau) = \infty$.

Definition 2.44 *Two functions* $\phi_1, \phi_2 \in \mathcal{K}$ *are said to be of the same order of magnitude if there exist positive constants* k_1 *and* k_2 *such that*

$$k_1\phi_1(r) \leq \phi_2(r) \leq k_2\phi_1(r),$$

for all $r \in [0, \infty)$.

Definition 2.45 *A function* $V(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}$ *with* $V(t, 0) = 0$ *for all* $t \in \mathbb{R}^+$ *is said to be* ***locally positive definite*** *if there exists a subset* $\mathcal{V}$ *of* $\mathbb{R}^n$ *containing* 0 *in its interior and a continuous function* $\phi \in \mathcal{K}$ *such that* $V(t, x) \geq \phi(\|x\|)$ *for all* $t \in \mathbb{R}^+$, $x \in \mathcal{V}$.

$V(t, x)$ is called locally negative definite if $-V(t, x)$ is positive definite.

Definition 2.46 *A function* $V(t, x) : \mathbb{R}^+ \times \mathcal{V} \mapsto \mathbb{R}$ *with* $V(t, 0) = 0$ *for all* $t \in \mathbb{R}^+$ *is said to be* ***locally positive semi-definite*** *if if there exists a subset* $\mathcal{V}$ *of* $\mathbb{R}^n$ *containing* 0 *in its interior such that* $V(t, x) \geq 0$ *for all* $t \in \mathbb{R}^+$, $x \in \mathcal{V}$.

$V(t, x)$ is called negative semi-definite if $-V(t, x)$ is positive semi-definite.

Definition 2.47 *A function* $V(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}$ *with* $V(t, 0) = 0$ *for all* $t \in \mathbb{R}^+$ *is said to be* ***decrescent*** *if there exists a subset* $\mathcal{V}$ *of* $\mathbb{R}^n$ *containing* 0 *in its interior and a continuous function* $\phi \in \mathcal{K}$ *such that* $V(t, x) \leq \phi(\|x\|)$ *for all* $t \in \mathbb{R}^+$ *and for all* $x \in \mathcal{V}$.

Definition 2.48 *A function* $V(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}$ *with* $V(t, 0) = 0$ *for all* $t \in \mathbb{R}^+$ *is said to be* ***positive definite*** *if there exists a continuous function* $\phi \in \mathcal{K}$ *such that* $V(t, x) \geq \phi(\|x\|)$ *for all* $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}^n$.

Definition 2.49 *A function $V(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}$ with $V(t,0) = 0$ for all $t \in \mathbb{R}^+$ is said to be a* ***radially unbounded*** *positive definite function if there exists a $\phi \in \mathcal{K}_\infty$ such that*

$$V(t,x) \geq \phi(\|x\|) \text{ for all } t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^n.$$

In what follows, without loss of generality, we assume that $x_e = 0$ is an equilibrium point of (2.29). Also, we define $\dot{V}$ as the time derivative of the function $V(t,x)$ along the solution of (2.29), that is,

$$\dot{V} = \frac{\partial V}{\partial t} + (\nabla V) f(t,x),$$

where

$$\nabla V = \frac{\partial V}{\partial x} = \begin{pmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \cdots & \frac{\partial V}{\partial x_n} \end{pmatrix}$$

is the gradient of V with respect to x.

The following theorem states the second method of Lyapunov.

Theorem 2.50 *Suppose that there exists a locally positive definite function*

$$V(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}$$

for some $r > 0$ with continuous first-order partial derivatives with respect to t and x and $V(t,0) = 0$ for all $t \in \mathbb{R}^+$. Then the following statements are true:

(*i*) *If $\dot{V}(t,x) \leq 0$ for all $t \in \mathbb{R}^+$ and all x in some open neighborhood of 0, then $x_e = 0$ is stable.*

(*ii*) *If V is decrescent and $\dot{V}(t,x) \leq 0$ for all $t \in \mathbb{R}^+$ and all x in some open neighborhood of 0, then $x_e = 0$ is uniformly stable.*

(*iii*) *If V is decrescent and $\dot{V}(t,x) < 0$ for all $t \in \mathbb{R}^+$ and all x in some open neighborhood of 0, then $x_e = 0$ is uniformly asymptotically stable.*

(*iv*) *If V is decrescent and there exist functions $\phi_1, \phi_2, \phi_3 \in \mathcal{K}$ of the same order of magnitude such that*

$$\phi_1(\|x\|) \leq V(t,x) \leq \phi_2(\|x\|) \text{ and } \dot{V}(t,x) \leq -\phi_3(\|x\|)$$

for all $t \in \mathbb{R}^+$ and for all x in some open neighborhood of the origin, then $x_e = 0$ is exponentially stable.

Let us examine statement (*ii*) of Theorem 2.50. If we remove the restriction of V being decrescent, we obtain statement (*i*). Therefore, one might be tempted to expect that by removing the condition of V being decrescent in statement (*iii*), we obtain a condition for asymptotic stabilty, that is, $\dot{V} < 0$ implies that $x_e = 0$ is asymptotic stability. Such an intuitive conclusion is not true, as demonstrated by a counterexample in [94], see also [44, Sect. 53], where a first-order differential equation and a positive definite, non-decrescent function $V(t, x)$ are used to show that $\dot{V} < 0$ alone does not imply that $x_e = 0$ is asymptotic stable.

The condition in statement (*iii*) of the above theorem, namely, $V(t, x)$, is decrescent, and $\dot{V}(t, x) < 0$ is also equivalent to the existence of functions $\phi_1, \phi_2, \phi_3 \in \mathcal{K}$, where ϕ_1, ϕ_2, ϕ_3 do not have to be of the same order of magnitude, such that

$$\phi_1(\|x\|) \leq V(t, x) \leq \phi_2(\|x\|) \text{ and } \dot{V}(t, x) \leq -\phi_3(\|x\|)$$

for all $t \in \mathbb{R}^+$ and $x \in \mathcal{B}(r)$.

We recognize that, in the above theorem, the state x is restricted to a neighborhood of the origin. As such, the results (*i*)–(*iv*) of Theorem 2.50 are referred to as local results. The following theorems are concerned with the global results.

