Chapter VIII.

Applications of the theory of the top.

Part A. Astronomical applications.

§1. The precession of the Earth's axis, treated in association with an idea of Gaufs.

Corresponding to the dominant position of astronomical applications in the older mathematical literature, the problem of the rotation of the Earth has been of conspicuous influence on the development of the theory of the top, as is proven, for example, by our borrowed nomenclature: regular precession, nutation, and line of nodes. We find the names of almost all the classical mathematicians associated with the history of this problem, beginning with Newton and continuing with Euler, d'Alembert, Laplace, Lagrange, and Poisson.

The theory of astronomical precession is very simple in the first approximation, and very complicated if an exhaustive treatment is attempted. The latter standpoint is taken in textbooks on astronomy;*) we must essentially adopt the former. To give the nonastronomical reader a glimpse of the laborious and admirable methods of astronomy, we present a few results of the more precise theory at the conclusion of this part of the chapter.

The difficulty increases enormously if we abandon the grounds of abstract dynamics and no longer regard the Earth as absolutely rigid. The debates that then occur are in no way closed at the present time. We will reserve this matter for the following part of the chapter, and first hold fast to the *assumption of rigidity.*

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*) We refer in the following to Tisserand, Mécanique céleste, t. II, Chaps. 22–27. In §194, p. 442, Tisserand reports on the history of the problem and the contributions of the named classical mathematicians to his research.[222]

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Our method is modeled after a procedure given by G a u f s for the calculation of the secular perturbations of the planetary orbits. It has the advantage of great intuitiveness, and provides the individual components of the solution stepwise, according to the order of their importance. It appears not to have been applied to the present problem. Gaufs himself introduced his method with the remark that "the secular variations of a planetary orbit due to the perturbation of another planet are the same, whether the perturbing planet actually describes its elliptical orbit according to Kepler's law, or whether its mass is assumed to be distributed on the circumference of the ellipse in such a measure that equally large shares of the total mass are given to segments of the ellipse that are described in equally large times."*)

We wish to appropriate this idea and broaden it: we will distribute not only the mass of the perturbing body along its orbit, but also, where it is later desirable (§2), the mass of the perturbed body, which we will then treat as a rigid ring; we will learn to find not only the secular perturbations, but also, on the basis of a different mass distribution, the periodic perturbations (§3).

As Gaufs presented his method, it serves for the *exact* determination of the secular perturbations (at least those of the first order). In that we forgo the precision intended by Gaufs, we will simplify, in that we first disregard the eccentricities of the orbits; that is, for us, the orbits of the Sun and the Moon. We therefore assume that these orbits are circular. The nonuniformity of the mass distribution in the quotation of Gaufs, which indeed corresponds to the nonuniform motion on the ellipse, is then eliminated, and gives way to a uniform distribution on the circumference of the circle.

The most important element of the rotational phenomena of the Earth is its *precessional motion.* The approximate kinematic relations for this motion are already known (page 50): the axis of the Earth forms an angle of $23\,{}^1\!/_2^\circ$ with respect to the normal to the ecliptic (more precisely, at the present time, $23^\circ\,27'\,7''$, which number, however, is slowly changing), and rotates about the named normal at this angle once in approximately 26 000 years. Together with the daily rotation of the Earth, this motion of the axis represents a regular precession in the previous

*) Determinatio attractionis etc., Ges. W. Bd. 3, p. 331 and 357. It is this same treatise that contains the single direct communication of Gaufs on his theory of elliptic integrals.[223]

sense, and, in particular, a retrograde precession. If we consider the process from the side of the ecliptic toward which the north pole of the Earth points, then the rotation of the Earth about its figure axis occurs in the counterclockwise sense, and the rotation of the Earth's axis about the normal to the ecliptic occurs in the clockwise sense (see the adjacent figure; the three arrows that are assigned to the figure axis of the Earth F, the normal N, and the plane of the ecliptic E denote the direction of the Earth's rotation, the direction of the precession of the Earth's axis, and the apparent direction of the motion of the sun, respectively); the narrow polhode cone, whose size was determined on page 50, rolls in the interior of the herpolhode cone (cf. Fig. 8 of page 52, as well as Fig. 100a).

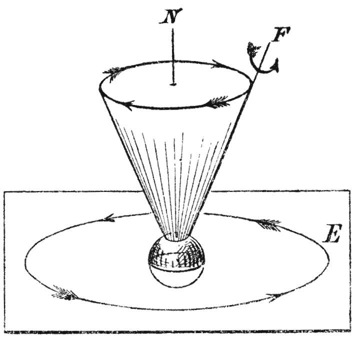


Fig. 95.

These relations, just like the number 26 000, are naturally not directly accessible to observation. The period of 26 000 years refers, rather, to the intersection points of the ecliptic with the equatorial plane, which are well known as the *vernal and autumnal equinox points* (*points of equal day and night*); the line that connects these points is the line of nodes K. It follows from the precessional motion of the Earth's axis that these points also rotate in the clockwise sense—that is, opposite to the sense of the apparent motion of the sun about the normal to the ecliptic—and indeed, as observation shows, by an amount of approximately $50''$ per year. The given approximate period of 26 000 years is calculated in reverse from this observation. The time of a complete revolution of the equinox points, and therefore the time in which the Earth's axis once encircles the normal to the ecliptic, is equal, namely, to

$$\frac{360^\circ}{50''} = \text{ca. } 26\,000 \text{ years.}$$

We now ask to what extent this phenomenon can be explained by our presently developed theory of the heavy symmetric top. That the precession is nothing other than an effect of gravity on the bulging toroidal mass of the rotating Earth was already recognized by N e w t o n,*) and supplied one of the most important and admirable verifications of his theory.[224]

*) Philosophiae naturalis principia mathematica. 1687. Book III, Prop. XXI, Theor. XVII.

Since the gravitational forces in question depend on the relative positions of the heavenly bodies, we may imagine the center of gravity of the Earth as fixed, and imagine the remaining heavenly bodies as moving with respect to the Earth. Of these bodies, we need only consider those that are distinguished by either their predominant size or small distance from the Earth; that is, only the Sun and the Moon. For a complete treatment of the rotational phenomena of the Earth, it would be necessary to consider the changing direction of the gravitational force due to the motion of the Sun and Moon in their orbits. We will return to the problem in this generality in the third section. We will there expand the temporally changing potential $V(t)$ of the Sun and Moon attractions into a trigonometric series with respect to the time t, and consider separately the terms that correspond to the orbital period of the Sun, the period of the lunar node motion, etc. The constant term of each series provides, in particular, the *secular effect* of the Sun and the Moon on the Earth, which gives as a resulting phenomenon the precessional motion of the Earth's axis that is of primary interest to us. We only indicate the more general consideration here; we now wish to use the intuitive procedure of Gaufs that directly separates the relevant secular component of the total attractive force.

We therefore imagine that the masses of the Sun and the Moon are distributed over their respective orbits with respect to the Earth, and, in particular, uniformly distributed, since we wish to assume that these orbits are circles. The radii of the circles correspond to the mean distances from the Earth to the Sun and the Moon. We must therefore investigate, instead of the actual Sun and Moon attractions, the attractions of the infinitely thin and uniformly dense "Sun-ring" and "Moon-ring." Further, we first wish to disregard the inclination of the Moon-ring to the ecliptic, which, as is well known, amounts to 5°, and imagine that the Moon-ring is rotated into the plane of the Sun-ring (see Fig. 96, where the customary astronomical signs ☉ for the Sun, ☾ for the Moon, and ♁ for the Earth are attached to the relevant rings). We also wish to make a simplifying assumption with regard to the nature of the Earth. We assume, as agreed, that it is rigid, and moreover is a body of revolution about the north–south axis with moments of inertia C and A, where, because of the bulge at the equator, $C > A$. For the calculation of all inertial effects, the particular form of the Earth in no way enters; any other body with the same moments of inertia C, A, and A, placed in the position of the Earth, would behave exactly as the Earth does with respect to all inertial rotational effects. But more: we claim that the calculation of the attractive forces of the Sun and the

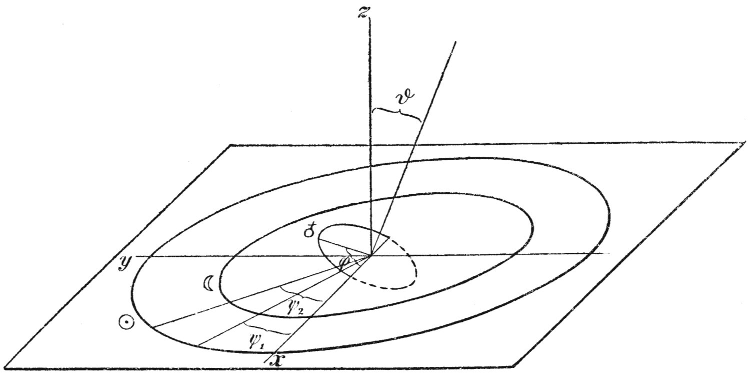


Fig. 96.

Moon also depends only on the values of the moments of inertia, in so far as we are satisfied with a certain approximation.

As a proof, we imagine the attractive potential of the actual Earth to an external, sufficiently distant point P; for example, a point on the Sun-ring or Moon-ring. This potential has the form $\sum \frac{m}{r}$, where m is a mass element of the Earth and the summation is extended over the entire mass of the Earth. Here one expands $1/r$ in powers of the ratios X/r_0, Y/r_0, and Z/r_0, where XYZ are understood as the coordinates of the mass element m, the origin of the coordinates is imagined at the midpoint (center of gravity) of the Earth, and r_0 signifies the distance of the point P from the center of the Earth. This series converges very rapidly, since the named ratios are at most equal, in our case, to the ratio of the radius of the Earth to the radius of the orbit of the Moon. One may thus truncate the series, if one strives for no great accuracy, with the terms of the lowest order. The terms of the first order vanish in the summation over the Earth, since the origin of the coordinates is chosen at the center of gravity. The terms of the second order give as coefficients, after the execution of the summations, the quantities $\sum mX^2$, $\sum mXY, \dots$; that is, the moments of inertia and products of inertia (or centrifugal moments) of the Earth. If, in particular, one lets the coordinate axes coincide with the principal inertial axes, then the number of quadratic terms is reduced to three, and their coefficients will be the three principal moments of inertia. Thus it follows, however, that in the first approximation (that is, for the consideration of merely the terms of the lowest order), all bodies with equal positions of the principal axes and equal values of the principal moments of inertia must also behave in the same manner with respect to gravitational effects. We can, in this respect, substitute for the Earth another arbitrary body, if only the ellipsoid of inertia of this body is identical to that of the Earth.

For many purposes, it is common and useful to imagine that the Earth is replaced by an ideal ellipsoid of revolution. In our case, however, another choice is preferred: we imagine a perfect, homogeneous sphere that is supplied at the equator with a belt of uniformly distributed mass. Let a be the moment of inertia of the sphere about one of its diameters and m be the mass distributed on our belt, the "Earth-ring." In order to have, for our purpose, a complete substitute for the actual Earth, we must arrange that the combination of the sphere and the ring possesses the same principal moments of inertia as the actual Earth. The moment of inertia of the ring about the north–south axis is mR^2, and its moment of inertia about an equatorial axis is $\frac{1}{2}\, mR^2$, understanding by R the radius of the Earth. Thus we must arrange that

$$mR^2 + a = C,$$
$$\frac{1}{2}\, mR^2 + a = A;$$

we must therefore choose

$$m = \frac{2(C-A)}{R^2}, \quad a = 2A - C. \tag{1}$$

It is clear from symmetry considerations that the sphere with moment of inertia a does not come into question in the calculation of the turning-moment of the attractive forces of the Sun-ring and the Moon-ring. Mechanical intuition immediately shows, moreover, that the Sun- and Moon-rings will strive to turn the Earth-ring in the plane of the ecliptic in the same manner. The relevant turning-force has the line of nodes as its axis, and acts about this axis in the clockwise sense as seen from the side of the line of nodes that bears the vernal equinox point, just as the gravity force does for a symmetric top whose center of gravity lies beneath the support point. We wish to calculate the magnitude of this turning-force.

Let m_1 be the mass and r_1 be the radius of the Sun-ring, and let ψ_1 be an angle, measured from the line of nodes, that distinguishes the individual points of the Sun-ring (see Fig. 96). The quantities m_2, r_2, ψ_2 have the analogous meanings for the Moon-ring. Finally, the same quantities for the equatorial Earth-ring are m (see above), R (Earth radius), and φ (an angle in the equatorial plane measured from the line of nodes).

The angle between the Earth-ring and the ecliptic is denoted by ϑ ($=$ ca. $23\,^1\!/\!_2°$). We define rectangular coordinates x, y, z by letting the z-direction coincide with the normal to the ecliptic and the x-direction coincide with the line of nodes; we then have, for the Sun-ring and the Moon-ring,

$$x_1 = r_1\cos\psi_1, \quad y_1 = r_1\sin\psi_1, \quad z_1 = 0,$$
$$x_2 = r_2\cos\psi_2, \quad y_2 = r_2\sin\psi_2, \quad z_2 = 0,$$

while for the Earth-ring we have

$$x = R\cos\varphi, \quad y = R\sin\varphi\cos\vartheta, \quad z = R\sin\varphi\sin\vartheta.$$

In order to form the attractive potential of the Sun-ring on the Earth-ring, we calculate

$$\frac{1}{r} = \{(x_1-x)^2+(y_1-y)^2+(z_1-z)^2\}^{-\frac{1}{2}} = \{r_1^2+R^2-2Rr_1 s\}^{-\frac{1}{2}},$$
$$s = \frac{xx_1+yy_1+zz_1}{Rr_1} = \cos\psi_1\cos\varphi + \sin\psi_1\sin\varphi\cos\vartheta,$$

and expand $\dfrac{1}{r}$ in powers of the small quantity $\dfrac{R}{r_1}$. If we write only the terms up to the second power inclusive, then

$$\frac{1}{r} = \frac{1}{r_1}\Big(1+\frac{Rs}{r_1}+\frac{3}{2}\Big(\frac{Rs}{r_1}\Big)^2-\frac{1}{2}\Big(\frac{R}{r_1}\Big)^2+\cdots\Big).$$

This expression is to be integrated with respect to ψ_1 and φ; that is, over the Sun- and the Earth-rings. We find

$$\int_0^{2\pi} s\,d\psi_1 = 0, \qquad \int_0^{2\pi} s^2\,d\psi_1 = \pi(\cos^2\varphi+\sin^2\varphi\cos^2\vartheta),$$
$$\int_0^{2\pi} d\varphi\int_0^{2\pi} s^2\,d\psi_1 = \pi^2(1+\cos^2\vartheta).$$

Denoting the gravitational constant by f, the desired potential is therefore

$$V_1 = f\iint\frac{dm_1\,dm}{r} = f\,\frac{m_1 m}{(2\pi)^2}\int_0^{2\pi} d\psi_1\int_0^{2\pi}\frac{1}{r}\,d\varphi$$
$$= f\,\frac{m_1 m}{r_1}\Big(1+\frac{3}{8}\frac{R^2}{r_1^2}(1+\cos^2\vartheta)-\frac{1}{2}\frac{R^2}{r_1^2}+\cdots\Big).$$

This expression depends, as we see, only on the angle ϑ. The attractive force therefore acts only to change the angle ϑ; that is, acts only to turn about the line of nodes, as we already recognized above. The magnitude of this turning-force is, in the first approximation (that is, for the previously named omission of the higher powers of $\frac{R}{r_1}$),

$$\frac{\partial V_1}{\partial \vartheta} = -\frac{3}{4} f \frac{m_1 m R^2}{r_1^3} \sin\vartheta \cos\vartheta. \tag{2}$$

Finally, we express the mass m of the Earth-ring in terms of the moments of inertia A and C of the Earth (see equation (1)) and obtain

$$\frac{\partial V_1}{\partial \vartheta} = -\frac{3}{2} f \frac{m_1(C-A)}{r_1^3} \sin\vartheta \cos\vartheta. \tag{2'}$$

In the same manner, the turning-moment of the Moon-ring is

$$\frac{\partial V_2}{\partial \vartheta} = -\frac{3}{2} f \frac{m_2(C-A)}{r_2^3} \sin\vartheta \cos\vartheta. \tag{2''}$$

The desired turning-force is thus equal to the sum of these two expressions; that is, equal to

$$P \cos\vartheta \sin\vartheta,$$

where we have used the abbreviation

$$P = -\frac{3}{2} f(C-A)\left\{\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right\}. \tag{3}$$

We have thus found a value for the external turning-force in the present case ($P \sin\vartheta \cos\vartheta$, $P < 0$) that is entirely analogous to the previous value for the heavy symmetric top whose center of gravity lies beneath the support point ($P \sin\vartheta$, $P < 0$).

We now make it clear to ourselves *that under the influence of this turning-force, regular precession represents, as previously, a possible form of motion.* At the same time, we note that regular precession is not, any more than it was previously, the most general possible form of motion. (The question whether the motion of the Earth is the particular *regular* precession or the general *pseudoregular* precession forms the proper subject of the following geophysical part of this chapter. In that we refer the reader to this second part, we will in the present part treat of the motion of the Earth, and likewise the motion of the Moon-ring, as a regular precession.) We rely most simply on the D'Alembert principle (Chap. III, §4): in every possible or "natural" motion of the top, the inertial action always maintains equilibrium with the external turning-force. The inertial action of the symmetric top for regular precession was found on page 175 to be

$$K = -C\mu\nu \sin\vartheta - (C-A)\nu^2 \sin\vartheta \cos\vartheta; \tag{4}$$

this moment has the line of nodes as its axis, just as the external turning-force $P \sin\vartheta \cos\vartheta$ does in the present case. The stated principle therefore demands that

$$K + P \sin\vartheta \cos\vartheta = 0. \tag{5}$$

In equation (4), ν signifies the precessional velocity; that is, the angular velocity with which the axis of the Earth turns about the normal to the ecliptic; μ is the angular velocity of the Earth for its daily rotation, measured from the line of nodes. We must regard the quantity ν as unknown. Our equation yields two values for ν (as previously for the precessional motion of the symmetric top on page 178); since P (see below) is very small, one of these values will likewise be very small, and the other will be of the order of magnitude of μ. In our case, only the first value comes into consideration as the precessional velocity, since observations show unambiguously that ν is considerably smaller than μ. At the same time, the smallness of the ratio $\nu : \mu$ justifies us in neglecting the second term of equation (4) in comparison with the first, and in writing equation (5) more simply as

$$C\mu\nu = P\cos\vartheta. \tag{5$'$}$$

The theoretical value for ν is thus

$$\nu = \frac{P\cos\vartheta}{C\mu} = -\frac{3}{2}\frac{f}{\mu}\frac{C-A}{C}\left\{\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right\}\cos\vartheta. \tag{6}$$

The right-hand side may be formulated more conveniently for numerical calculation if we transform it with the help of Kepler's third law. The most precise expression of this law is the well-known equation

$$f\,\frac{(m+m')}{a^3} = \left(\frac{2\pi}{T}\right)^2;$$

here m and m' signify the two masses of the two-body problem, a the semi-major axis of the Kepler ellipse, and T the period. If we disregard the eccentricity, then a becomes identical with the mean distance r. There follows for the motion of the Earth around the Sun, since the mass of the Earth may be neglected in comparison with that of the Sun,

$$f\,\frac{m_1}{r_1^3} = \left(\frac{2\pi}{T_1}\right)^2, \tag{7}$$

and for the motion of the Moon about the Earth,

$$f\,\frac{M+m_2}{r_2^3} = \left(\frac{2\pi}{T_2}\right)^2, \quad \text{or} \quad f\,\frac{m_2}{r_2^3} = \frac{m_2}{M+m_2}\left(\frac{2\pi}{T_2}\right)^2. \tag{7$'$}$$

Equation (6) is thus written as

$$\nu = -6\pi^2\,\frac{C-A}{\mu C}\left(\frac{1}{T_1^2} + \frac{m_2}{M+m_2}\frac{1}{T_2^2}\right)\cos\vartheta. \tag{6$'$}$$

We wish to draw some numerical conclusions from this formula. First, let us compare the component (ν_1) of the precession that is produced by the Sun with the component (ν_2) that is produced by the Moon. We have, evidently,

$$\frac{\nu_1}{\nu_2} = \left(\frac{M}{m_2} + 1\right)\frac{T_2^2}{T_1^2}.$$

Here $T_2 : T_1$ is the ratio of the (sidereal) period of the Moon to the (sidereal) year; that is, approximately $27\,{}^1\!/_3 : 365\,{}^1\!/_4$. For the ratio of the mass of the Earth to the mass of the Moon, we will adopt the value 82. As as result, there follows

$$\frac{\nu_1}{\nu_2} = 0{,}47, \quad \text{or} \quad \frac{\nu_2}{\nu_1} = 2{,}13.$$

The contribution of the Moon to the precession phenomenon is thus, because of its smaller distance and in spite of its smaller mass, more than twice as large as that of the Sun.

We now calculate the two components individually. We have

$$\nu_1 = -6\pi^2 \frac{C-A}{C} \frac{\cos\vartheta}{\mu T_1^2}, \quad \nu_2 = 2{,}13 \cdot \nu_1. \tag{8}$$

The quantity μ, the angular velocity of the rotation of the Earth, is equal to -2π divided by the length of the sidereal day,*) and μT_1 is therefore equal to -2π multiplied by the number of sidereal days in one year. This number is well known to be about 1 greater than the number of solar days in one year. Thus $\mu T_1 = -2\pi \cdot 366\,{}^1\!/_4$. (The negative sign depends on the fact that the rotation of the Earth occurs in the counterclockwise sense.) We must further know the value of $\dfrac{C-A}{C}$. Allowing ourselves to be guilty of a certain circular reasoning (see §4), we will accept for this ratio the value $\dfrac{1}{305}$. If we take the unit of time as the year, then the final result, expressed in arc seconds, is

$$\nu_1 = 3\pi \cdot \frac{\cos 23{,}5^\circ}{305 \cdot 366\,{}^1\!/_4} = 16''. \tag{9}$$

*) This result is not entirely exact. Since the angular velocity μ, just like the Euler angle φ, whose temporal differential quotient it is, is measured with respect to the line of nodes and this line is displaced because of the precession in the opposite sense of the rotation of the Earth, μ will in reality be somewhat larger. The above result actually refers to the true angular velocity r, the third component of the rotation vector (p, q, r). Since, however, $r = \varphi' + \cos\vartheta \cdot \psi'$, and since further $\varphi' = \mu$ and $\psi' = \nu$, the difference between r and μ is equal to $\nu\cos\vartheta$, which quantity, because of the smallness of ν, does not come into consideration for our purpose.

The line of nodes thus rotates $16''$ *forward in the course of one year due to the Sun attraction alone.*

Further, it follows from equation (8) that

$$\nu_2 = 2{,}13 \cdot 16'' = 34''. \tag{9'}$$

Due to the Moon attraction alone, the line of nodes rotates $34''$ *in one year. The total amount of the precession is thus*

$$\nu_1 + \nu_2 = 50''. \tag{10}$$

So much for the explanation and approximate determination of the magnitude of the precession. For comparison with the sequel, we wish to write the motion of the Earth's axis in terms of the Euler angles ψ and ϑ. The present degree of approximation corresponds to the representation

$$\begin{cases} \psi = \psi_0 + 50'' \cdot t, \\ \vartheta = 23^\circ\, 27'\, 7''. \end{cases} \tag{11}$$

The quantity ψ_0 remains undetermined; it depends on the point of the ecliptic from which we measure the angle ψ.

§2. The regression of the lunar nodes. First extension of the Gaufsian method.

It is well known that the orbit of the Moon does not coincide exactly with the plane of the ecliptic; it forms, as previously mentioned, an angle of approximately 5° with respect to this plane (more precisely, an angle that oscillates between $5^\circ\, 0'$ and $5^\circ\, 18'$). The intersection points of the orbit with the ecliptic are the *lunar nodes*, and the line that connects these points is called the *line of nodes of the Moon*. This line of nodes now executes, under the influence of the Sun's attraction, a retrograde motion with respect to the sense of the motion of the Moon; the lunar line of nodes rotates about the normal to the ecliptic in the clockwise sense, just as the line of nodes of the Earth does, but with the considerably greater angular velocity of one rotation in approximately $18\,{}^2\!/_3$ years.

We can also bring this nodal motion into relation with our theory of the top, and can thus determine its numerical value. We must, however, assume essential points of the lunar theory as known. We must know from the outset, in particular, that the principal perturbation of the orbit of the Moon by the Sun consists of a motion of the lunar nodes without a change in the orbital inclination with respect to the ecliptic. We must further know that the (rather large) eccentricity of the orbit of the Moon, which we will perforce disregard, will not appreciably influence the magnitude of the nodal motion, so that the nodal motion, on the one hand, and the perturbation of the Moon's orbit produced by its eccentricity, on the other hand, can be computed separately. Our consideration also wants, mathematically speaking, an existence proof for the motion of the lunar nodes; what we can obtain from the theory of the top is merely the calculation of the magnitude of this motion under the assumption of its existence.

We hold fast in the following to our previous representation of a Sun-ring and a Moon-ring, both of which we imagine as rigid and circular. The "Earth-ring," whose attraction we will consider after the fact, has too small a mass to come into perceptible consideration for our present purpose, so that we will first restrict ourselves to the attractive force of the Sun-ring. We imagine that the Moon-ring continuously rotates about its normal as a rigid body, with the velocity corresponding to the orbital motion of the Moon about the Earth. We are then faced with the following simple problem from the theory of the top: *the rotating Moon-ring stands under the influence of the attraction of the Sun-ring, which strives to draw it into the plane of the ecliptic. It describes, under the influence of this attraction, a regular precession about the normal to the ecliptic; what is its precessional velocity?*

With this formulation we have gone one step beyond Gaufs himself in the application of the Gaufsian method. While Gaufs distributed only the mass of the perturbing (the attracting) body on its orbit, we have also replaced the mass of the perturbed (the attracted) body by a continuous mass distribution. While, however, it is indifferent whether we think of the attracting mass (the Sun-ring) as in motion or at rest, it is essential that we consider the attracted mass (the Moon-ring) as a rotating ring. For it is directly this rotational motion that is, according to the fundamental principles of the theory of the top, necessary to maintain the inclination of the Moon's orbit with respect to the ecliptic in the presence of the turning-moment exerted by the Sun-ring.

We first form the attractive potential of the Sun-ring on the Moon-ring, and thus derive the turning-force that acts about the line of nodes of the Moon-ring. This turning-force is, according to equation (2) of the previous section,

$$\frac{\partial V}{\partial \vartheta_2} = -\frac{3}{4} f \frac{m_1 m_2 r_2^2}{r_1^3} \sin\vartheta_2 \cos\vartheta_2; \tag{1}$$

in fact, we need only replace the quantities m, ϑ, and R that refer to the Earth-ring in the named equation by the quantities m_2, $\vartheta_2 = 5°$, and r_2 that refer to the Moon-ring. If we write the right-hand side of (1) as $P_2 \sin\vartheta_2 \cos\vartheta_2$, then, with consideration of equation (7) of the previous section,

$$(2) \qquad P_2 = -\frac{3}{4} f \frac{m_1 m_2 r_2^2}{r_1^3} = -\frac{3}{4} m_2 r_2^2 \left(\frac{2\pi}{T_1}\right)^2 = -\frac{3\pi^2}{T_1^2} C_2,$$

where $C_2 = m_2 r_2^2$ now signifies the moment of inertia of the Moon-ring about its figure axis.

A possible precessional motion of the Moon-ring with a long period will again be defined with sufficient precision by equation (5′) of the previous section, which we write, understanding by N the unknown precessional velocity and by M the rotational velocity of the Moon-ring, as

$$(3) \qquad C_2 \mathsf{M}\mathsf{N} = P_2 \cos\vartheta_2;$$

there follows

$$(4) \qquad \mathsf{N} = -\frac{3\pi^2}{T_1^2 \mathsf{M}} \cos\vartheta_2.$$

Now M signifies the angular velocity of the Moon-ring with respect to its line of nodes; it is equal to the angular velocity with which, as seen from the Earth, the Moon progresses in its orbit with respect to the Moon nodes. The corresponding period is called the draconian period; it is equal to 27,2 days.*) Thus

$$\mathsf{M} = -\frac{2\pi}{27{,}2} \quad \text{and} \quad \mathsf{M} T_1 = -2\pi \frac{365{,}25}{27{,}2}.$$

If we take the year as the unit of time, then, expressed in degree measure,

$$(5) \qquad \mathsf{N} = \frac{3}{2} \frac{27{,}2}{365{,}25} \cos 5° \cdot 180° = 20{,}0°.$$

This is the number of degrees that the lunar nodes would regress in a year; the complete period of the lunar nodes would thus amount to

$$(6) \qquad \frac{360}{\mathsf{N}} = 18 \text{ years}.$$

The value given above was $18\,{}^2\!/_3$ years, or, more precisely, 6793 days; this corresponds to the more precise value of $19\,{}^1\!/_3{}°$ for N. The difference

*) The relation of this angular velocity to the true or sidereal angular velocity of the Moon is the same as the relation between μ and r above. If we denote the sidereal angular velocity (that is, the quantity 2π divided by the sidereal month) by R, then $R = \mathsf{M} + \mathsf{N} \cos 5°$.

cannot surprise us, considering the crudeness of our representation of the Moon-ring and our disregard of the eccentricity of the Moon's orbit.

We wish to determine in a supplementary manner the influence of the *Earth attraction* on the motion of the lunar nodes, at least in a rough approximation. It is clear that the Earth can disturb the plane of the Moon's orbit only in so far as the Earth deviates from a spherical form, and that, for the decomposition of the Earth into an "Earth-sphere" and an "Earth-ring" that was discussed in the previous section, only the Earth-ring is to be considered. This Earth-ring with mass m now seeks, just as the Sun-ring does, to rotate the Moon-ring in a plane, and therefore here in the plane of the equator of the Earth. We conclude, as above, that under the influence of this turning-moment and due to the angular velocity of the Moon-ring, a regular precession about the normal of the named plane, and therefore here about the north–south axis of the Earth, is a possible form of motion of the Moon-ring, where we disregard the eigenmotion of the Earth's axis investigated in the previous section. We wish to determine the precessional velocity and the period of this precession. In that we find, as was predicted from the small mass of the Earth-ring, that this precessional velocity is very small, and that the precessional period will thus be very long compared with the corresponding velocity and period of the lunar node motion caused by the Sun, it is shown that the effect of the Earth changes the lunar node motion only in a small manner and in a secular form, and that we may neglect the Earth attraction in the previous calculation of the motion of the lunar nodes. The type of this (very insignificant) change consists not in a simple acceleration or deceleration of the nodal motion that is effected by the Sun, but rather in a change of the inclination of the Moon orbit with respect to the ecliptic, since, as mentioned, the precessional motion effected by the Earth occurs about a different axis from that effected by the Sun.

The turning-moment of the Earth-ring on the Moon-ring depends on the angle of inclination of the Moon-ring with respect to the equatorial plane of the Earth. This angle changes due to the nodal motion that is effected by the Sun, and oscillates by $\pm 5°$ in $18\,{}^2\!/_3$ years. It is simplest and most natural to replace this inclination angle by its mean value; that is, by the angle $\vartheta = 23{,}5°$ at which the equatorial plane of the Earth is inclined to the ecliptic. In that we do this, we thus disregard, as in the first section, the inclination of the Moon's orbit with respect to the ecliptic, and rather imagine that the Moon-ring rotates in the ecliptic.

We can now take the turning-moment of the Earth's attraction on the Moon-ring directly from equation ($2''$) of the previous section. The formula there signifies the turning-moment that the Moon-ring, rotating in the ecliptic, exerts on the Earth-ring. Equally as large, however, is the turning-moment now in question. If we set this moment equal to $P_2' \sin\vartheta \cos\vartheta$, then, according to the named equation,

$$P_2' = -\frac{3}{2}\, f\, \frac{m_2(C-A)}{r_2^3}.$$

We compare the product $P_2' \cos\vartheta$ with the product $P_2 \cos\vartheta_2$, understanding by P_2 the value given in equation (2) of this section. According to equation (3) of this section, the angular velocity with which the lunar nodes would rotate about the north–south axis of the Earth as a result of the Earth-ring is to the angular velocity with which it rotates in the ecliptic as a result of the Sun attraction as $P_2' \cos\vartheta$ is to $P_2 \cos\vartheta_2$. If, as above, we name the two velocities N' and N, then

$$\frac{\mathsf{N}'}{\mathsf{N}} = \frac{P_2' \cos\vartheta}{P_2 \cos\vartheta_2} = \frac{2(C-A)}{m_1 r_2^2}\, \frac{r_1^3}{r_2^3}\, \frac{\cos\vartheta}{\cos\vartheta_2}.$$

According to Kepler's third law (equations (7) and ($7'$) of §1), we may set

$$\frac{r_1^3}{r_2^3} = \frac{m_1}{M+m_2}\, \frac{T_1^2}{T_2^2},$$

and thus obtain

$$\frac{\mathsf{N}'}{\mathsf{N}} = \frac{2(C-A)}{(M+m_2)r_2^2}\, \frac{T_1^2}{T_2^2}\, \frac{\cos\vartheta}{\cos\vartheta_2} = 2\, \frac{C-A}{C}\, \frac{C}{(M+m_2)r_2^2}\, \frac{T_1^2}{T_2^2}\, \frac{\cos\vartheta}{\cos\vartheta_2}.$$

Here an approximate value of C will be used in the numerator of the third factor on the right. We momentarily regard the Earth as a sphere of uniform density, so that we may assume, according to a well-known formula, $C = \frac{2}{5} M R^2$; there follows, finally,

$$\frac{\mathsf{N}'}{\mathsf{N}} = \frac{4}{5}\, \frac{C-A}{C}\, \frac{M}{M+m_2}\, \frac{R^2}{r_2^2}\, \frac{T_1^2}{T_2^2}\, \frac{\cos\vartheta}{\cos\vartheta_2}.$$

All the factors of this expression are known numbers. The ratio R/r_2, for example, is equal to ca. $1/60$, while the ratio $M/M+m_2$ can be taken with sufficient accuracy equal to 1. With the use of the previously given other numerical values, there follows

$$\frac{\mathsf{N}'}{\mathsf{N}} = \frac{4}{5}\, \frac{1}{305} \left(\frac{1}{60}\right)^2 \left(\frac{365{,}25}{27{,}3}\right)^2 \frac{\cos 23{,}5^\circ}{\cos 5^\circ} = 1{,}2 \cdot 10^{-4}.$$

The velocity N' is therefore extraordinarily small compared with the velocity N. Conversely, the precessional period corresponding to N' is extraordinarily large compared with the period of the motion of the lunar nodes in the ecliptic, which amounts to $18\,{}^2\!/_3$ years. The precessional period corresponding to N' would be, namely,

$$\frac{18\,{}^2\!/_3 \cdot 10^4}{1{,}2} = 156\,000 \text{ years.}$$

The magnitude of this number shows immediately that our consideration has only the significance of an *estimate*, and not a *permissible calculation*. For, on the one hand, the elements of the Moon's orbit change during the named time duration in a significant and not to be predetermined measure, while they were taken as constant in our calculation. On the other hand, the position of the Earth-ring in space indeed changes completely because of the nodal motion of the Earth, while in our calculation we must assume that the position of the Earth-ring and its applied turning-moment are invariable. This assumption is permissible only for a time duration that is small compared with the precession period (26 000 years) of the Earth nodes, and, in contrast, is completely indefensible for the time duration found here, which is even greater than 26 000 years.

Nevertheless, the present calculation proves as much as we wished to show: namely, that the lunar node motion effected by the Earth-ring is to be neglected, and that the Sun's attraction is to be considered as the decisive factor here.

