

Chapter 2

Dynamic Strategic Interactions in Economic Systems

The recent globalization and emergence of multinational corporations turned many major economic activities into dynamic interactive endeavors. The number of decision makers involved is relatively small and it leads to significant strategic interdependence. With human life being lived over time, and institutions like markets, firms, and governments changing over time, the economic system is definitely a dynamic interactive entity. Section 2.1 provides a general overview of dynamic interactive economic systems. Market outcomes under open-loop equilibria are investigated in Sect. 2.2 and those under feedback equilibria are examined in Sect. 2.3. An extension of the analysis to a stochastic framework is provided in Sect. 2.4.

2.1 Dynamic Interactive Economic System

In this section, we provide the formulation of dynamic interactive economic systems, the typical dynamic game paradigm in economic analysis, and the characterization of market equilibria.

2.1.1 Basic Formulation

A fruitful way of modeling a dynamic interactive situation is by differential games. Differential games study a class of decision problems under which the evolution of the state is described by a differential equation and the players act throughout a time interval.

In economic analysis the general form of n -person differential games can be characterized as follows. Economic agent i seeks to maximize its objective

$$\begin{aligned} & \int_{t_0}^T g^i[s, x(s), u_1(s), u_2(s), \dots, u_n(s)] \exp\left[-\int_{t_0}^s r(y) dy\right] ds \\ & + \exp\left[-\int_{t_0}^T r(y) dy\right] q^i(x(T)), \quad \text{for } i \in N = \{1, 2, \dots, n\}, \end{aligned} \quad (2.1)$$

where $r(y)$ is the discount rate, $x(s) \in X \subset R^m$ denotes the state variables of the game, $q^i(x(T))$ is agent i 's valuation of the state at terminal time T , and $u_i \in U^i$ is the control of agent i , for $i \in N$.

The state variable evolves according to the dynamics

$$\dot{x}(s) = f[s, x(s), u_1(s), u_2(s), \dots, u_n(s)], \quad x(t_0) = x_0, \quad (2.2)$$

where $x(s) \in X \subset R^m$ denotes the state variables of the game and $u_i \in U^i$ is the control of agent i , for $i \in N$. The functions $f[s, x, u_1, u_2, \dots, u_n]$, $g^i[s, \cdot, u_1, u_2, \dots, u_n]$, and $q^i(\cdot)$, for $i \in N$, and $s \in [t_0, T]$ are differentiable functions.

Examples of economic state variables include capital stock, resource biomass or deposits, the level of technology, market shares, economic assets, equity, prices, pollutants, and company goodwill. Examples of controls include investment, resource extraction rate, research and development (R&D) efforts, advertising rate, output produced, input used, taxes, subsidy, and expenditures.

In many economic situations, the terminal time of the game, T , is either very far in the future or unknown to the agents. For example, the value of a publicly listed firm is the present value of its discounted expected future earnings. Nobody knows when the firm will be out of business. As argued by Dockner et al. (2000), in this case setting $T = \infty$ may very well be the best approximation for the true game horizon. Even if the firm's management restricts itself to considering profit maximization over the next year, it should value its asset positions at the end of the year by the earning potential of these assets in the years to come.

In the case when the terminal horizon T approaches infinity, an autonomous game structure with constant discounting will replace (2.1) and (2.2). In particular, the game becomes

$$\max_{u_i} \int_{t_0}^{\infty} g^i[x(s), u_1(s), u_2(s), \dots, u_n(s)] \exp[-r(s - t_0)] ds, \quad \text{for } i \in N, \quad (2.3)$$

subject to the state dynamics

$$\dot{x}(s) = f[x(s), u_1(s), u_2(s), \dots, u_n(s)], \quad x(t_0) = x_0, \quad (2.4)$$

where r is a constant discount rate.

Since time s does not appear explicitly in the agent's payoff $g^i[x(s), u_1(s), u_2(s), \dots, u_n(s)]$ and the state dynamics $f[x(s), u_1(s), u_2(s), \dots, u_n(s)]$, the problem is an autonomous problem.

Theoretical research and the applications of differential games proceeded apace in the past in many areas of economics. An in-depth survey and analysis on differential games in economics and management science can be found in Dockner et al. (2000). A detailed account and a comprehensive list of differential games in marketing can be found in Zaccour (2003). A thorough survey of models of dynamic games in economics is given in Long (2010).

