Preface

Preface to Volume 1

As an undergraduate student at a good engineering school, I had never heard of stochastic processes or Lie groups (even though I double majored in Mathematics). As a faculty member in engineering I encountered many problems where the recurring themes were "noise" and "geometry." When I went to read up on both topics I found fairly little at this intersection. Now, to be certain, there are many wonderful texts on one of these subjects or the other. And to be fair, there are several advanced treatments on their intersection. However, for the engineer or scientists who has the modest goal of modeling a stochastic (i.e., time-evolving and random) mechanical system with equations with an eye towards numerically simulating the system's behavior rather than proving theorems, very few books are out there. This is because mechanical systems (such as robots, biological macromolecules, spinning tops, satellites, automobiles, etc.) move in multiple spatial dimensions, and the configuration space that describes allowable motions of objects made up of rigid components does not fit into the usual framework of linear systems theory. Rather, the configuration space manifold is usually either a Lie group or a homogeneous space.[1]

My mission then became clear: write a book on stochastic modeling of (possibly complicated) mechanical systems that a well-motivated first-year graduate student or undergraduate at the senior level in engineering or the physical sciences could pick up and read cover-to-cover without having to carry around twenty other books. The key point that I tried to keep in mind when writing this book was that the art of mathematical modeling is very different than the art of proving theorems. The emphasis here is on "how to calculate" quantities (mostly analytically by hand and occasionally numerically by computer) rather than "how to prove." Therefore, some topics that are treated at great detail in mathematics books are covered at a superficial level here, and some concrete analytical calculations that are glossed over in mathematics books are explained in detail here. In other words the goal here is not to expand the frontiers of mathematics, but rather to translate known results to a broader audience.

The following quotes from Felix Klein[2] in regard to the modern mathematics of his day came to mind often during the writing process:

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[1] The reader is not expected to know what these concepts mean at this point.

[2] F. Klein, *Development of Mathematics in the 19th Century*, translated by M. Ackerman as part of *Lie Groups: History, Frontiers and Applications*, Vol. IX, Math Sci Press, 1979.

> The exposition, intended for a few specialized colleagues, refrains from indicating any connection with more general questions. Hence it is barely accessible to colleagues in neighboring fields and totally inaccessible to a larger circle...
> In fact, the physicist can use little, and the engineer none at all, of these theories in his tasks.

The later of these was also referenced in Arnol'd's classic book[3] as an example of how work that is initially viewed as esoteric can become central to applied fields.

In order to emphasize the point that this book is for practitioners, as I present results they generally are not in "definition-proof-theorem" format. Rather, results and derivations are presented in a flowing style. Section headings punctuate results so that the presentation (hopefully) does not ramble on too much.

Another difference between this book and one on pure mathematics is that while pathological examples can be viewed as the fundamental motivation for many mathematical concepts (e.g, the behavior of $\sin\frac{1}{x}$ as $x \to 0$), in most applications most functions and the domains on which they are defined do not exhibit pathologies. And so practitioners can afford to be less precise than pure mathematicians.

A final major difference between this presentation and those written by mathematicians is that rather than the usual "top-down" approach in which examples follow definitions and theorems, the approach here is "bottom-up" in the sense that examples are used to motivate concepts throughout this book and the companion volume. Then after the reader gains familiarity with the concepts, definitions are provided to capture the essence of the examples.

To help with the issue of motivation and to illustrate the art of mathematical modeling, case studies from a variety of different engineering and scientific fields are presented. In fact, so much material is covered that this book has been split into two volumes. Volume 1 (which is what you are reading now) focuses on basic stochastic theory and geometric methods. The usefulness of some of these methods may not be clear until the second volume. For example, some results pertaining to differential forms and differential geometry that are presented in Volume 1 are not applied to stochastic models until they find applications in Volume 2 in the form of Integral Geometry (also called Geometric Probability) and in Multivariate Statistical Analysis. Volume 2 serves as an in-depth (but accessible) treatment of Lie groups, and the extension of statistical and information-theoretic techniques to that domain.