§3. The astronomical nutation of the Earth's axis. Generalization of the Gaufsian method to periodic perturbations.

As we now turn to the *nutation of the Earth's axis* that was discovered by Bradley in 1747,[225] we emphasize in advance that this "astronomical" nutation has nothing in common, in *kinetic* respects, with the motion previously designated as the nutation of the axis of the top. The nutation of the general theory of the top (cf., in particular, Chap. V, §2) is due to the initial state of the motion not corresponding exactly to a regular precession, and the fact that the figure axis generally describes a cone in space even in the absence of all external forces. The astronomical nutation, in contrast, has its origin in periodically changing forces that act on the rotating Earth, which naturally cause a synchronous periodic motion of the Earth's axis. In association with a well-known and important general distinction in mechanics, we can say concisely that *the previous nutation was a free oscillation, and the present nutation is a forced oscillation.*

The similarity of the two motions, which may justify the choice of the same designation, is only of a *kinematic* nature. In both cases, the nutation is an oscillation that is very short with respect to the period of the precession. The period of the free nutation in the general theory of the top is $2\pi A/N$, and that of the precession is $2\pi N/P$ (see, for example, page 305, equations (13) and (15)), so that the ratio of the two periods is the often named quantity AP/N^2, which we may assume, as a rule, to be a small number (for example, $< 1/100$). On the other hand, the astronomical nutation arises from the motion of the lunar nodes, and thus has a period of $18\,{}^2\!/_3$ years, while the period of the precession of the Earth's axis was calculated as 26 000 years, so that the ratio of these two periods is also very small, even $< 1/1000$.

In order to be able to associate the theory of the astronomical nutation with our considerations thus far, we must first broaden our adopted method of Gaufs once more. In its original form, this method serves for the calculation of *secular perturbations.* We will see, however, that it will also provide, with a slight modification, for *periodic perturbations.*

We first formulate the problem of the rotation of the Earth in the most general manner. We have on the one hand the Earth, and on the other hand the Sun and the Moon, which describe known relative orbits about the Earth, and correspondingly exert time-varying gravitational attractions. One finds the totality of the attractive forces most simply by differentiating the *attractive potential* with respect to the coordinates. The potential is calculated, as it always is for perturbation problems, from the relative positions of the relevant bodies *under the preliminary disregard of the perturbations to be found in the course of the calculation itself.* Since the perturbations are, as a rule, small in proportion to the principal motion, only a small error will thus arise. If, in contrast, one would adopt the disturbed motion for the calculation of the attractive potential, then one would, in addition to the so-called perturbations of the first order that we seek in the following, determine at the same time the "perturbations of the second order." Even if one wishes to know the latter, a stepwise approach and a temporary restriction to the perturbations of the first order is always recommended.

In our case, we understand by the undisturbed motion of the Earth its uniform rotation about the figure axis that is inclined to the ecliptic.

One will naturally decompose the potential V of the Sun and Moon attraction on the Earth into periodic and aperiodic components. The periodic components of the Sun potential V_1 will have the period of one year, and the periodic components of the Moon potential V_2 will have partly the period of one month and partly the period of the lunar nodes, etc. *Harmonic analysis* provides a methodical means of separating these components from one another. As is well known, one finds the coefficients of the trigonometric series in the form of certain integrals. The *aperiodic part of* V_1 is equal to $\frac{1}{T_1}\int V_1(t)\,dt$, extended over the time of a complete Sun orbit. This formula, however, may be interpreted as the potential of the relative orbit of the Sun that is bestowed, in an appropriate manner, with mass. Let dm be the mass element that we assign to the orbital element ds that is traversed with velocity $\frac{ds}{dt}$. Since the potential $V_1(t)$ corresponds to the entire mass of the Sun m_1, the potential of the named mass element will be $\frac{dm}{m_1}V_1(t)$, and the total potential of the Sun orbit provided with mass will be $\frac{1}{m_1}\int V_1(t)\,dm$. If this potential is to agree with the named coefficient of the trigonometric series, the mass distribution must be arranged so that the mass element

$$dm = \frac{m_1\,dt}{T_1} \tag{1}$$

is assigned to the element ds of the orbit. The total mass borne by the orbit is then precisely equal to the total mass m_1 of the Sun. We thus have exactly the original starting point of Gaufs. If, moreover, the trajectory is assumed to be circular, then the mass distribution is uniform. This was our standpoint for the previous treatment of the precession, which in fact arises from the constant or mean components of the Sun and Moon attractions.[226]

We now consider the *periodic components*. In that we again argue for the Sun, let T_1/n be the considered period, understanding by n a whole number. The coefficients of the two terms in the trigonometric series with this period are

$$\frac{2}{T_1}\int V_1(t)\,\cos 2\pi\,\frac{nt}{T_1}\,dt, \qquad \frac{2}{T_1}\int V_1(t)\,\sin 2\pi\,\frac{nt}{T_1}\,dt. \tag{2}$$

We again conceive them as attractions of the Sun's orbit bestowed with mass, where now, however, the mass $m_1 \frac{2}{T_1} \begin{matrix}\cos\\ \sin\end{matrix} 2\pi \frac{nt}{T_1}\, dt$ is assigned to the orbital element ds, understanding by t and dt the time and the time interval, respectively, in which the element ds is traversed by the Sun. The total mass assigned to the distribution is now zero, since we must employ, in addition to the positive masses, equally many "negative" masses. The density is not uniform even for a circular orbit, but rather is harmonically variable. The adjacent schematic figures illustrate these conditions in the cases $n = 0$ and $n = 2$.

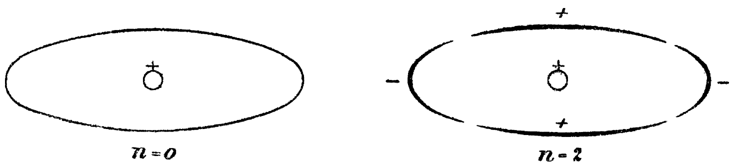


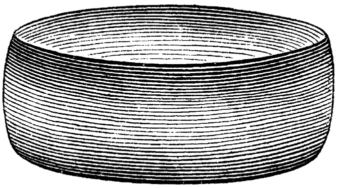
Fig. 97.

The periodic turning-forces acting on the body of the Earth result from the calculated trigonometric coefficients by differentiation with respect to the spatial coordinates and multiplication by $\begin{matrix}\sin\\ \cos\end{matrix} \frac{2\pi nt}{T_1}$. These turning-forces will produce perturbations of the Earth's axis with the same period T_1/n. We do not enter into this calculation here; it can be carried out according to the example to be given further below. The only perturbations of practical importance are the perturbations with the period $T_1/2$ and the perturbations caused by the Moon with the corresponding period $T_2/2$. The oscillation amplitude exceeds $1''$ for only one of these terms (see the formulas at the conclusion of the next section). The amplitudes of the terms with periods T_1, $T_1/3$, ... , T_2, $T_2/3$, ... are so small that they vanish even for the requirements of astronomical precision.

It is otherwise for the perturbations that have the period of the motion of the lunar nodes.

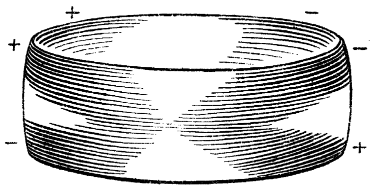
We first see how our method is to be formulated for these perturbations.

As we previously generated the Sun-ring and the Moon-ring by the simultaneous consideration of the changing positions of the Sun and Moon, we will now obtain a "Moon-ring surface" by representing the collected positions that the inclined Moon-ring takes in its precessional motion. The Moon-ring surface that arises from the rotation of the Moon-ring about the normal to the ecliptic is evidently a doubly covered spherical zone of radius r_2 and height $2r_2 \sin 5°$.*) The following two figures represent the mass distributions that we assign to our Moon-ring surface in the cases $n = 0$ and $n = 1$. It may be explicitly emphasized that the intention of the introduction of our Moon-ring surface and the drawing of the following figures is nothing other than the primary intention of the Gaufsian method: to illustrate the meaning of the calculations on a geometric basis. The calculations themselves are not fundamentally simplified, but rather are exactly the same as those that we would have to perform in a purely analytic procedure.



$n = 0$

Fig. 98 a.



$n = 1$

Fig. 98 b.

a) *In the case* $n = 0$ (secular perturbation), the mass distribution is to be chosen so that each element of the Moon-ring surface is assigned a mass $d\mu$, which, by analogy with equation (1), is equal to the product of the mass of the ring-element that sweeps out the element of the Moon-ring surface and the ratio dt/T, where dt is the sweeping time and T is the entire period of the lunar nodes. We wish to

*) We imagine the spherical zone as *doubly covered* (that is, consisting of *two shells* that are connected along their upper and lower edges) because each position of the spherical zone will be swept over *twice* by the rotating moon ring, once by the half-arc drawn in the foreground of Fig. 99, and once again by the half-arc that is not shown in the rear of this figure. The representation will be simplest if we ascribe a certain thickness to the spherical zone, conceive the exterior surface of the zone as one shell and the interior surface as the other, and stipulate that the Moon-ring passes over at each position from one to the other of the two shells at the upper or lower edge of the spherical zone. This is in conformity with the subsequent choice of our coordinates α, β; if, in the following, we integrate α and β from 0 to 2π, then we sweep over each position of the spherical zone twice, and therefore each of the two shells once; the one shell thus corresponds to coordinate values $-\pi/2 < \alpha < +\pi/2$, $0 < \beta < 2\pi$, and the other shell to the values $+\pi/2 < \alpha < +3\pi/2$, $0 < \beta < 2\pi$.

measure an angle α in the plane of the Moon-ring in such a manner that we calculate the lunar line of nodes OM (cf. Fig. 99) as $\alpha = 0$; each point P of the Moon-ring is then characterized by the central angle $\alpha = MOP$. On the other hand, we wish to designate as β the angle that the lunar line of nodes OM forms with an arbitrary fixed initial ray OA in the ecliptic; ψ is the angle that the nodal line of the Earth forms with the same ray OA. The angles α and β then represent skew-

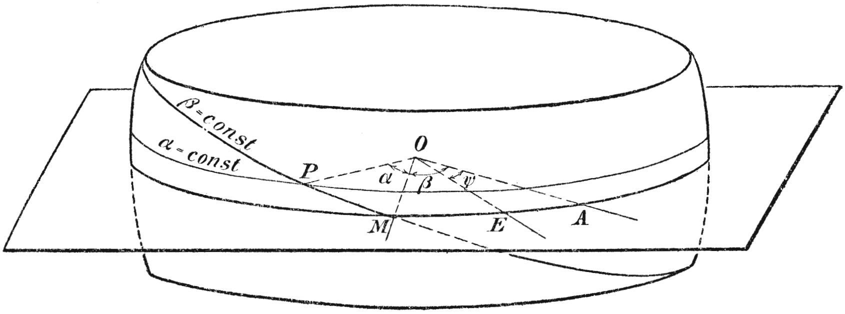


Fig. 99.

spherical coordinates on our spherical zone through which the position of any point of the spherical surface can be fixed, and which divide the spherical zone into parallelogram elements. The mass $d\mu$ allocated to such an element is now to be set equal to the product of the mass $m_2\,d\alpha/2\pi$ of the Moon-ring element that corresponds to the angle $d\alpha$ and the above-named ratio dt/T, which for a uniform rotation of the lunar nodes is equal to $d\beta/2\pi$; one thus has

$$d\mu = \frac{m_2}{4\pi^2}\,d\alpha\,d\beta. \tag{3}$$

The total mass assigned to the distribution, which follows from $d\mu$ by integration with respect to α and β, each between 0 and 2π, is naturally equal to the mass m_2 of the Moon-ring.

The density of the distribution—that is, the mass per unit area of the Moon-ring surface (calculated for both shells together)—is, as one easily understands from the inclined position of the Moon-ring, not uniformly disposed, but rather accumulates infinitely on the edges of the Moon-ring surface (for $\alpha = \pm\pi/2$). Along the latitude-circles, in contrast, the density is constant. It is attempted to indicate these relations in Fig. 98 by the strength of the hatching.[227]

b) *In the case* $n = 1$ (periodic perturbation), the assumed mass distribution on the Moon-ring surface is not uniform along the latitude-circles, but rather is harmonically variable. The factor $2\cos\beta$ or $2\sin\beta$, namely, is added to the previously determined mass (cf. formula (2) for the coefficients of the trigonometric series). Thus

$$d\mu = \frac{m_2}{2\pi^2} \begin{matrix}\cos\\ \sin\end{matrix}\, \beta \, d\alpha \, d\beta. \tag{4}$$

The total mass of the distribution, which is again obtained by integration with respect to α and β between 0 and 2π, is now equal to zero.

Here too the density, which we calculate as the algebraic sum of the mass per unit area of the two shells, accumulates toward the edges, and is oppositely equal in the neighboring octants of the spherical zone. These relations are indicated in Fig. 98b partly by the strength of the hatching, and partly by the addition of the signs.

After having thus explained Fig. 98, we form the potential that corresponds to the given assignments of mass; the potential that corresponds to the assignment (3) is called U, and the potentials that correspond to the assignments (4) are called w_1 and w_2 (w_1 corresponding to $\cos\beta$ and w_2 corresponding to $\sin\beta$). These potentials are nothing other than the first coefficients in the expansion of the attractive potential $V_2(t)$ exerted by the Moon on the Earth with respect to the period of the lunar nodes; the potential $V_2(t)$ is expressed, namely, in terms of U, w_1, w_2, and the angular velocity N of the lunar nodes as

$$V_2(t) = U + w_1 \cos \mathsf{N}t + w_2 \sin \mathsf{N}t + \cdots,$$

where we also write concisely

$$V_2(t) = U + W + \cdots, \quad W = w_1 \cos \mathsf{N}t + w_2 \sin \mathsf{N}t.$$

The constant term U corresponds to the value $n = 0$ of the index of the expansion, and the temporally variable term W comprises the two terms of the expansion that correspond to the value $n = 1$ of the index.

We cannot obtain anything essentially new from the value of U, but must rather return to the contribution of the Moon to the precessional motion of the Earth, which was calculated in the first section. We develop this calculation again only in order to convince ourselves that the previously neglected inclination of the Moon orbit with respect to the ecliptic does not essentially influence the phenomenon of the precession. The explanation and predictive calculation of the astronomical nutation will follow, in contrast, from the value of W.

The case $n = 0$. The potential of an element $d\mu$ of the Moon-ring surface on an element dm of the Earth-ring is, understanding by f the gravitational constant, $f\,d\mu\,dm/r$; thus the potential of the entire Moon-ring surface on the Earth-ring becomes

$$U = f \iint \frac{d\mu\,dm}{r}. \tag{5}$$

If x, y, z and x_2, y_2, z_2 are the coordinates of the points of the Earth-ring and the Moon-ring surface, respectively, then we set, as previously,

$$x = R\cos\varphi, \quad y = R\sin\varphi\cos\vartheta, \quad z = R\sin\varphi\sin\vartheta.$$

If, further, the nodal line of the Moon coincides directly with the nodal line of the Earth, then we can write, with respect to the same coordinate system,

$$x_2 = r_2\cos\alpha, \quad y_2 = r_2\sin\alpha\cos 5^\circ, \quad z_2 = r_2\sin\alpha\sin 5^\circ.$$

These coordinates correspond to the particular position $\beta = \psi$ of the Moon-ring (cf. Fig. 99). For arbitrary β, the value of z_2 remains as given, but the coordinates x_2, y_2 are obtained from the previous according to the coordinate transformation rule, where the rotation angle $\beta - \psi$ enters. The generally valid expressions are

$$\begin{aligned} x_2 &= r_2\bigl(\cos\alpha\cos(\beta-\psi) - \sin\alpha\cos 5^\circ\sin(\beta-\psi)\bigr),\\ y_2 &= r_2\bigl(\cos\alpha\sin(\beta-\psi) + \sin\alpha\cos 5^\circ\cos(\beta-\psi)\bigr),\\ z_2 &= r_2\sin\alpha\sin 5^\circ. \end{aligned}$$

We thus calculate

$$r^2 = (x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2 = R^2 + r_2^2 - 2Rr_2 s,$$

where s signifies

$$\left\{\begin{aligned} s = \frac{xx_2 + yy_2 + zz_2}{Rr_2} &= \\ \cos\varphi\bigl(\cos\alpha\cos(\beta-\psi) &- \sin\alpha\cos 5^\circ\sin(\beta-\psi)\bigr)\\ + \sin\varphi\cos\vartheta\bigl(\cos\alpha\sin(\beta-\psi) &+ \sin\alpha\cos 5^\circ\cos(\beta-\psi)\bigr)\\ + \sin\varphi\sin\vartheta\sin\alpha\sin 5^\circ&. \end{aligned}\right. \tag{6}$$

By expansion in powers of r_2 there follows

$$\frac{1}{r} = \frac{1}{r_2}\Bigl(1 + \frac{R}{r_2}s - \frac{1}{2}\frac{R^2}{r_2^2} + \frac{3}{2}\frac{R^2}{r_2^2}s^2 + \cdots\Bigr). \tag{7}$$

We integrate this expression with respect to $d\mu$ and dm, in that we take $d\mu$ from (3) and set dm equal to $\frac{m}{2\pi}\,d\varphi$. First, $\int s\,d\varphi = 0$; further, the first and third terms on the right-hand side of (7) yield contributions to our potential that are free of the angles ϑ and ψ that determine the position of the Earth-ring. Since we will later have to differentiate the potential with respect to these angles, these terms will also vanish. We thus do not write the first three terms or the higher terms of the expansion, and set

$$U = \cdots + \frac{3}{16} f \frac{mm_2}{\pi^3} \frac{R^2}{r_2^3} \cdot \int d\alpha \int d\beta \int d\varphi \, s^2 + \cdots .$$

One now easily calculates that

$$\int d\alpha \int d\beta \int d\varphi \, s^2 = \pi^3 \{(1 + \cos^2 \vartheta)(1 + \cos^2 5^\circ) + 2 \sin^2 \vartheta \sin^2 5^\circ\}.$$

Thus U becomes, if we introduce for the mass of the Earth-ring its value from equation (1) of §1,

$$U = \frac{3}{8} f \frac{m_2(C - A)}{r_2^3} \{(1 + \cos^2 \vartheta)(1 + \cos^2 5^\circ) + 2 \sin^2 \vartheta \sin^2 5^\circ\}.$$

The corresponding turning-moment on the Earth-ring is now found by differentiation with respect to ϑ, and is

$$\begin{aligned} \frac{\partial U}{\partial \vartheta} &= -\frac{3}{4} f \frac{m_2(C - A)}{r_2^3} \{1 + \cos^2 5^\circ - 2 \sin^2 5^\circ\} \sin \vartheta \cos \vartheta \\ &= -\frac{3}{2} f \frac{m_2(C - A)}{r_2^3} \left\{1 - \frac{3}{2} \sin^2 5^\circ\right\} \sin \vartheta \cos \vartheta. \end{aligned}$$

This value may be compared directly with the value derived in equation ($2''$) of the first section for the same turning-moment. It differs from that value, as one sees, only by the appearance of the additional factor

$$1 - \frac{3}{2} \sin^2 5^\circ = 1 - 0{,}012.$$

For numerical calculation, however, this difference plays no role, in so far as we wish to give, as in the first section, only the total seconds of the yearly precession. Thus the further treatment would follow exactly as there, and we can confirm all previous results as sufficiently precise when the inclination of the orbit of the Moon is considered.

b) *The case* $n = 1$. We also begin here from formula (5), where now, however, we understand by $d\mu$ the mass distribution defined by (4), and name the corresponding potentials, as agreed, w_1 and w_2. The quantity dm is, as above, equal to $\frac{m}{2\pi} d\varphi$, and the expansion (7) is inserted for $\frac{1}{r}$. In that we again suppress those terms that vanish in the integration or in the later differentiation with respect to ϑ and ψ, we write

$$\left.\begin{matrix} w_1 \\ w_2 \end{matrix}\right\} = \cdots + \frac{3}{8} f \frac{mm_2}{\pi^3} \frac{R^2}{r_2^3} \int \begin{matrix} \cos \\ \sin \end{matrix} \beta \, d\beta \int d\alpha \int d\varphi \, s^2 + \cdots .$$

If we first execute the integration with respect to α and φ, then we obtain from (6)

$$\int d\alpha \int d\varphi\, s^2 = \pi^2\{\cos^2(\beta-\psi) + \cos^2 5^\circ \sin^2(\beta-\psi) + \cos^2\vartheta \sin^2(\beta-\psi) \\ + (\cos\vartheta\cos 5^\circ \cos(\beta-\psi) + \sin\vartheta \sin 5^\circ)^2\};$$

if we multiply this by $\cos\beta$ or $\sin\beta$ and integrate with respect to β, then all the terms vanish that are of odd dimension in $\frac{\cos}{\sin}\,\beta$ after the decomposition of $\frac{\cos}{\sin}\,(\beta-\psi)$. As the single nonvanishing term there remains

$$2\pi^2 \sin\vartheta\cos\vartheta\sin 5^\circ \cos 5^\circ \int \frac{\cos}{\sin}\,\beta\cos(\beta-\psi)\,d\beta \\ = 2\pi^2 \sin\vartheta\cos\vartheta\sin 5^\circ \cos 5^\circ\, \frac{\cos}{\sin}\,\psi.$$

Thus

$$\left.\begin{matrix} w_1 \\ w_2 \end{matrix}\right\} = \frac{3}{4}\, f\, \frac{m m_2 R^2}{r_2^3} \sin\vartheta\cos\vartheta\sin 5^\circ \cos 5^\circ\, \frac{\cos}{\sin}\,\psi.$$

The potential of the Moon-ring surface has thus been found for the two mass assignments that are schematically illustrated in Fig. 98b, or, as we can also say, for the two coefficients of the trigonometric expansion that correspond to the terms with the full period of the lunar nodes. The sum of these terms, which, as agreed, should be called W, now becomes

$$W = \frac{3}{4}\, f\, \frac{m m_2 R^2}{r_2^3}\, \sin\vartheta\cos\vartheta\sin 5^\circ \cos 5^\circ \cos(\mathsf{N}t-\psi). \tag{8}$$

We transform this expression slightly, in that we consider, on the one hand, the definition of m (eqn. (1) of §1), and, on the other hand, the third Kepler law, and obtain

$$W = \frac{3}{2}\,\frac{m_2}{M+m_2}\left(\frac{2\pi}{T_2}\right)^2 (C-A)\sin\vartheta\cos\vartheta\sin 5^\circ\cos 5^\circ\cos(\mathsf{N}t-\psi). \tag{9}$$

From the potential W, we now derive the turning-moment that acts on the Earth-ring. Since W depends on ϑ as well as ψ, we obtain by differentiation with respect to ϑ a turning-moment that acts about the line of nodes, and by differentiation with respect to ψ a second turning-moment that acts about the normal to the ecliptic. There follows, namely,

$$\left\{\begin{aligned} \frac{\partial W}{\partial\vartheta} &= \frac{3}{2}\,\frac{m_2}{M+m_2}\left(\frac{2\pi}{T_2}\right)^2 (C-A)\,\cos 2\vartheta\,\sin 5^\circ\,\cos 5^\circ\,\cos(\mathsf{N}t-\psi), \\ \frac{\partial W}{\partial\psi} &= \frac{3}{4}\,\frac{m_2}{M+m_2}\left(\frac{2\pi}{T_2}\right)^2 (C-A)\,\sin 2\vartheta\,\sin 5^\circ\,\cos 5^\circ\,\sin(\mathsf{N}t-\psi). \end{aligned}\right. \tag{10}$$

We now see before us the following top problem: *the Earth stands under the influence of the just-named turning moments; what is its* *motion?* For the further treatment of this problem, we naturally have to consider not the Earth-ring alone, as we did for the calculation of the attractive potential, but rather the entire body of the Earth.

The so-defined top problem differs from all previous problems in two respects: on the one hand, there is added to the turning-moment about the line of nodes, which is also present in the case of the usual heavy top, a second turning-moment about the "vertical" (here, the normal to the ecliptic). On the other hand, the two turning-moments vary not only with the position of the top, but also, rather, with time. The time dependence of this variability obviously determines the time dependence with which the Earth follows the turning moments. While the oscillation period for the previous *free* nutation that was investigated in the general theory of the top was determined by the mass distribution and the state of motion of the top itself, the period of the present *forced* nutation is prescribed by the alternation of the external force, and coincides, in our case, with the period of the lunar nodes.

In general, one can say that the problem of forced oscillations, if one disregards particular occurrences (resonance, etc.), is simpler than that of free oscillations, because the period of the oscillation need not first be discovered from the nature of the oscillating system, but rather is known from the outset. If the problem appears somewhat complicated in our case, this is due only to the combined character of the applied forces. Moreover, the method that we will adopt is exemplary for the treatment of any kind of forced oscillation, in case the oscillation is sufficiently small. The forced oscillation can always be superposed with the free oscillation, which we may, however, disregard in the present case, since we will speak of the possibility of such free oscillation in the second part of this chapter.

Mathematically speaking, the deferment of the free oscillation signifies that we will be satisfied with a *particular* integral of the present dynamic problem; namely, with the integral that is purely periodic with the period of the forcing, and for just this reason is called the *forced* oscillation. The *general* integral results by the addition of the most general *free* oscillation; that is, the general solution that corresponds to the force-free case. This addition is rigorous if the problem is governed by linear differential equations, and is valid with a certain degree of approximation if, as in the present case, the differential equations of the problem can be reduced to linear equations by the neglect of small quantities.

For the calculation of the forced oscillation of the body of the Earth, we will use the *Lagrange equations in the coordinates* ϑ, ψ, φ. On the right-hand sides of these equations stand the components of the external force with respect to these coordinates; that is, in our case,

$$\frac{\partial W}{\partial \vartheta}, \quad \frac{\partial W}{\partial \psi}, \quad \frac{\partial W}{\partial \varphi} = 0.$$

The well-known expression for the *vis viva* is

$$T = \frac{A}{2}(\vartheta'^2 + \sin^2\vartheta\,\psi'^2) + \frac{C}{2}(\varphi' + \cos\vartheta\,\psi')^2,$$

which yields

$$\frac{\partial T}{\partial \vartheta} = A\sin\vartheta\cos\vartheta\,\psi'^2 - C(\varphi' + \cos\vartheta\,\psi')\sin\vartheta\,\psi', \quad \frac{\partial T}{\partial \psi} = \frac{\partial T}{\partial \varphi} = 0,$$

$$\frac{\partial T}{\partial \vartheta'} = [\Theta] = A\vartheta', \quad \frac{\partial T}{\partial \psi'} = [\Psi] = A\sin^2\vartheta\,\psi' + C\cos\vartheta(\varphi' + \cos\vartheta\,\psi'),$$

$$\frac{\partial T}{\partial \varphi'} = [\Phi] = C(\varphi' + \cos\vartheta\,\psi').$$

The Lagrange equations are now

$$A\vartheta'' - A\sin\vartheta\cos\vartheta\,\psi'^2 + C(\varphi' + \cos\vartheta\,\psi')\sin\vartheta\,\psi' = \frac{\partial W}{\partial \vartheta},$$

$$\frac{d}{dt}\big(A\sin^2\vartheta\,\psi' + C\cos\vartheta(\varphi' + \cos\vartheta\,\psi')\big) = \frac{\partial W}{\partial \psi},$$

while the third equation yields $[\Phi] = \text{const}$. Since $[\Phi] = Cr$, where r is the rotational velocity of the Earth about its figure axis and $2\pi/r$ is the duration of the sidereal day, r also becomes constant, and thus the length of the sidereal day is not influenced by the presently considered lunar perturbations.

We introduce the angular velocity $r = \varphi' + \cos\vartheta\,\psi'$ into the previous equations, and write these equations more simply as

$$A\vartheta'' - A\sin\vartheta\cos\vartheta\,\psi'^2 + C\sin\vartheta\,r\,\psi' = \frac{\partial W}{\partial \vartheta},$$

$$\frac{d}{dt}(A\sin^2\vartheta\,\psi' + C\cos\vartheta\,r) = \frac{\partial W}{\partial \psi}.$$

We now consider that the angular rates ψ' and ϑ' are, according to observation, extraordinarily slower and have an extraordinarily smaller amplitude than the rotation r, and that r is therefore extraordinarily large compared with φ' and ϑ'. Correspondingly, we will strike all terms on the left-hand sides that do not possess r as a factor, and simplify the preceding equations as

$$C \sin\vartheta\, r\, \psi' = \frac{\partial W}{\partial \vartheta},$$
$$-\, C \sin\vartheta\, r\, \vartheta' = \frac{\partial W}{\partial \psi}.$$

If we insert on the right-hand sides the values from (10), there follow

$$\psi' = \quad \frac{3}{2}\frac{m_2}{M+m_2}\left(\frac{2\pi}{T_2}\right)^2 \frac{C-A}{Cr} \sin 5^\circ \cos 5^\circ \,\frac{\cos 2\vartheta}{\sin\vartheta}\, \cos(\mathsf{N}t-\psi),$$
$$\vartheta' = -\frac{3}{2}\frac{m_2}{M+m_2}\left(\frac{2\pi}{T_2}\right)^2 \frac{C-A}{Cr} \sin 5^\circ \cos 5^\circ \; \cos\vartheta \;\sin(\mathsf{N}t-\psi).$$

Here we can once again make a simplification, in that we insert on the right-hand side the values of ψ and ϑ found in the first approximation (see equations (11) of §1); namely, $\psi = \psi_0 + 50''t = \psi_0 + \nu t$, $\vartheta = 23^\circ\, 27'\, 7'' = \vartheta_0$. The integration with respect to t may then be executed easily, and yields

(11)
$$\begin{cases} \vartheta = \dfrac{3}{2}\dfrac{m_2}{M+m_2}\left(\dfrac{2\pi}{T_2}\right)^2 \dfrac{C-A}{Cr}\dfrac{\sin 5^\circ \cos 5^\circ}{\mathsf{N}-\nu} \;\cos\vartheta_0 \;\cos(\mathsf{N}t-\nu t-\psi_0), \\ \psi = \dfrac{3}{2}\dfrac{m_2}{M+m_2}\left(\dfrac{2\pi}{T_2}\right)^2 \dfrac{C-A}{Cr}\dfrac{\sin 5^\circ \cos 5^\circ}{\mathsf{N}-\nu} \;\dfrac{\cos 2\vartheta_0}{\sin\vartheta_0}\, \sin(\mathsf{N}t-\nu t-\psi_0). \end{cases}$$

In these equations, the theoretical representation of the astronomical nutation is achieved. As we see, both the angle ϑ and the angle ψ are subjected to a harmonic oscillation, whose period $2\pi/\mathsf{N}$ coincides with that of the lunar nodes. (We can, in particular, neglect the angular velocity ν of the Earth-nodes with respect to that of the lunar nodes N without further consideration.) In order to find the numerical value of the amplitudes, which may be called a and b, we first calculate

$$\frac{b}{a} = 2 \operatorname{ctg} 2\vartheta_0 = 1{,}9.$$

If we further take the year as the unit of time and use the previously given values

$$\frac{M}{m_2} = 82, \quad \frac{C-A}{C} = \frac{1}{305}, \quad T_2 = \frac{27\,{}^1\!/\!_3}{365\,{}^1\!/\!_4}, \quad r = -2\pi \cdot 366\,{}^1\!/\!_4, \quad \mathsf{N} = \frac{2\pi}{18\,{}^2\!/\!_3},$$

then the amplitude of ϑ, expressed in seconds, becomes

$$a = \frac{3}{2}\cdot\frac{1}{83}\cdot\frac{(365\,{}^1\!/\!_4)^2}{366\,{}^1\!/\!_4}\cdot\frac{18\,{}^2\!/\!_3}{(27\,{}^1\!/\!_3)^2}\cdot\frac{0{,}087\cdot 0{,}917}{305}\cdot\frac{360\cdot 60\cdot 60}{2\pi} = 9''.$$

There follows

$$b = 1{,}9 \cdot a = 17''.$$

The axis of the Earth thus describes on the firmament a small ellipse that is called, after its discoverer, the B r a d l e y ellipse. The major axis of this ellipse amounts to $a = 9''$; it is directed toward the pole of the ecliptic. An elementary geometric deliberation shows that the minor axis is $b \sin \vartheta_0 = 7''$.

We wish, finally, to supplement the representation of the motion of the axis of the Earth that was given at the conclusion of the first section (equation (11) of page 643) by the addition of the nutational terms. The representation is then

$$\left\{\begin{aligned} \psi &= \psi_0 + 50''\, t + 17'' \sin(\mathsf{N}t - \psi_0), \\ \vartheta &= 23^\circ\, 27' + 9'' \cos(\mathsf{N}t - \psi_0). \end{aligned}\right. \tag{12}$$

§4. Concluding remarks on the problem of precession and nutation. The determination of the mass of the Moon and the ellipticity of the Earth.

With the corrections considered until now, the subject is still far from closed. First, one can pursue further the influence of the lunar node motion, and calculate the terms of period $\frac{4\pi}{\mathsf{N}}$, $\frac{6\pi}{\mathsf{N}}$, etc. The first two of these terms are actually considered in practice, even though their amplitudes amount to only the tenth and fifth part of a second, respectively. Then, however, the eccentricities of the orbits of the Sun and, particularly, the Moon are to be considered. These eccentricities influence not only the periodic precession terms, but also the secular precession term. The resulting correction of the precessional velocity amounts, however, to less than $1''$.

We further wish to point out once again the previously discussed but not calculated influences that are caused by the changing position of the Sun and Moon in their orbits, and which have as their periods an aliquot part of the Sun's or the Moon's orbital period.

Finally, it is to be considered that all the elements that enter into our calculations, such as the eccentricity of the orbit of the Sun, the position of the ecliptic with respect to the fixed stars, etc., are subject to secular changes, changes that one develops in the usual manner as a series that progresses in powers of t. It follows, in particular, that the precessional angle is not simply proportional to time, but rather is represented, in its turn, by a power series in t. The coefficient of t^2 in this series is already extremely small, ca. $10^{-4} \cdot 1''$; nevertheless, its presence is sufficient to make results that refer to a large number of years and are inferred only from the first term (νt), as, for example, the calculation of the period of 26 000 years given at the beginning of this part of the chapter, appear illusory to a certain extent.

With consideration of these various influences, the formulas for the motion of the axis of the Earth become essentially more complicated. The precession is no longer uniform, but is rather somewhat accelerated or decelerated due to the latterly named conditions. In addition, a series of secondary nutations (for example, nutations with the half-period of the lunar nodes, the half-period of the Sun and Moon orbits, etc.) will be superposed on the primary nutation discussed so far. In order to give an illustration of the resulting formulas, we place as a counterpart to the approximate formulas at the conclusion of the previous section the following more complete description of the motion of the axis of the Earth. This is taken, with changes of notation, from the work of T i s s e r a n d;*) the origin and the meaning of the individual terms will be clear from the preceding.

$$\begin{aligned}
\psi &= 50'',37140\,t - 0'',00010881\,t^2 \\
&\quad - 17'',251 \sin \mathsf{N}t + 0'',207 \sin 2\mathsf{N}t \\
&\quad - 1'',269 \sin \frac{4\pi t}{T_1} - 0'',204 \sin \frac{4\pi t}{T_2}, \\
\vartheta &= 23^\circ\, 27'\, 32'',0 + 0'',00000719\,t^2 \\
&\quad + 9'',223 \cos \mathsf{N}t - 0'',090 \cos 2\mathsf{N}t \\
&\quad + 0'',551 \cos \frac{4\pi t}{T_1} + 0'',089 \cos \frac{4\pi t}{T_2}.
\end{aligned}$$

Even this more complete formula is not to be claimed as exact, and may be extended to an arbitrarily long time no more than our previous representation. Its purpose, rather, is only to make possible, under the current values of the astronomical constants, the predetermination of the position of the axis of the Earth for a time that is sufficiently long for the requirements of calculating astronomers. Other authors**) give still longer formulas.