Theorem 2.51 *Assume that the solution of* (2.29) *exists and is unique for each* $x_0 \in \mathbb{R}^n$. *Suppose that there exists a decrescent, radially unbounded positive definite function* $V(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^+$ *with continuous first-order partial derivatives with respect to* t *and* x *and* $V(t, 0) = 0$ *for all* $t \in \mathbb{R}^+$. *Then the following statements are true:*

(*i*) *If* $\dot{V}(t, x) < 0$ *for all* $t \in \mathbb{R}^+$ *and all* $x \in \mathbb{R}^n$, *then* $x_e = 0$ *is uniformly globally asymptotically stable.*

(*ii*) *If there exist functions* $\phi_1, \phi_2, \phi_3 \in \mathcal{K}_\infty$ *of the same order of magnitude such that*

$$\phi_1(\|x\|) \leq V(t, x) \leq \phi_2(\|x\|) \text{ and } \dot{V}(t, x) \leq -\phi_3(\|x\|)$$

for all $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}^n$, *then* $x_e = 0$ *is globally exponentially stable.*

The condition in statement (*i*) of the above theorem, namely, $\dot{V}(t, x) < 0$, is also equivalent to the existence of functions $\phi_1, \phi_2 \in \mathcal{K}_\infty$ and $\phi_3 \in \mathcal{K}$ such that

$$\phi_1(\|x\|) \leq V(t, x) \leq \phi_2(\|x\|) \text{ and } \dot{V}(t, x) \leq -\phi_3(\|x\|)$$

for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^n$.

Theorem 2.52 *Let the solution of* (2.29) *be unique for each* $x_0 \in \mathbb{R}^n$. *Suppose that there exists a decrescent, positive definite function* $V(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^+$ *with continuous first-order partial derivatives with respect to* t *and* x *and* $V(t, 0) = 0$ *for all* $t \in \mathbb{R}^+$. *Then the following statements are true:*

(*i*) *If there exists a* $R > 0$ *such that* $\dot{V}(t, x) \leq 0$ *for all* $t \in R^+$ *and all* $x \in \mathbb{R}^n$ *with* $\|x\| > R$, *then* $x_e = 0$ *is uniformly bounded.*

(*ii*) *If there exists a* $R > 0$ *and* $\phi \in \mathcal{K}$ *such that* $\dot{V}(t, x) \leq -\phi(\|x\|)$ *for all* $t \in R^+$ *and all* $x \in \mathbb{R}^n$ *with* $\|x\| > R$*, then* $x_e = 0$ *is uniformly ultimately bounded.*

Theorems 2.50–2.52 also hold for the autonomous system (2.30) because it is a special case of (2.29)). In the case of (2.30), however, $V(t, x) = V(x)$, that is, it does not explicitly depend on time t, and all references to the words "decrescent" and "uniform" could be deleted. This is because $V(x)$ is always decrescent and the stability (respectively, asymptotic stability) of the equilibrium $x_e = 0$ of (2.30) implies uniform stability (respectively, uniform asymptotic stability). Also, for (2.30), we can obtain a stronger result than Theorem 2.51 for global asymptotic stability. Before we state this result, let us have the following definition.

Definition 2.53 *A set* $\Omega \in \mathbb{R}^n$ *is* ***invariant*** *with respect to* (2.30) *if every solution of* (2.30) *starting in* Ω *remains in* Ω *for all* t*.*

Theorem 2.54 *Let the solution of* (2.30) *be unique for each* $x_0 \in \mathbb{R}^n$*. Suppose that there exists a positive definite and radially unbounded function* $V(x) : \mathbb{R}^n \mapsto \mathbb{R}^+$ *with continuous first-order partial derivatives with respect to* x *and* $V(0) = 0$*. If*

(*i*) $\dot{V} \leq 0$ *for all* $x \in \mathbb{R}^n$*.*

(*ii*) *The origin* $x = 0$ *is the only invariant subset of the set*

$$\Omega = \{x \in \mathbb{R}^n \mid \dot{V} = 0\},$$

then the equilibrium $x_e = 0$ *of* (2.30) *is globally asymptotically stable.*

All the above theorems are referred to as Lyapunov-type theorems. The function $V(t, x)$ or $V(x)$ that satisfies any Lyapunov-type theorem is referred to as a Lyapunov function.

Lyapunov functions can also be used to predict the instability properties of an equilibrium point x_e. Several instability theorems based on the second method of Lyapunov are given in [44].

2.8 External stability

Securing the stability of the equilibrium point of a given system or physical process is central to any control system design. In this regard, the classical concept of internal stability or otherwise often known as Lyapunov stability of a given system, as discussed in Sect. 2.7, dwells on various notions of the stability of an equilibrium point. It is widely discussed in many text books. On the other hand, another classical notion well known in the context of linear systems is the concept of bounded input bounded output (BIBO) stability or input–output stability. It is rooted in the requirement that a "small" excitation should cause only a "small" response. Motivated by this, the notion of *external stability* is indeed an attempt to bring a notion similar to BIBO to nonlinear systems as well by defining an appropriate measure of "smallness". Most of the literature considers the L_p (or ℓ_p) norm as an appropriate measure of "smallness". Thus, external stability seeks the controlled output be in the L_p or ℓ_p space for $p \in [1, \infty]$ whenever the external input or disturbance of a system is in the L_p or ℓ_p space. Moreover, one can also define the notion of system gain as the induced norm of the mapping from the external input to the controlled output. Owing to the use of the L_p (or ℓ_p) norm as an appropriate measure, external stability is also known as L_p stability for continuous-time systems or ℓ_p stability for discrete-time systems. Thus, the notions of input–output stability, external stability, and L_p or ℓ_p stability tantamount to the same.

To distinguish internal stability, or otherwise called Lyapunov stability, from input–output stability, it is worth quoting here two paragraphs from J. C. Willem's book, *The Analysis of Feedback Systems* ([204], pp. 102–103), from which one may gain a historical view on how the Lyapunov stability and input–output stability are distinguished and get separated.

"Lyapunov stability considers stability as an internal property of a system, and inputs and outputs do not play a role. This formulation accounts for the early development and great historical importance of this type of stability. The study of systems without inputs and outputs is indeed basic to classical dynamics. The traditional question of the stability of the solar system, for example remains a long standing challenge and does not involve inputs in any way. It is thus more than natural that stability of control systems has been studied in this context; namely, as a condition on undriven classical *dynamical systems*. This is in spite of the fact that its founders, Lyapunov and Poincaré, were not primarily interested in control. It should be noted that this dynamical-system point of view is supported by the work of Maxwell and much of the subsequent work on regulators. Although Lyapunov stability remains important and very useful in many control applications, its basic philosophy can often be challenged and is somewhat out of line with the modern approach to systems, where inputs and outputs are the fundamental variables and the state is merely an auxiliary variable that essentially represents the contents of a memory bank. The development and success of input–output stability should thus come as no surprise. This does not exclude that for many applications Lyapunov stability does represent a very satisfactory type of stability, and thus its study will remain both important and fruitful."