I have organized Volume 1 into the following 9 chapters and an appendix: Chapter 1 provides an introduction and overview of the kinds of the problems that can be addressed using the mathematical modeling methods of this book. Chapter 2 reviews every aspect of the Gaussian distribution, and uses this as the quintessential example of a probability density function. Chapter 3 discusses probability and information theory and introduces notation that will be used throughout these volumes. Chapter 4 is an overview of white noise, stochastic differential equations (SDEs), and Fokker–Planck equations on the real line and in Euclidean space. The relationship between Itô and Stratonovich SDEs is explained and examples illustrate the conversions between these forms on multi-dimensional examples in Cartesian and curvilinear coordinate systems. Chapter 5 provides an introduction to Geometry including elementary projective, algebraic, and differential geometry of curves and surfaces. That chapter begins with some concrete examples that are described in detail. Chapter 6 introduces differential forms and the generalized Stokes theorem. Chapter 7 generalizes the treatment of surfaces and

[3]See Arnol'd, VI, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, Berlin, 1978.

polyhedra to manifolds and polytopes. Geometry is first described using a coordinate-dependent presentation that some differential geometers may find old fashioned, but it is nonetheless fully rigorous and general, and far more accessible to the engineer and scientist than the elegant and powerful (but cryptic) coordinate-free descriptions. Chapter 8 discusses stochastic processes in manifolds and related probability flows. Chapter 9 summarizes the current volume and introduces Volume 2. The appendix provides a comprehensive review of concepts from linear algebra, multivariate calculus, and systems of first-order ordinary differential equations. To the engineering or physical science student at the senior level or higher, some of this material will be known already. But for those who have not seen it before, it is presented in a self-contained manner. In addition, exercises at the end of each chapter in Volume 1 reinforce the main points. There are more than 150 exercises in Volume 1. Volume 2 also has many exercises. Over time I plan to build up a full solution set that will be uploaded to the publisher's webpage, and will be accessible to instructors. This will provide many more worked examples than space limits allow within the volumes.

Volume 1 can be used as a textbook in several ways. Chapters 2–4 together with the appendix can serve as a one-semester course on continuous-time stochastic processes. Chapters 5–8 can serve as a one-semester course on elementary differential geometry. Or, if chapters are read sequentially, the whole book can be used for self-study. Each chapter is meant to be relatively self contained, with its own references to the literature. Altogether there are approximately 250 references that can be used to facilitate further study.

The stochastic models addressed here are equations of motion for physical systems that are forced by noise. The time-evolving statistical properties of these models are studied extensively. Information theory is concerned with communicating and extracting content in the presence of noise. Lie groups either can be thought of as continuous sets of symmetry operations, or as smooth high-dimensional surfaces which have an associated operator. That is, the same mathematical object can be viewed from either a more algebraic or more geometric perspective.

Whereas the emphasis of Volume 1 is on basic theory of continuous-time stochastic processes and differential geometric methods, Volume 2 provides an in-depth introduction to matrix Lie groups, stochastic processes that evolve on Lie groups, and information-theoretic inequalities involving groups. Volume 1 only has a smattering of information theory and Lie groups. Volume 2 emphasizes information theory and Lie groups to a much larger degree.

Information theory consists of several branches. The branch originating from Shannon's mathematical theory of communication is covered in numerous engineering textbooks with minor variants on the titles "Information Theory" or "Communications Theory." A second branch of information theory, due to Wiener, is concerned with filtering of noisy data and extracting a signal (such as in radar detection of flying objects). The third branch originated from the field of mathematical statistics in which people like Fisher, de Bruijn, Cramér, and Rao developed concepts in statistical estimation. It is primarily this third branch that is addressed in Volume 1, and so very little of the classical engineering information theory is found here. However, Shannon's theory is reviewed in detail in Volume 2, where connections between many aspects of information and group theory are explored. And Wiener's filtering ideas (which have a strong connection with Fourier analysis) find natural applications in the context of deconvolving functions on Lie groups (an advanced topic that is also deferred to Volume 2).