At the conclusion of this part of the chapter, we must still discuss a certain circular conclusion of which we were guilty in the preceding numerical calculations. It concerns the *ratio M/m_2 of the mass of the Earth to the mass of the Moon*, and the so-called *ellipticity of the Earth* (concerning the terminology, cf. §7 of the present chapter); that is, the ratio $(C - A)/A$. While we previously adopted certain numerical values for these quantities in order to calculate the magnitude of the

*) l. c. tome 2, §192, eqns. (m) and (n). In addition, we have suppressed two of the T i s s e r a n d terms that have found no explanation in the preceding.

**) For example, Th. O p p o l z e r, Bahnbestimmung der Kometen und Planeten, Leipzig 1870 and 1882, Bd. I, erster Teil.[228]

precessional velocity and the amplitude of the nutation, the situation is, in reality, that the most reliable numerical values of these ratios follow just from the observation of the precession and the nutation. Thus it is not possible, in a logical manner, to calculate in advance the precession and the nutation. In addition, there is still the fundamental physical assumption that one may regard the Earth as a rigid body for the effects computed here, an assumption to which we will return in the following part of this chapter.

We saw above that the two quantities $(C-A)/C$ and M/m_2 enter into the theoretical expression for the precessional velocity ν (equation ($6'$) on page 641) as well as the expression for the nutation amplitudes a and b (equation (11) on page 660). If we therefore take the two most observationally precise values—for example, ν and a—then we have two equations for the determination of the two unknowns $(C-A)/C$ and M/m_2. One finds, in such a manner, that the currently most trustworthy values of these two unknowns are*)

$$\frac{C-A}{C} = \frac{1}{304{,}9}, \quad \frac{M}{m_2} = 81{,}58.$$

These correspond to the abbreviated numerical values $1/305$ and 82 that were used above. There follows for the so-called ellipticity, with the same approximation, $(C-A)/A = 1/304$.

Moreover, numerical results obtained by other means (for example, from geodetic measurements of the Earth and from the mutual perturbations of the orbits of the Earth and Moon) agree with the given numbers, in so far as one can expect considering the greater uncertainty of the latter means of determination.[230]

B. Geophysical Applications.

§5. The Euler period of the pole oscillations; theoretical treatment.

It is well known from the preceding that the *pure precessional motion* of the top under the influence of gravity represents an exceptional case, and that the motion will generally be overlaid with a periodic oscillation of the figure axis. This oscillation, which becomes imperceptibly small

*) cf. Newcomb, Fundamental Constants of Astronomy. Washington 1895, p. 133.[229]

for a sufficiently strong eigenimpulse, was designated simply as a *nutation*. We will now designate it, in distinction to the nutation discussed in the previous part of this chapter, as a *free nutation*. Our "*pseudoregular precession*" arises from the composition of a uniform precession and this free nutation.

The question now presses upon us: is the precession of the Earth's axis that was calculated in the preceding part of this chapter accompanied by an oscillation that is not caused by external forces, but rather represents a free oscillation of the system? Or, more concisely, *is the rotational motion of the Earth*, disregarding all forced oscillations, *a regular precession or a pseudoregular precession?*

The answer to this question requires the collaboration of theory and observation. We first give the theory.

The words "axis of the Earth" are ambiguous. One signifies by these words, on the one hand, the *figure axis* of the Earth—that is, the principal axis of inertia of the Earth that approximately coincides with the line that connects the north and south poles, and is therefore an axis that is *fixed in the body of the Earth*—and, on the other other hand, the *instantaneous rotation axis* of the Earth, and therefore a line that precisely connects the instantaneous north and south poles and is thus instantaneously fixed *in space*. The noncoincidence of the two meanings is directly the subject of the following discussion, in which we must distinguish well between the figure axis and the rotation axis.

The motion of the *figure axis* for pseudoregular precession was discussed on page 291. It was determined in terms of the angles ϑ and ψ by the approximate equations (see page 303, equations (11))

$$(1)\qquad \begin{cases} \cos\vartheta = \cos\vartheta_0 + a\sin\vartheta_0 \,\sin\dfrac{N}{A}t, \\ \quad\psi = \dfrac{P}{N}t + \dfrac{a}{\sin\vartheta_0}\,\cos\dfrac{N}{A}t, \end{cases}$$

where a is expressed in terms of the quantity n' defined on page 296 as

$$(1')\qquad a = \frac{n'}{N\sin\vartheta_0} - \frac{AP}{N^2}\sin\vartheta_0.$$

The first terms on the right-hand sides of (1) give the precessional component of the motion, and do not come into consideration in the following. We only remark that the quantity P, which for the top was equal to MgE, is to be replaced by $P\cos\vartheta_0$, where P is determined by the expression (3) of page 640. The second terms give the free nutation, and interest us here exclusively. They represent a *circular* *oscillation* (cf. page 305); that is, a motion in which the intersection of the figure axis with the firmament describes, if one disregards the precessional motion and the forced oscillations that are considered in the previous section, a small circle on the heavens. The apparent magnitude of the radius is a, and depends on the initial position of the impulse, to which the quantity n' in equation $(1')$ refers. The angle ϑ_0 signifies the mean inclination of the figure axis with respect to the normal to the ecliptic during this circular oscillation. The oscillation period τ—that is, the time in which the circle is traversed once—is determined by the equation

$$\frac{2\pi}{\tau} = \frac{N}{A} = \frac{C}{A} r,$$

where r, the angular velocity of the rotation of the Earth, is equal to 2π divided by the length of the sidereal day. If we take the latter as the unit of time, then $r = 2\pi$ and $\tau = A/C$. Since C is only slightly greater than A, *the oscillation period is slightly smaller than one sidereal day.*

This result was to be predicted. If, namely, the figure axis does not coincide with the rotation axis, then the former will rotate about the latter in a circular cone. If the rotation axis were to stand completely still, then the period would amount to exactly one day; if the position of the rotation axis changes slowly, then the period deviates only slightly from one day.

The approximately one-day oscillation of the figure axis may not be verified by observation, however, since the observation of the heavens necessarily refers to the change of the rotation axis. To the latter we now turn.

We must distinguish between the *change of the rotation axis with respect to space* and its *change with respect to the body of the Earth.* The former is determined by the components π, κ, ϱ of the rotation vector, and the latter by the components p, q, r; the two sets of components are related to the Euler angles φ, ψ, ϑ in equations (7) and (8) of page 45. The components π, κ, ϱ are the coordinates of a point on the *herpolhode*, and the components p, q, r are the coordinates of a point on the *polhode.*

The values of π, κ, ϱ are

$$(2)\qquad \begin{cases} \pi = \vartheta' \cos\psi + \varphi' \sin\vartheta \sin\psi, \\ \kappa = \vartheta' \sin\psi - \varphi' \sin\vartheta \cos\psi, \\ \varrho = \psi' + \cos\vartheta\, \varphi'; \end{cases}$$

they are referred to the fixed spatial coordinate system x, y, z whose third axis coincides, in our case, with the normal to the ecliptic (since we measure the angle ϑ from this axis), and whose first axis is the ray $\psi = 0$ lying in the plane of the ecliptic (according to the general stipulation for the measurement of the angle ψ). It is more convenient, however, to use a coordinate system whose third axis coincides with the mean position of the figure axis, and that is therefore inclined by the angle ϑ_0 with respect to the ecliptic. The first axis of the new system may coincide with first axis of the old. If we denote the components of the rotation vector in this new system by π_1, κ_1, ϱ_1, then, evidently,

$$\begin{aligned}
\pi_1 &= \pi, \\
\kappa_1 &= \quad \kappa \cos\vartheta_0 + \varrho \sin\vartheta_0, \\
\varrho_1 &= -\ \kappa \sin\vartheta_0 + \varrho \cos\vartheta_0.
\end{aligned}$$

If we substitute from (2), there follow

$$(3)\ \begin{cases}
\pi_1 = \vartheta' \cos\psi + \varphi' \sin\vartheta \sin\psi, \\
\kappa_1 = \quad \vartheta' \cos\vartheta_0 \sin\psi - \varphi'(\sin\vartheta \cos\vartheta_0 \cos\psi - \sin\vartheta_0 \cos\vartheta) + \psi' \sin\vartheta_0, \\
\varrho_1 = -\vartheta' \sin\vartheta_0 \sin\psi + \varphi'(\sin\vartheta_0 \sin\vartheta \cos\psi + \cos\vartheta_0 \cos\vartheta) + \psi' \cos\vartheta_0.
\end{cases}$$

Now it is to be considered, according to (1), that $\vartheta - \vartheta_0$, ψ, and the differential quotients ϑ' and ψ' are small quantities; if we may disregard, in addition, the precession term Pt/N, which is not of interest to us here, then all those quantities will become of the order of the nutation amplitude a. If we expand $\cos\vartheta$, then we can write, instead of (1),

$$(4)\quad \begin{cases}
\vartheta - \vartheta_0 = -\, a \sin\left(\dfrac{N}{A}t\right), & \vartheta' = -\, \dfrac{aN}{A} \cos\left(\dfrac{N}{A}t\right), \\
\sin\vartheta_0\, \psi = \quad a \cos\left(\dfrac{N}{A}t\right), & \sin\vartheta_0\, \psi' = -\, \dfrac{aN}{A} \sin\left(\dfrac{N}{A}t\right).
\end{cases}$$

In equations (3), only the terms of the lowest order in the small quantities should be retained. We thus set $\cos\psi = 1$, $\sin\psi = \psi$, $\sin\vartheta \sin\psi = \sin\vartheta_0 \cdot \psi$, $\vartheta' \sin\psi = 0$, etc., and obtain

$$\begin{aligned}
\pi_1 &= \vartheta' + \varphi' \sin\vartheta_0 \cdot \psi, \\
\kappa_1 &= -\, \varphi'(\vartheta - \vartheta_0) + \psi' \sin\vartheta_0, \\
\varrho_1 &= \varphi' + \psi' \cos\vartheta_0.
\end{aligned}$$

We further note that $\varphi' + \cos\vartheta_0\, \psi'$ is equal, according to equation (7) of page 45, to the angular velocity r of the rotation of the Earth, and therefore equal to 2π if we again choose the sidereal day as the unit of time. The last equation is then $\varrho_1 = 2\pi$; in the two first equations we may directly take $\varphi' = 2\pi$, since φ' is multiplied in these two equations by the small quantities ψ and $\vartheta - \vartheta_0$. Thus

$$\pi_1 = \vartheta' + 2\pi \sin\vartheta_0\, \psi,$$
$$\kappa_1 = -2\pi(\vartheta - \vartheta_0) + \psi' \sin\vartheta_0,$$
$$\varrho_1 = 2\pi.$$

If we now insert the values of ψ, ψ', etc. from (4) and consider that $N = Cr = 2\pi C$, we finally obtain the *representation of the herpolhode* as

$$\left\{\begin{aligned} \pi_1 &= -2\pi a\,\frac{C-A}{A}\cos 2\pi\frac{C}{A}t, \\ \kappa_1 &= -2\pi a\,\frac{C-A}{A}\sin 2\pi\frac{C}{A}t, \\ \varrho_1 &= 2\pi. \end{aligned}\right. \tag{5}$$

We thus recognize that *the rotation axis describes a circular cone in space about the direction of our third coordinate axis ϱ; that is, about the mean position of the figure axis. The period with which the rotation axis once traverses this cone is again $\tau = A/C$, and is therefore slightly smaller than a sidereal day.*

We can also say that the intersection point of the rotation axis with the firmament traverses a circle in this same time. The apparent radius of this circle, measured by the angle under which it is seen from the Earth, is (replacing the trigonometric tangent by the arc)

$$\frac{\sqrt{\pi_1^2 + \kappa_1^2}}{\varrho_1} = a\,\frac{C-A}{A}.$$

This radius is considerably smaller than the apparent radius a of the circle that the intersection point of the figure axis describes on the firmament. We found, namely (cf. page 663),

$$\frac{C}{C-A} = 305, \quad \text{and therefore} \quad \frac{A}{C-A} = 304. \tag{6}$$

The oscillation of the rotation axis in space hardly amounts to the 300th part of that of the figure axis. Since it follows from observations, as we will see, that the angular magnitude a lies firmly on the boundary of detectability, the angular magnitude $a\,\dfrac{C-A}{A} = a/304$ will completely escape observation. *One may therefore assume, for all practical purposes, that the rotation axis stands perfectly still in space.*

The above representation of the herpolhode curve is naturally not entirely exact, since we firstly omitted the higher-order terms, and secondly neglected the precession term. Had we considered the latter, then we would have obtained, instead of a circle on the firmament, a very slightly looped cycloid.

More interesting is the study of the *polhode.* Its coordinates p, q, r are given by equation (7) of page 45 as

$$\begin{aligned} p &= \quad \vartheta' \cos\varphi + \psi' \sin\vartheta \sin\varphi, \\ q &= -\,\vartheta' \sin\varphi + \psi' \sin\vartheta \cos\varphi, \\ r &= \varphi' + \cos\vartheta\, \psi'. \end{aligned}$$

The last coordinate is constant; it is equal, for our choice of the unit of time, to 2π. In the first two equations, we insert for ϑ' and ψ' the values from (4), and write, with the neglect of small quantities of higher order, $\sin\vartheta = \sin\vartheta_0$, $\varphi = 2\pi t$, $N = 2\pi C$; we obtain

$$\begin{aligned} p &= -\,2\pi a\frac{C}{A}\left(\cos 2\pi\frac{C}{A}t\ \cos 2\pi t + \sin 2\pi\frac{C}{A}t\ \sin 2\pi t\right), \\ q &= \quad 2\pi a\frac{C}{A}\left(\cos 2\pi\frac{C}{A}t\ \sin 2\pi t - \sin 2\pi\frac{C}{A}t\ \cos 2\pi t\right), \\ r &= 2\pi, \end{aligned}$$

or

$$(7) \qquad \begin{cases} p = -2\pi a\dfrac{C}{A}\cos 2\pi\dfrac{C-A}{A}t, \\ q = -2\pi a\dfrac{C}{A}\sin 2\pi\dfrac{C-A}{A}t, \\ r = 2\pi. \end{cases}$$

This is the desired *representation of the polhode.* It shows us *that the rotation axis also describes a circular cone in the body of the Earth. The angle at the peak of the cone* between the figure axis and the generators of the cone is (replacing the trigonometric tangent by the arc)

$$\frac{\sqrt{p^2+q^2}}{r} = a\,\frac{C}{A}.$$

This angle is therefore $C/(C-A) = 305$ times the corresponding angle of the herpolhode cone. *The time in which the rotation axis traverses the polhode cone once* amounts to $A/(C-A) = 304$ sidereal days, or about 10 months. This time is called the *Euler period* or the *Euler cycle*, since Euler*) first gave the necessary theoretical preliminaries for the calculation of this period.

*) Mechanica sive motus scientia. Petersburg 1736, third Part, Ch. XVI, §839 ff. Theoria motus corporum solidorum seu rigidorum, Greifswald 1765, Ch. XII, §§711, 717–732. The numerical value 304, however, appears not to be present in Euler.

Equations (7) are naturally not entirely complete, since we have disregarded the precession terms in their derivation; if we would consider the latter, then certain easily assignable terms of very small magnitude and of a period equal to one sidereal day would be added to the values of p and q given above.

Moreover, the above values of p and q can also be taken immediately from the Euler equations, if one bears in mind that the motion in question is a free nutation, and correspondingly prescinds the external forces (the Sun and Moon attractions) in the calculation. For $A = B$ and $r = \text{const.} = 2\pi$, the Euler equations are (cf. page 140)

$$A\,\frac{dp}{dt} = 2\pi(A - C)q, \quad A\,\frac{dq}{dt} = 2\pi(C - A)p,$$

which give, after integration (cf. page 151, equation (6′)),

$$p + iq = ce^{2\pi i\,\frac{C-A}{A}\,t}.$$

One need only resolve this expression into real and imaginary parts in order to essentially recover (namely, up to the changed notation of the constants of integration) equations (7).

It is useful to illustrate the free nutation with the figures of the polhode and herpolhode cones in the sense of P o i n s o t, and to compare with the analogous figures that represent the forced (by the Sun and Moon attractions) precession of the axis of the Earth. This is done in Figs. 100a and 100b.

In Fig. 100a (forced precession), the motion of the axis of the Earth occurs about the normal N to the ecliptic in the repeatedly named approximate time period of 26 000 years. The angle at the apex of the herpolhode cone (actually the angle between the normal N and the *rotation axis*, which we can also take without appreciable error, however, as the angle between the normal N and the *figure axis*) is $23\,{}^1\!/_2{}^\circ$. The opening angle of the polhode cone was calculated on page 49, and is equal, according to equation (2) of that page, to $\sin 23\,{}^1\!/_2{}^\circ/\,365 \cdot 26\,000$ = approximately $0{,}01''$; the smallness of the polhode cone was illustrated in the cited place by the statement that it intersects the surface of the Earth in a circle that is described about the north pole with a radius of only 27 cm. We therefore have a *rather broad herpolhode cone and an extremely acute polhode cone.* We could naturally not express the actual quantitative ratio of the two cones in Fig. 100a; the polhode cone is drawn proportionally almost 10^6 times too broad. We must imagine that the polhode cone, fixed in the Earth and taking part in the Earth's rotation, rotates once per day in the *counterclockwise* sense as seen from F, and rolls without sliding on the interior of the herpolhode cone. Because of its ex-

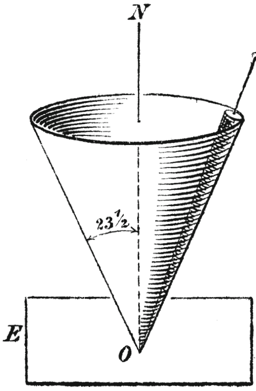


Fig. 100 a.

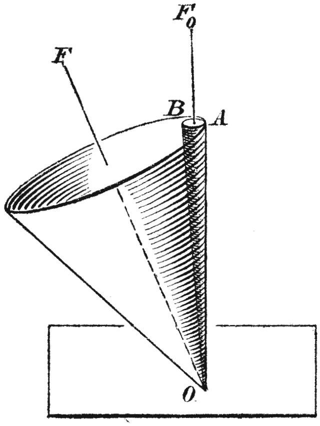


Fig. 100 b.

traordinary smallness, the polhode cone traverses the shell of the herpolhode cone only once in 26 000 years. The sense of the rolling follows from the rotational sense of the polhode cone; the rolling progresses in the figure across the front from right to left, and therefore in the *clockwise* sense as seen from N.

We now consider Fig. 100b (free nutation). The motion here occurs about the mean position of the figure axis (the vertically drawn line OF_0, as opposed to the instantaneous position of the figure axis OF). The angle at the apex of the polhode cone, according to the preceding, is equal to $a\,\dfrac{C}{A}$, and the angle at the apex of the herpolhode cone is $a\,\dfrac{C-A}{A}$; the ratio of these two angles was found to be 305. *The herpolhode cone is now considerably more pointed than the polhode cone*; the numerical proportions of the two cones in this figure could also not be expressed correctly, and the herpolhode cone must be drawn proportionally as much too obtuse. According to our formulas, the absolute magnitudes of the cone opening angles depend on the quantity a, about which only observations can give information. We are therefore uncertain, for the time being, about the actual form of the polhode and herpolhode cones, and have thus given the polhode cone in Fig. 100b somewhat the same size as the herpolhode cone in Fig. 100a. Since observations give an extremely small value of a, the polhode cone is actually extremely acute, and the herpolhode cone is correspondingly 300 times still more acute. Fig. 100b can thus give only a rough qualitative illustration of the proportions. We must now imagine that the relatively broad polhode cone that surrounds the thin herpolhode cone rotates with the angular velocity of the Earth, and rolls without sliding on the herpolhode cone. The rotational sense of the polhode cone is again *counterclockwise* as seen from F. It follows that the rolling seen from F_0 likewise follows in the *counterclockwise* sense. The contact line of the two cones represents the position of the rotation axis in space as well as in the Earth. *The rotation axis thus circulates in space in somewhat less than a sidereal day.* If, namely, the contact line returns to its original position on the herpolhode cone (OA in the figure) after one traversal of the herpolhode cone, it coincides with the generator OB of the polhode cone that we obtain if make the arc AB on the polhode cone equal to the circumference of the herpolhode cone at the distance OA from O. The ray OA, considered as a generator of the polhode cone, is consequently not returned to its original position; the time duration of the circuit of the rotation axis on the herpolhode cone will thus be somewhat smaller than the time duration in which a ray of the polhode cone circulates once, which is equal, in turn, to a sidereal day. *On the polhode cone, on the other hand, the rotation axis circulates much more slowly.* Since, namely, it advances during a sidereal day little more than the small segment AB on the polhode cone in the sense of the Earth's rotation, the traversal of the rotation axis through the entire circumference of the polhode cone lasts a considerable number of sidereal days. This number was designated above as the Euler cycle, and was found to be 304. According to the figure, and in conformity with the calculation above, the ratio between the circulation time of the rotation axis in the Earth and the circulation time in space is equal to the ratio of the circumference of the polhode cone to that of the herpolhode cone, measured at an equal distance from the apices of the cones.

In our calculations as well as in our drawings, we have with good reason separated the treatment of the forced precession from that of the free nutation, and have left the forced nutation (which one could likewise accompany with the P o i n s o t representation) entirely to the side.

In reality, a superposition of these different motions naturally occurs, and thus a superposition of the formulas, and, in a certain sense, a superposition of the figures. Unfortunately, the Poinsot representation of the rolling cones loses its primary merit of immediate clarity for such a composite motion. If we would represent, namely, the precession and the nutation in *one* figure by means of *one* pair of rolling cones, then we must supply the herpolhode cone with extraordinarily small undulations that engage corresponding undulations of the polhode cone. For the intuitive understanding of the process, however, nothing would thus be achieved.

In the interest of the following discussion, we finally go over from the now well-known polhode cone for free nutation to the circle in which the polhode cone intersects the surface of the Earth. We distinguish the intersection of the rotation axis with the surface of the Earth, the "instantaneous Earth-pole," from the intersection of the figure axis with the surface of the Earth, the "geometric Earth-pole." Our circle is the geometric locus of the instantaneous pole, and its midpoint coincides with the geometric pole. According to the preceding theory, we must expect that the instantaneous pole encircles the geometric pole in the sense of the rotation of the Earth with the period of the Euler cycle, and therefore in about 10 months. The radius of the circle as seen from the midpoint of the Earth is, according to the preceding, $a\,\dfrac{C}{A}$.

We will report in the following section on how such a motion of the instantaneous pole is made perceptible in observations of the pole oscillations. If we estimate here the chance of observability, we see that this is now much more favorable than the previous chance of verifying the spatial motion of the rotation axis. The period and the magnitude of the motion of the instantaneous pole are about 300 times as large as the period and magnitude of the motion of the intersection of the rotation axis on the firmament. Although, as we remarked, the previous motion was imperceptible, the present motion need not be.

§6. Observational verification of the pole oscillations; the Chandler period.

The possible pole oscillations that are demonstrated in the preceding section are betrayed in observations by a variability in the *latitude* of the observation location. It is equally valid, for this purpose, whether one defines the latitude as geographic (the complement of the angle that the plumb line of the observation location forms with the rotation axis of the Earth) or as geocentric (the complement of the angle that the line connecting the observation location and the midpoint of the Earth encloses with the rotation axis). In both cases, it is a matter of an angle between a line that is fixed in the Earth and the rotation axis that is variable in the Earth. According to whether the latter approaches or recedes from the observation location, the latitude of the location will decrease or increase.

In fact, a latitude oscillation that could not be explained by observational errors has been repeatedly conjectured previously by Peters (1842) and Nyrén (1871) at the Pulkovo observatory, and by Clerk Maxwell at the Greenwich observatory in the years 1851 to 1854.[231] The amplitude of the oscillation was in the tenths of a second, and the results for the period were inconsistent. The existence of a latitude oscillation was elevated to a certainty for the first time by the particularly precise observations of Küstner at the Berlin observatory in the year 1885.[232] We will return below to the very detailed work in which Chandler*) subjected the complete presently available observational data to a thorough discussion.[233]

New light was thrown on the entire question in the year 1891, when an astronomical expedition was dispatched to Honolulu for the purpose of latitude measurements that would be compared with simultaneous observations in Berlin.[234] Honolulu lies approximately on the meridian opposite (171° to the west) to Berlin. If the latitude oscillation actually has its basis in the change of the rotation axis of the Earth, then they must be expressed in the opposite sense at the two stations (cf. Fig. 101). The latitude in Berlin must increase if it decreases in Honolulu; a maximum of the latitude in Berlin must coincide with a minimum in Honolulu, etc. How completely this expectation is confirmed is shown in the following two

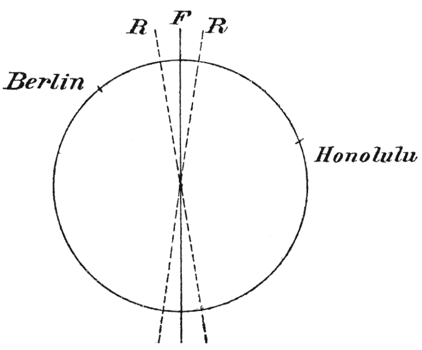


Fig. 101.

*) Astronomical Journal Vol. XI, XII, XV, XIX, XXI, XXII (1891–1902).

diagrams (Fig. 102).*) The abscissa of the figures signifies the time during the years 1891 and 1892, and the ordinate gives the deviation of the geographic latitude from its mean value, in a measure that is evident from the written numbers. The amplitude of the oscillation is,

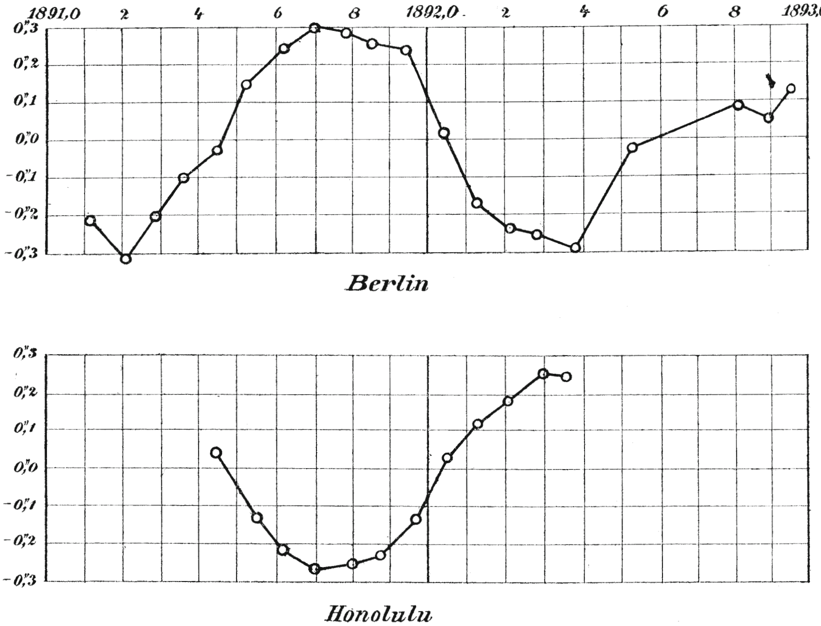


Fig. 102.

as we see, approximately equal for the two stations; it lies between 0″,2 and 0″,3. But above all we see that *the phase for the two stations is precisely opposite.* It is most obviously substantiated through the latter fact *that the latitude oscillation has its basis in the change of position of the rotation axis, and that this axis therefore executes a certain motion relative to the Earth.*

Two stations lying on opposite meridians, such as Berlin and Honolulu, obviously give only one component of the motion, the component with respect to the meridional plane through the two stations. For a complete knowledge of the motion, in contrast, two stations whose meridians form an approximate right angle will suffice. If more stations

*) We take these figures from the Proceedings of the 1895 Berlin Conference of the *Internationale Erdmessung*, Berlin 1896, plate 4.

are available, in particular stations that lie on opposite meridians, then their results can be confirmed with respect to each other.

Fig. 103 represents the positions of the observation locations that are the basis for the determination of the pole oscillations that was initiated by the Permanent Commission of the International Geodetic Association.[235] The majority of the European stations lie at an approximate right angle, as seen from the North Pole, to the principal American stations. The collected observational material will be processed by Th. A l b r e c h t[236] in Potsdam, and continuously released from the Central Bureau of the International Geodetic Association. We take Fig. 104, which summarizes the observational results from 1890 to 1900, from the latest report.*)

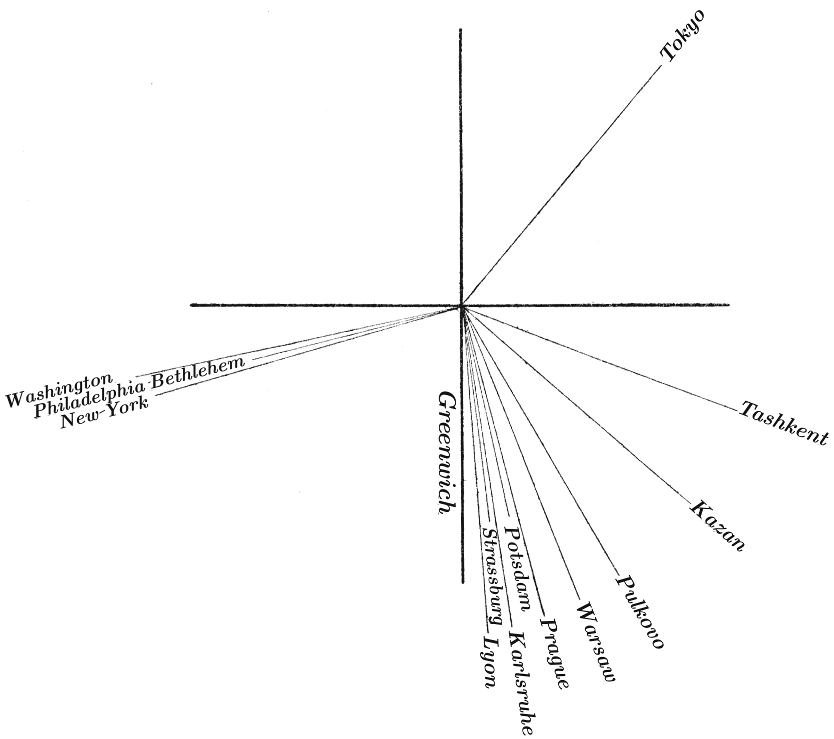


Fig. 103.

This figure represents the path of the pole in the named time interval; for clarity of the drawing, the dotted line corresponds to the first five

*) Berlin 1900. Earlier communications are found in the proceedings of the named commission from the 1896 Conference in Lausanne. Cf. also the Astron. Nachr. Nr. 3808 for the years after 1900.

years, and the solid line to the last five years. The inscribed numbers signify the dates (year and tenths of a year) for which the observations at the collected stations were reduced. The mean error for a single pair of coordinates of the instantaneous pole is given as $0'',03$. This relatively small mean error is achieved, however, only by deriving each coordinate from a large number of individual observations, which themselves have a much larger mean error. The origin of the coordinate system corresponds to the mean position of the instantaneous pole, or, as we can also say, to the geometric pole.

Motion of the North pole of the Earth's axis.

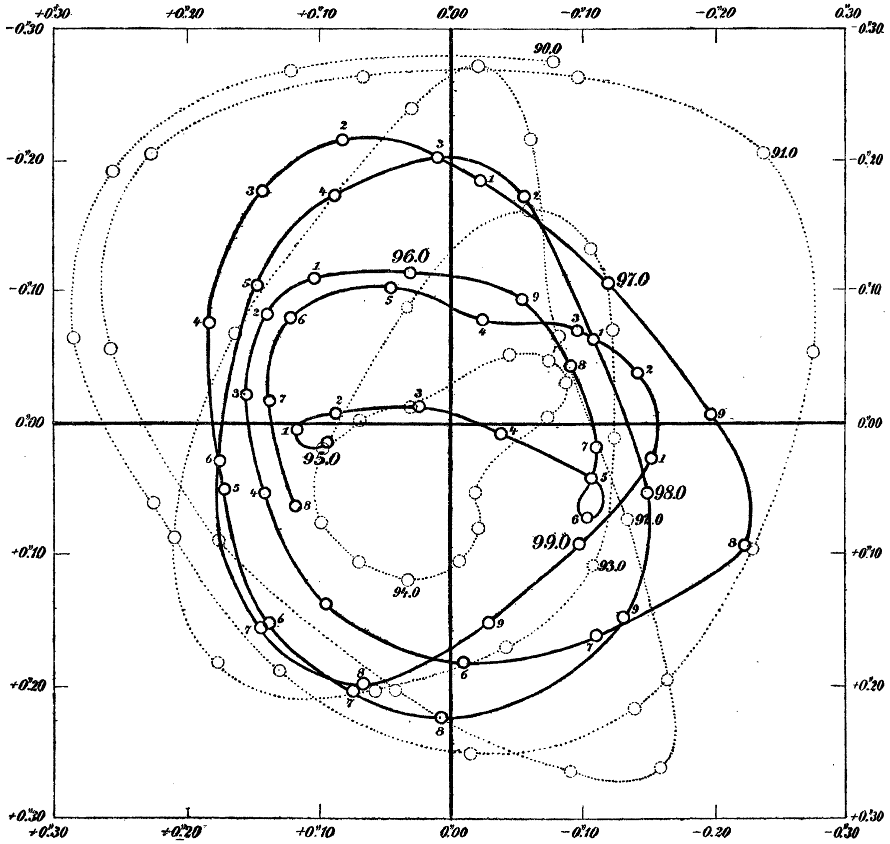


Fig. 104.

We now compare this figure with the preceding theory of the pole oscillations.

It is apparent at first glance that the pole trajectory satisfies no simple mathematical law with precision, but has an apparently accidental character and is greatly contorted. The simple relations of celestial mechanics are evidently no longer decisive for the present problem; we find ourselves here, rather, in the complicated domain of geophysics. According to the abstract theory of the preceding section, the trajectory should be a *circle*; this is, in reality, out of the question; only at the beginning of the observational time interval is a circular form somewhat approximated. In fact, we will soon recognize a series of incalculable perturbations that influence the pole motion and remove it from theoretical regularity.

On the contrary, it is to be emphasized that the *sense of the pole motion* thoroughly coincides with the theoretically required sense of the Earth rotation, if we remove a temporary irregularity in the interval from 95,0 to 95,6. In this range, either the temporary perturbations that will be discussed later have occurred to such a degree that the pole trajectory in the figure is drawn over the coordinate origin by these perturbations, or the loop is attributable to observational error, which is by no means to be excluded, since a correction in the coordinates of about the given mean error suffices to remove the entire irregularity.

For what concerns the *amplitude of the pole oscillation* (that is, the radius of the pole trajectory), this amounts in degree measure to a maximum of about $1/4''$ and in the mean to perhaps $1/8''$. The undetermined quantity a in the formulas of the preceding section is thus to be set equal, in the mean, to approximately $1/8''$. The mean distance e from the geometric pole to the instantaneous pole on the surface of the Earth follows as approximately

$$e = a \times \text{Earth radius} = \frac{\pi}{180 \cdot 60 \cdot 60} \frac{1}{8} \cdot \frac{2}{\pi} 10^7 = \text{circa 4 m.}$$

In the years 1890 to 1895, this mean distance decreased on the average, and from 1895 to 1900 it increased; it then became smaller, but now (1902) has already gone over again into an increase, as emerges from the supplementary publication to our figure in the Astron. Nachr. (cf. the preceding footnote).