2.1.2 Typical Dynamic Economic Game Paradigms

In this section we present the general model structures and specific examples of some typical dynamic economic game paradigms.

2.1.2.1 Investment Games

A general structure of investment games can be characterized as follows. Let $K^i(s)$ be the physical capital stock of firm $i \in N$ at time s . Each firm in the industry accumulates capital according to the equation

$$\dot{K}^i(s) = I^i(s) - \delta^i K^i(s), \quad \text{for } i \in N, \quad (2.5)$$

where $I^i(s)$ is the gross investment of firm i at time s and $\delta^i \geq 0$ is the constant rate of depreciation. At initial time t_0 , the capital stock $K^i(t_0) = K_0^i$, for $i \in N$, is given.

Denote the output of firm i by $q_i(s)$, the industry output by $Q(s) = \sum_{j=1}^n q_j(s)$, and the output price by $P[Q(s)]$.

The output of firm i is governed by the production function $q_i(s) = f^j[L^i(s), K^i(s)]$, where $L^i(s)$ is the quantity of noncapital input (like labor) employed at time s . The cost of production is $c_i\{f^i[K^i(s), L^i(s)]\}$.

At time instant s , the operating profit of firm i becomes

$$P\left(\sum_{j=1}^n f^j[L^j(s), K^j(s)]\right) f^i[L^i(s), K^i(s)] - c^i\{f[L^i(s), K^i(s)]\},$$

for $i \in N$. (2.6)

The optimal choice of noncapital input by firm i satisfies

$$\begin{aligned} & P'\left(\sum_{j=1}^n f^j[L^j(s), K^j(s)]\right) f_{L^i}^i[L^i(s), K^i(s)] f^i[L^i(s), K^i(s)] \\ & + P\left(\sum_{j=1}^n f^j[L^j(s), K^j(s)]\right) f_{L^i}^i[L^i(s), K^i(s)] \\ & - c_{L^i}^i\{f[L^i(s), K^i(s)]\} = 0, \end{aligned} \quad (2.7)$$

for $i \in N$.

If an instantaneous industry equilibrium exists, the optimal choice of noncapital inputs by these n firms can be found by solving (2.7) and be expressed as

$$L^{i*}(s) = \ell^i[K^1(s), K^2(s), \dots, K^n(s)] = \ell^i[K(s)], \quad \text{for } i \in N. \quad (2.8)$$

Upon substituting $L^{i*}(s)$ in (2.8) into the firm's instantaneous operating profit in (2.6) yields

$$\begin{aligned} \pi^i[K(s)] = & P \left(\sum_{j=1}^n f^j[\ell^j(K(s)), K^j(s)] \right) f^i[\ell^i(K(s)), K^i(s)] \\ & - c^i \{ f[\ell^i(K(s)), K^i(s)] \}, \quad \text{for } i \in N. \end{aligned} \quad (2.9)$$

The cost of investment is $m^i[I^i(s)]$. Firms will choose an investment path over the time period $[t_0, T]$ to maximize their future streams of profits. The present value of future profits of firm i can then be expressed as

$$\int_{t_0}^T (\pi^i[K(s)] - m^i[I(s)]) \exp \left[- \int_{t_0}^s r(y) dy \right] ds, \quad \text{for } i \in N. \quad (2.10)$$

The maximization of (2.10) by firm $i \in N$ subject to (2.5) forms a differential game.

If the time horizon approaches infinity, that is, $T = \infty$, an infinite-horizon version of the game in (2.5) and (2.10) can be set up as

$$\int_{t_0}^{\infty} (\pi^i[K(s)] - m^i[I(s)]) \exp(-rs) ds, \quad \text{for } i \in N, \quad (2.11)$$

subject to (2.15).

Example 2.1 Consider a specific example of investment games in which the demand function is given as

$$\begin{aligned} P[Q(s)] &= a - Q(s), \\ m^i[I^i(s)] &= a_i [I(s)]^2, \\ c^i \{ f[\ell^i(K(s)), K^i(s)] \} &= b_i K^i(s) + \hat{b}_i [K^i(s)]^2. \end{aligned}$$

The interest rate is r .

With these specifications, different investment games with a linear quadratic type can be formulated.