"Input–output stability is, from an engineering point of view, a very significant and important type of stability. The informal definition of stability given for instance by Nyquist in his classic paper on *Regeneration Theory* is essentially that of input–output stability. It is intimately related to the idea of *stability under constant disturbances* and thus has some classical—although not system oriented—foundations. The concept of input–output stability stands in direct competition with the idea of stability in the sense of Lyapunov. Input–output stability considers the disturbance entering the system as a constantly acting input, where as stability in the sense of Lyapunov considers the initial conditions as the disturbance to the system. Which of these two types of stability is to be preferred clearly depends on the particular application. In a sense, input–output stability protects against noise disturbances, whereas Lyapunov stability protects against a single impulse-like disturbance."

Our concern in this section is to recall several definitions all pertaining to input–output stability or external stability or some other notions of stability related to it.

Throughout this section, we consider a system Σ of the form,

$$\begin{cases} \rho x = f(x,d), & x(0) = x_0 \\ z = g(x,d), \end{cases} \tag{2.31}$$

with $f(0,0) = 0$ and $g(0,0) = 0$, where, as before, ρ denotes the time derivative ($\rho x = \frac{\mathrm{d}}{\mathrm{d}t}x$) or the shift operator ($(\rho x)(k) = x(k+1)$) for continuous-time and discrete-time systems, respectively. Here x is the state, d is the input, and z is the controlled output.

At first, we start with various notions of L_p or ℓ_p stability where initial conditions are fixed; in fact as is customary, the initial conditions are fixed at the origin.

Definition 2.55 *Let $p \in [1,\infty]$. Consider a continuous-time system Σ as given in (2.31). Then, it is said to be **L_p stable with fixed initial condition and without finite gain** if, given any input $d \in L_p$ and $x(0) = 0$, there exists a unique solution x such that the controlled output $z \in L_p$.*

*Similarly, consider the discrete-time system Σ as given in (2.31). Then, it is said to be **ℓ_p stable with fixed initial condition and without finite gain** if, given any input $d \in \ell_p$ and $x(0) = 0$, the unique solution x is such that the controlled output $z \in \ell_p$.*

Remark 2.56 *We note that if the continuous-time system (2.31) has an L_p input, the classical condition of f being locally Lipschitz might not suffice to ensure the existence and uniqueness of solution. This can be shown in the following example. Consider*

$$\dot{x} = (1-x)d^3, \qquad x(0) = 0.$$

The above system has a right hand side which obviously is nicely differentiable. Let

$$d = \begin{cases} \left(\frac{1}{1-t}\right)^{\frac{1}{3}} & t < t_0, \\ 0 & t \geqslant t_0, \end{cases}$$

where $t_0 > 1$. It is easy to verify that we have $d \in L_1$. One solution with this input is

$$x(t) = \mathrm{sat}_{t_0}(t).$$

Another solution is

$$x(t) = \mathrm{sat}_1(t),$$

where sat_Δ *is a standard saturation function with saturation level Δ as defined in Definition 2.19. Both of these solutions are so-called weak solutions in the sense that*

$$x(t) = \int_0^t (1 - x(\tau))d^3(\tau)\, d\tau.$$

We should note that if we impose that $d \in L_p \cap L_\infty$, then the solution is unique.

Definition 2.57 *Let $p \in [1, \infty]$. Consider a continuous-time system Σ as given in (2.31). Then, it is said to be* ***L_p stable with fixed initial condition with finite gain and with zero bias*** *if, given any input $d \in L_p$ and $x(0) = 0$, there exists a unique solution x such that the controlled output $z \in L_p$ and, moreover, if there exists a positive constant γ_p such that the following holds:*

$$\|z\|_p \leq \gamma_p \|d\|_p, \quad \text{for all } d \in L_p.$$

Furthermore, the infimum over all such γ_p's is called the L_p gain of the system.

Similarly, consider the discrete-time system Σ as given in (2.31). Then, it is said to be ***ℓ_p stable with fixed initial condition with finite gain and with zero bias*** *if, given any input $d \in \ell_p$ and $x(0) = 0$, the unique solution x is such that the controlled output $z \in \ell_p$ and, moreover, if there exists a positive constant γ_p such that the following holds:*

$$\|z\|_p \leq \gamma_p \|d\|_p, \quad \text{for all } d \in \ell_p.$$

Furthermore, the infimum of such γ_p's is called the l_p gain of the system.

Definition 2.58 *Let $p \in [1, \infty]$. Consider a continuous-time system Σ as given in (2.31). Then, it is said to be* ***L_p stable with fixed initial condition with finite gain and with bias*** *if, given any input $d \in L_p$ and $x(0) = 0$, there exists a unique solution x such that the controlled output $z \in L_p$ and, moreover, if there exist positive constants γ_p and b_p such that the following holds:*

$$\|z\|_p \leq \gamma_p \|d\|_p + b_p, \quad \text{for all } d \in L_p.$$

Furthermore, the infimum over all such γ_p's is called the L_p gain of the system.

Similarly, consider the discrete-time system Σ as given in (2.31). Then, it is said to be ***ℓ_p stable with fixed initial condition with finite gain and with bias*** *if, given any input $d \in \ell_p$ and $x(0) = 0$, the unique solution x is such that the controlled output $z \in \ell_p$, and moreover, if there exist positive constants γ_p and b_p such that the following holds:*

$$\|z\|_p \leq \gamma_p \|d\|_p + b_p, \quad \text{for all } d \in \ell_p.$$

Furthermore, the infimum of such γ_p's is called the ℓ_p-gain of the system.

Remark 2.59 *Definitions 2.57 and 2.58 are equivalent for linear systems as one implies the other and conversely.*

Remark 2.60 *In the literature, L_p or ℓ_p stability with fixed initial condition and without finite gain is often simply referred to as L_p or ℓ_p stability. Also, L_p or ℓ_p stability with fixed initial condition and with finite gain is often simply referred to as L_p or ℓ_p stability with finite gain. In this book, we do the same.*

In the definitions of L_p or ℓ_p stability given above, we assumed that the initial conditions of the given system are fixed at zero, that is, the system is at rest. Nevertheless, one can modify easily the above definitions by setting the initial conditions at any fixed nonzero point. This is done in the literature by Shi [149]. In this regard, we observe here one important aspect as pointed out by Shi and others (see [151]); that is, if a given system is L_p-stable in some sense for one fixed initial condition, it does not necessarily imply that it is L_p stable in the same sense for another fixed initial condition. The following example illustrates this.