The primary interest, however, is concentrated on the question of the *period of the pole motion*. Here an initially astonishing departure from the theory appears, which is all the more noteworthy as it appears to be thoroughly substantiated. While the theory demands a period of approximately 10 months, an examination of Fig. 104 yields a period of about 14 months. To see this, we proceed rather roughly, but with sufficient precision for our purpose, as follows. We first imagine the obviously irregular loop from 95,0 to 95,6 extended downward, so that it gives, together with the adjacent curve segments, a counterclockwise circuit about the coordinate origin that is equivalent to the other circuits, and then count the number of circuits from 90,0 to 99,4. There are directly 8 of these circuits that surround the pole in 9,4 years. Thus the duration of one circuit, or the period of the pole oscillation, amounts to

$$\frac{9{,}4}{8} \cdot 12 = 14{,}1 \text{ months.}$$

While we expected to find the ten-month Euler period, the observations indicate an essentially longer period.

The credit for discovering the longer period that emerges here belongs to the American astronomer C h a n d l e r. In the previously cited comprehensive works, Chandler examined the collected observational data from 1840 to 1891 purely numerically, without theoretical prejudice, and was thus led to a period of 427 days = ca. 14 months, a period that is called, in contrast to the *Euler* period, the *Chandler* period.

Without first entering into the theoretical explanation of this period, we wish to provide an image, through only the discussion of the observations recorded in Fig. 104, of the extent to which the pole oscillation can be represented by the assumption of a 14-month period. We will not use the laborious and fundamental calculational procedure of C h a n d l e r, but rather an obvious graphical procedure.

Let $w = x + iy$ be the vector from the coordinate origin to the current position of the instantaneous pole. Were the motion of the pole entirely created by *one* period τ_1 (for example, 14 months), then we could simply write

$$w = a\,e^{2\pi i \frac{t}{\tau_1}} + a'\,e^{-2\pi i \frac{t}{\tau_1}}; \tag{1}$$

were the motion, moreover, a purely circular motion, then one of the two constants a and a' (say, a') would vanish; at the same time, the radius of the pole trajectory would be determined by the absolute value of the other constant a. We can, however, directly account for the case of an elliptical pole oscillation by assuming that a and a' are, in general, nonvanishing complex quantities.

The complexity of Fig. 104 shows immediately that the representation by *one* period does not suffice. We thus make the more general assumption

$$w = a e^{2\pi i \frac{t}{\tau_1}} + a' e^{-2\pi i \frac{t}{\tau_1}} + b e^{2\pi i \frac{t}{\tau_2}} + b' e^{-2\pi i \frac{t}{\tau_2}} + \cdots , \tag{2}$$

in that we seek to represent the actually observed motion through the superposition of two (or more) oscillatory motions. It is very easy to eliminate the known 14-month period from the pole motion. For this purpose we form, according to equation (2),

$$\begin{aligned} w_{t+\tau_1} - w_t &= b\Big(e^{2\pi i \frac{t+\tau_1}{\tau_2}} - e^{2\pi i \frac{t}{\tau_2}}\Big) + b'\Big(e^{-2\pi i \frac{t+\tau_1}{\tau_2}} - e^{-2\pi i \frac{t}{\tau_2}}\Big) + \cdots \\ &= b\Big(e^{2\pi i \frac{\tau_1}{\tau_2}} - 1\Big) e^{2\pi i \frac{t}{\tau_2}} + b'\Big(e^{-2\pi i \frac{\tau_1}{\tau_2}} - 1\Big) e^{-2\pi i \frac{t}{\tau_2}} + \cdots \\ &= c e^{2\pi i \frac{t}{\tau_2}} + c' e^{-2\pi i \frac{t}{\tau_2}} + \cdots , \end{aligned}$$

where c and c', just like the previous b and b', signify unknown complex quantities. If, therefore, a period τ_2 is included in the pole motion in addition to τ_1, then this additional period must be directly expressed in the difference $w_{t+\tau_1} - w_t$, in which the primary period τ_1 is eliminated, just as the period τ_1 is expressed in w itself.

One determines the difference $w_{t+\tau_1} - w_t$ in the simplest manner by the following *graphical construction* on the pole trajectory in Fig. 104.*) Let $\tau_1 = 14$ months $= 1{,}17$ years. One then compares, for example, the location of the pole for the time 90,0 with the location for the time 91,17. The line that connects these two locations gives us the magnitude, direction, and sense of the vector $w_{t+\tau_1} - w_t$ for $t = 90{,}0$. It is only necessary to carry over this segment with a parallel straightedge from Fig. 104 into a new figure (Fig. 105a). We thus obtain a vector emanating from the coordinate origin of this new figure, of which only the endpoint is marked, and which is denoted by the number 90,0. We proceed in a similar

*) This graphical procedure may be new and useful for many similar cases. Cf. also, for what concerns the analytic rules for finding "hidden periodicities," the report of H. Burkhardt: Entwickelungen nach oscillierenden Funktionen. Jahresbericht der deutschen Mathematiker-Vereinigung, Bd. 10 (1901), pp. 312–332. More recently, A. Schuster has given a general method (the construction of a so-called "periodgraph") through which the question may be decisively settled. Cf. Nature, Vol. 66 (1902), pp. 614–618.[237]

manner for the two locations 90,1 and 91,27 in Fig. 104, and thus obtain in Fig. 105a a point that corresponds to the difference $w_{t+\tau_1} - w_t$ for $t = 90{,}1$, and is denoted by 1. Proceeding in this manner, we derive from the pole trajectory in Fig. 104 a new pole trajectory that is free of the primary term of the motion. The so-derived pole trajectory is, as one sees, still more convoluted and irregular than the original. In order that the figure should not become too unclear, the derived pole trajectory is separated into two parts. Fig. 105a gives the time interval from 90,0 to 94,0, and Fig. 105b gives the time interval from 94,0 to 98,0. In total, therefore, the positions between 90,0 and 99,17 are used for the derived pole trajectory. For clarity, half of the

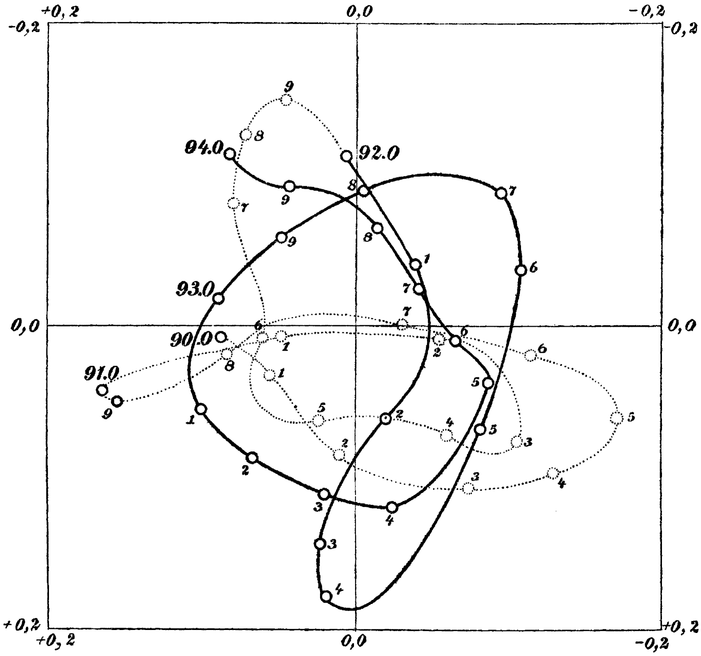


Fig. 105 a.

derived pole trajectory in each figure is dotted, and the other half solid.

A look at the derived pole trajectory shows that its dimensions are smaller than those of the original figure. While the amplitude in Fig. 104 reaches approximately $0'',30$, the amplitude in Figs. 105a and 105b exceeds $0'',15$ at only a few positions. This result is in no way self-evident, since the amplitude can just as well increase as decrease in the formation of the difference between $w_{t+\tau_1}$ and w_t.

We thus conclude that a period of 14 months is in fact present in the pole oscillation, and that approximately half of the total observed oscillation is attributable to a motion of this period.

We further conclude from the previously mentioned greater convolutedness of the new figures *that the circumstances that cause the residual amount of the pole oscillation, after the removal of the 14-month period, are less regular and less simple than those that determine the character and the period of the primary motion.* If one considers, in particular, the time interval from 96,5 to 97,5 in Fig. 105b, one

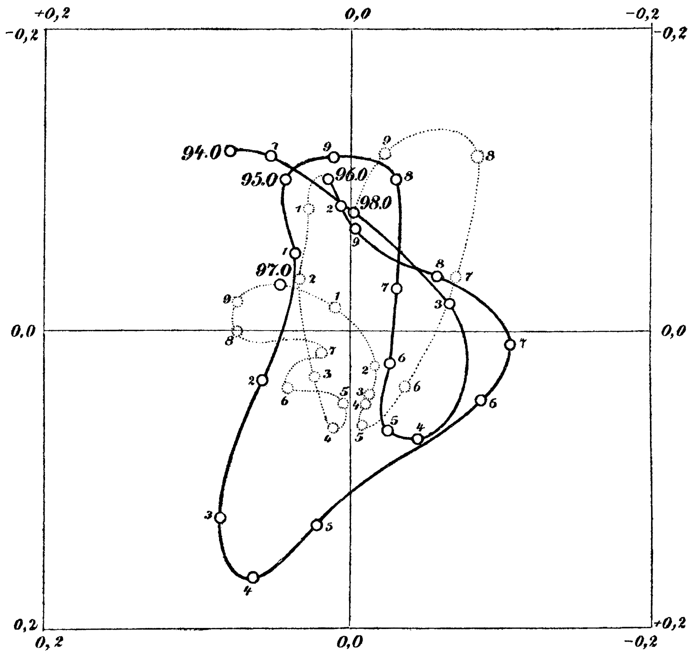


Fig. 105 b.

acquires the impression that the zigzags of this segment of the trajectory run randomly, and will entertain the suspicion that they may be completely attributable to observational error. The mean error of the pole trajectory that is derived by our construction turns out to be somewhat greater than that of the original trajectory ($0'',03$), so that the positions of the points of the derived pole trajectory are determined only up to about $0'',05$.

A certain regularity, however, cannot be mistaken in the derived pole trajectory. In both Fig. 105a and Fig. 105b, the points denoted by the same numbers lie, as a rule, rather close to one another. The full-year numbers, for example, are all found in the upper left quadrant of the figures, or in its immediate vicinity. To make this still more clear, one could insert in Fig. 105 the polygons of the various tenths of the years, by connecting with straight lines the positions of the pole with equal numbers 1, 2, ... in the different years. All these polygons would turn out to be relatively small. After the course of a year, the "derived pole" therefore returns, by and large, to its previous position. *A period of a full year thus appears to be present in the derived pole trajectory.*

If we seek to make the presence of the yearly period probable in the same way as above for the 14-month period, by means of counting the circuits, then we must make use in elevated measure of the arbitrariness that was applied there to pull individual loops over the coordinate origin and thus smooth the curve. We consider, for example, the solid part of the two figures from 92,0 to 96,0, which exhibit the most relatively clear elongations. We imagine the path from 92,0 to 92,4 pulled to the left across the midpoint of the figure. This path then gives, together with the segment from 92,4 to 92,8, a first circuit about the coordinate origin. We have a second rather regular circuit in the trajectory from 92,8 to 93,8. As we go over to the other figure, we imagine that the loop from 94,0 to 94,5 is pulled downward and to the left; we can then count a third circuit until 94,9. The time from 94,9 to 96,0 gives a fourth full circuit. We thus have in total, if we allow the arbitrary displacements of the trajectory as valid, directly four circuits in four years, and thus a yearly periodicity of the derived path.

This result has already been discussed by Chandler on the basis of his calculational reduction of the observations. On a similar basis, van de Sande Bakhuyzen later sought to represent the yearly component of the pole motion formulaically.[*])[238] He found, after subtraction of the 14-month pole oscillation, that the mean yearly pole trajectory is an ellipse whose major axis is equal to $0'',104$ and is directed toward the 19th meridian east of Greenwich; its minor axis amounts to $0'',044$. The ellipse runs, as is also to be seen from Figs. 105a

*) Akademie von Wetenschappen, Amsterdam, August 1900.

and 105b, in the direction from west to east; the pole passes the 19th meridian at the beginning of October.

It is natural to repeat our procedure, and to derive a new figure from Figs. 105a and 105b by elimination of the yearly period. If w now signifies the vector represented in Figs. 105 and τ_2 signifies the period of one year, then we form in a graphical manner, as above, $w_{t+\tau_2} - w_t$ and insert the result into a new figure. Only the solid part of Figs. 105 (from 92,0 to 96,0) has been adopted for further consideration, so that the resulting new Fig. 106 ranges from the marks 92,0 to 95,0, where again half the figure (from 92,0 to 93,5) is solid, and half (from 93,5 to 95,0) is dotted. Were a further period in addition to τ_1 and τ_2 contained in the pole motion, then this must become visible in our Fig. 106. It appears, however, that this is not so; *Fig. 106, rather, gives the impression that here it is a matter of individual perturbations that cause corresponding individual protrusions in the trajectory, which are followed by a retreat into the equilibrium position.* We have one such protrusion in the figure for 92,1, a second for 93,2, etc. As the comparison of Figs. 105 and 106 shows, an essential diminishment of the dimensions no longer occurs as a result of our procedure. Thus the magnitude of our figure does not illustrate the reality of the 12-month period, as it did previously for the 14-month period. *We must therefore conclude that the perturbations that are made apparent in Fig. 106 are of approximately the same order of magnitude as the influences that cause the assumed yearly circulation of the pole.* In any case, there remains a considerable remainder in the pole motion that is explained by neither a 14-month nor a 12-month oscillation. Of the 10-month Euler oscillation, there is absolutely nothing to be seen.

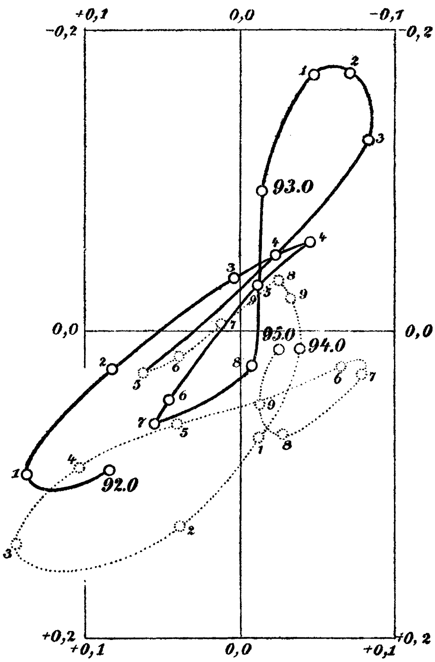


Fig. 106.

It is not impossible that the conclusions drawn here in a graphical manner are partly premature, and that they are to be modified by the further observations of the pole oscillations that are currently in progress. The last figure, in particular, must be held as rather problematic, due to the cumulative uncertainty in the repetition of our graphical process. A certain arbitrariness is also present in our derivation procedure, in so far as we need not necessarily compare the vectors w_t and $w_{t+\tau}$, but rather could just as well have taken the difference between $w_{t+2\tau}$ or $w_{t+3\tau}$ and w. The derived figures would then be different. Our results, however, coincide with the calculational results of Chandler et al.; in any case, our procedure appears to suffice for the more orienting than conclusive discussion here, and for the current precision of the relevant observations. For a conclusive judgment of the matter, a probabilistic investigation in the sense of A. Schuster (cf. the footnote on page 679) must be used to decide whether a found periodicity is to be regarded as actual or accidental.

The primary result of these considerations, the 14-month Chandler period, appears to find another certain support in the phenomenon of the tides. It it clear that the change in the location of the Earth's rotation axis must influence the motion of the ocean because of the changed centrifugal effects, and that a periodic change in position of the rotation axis must have, as a consequence, an oscillation of the mean ocean surface with the same period, assuming that the influence on the latter is sufficiently strong. Mr. van de Sande Bakhuyzen*) and Mr. Christie**) believe that this assumption is confirmed, and can demonstrate a 14-month variability of a few centimeters in the Dutch and American tidal observations.

In summary, we conclude two things from the transmitted observations. First, the pole oscillations are established without doubt, and the rotation axis of the Earth can no longer be regarded as "*der ruhende Pol in der Erscheinungen Flucht*";[240] second, the pole oscillations do not

*) Astronom. Nach. Nr. 3261.

**) Bulletin of the Phil. Soc. of Washington, 1895, Vol. XII, p. 103.[239]

have the simple regularity and the period that we would have expected according to the discussion of the preceding section.

§7. The explanation of the fourteen-month Chandler period and the elasticity of the Earth.

The old dispute over whether the interior of the Earth is molten or solid continues today in the sense of deciding whether the interior of the Earth, taken as a whole, behaves as a *solid body.* In order to avoid a mere debate over words, one must define the terms fluid and solid: a medium will be called a fluid if, under given circumstances, appreciable relative displacements of its parts can occur in its interior; it will be be called a solid if such displacements are impossible. It can remain unresolved, in the latter case, whether the deformability of the parts is caused by a kind of elastic binding (by solidity in the usual sense) or by a particularly high degree of viscosity; a fluid of sufficient viscosity (for example, asphalt) behaves perceptibly as a solid body with respect to external effects of not too long a duration, and exhibits no meaningful displacements of its parts with respect to one another. We can speak in this connection of the term *effective solidity* that is used in the English literature to denote a behavior that, under given circumstances, is analogous to a solid body of a specific degree of elasticity.

On the other hand, nothing more about its physical state should be implied by the statement that the interior of the Earth is solid. This state may differ from all that we otherwise know of fluids or solids due to the extraordinary temperatures and pressures that obtain in the interior of the Earth. Certain critical states in which the aggregate phases of matter are continuously transformed into one another may already be created in laboratory experiments; the state of the interior of the Earth, however, lies far beyond these critical boundaries. The correct standpoint is obviously not to predict the state of the interior of the Earth by risky analogies and extrapolations from laboratory experiments, but rather to deduce in reverse the average or effective state from the actual behavior of the Earth, as it is shown, for example, in the pole oscillations.

Also, it should not be claimed from the calculation of a certain elastic modulus that the entire Earth has the nature of a body with the relevant elasticity. The currently most probable and reigning view is that the constitution of the Earth is *inhomogeneous*; that it consists, namely, of a dense and solid core (iron) and a less dense and more compliant shell (stone mantle), which are separated from one another by a not very extended layer of a viscous fluid magma (cf. the theory of E. Wiechert to be cited below[241]). We wish to include the possibility of such inhomogeneity in the word "effective" elasticity or solidity. The calculated modulus of elasticity then signifies the value of the modulus of a homogeneous elastic body that behaves just as the probably inhomogeneous Earth does with respect to the elastic effects in question.

We do not intend to enter in more detail into the discussion of the interior of the Earth, but rather emphasize only a few points of historical development.*) In the interest of the theory of volcanism, geologists have long advocated for a molten fluid interior of the Earth. The first who claimed the contrary with a scientific basis appears to be Hopkins.**) Hopkins investigated the precession and nutation of a fluid-filled spherical shell, and found that such a shell would behave quite differently from the Earth. The later and deeper investigations of Lord Kelvin***) showed that the reasoning of Hopkins was defective, and that his results are also to be corrected in essential points. Kelvin considered an oblate ellipsoidal shell instead of a spherical shell, and showed that *a completely rigid shell* would give a difference between observation and calculation in the more rapid nutations (the half-year and particularly the half-month nutations; cf. page 651) but not in the precession and the 18 2/3-year nutation,†) and that, on the contrary, all

*) For more detailed information, cf. the representation in Chap. 15 of the excellent popular scientific work of G. H. Darwin, The Tides, London 1898,[242] German edition by A. Pockels, Leipzig 1902, or the recently appearing *Kosmische Physik* by Sw. Arrhenius, Leipzig 1903.[243]

**) Researches in physical geology, Philosophical Transactions London R. Soc. 1839, 1840, 1842.[244]

***) Mathematical and Physical Papers, Vol. 3, art. 45; cf., in particular, §§21–38, summarized in Popular Lectures, Vol. 3, p. 238.[245]

†) We have confirmed a related remark of Lord Kelvin with the models of the Göttingen mathematical collection: a top whose rotary mass is replaced by a fluid-filled *oblate* ellipsoid of revolution behaves *stably* when set into rotation about its axis on a horizontal support surface, and executes a precessional motion similar to that of a solid top (on the course of the shorter nutation, the observations of our model give no clear conclusion). In contrast, a top whose rotary mass consists

these phenomena could proceed as in reality for a *somewhat compliant shell*. The astronomical facts therefore disprove only the assumption of a fluid interior in a rigid shell, an assumption that indeed is also untenable on physical grounds, since we know no material that would be fully incompliant as a thin shell. On the other hand, however, the assumption of a fluid interior in a compliant shell is refuted by the phenomenon of the tides. A thin crust of the Earth with the elastic compliability of the materials known to us would follow the deforming influence of the tidal forces almost as willingly as the water of the sea. There would then be, however, no relative motion of the water with respect to the land under the influence of these forces, but only a common rise and fall of the sea and the continents that would escape immediate perception. Thus there remains only the assumption that the Earth is, in the mean, effectively solid (solid in the sense of the prefaced explanation). This assumption is fully compatible with the existence of peripheral cavities that are full of a type of fluid magma, or also the existence of a complete circumferential fluid layer, if this fluid layer is thin in proportion to the effectively solid core and the solid crust of the Earth, so that not only the requirements of the geological theories, but also, in particular, the important results of pendulum experiments can be supported. (Cf. here too W i e c h e r t's theory of the interior of the Earth.)

It should not be claimed at the same time that the Earth is effectively rigid. L o r d K e l v i n has investigated*) the degree of elastic deformability of the body of the Earth on the basis of estimating the actual height of the tides. While, as we just said, the tidal height must be reduced to zero for a primarily fluid and therefore completely compliant Earth, any finite degree of elastic deformability will account for a certain fraction of the height that must be formed by a fully rigid Earth. Kelvin estimates from this investigation that the compliability

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of a fluid-filled *prolate* ellipsoid proves to be completely *labile* under the same conditions. Since the Earth is an oblate ellipsoid, one understands that its precessional motion would not deviate essentially from that of a completely rigid body if it were filled with fluid, assuming that the crust of the Earth is, as we may assume without perceptible error in our model, absolutely rigid.

*) T h o m s o n and T a i t, Natural Philosophy II, art. 843.

of the Earth is smaller than that of glass, and approximately equal to that of steel. We will find a much sharper basis for such an estimation in the following, when we turn to the explanation of the Chandler period.

To S. Newcomb*) belongs the credit for recognizing that the period of the free nutation depends on the degree of compliability of the body of the Earth. The Euler period of 10 months corresponds to the assumption of complete rigidity; a calculation with any finite degree of elasticity gives, in contrast, a different and indeed longer period. In reverse, the Chandler period allows the assignment of a degree of elasticity for which the period of the free nutation would directly take on the observed duration of 14 months.

In the literature, these phenomena are investigated most fundamentally in a work by S. S. Hough**) that begins from the differential equations of elasticity for a rotating spheroid. We will obtain the results of Hough in a much simpler way by making use of a theorem of Chap. VII, §8 (page 607). In that section, the period of the free nutation, or, equivalently, the period of the force-free precession, is calculated for a deformable spheroidal top under the assumption that the deformation caused by the centrifugal effect of the rotation is opposed by elastic resistance only. This assumption does not apply for a body of the dimensions of the Earth, since here it is essential to consider the mutual gravitational forces among the elements.

We must therefore begin with some preliminary remarks on these gravitational forces, and on the way in which they are combined with the effect of the elastic forces. We arrange the following discussion in a series of individual problems.

First problem. A homogeneous, incompressible fluid mass stands under the influence of the mutual gravitation of its parts, and would form, in the state of rest, a sphere of radius R; it is set into rotation about a fixed axis with angular velocity ω. A possible equilibrium

*) On the dynamics of the Earth's rotation with respect to the periodic variations of Latitude, Monthly Notices Astr. Soc. London (1892), Vol. 52, p. 336,[246] and Remarks on Mr. Chandler's Law of Variation of Terrestrial Latitude, Astronomical Journal, Vols. 11, 12, 19.

**) On the Rotation of an elastic Spheroid, Philos. Transactions R. Soc. London (1896) Vol. 187, p. 319.[247]

form of the fluid is then an oblate ellipsoid of revolution that has the rotation axis as the symmetry axis (the Maclaurin ellipsoid[248]). *The ellipticity of this ellipsoid is given, under the assumption that it is small, by the formula*

$$\varepsilon_1 = \frac{5}{4}\frac{\omega^2 R}{g}, \tag{1}$$

where g signifies the gravitational acceleration at the surface of our fluid.

The numerical values for the Earth, expressed in meters and seconds, are

$$\omega = \frac{2\pi}{24 \cdot 60 \cdot 60}, \quad R = \frac{2}{\pi}\,10^7, \quad g = 9{,}81, \quad \frac{\omega^2 R}{g} = \frac{1}{289}, \quad \varepsilon_1 = \frac{1}{231}. \tag{2}$$

If we wish not to call upon the (very well known) formula for the potential of an ellipsoid, we can derive equation (1) advantageously by taking up our previous representation of the "Earth-ring." Just as we did previously for the rigid Earth, we now replace our ellipsoidal fluid by a sphere (radius R, mass M) and a ring (radius R, mass m) that lies in the equatorial plane of the ellipsoid. As we saw in §1 of this chapter, the gravitational potential of the combination of the ring and the sphere will be equal, up to terms of the second order inclusive, to the potential of any other mass distribution with the same principal moments of inertia. Let the moments of inertia of our fluid mass about the rotation axis (z-axis) and about two perpendicular axes (y- and x-axes) be C, A, and A. By the ellipticity we understand, as previously, the ratio*)

$$\varepsilon = \frac{C - A}{A}.$$

The mass m of the ring is to be chosen, according to equation (1) of §1, as

$$m = \frac{2(C - A)}{R^2} = \frac{2A}{R^2}\,\varepsilon = \frac{4}{5}\,M\varepsilon, \tag{3}$$

*) In addition to this definition, the definition

$$\varepsilon = \frac{a - b}{a}$$

also appears in the literature, where a is the major axis and b is the minor axis of the ellipsoid. One is easily convinced, with consideration of page 600, that for a *homogeneous* mass distribution this definition coincides, up to the higher powers of ε, with ours.

where we have introduced for A the approximate value $A = \frac{2}{5}MR^2$, which is the moment of inertia of a sphere of radius R.

If r is the distance of an arbitrary external point P from the midpoint of the fluid mass, then the potential of the sphere and ring is

$$V = f\Big(\frac{M}{r} + \frac{m}{r}\frac{1}{2\pi}\int\frac{d\varphi}{\sqrt{1+(R/r)^2-2(R/r)s}}\Big),$$

where the integration is extended over the circumference of the ring. Here s signifies, as on page 639, the abbreviation

$$s = \frac{xx' + yy' + zz'}{rR},$$

where x, y, z are the coordinates of P and x', y', z' are the coordinates of a point of the ring. If we lay the xz-plane through the point P and the xy-plane through the ring, and denote the angle between OP and OX by Θ, then

$$\begin{aligned} x &= r\cos\Theta, & y &= 0, & z &= r\sin\Theta, \\ x' &= R\cos\varphi, & y' &= R\sin\varphi, & z' &= 0, \end{aligned}$$

and thus

$$s = \cos\Theta\cos\varphi.$$

The power expansion of the square root yields, as on page 639,

$$\frac{1}{\sqrt{1+(R/r)^2-2(R/r)s}} = 1 + \frac{R}{r}s + \Big(\frac{R}{r}\Big)^2\Big(\frac{3}{2}s^2 - \frac{1}{2}\Big) + \cdots,$$

and the execution of the integration gives

$$(4)\quad \begin{cases} \displaystyle\frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{\sqrt{1+(R/r)^2-2(R/r)s}} &= 1 + \Big(\frac{R}{r}\Big)^2\Big(\frac{3}{4}\cos^2\Theta - \frac{1}{2}\Big) + \cdots \\ &= 1 + \frac{3}{4}\Big(\frac{R}{r}\Big)^2\Big(\cos^2\Theta - \frac{2}{3}\Big) + \cdots. \end{cases}$$

The potential of the attraction then becomes, if we insert for m the value from (3),

$$V = fM\Big(\frac{1}{r} + \frac{4}{5}\frac{\varepsilon}{r} + \frac{3}{5}\varepsilon\frac{R^2}{r^3}\Big(\cos^2\Theta - \frac{2}{3}\Big) + \cdots\Big).$$

We seek the value of V on the surface of the fluid mass, whose equation we may write as[249]

$$(5)\qquad r = R\Big(1 + \varepsilon\Big(\cos^2\Theta - \frac{2}{3}\Big)\Big)$$

(cf. equation (2) of page 601, where we have taken for the mean radius, there denoted by m, the approximate value R). Because of the smallness of ε, we can write

$$\frac{1}{r} = \frac{1}{R}\Big(1 - \varepsilon\Big(\cos^2\Theta - \frac{2}{3}\Big)\Big).$$

We insert this value into the first term of the expression for V; we may further directly set $r = R$ in the subsequent terms of the potential, which contain ε and are therefore to be considered as small. There follows

$$V = \frac{fM}{R}\Big(1 + \frac{4}{5}\,\varepsilon - \varepsilon\Big(1 - \frac{3}{5}\Big)\Big(\cos^2\Theta - \frac{2}{3}\Big) + \cdots\Big). \tag{6}$$

The surface of the rotating fluid must be a surface of constant pressure. It follows, according to the fundamental principles of hydrodynamics, that the potential energy of the forces acting on a unit mass at the surface of the fluid must also be constant. These forces are, on the one hand, gravity, and, on the other hand, the centrifugal force. The potential energy of the latter, calculated for a unit mass at an arbitrary point of the rotating ellipsoid, is

$$U = \frac{1}{2}\,\omega^2(x^2 + y^2) = \frac{1}{2}\,\omega^2 r^2 \cos^2\Theta;$$

at the surface of the ellipsoid, where r is approximately R, the potential energy U becomes, with a small formal change,

$$U = \frac{1}{3}\,\omega^2 R^2 + \frac{1}{2}\,\omega^2 R^2\Big(\cos^2\Theta - \frac{2}{3}\Big). \tag{7}$$

Thus if the sum $V + U$ on the surface of the fluid is to have a constant value—that is, a value independent of Θ—it is necessary that the factors of $\Big(\cos^2\Theta - \frac{2}{3}\Big)$ in (6) and (7) should be oppositely equal. This gives the equation

$$\frac{2}{5}\,\frac{fM}{R}\varepsilon = \frac{1}{2}\,\omega^2 R^2, \tag{8}$$

from which follows

$$\varepsilon = \frac{5}{4}\frac{\omega^2 R^2}{fM}.$$

If we introduce the gravitational acceleration g at the surface of our fluid mass—namely, $g = fM/R^2$ (approximately)—then we directly obtain the value of ε given above in equation (1).

Equation (1) was given by Clairaut, and is a fundamental formula in the theory of the figure of the Earth.[250] It assumes that the density of the fluid is constant, and that the centrifugal force is held in equilibrium only by gravity.

As is well known, one designates the functions of Θ that appear as coefficients of the various powers of R/r in the expansion (4) of the reciprocal distance between the two points as *spherical functions.**) The ex-

*) More precisely said, as *spherical surface functions.* A *spatial spherical function* is any expression that is homogeneous in the rectangular coordinates x, y, z and

pression $\cos^2\Theta - 2/3$ is such a function, and indeed a "spherical function of the second order." The series (6) represents, we can say, the expansion of the potential in spherical functions. Further, we have also ordered the expression for U in equation (7) in spherical functions. In the equilibrium condition (8), finally, we have compared the two terms of V and U that contain our spherical function of the second order with one another, and thus arrived at the calculation of the ellipticity. If we denote by U_2 and V_2 the considered terms in the expansion of U and V, where we calculate V_2 with the ellipticity 1, and, with consideration that the centrifugal force represents a perturbation of a small magnitude, completely disregard the ellipticity in the calculation of U_2, then we can write the equilibrium condition schematically as

$$\varepsilon V_2 = U_2, \quad V_2 = \frac{2}{5}\frac{fM}{R}\left(\cos^2\Theta - \frac{2}{3}\right). \tag{9}$$

For a more detailed development of the theory, particularly for substantial values of the ellipticity, we must refer to the literature.*)

Second problem. A solid homogeneous elastic sphere of radius R and density ϱ rotates with the angular velocity ω about one of its diameters. It is thus transformed into an ellipsoid of revolution that has the rotation axis as its axis of symmetry. The elastic behavior of the material is determined by the modulus of elasticity (E) and the assumption that the material is incompressible, and thus that the Poisson ratio of transverse contraction to longitudinal extension has the special value $1/2$. The latter assumption simplifies the calculations, and has no significant effect on the result.

The ellipticity of the resulting ellipsoid is then

$$\varepsilon_2 = \frac{15}{38}\frac{\varrho\omega^2R^2}{E}. \tag{10}$$

In order to first give a numerical example that is related to the proportions of the Earth, we choose, in CGS units,

$$\varrho = 5{,}5, \quad \omega = \frac{2\pi}{24\cdot 60\cdot 60}, \quad R = \frac{2}{\pi}\,10^9,$$

satisfies the potential equation $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = 0$. A spherical surface function results from a spatial spherical function if one sets $x^2 + y^2 + z^2 = \text{const.}$ in the expression for the latter.

*) H. Lamb, Hydrodynamics, Cambridge 1895, Chap. XII, p. 580. W. Wien, Hydrodynamik, Leipzig 1900, Kap. VIII, p. 303. Thomson and Tait, Natural Philosophy II, Cambridge 1895, art. 771, 793 ff. A more detailed literature review is given by A. E. H. Love, Encykl. d. math. Wissensch. Bd. IV, Art. 16, Nr. 4.

and take E equal to the elastic modulus of steel; that is, approximately $2{,}2 \cdot 10^6$ (kg weight/cm^2) $= 2{,}2 \cdot 981 \cdot 10^9$ CGS units. Then

$$\frac{\varrho\omega^2 R^2}{E} = \frac{1}{184}, \quad \varepsilon_2 = \frac{1}{465}. \tag{11}$$

For the derivation of equation (10), we must return to the foundations of the theory of elasticity.

If u, v, w are the displacements of a point in the interior of the sphere with respect to the coordinate axes, which are chosen just as in problem 1, there first follows, because of the assumed incompressibility,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{12}$$

The elastic differential equations for an incompressible material that is subjected to centrifugal forces take the form

$$\begin{cases} \frac{E}{3}\Delta u + \frac{\partial p}{\partial x} + \varrho\frac{\partial U_2}{\partial x} = 0, \\ \frac{E}{3}\Delta v + \frac{\partial p}{\partial y} + \varrho\frac{\partial U_2}{\partial y} = 0, \\ \frac{E}{3}\Delta w + \frac{\partial p}{\partial z} + \varrho\frac{\partial U_2}{\partial z} = 0. \end{cases} \tag{13}$$

Here Δ signifies, as usual, the abbreviation for the second differential parameter and p is the isotropic pressure that changes from point to point, which is determined so that the condition (12) is satisfied. The quantity U_2 is the Θ-dependent potential

$$U_2 = \frac{1}{2}\,\omega^2 r^2\left(\cos^2\Theta - \frac{2}{3}\right) = \frac{\omega^2}{6}(x^2 + y^2 - 2z^2) \tag{7'}$$

of the centrifugal force (see equation (7)). As one sees, U_2 is a spatial spherical function of the second order (cf. the footnote on page 691). We can disregard the term $\frac{1}{3}\,\omega^2 r^2$ that is added to U_2 in (7), since this term can change only the size and not the form of the sphere, and is completely without influence for an incompressible material.