Example 2.2 A knowledge investment game, with knowledge being a public good, can be formulated as follows. The level of knowledge $K(s)$ will change according the accumulation equation

$$\dot{K}(s) = \sum_{j=1}^n I^j(s) - \delta K(s), \quad K(t_0) = K_0.$$

Economic agent i 's cost of investment in the public knowledge capital is

$$m^i[I^i(s)] = \rho I^i(s) + \frac{1}{2}[I^i(s)]^2, \quad \text{for } i \in N.$$

Economic agent i 's instantaneous operating net revenue is

$$\pi^i[K(s)] = K(s)[a^i - K(s)].$$

Once again the interest rate is r .

Dockner (1992), Fershtman and Muller (1984, 1986), Fudenberg and Tirole (1983, 1986, 1991), Reynolds (1987, 1991), and Spence (1979) presented various examples of this class of investment games.

2.1.2.2 Renewable Resource Extraction Games

A general structure of renewable resource extraction games can be characterized as follows. Consider an economy endowed with a single renewable resource, with $n \geq 2$ resource extractors (firms). Let $u_i(s)$ denote the quantity of the resource extracted by firm i at time s , for $i \in N$, where each firm controls its rate of extraction. Let U^i be the set of admissible extraction rates and $x(s)$ the size of the resource stock at time s . In particular, we have $U^i \in R^+$, $x(s) > 0$, and $U^i = \{0\}$ for $x(s) = 0$. The growth dynamics of the renewable resource stock becomes

$$\dot{x}(s) = f[s, x(s)] - \sum_{j=1}^n u_j(s), \quad x(t_0) = x_0 > 0, \quad (2.12)$$

where $f[s, x(s)]$ is the natural rate of evolution of the resource.

The extraction cost for firm $i \in N$ depends on the quantity of the resource extracted $u_i(s)$, the resource stock size $x(s)$, and some other input parameters. In particular, the extraction cost can be specified as

$$C^i[u_i(s), x(s)]. \quad (2.13)$$

The cost per unit of the resource extracted by firm i is negatively related to the size of the resource stock.

The market price of the resource depends on the total amount of the resource extracted and supplied to the market. The price-output relationship at time s is given by the following downward-sloping demand curve:

$$p = P[s, Q(s)], \quad (2.14)$$

where p is the market price of the resource and $Q(s) = \sum_{j=1}^n u_j(s)$ is the total amount of the resource extracted and marketed at time s . The firm's horizon is $[t_0, T]$, and at time T a terminal payment $q^i[x(T)]$ will be given to firm i .

Firm i seeks to maximize the present value of its profits

$$\begin{aligned} & \int_{t_0}^T (P[s, Q(s)]u_i(s) - C^i[u_i(s), x(s)]) \exp\left[-\int_{t_0}^s r(y) dy\right] ds \\ & + \exp\left[-\int_{t_0}^T r(y) dy\right] q^i[x(T)], \quad \text{for } i \in N, \end{aligned} \quad (2.15)$$

subject to (2.12).

If the time horizon approaches infinity, that is, $T = \infty$, an infinite-horizon version of the game in (2.12) and (2.15) can be set up as

$$\int_{t_0}^{\infty} (P[Q(s)]u_i(s) - C^i[u_i(s), x(s)]) \exp[-r(s - t_0)] ds, \quad \text{for } i \in N, \quad (2.16)$$

subject to

$$\dot{x}(s) = f[x(s), u_1(s), u_2(s), \dots, u_n(s)], \quad x(t_0) = x_0 > 0. \quad (2.17)$$

Example 2.3 Consider the deterministic version of the Jørgensen and Yeung (1996) renewable resource game in which the growth dynamics is governed by

$$\dot{x}(s) = ax(s)^{1/2} - bx(s) - \sum_{j=1}^n u_j(s) \quad \text{and} \quad x(t_0) = x_0 > 0. \quad (2.18)$$

The natural growth function is $ax^{1/2} - bx = x[ax^{-1/2} - b]$. This function represents that pure compensation, viz., the proportional growth rate $ax^{-1/2}$ is a decreasing function of x .

The extraction cost for firm $i \in N$ depends on the quantity of the resource extracted $u_i(s)$, the resource stock size $x(s)$, and a parameter c . In particular, the extraction cost can be specified as follows:

$$C^i[u_i(s), x(s)] = \frac{c}{x(s)^{1/2}} u_i(s).$$

This specification implies that the cost per unit of the resource extracted by firm i , $c/x(s)^{1/2}$, decreases when $x(s)$ increases. The above cost structure was also adopted by Jørgensen and Yeung (1996). A decreasing unit cost follows from two assumptions: (i) The cost of extraction is proportional to the extraction effort and (ii) the amount of the resource extracted, seen as the output of a production function of two inputs (effort and stock level), is increasing in both inputs (cf. Clark 1990).