Example 2.61 Consider the double integrator with a linear feedback control law and external input d:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \sigma(-x_1 - x_2) + \delta\sigma(d) \\ \quad z = x_1, \end{cases} \tag{2.32}$$

where $\sigma(.)$ is a standard saturation function. This system is globally asymptotically stable and locally exponentially stable if $d = 0$. Hence, given the zero initial condition $x(0) = 0$, there exists a sufficiently small $\delta > 0$ such that for all $d \in L_p$, we have $x \in L_p$ for all $p \in [1, \infty]$. However, it is shown in Chap. 14 and in [169] that if $p > 2$, then, no matter how small δ is, there exist a $d^* \in L_p$ and an initial state (x_1^*, x_2^*) such that the state trajectory of the closed-loop system diverges to infinity. Thus, $z \notin L_p$.

To create an example for any $p \geqslant 1$ where L_p stability depends on the choice of the initial condition, we simply modify the above system. Consider the same double integrator but with the input passing through a nonlinear element $(\cdot)^{p/3}$ for some $p \geqslant 1$ and with a nonlinear output:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \sigma(-x_1 - x_2) + \delta\sigma(d^{p/3}) \\ \ z = |x_1|^{3/p}. \end{cases} \tag{2.33}$$

Consider the initial condition (x_1^*, x_2^*) and define $d = |d^*|^{3/p} \operatorname{sgn}(d^*)$ where $d^* \in L_3$ is an external signal for the system (2.32), which drives the state from initial condition (x_1^*, x_2^*) to infinity. Since $d^* \in L_3$, obviously $d \in L_p$. By construction, for initial condition (x_1^*, x_2^*), the state x of (2.33) diverges, and hence, $z \notin L_p$. This establishes that the system (2.33) is not L_p stable for initial condition (x_1^*, x_2^*).

On the other hand, we claim that the system (2.33) is L_p stable for zero initial conditions. After all, for any $d \in L_p$, we have $d^* = d^{p/3} \in L_3$. We know that system (2.32) with zero initial conditions and input d^* yields a state $x \in L_3$. But this clearly yields that the system (2.33) with zero initial conditions and input d yields a state $x \in L_3$ and an output $z \in L_p$. Hence, the system (2.33) is L_p stable for zero initial condition.

It is clear from the above discussion that the initial conditions of the given system play a dominant role in achieving or not achieving external stability (in one sense or other), and hence, any definition of external stability must take into account the initial conditions. Motivated by this, Shi [149] not only defines external stability by setting the initial conditions at a fixed point but also when initial conditions are arbitrary. That is, Shi [149] defines what is now known as external stability with arbitrary initial conditions. These definitions are recalled below.

Definition 2.62 *Let $p \in [1, \infty]$. Consider a continuous-time system Σ as given in (2.31). Then, it is said to be* ***L_p stable with arbitrary initial conditions and without finite gain*** *if for any input $d \in L_p$ and for any arbitrary initial condition $x_0 \in \mathbb{R}^n$, there exists a unique solution x such that the controlled output $z \in L_p$.*

Similarly, consider the discrete-time system Σ as given in (2.31). Then, it is said to be ***ℓ_p stable with arbitrary initial conditions and without finite gain*** *if for any input $d \in \ell_p$ and for any arbitrary initial condition $x_0 \in \mathbb{R}^n$, the unique solution x is such that the controlled output $z \in \ell_p$.*

Following the above definitions, a recent paper [126] defines external stability with arbitrary initial conditions with finite gain and with bias.

Definition 2.63 *Let $p \in [1, \infty]$. Consider a continuous-time system Σ as given in (2.31). Then, it is said to be* ***L_p stable with arbitrary initial conditions with finite gain and with bias*** *if the following hold:*

(*i*) *There exists a unique solution x for any $x(0) = x_0 \in \mathbb{R}^n$.*

(*ii*) *There exists a positive constant γ_p and a class $\mathcal{K}$-function b_p such that for any $x(0) = x_0 \in \mathbb{R}^n$, the following holds:*

$$\|z\|_p \le \gamma_p \|d\|_p + b_p(\|x_0\|), \quad \text{for all } d \in L_p.$$

Furthermore, the infimum of such γ_p's is called the L_p gain of the system.

Similarly, consider the discrete-time system Σ as in (2.31). Then, it is said to be ***ℓ_p stable with arbitrary initial conditions with finite gain and with bias*** *if there exists a positive constant γ_p and a class $\mathcal{K}$-function b_p such that for any $x(0) = x_0 \in \mathbb{R}^n$, the following holds:*

$$\|z\|_p \le \gamma_p \|d\|_p + b_p(\|x_0\|), \quad \text{for all } d \in \ell_p.$$

Furthermore, the infimum of such γ_p's is called the ℓ_p gain of the system.

More recently, some other notions of external stability are introduced. These notions are similar to the definitions of L_p or ℓ_p stability with arbitrary initial conditions as they incorporate within them in some sense or other the notion of internal stability in the absence of disturbance or external input d. One such definition is introduced in [156] and is called input-to-state stability (ISS). It makes an attempt to marry both the notions of internal Lyapunov stability and the L_∞ stability or ℓ_∞ stability. We first need to recall the definition of a $\mathcal{KL}$-function before we recall the definition of ISS.

Definition 2.64 *A continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be of class $\mathcal{L}$ if it is monotonically decreasing and $\lim_{r\to\infty} \psi(r) = 0$. A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a class $\mathcal{KL}$-function if it is class $\mathcal{K}$ with respect to the first argument and class $\mathcal{L}$ with respect to the second argument.*

Definition 2.65 *Consider the continuous-time system Σ as given in (2.31) where $f : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^n$. Then, Σ is said to be* ***input-to-state stable (ISS)*** *if there exist a class $\mathcal{KL}$-function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and a class $\mathcal{K}$-function α such that, for each input $d \in L_\infty$ and for each initial condition $x(0) = x_0$ with $x_0 \in \mathbb{R}^n$, there exists a unique solution $x(t; x_0, d)$ of Σ satisfying*

$$\|x(t; x_0, d)\| \leqslant \beta(\|x_0\|, t) + \alpha(\|d\|_{L_\infty}) \tag{2.34}$$

for each $t \geqslant 0$.