Volume 2 is a more formal and more advanced presentation that builds on the basics covered in Volume 1. It is composed of three parts. Part 1 begins with a detailed treatment of Lie groups including elementary algebraic, differential geometric, and functional analytic properties. Classical variational calculus techniques are reviewed, and the coordinate-free extension of these concepts to Lie groups (in the form of the Euler–Poincaré equation) are derived and used in examples. In addition, the basic concepts of group representation theory are reviewed along with the concepts of convolution of functions and Fourier expansions on Lie groups. Connections with multivariate statistical analysis and integral geometry are also explored. Part 2 of Volume 2 is concerned with the connections between information theory and group theory. An extension of the de Bruijn inequality to the context of Lie groups is examined. Classical communication-theory problems are reviewed, and information inequalities that have parallels in group theory are explained. Geometric and algebraic problems in coding theory are also examined. A number of connections to problems in engineering and biology are provided. For example, it is shown how a spherical optical encoder developed by the author and coworkers[4] can be viewed as a decoding problem on the rotation group, $SO(3)$. Also, the problem of noise in coherent optical communication systems is formulated and the resulting Fokker–Planck equation is shown to be quite similar to that of the stochastic Kinematic cart that is described in the introductory chapter of Volume 1. This leads to Part 3 of Volume 2, which brings the discussion back to issues close to those in Volume 1. Namely, stochastic differential equations and Fokker–Planck equations are revisited. In Volume 2 all of these equations evolve on Lie groups (particularly the rotation and rigid-body-motion groups). The differential geometric techniques that are presented in Volume 1 are applied heavily in this setting. Several closely related (though not identical) concepts of "mean" and "covariance" of probability densities on Lie groups are reviewed, and their propagation under iterated convolutions is studied. As far as the descriptions of probability densities on Lie groups are concerned, closed-form Gaussian-like approximations are possible in some contexts, and Fourier-based solutions are more convenient in others. The coordinate-based tools needed for realizing these expressions as concrete quantities (which can in principle be implemented numerically) are provided in Volume 2.

During a lecture I attended while writing this book, an executive from a famous computer manufacturer said that traditionally technical people have been trained to be "I-shaped," meaning an education that is very deep in one area, but not broad. The executive went on to say that he now hires people who are "T-shaped," meaning that they have a broad but generally shallow background that allows them to communicate with others, but in addition have depth in one area. From this viewpoint, the present book and its companion volume are "ΠI-shaped," with a broad discussion of geometry that is used to investigate three areas of knowledge relatively deeply: stochastic models, information theory, and Lie groups.

It has been a joy to write these books. It has clarified many issues in my own mind. And I hope that you find them both interesting and useful. And while I have worked hard to eliminate errors, there will no doubt be some that escaped my attention. Therefore I welcome any comments/corrections and plan to keep an updated online erratum page which can be found by searching for my name on the web.

There are so many people without whom this book would not have been completed. First, I must thank John J. Benedetto for inviting me to contribute to this series that he is editing, and Tom Grasso at Birkhäuser for making the process flow smoothly.

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[4]Stein, D., Scheinerman, E.R., Chirikjian, G.S., "Mathematical models of binary spherical-motion encoders," *IEEE-ASME Trans. Mechatron.*, 8(2), 234-244, 2003.

A debt of gratitude is owed to a number of people who have worked (and maybe suffered) through early drafts of this book. These include my students Kevin Wolfe, Michael Kutzer, and Matt Moses who received very rough drafts, and whose comments and questions were very useful in improving the presentation and content. I would also like to thank all of my current and former students and colleagues for providing a stimulating environment in which to work.