The differential equations (12) and (13) are still to be supplemented by the condition that the surface of the sphere is a force-free surface. This condition states that the stresses with respect to all three coordinate directions must vanish on each surface element. Written in terms of the displacements u, v, w, the condition for the x-direction is

$$\left(\frac{2}{3}E\,\frac{\partial u}{\partial x} + p\right)\cos(n, x) + \frac{E}{3}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos(n, y) + \frac{E}{3}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\cos(n, z) = 0;$$

since for the sphere $\cos(n,x) : \cos(n,y) : \cos(n,z) = x : y : z$, we can write

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial x} + z\frac{\partial w}{\partial x} = -\frac{3p}{E}x. \tag{14}$$

The conditions with respect to the y- and z-directions follow from (14) by the cyclic interchange of $(x\,y\,z)$ and $(u\,v\,w)$.

We now claim that equations (12) through (14) are satisfied, with the appropriate choice of the constants α, β, γ, by the assumption

$$\begin{cases} u = \alpha\dfrac{\partial U_2}{\partial x} + \beta r^2\dfrac{\partial U_2}{\partial x} + \gamma\dfrac{\partial}{\partial x}(r^2U_2), \\ v = \alpha\dfrac{\partial U_2}{\partial y} + \beta r^2\dfrac{\partial U_2}{\partial y} + \gamma\dfrac{\partial}{\partial y}(r^2U_2), \\ w = \alpha\dfrac{\partial U_2}{\partial z} + \beta r^2\dfrac{\partial U_2}{\partial z} + \gamma\dfrac{\partial}{\partial z}(r^2U_2), \end{cases} \tag{15}$$

which thus constitutes the complete solution of the posed problem.

In the following proof, we will make frequent use of the rules

$$\Delta U_2 = 0, \quad \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)U_2 = 2U_2, \quad \Delta(r^2U_2) = 14U_2,$$
$$\Delta\left(r^2\frac{\partial U_2}{\partial x}\right) = 10\,\frac{\partial U_2}{\partial x}, \ \text{ etc.,}$$

which follow immediately from the definition and homogeneity of the spherical functions.

We first insert the assumption (15) into equation (12) and obtain

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = (4\beta + 14\gamma)U_2 = 0;$$

thus

$$\beta = -\frac{7}{2}\gamma \tag{16}$$

is to be chosen. We next go over to the first of equations (13) and calculate, according to (15) and (16),

$$\Delta u = (10\beta + 14\gamma)\frac{\partial U_2}{\partial x} = -21\gamma\frac{\partial U_2}{\partial x}.$$

Thus, according to (13),

$$\frac{\partial p}{\partial x} = (7E\gamma - \varrho)\frac{\partial U_2}{\partial x},$$

and, correspondingly,

$$\frac{\partial p}{\partial y} = (7E\gamma - \varrho)\frac{\partial U_2}{\partial y}, \quad \frac{\partial p}{\partial z} = (7E\gamma - \varrho)\frac{\partial U_2}{\partial z}.$$

We conclude, if we disregard the addition of a constant of integration, that

$$p = (7E\gamma - \varrho)U_2. \tag{17}$$

Finally, we must consider the surface condition (14). We first form, from (15),

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\alpha\frac{\partial U_2}{\partial x}+3\beta r^2\frac{\partial U_2}{\partial x}+3\gamma r^2\frac{\partial U_2}{\partial x}+6\gamma xU_2,$$
$$x\frac{\partial u}{\partial x}+y\frac{\partial v}{\partial x}+z\frac{\partial w}{\partial x}=\alpha\frac{\partial U_2}{\partial x}+\beta r^2\frac{\partial U_2}{\partial x}+4\beta xU_2+3\gamma r^2\frac{\partial U_2}{\partial x}+6\gamma xU_2;$$

equation (14) therefore demands, with consideration of (17), that

$$(2\alpha+4\beta r^2+6\gamma r^2)\frac{\partial U_2}{\partial x}+(4\beta+12\gamma)xU_2=-3\Big(7\gamma-\frac{\varrho}{E}\Big)xU_2.$$

We insert for β the value from (16) and write, adjoining the corresponding equations for the y- and z-directions,

$$(2\alpha-8\gamma r^2)\frac{\partial U_2}{\partial x}+\Big(19\gamma-\frac{3\varrho}{E}\Big)xU_2=0,$$
$$(2\alpha-8\gamma r^2)\frac{\partial U_2}{\partial y}+\Big(19\gamma-\frac{3\varrho}{E}\Big)yU_2=0,$$
$$(2\alpha-8\gamma r^2)\frac{\partial U_2}{\partial z}+\Big(19\gamma-\frac{3\varrho}{E}\Big)zU_2=0.$$

If the expression (7′) for U_2 is now introduced, then the first two equations become identical after the removal from each of a common factor. Our three equations are thus reduced to the two equations

$$2(2\alpha-8\gamma r^2)+\Big(19\gamma-\frac{3\varrho}{E}\Big)(x^2+y^2-2z^2)=0,$$
$$-4(2\alpha-8\gamma r^2)+\Big(19\gamma-\frac{3\varrho}{E}\Big)(x^2+y^2-2z^2)=0,$$

which should be satisfied at all points of the spherical surface $r=R$. This is possible only if

$$2\alpha-8\gamma R^2=0,\quad 19\gamma-\frac{3\varrho}{E}=0. \tag{18}$$

The required values of our coefficients α, β, γ are determined by these equations and equation (16); they are

$$\gamma=\frac{3}{19}\frac{\varrho}{E},\quad \beta=-\frac{21}{38}\frac{\varrho}{E},\quad \alpha=\frac{12}{19}\frac{\varrho}{E}R^2. \tag{19}$$

At the same time, the correctness of the assumption (15) is proven by this determination.

It is now easy to calculate the ellipticity of the ellipsoid that results from the deformation. We form, for this purpose, the expression for the radial displacement of a point on the surface of the sphere; namely,

$$\delta R=\frac{1}{R}(ux+vy+wz).$$

According to (15),

$$\delta R = \left(\frac{2\alpha}{R} + 2\beta R + 4\gamma R\right) U_2,$$

and therefore, because of (19),

$$\delta R = \frac{15}{19}\frac{\varrho}{E} R U_2. \tag{20}$$

On the other hand, this same displacement, if we calculate it from the ellipsoid equation (5), is

$$\delta R = r - R = R\varepsilon\left(\cos^2\Theta - \frac{2}{3}\right). \tag{21}$$

If we take the surface value of U_2 from ($7'$), then the value $\varepsilon = \varepsilon_2$ given in (10) in fact follows by comparison of (20) and (21).

We wish to represent the result of the comparison of (20) and (21) in the soon to be useful form

$$\varepsilon W_2 = U_2, \quad W_2 = \frac{19}{15}\frac{E}{\varrho}\left(\cos^2\Theta - \frac{2}{3}\right). \tag{22}$$

The quantity W_2 is a spherical function through which the elastic behavior of our sphere is characterized. Equation (22) (and similarly equation (9)) can be called a constraint equation, since it gives the oblateness of the outer surface through which the centrifugal forces will be directly canceled.

The preceding results are derived under more general assumptions by Lord Kelvin.*) Lord Kelvin considers, instead of an incompressible elastic body, a *general* elastic body, and instead of the particular spherical function U_2, a perturbation function U_n of order n; he also considers the increase of density toward the center that is verifiable for the Earth. By presuming the simplest assumptions, we succeeded in significantly shortening the Kelvin calculation.

Third problem. We again consider a solid sphere of elastic material that rotates with the angular velocity ω. In addition to the elastic resistance to a change of form, we consider the resistance produced by the mutual gravitation of the parts of the sphere. For the same elastic behavior as in the previous case, the ellipticity must now be smaller, since the resistance to the change of the spherical form is increased. *We claim that the ellipticity is now calculated from the formula***)

*) Thomson and Tait, Natural Philosophy, Part II, particularly art. 834. Sir W. Thomson, Mathem. and Phys. Papers, Vol. III, art. 45. Cf. also A. E. H. Love, Elasticity, Cambridge 1892, Chap. X.

**) Thomson and Tait, Natural Philosophy, art. 840.

$$\frac{1}{\varepsilon_3} = \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}, \tag{23}$$

where ε_1 and ε_2 are determined by equations (1) and (10).

The proof is contained in equations (9) and (22). When gravitation alone or elasticity alone acted against the centrifugal forces, we found

$$\varepsilon_1 V_2 = U_2 \quad \text{or} \quad \varepsilon_2 W_2 = U_2. \tag{24}$$

If the two resistances together produce equilibrium against the centrifugal forces, the equilibrium condition is

$$\varepsilon_3 V_2 + \varepsilon_3 W_2 = U_2,$$

where ε_3 is the ellipticity that is now present.

If we divide this equation by $\varepsilon_3 U_2$ and express the ratios V_2/U_2 and W_2/U_2 in terms of ε_1 and ε_2 according to (24), we obtain exactly the formula (23) that is to be proved.

For an experimental sphere of moderate dimensions, ε_1 is extraordinarily large compared with ε_2. From (1) and (10), namely, there follows

$$\frac{\varepsilon_1}{\varepsilon_2} = \frac{19}{6}\frac{E}{\varrho R g}.$$

The quantity g, which signifies the gravitational acceleration at the surface of our test sphere, is always smaller than the gravitational acceleration of the Earth in the same proportion as the radius of the sphere is smaller than the radius of the Earth (equal mean density of the sphere and the Earth assumed). From the smallness of g as well as the bigness of E, it follows that ε_2 will be negligible compared with ε_1, and therefore that $1/\varepsilon_1$ will be negligible compared with $1/\varepsilon_2$. Equation (23) states, in this case, that $\varepsilon_3 = \varepsilon_2$; that is, that one can disregard the influence of gravity on the centrifugal deformation under laboratory conditions. With the enlargement of the dimensions of the sphere, however, $\varepsilon_1/\varepsilon_2$ decreases quadratically; for a sphere of the size of the Earth and the elasticity of steel, $1/\varepsilon_1 = 231$ is no longer to be neglected with respect to $1/\varepsilon_2 = 465$. The ellipticity of such a sphere is therefore to be calculated according to (23) and would amount to about 1/700, which is essentially smaller than it would be if elasticity alone acted against the centrifugal effect.

One must not believe, however, that the observable ellipticity of the Earth would be determined by the common action of gravitation and elasticity in the sense discussed here, so that its calculation must follow from formula (23) with the adoption of an appropriate degree of elasticity E. We must imagine, rather, that the Earth was once, as the Sun is now, in a molten fluid state. In this state, only gravitation could maintain equilibrium against the centrifugal action. The ellipticity must therefore have amounted to $\varepsilon_1 = 5\omega^2 R/4g$. With the gradual cooling of the Earth, rigidity then appeared, and indeed, according to Lord Kelvin's depiction of this sequence of events, by a relatively rapid process. The ellipticity of the now rigid form of the Earth essentially coincides, one may assume, with the earlier fluid form. In this form, the Earth is *stress-free* for an unchanging rotation ω. The natural state of the Earth is this oblate form; elastic forces occur only in so far as a change of this original form is caused by a change of the rotational properties or by other forces, in which case the elastic forces would act to restore this stress-free form.

It thus follows that one can extract nothing about the elastic properties of the body of the Earth directly from the currently observable ellipticity. The situation here is different from that of the previously mentioned experimental sphere, whose natural state without rotation is the spherical form, and for which elastic resistance thus appears if this spherical form is changed by centrifugal effects. As a result, the form of the rotating experimental sphere of moderate dimensions will be influenced predominantly, as we saw, by the elastic forces; the actual form of the Earth, in contrast, gives evidence only of the gravitational action for the normal rotational velocity ω.

Fourth problem. In order to take a further step toward the present properties of the Earth, we now begin with an oblate ellipsoid of ellipticity ε_1 that rotates with angular velocity ω; the ellipsoid consists of a gravitating elastic material, and is in stress-free equilibrium with this form and this motion. We ask for the ellipticity ε that it would assume *if the rotation ceased.* This ellipticity will in any case be smaller than ε_1; gravity will abet the diminishment of the ellipticity, and elasticity will resist it.

We claim that the desired ellipticity ε is expressed in terms of the previously calculated ellipticities ε_1 and ε_2 (equations (1) and (10)) *as*

$$\varepsilon = \frac{\varepsilon_1^2}{\varepsilon_1 + \varepsilon_2}. \tag{25}$$

We note, in addition, the difference between the ellipticity of the stress-free state with rotation ω and the ellipticity of the state when the rotation has ceased. This difference is called ε'; it is, according to (25),

$$\varepsilon' = \varepsilon_1 - \varepsilon = \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2}. \tag{26}$$

We conduct the proof in a twofold manner.

a) In the natural state of the ellipsoid (rotation ω, ellipticity ε_1), equilibrium obtains between the centrifugal forces and the gravitational forces. For this state, according to (9),

$$\varepsilon_1 V_2 = U_2.$$

In the deformed state (rotation 0, ellipticity ε), we have, in contrast, equilibrium between gravitation and elasticity. Since the elastic forces tend to deform the body toward the stress-free state (ellipticity ε_1), the elastic effects are now to be measured by the difference in ellipticity ε', and will be given by $\varepsilon' W_2$. Equilibrium between gravitation and elasticity requires that

$$\varepsilon V_2 = \varepsilon' W_2, \quad \text{or} \quad \varepsilon V_2 = (\varepsilon_1 - \varepsilon) W_2.$$

We divide this equation by U_2 and set, according to (9) and (22), $V_2/U_2 = 1/\varepsilon_1$, $W_2/U_2 = 1/\varepsilon_2$. Then

$$\varepsilon\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right) = \frac{\varepsilon_1}{\varepsilon_2}, \tag{27}$$

which coincides with equation (25).

If we begin directly from the solution to our third problem above, then we can also reach the conclusion in the following manner.

b) In the stress-free state ε_1 of the rotating ellipsoid, we imagine that the centrifugal forces cancel the gravitational forces. In order to go over to the rotationless state ε, we must apply to our ellipsoid the centrifugal forces in the reversed (centripetal) sense, and also apply the difference of the gravitational forces with respect to the previous state in the reversed (or centrifugal) sense. The elastic forces that tend to produce the state ε_1 act in the same (centrifugal) sense. Thus the elastic forces and the difference of the gravitational forces act together against the reversed centrifugal forces in the passage from the state ε_1 to the state ε_2; that is, in the ellipticity change ε'. The resulting change ε' of the ellipticity can thus be calculated immediately according to equation (23); we obtain

$$\frac{1}{\varepsilon'} = \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}, \tag{28}$$

which coincides with (26) —.

After settling these four preliminary problems, we now enter into the actual subject of this section, the *explanation of the Chandler period.* We recall the result of Chap. VII, §8 concerning the nutation period*) of a deformable top with an approximately spherical form. We saw on page 607 that this period is calculated from the "original" ellipticity ε that the top would have in the rest state of zero rotation, and not from the ellipticity that refers to the spheroid in rotation, which we named $\mathsf{E} = \varepsilon + \varepsilon'$. The formula is (cf. equation (12) of page 607)

$$\frac{\text{nutation period}}{\text{rotation period}} = \frac{1}{\varepsilon}. \tag{29}$$

In the case of the Earth, the rotation period is equal to one day, and ε signifies the ellipticity that the Earth would assume for zero rotation, and is therefore to be calculated according to equation (25). The ellipticity E of the rotating top is to be identified, in the case of the Earth, with ε_1; the difference of the two ellipticities was denoted by ε' in the preceding as well as previously. The distinction is that we previously thought that this additional ellipticity ε' was caused by the elastic properties of the top alone, while it is caused for the Earth by the elastic and gravitational actions taken together in the sense of equation (28). The applicability of our previous deliberations is not affected; only the presence of a deformation was assumed in that calculation, not the circumstances under which this deformation came into being. It matters equally little that we previously regarded the top as stress-free in the rotationless state ε, while, in contrast, the Earth is stress-free in the state $\varepsilon_1 = \varepsilon + \varepsilon'$, and in the imagined rotationless state ε is subjected to elastic stresses in such a measure that it would perhaps burst. For it is indifferent in the determination of the motion of a body whether a force system is added to the body or taken from it, if this force system holds the body in equilibrium.

*) With respect to the terminology (free nutation = force-free precession), cf. the beginning of §8, page 599 and below.

According to equations (29) and (25), the period of the free nutation of the Earth, expressed in days, is equal to

$$\frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1^2} = \frac{1}{\varepsilon_1}\Big(1 + \frac{\varepsilon_2}{\varepsilon_1}\Big). \tag{30}$$

Were the material of the Earth absolutely rigid (its modulus of elasticity infinitely large, and therefore $\varepsilon_2 = 0$), then the nutation period would be calculated from the ellipticity of the *rotating* Earth, and would equal $1/\varepsilon_1$. If the Earth, however, were elastically compliant (elastic modulus finite, $\varepsilon_2 > 0$), then the nutation period would depend on the ellipticity of the *hypothetically rotationless* Earth, and an additional term would be added to $1/\varepsilon_1$. *The elastic compliability of the Earth therefore lengthens the period of the free nutation*, and indeed in the ratio $1 + \varepsilon_2/\varepsilon_1 : 1$. If we assume, for example, that the Earth has the elastic modulus of steel, then this ratio of lengthening is

$$1 + \frac{231}{465} = 1{,}5.$$

If we set the nutation period of the absolutely rigid Earth equal to 10 months (the Euler period), then the nutation period of an Earth with the elasticity of steel has a period of 15 months. Since, on the other hand, observation has given a nutation period of 14 months (the Chandler period), and therefore a lengthening ratio with respect to the Euler period of 1,4, we conclude that

$$1 + \frac{\varepsilon_2}{\varepsilon_1} = 1{,}4$$

for the material of the Earth. We can thus extract the effective degree of elasticity of the Earth. There follows

$$\varepsilon_2 = 0{,}4\,\varepsilon_1 = \frac{0{,}4}{231} = \frac{1}{578}.$$

According to equation (10), the quantity ε_2 is inversely proportional to the elastic modulus of the relevant material, and according to equation (11), $\varepsilon_2 = 1/465$ for steel. The elastic modulus of the Earth is thus calculated as 578/465 (that is, 1,24) times the elastic modulus of steel. *We must therefore ascribe to the Earth only a very slight degree of elastic compliability in order to explain in this manner the elongation of the Euler period into the Chandler period; the Earth must be, in its mean elastic behavior, somewhat less compliant than steel, or have a somewhat higher modulus of elasticity.*

This manner of conclusion still begs a point of explanation. The given value (30) for the nutation period of the Earth's axis gives, under the assumption of rigidity ($\varepsilon_2 = 0$), the period $1/\varepsilon_1 = 231$ days. This period is different from the Euler period, and does not even amount to 8 months. The basis for this difference naturally lies in the fact that the actual ellipticity of the Earth is different from that which we calculated by hydrodynamics under the assumption of a homogeneous mass distribution. We have obviously been guilty of a certain inconsistency in the preceding, in that we determined the enlargement ratio $1 + \varepsilon_2/\varepsilon_1$ from the *theoretical formula* (30), and, in contrast, set the nutation period for the rigid behavior of the Earth equal to the Euler period, which was taken directly from the astronomical *observations* of the precession of the Earth. This inconsistency may be justified after the fact in the following manner.

The theoretical value $\varepsilon_1 = 1/231$ takes no account of the nonuniform mass distribution in the interior of the Earth, and represents only an *upper limit* for the ellipticity of the Earth, whose mean density is empirically much larger than its surface density. In fact, the actual ellipticity of the Earth (1/304 according to astronomical observations; cf. page 663) or the actual oblateness (1/298 according to geodetic measurements) is indeed smaller than the calculated value 1/231 for a homogeneous mass distribution. It is also clear that the concept of the ellipticity itself becomes undetermined for an inhomogeneous mass distribution, in so far as the two definitions on page 689 must then lead to different numerical values. We can denote the definition given in the text as the *mass ellipticity*, and the alternative definition in the footnote as the *oblateness* or the *surface ellipticity.* Numerous investigations of Radau, Callandreau, Poincaré, and older researchers are concerned with the question of what law of density increase (assumed as continuous) toward the interior of the Earth must be supplied in order to be compatible with the empirical mass and surface ellipticities, as well as with the empirical mean Earth density.[251] We refer for this subject to the summarizing presentation of Tisserand,*) to whose report, however, is to be added that E. Wiechert**) has more recently combined the collected astronomical, geodetic, and physical data into a noteworthy theory of the interior of the Earth, to which repeated

*) Tisserand, Mécanique céleste, t. 2, chap. XIV, in particular art. 110–112.

**) E. Wiechert, Die Massenverteilung im Innern der Erde, Göttinger Nachrichtern 1897, p. 221. Cf. also G. H. Darwin, Monthly Notices of the R. Astr. Soc. London. Vol. 60 (1899) No. 2. The results of Wiechert and Darwin are compared by F. R. Helmert, Sitzungsberichte der Akademie d. Wiss. Berlin 1901, p. 328.

reference has already been made. In this theory, the density of the Earth is assumed as stepwise variable; the Earth, namely, is taken to consist of a more dense metal core and a less dense stone mantle, which are separated from one another by an intermediate viscous layer. The size and mass ratios of the core and the mantle are determined so that the mass and surface ellipticities turn out correctly, and so that the surface of the mantle corresponds exactly, and the surface of the core approximately, to stress-free hydrodynamic equilibrium. We mention this work here in order to make it understandable that through appropriate assumptions on the mass distribution, the theoretical limiting value 1/231 can actually be transformed into the observed mass ellipticity value of 1/304, as well as into the observed value of the oblateness, and that therefore, in particular, the mass ellipticity of the inhomogeneous Earth is diminished in the ratio 231/304. The nutation period corresponding to the rigid constitution, which indeed was inversely proportional to the mass-ellipticity, must be enlarged, and it is natural to assume that the nutational period corresponding to the actual elastic constitution is enlarged in just this ratio with respect to the value that it would have for a homogeneous mass distribution. This assumption lies implicitly at the basis of our above explanation of the Chandler period, in which we began not from the nutation period $1/\varepsilon_1$ that is valid for a homogeneous mass distribution, but rather from the greater Euler period of 304 days due to inhomogeneity, which we then multiplied, because of the elasticity of the Earth, by the enlargement ratio $1+\varepsilon_2/\varepsilon_1$ calculated from the theoretical values of ε_1 and ε_2. The same assumption is made, for want of another more certain basis, in the work of H o u g h cited above.

Thus it should not be denied that our result is originally drawn from a homogeneous ellipsoid, and that the passage to the inhomogeneous Earth is necessarily bound with some uncertainty.

This uncertainty, however, concerns only the quantitative results, and not the qualitative. It is very well possible that one may find, with consideration of the inhomogeneous density distribution in the interior of the Earth, a mean modulus of elasticity that is somewhat different from the numerical value given above. In contrast, the general result confidently remains that the period of the free nutation is increased by the elasticity of the Earth, and that for a certain degree of compliability, the Euler period goes over into the Chandler period for an arbitrary mass distribution and an arbitrary structure of the interior of the Earth.

Finally, a contribution to the full understanding of the pole oscillations (or, more precisely said, the 14-month period of the same) will be made if we carry over the depiction of the motion of a deformable top, given generally in §8 of the preceding chapter, to the proportions of the Earth.

For the normal position of the rotation axis, in which the rotation axis coincides with the polar principal inertial axis, the Earth rotates uniformly about this axis with the oblateness $1/298$. The difference of the equatorial and the polar Earth radii thus amounts to $R/298$, or approximately 21 km, where R signifies the mean radius of the Earth. Now let the rotation axis be deflected by any circumstance. The ellipticity of the Earth thus remains the same (cf. page 603), but the position of the principal inertial axes does not (cf. Fig. 90 on page 602). The mass of the Earth deflects, for a fixed form of the Earth ellipsoid, toward the side of the deflected rotation axis. The mass distribution, however, is not symmetric about the rotation axis, but rather about an axis (the instantaneous principal inertial axis) that lies between the original principal inertial axis and the instantaneous rotation axis. And indeed this axis divides the angle (δ in Fig. 90) between the original principal inertial axis and the instantaneous rotation axis, according to equation (6) of page 603, in the ratio $\varepsilon'/(\varepsilon + \varepsilon')$. Since ε' was equal to $\varepsilon_1 - \varepsilon$, the named ratio can also be written as $1 - \varepsilon/\varepsilon_1$. Now $1/\varepsilon$ determines the duration of the Chandler period, and $1/\varepsilon_1$ that of the Euler period. Thus

$$\frac{\varepsilon}{\varepsilon_1} = \frac{10}{14} \text{ and } 1 - \frac{\varepsilon}{\varepsilon_1} = \frac{2}{7}.$$

If e is the deflection of the instantaneous rotation pole on the surface of the Earth, then the deviation of the instantaneous inertia pole is $2e/7$. On page 677, we concluded from observations that e amounts, in the mean, to 4 m; the deviation of the principal inertia pole will thus be only 1,1 m. If the instantaneous rotation pole were simply to describe a circle of radius 4 m in 14 months about the original principal inertia pole (the geometric pole), then the instantaneous inertia pole must traverse a circle of radius 1,1 m about the same mean point in the same time and in the same rotation sense.

The displacement that a point on the surface of the Earth would thus experience, and which consists in part of an elevation and in part of a descent, is extremely small. We can extract it immediately from equations (3) and (5) of pages 601 and 602. In equation (3), r signifies the distance between a point on the surface of the spheroid and the midpoint of the spheroid for the normal rotation ω about the original principal inertial axis; equation (5) gives the same distance for the deflected rotation axis. The difference of the two represents the displacement of the point as a result of the deformation of the spheroid; it amounts, if we set the previously designated radius m to the mean radius of the Earth and treat the angular deviation δ as a small quantity, to

$$(3) - (5) = R\varepsilon'(\cos^2\Theta - \cos^2(\Theta+\delta)) = R\varepsilon'\delta\sin 2\Theta.$$

In Fig. 90 of page 602, this quantity is represented by the thickness of the strip between the original elliptical outline and the outline of the ellipse that is deformed and rotated by ϑ. The greatest displacement is found, according to Fig. 90 and the present formula, for $\Theta = 45°$, where $\sin 2\Theta = 1$. For this latitude we can represent the displacement, if we denote by e the quantity $R\delta$ that measures the deflection of the rotation pole on the surface of the Earth, as

$$\varepsilon' e = (\varepsilon_1 - \varepsilon)e = \varepsilon_1\Big(1 - \frac{\varepsilon}{\varepsilon_1}\Big)e = \frac{1}{304}\Big(1 - \frac{10}{14}\Big)e < e\cdot 10^{-3}.$$

Since e amounts to only 4 m, the greatest displacement of a point on the surface of the Earth will be less than 4 mm.

The smallness of this displacement implies a corresponding smallness of the plumb line oscillation that is produced by the deformation of the Earth. We determine, once from equation (3) of page 601 and once again from equation (5) of page 602, the angle that the normal to the ellipsoid forms with the line that connects the relevant location to the midpoint of the Earth. This angle is (with the interchange of the angle and the tangent),

$$\frac{1}{r}\frac{dr}{d\Theta},$$

and becomes, in the first approximation,

$$\begin{aligned} &\text{according to (3)} \cdots && -(\varepsilon+\varepsilon')\sin 2\Theta,\\ &\text{according to (5)} && -\varepsilon\sin 2\Theta - \varepsilon'\sin 2(\Theta+\delta). \end{aligned}$$

The difference of the two angles, which is equal to the change of direction of the plumb line, is thus

$$\varepsilon'(\sin 2(\Theta+\delta) - \sin 2\Theta) = 2\varepsilon'\delta\cos 2\Theta.$$

The greatest plumb-line oscillation thus occurs, in agreement with Fig. 90, for $\Theta = 0$ and $\pi/2$; that is, at the poles and at the equator. It amounts to

$$\pm 2\varepsilon'\delta.$$

Since we just found that $\varepsilon' < 10^{-3}$, the greatest plumb-line oscillation will be smaller than the 500th part of the deviation of the rotation axis. Since the latter, according to Fig. 104 of page 676, is smaller than $0'',3$, the plumb-line deviation will in any case be smaller than $0'',0006$, a magnitude that is by no means accessible to observation.

Finally, the possible influence of the water of the oceans on the duration of the nutation period may be pointed out. Were the surface of the Earth fully covered with water, then this water, since it would be free to follow the influence of the centrifugal forces, would form a symmetric ring about the instantaneous rotation axis. We would then have to distinguish, in association with Fig. 90, the surface of the fluid, which will rotate for a deflection of the rotation axis by the full angle δ with respect to its original position, and the surface of the solid core of the Earth, which will be displaced toward the side of the rotation axis by only the fractional part of δ that is determined on page 603. The more extensive deformation of the fluid surface would lengthen, in its turn, the period of the free nutation; thus a part of the deviation between the Chandler and the Euler periods must be explained by the behavior of the fluid covering, and only the remainder through the elasticity of the Earth. The compliance of the Earth would thus be still smaller, or its mean elastic modulus still greater, than was found above, where we ascribed in the calculation the entire deviation to the elasticity of the Earth. In reality, however, the Earth's surface is covered not completely by water, but only $^2/_3$ so, and the mobility of the water will be restricted in a complicated manner by the form of the continents. Thus it is hardly possible that the influence of the oceans on the nutation period of the Earth's axis can be estimated a priori in an objectionless manner. One must wait, rather, until sufficient observational data on the water motion that corresponds to the pole oscillations are at hand. It was already suggested on page 684 that the tidal flow shows a 14-month period; if the existence of the same is established with certainty and its magnitude is determined approximately, the exercise of determining the influence of this flow on the problem of the free nutation of the axis of the Earth will then arise.

§8. Pole oscillations of yearly period. Mass transport and flow friction.

With the explanation of the Chandler period, only one aspect of the problem of the pole oscillations is settled. It is to be determined further why an additional yearly period occurs, which we read from Figs. 105a and 105b of pages 680 and 681, and also why a residuum of apparently irregular and accidental perturbations (Fig. 106 of page 683) remains after the removal of the oscillation with this yearly period. The important general question is still to be answered: why is the free nutation of the Earth's axis so complicated and irregular, while the forced nutation (cf. §3 of this chapter) conforms so rigorously to a mathematical law?

The basis for this appears to be the fact that the Earth is indeed *effectively rigid* in the sense of the preceding section, but not *actually rigid*, and that its parts can be displaced to a certain degree with respect to one another. In particular, the presence of the yearly period suggests the hypothesis that such a displacement or a "mass transport" is caused by the heat of the sun, and may therefore have a meteorological origin. For the explanation of the yearly pole oscillation, one must draw upon various meteorological influences: the yearly change in snow and ice deposits, the sea storms of yearly period and the resulting water transport, and the oscillation in the level of the atmosphere. The latter can apparently be determined most quantitatively on the basis of the well-known isobaric maps for the greater part of the Earth, and yields mass displacements of astonishingly large amounts.

We take the following information from an investigation by R. S p i t a l e r.*) It is well known that the air pressure in the winter is higher, in the mean, than in the summer. Thus the air pressure difference between January and July in the northern hemisphere is in the mean positive, and in the southern hemisphere is negative. The distribution of the pressure difference is naturally not uniform, but rather differs essentially according to whether the considered region is fixed land or ocean, and indeed different in the sense that the water covering noticeably offsets the air pressure oscillation. In conformity with this deliberation, the isobaric maps show that the pressure excess between January and July in the northern hemisphere is concentrated over the Asiatic land mass, while the pressure excess between July and January in the southern hemisphere is grouped in islands over the three regions of South Africa, South America, and Australia. Nothing is known with certainty about the polar regions. With respect to the magnitude of the

*) Die periodischen Luftmassenverschiebungen und ihr Einfluſs auf die Lagenänderung der Erdachse. Petermanns Mitteilungen, Ergänzungsheft Nr. 137 (1901).[252]

pressure difference, an analysis of the isobaric maps shows the following. In the northern hemisphere between 0° and 80° latitude, there is an air mass excess in January with respect to July that is equal to 192,5 cubic kilometers of mercury, and in the southern hemisphere between 0° and 50° latitude an air mass excess in July with respect to January of 402,2 cubic kilometers of mercury! Since a cubic kilometer of mercury has a mass of $13{,}6 \cdot 10^{12}$ kg, we are concerned here with mass differences that are already somewhat comparable to the total mass of the Earth (equal to the mean density times $4\pi R^3/3 =$ ca. $6 \cdot 10^{24}$ kg).*)

With respect to the mass transport caused by ocean currents, we refer to an estimation by J. Lamp.**)

In addition to the meteorological influences, mass transports of shorter periods occur as a result of tidal flows; aperiodic mass displacements of smaller amounts are due to earthquakes, volcanic eruptions, deposits of rivers, and the secular rising and sinking of the crust of the Earth.***)

We must now ask ourselves how such mass transports influence the motion of the Earth. We can distinguish an *indirect* and a *direct* influence: an indirect influence through which the changed mass distribution influences the principal inertial axes of the Earth and thus also influences the position of the rotation axis in the Earth, and a direct influence in which the production of the mass transport consumes a part of the total impulse, and thus modifies in magnitude and direction the remaining impulse for the rotation of the Earth.

We first give a general depiction of the relevant relations.

The *indirect influence* of a mass transport is determined as follows.

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*) The corresponding air pressure differences are in no way large. If we imagine, for example, the total mass of 192,5 km^3 of mercury distributed uniformly on the spherical zone between 0° and 80° north latitude, there follows a covering of only 0,78 mm in height. The named mass excess on the northern spherical zone thus corresponds to a barometric state that is 0,78 mm higher in January than in July. In the same manner, the mass excess of 402,2 km^3 of mercury in the southern spherical zone corresponds to a mean barometric state that is 2,08 mm higher in July than in January.

**) Über Niveauschwankungen der Ozeane als eine mögliche Ursache der Veränderlichkeit der Polhöhe. Astron. Nachrichten 126 (1891), Nr. 3014.[253]

***) For more details on this subject, cf. Helmert, Die mathem. und physikalischen Theorien der höheren Geodäsie, II, Kap. 5, Leipzig 1884.[254]

The position of the moving mass with respect to the body of the Earth is regarded as known, and one calculates from this mass position the positions of the principal inertial axes of the Earth, and, in particular, the position of the inertia pole.[255] If the latter happens to coincide with the instantaneous rotation pole before the mass transport, it will differ during and after the transport. If one is allowed to assume that the Earth is a symmetric top with equal equatorial moments both after and before the transport, which is permissible to a very high degree of approximation, then the motion of the rotation pole after the mass transport consists of a rotation around the inertia pole. The period of this motion is—under the assumption of the previously calculated elasticity—fourteen months. The radius of the circle depends primarily on the velocity of the mass transport; the motion persists, theoretically, until it is changed by a new mass transport.

For the discussion of the *direct influence* of the mass transport on the impulse, we wish to assume that our mass transport is produced by internal forces, and therefore by forces that satisfy the law of equality of action and reaction inside the mass system of the Earth. Our fundamental impulse theorem of page 113 then holds just as unrestrictedly for the nonrigid Earth as for a rigid body (cf. a remark on page 111). This theorem states that the magnitude and direction of the total impulse of the mass system of the Earth remains constant in space. The total impulse is divided, however, into the impulse of the mass transport and that of the Earth's rotation. If the former be variable, then so must the latter be. In general, the rotation axis of the Earth and the position of the instantaneous pole of the Earth change with the impulse of the Earth's rotation. If the rotation axis coincides with the geometric pole before the mass transport, then it will be removed from it during the transport; if it originally moves in a circle about the geometric pole, then the radius of this circle is diminished or enlarged by the mass transport.