Definition 2.66 *Consider the discrete-time system Σ as given in (2.31) where $f : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^n$. Then, Σ is said to be* ***input-to-state stable (ISS)*** *if there exist a class $\mathcal{KL}$-function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and a class $\mathcal{K}$-function α such that, for each input $d \in \ell_\infty$ and for each initial condition $x(0) = x_0$ with $x_0 \in \mathbb{R}^n$, the unique solution $x(k; x_0, d)$ of Σ satisfies*

$$\|x(k; x_0, d)\| \leqslant \beta(\|x_0\|, k) + \alpha(\|d\|_{\ell_\infty}) \tag{2.35}$$

for each integer $k \in \mathbb{Z}^+$.

Remark 2.67 *Note that, by causality, the same definition would result if one would replace (2.35) by*

$$\|x(k; x_0, d)\| \leqslant \beta(\|x_0\|, k) + \alpha(\|d_{[k-1]}\|_{\ell_\infty}),$$

where $k \geqslant 1$ and, for each $r \geqslant 0$, $d_{[r]}$ denotes the truncation of d at r; that is,

$$d_{[r]}(j) = \begin{cases} d(j) & \text{if } j \leqslant r, \\ 0 & \text{if } j > r. \end{cases}$$

Remark 2.68 *By definition, an immediate consequence of an ISS system is that, for any arbitrarily fixed initial state $x_0 \in \mathbb{R}^n$, any bounded input d must produce a bounded state. Moreover, when the input d is identically zero, the ISS implies the global asymptotic stability of the zero equilibrium point.*

Before we present another notion of external stability which is married in some sense or other to the notion of internal stability, let us next recall a well-known fact in linear system theory that global asymptotic stability implies that any vanishing external input produces a vanishing state and as such, external inputs that vanish as time progresses affect only the transient behavior of a given system. In general, such a behavior is not true for nonlinear systems. This is what is behind the notion of converging input converging state (CICS) stability as recalled shortly.

Let us first introduce some notation.

Definition 2.69 *The set of function $f : \mathbb{R}^+ \to \mathbb{R}^s$ with the property that*

$$\lim_{t \to \infty} f(t) = 0$$

is denoted by $\mathcal{C}_0$. Similarly, the set of function $f : \mathbb{Z}^+ \to \mathbb{R}^s$ with the property that

$$\lim_{k \to \infty} f(k) = 0$$

is denotd by c_0.

We have the following CICS definitions.

Definition 2.70 *Consider the continuous-time system Σ as given in (2.31) where $f : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^n$. We say that the system Σ satisfies the CICS stability if for each input $d \in \mathscr{C}_0$ and for any initial condition $x(0) = x_0$ with $x_0 \in \mathbb{R}^n$, there exists a unique solution $x(t; x_0, d)$ that satisfies $x(\cdot; x_0, d) \in \mathscr{C}_0$.*

Definition 2.71 *Consider the discrete-time system Σ as given in (2.31) where $f : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^n$. We say that the system Σ satisfies the CICS stability if for each input $d \in c_0$ and for any initial condition $x(0) = x_0$ with $x_0 \in \mathbb{R}^n$, the solution $x(k; x_0, d)$ of Σ satisfies $x(\cdot; x_0, d) \in c_0$.*

Remark 2.72 *We note that, in the absence of any disturbance d, CICS stability implies global attractivity of the origin.*

Remark 2.73 *Various concepts of external stability are formulated above. In these formulations, the space of external signals or disturbances has no restrictions, that is, it is the entire such possible space. In that sense, these results are global results even though we did not explicitly add "global" in the terminology. On the other hand, one can restrict the space of external signals to a certain proper subset of the entire possible space. Such a restriction can be useful to define semi-global rather than global external stability concepts, the study of which will be undertaken in detail in subsequent chapters. An example of a version of local external stability will be seen in the next section.*

2.9 Relationship between internal stability and external stability

Sections 2.7 and 2.8 deal respectively with various definitions of internal stability and external stability. One natural question that arises at this stage is whether there exists any relationship between these two notions of stability. We remark that a well-known result in linear system theory is that any asymptotically stable system has very good external stability properties. That is, for linear systems under some mild conditions, the notions of internal stability and external stability are very highly coupled and in fact simply coalesce. However, for nonlinear systems, as readers will observe in subsequent chapters, in general, just having internal stability in one sense or other does not necessarily imply external stability in some sense or another. One can generate several examples to illustrate this. To quote an example as given in Liu et al. [89, 90], consider a nonlinear system (a linear system subject to actuator saturation),

$$\dot{x}_1 = \sigma(-3x_1 + 7x_2 + d_1), \quad \dot{x}_2 = \sigma(-x_1 + 2x_2 + d_2),$$

where d_1 and d_2 are some inputs and $\sigma(.)$ is a standard saturation function with ± 1 as the saturation level. It is easy to see that in the absence of d_1 and d_2, such a system is locally asymtotically stable with the origin as its equilibrium point. On the other hand, one can find for some finite T some input functions $d_1(t)$ and $d_2(t)$ on the interval $[0, T]$ such that the origin $[0, 0]'$ of the considered system at $t = 0$ can be steered to $[1, 1]'$ at $t = T$. By defining $d_1 = 0$ and $d_2 = 0$ on (T, ∞), we have the solution of the considered system as $x_1 = t - T + 1$ and $x_2 = t - T + 1$. Thus, when we consider $d_1(t)$ and $d_2(t)$ as external input or disturbance signals, the considered system is not L_p stable for any $1 \leq p \leq \infty$ in the traditional sense of L_p stability with fixed initial conditions.

Then, another immediate query arises: Suppose the given system is externally stable in some sense. Does such an external stability have any consequences for internal stability of the given system in the absence of external input signals? In this regard, we have already remarked that having ISS implies global asymptotic stability of zero equilibrium point of a given nonlinear system in the absence of any external input signals. Additionally, if a nonlinear system is CICS stable, then it also has the property of global attractivity of the origin in the absence of any external input signals. However, it is not clear yet, whether having external stability in the classical sense of L_p stability with fixed initial conditions has any consequences for internal stability in some sense or other. It turns out that under some mild conditions on the given nonlinear system, external stability does imply certain properties of internal stability. We pursue such properties in this section based on the recent work of [191].