Mathematicians Ernie Kalnins, Peter T. Kim, Willard Miller, Jr., and Julie Mitchell provided comments that helped significantly in identifying mathematical errors, fine-tuning definitions, and organizing topics. I am thankful to Tamás Kalmár-Nagy, Jennifer Losaw, Tilak Ratnanather, and Jon Selig for finding several important typographical errors. John Oprea went way above and beyond the call of duty to read and provide detailed comments on two drafts that led to a significant reorganization of the material. Andrew D. Lewis provided some very useful comments and the picture of a torus that appears in Chapter 5. Andrew Douglas, Tak Igusa, and Frank C. Park each provided some useful and/or encouraging comments. Wooram Park helped with some of the figures.

I would like to thank William N. Sharpe, Jr. for hiring me many years ago straight out of graduate school (even after knowing me as an undergraduate), and Nick Jones, the Benjamin T. Rome Dean of the JHU Whiting School of Engineering, for allowing me to have the sabbatical during the 2008 calendar year that was used to write this book after my service as department chair finished.

I would also like to thank the faculty and staff of the Institute for Mathematics and Its Applications (IMA) at the University of Minnesota for the three week-long workshops that I attended there during part of the time while I was writing this book. Some of the topics discussed here percolated through my mind during that time.

Last but not least, I would like to thank my family. Writing a single-author book can be a solitary experience. And so it is important to have surroundings that are "fuuuun."

Baltimore, Maryland

Gregory Chirikjian
May 2009

Preface to Volume 2

This book, Volume 2, builds on the fundamental results and differential-geometric terminology established in Volume 1. The goal of Volume 2 is to bring together three fields of study that are usually treated as disjoint subjects: stochastic models, information theory, and Lie groups.[5]

Stochastic phenomena appear frequently in engineering and physics. From one perspective, stochasticity (i.e., randomness) can be viewed as an inherent characteristic of the physical world, and developing mathematical models that describe this inherent stochasticity allows us to understand truly random phenomena. Alternatively, stochastic modeling can be thought of as a way to sweep the true complexity of the physical world under the rug by calling a phenomenon random when it is too complicated to model deterministically. The benefit of a stochastic model in that case is that the computational effort can be far less than that required to describe a complex deterministic system. Stochastic models can be used to generate estimates of the average behavior of very complicated deterministic systems together with the variance of these estimates. For example, a rigid model of a macromolecule being subjected to impacts by the surrounding solvent molecules could be modeled using a molecular dynamics simulation involving millions of molecules. However, the details of the interactions can be quite computationally intensive. In contrast, viewing the macromolecule as being forced by Gaussian white noise (i.e., increments of a Wiener process) allows for the relatively simple description of the behavior as a stochastic differential equation (SDE) or the corresponding Fokker–Planck equation (FPE). The relationship between SDEs and FPEs was explored in detail in Volume 1, which can be viewed as the "prequel" to the current volume.

In Volume 1 the probabilistic foundations of information theory (e.g., the definitions of and properties of entropy, mutual information, entropy-power inequality, etc.) were reviewed, but almost none of Shannon's mathematical theory of communication (which is the information theory known to engineers) was described. Shannon's information theory is concerned with the passage of data through noisy environments (called channels) as efficiently as possible. Such channels might be copper wires, fiber optic cables, the atmosphere (for radio, laser, and microwave transmission), the ocean (for acoustic/sonar signals), and so forth. This means that the data should be coded in some way so as not to be corrupted by random noise in the environment. The simplest robust scheme would be to repeat the message many times, but this would not be efficient. If the noise characteristics of the channel are known, then the data can be coded (or packaged) before it is sent so as to reach the receiver with high probability and at a high rate. Of course, there is a trade-off between the speed of transmission and the probability that the original message is actually received. One model for the way data is corrupted during transmission is by Gaussian noise. The communication channels that subject data to this kind of noise are called Gaussian channels. From this description it should be clear that stochastic models of information channels go hand-in-hand with the design of codes for transmission of data through known channels. And when the physical nature of the channel (or the action of intelligent agents that transmit, receive, and process information in the physical world) is of interest, Lie groups enter in several ways.