We give now some analytic developments, in which we first consider the two distinguished influences separately, and divide the subject matter into a series of individual problems.

First problem: a mass or the center of gravity of a not too extended mass system m is displaced from the position $X_0\,Y_0\,Z_0$ on the Earth to the position $X\,Y\,Z$. How does the polar principal inertial axis change?

Let the coordinate axes be the principal inertial axes for the original position of m. The principal moments of inertia are called A, $B = A$, C; the products of inertia (cf. page 98) are zero. For the changed position of m, we write the moments and products of inertia with respect to the coordinate axes in the form

$$\begin{aligned} \overline{A} &= A + a, & \overline{B} &= A + b, & \overline{C} &= C + c, \\ \overline{E} &= \phantom{A+{}} e, & \overline{F} &= \phantom{A+{}} f, & \overline{G} &= \phantom{C+{}} g. \end{aligned}$$

The quantities $a, \dots, e, \dots$ have the meanings

$$\text{(1)} \qquad \begin{cases} a = m(Y^2 + Z^2 - Y_0^2 - Z_0^2), \; \dots \\ e = m(YZ - Y_0 Z_0), \; \dots, \end{cases}$$

and are treated as small quantities in comparison with A and C. For the determination of the changed position of the principal inertial axes, we begin, according to page 100, with the surface of the second order

$$(A + a)\xi^2 + (A + b)\eta^2 + (C + c)\zeta^2 - 2e\eta\zeta - 2f\zeta\xi - 2g\xi\eta = 1.$$

The principal axes of this surface, which are at the same time the principal inertial axes, are determined by the equations

$$\begin{aligned} (A + a - \lambda)\xi - g\eta - f\zeta &= 0, \\ -g\xi + (A + b - \lambda)\eta - e\zeta &= 0, \\ -f\xi - e\eta + (C + c - \lambda)\zeta &= 0. \end{aligned}$$

Here λ is to be chosen so that the three equations are compatible with one another. If this is done, then the ratios $\xi : \eta : \zeta$ determine the position of each of the three principal axes. It is convenient for the following to conceive ξ, η, ζ as the direction cosines of the relevant principal axis, and therefore to choose their absolute values so that $\xi^2 + \eta^2 + \zeta^2 = 1$.

We are particularly interested in the polar principal inertial axis, and may assume that this axis deviates only slightly from its original direction, the Z-axis. (For the equatorial principal inertial axes, the corresponding assumption would be impermissible, since their positions in the equatorial plane are originally undetermined, and thus can be changed significantly by a small mass transport.) We will therefore assume that ξ and η are small, and take ζ equal to 1. Quantities such as $f\xi$ and $a\xi$ are then to be struck; our third equation gives simply $\lambda = C + c$, and our first two equations become

$$\text{(2)} \qquad (A - C)\xi = f, \quad (A - C)\eta = e.$$

The change of direction of the principal axis in question is thus known on the basis of the values of f and e given in (1). The quantities ξ and

η can be regarded, at the same time, as the x- and y-coordinates of the inertia pole measured by the corresponding geocentric angles. If we multiply ξ and η by the radius R of the Earth, then we directly obtain the displacement of the inertia pole on the surface of the Earth.

In order to give a numerical example, we wish to assume that the mass m is displaced on a meridian, which we can take as the XZ-plane, from the latitude Θ_0 to the latitude Θ. Then

$$e = 0, \quad f = \frac{mR^2}{2}(\sin 2\Theta - \sin 2\Theta_0), \quad \eta = 0.$$

We can write the expression (2) for ξ as

$$\xi = \frac{A}{A - C}\,\frac{f}{A}.$$

The quantity A would be calculated for a homogeneous mass distribution as $2MR^2/5$, understanding by M the mass of the Earth; the actual mass distribution, however, corresponds better to the assumption $A = MR^2/3$.*) With use of the known numerical value of $A/(C - A)$, there follows

$$\xi = -456\,\frac{m}{M}\,(\sin 2\Theta - \sin 2\Theta_0).$$

If, for example, $\Theta_0 = -45°$ and $\Theta = +45°$, then the mass that is necessary to produce a deviation of the principal inertial axis by $1''$ is

$$m = \frac{\pi M}{180 \cdot 60 \cdot 60 \cdot 912} = \frac{1}{2}\,10^{-8}\,M.$$

The pole is naturally deflected in the same sense as the mass transport.

Second problem: The impulse of a mass displacement is given in magnitude and position with respect to the body of the Earth by a possibly time-dependent vector $\lambda\,\mu\,\nu$. It is assumed that the principal inertial axes are not changed by this mass displacement (see below). What influence does the mass displacement have on the position of the rotation axis?

If we disregard external forces and assume that the mass transport is caused only by internal forces, then the total impulse remains constant in space. This impulse has, with respect to the moving Earth frame, the components $L + \lambda$, $M + \mu$, $N + \nu$, where $L\,M\,N$ denotes the impulse of the Earth's rotation. According to page 140, the Euler equations obtain in the form

*) Cf. Helmert, l. c., II, p. 473.

$$\begin{aligned}
\frac{d(L+\lambda)}{dt} &= \quad r(M+\mu) - q(N+\nu),\\
\frac{d(M+\mu)}{dt} &= -r(L+\lambda) + p(N+\nu),\\
\frac{d(N+\nu)}{dt} &= \quad q(L+\lambda) - p(M+\mu).
\end{aligned}$$

We thus write

$$\text{(3)}\qquad \begin{cases}
\dfrac{dL}{dt} = \quad rM - qN + \mathsf{\Lambda},\\
\dfrac{dM}{dt} = -rL + pN + \mathsf{M},\\
\dfrac{dN}{dt} = \quad qL - pM + \mathsf{N},
\end{cases}$$

in that we set

$$\mathsf{\Lambda} = -\frac{d\lambda}{dt} + r\mu - q\nu, \quad \mathsf{M} = -\frac{d\mu}{dt} - r\lambda + p\nu, \quad \mathsf{N} = -\frac{d\nu}{dt} + q\lambda - p\mu.$$

The quantities λ, μ, ν are small; we can thus replace p, q, r in the preceding equations by their approximate values in the unperturbed rotation of the Earth; that is, by the values $p = 0$, $q = 0$, $r = \omega$. The equations thus simplify to

$$\text{(4)}\qquad \mathsf{\Lambda} = -\frac{d\lambda}{dt} + \omega\mu, \quad \mathsf{M} = -\frac{d\mu}{dt} - \omega\lambda, \quad \mathsf{N} = -\frac{d\nu}{dt}.$$

The quantities $\mathsf{\Lambda}$, M, N are thus, just as the quantities λ, μ, ν are, known functions of time. Equations (3) may be interpreted as follows: our mass transport with impulse $\lambda\,\mu\,\nu$ influences the rotation of the Earth as if a given time-dependent turning-force $\mathsf{\Lambda}\,\mathsf{M}\,\mathsf{N}$ acted upon the body of the Earth.

Since we assume that the positions of the principal axes are not influenced by the mass transport, these axes are fixed in the body of the Earth, and we can set $L = Ap$, $M = Aq$, $N = Cr$ in (3); in addition, we can replace r by its approximate value ω in the terms that are accompanied by the small factor p or q. The first two of equations (3) are thus

$$\text{(5)}\qquad \begin{cases}
A\,\dfrac{dp}{dt} = (A-C)\omega q + \mathsf{\Lambda},\\
A\,\dfrac{dq}{dt} = (C-A)\omega p + \mathsf{M}.
\end{cases}$$

The third equation is unimportant for the following.

Through multiplication by 1 and i, we combine equations (5) into a complex equation

$$\text{(6)}\qquad A\,\frac{d(p+iq)}{dt} = (C-A)i\omega(p+iq) + \mathsf{\Lambda} + i\mathsf{M},$$

and assume that the mass transport is periodic, so that λ, μ, ν, and therefore also $\mathsf{\Lambda}$, M, N, are periodic functions of time. We expand these functions in a Fourier series according to multiples of the period, and consider a single term of this series. We can thus assume for $\mathsf{\Lambda} + i\mathsf{M}$, in complete generality,

$$\mathsf{\Lambda} + i\mathsf{M} = ae^{i\alpha t} + a'e^{-i\alpha t}.$$

The corresponding general integral of (6) is then

$$p + iq = be^{i\alpha t} + b'e^{-i\alpha t} + ce^{i\beta t}, \tag{7}$$

where c is the constant of integration, and where we set, as abbreviations,

$$\begin{cases} \beta = \dfrac{C-A}{A}\omega, \\ b = \dfrac{a}{i\alpha A - i\omega(C-A)} = \dfrac{ia}{A}\dfrac{1}{\beta - \alpha}, \\ b' = \dfrac{a'}{-i\alpha A - i\omega(C-A)} = \dfrac{ia'}{A}\dfrac{1}{\beta + \alpha}. \end{cases} \tag{8}$$

The quantity c is in general complex, just as the previous a and a' are. The first two terms on the right-hand side of (7) represent the *forced* oscillation of the rotation axis caused by the mass transport, and the last term represents the *free* oscillation. The first terms naturally have the period of the mass transport, and the last term has the period of $A/(C-A)$ days indicated by the value of β. If we set the latter equal not to the Euler period, but rather to the Chandler period, then we consider in the simplest manner, in the sense of the previous section, the elastic compliability of the Earth, which is naturally relevant in this place as well.

As always in oscillation problems, we encounter here a certain resonance phenomenon; that is, a strengthening of the amplitude in the case of coincidence between the free and forced vibrations. This coincidence occurs, in our notation, if $\alpha = \pm\beta$, in which case either b or b' becomes infinitely large. We best measure the strengthening at a certain frequency by comparison with a very slow oscillation ($\alpha = 0$). According to (8), there follow, for the coefficient b_0 for a very small frequency and for the ratio of the coefficient b_α for an arbitrary frequency to that for a very small frequency,

$$b_0 = \frac{ia}{A}\frac{1}{\beta}$$

and

$$\frac{b_\alpha}{b_0} = \frac{1}{1 - \alpha/\beta}. \tag{9}$$

(The same formula holds for the coefficient b' if we interchange $+\alpha$ with $-\alpha$; the following remarks that we attach to the value of b follow just as well from the corresponding formula for b' if we consider negative frequency, and therefore assign the opposite sense to the mass transfer.)

If, for example, the mass transport has the period of a year and we take the period of the free oscillation, as agreed, as the Chandler period, then $\alpha/\beta = 14/12$, and (disregarding the sign) $b_\alpha/b_0 = 6$. *The circumstance that the yearly period is not very far removed from the period of the pole oscillation has the consequence that a mass transport of a yearly period produces a sixfold stronger deviation than a process in which the same turning-force is applied infinitely slowly to the Earth.* If the mass transport, on the other hand, has a very short period (α very large), then α/β will be large and b_α/b_0 small. For example, we wish to refer to the significant mass transport that occurs relative to the rotating Earth due to the phenomenon of the tides.*) Here α/β is approximately equal to 840 and b_α/b_0 is approximately equal to $1/840$. *A mass transport of so short a period produces, for equal magnitude of the applied turning-force, only a vanishingly small effect on the rotation axis compared with an infinitely slow transport.* The mass system of the Earth is too inert to follow an influence with so short a period; it follows a perturbation all the more willingly and yieldingly as the disturbance period lies nearer to the natural period of the pole oscillation.

Moreover, the same resonance phenomenon occurs if we assume in the calculation, as in the first problem of this section, only the indirect action of the mass transport; that is, its influence on the mass distribution, in that the inertia pole of the Earth is also displaced periodically by a periodic mass transport, and thus an ever stronger oscillation of the rotation pole occurs as the period of the mass transport lies nearer to the natural period of the pole oscillation. We will have occasion to return to this in a third problem below.

Because of the small difference between the yearly period of the meteorological mass transports and the free oscillation period of the pole, the possibility exists, in any case, that a relatively weak meteorological

*) The mass transport considered here is indeed caused by external forces (the Moon attraction), so that the present discussion is not immediately valid.

mass transport can have a relatively strong pole oscillation as a result, a possibility that is to be kept in mind for the study of the pole oscillation of yearly period.

There is a class of mass transports for which the effect on the impulse that is treated here separately actually occurs separately, and the effect on the mass distribution vanishes. We speak of "a cyclic mass transport" if the displaced mass is immediately replaced by a new mass of the same density. Evidently, a cyclic mass transport gives no occasion for a displacement of the principal inertial axes, while it influences, on the other hand, the impulse of the Earth rotation according to the measure of its velocity and its quantity. This case permits of a very elegant treatment, particularly if the impulse of the mass transport remains constant with respect to the body of the Earth; it has been investigated in a series of works by V. Volterra.*)

It has not yet been possible, however, to demonstrate the existence of actual cyclic mass transports that are of sufficient intensity or sufficient duration to have a perceptible influence on the pole oscillations. In particular, Volterra's attempt to explain the pole oscillation with the Chandler period on this basis does not appear promising. The cyclical motions that Volterra must postulate in order to arrive at the Chandler period are of a purely hypothetical nature, and are not made probable by geophysical experience. Moreover, we will see in the following that the direct action of a mass transport on the impulse is generally small compared with its indirect action on the principal axes, and that a noncyclic mass transport generally influences the rotation of the Earth more than a cyclic mass transport of the same strength. Thus the Volterra investigations appear to have a more general mathematical interest than an immediate geophysical interest.

Purely theoretically, without consideration of geophysical questions, the motion of a top with an interior cyclic motion has been treated previously by A. Wangerin.**)

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*) Astronom. Nachr. Vol. 138 (1895), p. 33; Atti d. R. Accademia di Torino, Vol. 30 and 31 (1895). The explanations of G. Peano, ibid., are in the same direction. Volterra summarizes his investigations in Acta Mathematica, Vol. 22 (1898).[256]

**) Halle 1889, University publication. The problem is taken up in a mathematically generalized form by V. Volterra, Rend. d. R. Accademia dei Lincei (5) Vol. 4 (1895) and by E. Jahnke, Liouville's Journal (5) t. 5 (1899).[257]

Combining the developments given for our first and second problems, we now consider simultaneously the direct effect of a mass transport on the impulse and the indirect effect of the same mass transport on the mass distribution of the Earth. We pose, correspondingly, the following

Third problem: A mass m is displaced from an initial position $X_0\,Y_0\,Z_0$ in a specified manner, so that its coordinates X, Y, Z with respect to the body of the Earth are known, and are, in particular, periodic functions of time. The inertia pole of the Earth is thus deflected in a determined manner, and the impulse of the Earth's rotation will be influenced at the same time as if a determined turning-force $\mathsf{\Lambda\,M\,N}$ *acted on the body of the Earth. Construct and integrate the differential equations of the rotational motion.*

From the given time-dependent coordinates X, Y, Z of the mass m, we first compute the vector

$$mX',\;\; mY',\;\; mZ',$$

and next compute its moments

$$(10)\quad \lambda = m(YZ' - ZY'),\;\; \mu = m(ZX' - XZ'),\;\; \nu = m(XY' - YX')$$

about the coordinate axes, which are the components of the turning-impulse of the mass transport with respect to the same axes.

The motion of the body of the Earth will be represented after the transport as well as before, under the assumption that exterior forces are not present, by equations (3), in which $\mathsf{\Lambda}$, M, N can be calculated with sufficient accuracy by means of equations (4) with the just-given λ, μ, ν. In fact, equations (3) of page 712 or equations (2′) of page 140, from which we deduced equations (3), are valid for an arbitrary rectilinear system of axes that are fixed in the top, whether or not these axes are the principal inertial axes. In contrast to the considerations for our second problem, our coordinate axes are now no longer the principal inertial axes; if we assume, for example, that they were the principal axes at the beginning of the motion, they will lose this property to the extent that the inertia pole is deflected by the mass transport. As a result, there appear in the place of the simple relations $L = Ap$, $M = Aq$, $N = Cr$ the general equations (2) of page 95 for the relation between the impulse vector and the rotation vector, which we can write, with consideration of the definitions of the quantities $a\,b\,c$, $e\,f\,g$ in equations (1), as

$$\begin{aligned} L &= (A+a)p - gq - fr, \\ M &= -gp + (A+b)q - er, \\ N &= -fp - eq + (C+c)r. \end{aligned}$$

We consider $a\,b\,c$, $e\,f\,g$, $p\,q$ as small quantities, and can thus simplify the previous equations as

$$L = Ap - fr, \quad M = Aq - er, \quad N = (C + c)r. \tag{11}$$

We must insert these values into equations (3). From the third of these equations, it first follows that dr/dt (in contrast to r itself) will be a small quantity, which one could also deduce from the fact that the unperturbed original motion consists of a *uniform* rotation $r = \omega =$ const. In the first two equations of (3), we further neglect all those terms of the second order in the small quantities, and replace r by its approximate value ω in the terms of the first order. There follow

$$\begin{cases} A\dfrac{dp}{dt} = (A - C)q\omega + \mathsf{\Lambda}', \\ A\dfrac{dq}{dt} = (C - A)p\omega + \mathsf{M}', \end{cases} \tag{12}$$

where the abbreviations

$$\begin{cases} \mathsf{\Lambda}' = \omega\dfrac{df}{dt} - \omega^2 e + \mathsf{\Lambda}, \\ \mathsf{M}' = \omega\dfrac{de}{dt} + \omega^2 f + \mathsf{M} \end{cases} \tag{13}$$

are used.

Equations (12) have completely the same form as equations (5); the quantities $\mathsf{\Lambda}'$, M' here, just like $\mathsf{\Lambda}$, M there, are known functions of time if the time dependence of the mass transport is known. The quantities $\mathsf{\Lambda}'$, M' bring together the direct action on the impulse and the indirect action of the mass transport, and may be interpreted once more as an apparent turning-force acting on the body of the Earth. The analytic expression

$$\begin{aligned} \mathsf{\Lambda}' &= -\frac{d}{dt}(\lambda - \omega f) + \omega(\mu - \omega e), \\ \mathsf{M}' &= -\frac{d}{dt}(\mu - \omega e) - \omega(\lambda - \omega f) \end{aligned} \tag{14}$$

of this apparent turning-force, which results immediately from the defining equations (4) and (13) for $\mathsf{\Lambda}$, M and $\mathsf{\Lambda}'$, M', is also noteworthy. In order to consider the indirect effect of the mass transport in addition to the direct effect, one must simply replace λ, μ by $\lambda - \omega f$, $\mu - \omega e$.

The further treatment of equations (12), their integration and the discussion of their solution, is no different from the above treatment of equations (5); in particular, the above-emphasized resonance phenomenon occurs if the mass transport is periodic and its period lies near the period of the free oscillation of the Earth's axis.

We first wish to decide the question whether, for a periodic mass transport, the direct or the indirect—that is, the effect on the impulse or the effect on the mass distribution—is more significant, in order to arrive at a further simplification of equations (12) that is useful for numerical calculation. We have, for this purpose, merely to examine, according to equations (14), the ratio of the pairs λ, μ and ωf, ωe.

The rule according to which the mass transport runs temporally in the body of the Earth may be expressed by the most conveniently chosen equations

$$\begin{aligned} X &= X_0 + a\sin\alpha t, \\ Y &= Y_0 + b\sin\alpha t, \\ Z &= Z_0. \end{aligned}$$

The mass in question thus oscillates about its initial and mean position $X_0\,Y_0\,Z_0$ with the period $2\pi/\alpha$. We calculate, according to (1),

$$e = mZ_0 b\sin\alpha t, \qquad f = mZ_0 a\sin\alpha t,$$

and, according to (10),

$$\lambda = -mZ_0\, b\alpha\cos\alpha t, \qquad \mu = mZ_0\, a\alpha\cos\alpha t.$$

There follow the ratios

$$\frac{\lambda}{\omega f} = -\frac{\alpha}{\omega}\,\frac{b}{a}\,\frac{\cos\alpha t}{\sin\alpha t}, \qquad \frac{\mu}{\omega e} = \frac{\alpha}{\omega}\,\frac{a}{b}\,\frac{\cos\alpha t}{\sin\alpha t}.$$

We do not wish to make a detailed assumption about the amplitudes a and b; we will assume, however, that they are of the same order of magnitude. The orders of magnitude of the preceding ratios are then given in the mean by the factor α/ω. Now the quantities α and ω are inversely proportional to the period of the mass transport and the period of the rotation of the Earth, respectively. The ratio α/ω will thus be equal to the reciprocal number of days of the period of the mass transport. We saw previously that only mass transports with a period that lies near the natural period of the pole oscillation can exert a strong influence on the pole oscillation. Thus for all mass transports that interest us, α/ω is a small number; for the meteorological mass transports, for example, it is equal to 1/365. It follows that *for these mass transports the direct action is considerably smaller than the indirect action*, so that we can strike λ and μ in equations (14) with respect to ωe and ωf. At the same time, we can also strike df/dt with respect to ωe and de/dt with respect to ωf, since the ratio of these pairs of quantities is again determined by the value α/ω. Equations (4) simplify, on the basis of these omissions, to

$$\mathsf{\Lambda}' = -\,\omega^2 e, \quad \mathsf{M}' = +\,\omega^2 f,$$

where we can also write, according to equations (2), if we introduce the angular deviations of the principal axes,

$$\mathsf{\Lambda}' = -\,\omega^2 (A - C)\eta, \quad \mathsf{M}' = +\,\omega^2 (A - C)\xi. \tag{15}$$

This simplification is derived here on the basis of a very special assumption about the mass transport. One easily sees, however, that analogous conclusions are also possible for a more general assumption in which one gives each of X, Y, Z by a Fourier series, to whose first terms we have restricted ourselves above; for the terms with a longer period (that is, long compared to the period of the rotation of the Earth), the indirect influence caused by the change of the mass distribution dominates, while for the terms of shorter period (that is, short compared to the free oscillation of the Earth's axis), the direct influence on the impulse as well as the indirect influence on the Earth's rotation will be insignificant. The direct influence may be decisive only for sudden mass transports, and it indeed remains doubtful whether such mass transports with significant strength are present in reality.

If we insert the values (15) into our differential equations (12), then these equations become

$$\begin{cases} \dfrac{dp}{dt} = \dfrac{A - C}{A}\,\omega(q - \omega\eta), \\[2ex] \dfrac{dq}{dt} = \dfrac{C - A}{A}\,\omega(p - \omega\xi). \end{cases} \tag{16}$$

Here we wish to introduce, in addition to the coordinates ξ, η of the inertia pole, the similarly measured coordinates of the rotation pole, which may be called u, v. The coordinates u, v signify the direction cosines of the rotation axis with respect to the coordinate axes X and Y, and are therefore equal to p/r and q/r, respectively, where we can also take, with sufficient accuracy, p/ω and q/ω. If we also use, as in equation (8), the abbreviation β for the frequency of the free oscillation of the axis of the Earth, then our equations become

$$\begin{cases} \dfrac{du}{dt} = -\,\beta(v - \eta), \\[2ex] \dfrac{dv}{dt} = +\,\beta(u - \xi). \end{cases} \tag{17}$$

These equations simply state *that the rotation pole will rotate at each instant about the inertia pole with angular velocity β*. The sense of the rotation coincides with the sense of the rotation of the Earth; the coordinate system is imagined to be chosen so that the positive X-axis is transformed into the positive Y-axis along the shortest path by the rotation of the Earth.

For the purpose of integration, we combine equations (7) in complex form as

$$\frac{d(u+iv)}{dt} = i\beta\big((u+iv)-(\xi+i\eta)\big), \tag{18}$$

and assume that the inertia pole executes an elliptical oscillation about its mean position as a result of the mass transport. We can then make for $\xi + i\eta$, as previously for $\mathsf{\Lambda} + i\mathsf{M}$, the assumption

$$\xi + i\eta = ae^{i\alpha t} + a'e^{-i\alpha t}. \tag{19}$$

There corresponds as the particular integral of (18), which represents the oscillation that is *forced* by the mass transport (we can disregard in the following the *f r e e* oscillation with the period $2\pi/\beta = 14$ months),

$$\begin{cases} u+iv = be^{i\alpha t} + b'e^{-i\alpha t}, \\ b = \dfrac{a\beta}{\beta-\alpha}, \quad b' = \dfrac{a'\beta}{\beta+\alpha}. \end{cases} \tag{20}$$

Equations (20) represent, just as equation (19) does, an elliptical oscillation. In order to conveniently visualize the relative position and magnitude of the two ellipses, we can imagine the coordinate directions chosen so that they coincide with the principal axes of the ellipse (19). Then a and a' are real, and, according to (20), b and b' are also real. If we name the principal axes of the two ellipses h, k and H, K, respectively, then we can write, instead of (19) and (20),

$$\xi + i\eta = h\cos\alpha t + ik\sin\alpha t, \quad h = a + a', \quad k = a - a', \tag{19$'$}$$

$$u + iv = H\cos\alpha t + iK\sin\alpha t, \quad H = b + b', \quad K = b - b'. \tag{20$'$}$$

One thus recognizes that the directions of the principal axes of the two ellipses coincide; for what concerns their magnitude, it follows from (20) and the definitions of h, k, H, K that

$$H = \frac{\beta(\beta h + \alpha k)}{\beta^2 - \alpha^2}, \quad K = \frac{\beta(\beta k + \alpha h)}{\beta^2 - \alpha^2}. \tag{21}$$

It is noted that H, K are to be calculated with signs, and that one must also, if necessary, bestow on h a negative sign in order to express by (19$'$) the correct rotation sense of the inertia pole. The inversion of equations (21) gives

$$h = H - \frac{\alpha}{\beta}K, \quad k = K - \frac{\alpha}{\beta}H. \tag{22}$$

We may first give some numerical examples and figures. We assume that the relevant mass transport is of meteorological origin, and thus has the period of one year. In order to include the elasticity of the Earth into the calculation (cf. page 713), we regard the period of the free oscillation as the Chandler period. Then α/β is approximately equal to 7/6. The inertia pole may execute a linear oscillation, and therefore, for example, $h = 0$ and $\eta = k \sin \alpha t$. From (21) there follows

$$H = -\frac{42}{13}\,k = -3{,}2\,k; \quad K = -\frac{36}{13}\,k = -2{,}8\,k,$$

and from (20′),

$$u = -3{,}2\,k\,\cos\alpha t, \quad v = -2{,}8\,k\,\sin\alpha t.$$

This case is illustrated in Fig. 107a. We have designated the corresponding points of the inertia pole and the rotation pole (that is, points occupied at the same time) with equal numbers.

The previous assumption with respect to the trajectory of the inertia pole is retained in Fig. 107b. We have chosen, however, the Euler period

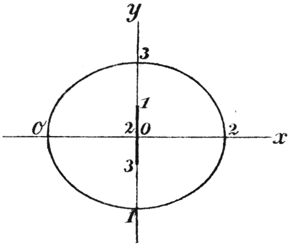

Fig. 107 a.

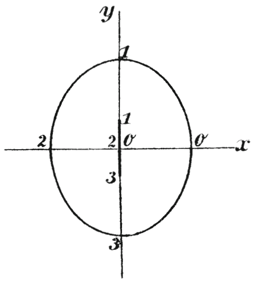

Fig. 107 b.

as the period of the free oscillation, as would be the case for an absolutely rigid Earth. Then $\alpha/\beta = 5/6$, and

$$H = \frac{30}{11}\,k = 2{,}7\,k; \quad K = \frac{36}{11}\,k = 3{,}3\,k,$$
$$u = +2{,}7\,k\,\cos\alpha t, \quad v = +3{,}3\,k\,\sin\alpha t.$$

Both Fig. 107a and Fig. 107b express the repeatedly emphasized resonance effect, due to which the motion of the rotation pole will be essentially more extensive than that of the inertia pole when the periods of the free and forced oscillations are not very different. That the rotation pole is found on the side opposite the inertia pole in Fig. 107a and on the same side in Fig. 107b (has opposite or equal phase) corresponds to a general rule of oscillations: the opposite phase always occurs in the case $\alpha > \beta$, and the same phase in the case $\alpha < \beta$. The passage between the ellipses is mediated by the case $\alpha = \beta$, where our ellipse (see equations (21)) is transformed into a circle of infinitely large radius. In the case $\alpha = 0$ (infinitely long period, or secular mass transport) the elliptical oscillation degenerates into a linear oscillation, since (cf. (21)) $H = 0$, $K = k$; the rotation pole then follows exactly the path of the inertia pole. In the case $\alpha = \infty$ (infinitely rapid oscillation), the rotation pole may not at all follow the action of the mass transport; here, according to (21), $H = K = 0$. If one imagines in Fig. 107b that the degenerate line of the inertia pole is transformed into the infinite circle through a continuous series of broadened ellipses, to which the ellipse of the rotation pole constructed in this figure also belongs, and in Fig. 107a that this infinite circle is transformed through a continuous series of narrowed ellipses, of which one coincides with the ellipse drawn in this figure, then one has the complete image of the possible trajectories of the rotation pole for arbitrary values of the ratio α/β.

The relations are no longer so transparent, however, if we assume that the trajectory of the inertia pole itself is elliptic, and therefore add to the just considered linear oscillation a second oscillation perpendicular to it and shifted in phase. It can then occur, in particular, that the resonance will be concealed, to a certain extent, by interference; the multiplicity of the relative positions of the two ellipses that are possible according to equations (21) will then become extraordinarily large. —

After dispatching the three posed problems, we now return to the actual conditions that are present for the Earth, and, in particular, to the air mass transport that was discussed at the beginning of this section. As Mr. Spitaler calculates on the basis of the air pressure maps (by mechanical quadrature over the surface of the Earth), air transport will deflect the inertia pole

in January by $0'',055$ toward $100°$ west of Greenwich,

in July „ $0'',041$ „ $68°$ east of Greenwich.

The inertia pole thus moves at these two times by approximately equal angles toward approximately opposite meridians. The amplitudes for April and October are not calculated, but rather only estimated; they are directed approximately toward the meridians 180° and 0° and are presumably smaller than those given above. The inertia pole therefore runs in the direction from east to west; that is, in the opposite sense from the rotation of the Earth. The exact form of the trajectory may not be established from these data, and it is thus not possible to determine the corresponding path of the rotation pole.

The reverse path, in contrast, is passable. According to page 682, the rotation pole oscillation of yearly period is described as an ellipse with principal axes $0'',104$ and $0'',044$, whose major axis is directed toward the meridian 19° east of Greenwich, and which is traversed in the sense of the rotation of the Earth. We thus set $H = 0'',104$, $K = 0'',044$, and calculate, according to equations (22) with $\alpha/\beta = 7/6$,

$$h = 0'',053, \quad k = -0'',077.$$

The so-determined ellipse will (due to the sign of k) be traversed in the sense opposite to that of the previous ellipse; the positions of the major and minor axes are opposite to those of the previous ellipse.

The ellipse H, K of the rotation pole and the theoretically corresponding ellipse h, k of the inertia pole are drawn in Fig. 108. Corresponding positions of the two ellipses are marked by the same month names. Further, the position of the inertia pole and its sense of motion according to the calculations of Spitaler for the times January and July are registered in the figure. The relevant points are made recognizable by small circles. One sees from the figure that a general agreement is present between these points and the theoretically determined simultaneous positions of the inertia pole, at least in order of magnitude. The actual differences in their positions can be explained either by our still rather complete ignorance of the arctic air pressure values, or by the fact that other meteorological processes (water transport, etc.) in addition to air transport influence the yearly trajectory of the rotation pole.

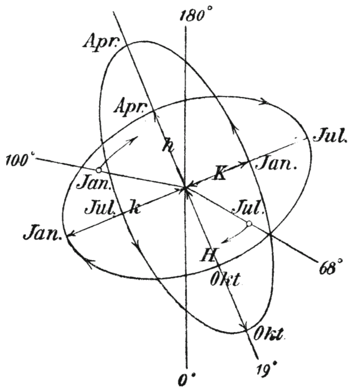

Fig. 108.

All in all, there is a good basis for the assumption that it will be possible, with further enrichment of the observational data, to satisfactorily explain the yearly component of the pole oscillations from meteorological mass transports.

Not as favorable stand the prospects for the explanation of the remainder of the aperiodic pole oscillations that are represented in Fig. 106. Secular mass transports of reasonably probable amounts give, at most, only very small effects on the inertia and rotation poles.*) Also, the general form of Fig. 106, in so far as we can speak of it with real significance, gives the impression that it consists of aperiodic pole oscillations of longer or shorter duration, followed by perturbations acting in the reverse sense.

Perturbations of this character would be produced, in our previous terminology, by the direct influence on the impulse of the rotational motion if a mass transport on the Earth occurred rather suddenly and then came again to rest, so that the impulse of the mass displacement is first generated and then annihilated, and the corresponding impulse change of the Earth's rotation occurs first in one and then in the opposite sense. Since we generally have, however, no indication for the assumption that such mass transports of sufficient strength are possible on the Earth, we consider it useless to carry out the just indicated representation further. —

With respect to the general analytic developments of this section, it is emphasized that for the treatment of the body of the Earth with a variable mass distribution, the fundamental equations (3) result immediately from our conception of the Euler equations, even for the cases in which the coordinate axes are not or do not remain the principal axes of the body of the Earth. With the establishment of the chosen coordinates, we then arrived at the simplest equations (17) by mere specialization to the particular present case. The problem is treated most thoroughly in the literature by G. H. Darwin.**) Darwin does not, as we do, adopt coordinate axes that are fixed in the body of the Earth, but rather the moving instantaneous principal axes in the body of the Earth, and likewise arrives in this manner at the final equations (17). The calculations at the basis of Fig. 107b were first given by R. Radau,***) on which account the ellipse of that figure is occasionally designated as the Radau ellipse. F. R. Helmert†) has discussed the relation between the ellipses of the inertia pole and the rotation pole under more general assumptions.

Our presentation of the pole oscillations would be incomplete, however, if we did not mention, in addition to the *centrifugal* effect of the

*) Cf. Tisserand, Mécanique céleste II, Chap. 29, art. 208 and Chap. 30, art. 218.

**) G. H. Darwin: On the influence of Geological Changes on the Earth's Rotation. London, Phil. Trans. 167 (1877), with an appendix by Lord Kelvin.

***) R. Radau, Comptes Rendus 111 (1890) and Bulletin Astronomique 7 (1890).

†) F. R. Helmert, Astronom. Nachr. 126 (1891), Nr. 3014.

mass transports on the rotation pole, certain *centripetal* tendencies that are caused by the appearance of frictional influences, and which, in a certain manner, can calm and simplify the motion of the rotation pole, just as the previous influences disturb and complicate it.

We think, in the first place, of the friction associated with the *tides*, and, in particular, the usual tides produced by the Moon or Sun attraction. Immanuel Kant emphasized the presence of such friction as early as 1754, and thus deduced the necessity of a secular elongation of the sidereal day.[258] We need not discuss in detail how this friction occurs;*) the following somewhat grotesque representation suffices for our purpose. Two diametral tidal bulges accumulate in the water that covers the surface of the Earth; the Earth continues to rotate under them, while the tidal bulges themselves stand still, or, according to the measure of the motion of the Moon, slowly change their relative position. Through the viscosity of the water, the bulges apply a turning-moment on the Earth that opposes its rotation. If the moon stood exactly fixed in the instantaneous equator of the Earth and the symmetry of the tidal motion were not disturbed by the continents, the axis of the turning-moment would coincide with the instantaneous rotation axis, and its magnitude would be proportional to the magnitude of the rotation. We wish to regard this simplest imaginable determination of the turning-moment of the tidal friction as approximately and in the mean valid. We can, for example, compare the two tidal bulges with the two shoes of a railroad brake that are applied to the rotating wheel and slow its rotation.