To be specific, in this section, for a nonlinear system that is L_p stable, our interest is to investigate the internal stability of the autonomous system (i.e, the given nonlinear system with the input zero). Our work here in this respect evolves along two main lines. The first line starts with L_p stability without finite gain. An important prior result in this direction is that in [89], which under a fairly restrictive condition on the structural property of the system, shows that L_p stability without finite gain implies global attractivity of the equilibrium point. Indeed, it turns out that this result of [89] can be obtained under weaker conditions. We show here that under mild conditions, global L_p stability without finite gain ensures attractivity of the equilibrium point in the absence of input and attractivity of the origin with any L_p input.

The other line emanates from L_p stability with finite gain. There is a large body of work in the literature in this direction; see, for instance, [25, 46, 89, 190]. Along this line of research, the objective here is to conclude local asymptotic stability of the equilibrium point based on L_p stability with finite gain. It was shown in [46] that under a uniform reachability condition, global L_p stability with finite gain implies local asymptotic stability of the equilibrium point. In [190], the notion of small-signal L_p stability with finite gain was introduced and its connection to attractivity of the equilibrium point was established. This concept of small-signal L_p stability was extended in [25] by so-called gain-over-set stability, and it was shown that finite-gain L_p stability over a set in L_p space yields local asymptotic stability of the equilibrium point. We prove here a result on the relationship between Lyapunov stability and local L_p stability with finite gain, which further extends, to some extent, the result in [25].

We consider a nonlinear system

$$\Sigma_1: \quad \dot{x} = f(x,d), \quad x(0) = x_0, \tag{2.36}$$

where $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$. We assume that for all $x_0 \in \mathbb{R}^n$, Σ_1 has a unique solution defined on $[0,\infty)$, which is absolutely continuous on any compact interval. Moreover, we assume that $f(x,d)$ is continuous with respect to x. Let $x(t;t_0,x_0,d)$ denote the trajectory of Σ_1 initialized at time t_0 with input d and initial condition x_0. If $t_0 = 0$, we will use $x(t;x_0,d)$ instead of $x(t;t_0,x_0,d)$.

We shall investigate the internal stability of the unforced system

$$\Sigma_2: \quad \dot{x} = f(x,0), \quad x(0) = x_0, \tag{2.37}$$

under the assumption that Σ_1 is L_p stable in some sense.

We first recall some preliminaries. We defined several types of L_p stability earlier in Sect. 2.8. These were basically all global definitions even though, for brevity, we did not explicitly use the word "global." Here global is in the sense that $d \in L_p$ is not bounded in size. Below, we define a local version of L_p stability.

Definition 2.74 *The system Σ_1 is said to be locally L_p stable with fized initial condition and with finite gain if there exists a δ and a γ such that for $x_0 = 0$ and any d with $\|d\|_{L_p} \leqslant \delta$, a unique solution exists and $\|x(t;0,d)\|_{L_p} \leqslant \gamma\|d\|_{L_p}$.*

The region or domain of attraction as defined in Definition 2.29 is denoted by $\mathcal{A}(\Sigma_2)$ for the system Σ_2, that is,

$$\mathcal{A}(\Sigma_2) = \{x_0 \in \mathbb{R}^n \mid x(t;x_0,0) \to 0 \text{ as } t \to \infty\}. \tag{2.38}$$

Definition 2.75 *A point $\xi \in \mathbb{R}^n$ is an L_p-reachable point of system Σ_1 if there exist a finite T, and an M, and a measurable input $d : [0,T] \to \mathbb{R}^m$ such that $x(T;0,d) = \xi$ and*

$$\int_0^T \|d(t)\|^p dt \leqslant M.$$

The set of all L_p-reachable points of Σ_1 is called the L_p-reachable set of Σ_1, which is denoted as $\mathcal{R}_p(\Sigma_1)$.

Remark 2.76 *The requirement in Definition 2.75 is a weak condition that ensures that the integral of $\|d(t)\|^p$ over the interval $[0, T]$ is finite. For example, any x_0 that is reachable via a signal $d(t)$ that is essentially bounded on $[0, T]$ is L_p-reachable for any $p \in [1, \infty)$.*

The following definition of small-signal local L_p reachability is adapted from [25].

Definition 2.77 *The system Σ_1 is said to be small-signal locally L_p reachable if for any $\varepsilon > 0$, there exists a δ such that for any $\xi \in \mathbb{R}^n$ with $\|\xi\| \leqslant \delta$, we can find a finite time T and a measurable input $d : [0, T] \to \mathbb{R}^m$ such that $x(T; 0, d) = \xi$ and $\|d\|_p \leqslant \varepsilon$.*

We have the following result.

Theorem 2.78 *Suppose the system Σ_1 is globally L_p stable with fixed initial condition without finite gain for some $p \in [1, \infty)$. Then $\mathcal{A}(\Sigma_2) \supseteq \mathcal{R}_p(\Sigma_1)$.*

In order to prove Theorem 2.78, we need the following lemma:

Lemma 2.79 *Consider the system Σ_2. If $x(t; x_0, 0) \in L_p$ for some $p \in [1, \infty)$, then $x(t; x_0, 0) \to 0$.*

Proof of Lemma 2.79 : For simplicity, we denote in this proof, $x(t; x_0, d)$ and $f(x(t), 0)$ by $x(t)$ and $f(x(t))$, respectively. Suppose, for the sake of establishing a contradiction, that $x(t) \to 0$ does not hold. Then there exists a $\delta > 0$ such that, for any arbitrarily large $T \geqslant 0$, there is a $\tau \geqslant T$ such that $\|x(\tau)\| \geqslant 2\delta$. Let m be a bound on $\|f(x)\|$ on the closed ball $\mathcal{B}(2\delta)$. This bound exists due to continuity of $f(x)$ with respect to x.