[5]This material is based upon work supported by the National Science Foundation under Grant No. IIS-0915542.
Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

Lie groups have been studied intensively over the past century. These are continuous sets of operations that can be composed and inverted and for which an identity element exists and the associative law holds. The Lie groups of most interest in engineering applications are the rotation and rigid-body-motion groups in two- and three-dimensional space. In physics, other Lie groups such as the special unitary groups and the Galilean group describe invariants/symmetries of systems of interest. For example, all of the equations of classical mechanics are invariant under Galilean transformations, and unitary groups describe symmetries in quantum mechanics and particle physics. Strong connections have existed between Lie groups and stochastic processes for many decades. For example, in the 1930s, Perrin studied the rotational Brownian motion of rigid molecules; that is, the orientation of a molecule follows a stochastic path induced by its environment. In the estimation of the position and orientation of a robot, satellite, or submarine from noisy and/or incomplete data, measurements corrupted by noise are used to determine a best guess of the kinematic state of the system. Such topics have been investigated since the early 1970s and are still being studied today.

Although strong historical connections exist between information theory and stochastic models and between stochastic models and Lie groups, direct connections between information theory and Lie groups are far less explored. In coding theory, finite groups arise in several contexts (e.g., as groups of symmetry operations acting on sphere packings, Abelian groups over finite fields, coset codes, etc.). Although these concepts are reviewed in this book, they are not the focus. Lie groups (as opposed to finite groups) are connected to information theory in several ways that have not been explored much in the literature. Indeed, one of the main contributions of this book is the exploration of the connection between Lie groups and information theory. In some cases, these connections are through a stochastic model. In other cases, the connections are direct. For example, Lie groups have associated with them certain algebraic structures such as subgroups and coset spaces. Probability density functions (pdfs) over a Lie group can be marginalized over a subgroup or coset space and concepts of conditional and marginal entropy and mutual information can be defined. Additionally, the group operation can be used to define a convolution operation. It is shown in this book that the entropy of the convolution of two pdfs on a Lie group is no less than the entropy of the individual pdfs and that a version of the famous data processing inequality holds for Lie groups.

These rather abstract connections between information-theoretic concepts and Lie groups are supported by certain applications. For example, consider a mobile robot, bacterium, or animal that wonders around foraging for resources uses sensory information to update its position and orientation in space. This “infotaxis” (information-driven motion) is defined by the processing of sensory information resulting in action in the physical world and, specifically, the action that results is a trajectory in the Lie group of rigid-body motions. As a second example, when information is transmitted as pulses through a fiber optic cable using a certain transmission and reception strategy, the rate of information transmission is limited by the fact that lasers are imperfect physical devices and the pulses are not perfectly sharp. This is due to spontaneous (and uncontrolled) emission of photons in the laser cavity. The resulting “phase noise” leads to a blurring of the pulses and a reduction in the rate of reliable information transmission. As it turns out, the stochastic model that describes this noise leads to SDEs and FPEs that evolve on the Euclidean group of the plane. Additionally, characteristics of these pdfs indicate the rate at which information can be reliably transmitted. Other examples of the connection between Lie groups and information theory arise in biomolecular applications. DNA is known to carry the genetic information in all known living organisms. This information is described by the classical discrete information theory, and many books on bioinformatics address this. However, there is also information (of the continuous kind) that describes fluctuations in the shape of biomolecules. For example, the spatial packing density for genetic information is a function of how DNA molecules are packaged in chromosomes. It is possible to ask questions about how the sequential content of DNA impacts its flexibility and how the resulting ensemble of Brownian-motion-generated conformations differ from each other. This type of question is at the interface of information theory and statistical mechanics, where concepts of entropy and the Lie groups describing the motion of molecules come together.

This book is structured as follows:

In Chapter 10 the concept of Lie groups and their relationship to Lie algebras are defined rigorously and many examples are provided in the concrete setting of matrices.