The further consequence of the influence of tidal friction is thus reduced to a top problem that was already treated in Chap. VII, §7 as the problem of air resistance: a force-free top stands under the influence of a turning-force whose axis is the instantaneous rotation axis, and whose magnitude is negatively proportional to the instantaneous rotation. We saw that the rotation of such a top is gradually annihilated, and that, at the same time, the rotation axis spirals asymptotically to the axis of the greatest principal moment of inertia (cf. page 588 and the figure of page 589). For the Earth, the axis of the greatest moment of inertia is

*) Cf. Chaps. 16 and 17 of the work of G. H. Darwin for further literature. In particular, the astonishing cosmological effects of tidal friction are indicated there.[259]

the polar principal axis. It may thus appear that tidal friction is a countervailing and damping effect on the pole oscillations, and that we owe it to this effect that, in spite of temporary perturbations, the rotation pole remains, empirically, so near in the mean to the inertia pole.

However, a numerical calculation shows that this effect is entirely to be neglected. We begin from equations (1) and (6) of pages 587 and 588. In equation (6), β signifies the angle that is enclosed by the instantaneous rotation axis and the axis of the greatest principal moment of inertia at time t, and β_0 is the same angle at $t = 0$. If we restrict ourselves to small angles β and β_0, we can write equation (6) as

$$\frac{\beta}{\beta_0} = e^{-\lambda t\left(\frac{1}{A} - \frac{1}{C}\right)} = e^{-\frac{\lambda t}{C}\varepsilon},$$

where ε (approximately equal to 1/300) denotes, as previously, the ellipticity of the Earth. According to the cited equation (1), on the other hand,

$$\frac{r}{r_0} = e^{-\frac{\lambda t}{C}}.$$

The two equations combined give

$$\frac{\beta}{\beta_0} = \left(\frac{r}{r_0}\right)^{\varepsilon}.$$

Thus in the time that the tidal friction reduces an originally present deviation β_0 of the rotation axis to half its amount $\left(\beta = \frac{1}{2}\,\beta_0\right)$, it reduces, at the same time, the originally present rotation of the Earth r_0 by the fraction

$$\left(\frac{1}{2}\right)^{1/\varepsilon} = 2^{-300} = \frac{1}{2} \cdot 10^{-90}$$

of itself. In other words, *the rotation of the Earth must, due to tidal friction, have come as well as completely to rest before half of the originally present deviation of the rotation axis is canceled.* In such a manner, it is understood that tidal friction does not at all come into consideration for the question of the pole oscillations (equally little as the mass transports of the usual Moon and Sun tides; cf. page 714), and also cannot (cf. page 593) be drawn upon for the explanation of the secular changes of the rotation axis, as has frequently been postulated in geology.

There is, however, yet another type of flow and another type of flow friction that may deflect the rotation pole more effectively back toward the inertia pole; namely, the flow that is produced by the pole oscillation itself (cf. page 684, where we discussed, in particular, the fourteen-month component of this flow). This flow also will be accompanied by friction, and indeed one can assume here that the friction acts against the *change of the rotation axis*, and that its axis stands *perpendicular* to the rotation axis, while the friction for the usual Moon and Sun tides depends on the instantaneous magnitude of the *rotation itself*, and its axis *coincides* with the instantaneous rotation axis.

If we wish to form the simplest possible, if again somewhat rough, representation for the formation of this flow, then we can speak in the following manner. The position of the rotation axis in the body of the Earth is specified at a given time by the quantities p, q, r; this position corresponds, if the effect of the continents is disregarded, to a disposition of the water covering in which a fluid belt is formed about the instantaneous equator that is perpendicular to the rotation axis. At a following point of time, the position of the rotation axis will be given by $p + p'\,dt$, $q + q'\,dt$, $r + r'\,dt$; the fluid belt now lies about the new equator, and is rotated with respect to its previous location. We transform the belt from its first to its second position if we rotate it about the common perpendicular to the first and second positions of the rotation axis, and indeed through an angle that is equal to the deviation angle of the rotation axis. The fluid friction opposes this rotation; we assume for simplicity that the moment of the fluid friction acts about the same axis and is proportional to the magnitude of the rotational velocity. The axis of the fluid friction is then calculated by the subdeterminants of the schema

$$\begin{vmatrix} p & q & r \\ p' & q' & r' \end{vmatrix}.$$

The components of the fluid friction will thus be proportional to the expressions

$$qr' - rq', \quad rp' - pr', \quad pq' - qp'.$$

If we consider that the quantities p, q, p', q', r' are small, and that r is approximately equal to ω, then we can write, with the omission of small quantities of the second order,

$$-\omega q', \quad \omega p', \quad 0.$$

With the use of a positive factor of proportionality λ, we correspondingly set the components of the fluid friction equal to the quantities

$$-\lambda A q', \quad +\lambda A p', \quad 0.$$

In fact, one easily recognizes that through this assumption, the above agreements on the magnitude, axis, and sense of the frictional moment will be answered if the rotational velocity of the Earth ω is calculated as positive, and the coordinate axes therefore have the position given on page 720. The moment of inertia A was added to the previous expressions as a factor so that the quantity λ has the dimension of a pure number, as is convenient for the sequel.

In order to determine the influence of this fluid friction on the pole oscillations, we return to the Euler equations, to which we add, on the right-hand side, the just-determined components of the fluid friction. In the first approximation, the equation for the component r will not be changed. We can thus regard this component as a constant even with the consideration of friction, and set it equal to ω; in other words, the length of the sidereal day will not be increased by the friction now in question, within the bounds of precision established by us. The Euler equations for the components p and q of the rotation vector are, if we disregard the perturbation of the motion by the mass transport and consider only the free oscillation of the Earth's axis,

$$Ap' = (A - C)\omega q - \lambda A q',$$
$$Aq' = (C - A)\omega p + \lambda A p'.$$

For the purpose of integration, we combine these equations, in the frequently written manner, into the complex equation

$$A(p' + iq') = (C - A)i\omega(p + iq) + i\lambda A(p' + iq'),$$

where we can also write, with the introduction of the ellipticity ε,

$$(1 - i\lambda)(p' + iq') = \varepsilon i\omega(p + iq).$$

The number λ will, in any case, be small compared with 1, since in the opposite case a periodic pole oscillation could generally not exist. Thus we can also recast the equation without perceptible error as

$$\frac{p' + iq'}{p + iq} = \varepsilon i\omega(1 + i\lambda),$$

and integrate it as

$$p + iq = a\, e^{-\varepsilon\omega\lambda t + \varepsilon i\omega t}.$$

The quantity a is the constant of integration, which depends on the initial position of the Earth's axis; that is, on the consideration of previous perturbations.

It now follows that the friction leaves the period of the pole oscillation unchanged (unchanged up to quantities of the second order); its frequency will here too be determined by the product $\varepsilon\omega$. In contrast, the oscillation is now, because of the friction, *damped.* The damping factor for the duration of a free oscillation is, according to the preceding formula, equal to $e^{-2\pi\lambda}$. Because of this damping, the rotation pole will evidently approach the inertia pole; it is also clear that the previously emphasized resonance effect will be mitigated, so that the amplitude of the rotation pole will no longer become infinitely large for coincidence of the free and forced oscillations, but rather will take on a finite value determined by the value of the damping factor.

We are unfortunately in complete uncertainty concerning the numerical magnitude of the damping, and in particular concerning the damping constant λ. Since we were previously able to say nothing about the magnitude of the relevant tides (cf. page 706), it will be even less possible to numerically estimate the magnitude of their frictional effect.

We wish to remark that the deformation of the Earth discussed in the previous section is very probably accompanied by an energy loss, and will thus likewise provide a contribution to the damping of the free oscillation. At least there is no elastic body known to us in which an initially generated deformational oscillation does not soon expire; we attribute this circumstance to the occurrence of interior frictional processes or elastic aftereffects. It would be most highly unphysical to assume that it should be otherwise for the body of the Earth. As a result, it appears fitting to consider interior friction of the Earth, in addition to tidal friction, as a possible cause of damping for the pole oscillations.

Until the present time, the damping effect of the different possible energy losses (which is indeed properly considered in analogous cases for other mechanical problems*)) has always been neglected in the calculational treatment of the pole oscillations, in that the pole trajectory is represented by a Fourier series that progresses in pure, undamped trigonometric functions of time (cf. the citations to *Chandler* on page 673 and *van de Sande Bakhuyzen* on page 682). Our graphical reduction of the pole trajectory in §6 of this chapter is also based on this assumption, and is to be modified if we would consider damping, or if we wished to identify the magnitude of the damping as well as the different periods

*) Cf., for example, R o u t h, Dynamik starrer Körper, Bd. II (German edition, Leipzig 1898) Kap. VII §331–333.

hidden in the experimentally observed pole trajectory. Since a predictive theoretical calculation of the damping constant λ seems rather hopeless, one should perhaps seek, in the manner just indicated, to obtain a result from the pole oscillation itself.

The preceding representation of the effect of the pole oscillation tides is naturally quite idealized; because of the influence of the continents on the tidal motion, the phenomena will be much more complicated in reality. It is thus desirable to determine the effect of an arbitrary energy-dissipating circumstance, or at least its sense, by an entirely general consideration, without particular assumptions, that also includes the case of deformation friction in the interior of the Earth.

For the pole oscillation and the flows and deformations generated by it, as well as for the corresponding flow friction and deformation friction, only internal forces come into play. These forces leave the total impulse of the mass system, which we call the Earth, unchanged (in contrast to the previously considered flow friction that is produced by the external attractive forces of the Sun and Moon). For the components of the total impulse, the equation

$$L^2 + M^2 + N^2 = \text{const.}$$

thus obtains, which we can interpret as the equation of a sphere. On the other hand, the *vis viva* of the system will be diminished by friction, in that a portion of the *vis viva* will be transformed into heat. If we permit ourselves to carry over the expression for the *vis viva* of the rigid top to our moving system, then we can write

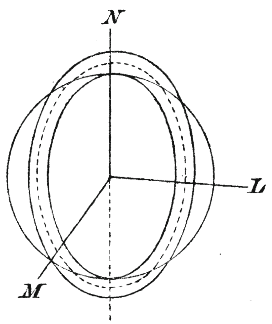

Fig. 109.

$$\frac{L^2 + M^2}{A} + \frac{N^2}{C} = 2T,$$

and can interpret this equation, for any value of T, as an ellipsoid of revolution in the coordinates L, M, and N. And indeed this equation represents a prolate ellipsoid of revolution (because $C > A$), which, remaining similar to itself, gradually contracts (because of the gradual decrease of T). The endpoint of the impulse vector L, M, N must lie on the intersection curve of the two surfaces (sphere and ellipsoid); this intersection curve contracts, however, to a point on the N-axis as our ellipsoid is gradually diminished (cf. Fig. 109). The impulse vector and the rotation axis thus simultaneously approach the polar principal axis of inertia, and the state of motion goes over into a simple uniform rotation about this axis.

In so far as this deliberation is applicable to the case of tidal friction or other dissipative influences, we may claim that such influences counteract any induced disturbance of the simplest state of motion of the Earth, and that the position of the rotation pole will be stabilized on the surface of the Earth.

§9. The proof of the rotation of the Earth by the top-effect. Foucault's gyroscope and Gilbert's barogyroscope.

Having carried out in the year 1851 his brilliant pendulum experiment for the proof of the rotation of the Earth, Léon Foucault undertook in the following year to accomplish the same purpose with the top-effect. He used a top in a Cardanic suspension (cf. the schematic Fig. 2 of page 2), whose individual components (rotor, inner and outer rings) were adjusted with great care, so that the intersection of their axes of rotation was simultaneously the center of gravity of each component. Foucault's experimental disposition was twofold: in the first case,*) he gave the top *three degrees of freedom*, in that he let the outer ring rotate on conical bearings about a vertical axis. These bearings served not so much for supporting the top as for the prevention of its lateral motion; the weight of the top was borne by a torsionless thread from which the outer ring was suspended. The inner ring was supported by knife edges that rested on bearing surfaces fixed to the outer ring.[260] In the second case,**) he held the inner ring fixed with respect to the outer, and thus operated with a top of only *two degrees of freedom*, which, due to its attachment to the Earth, was constrained to move in a certain manner.

According to Foucault, the rotor of the top with three degrees of freedom and a strong rotation retains *the original direction of its axis in absolute space*; or, otherwise expressed, this axis is constantly directed *to the same point of the firmament.* As seen from the Earth, therefore,

*) Sur une nouvelle démonstration expérimentale du mouvement de la Terre, Comptes Rendus t. 35, Paris 1852, p. 421.

**) Sur le phénomènes d'orientation des corps tournants entraînés par une axe fixe à la surface de la Terre—Nouveau signes sensibles du mouvement diurne. l. c. page 424.[261]

each of its points moves parallel to the direction of the equator. Geometric considerations of the simplest kind then show the correctness of the following statements.

If the axis points toward the zenith at the beginning of the experiment, then it forms the angle $\omega \cos \varphi \, \Delta t$ (ω = the angular velocity of the rotation of the Earth, φ = the geographic latitude) with the plumb line after the observation interval Δt, since the original zenith describes this arc about the pole of the heavens in the same time interval. If, on the other hand, the axis of the rotor is originally horizontal and in the direction of the meridian, then it remains horizontal for a sufficiently short observation time, and forms the angle $\omega \sin \varphi \, \Delta t$ with the meridian after the time Δt, since a star on the horizon in the direction of the meridian has the polar angle φ (or $\pi - \varphi$) and describes an arc $\omega \sin \varphi \, \Delta t$ in the horizontal direction during the time Δt. The same expression $\omega \sin \varphi \, \Delta t$, which, moreover, also appears for the Foucault pendulum, is valid for the horizontal component of the angular change for an arbitrary horizontal initial position of the rotor axis. If we ask, namely, for the apparent motion of a star on the horizon at an arbitrary azimuth, then this motion consists of a rotation ω about the polar axis, which we can decompose into a rotation $\omega \sin \varphi$ about the plumb line and a rotation $\omega \cos \varphi$ about the meridian. The former component produces the horizontal motion of the star, which will thus amount to $\omega \sin \varphi \, \Delta t$ during the observation time Δt; the latter component gives the elevation change of the star. The former component, and thus also the horizontal motion of the axis of the top, is independent of the azimuth of the initial position.

The experimental disposition of Foucault is tied to the latter circumstance. One notices that for a horizontal initial position, the motion of the axis of the top is decomposed by the Cardanic suspension itself into its two components; the motion of the outer ring reproduces the horizontal component of the motion of the axis of the top, while the motion of the inner ring determines the change in the elevation of the axis of the top.[262] Foucault used a microscope to observe the outer ring, whose rotation should be equal to $\omega \sin \varphi \, \Delta t$. Foucault gives 8 to 10 minutes as the largest possible value of the observation time. If we therefore calculate the expected deflection with $\Delta t = 8$ min. and $\varphi = 49^\circ$ (the approximate latitude of Paris), there follows, in degree measure,

$$\omega \sin \varphi \, \Delta t = \frac{360 \cdot 8}{24 \cdot 60} \, 0{,}75 = 1^\circ{,}5.$$

It must surely be possible to detect this rather considerable rotation under the microscope.

This contrasts, to a certain extent, with the circumstance that Foucault speaks only of the direction of the rotation, which resulted correctly in his experiment, and therefore gave the direction of the rotation of the Earth in the opposite sense, but does not give numerical values for his observations. We know not to what extent these observations coincide with the theoretical values. As long, however, as a quantitative agreement is not proven, or as long as the sources of error that cause the disagreement are unknown, the experiment can hardly be claimed as an irrefutable proof of the rotation of the Earth; it could indeed be, in the present case, that the sources of error influence the deflection more strongly than the rotation of the Earth itself, and that the correct sense of the result is produced only speciously by an accidental grouping of the various errors.

Imperfect centering of the apparatus and friction in the bearings are particularly important sources of error here. For the *pendulum experiment* of Foucault and Gaufs, all sources of error have been examined quantitatively in an exemplary manner by K a m e r l i n g h O n n e s;*)[263] for the Foucault *top experiment*, in contrast, such an examination has never been undertaken.

The great historical significance of the Foucault top experiment thus appears to us to lie less in the proof of the Earth's rotation than in the fact that this experiment attracted general attention to the top-effect, and that the awareness of the top-effect was essentially advanced by the brilliant, formula-free immediateness of the Foucault conception.[264]

Before we examine the theory of this experiment critically, we first wish to report in more detail on the second experimental disposition of Foucault, the *top with two degrees of freedom.* The axis of the rotor is now no longer fixed in space; *the axis rather seeks, according to Foucault, to place itself as parallel to the axis of rotation of the Earth as the particular circumstances of the experiment permit.* Foucault thus speaks of the *tendency of the rotation axes to parallelism*;**) parallelism of the axes is understood not only as coincidence of the direction of the axes,

*) Dissertation Groningen 1879. Nieuve bewijzen voor de aswenteling der aarde.

**) Sur la tendance des rotations au parallélisme. Comptes Rendus l. c. p. 602.

but rather, at the same time, coincidence of the rotation sense—one can thus speak, more precisely, of the *tendency to equi-orientational or homologous parallelism.*

At approximately the same time as Foucault, G. Sire*) made the same law the subject of a communication to the Paris Academy, and applied it to the proof of the rotation of the Earth, without carrying out an experiment himself. The theoretical deliberations of Sire on which the law is based, however, are not unobjectionable, since they suffer a certain ambiguity of the word axis (figure axis, rotation axis, impulse axis). A somewhat similar objection may be made against the brilliantly written works of Foucault; these works are intended, because of their brevity, to be more of a descriptive than a demonstrative nature. (Cf. our critique of the popular top literature in Chap. V, §3 sub 2.)

Foucault attempted to use the named law as a basis for the behavior of the rotor axis in the two special cases in which the rotor axis was free to move only in the horizontal plane or only in the vertical plane through the meridian of the observation location. The restriction of the motion is accomplished in both cases by clamping the inner ring at a right angle with respect to the outer ring. In the first case, the rotation axis of the outer ring was placed in the direction of the plumb line; in the second case, the rotation axis of the outer ring was placed perpendicular to the meridional plane of the observation location.

In the first case, where the axis of the rotor cannot leave the horizontal plane, an actual parallelism between it and the axis of the Earth is not possible; the axis of the rotor then strives to the direction that forms the smallest angle with the axis of the Earth; that is, the direction of the meridian. And indeed, the side of the axis from which the rotor is seen to turn in the counterclockwise direction points to the north, since the Earth rotates about the north pole in the same sense. *Our horizontally mobile rotor thus behaves similarly to the magnetic needle in a declination compass* (naturally with the difference that in Foucault's experiment the astronomical meridian takes the place of the magnetic meridian). In association with this analogy, we can denote the side of the axis from which the rotor is seen to turn in the counterclockwise sense as the *north pole* of the rotor, and the opposite side as the *south pole.*

In the second case, in which the axis of the rotor is free to move in

*) A later publication of Sire is found in the Bibliothèque universelle de Genève, Arch. d. scienc. phys. et natur., t. 1 (1858), p. 105.

the meridional plane, exact parallelism of the rotor axis with the axis of the Earth is not only sought, but rather (for a sufficiently long maintenance of the rotor rotation) is achieved. The axis of the rotor moves, if it is initially horizontal, in such a manner that, in the northern hemisphere, its "north pole" rises up out of the horizontal plane, and the line that connects the north and south poles of the rotor is directed parallel to the line that connects the north and south poles of the Earth. *Our rotor axis that moves in the meridional plane can thus be compared with the magnetic needle in an inclination compass* (naturally again with the difference that the exact geographical latitude lines take the place of the quite irregular lines of equal inclination on the surface of the Earth). An essential difference is *that the "north pole" of the rotor ascends in the northern hemisphere, while the north pole of the inclination needle descends.*[265]

The theoretical possibility thus exists, as Foucault emphasizes, of deriving the position of the meridian and the position of the Earth's axis at an arbitrary location merely from observations of the top, without astronomical or magnetic observations. It is obvious that the approach of the axis of the top to the meridian or to the direction of the Earth's axis must occur not monotonically, but rather with oscillations. The turning-force that leads the horizontally mobile axis of the top to the meridian, for example, produces a certain angular acceleration and angular velocity about the vertical axis. While the turning-force vanishes for the meridional position of the axis of the top, the angular velocity does not vanish at the same time. This leads the axis of the top across the equilibrium position, so that the restoring force is reversed in sense, and acts to first decelerate and then accelerate in the reversed sense. The axis of the top must therefore oscillate about the meridian—likewise in analogy with the magnetic needle. If the axis of the top is initially in the direction of the meridian, but so that its "north pole" points to the south, then this position is also an equilibrium position, since the turning-force on the axis of the top vanishes, but is evidently unstable: for a small deviation from this position, the turning-force strives to make the deviation larger, and the north end of the axis turns over to the north.

Foucault disputed that the position of the meridian or the location of the Earth's axis could be determined with sufficient accuracy in the prescribed manner. It appears that Foucault also examined the second experiment more for its general possibility than for its exact implementability. —

We mention further that Foucault coined, in association with his experiments, the now commonly used word *gyroscope.* This word expresses in a forceful way the result of the Foucault experiment; *namely, that the top is a means to make an existing rotational motion* (or gyration) *recognizable*, just as the electroscope denotes a means for making the presence of electrical charges visible. If it were also possible to establish the magnitude of an existing rotational motion through a quantitative measurement of the motion of the top, then one might even bestow to the top the more far-reaching designation of a "gyrometer."

On the other hand, it appears to us inappropriate to generalize the designation "gyroscope," and to use it as an equivalent to the word top, as is frequently done in the literature. In fact, the designation gyroscope expresses only one particular application of the multifaceted interest and importance of the concept of the top, and there is no reason to abandon the characteristic designation *top* (*turbo, toupie, Kreisel*). —

We must now deepen the theory of the Foucault experiments; we begin with the top of three degrees of freedom. It is far from our desire to accompany this experiment with extensive analytic developments from the theory of relative motion,*) developments whose final result, according to their fundamental assumptions, can be nothing other than the confirmation of Foucault's statement that the axis of the top essentially retains its position in absolute space. The difficulty and obscureness of these developments have their basis only in the fact that it is not always assumed that the gyroscope is vanishingly small with respect to the Earth, or that its rotational velocity is infinitely large compared to

*) These developments are treated, for example, in the works of Quet (Liouville's Journal t. 18 (1853)), Lottner (Crelle's Journal Bd. 54 (1857)), and Bour (Louiville's Journal (2) t. 8 (1863)).[266] A summary is given by Gilbert: Étude historique et critique sur le problème de la rotation (from the Annales de la Société Scientifique de Bruxelles, t. 2 (1878)). We further mention the great work of Gilbert: Mémoire sur l'application de la méthode de Lagrange à divers problèmes du mouvement relatif (ibidem, t. 6 and 7, 1881–1883) and the work of Budde: Allgemeine Mechanik der Punkte und starren Systeme (Berlin 1890, 1891, Bd. 2, Nr. 294).

that of the Earth. We refer for this matter to the relevant critique of E. Guyou,*) to which we would add only that one may neglect the inertial effect of the inner and outer rings compared to that of the rotor with the same enormous degree of approximation with which one neglects the rotational velocity of the Earth compared to that of the top, as we will discuss below.[267]

We first wish to assume explicitly that it may be permitted to consider the outer and inner rings as *massless*, and *to disregard the effects of friction.* We then have to speak of the rotor alone. The rotor is free to move about its center of gravity and is free of external forces, since the gravitational force applied at the support point does not come into consideration. The motion of the rotor thus consists, generally speaking, of a *regular precession* with respect to absolute space. The motion of the center of gravity due to the rotation of the Earth in no way influences this rotational motion. For the motion of the center of gravity and the rotation about the center of gravity are, as is well known, two processes that are superposed smoothly in the absence of external forces, without disturbing one another in any way.

This would also hold if the center of gravity were not led approximately uniformly in a straight line, as it in fact is by the rotation of the Earth in the time duration of a few minutes, but rather *if the center of gravity were led in an arbitrary manner along a path with an arbitrarily sharp curvature.* Indeed, it would be valid not only for a rapid spin of the rotor, but rather equally well *for a nonspinning rotor*, always under the assumption of the frictionlessness of the guidance and the masslessness of the rings. In fact, the force-free motion of the symmetric top is a regular precession for an arbitrary magnitude of the eigenrotation; the greater or lesser eigenrotation that is imparted to the top merely determines whether the resulting precession cone has, for a given lateral impact, a smaller or larger opening angle. Were it attained that the axis of the rotor stood instantaneously at rest in absolute space at the beginning of the experiment, then its direction with respect to absolute space would be precisely retained, and the opening angle of the precession cone would be and remain exactly zero, completely independently of whether the rotor

*) Sur une solution élémentaire du problème du gyroscope de Foucault, Comptes Rendus, t. 106, Paris 1888, p. 1143.

turned or not and whether the center of gravity of the apparatus moved or not; for after we have assumed away friction, there is nothing that can cause the initially stationary axis of the rotor to rotate. We would thus have the stability of the axis of the top that is claimed by Foucault, without the large eigenrotation that is regarded by Foucault as an indispensable means.

The just-assumed initial state of the axis of the top is not, however, to be achieved experimentally. The experimenter can judge the initial state of rest of the axis of the top only from the standpoint of the moving Earth; he seeks to effect not absolute rest in space, but rather rest relative to the Earth. We now assume that the latter is precisely achieved, and consider the form of motion of the axis of the top if, in particular, the top is not spun.

In the initial state of the Foucault experiment, the axis of the top is horizontal; the angle between the axis of the top and the rotation axis of the Earth is α, which is contained between φ and $\pi - \varphi$ (φ = the geographical latitude of the location), and depends on the azimuth of the axis of the top with respect to the meridian. The initial state of velocity is a rotation about the axis of the Earth with magnitude ω (ω = the rotational velocity of the Earth). This rotation is decomposed into a component $\omega \cos\alpha$ about the figure axis and a component $\omega \sin\alpha$ about an equatorial axis of the rotor. The initial impulse of the top has, with respect to these same axes, the components $C\omega\cos\alpha$, $A\omega\sin\alpha$; it forms, in the case of the rotor (oblate ellipsoid of inertia), an angle $\beta < \alpha$ with the figure axis, which angle is determined by

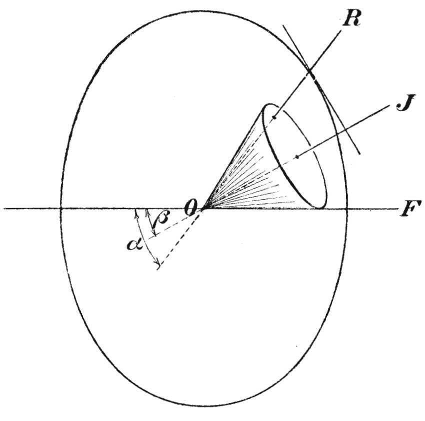

Fig. 110.

$$\text{(1)} \qquad \operatorname{tg}\beta = \frac{A}{C}\operatorname{tg}\alpha.$$

The angle β can also be found by a well-known construction (cf. page 106) that is indicated in Fig. 110. The direction of the impulse axis OJ follows from the direction of the rotation axis OR (which here coincides with the axis of the Earth) and the direction of the figure axis OF, in that one attaches to the trace of the ellipsoid of inertia in the plane ROF the tangent at the intersection point with OR, and drops the perpendicular to this tangent though O.

In the resulting motion, the figure axis describes a precession cone in space about the impulse axis with the just determined opening angle β. In contrast, a direction emanating from O that is fixed to the Earth describes, due to the rotation of the Earth, a cone about the rotation axis of the Earth. From the difference between the cones, it follows that the nonrotating top, judged from the Earth, would move, and would thus (for completely eliminated friction) function as a gyroscope in the sense of Foucault.

The assumption that the top is exactly at rest with respect to the Earth in its initial state is naturally also impermissible. Even in the most careful experiments, the axis of the top will have an initial angular velocity with respect to the Earth, which, with consideration of the smallness of the Earth's rotation, can very possibly be greater than the angular velocity corresponding to the latter. The initial impulse vector, which is determined by the composition of this angular velocity with the angular velocity of the Earth's rotation, can then have any arbitrary position, and the precession cone that the figure axis describes about this impulse vector can have any arbitrary opening angle. If, for example, the initial impact directly canceled the component of the Earth's rotation with respect to the figure axis of the top, then the eigenrotation of the top becomes accidentally zero, and the impulse vector would fall onto an equatorial axis; the precession cone would then degenerate into the plane normal to this axis. If, on the other hand, the component of the Earth's rotation with respect to the equatorial plane of the top is accidentally canceled by the initial impact, then the precession cone would become infinitely narrow and would coincide with the figure axis; the figure axis itself would then stand absolutely still in space. With consideration of such an uncontrollable small initial impulse, the further motion of the top would therefore become completely uncertain.

And one can now avoid this uncertainty if one imparts to the top an eigenrotation that is large with respect to the rotation of the Earth. A velocity of one rotation per second would already suffice, since this is $24 \cdot 60 \cdot 60$ times the velocity of the Earth's rotation. Foucault, in fact, worked with approximately this rotational velocity.*) The total impulse of the top, which is composed from this intentional

*) Cf. here the Instructions sur les expériences du gyroscope in the book Recueil des travaux scientifiques de L. Foucault, Paris 1878, p. 417.

eigenrotation, the unavoidable initial impulse, and the impulse of the Earth's rotation, will then have nearly the direction of the figure axis. At the same time, the precession cone will be so narrow that we can speak, with a precision sufficient for all experiments, of the absolute rest of the figure axis in space.

The purpose of the eigenrotation that is imparted to the top is thus to achieve, in the first place, *that the top will be free of the uncontrollable impulse of the initial actuation.* A completely determinable and well-defined motion of the top is then made possible, and *the precession cone will be made sufficiently narrow*, so that this motion will be made as simple as possible; it is perceptibly transformed, namely, into a simple rotation about the figure axis that is fixed in space. If Ω is the eigenrotation of the top, ω, as previously, the rotational velocity of the Earth, α the initial angle of the figure axis with respect to the rotation axis of the Earth, ω_0 the angular velocity relative to the Earth imparted to the rotor by unintentional impacts, and γ the angle that the figure axis forms with the axis of ω_0, then the opening angle β of the precession cone is determined, similarly to formula (1), by

$$\operatorname{tg}\beta = \frac{A}{C} \cdot \frac{\omega \sin\alpha + \omega_0 \sin\gamma}{\Omega + \omega\cos\alpha + \omega_0\cos\gamma}. \tag{2}$$

If Ω is large compared to ω and ω_0, then one has, perceptibly, $\beta = 0$. If we assume, as was done above, that ω_0 and ω are of the same order of magnitude, then we can say concisely that the opening angle of the cone will be of order ω/Ω, and that the direction of the figure axis is to be regarded as invariable if and only if one neglects quantities of order ω/Ω.

Thus the Foucault result of the stationarity of the rotor in space is confirmed as sufficiently exact under the omissions made thus far, and, at the same time, the proper role of the eigenrotation is revealed more clearly than in Foucault.

The influence of the *mass system of the suspension rings* should be studied next, or it should be demonstrated that they are without perceptible influence on the position of the rotor. As long as we assume that the suspension rings are massless, the axis of the rotor stands perceptibly still in space for a sufficient eigenrotation. Correspondingly, a diameter D of the inner ring (namely, that which coincides with the axis of the rotor) will, under this assumption, remain fixed in space. On the other hand, a diameter D' of the outer ring (namely, the diameter that falls along the plumb line of the observation location) will be constrained by its attachment to the rotating Earth to move in a completely determined manner. It is clear, however, that the motion of the system of our two rings is completely established by the motion of the two diameters D and D'; if the diameter D actually stands exactly still in space and the diameter D' exactly participates in the motion of the plumb line, then the motion of our two rings is *necessarily* determined. Their angular velocity will evidently be *of the order of magnitude of the angular velocity of the rotation of the Earth.* In more detail, we saw on page 732 that as long as the rotor deviates relative to the Earth only slightly from its horizontal initial position, the angular velocity of the outer ring will be equal to $\omega \sin\varphi$; that of the inner ring, as likewise follows from the previous deliberation, will be equal to $\omega \cos\varphi \sin\lambda$, where λ signifies the azimuth of the axis of the top with respect to the meridian of the observation location. Since the axis of the rotor subsequently maintains its position in space and is thus removed in the course of time from the horizontal of the observation location, these values of angular velocity continually change, but they always remain of the order of magnitude ω.

We now imagine that the described motion also occurs for nonvanishing masses of the rings. It is thus necessary that the rings be imparted at the beginning of the motion with the impulses $A_1\omega \cos\varphi \sin\lambda$, $A_2\omega \sin\varphi$, where A_1 and A_2 signify the moments of inertia of the inner and outer rings about one of their diameters, and that this impulse be changed in the manner that corresponds to the variability of the angular velocity. The impulse remains of the order of magnitude of $A_1\omega$ and $A_2\omega$. If we compose this impulse with the impulse of the eigenrotation of the rotor, which amounts in essence to $C\Omega$, there results a total vector that always deviates in direction and magnitude only slightly from the constant eigenimpulse of the rotor. The direction difference as well as the relative magnitude difference of the two vectors is, if we set the order of magnitude of the ratios of the moments of inertia A_1/C and A_2/C equal to 1 and thus calculate unfavorably, of the order of magnitude ω/Ω. (For the actual implementation of the Foucault gyroscope, the ratios A_1/C and A_2/C are indeed essentially smaller than 1.)

We can thus say that for our supposed enforced motion of the ring masses, as it would be determined by the stationarity of the rotor axis, there corresponds a total impulse that we may regard as constant in magnitude and direction, in so far as we neglect direction and magnitude changes of the order of ω/Ω. The total impulse therefore remains constant in the same sense and with the same degree of precision as the figure axis retains its position in space with the disregard of the ring masses. In fact, the invariability of the rotor axis was also only approximate, as was emphasized in association with equation (2), and occurs only with the neglect of quantities of order ω/Ω.

We thus conclude, in that we now go over from the previously considered enforced motion to the free motion of the mass system of the rotor and the inner and outer rings, that the free motion will be identical with the enforced, for the same choice of the initial state, in so far as we neglect differences of the order ω/Ω. For it is required in the free motion that the total impulse of the mass system remain constant in magnitude and direction. The considered enforced motion satisfies this requirement within the precision bound ω/Ω. The enforced motion thus coincides, within the same precision bound, with the natural motion of the mass system.

In other words, *the influence of the masses of the Cardanic suspension on the motion of the Foucault gyroscope with three degrees of freedom is only of the order of magnitude ω/Ω, and can in no way be detected in the observation.* It not only may, but rather must be neglected in a consistent manner, if one at all wishes to speak with Foucault of the invariability of the axis of the rotor. —

In order to avoid the occurrence of misunderstandings, we wish to emphasize explicitly that the angular change of the outer ring with respect to the Earth (or the horizontal component of the relative motion of the axis of the rotor) that is to be measured according to the experimental method of Foucault is not, in its turn, of the order of magnitude that is neglected here. This angular change amounts, namely, to $\omega \sin\varphi\, \Delta t$. Its ratio with respect to quantities of order ω/Ω is $\Omega \sin\varphi\, \Delta t$. Here $\Omega\Delta t$ signifies the rotation angle of the rotor during the observation time, and is thus an extraordinarily large multiple of 2π for a moderately rapid rotation and a typical observation time of 8 minutes. We thus recognize that the value of the gyroscopic effect to be observed is in no way obscured by the neglect of order ω/Ω. —

Friction may have a disproportionally larger influence than the masses of the suspension rings. We consider in part the friction in the guide bearings of the axis of the outer ring, and in part the resistance that is developed between the knife edges of the inner ring and their bearing surfaces on the outer ring. The investigation of this source of error, to which air resistance, air disturbances, the warming of the materials, etc. are still to be added, would certainly be more important for the actual understanding of the Foucault experiment than the unnecessarily general and mathematical considerations of relative motion that were mentioned on page 736.