For some τ such that $\|x(\tau)\| \geqslant 2\delta$, let $t_2 > \tau$ be the smallest value such that $\|x(t_2)\| = \delta$, and let t_1 be the largest value such that $t_1 < t_2$ and $\|x(t_1)\| = 2\delta$. Such t_1 and t_2 exist because $x(t)$ is absolutely continuous and $x \in L_p$. Since $\|x(t)\| \in \mathcal{B}(2\delta)$ for all $t \in [t_1, t_2]$, we have, due to the absolute continuity of the solution,

$$\|x(t_1)\| - \|x(t_2)\| \leqslant \|x(t_2) - x(t_1)\| = \left\| \int_{t_1}^{t_2} f(x(\tau))\, \mathrm{d}\tau \right\|$$
$$\leqslant \int_{t_1}^{t_2} \|f(x(\tau))\|\, \mathrm{d}\tau \leqslant (t_2 - t_1)m.$$

Hence, $t_2 - t_1 \geqslant (\|x(t_1)\| - \|x(t_2)\|)/m = \delta/m$. Clearly, $\|x(t)\| \geqslant \delta$ for all $t \in [\tau, t_2]$, and furthermore, $t_2 - \tau \geqslant t_2 - t_1 \geqslant \delta/m$. It follows that for each τ such that $\|x(\tau)\| \geqslant 2\delta$, we have $\|x(t)\| \geqslant \delta$ for all $t \in [\tau, \tau + \delta/m]$.

Let T be chosen large enough that

$$\int_T^\infty \|x(t)\|^p \,\mathrm{d}\tau < \frac{\delta^{p+1}}{m}. \tag{2.39}$$

Such a T must exist, since $x(t) \in L_p$. Let $\tau \geqslant T$ be chosen such that $\|x(\tau)\| \geqslant 2\delta$. We have

$$\int_T^\infty \|x(t)\|^p \,\mathrm{d}\tau \geqslant \int_\tau^{\tau+\delta/m} \|x(t)\|^p \,\mathrm{d}\tau \geqslant \frac{\delta^{p+1}}{m}.$$

This contradicts (2.39), which proves that $x(t) \to 0$. ■

Proof of Theorem 2.78 : For any $x_0 \in \mathcal{R}_p(\Sigma_1)$, there exist finite T, M, and an input $d_0(t)$ for $t \in [0, T]$ such that $x(T; 0, d_0) = x_0$ and

$$\int_0^T \|d_0(t)\|^p \mathrm{d}t \leqslant M.$$

Define

$$d(t) = \begin{cases} d_0(t), & t \in [0, T], \\ 0, & t > T. \end{cases}$$

Clearly, $d \in L_p$. Since Σ_1 is globally L_p stable with fixed initial condition without finite gain, we have $x(\cdot; 0, d) \in L_p$. On the other hand, $d(t) = 0$ for $t > T$ implies that after T the system Σ_1 is equivalent with system Σ_2 initialized at x_0, that is, $x(t; 0, d, 0) = x(t - T; x_0, 0)$ with $t > T$. Therefore, $x(t; x_0, 0) \in L_p$ over $[0, \infty)$. It follows from Lemma 2.79 that $x(t; x_0, 0) \to 0$ as $t \to \infty$. This completes the proof. ■

Corollary 2.80 *If $\mathcal{R}_p(\Sigma_1) = \mathbb{R}^n$, then the origin of Σ_2 is globally attractive.*

The next theorem shows that under a certain condition on the structure of $f(x, d)$, the origin of Σ_1 is attractive for any input $d \in L_p$.

Theorem 2.81 *Suppose that Σ_1 is globally L_p stable with fixed initial condition without finite gain for some $p \in [1, \infty)$. If there exist δ, m_1, m_2, and $q \in [0, p)$ such that for any x with $\|x\| \leqslant \delta$ and for any d,*

$$\|f(x, d)\| \leqslant m_1 + m_2\|d\|^q, \tag{2.40}$$

then for $x_0 = 0$ and any $d \in L_p$, $x(t, 0, d, 0) \to 0$ as $t \to \infty$.

Proof : Define a generalized saturation function $\overline{\sigma} : \mathbb{R}^n \to \mathbb{R}^n \in C^1$ as

$$\overline{\sigma}(x) = \begin{bmatrix} \overline{\sigma}_1(x_1) \\ \vdots \\ \overline{\sigma}_n(x_n) \end{bmatrix}, \quad \overline{\sigma}_i(x_i) = \begin{cases} -\frac{2\delta}{\pi}, & x_i < -\delta \\ \frac{2\delta}{\pi}\sin(\frac{\pi}{2\delta}x_i), & |x_i| \leqslant \delta \\ \frac{2\delta}{\pi}, & x_i > \delta. \end{cases}$$

Consider $\overline{x}(t) = \overline{\sigma}(x(t, 0, d, 0))$. Note that $\overline{x}(t)$ is still absolutely continuous on any compact interval. Let $\overline{x}_i$ and f_i denote the ith element of $\overline{x}$ and $f(x, d)$ respectively. We have

$$|\dot{\overline{x}}_i(t)| = \begin{cases} 0, & |x_i(t)| > \delta \\ |\cos(\frac{\pi}{2\delta}x_i)f_i(x(t), d(t))| \leqslant m_1 + m_2\|d(t)\|^q, & |x_i(t)| \leqslant \delta, \end{cases}$$

Therefore, $\|\dot{\overline{x}}(t)\| \leqslant \sqrt{n}(m_1 + m_2\|d\|^q)$ for all $t > 0$. Note that $\|d(t)\|^q \leqslant 1 + \|d(t)\|^p$, and hence, $\|d\|^q$ is locally uniformly integrable. Then it follows from [182] that $\overline{x}(t) \to 0$ as $t \to 0$. This implies that $x(t, 0, d, 0) \to 0$ as $t \to 0$. ■

Remark 2.82 *In [89], in order to prove the same result as in Theorem 2.81, the following condition was imposed on $f(x, d)$: there exist δ_1, K_1, K_2, and $\alpha \in [0, p]$ such that for $x \in \mathbb{R}^n$ with $\|x\| \leqslant \delta_1$ and $d \in \mathbb{R}^m$,*

$$\|f(x, d)\| \leqslant K_1(\|x\| + \|d\|) + K_2(\|x\|^\alpha + \|d\|^\alpha).$$

Theorem 2.81 shows that the dependence on $\|x\|$ in the upper bound of the above condition is not necessary.

An immediate consequence of Theorem 2.81 is the next theorem.

Theorem 2.83 *Suppose that Σ_1 is globally L_p stable with fixed initial condition without finite gain and $\mathcal{R}_p(\Sigma_1) = \mathbb{R}^n$ for some $p \in [1, \infty)$. If there exist δ, m_1, m_2 and $q \in [0, p]$ such that for any x with $\|x\| \leqslant \delta$ and for any d,*

$$\|f(x, d)\| \leqslant m_1 + m_2\|d\|^q,$$

then Σ_1 is globally L_p stable without finite gain with arbitrary initial condition. Moreover, for any $x_0 \in \mathbb{R}^n$ and any $d \in L_p$, $x(t; x_0, d) \to 0$ as $t \to \infty$.