Chapter 11 discusses functions on Lie groups and how concepts such as the derivative of functions and Taylor series extend from $\mathbb{R}^n$ to this setting.

Chapter 12 discusses integration on Lie groups and Fourier expansions.

Chapter 13 reviews classical variational calculus and its extensions to the case when the functionals of interest have arguments in a Lie group and Lie algebra.

Chapters 14 is an introduction to statistical mechanics via stochastic differential equations. The specific emphasis is on physical systems that can be modeled as multiple rigid bodies and hence have a configuration space that is a Lie group. Also in this chapter, concepts from ergodic theory are discussed.

In Chapter 15 the concept of entropy from statistical mechanics is modified for the context of robotic parts handling, and the relationship to the principal kinematic formula from the field of Integral Geometry is rederived, used, and modified. In particular, that chapter discusses the problem of automated assembly and how Sanderson's concept of parts entropy (which is the Shannon entropy of a random variable on a Euclidean group) provides a connection between Lie groups and information theory. Since parts occlude each other, knowing how much volume is available for a part to move in the group of rigid-body motions without bumping into another part is important in the computation of parts entropy. That is where the principal kinematic formula comes in.

Chapter 16 examines the relationship among matrix Lie groups, multivariate analysis, and random matrix theory. As it turns out, the covariance matrix for a multi-variate Gaussian distribution, which is a symmetric positive-definite matrix, can be thought of as a point in a quotient space of two groups of the form $GL^+(n,\mathbb{R})/SO(n,\mathbb{R})$.[6] As a result, if we know how to integrate over the group $GL^+(n,\mathbb{R})$ (which consists of all $n \times n$ matrices with real entries and positive determinant) and the rotation group in n-dimensional space, $SO(n,\mathbb{R})$, then from this knowledge we will know how to integrate over the space of all covariance matrices. This is important because the sample covariance obtained from any experiment is never exactly the same as the ideal covariance of the pdf describing the physical phenomenon being investigated. The field of multi-variate statistical analysis studies the distribution of possible covariance matrices; that is, whereas $\rho(\mathbf{x};\mathbf{0},\Sigma)$ describes a Gaussian distribution on $\mathbb{R}^n$ with zero mean and covariance Σ, in multivariate analysis the distribution of covariances, $f(\Sigma)$, is a pdf on the space $GL^+(n,\mathbb{R})/SO(n,\mathbb{R})$, called the Wishart distribution. Lie-theoretic terminology and results are useful in that context. This provides one link between Lie groups and classical probability and statistics, which are, in turn, linked to information theory. Connections between multi-variate analysis and the theory of random matrices are natural, and in recent years, random matrix theory has become a popular tool to model communication networks. This brings us back to information theory.

[6]The reader is not expected to know what this means yet.

Chapter 17 reviews classical information theory (the theory of communication) including Shannon's source-coding and channel-coding theorems. Several classical information-theory results such as the data processing inequality and the Shannon–Hartley theorem for the capacity of a continuous channel with Gaussian noise are reviewed. This chapter also shows how Lie groups enter into communication problems both as symmetries of equations describing physical channels (e.g., the (linear) telegraph equation, and (nonlinear) soliton-based communication strategies) as well as serving as the domain in which signals evolve (laser phase noise).

Chapter 18 discusses algebraic and geometric aspects of coding/decoding problems. For example, Hamming codes and the relationship to packing of spheres in high-dimensional Euclidean spaces is reviewed. It is also shown how one can design codes on Lie groups. For example, in a usual motor, a rotary encoder is used to measure the angle through which the motor turns. This can be viewed as a coding/decoding problem on the group $SO(2)$. For a spherical motor, such as the one co-invented by the author, an encoding strategy is needed in order to control it. This problem can be thought of as coding theory on the Lie group $SO(3)$. Similar problems exist in the design and use of fiducial patterns for use in medical imaging.