We can illustrate the influence of bearing friction in the grossest and roughest measure by a simple experiment. We consider a top whose center of gravity lies at the center of a Cardanic suspension (Fig. 2). We give the rotor a strong rotation and place its axis in an initially horizontal position. We then turn the frame slowly about the vertical. If the circumstances are favorable (that is, the friction in the bearings is small, the eigenrotation is strong, and the turning of the frame is slow), the axis of the rotor seems at first to retain its initial position, and the plane of the outer ring thus remains fixed in space. This result, however, is only the consequence of an imprecise observation. For a longer maintenance of the turning of the frame, or for intentionally increased friction in the bearings, we see that the axis of the rotor is slowly uprighted and that the inner ring thus tilts, while the outer ring apparently continues to retain, in essence, its initial position. If, on the other hand, we turn the frame about the horizontal axis of the inner ring, then we again notice that the axis of the rotor seems at first to remain at rest, so that the inner ring is stationary in its original horizontal plane. If the rotation of the frame in maintained longer or if the bearing friction is intentionally increased, however, we see that the axis of the rotor deviates laterally in the horizontal plane, so that the plane of the outer ring rotates about its axis.

The basis for these motions is obviously friction. If we turn the frame about the vertical, then the bearings of the outer ring move relative to its pintles, which are approximately held fixed by the rotor, and there results a frictional moment about the axis of the *outer* ring; this first sets into motion, as we see in the experiment, not the outer, but rather the *inner* ring. If, however, we turn the frame about the previously named horizontal axis, then the bearings of the inner ring slide with respect to its pintles, and a frictional moment thus appears about the axis of the *inner* ring; this moment sets into motion not the inner, but rather primarily the *outer* ring.

The explanation of these initially paradoxical phenomena may be taken, at least qualitatively, from the theory of the heavy top. Under the influence of a frictional moment about the rotation axis of the *inner* ring, our rotor with its support point at the center of gravity behaves, mutatis mutandis, like a heavy top. For the named frictional moment has, just as gravity has for the noncoincidence of the center of gravity and the support point, the horizontal line perpendicular to the figure axis (the "line of nodes") as its axis. The consequence is a pseudoregular precession of the rotor, in which the figure axis of the rotor deflects in the horizontal plane. The plane of the inner ring thus remains in the mean horizontal, and the plane of the *outer* ring will turn. The deliberation can also be carried over in a corresponding manner to the initially considered turning of the *outer* ring about the vertical axis, and then gives a precession of the rotor in a vertical plane, and therefore a rotation of the *inner* ring. We will return to the latter case in the following chapter, where we will give a thorough theory of the relevant phenomenon for the torpedo directional guidance apparatus.

These results carry over to the Foucault experiment as follows. What for us is the frame of the top, is for Foucault the Earth. Its rotation occurs about the polar axis. We decompose this rotation into its three components with respect to the plumb line (that is, the rotation axis of the outer ring), the originally horizontal rotation axis of the inner ring, and the figure axis of the rotor. The frictional resistances that correspond to the first two rotation components act on the rotor in the manner of our experiment; one component turns the inner ring and thus deflects the axis of the rotor in the vertical plane, and the other component turns the outer ring and thus effects a horizontal deflection of the rotor axis. Both circumstances perturb the apparent motion that the spatially fixed rotor should describe, according to Foucault, with respect to the Earth. The third rotation component with respect to the figure axis does not come into consideration; the corresponding frictional moment adds to the friction generated by the eigenrotation of the rotor axis in its bearings, and is to be neglected in comparison.

There are thus various frictional influences in the Foucault experiment that perturb the absolute rest of the axis of the rotor.

There now arises the question of how one can master these frictional influences. The means is again provided by a *sufficiently high eigenrotation of the rotor.* (F o u c a u l t himself leaves us somewhat in doubt concerning the role of the eigenrotation in his experiments, as was already mentioned above.) With respect to the initial motion of the rotor axis, we saw that the eigenrotation has the result *of making the precession cone of this axis sufficiently narrow.* With respect to friction, on the other hand, we must say that the eigenrotation has the purpose *of making the precessional velocity corresponding to the various frictional influences as small as possible.* The precessional motion considered now is completely different from the previous. In the precessional motion effected by friction that we derive by analogy with the heavy top, the axis of the rotor describes a degenerate cone (or a fan) in a plane, and indeed such a cone in the horizontal or the vertical plane, according to whether we consider the frictional moment about the rotation axis of the inner ring alone or the outer ring alone. (In actuality, the motions will naturally be superposed, and there is also to be added the minimal oscillation due to the previously considered precessional motion.) Since the precessional velocity of the heavy top is equal to P/N, where P signifies the moment of gravity and N the eigenimpulse of the top [cf., for example, page 305, equation (13)], the velocity of the precession effected by the bearing friction is equal, by analogy, to M/N, where M signifies one or the other frictional moment, and N is again the eigenimpulse. Through the enlargement of N, one can, in any case, make this precessional velocity so small that the axis of the top does not deviate perceptibly from its initial position in space during an observation time of a few minutes. Indeed, we see that if it is at all permitted to disregard friction, it is permitted only for a short time duration and a sufficiently large eigenimpulse. *Even if the eigenimpulse of the frictional moment cannot be eliminated, its effect can be delayed, so that it becomes inessential for not too long an observation time.*

It is not possible, however, to determine theoretically how large the eigenimpulse must be in order to achieve this, since this depends on the magnitude of the frictional moment M, and therefore on the construction of the bearings and knife edges. Here a precise experimental investigation of the error sources must be made, which appears to be wanting in Foucault's own work. The experimental genius of Foucault guarantees us that the frictional effect M for his apparatus was very small; how small it was, however, we cannot judge from his communications.

Another difficulty of the first Foucault experiment, the necessity of a very precise centering of the rotor,*) is avoided through a happy modification of the gyroscope, the so-called *barogyroscope*, which will be discussed below.

We next go over to the second Foucault experiment (the top with two degrees of freedom), and must prove here the two interesting theorems that a) a rotor that may move in the horizontal plane behaves like a declination needle, and b) a rotor that may move in the meridional plane behaves, mutatis mutandis, like an inclination needle.

The proof of both these theorems is immediately illuminated if we rely on the previously developed concept of the deviation resistance (cf. Chap. III, §6); detailed analytic developments, as they are given for this purpose by Gilbert,**) appear just as out of place here as in the previous case. The following simple considerations coincide in result with the Gilbert developments.

a) *Rotation axis of the outer ring along the plumb line, inner ring clamped at a right angle with respect to the outer ring, axis of the rotor sweeping the horizontal plane.* We decompose the Earth's rotation ω into its two components with respect to the plumb line and the meridian of the observation location. The first component is $\omega \sin \varphi$, where φ is the geographic latitude. This component does not influence, for sufficiently small friction in the pintles of the outer ring, the absolute position of the rotor; the rotor simply does not partake of this rotation, so that it naturally also prevents the inner and outer rings from following this rotation; the rotor behaves with respect to this component just as a top with three degrees of freedom behaves with respect to the entire rotation of the Earth. The second component is $\omega \cos \varphi$. If we imagine the position of the rotor in the horizontal plane as fixed for an instant, this component would lead the axis of the rotor on a circular cone about the meridian, and the top would describe a regular precession. Due to its inertia, the top resists this enforced motion with a moment whose axis is simultaneously perpendicular to the figure axis and the axis of the precession cone, and therefore falls, in our case, along the plumb line. According to page 175, equation (1), the magnitude of this moment is, if we insert for the precessional velocity there denoted

*) Cf. the Instructions sur les expériences du gyroscope cited above.

**) Cf. §XV and XVI of the work cited on p. 736: Mémoire sur l'application etc.

by ν the value $\omega \cos\varphi$, and introduce the eigenimpulse N of the rotor*) into the formula,

$$K = -\omega \cos\varphi \sin\vartheta \,(N - A\omega \cos\varphi \cos\vartheta). \tag{3}$$

Here ϑ is the angle between the figure axis of the top and the axis of the precession cone; that is, in our case, the angle between the figure axis and the meridian. For the determination of the sign, it is stipulated that we wish to measure ϑ from the northern side of the meridian, and that we reckon the (positive) figure axis as the side of the rotor axis about which the eigenrotation follows in the same sense as the rotation of the Earth about the line that connects the center of the Earth and the North Pole, and that we therefore imagine, with use of the manner of expression introduced on page 734, that the figure axis is drawn from the midpoint of the rotor to its "north pole." The product ωN in equation (3) is, on the basis of this stipulation, a *positive* quantity.

Moreover, the second term in the parentheses of (3) is unquestionably to be neglected with respect to the first. The order of magnitude of the second term is related to that of the first, namely, as $A\omega$ to N, or (with disregard of the difference between the equatorial and polar moments of inertia), as the velocity of the rotation of the Earth to the angular velocity of the rotor, or as the duration of one rotor rotation to the length of the day. We thus write, instead of (3),

$$K = -N\omega \cos\varphi \sin\vartheta. \tag{3$'$}$$

Should the assumed precessional motion of the rotor now be sustained under the unchanging inclination ϑ with respect to the meridian, a moment $-K$ about the plumb line that overcomes the inertial resistance K must be exerted. If this moment is not exerted, the rotor moves as if a moment $+K$ acted about the plumb line; this moment changes the angle ϑ. The plumb line is an equatorial principal axis for the rotor and for the outer ring; for the inner ring, in contrast, the plumb line is the figure axis itself. If we denote by A_1, C_1, A_2, C_2 the respective equatorial and polar principal moments of inertia of the inner and outer rings, then the sum of the relevant moments of inertia of the rotor and the

*) The eigenimpulse N is expressed (cf., for example, p. 222 above) in terms of the Euler angles φ, ψ, ϑ as $N = C(\varphi' + \cos\vartheta\,\psi')$; the angular velocities φ' and ψ' are, however, denoted in equation (1) of page 175 by μ and ν. The eigenimpulse is thus $N = C(\mu + \cos\vartheta \cdot \nu)$. The Euler angle φ obviously has nothing to do with the geographic latitude φ used in the text.

inner and outer rings about the plumb line will be $A + C_1 + A_2$. The equation of motion thus becomes

$$(A + C_1 + A_2)\vartheta'' = K = -N\omega\cos\varphi\sin\vartheta. \tag{4}$$

It is self-evident that the eigenimpulse of the rotor will not be simultaneously changed by the rotation of the Earth, since the axis of K stands perpendicular to the figure axis. Thus the quantity N in the preceding equation is a constant, which we can regard as the second equation that is necessary, in addition to (4), for the complete description of the motion of our top.[268]

Equation (4) now shows immediately that *the axis of the rotor is in equilibrium only if it is in the direction of the meridian.* For we have $\vartheta'' = 0$ only if $\vartheta = 0$ or $\vartheta = \pi$. *Of the two equilibrium positions $\vartheta = 0$ and $\vartheta = \pi$, the first is stable, and the second is labile.* Due to the sign of the right-hand side of (4), the axis of the rotor will, for a disturbance of the equilibrium position $\vartheta = 0$, be led back to this position by the resulting angular acceleration; for the equilibrium position $\vartheta = \pi$, in contrast, the axis of the rotor is removed farther from this position when it is disturbed. *In the stable equilibrium position, the eigenrotation of the rotor is in equi-orientational parallelism with the meridional component of the rotation of the Earth.* For we wished, in order to make the sign of ωN positive, to determine the angle ϑ between the meridian and the figure axis so that the eigenrotation about the figure axis followed in the same sense as the rotation of the Earth about its axis, or as the meridional component of the same about the northern half of the meridian from which we measured the angle ϑ. *The tendency to equi-orientational parallelism of the rotation axes that was emphasized by Foucault is shown in the appearance of the acceleration that directs the axis of the rotor toward the stable equilibrium position.*

The motion in question may be described most simply and completely if we compare it with the motion of a mathematical pendulum. In fact, (4) is nothing other than the usual differential equation of pendulum motion. We can write the latter, if we understand by l the length of the pendulum and by ϑ the angle that l encloses with the stable equilibrium position (that is, with the instantaneous direction of gravity), as

$$\vartheta'' = -\frac{g}{l}\sin\vartheta. \tag{4$'$}$$

In order to transform equations (4) and (4′) into one another, it is necessary only to choose l as

$$l = \frac{g(A + C_1 + A_2)}{N\omega\cos\varphi}. \tag{5}$$

This formula gives the length of the corresponding mathematical pendulum whose motion, for equal initial values of ϑ and ϑ', is exactly identical with the motion of our rotor. The length and the oscillation period of the pendulum will always be smaller as N is larger; correspondingly, the directional force of the Earth's rotation on our rotor increases with the magnitude of the eigenimpulse N.

The comparison with the declination needle may now be drawn in the most simple manner. Since the equation of motion of such a needle is $J\vartheta'' = -MH\sin\vartheta$, where J signifies the moment of inertia of the needle, M is the magnetic moment of the needle, and H is the horizontal component of the Earth's magnetic field, the length of the mathematical pendulum that corresponds to this magnetic needle is

$$l = \frac{gJ}{MH}. \tag{5'}$$

If one equates the pendulum lengths given in (5) and (5′), one recognizes how Foucault's identification of the spinning rotor with the magnetic needle may be realized quantitatively. If one imagines, for example, that the moment of inertia J of the needle coincides with the total moment of inertia $A+C_1+A_2$ that comes into consideration for the top apparatus, then one must simply choose the eigenimpulse of the rotor so that $N\omega\cos\varphi = MH$; *the motion of our rotor with eigenimpulse N will then be a congruent image, for equal initial values of ϑ and ϑ', with the motion of a magnetic needle with magnetic moment M.*

b) *Rotation axis of the outer ring perpendicular to the meridional plane, inner ring fixed perpendicular to the outer ring, rotor axis mobile in the meridional plane.*

Here we must refrain from the decomposition of the Earth's rotation into components, since the total rotation ω influences the position of the rotor in the meridional plane. If we imagine the angle ϑ between the axis of the rotor and the axis of the Earth's rotation as instantaneously fixed, then the rotor would describe a regular precession about the axis of the Earth. It would resist this precession, due to its inertia, with a moment K whose axis is simultaneously perpendicular to the axis of the rotor and that of the Earth's rotation, and therefore in the normal to the meridional plane; that is, along the rotation axis of the outer ring. According to equation (1) of page 175, the magnitude of the moment K is, if we now insert ω in place of ν and N in place of $C(\mu + \nu \cos\vartheta)$,

$$K = -\omega \sin\vartheta (N - A\omega \cos\vartheta). \tag{6}$$

Just as in a), we neglect the second term in the parentheses with respect to the first, so that (6) becomes more simply

$$K = -N\omega \sin\vartheta. \tag{6$'$}$$

If we stipulate, as in a), that we draw the figure axis from the midpoint of the rotor toward the side seen from which the rotation of the rotor occurs in the same sense as the rotation of the Earth about the North Pole, and we measure the angle ϑ from the northern half of the Earth axis to the so-defined figure axis, then the product $N\omega$ in the preceding equation will be positive.[269]

Should the rotor retain its position in the rotating meridional plane, a moment $-K$ about the rotation axis of the outer ring is necessary to overcome the inertial resistance of the rotor. If such a moment is not exerted, the angle ϑ between the Earth's axis and the figure axis must change, in such a measure that the product of the moment of inertia of the moving components and the angular acceleration is equal to K. This leads, just as above, to the differential equation

$$(A + C_1 + A_2)\vartheta'' = K = -N\omega \sin\vartheta. \tag{7}$$

From this equation, the conclusion immediately follows that *the rotor is in equilibrium in the meridional plane only if it has the direction of the Earth's axis; that is, only in the two positions $\vartheta = 0$ and $\vartheta = \pi$. The former position is a stable equilibrium position, and the latter is labile. In that the acceleration of the rotor determined by* (7) *leads toward the stable equilibrium position, it strives to direct the axis of the rotor parallel to the axis of the Earth in a homologous sense.*

The present motion is also congruent with the motion of a simple pendulum. *The corresponding pendulum length is now*

$$l = \frac{g(A + C_1 + A_2)}{N\omega}. \tag{8}$$

On the other hand, the motion of the inclination needle can also be identified with the pendulum motion. If J and M signify the moment of inertia and the magnetic moment of the needle and T is the total intensity of the Earth's magnetic field, then the length of the pendulum that corresponds to this inclination needle is

$$l = \frac{gJ}{MT}. \tag{8'}$$

From the comparison of (8) *and* (8′), *one recognizes how the behavior of our rotor axis that moves in the meridional plane may be identified quantitatively with the behavior of the magnetic needle in the inclination compass*, where, however, the difference between the two motions that was emphasized on page 735 is to be kept in mind.

c) The preceding results may be summarized and generalized if one assumes that the axis of the rotor moves in a *plane E that is oriented arbitrarily with respect to the Earth.* In order to achieve this arbitrary orientation, one can again fix the plane of the inner ring at a right angle with respect to that of the outer ring, and then place the rotation axis of the outer ring relative to the Earth so that it stands perpendicular to the plane E. If λ signifies the angle between the axis of the Earth's rotation and the plane E, then the effective component of the Earth's rotation is $\omega \cos \lambda$, and the length of the corresponding pendulum follows as

$$l = \frac{g(A + C_1 + A_2)}{\omega N \cos \lambda}. \tag{9}$$

This formula is transformed in cases a) and b), where, in particular, $\lambda = \varphi$ and $\lambda = 0$, into equations (5) and (8), respectively; it is due to G i l b e r t.*)

Two remarks may be added on the influence of friction and on the influence of the masses of the suspension system; these remarks refer at the same time to cases a) and b), as well as to the generalized case c).

Friction in the bearings of the outer ring will naturally also be perceptible for the gyroscope with two degrees of freedom. While the consideration above shows that the oscillation amplitude of the rotor axis must remain constant if friction is neglected, the amplitude will generally be damped by friction. Thus the rotor axis will approach the stable equilibrium position with an oscillation of decreasing amplitude, and will finally come to rest in this position if the eigenrotation of the rotor is

*) Equation (130) in the previously cited work Mémoire sur l'application etc. We differ in the text from G i l b e r t only in that we have suppressed a term with the factor ω^2 in the passage from (3) to (3′) and from (6) to (6′). With consideration of the degree of precision of the entirely theoretical assumptions, this term can have no significance.

maintained for a sufficiently long time. With respect to the quantitative relations, we may simply call upon the analogy with the simple pendulum or the magnetic needle. For a similar construction of the bearings, the effect of friction for the considered experiment will be similar to that for the oscillations of the simple pendulum and the magnetic needle, which, in their turn, are naturally always damped oscillations.

For what concerns the effects of the masses of the outer and inner rings, it may be surprising that we have expressed these effects in our last formulas, while we said in the discussion of the first Foucault experiment that they were to be neglected. This is explained by the fact that in the first Foucault experiment (with the neglect of friction) the rotor axis was perceptibly fixed in space and the angular velocities of the ring masses are only of the order of the rotation of the Earth, while in the second Foucault experiment, in contrast, the rotor axis experiences actual accelerations in which the masses of the inner and outer rings must take part. While in the first Foucault experiment the influence of the ring masses on the observed magnitude of the relative motion of the rotor is vanishingly small (of order ω/Ω; cf. page 742), this influence in the second Foucault experiment is of the same order of magnitude as the observed motion of the rotor itself, so that, for example, the moments of inertia C_1 and A_2 of the rings are to be added directly to the moment of inertia A of the rotor in formula (9). —

It remains, finally, to speak of a purposeful modification of the Foucault gyroscope, the previously named *barogyroscope* of Gilbert. As the name implies, both gravity and the rotation of the Earth come into play for this instrument. The device is illustrated in Fig. 111;[*]) we describe it by drawing the comparison with the Foucault gyroscope.

One sees in the figure the rotor D with its axis a, on which is found at E a gear for the actuation of the rotor and a sliding weight p on the lower elongation of the axis. We can designate the frame C as the inner ring; it rests at A and A' on knife edges. We can compare the bracket S with the outer ring of Foucault. It may rotate in the bushing H, and in this manner may be placed at an arbitrary azimuth; for any individual

[*]) The image is taken from the "Katalog mathem. Modelle, Apparate und Instrumente," by commission of the deutschen Mathem.-Vereinig., edited by W. Dyck, appendix page 79.

experiment, however, it is fixed, since friction in the bearing hinders any self-activated rotation of the bracket.

The apparatus is adjusted so that the axis AA' stands exactly horizontal at an arbitrary azimuth with respect to the meridian. The center of gravity of the rotor and frame is first brought, by the adjustment of the screws vv' and the added masses uu', to the line that connects the

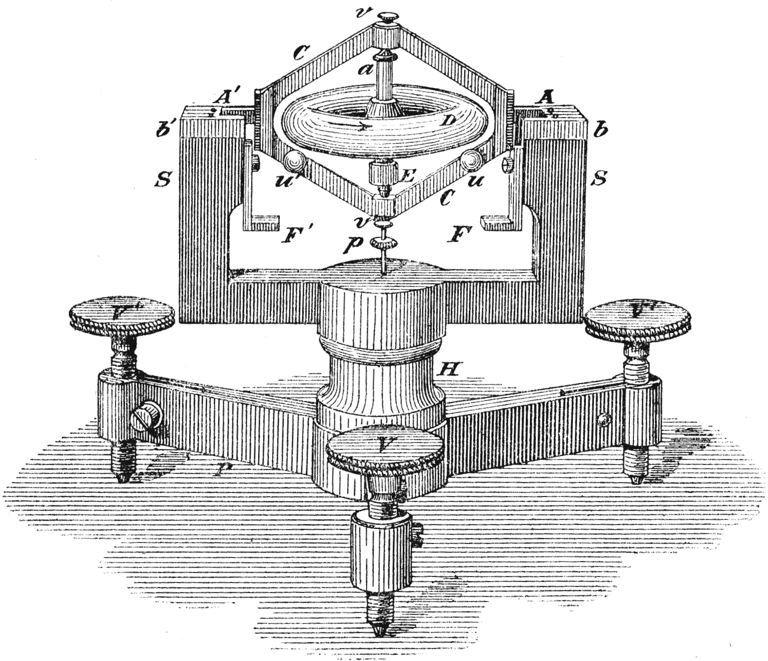

Fig. 111.

knife edges AA', so that the mobile part of the apparatus is in neutral equilibrium. The sliding weight p is then brought to the lower end of the needle that is fixed to the frame, so that the neutral equilibrium is changed to a (slightly) stable equilibrium.

After the rotor has been given a large eigenimpulse N (for G i l b e r t, 150 rotations per second), one observes that its axis moves in an oscillatory manner away from the vertical, and remains, after the oscillation has expired, at a certain inclination that depends, among other things, on the initially positioned azimuth of the frame.

The calculation of this inclination and theory of the experiment are again extremely simple if one begins from the concept of the deviation resistance.

We can speak qualitatively in the following manner. The rotor that moves in an arbitrary vertical plane is under completely similar conditions to those of the rotor in the second Foucault experiment (see above, under b) or c)); in order to fix its position within the vertical plane or its position with respect to the rotating Earth, it would be necessary to overcome the moment K, which has for its axis the connecting line AA' of the knife edges. Since a counter moment $-K$ is not exerted, the rotor moves as if a moment K acted on it about the named axis. This moment strives to place the figure axis of the rotor parallel to the rotation axis of the Earth; it therefore deflects the initially vertical axis of the rotor toward the direction of the projection of the Earth's axis of rotation onto the plane of motion of the rotor axis. In addition to this moment, however, the moment of gravity now acts, which deflects the figure axis of the rotor back toward the vertical. There will thus be a certain mean position between the vertical and the projection of the Earth's axis in which the two turning moments maintain equilibrium, and in which the axis of the rotor is thus at rest.

In order to complete the deliberation from the quantitative side, it is necessary to express the moment of gravity M, on the one hand, and the inertial effect K, on the other hand, in terms of quantities that can be observed in the apparatus.

Let m be the mass of the sliding weight p, and δ its distance from the center of gravity of the rotor. If χ signifies the angle that the figure axis of the rotor forms with the vertical, reckoned positive from the vertical toward the figure axis, then the moment arm of the gravitational force mg about the axis of the knife edges will be $\delta \sin \chi$, and thus the moment of gravity in the sense of the angle χ is

$$M = -mg\delta \sin \chi. \tag{10}$$

For the determination of the deviation resistance K, we again begin from equation (1) of page 175, introduce the eigenimpulse N of the rotor, and set for ν the component $\nu = \omega \cos \lambda$ of the Earth's rotation with respect to the plane of motion of the rotor axis, where λ signifies, as in c), the angle between the Earth's axis and this plane. We thus obtain, if we again neglect a term of the relative order of magnitude $A\omega/N$,

$$K = -N\omega \cos \lambda \sin \vartheta.$$

Here ϑ is the angle between the projection of the Earth's axis onto the plane of motion of the rotor axis and the rotor axis, reckoned as positive from the former to the latter. The moment K is reckoned in the same sense as ϑ. If we introduce the angle μ that the projection of the Earth's axis onto the plane of motion forms with the vertical, likewise reckoned as positive from the former to the latter, then we have $\vartheta = \mu + \chi$, and can write

$$K = -N\omega(\cos\chi\cos\lambda\sin\mu + \sin\chi\cos\lambda\cos\mu). \tag{11}$$

Here the products $\cos\lambda\sin\mu$ and $\cos\lambda\cos\mu$ are to be expressed in terms of quantities that permit of more direct measurement than the angles λ and μ. We choose as such quantities the geographic latitude φ of the observation location and the angle α ($< 180°$) that the plane of motion of the figure axis forms with the meridian of the observation location. On the unit sphere described about the midpoint of the rotor, we mark (cf. Fig. 112) the intersection point V with the vertical, the intersection point P with the parallel to the axis of the Earth, and finally the point Q that corresponds to the projection of the Earth's axis onto the plane of motion of the rotor axis. The corresponding spherical triangle PQV has a right angle at Q. Its hypotenuse is $\pi/2 - \varphi$, and its sides are λ and μ. The angle at V is equal to α.[270] According to Neper's rule, the two equations

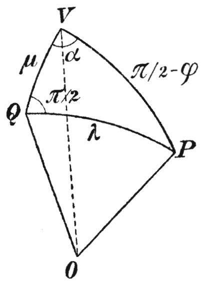

Fig. 112.

$$\begin{aligned} \sin\varphi &= \cos\lambda\cos\mu, \\ \cos\alpha &= \operatorname{tg}\varphi\ \operatorname{tg}\mu \end{aligned} \tag{12}$$

hold, from which follows, by multiplication,

$$\cos\alpha\cos\varphi = \cos\lambda\sin\mu. \tag{12$'$}$$

If we insert the values of $\cos\lambda\cos\mu$ and $\cos\lambda\sin\mu$ determined in (12) and (12′) into (11), there follows

$$K = -N\omega(\sin\chi\sin\varphi + \cos\chi\cos\varphi\cos\alpha). \tag{13}$$

We can now determine the equilibrium position, as well as the law of motion of the axis of the rotor, in the simplest manner.

The axis of the rotor is in *equilibrium* if the two moments M and K cancel each other. The equation $M = K$ therefore serves for the determination of the equilibrium position. If we denote the value of χ that corresponds to the equilibrium position by χ_0, then we have for the determination of χ_0, according to (10) and (13), the equation

$$mg\delta\sin\chi_0 = N\omega(\sin\chi_0\sin\varphi + \cos\chi_0\cos\varphi\cos\alpha),$$

or

$$\operatorname{tg}\chi_0 = \frac{N\omega\cos\varphi}{mg\delta - N\omega\sin\varphi}\cos\alpha. \tag{14}$$

More generally, we find the *acceleration equation* of the rotor axis if we set the product of the moments of inertia of the moving components and the acceleration of the angle ϑ, or, equivalently, the acceleration of angle χ, equal to the difference of the applied moments. If A is the equatorial moment of inertia of the rotor and A_1 is the moment of inertia of the frame about the connecting line of the knife edges, then the relevant moment of inertia for the change of the angle χ is the sum $A + A_1 + m\delta^2$. The equation of motion is thus

$$(A + A_1 + m\delta^2)\chi'' = K - M,$$

or, according to (10) and (13),

$$(A + A_1 + m\delta^2)\chi'' = -(N\omega \sin\varphi - mg\delta)\sin\chi - N\omega\cos\varphi\cos\alpha\cos\chi.$$

We can write this more conveniently, if we consider the definition of χ_0, as

$$(15)\quad (A+A_1+m\delta^2)\chi''=-\sqrt{(N\omega\sin\varphi-mg\delta)^2+(N\omega\cos\varphi\cos\alpha)^2}\,\sin(\chi-\chi_0).$$

This equation, evidently, can also be identified with the equation of motion of an ordinary pendulum (see above, equation (4′)). The corresponding pendulum length becomes, in analogy to equation (5),

$$l = \frac{g(A + A_1 + m\delta^2)}{\sqrt{(N\omega\sin\varphi - mg\delta)^2 + (N\omega\cos\varphi\cos\alpha)^2}}.$$

The axis of the barogyroscope therefore oscillates about the equilibrium position χ_0 with a period that corresponds to the so-defined pendulum length l. With consideration of the friction at the knife edges, the oscillations will gradually expire, and the final position of rest will coincide with the equilibrium position χ_0.

The formulas (14) and (15) are adopted by Gilbert as the calculational basis for the dimensioning of his apparatus. The Gilbert derivation*) of these formulas is essentially more complicated than that given here. We discuss the results in an obvious manner.

According to equation (14), the inclination χ_0 depends on the azimuth of the assembly, and will be zero, for example, if $\alpha = \pi/2$; that is, if the connecting line of the knife edges AA' lies in the meridian. This follows immediately from the fact that the projection of the Earth's axis onto the plane of motion of the rotor axis coincides with the vertical for this position of the apparatus, and that, therefore, the two moments M and K strive together to position the axis of the rotor vertically. In order, on the other hand, to obtain the most perceptible

*) Cf. the repeatedly cited work Mémoire sur l'application, §XVIII, eqn. (153). The results are summarized in the likewise cited Katalog mathem. Modelle etc., Nachtrag, p. 78

deflection χ_0, the connecting line of the knife edges is directed at a right angle to the meridian, so that the plane of motion of the rotor axis coincides with the meridional plane. In this plane, the needle fixed to the lower end of the rotor will deflect to the south or the north, according to whether this end is, in the manner of expression of page 734, a south or a north pole; that is, according to whether the rotation of the rotor about the needle follows in the opposite or in the same sense as the rotation of the Earth about the north pole of the Earth.

Moreover, equation (14) shows that the amplitude of the deflection depends only on the ratio $mg\delta/N\omega$, and that one would obtain the greatest amplitude (namely, a full right angle) if one were to choose this ratio directly equal to $\sin\varphi$. On the other hand, equation (15) shows that this choice is not the most advantageous in practice (completely disregarding that the support of the frame on the knife edges would make such a great displacement impossible). The length of the corresponding mathematical pendulum would then be, namely,

$$l = \frac{g(A + A_1 + m\delta^2)}{N\omega\cos\varphi\cos\alpha},$$

and the corresponding half oscillation period, for sufficiently small oscillations about the equilibrium position, would be equal to

$$\tau = \pi\sqrt{\frac{A + A_1 + m\delta^2}{N\omega\cos\varphi\cos\alpha}}.$$

For the purpose of a rough calculation, we assume the most favorable case $\cos\alpha = 1$, $\cos\varphi = 1$ (observation in the meridian at the equator) for the magnitude of the amplitude; further, the moment of inertia C of the rotor, which is approximately twice as large as the equatorial moment A of the rotor, may be taken, for example, equal to $A + A_1 + m\delta^2$. If n denotes the number of rotations of the rotor per second, then one has $N = 2\pi Cn$, and

$$l = \frac{g}{2\pi n\omega}, \qquad \tau = \pi\sqrt{\frac{1}{2\pi n\omega}}.$$

For Gilbert (see above), $n = 150$; the value of ω in seconds is then $2\pi/24\cdot 60\cdot 60$. Thus $2\pi n\omega$ will equal approximately $10/144$, and

$$l = 144 \text{ m}, \qquad \tau = 12 \text{ sec}.$$

This oscillation period is undesirably long, since the observation must be restricted to the first minutes after the actuation of the rotor, and since one wishes to obtain a judgment concerning the definitive equilibrium position of the rotor within this time. The strong oscillation of the axis of the rotor (in the assumed case the amplitude of the oscillation amounts to a right angle) would also be inconvenient for the observation of the mean equilibrium position. It is to be added that our value of τ is valid only for sufficiently small oscillation amplitudes, and that, in contrast, the oscillation period in our example would be significantly longer. Nevertheless, our calculation shows that one can arbitrarily enhance the deflection from the vertical through an appropriate choice of the circumstances, and can regulate it according to requirements.

Gilbert himself performed calculations for the numerical example $\varphi = 48^\circ\, 50'\, 39''$, $\alpha = 0^\circ$, $n = 200$, $m = 0{,}79$ gr., $\delta = 5$ cm.; with consideration of the dimensions of the rotor and the frame, there followed

$$\chi_0 = 7^\circ\, 37'\, 10'', \quad \tau = 3{,}76 \text{ sec.}$$

Here, therefore, a very large value of the deflection has been renounced in favor of a diminishment of the oscillation period, and the resulting more convenient observation of the final equilibrium position.

The Gilbert disposition appears to have many advantages compared to the original disposition of Foucault. In the sliding weight p and its distance δ from the center of gravity, one has, to a certain extent, a disposable parameter that can be chosen in a favorable manner for the observation. If one adjusts the proportions so that the final equilibrium position lies near the vertical, one eliminates many observational errors that would appear for large amplitudes. Further, the unavoidable errors in the centering of the moving masses, which for the Foucault gyroscope may be very disturbing, become relatively insignificant for the barogyroscope by means of the intentional addition of the overweight p. Concerning the quantitative coincidence between theory and experiment, however, G i l b e r t gives, just like F o u c a u l t, no more detailed information in the repeatedly cited treatise.[271]

In the preceding presentation, we have intentionally emphasized the probable error sources and the question of a possible quantitative confirmation more strongly than is otherwise common in textbooks of mechanics. Indeed, the present example appears eminently suited to showing how long is the path that extends between the mental conception of a dynamic process and its realization by a specific physical apparatus!

The genius of Foucault predicted, more through intuition than strict mechanical deduction, the behavior of the spinning rotor on the surface of the Earth. In order to complete his gyroscope, eight months of intense work was necessary. In spite of the applied labor and the unusual experimental capability of Foucault, the apparatus could operate only on the boundary at which the mechanical truth to be proven begins to be elevated above the disturbances and observational errors. How mistrustful Foucault himself believed one must be with respect to his apparatus emerges from a remark in the above cited "Instructions": One may not be fully convinced that the observed deflection of the gyroscope is actually due to the rotation of the Earth until one has not obtained the same sense of the deflection for the opposite rotation sense of the rotor. It thus appears that the sense of the deflection was never raised above all doubt; that the magnitude of the deflection would be given correctly with a certain precision appears all the more doubtful, especially if the experiment were made by an experimenter less skillful than Foucault.

The chances are more favorable, on the above-named grounds (the lack of the necessity for a particularly precise centering, the selection of more convenient experimental conditions through the appropriate specifications of the overweight), for the barogyroscope. Here too, however, an actual experimental proof that the sources of error do not excessively distort the observed deflection is required, a proof that we seek to no avail in Gilbert.

For a repetition of the Foucault or the Gilbert experiment, one would well, in any case, introduce electromagnetic actuation of the rotor, and thus eliminate the difficulties that arise from the gradual slowing of the eigenrotation.[272]

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