Proof : Since $\mathcal{R}_p(\Sigma_1) = \mathbb{R}^n$, for any $x_0 \in \mathbb{R}^n$, there exist finite T, M, and a measurable input $d_0 : [0, T] \to \mathbb{R}^m$ such that $x(T; 0, d_0) = x_0$ and

$$\int_0^T \|d_0(t)\|^p \, \mathrm{d}t \leqslant M.$$

For any $d \in L_p$, define

$$\bar{d}(t) = \begin{cases} d_0(t), & t \in [0, T] \\ d(t - T), & t > T. \end{cases}$$

Then we have $x(t; x_0, d) = x(t + T, 0, \bar{d}, 0)$. Clearly, $\bar{d} \in L_p$. This implies that $x(\cdot; 0, \bar{d}) \in L_p$, and hence, $x(\cdot; x_0, d) \in L_p$. This proves L_p stability with arbitrary initial condition and it follows from Theorem 2.81 that $x(t; 0, \bar{d}) \to 0$ as $t \to \infty$ and therefore $x(t; x_0, d) \to 0$ as $t \to \infty$. ■

In what follows, we prove a theorem that is a slight generalization of results of [25].

Theorem 2.84 *Suppose that Σ_1 is locally L_p stable with fixed initial condition and with finite gain and small-signal locally L_p reachable. Then the origin of Σ_2 is locally asymptotically stable.*

Proof : Let ε be an arbitrary positive real number. We need to show that there exists a $\delta > 0$ such that $\|x_0\| \leqslant \delta$ implies that $\|x(t; x_0, 0)\| \leqslant \varepsilon$ for all $t \geqslant 0$. Toward this end, let $\delta \leqslant \frac{\varepsilon}{2}$ be chosen such that for any $x_0 \in \mathbb{R}^n$ with $\|x_0\| \leqslant \delta$, there exist a finite time T and a measurable input $d : [0, T] \to \mathbb{R}^m$ such that

$$x(T; 0, d) = x_0 \quad \text{and} \quad \|d\|_{L_p} < \tfrac{\varepsilon}{2\gamma} \left(\tfrac{\varepsilon}{2M(\varepsilon)} \right)^{\frac{1}{p}}.$$

This is possible due to L_p local reachability.

Set $d(t) = 0$ for $t > T$. Since Σ_1 is locally L_p stable with finite gain, from Definition 2.74, there exists a γ such that

$$\int_T^\infty \|x(t; 0, d)\|^p \, \mathrm{d}t \leqslant \int_0^\infty \|x(t; 0, d)\|^p \, \mathrm{d}t \leqslant \gamma^p \|d\|_{L_p}^p < \tfrac{\varepsilon^{p+1}}{2^{p+1} M(\varepsilon)}.$$

For $t > T$, the system Σ_1 is equivalent to Σ_2 initialized at $x(0) = x_0$, that is, $x(t; 0, d) = x(t - T; x_0, 0)$. Hence, we have

$$\int_0^\infty \|x(t; x_0, 0)\|^p \, \mathrm{d}t < \frac{\varepsilon^{p+1}}{2^{p+1} M(\varepsilon)}. \tag{2.41}$$

It immediately follows from Lemma 2.79 that $x(t; x_0, 0) \to 0$ as $t \to \infty$.

We proceed to show that $\|x(t; x_0, 0)\| < \varepsilon$ for all $t \geqslant 0$. Suppose, for the sake of establishing a contradiction, that there exists a τ such that $\|x(\tau; x_0, 0)\| \geqslant \varepsilon$. Let $t_1 < \tau$ be the largest value such that $\|x(t_1; x_0, 0)\| = \varepsilon/2$, and let $t_2 \leqslant \tau$ be the smallest value such that $t_2 > t_1$ and $\|x(t_2; x_0, 0)\| = \varepsilon$. Such t_1 and t_2 exist because $\|x_0\| \leqslant \frac{\varepsilon}{2}$. Then $\varepsilon/2 \leqslant \|x(t; x_0, 0)\| \leqslant \varepsilon$ for all $t \in [t_1, t_2]$. Let $M(\varepsilon)$ be a bound on $f(x, 0)$ for $\|x\| \leqslant \varepsilon$. We have, owing to the absolute continuity of $x(t; x_0, 0)$,

$$\begin{aligned}\|x(t_2; x_0, 0)\| - \|x(t_1; x_0, 0)\| &\leqslant \|x(t_2; x_0, 0) - x(t_1; x_0, 0)\| \\ &\leqslant \left\| \int_{t_1}^{t_2} f(x(t), 0) \, \mathrm{d}t \right\| \leqslant \int_{t_1}^{t_2} M(\varepsilon) \, \mathrm{d}t \leqslant M(\varepsilon)(t_2 - t_1).\end{aligned}$$

This gives that $t_2 - t_1 \geqslant \frac{\varepsilon}{2M(\varepsilon)}$ and, hence, that

$$\int_0^\infty \|x(t; x_0, 0)\|^p \, \mathrm{d}t \geqslant \int_{t_1}^{t_2} \|x(t; x_0, 0)\|^p \, \mathrm{d}t \geqslant \int_{t_1}^{t_2} \left(\frac{\varepsilon}{2}\right)^p \mathrm{d}t = \frac{\varepsilon^{p+1}}{2^{p+1} M(\varepsilon)},$$

which contradicts (2.41). Hence, $\|x(t; x_0, 0)\| < \varepsilon$ for all $t \geqslant 0$, which completes the proof. ∎

Remark 2.85 *Compared with the result in [25], Theorem 2.84 only requires a finite gain within an arbitrary small neighborhood of the origin of L_p space.*

Remark 2.86 *We assume here that $f(x, d)$ is continuous with respect to x, which covers a large class of dynamical systems. In fact, it can be seen from the proof that we only need continuity of $f(x, d)$ with respect to x at $x = 0$.*

Internal and External Stabilization of Linear Systems
with Constraints

Saberi, A.; Stoorvogel, A.A.; Sannuti, P.

2012, XVI, 837 p. 55 illus., 30 illus. in color., Hardcover
ISBN: 978-0-8176-4786-5
A product of Birkhäuser Basel