Chapter 19 introduces the author's observations about how classical information theory extends to the case when the random variables under investigation live in a Lie group rather than in Euclidean space. It discusses how inequalities of classical information theory (the de Bruijn inequality, the data processing inequality, and the Cramér–Rao bound) extend to the setting of Lie groups.

Chapter 20 is a return to the sorts of stochastic processes discussed in Volume 1, but with a focus on Lie groups. Properties of Fokker–Planck equations on Lie groups are discussed, and the properties and mathematical machinery developed in Chapters 10–12 are used to analyze the properties of solutions of these Fokker–Planck equations. This includes several concrete physical problems (rotational Brownian motion, conformational fluctuations of DNA, bacterial chemotaxis).

Chapter 21 applies the properties of the stochastic models from Chapter 20 by introducing the concept of a Gaussian channel on a Lie group and the concept of injecting noise through fiber bundles (which are a differential geometric structure that was briefly discussed in Chapter 7 and the description of which is expanded here). The nonholonomic kinematic cart is again used as the prototypical example of a system to which this methodology can be applied.

Chapter 22 provides a summary of this book and discusses other application areas that are ripe for future work. These include so-called "infotaxis" (or information-driven motion), connections between statistical mechanics and conformational aspects of biomolecular information theory, and medical imaging problems.

According to folklore, when Claude Shannon (the engineer) was pondering what name to give to the quantity that he was studying, John von Neumann (the mathematician) advocated the use of the word "entropy" and is purported to have said to him:

> No one really knows what entropy is, so in a debate you will always have the advantage.

Regardless of whether or not I understand entropy, I'd like to think that both of these men (as well as Wiener, Itô, and Stratonovich) would be happy to see the confluence of information-theoretic, geometric, stochastic, and Lie-algebraic ideas that are summarized here for a wide audience.

Perhaps the most important issue that needs to be addressed in this preface is a question that has been asked by some students, which is "Dr. C, why did you make this two volumes? Now I have two books to buy!" There are several answers to this question:

- Contrary to intuition, the cost of a single large book can be quite prohibitive relative to the cost of each of these volumes. By cutting the book into two volumes, readers who are only interested in topics in one of the volumes do not need to pay the extra price that would have accompanied a single large volume. Students may be more interested in Volume 1, whereas researchers who already know the definitions and basic principles in Volume 1 may be more interested Volume 2.
- Each volume is just the right size that it can be slipped into a backpack and read (or at least skimmed) on an intercontinental airplane trip. For the busy researcher, there is little time in daily life to sit down and actually read, whereas the quiet undisturbed time of a trip is ideal for reading. A larger book would not travel as well.
- When reading a large book contained within two hard covers, it can be quite annoying when reading p. 826 to be required to access necessary definitions on p. 17. The process of flipping back and forth a massive number of pages to connect the necessary concepts can impede the learning process and add wear and tear to the book. In contrast, if p. 17 is in Volume 1 and the page that would have been p. 826 is actually p. 399 in Volume 2, then both pages can be viewed *simultaneously* without flipping.

Finally, I have many people to thank. The students who took my class for credit during the Spring 2010 semester worked hard to find errors: Martin Kendal Ackerman, Graham Beck, Mike Kutzer, Mike Mashner, Matt Moses, Valentina Staneva, Roberto Tron, Kevin Wolfe, and Yan Yan. Comments by Garrett Jenkinson, Manu Madhav, Shahin Sefati, John Swensen, and Tom Wedlick were also helpful in modifying the presentation. Useful comments by Andrea Censi, Ming Liao, and Rolf Schneider have also improved the presentation and helped to eliminate errors. Any errors that are found after publication will be posted as a list of errata on my lab's webpage, together with an addendum including additional exercises and pointers to the ever-growing literature, as was done for Volume 1. Last but not least, I have my whole family to thank for their love and support.

Baltimore, Maryland *Gregory Chirikjian*
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