The market price of the resource depends on the total amount extracted and supplied to the market. The price-output relationship at time s is given by the following downward-sloping inverse demand curve:

$$P(s) = Q(s)^{-1/2},$$

where $Q(s) = \sum_{i \in N} u_i(s)$ is the total amount of the resource extracted and marketed at time s .

The objective of extractor $i \in N$ is to maximize the present value of the stream of future profits

$$\int_{t_0}^T \left[\left(\sum_{j=1}^n u_j(s) \right)^{-1/2} u_i(s) - \frac{c}{x(s)^{1/2}} u_i(s) \right] e^{-r(s-t_0)} ds + e^{-r(T-t_0)} x(T)^{1/2},$$

for $i \in N$,

(2.19)

subject to the stock dynamics of (2.18).

Chiarella et al. (1984), Reinganum and Stokey (1985), Clemhout and Wan (1985a, 1994), Dockner and Kaitala (1989), Plourde and Yeung (1989), Jørgensen and Sorger (1990), Fischer and Mirman (1992), Kaitala (1993), and Dockner et al. (1989) presented specific dynamic resource extraction games.

2.1.2.3 Marketing Games

Three major types of marketing games are presented below.

(i) Market Share Models

Market share models derive their name from the fact that the state variables of the game are the firm's market shares. In an n -firm oligopoly, let $x_i(s)$ denote the market share of firm $i \in N$. The state space X is represented by

$$X = \left\{ x^i(s) \in R \mid x^i(s) \in [0, 1], i \in N, \sum_{j=1}^n x^j(s) = 1 \right\}. \quad (2.20)$$

Let $u_i(s) \in R^m$ denote the advertising efforts of firm i at time s ; a general version of the market shares dynamics can be expressed as

$$\begin{aligned} \dot{x}^i(s) &= [1 - x^i(s)] f^i[u_i(s)] - x^i(s) \sum_{\substack{j=1 \\ j \neq i}}^n f^j[u_j(s)] \\ &= f^i[u_i(s)] - x^i(s) \sum_{j=1}^n f^j[u_j(s)], \quad \text{for } i \in N. \end{aligned} \quad (2.21)$$

The advertising response function $f^i[u_i(s)]$ is positive for positive advertising efforts. A diminishing (or nonincreasing) marginal product of advertising efforts is assumed, leading to the second-order derivative of $f^i[u_i(s)]$ being nonpositive.

Instead of market shares, the state variable may also represent the sales rates in a market where sales are fixed, say at level $\bar{m}$, and therefore

$$\sum_{j=1}^n \varpi^j(s) = \bar{m}. \quad (2.22)$$

Equation (2.22) reflects a market at its maturity stage with a stationary total sales volume of $\bar{m}$. The market of firm i is then $\varpi^i(s)/\bar{m} = x^i(s)$. The dynamics of the change in sales rates can be formulated as

$$\begin{aligned} \dot{\varpi}^i(s) &= [\bar{m} - \varpi^i(s)] f^i[u_i(s)] - \varpi^i(s) \sum_{\substack{j=1 \\ j \neq i}}^n f^j[u_j(s)] \\ &= \bar{m} f^i[u_i(s)] - \varpi^i(s) \sum_{j=1}^n f^j[u_j(s)], \quad \text{for } i \in N. \end{aligned} \quad (2.23)$$

Firm i 's cost of advertising efforts is $c^i[u_i(s)]$ and the gross profit of a unit of sales is P_i . The terminal valuation of the sales (or market shares) yields firm i a value $q^i[\varpi^i(T)]$.

The profit to firm i can be expressed as

$$\begin{aligned} &\int_{t_0}^T (P_i \varpi^i(s) - c^i[u_i(s)]) \exp\left[-\int_{t_0}^s r(y) dy\right] ds \\ &+ \exp\left[-\int_{t_0}^T r(y) dy\right] q^i[\varpi^i(T)]. \end{aligned} \quad (2.24)$$

If the time horizon approaches infinity, that is, $T = \infty$, an infinite-horizon version of the game can be set up as

$$\max_{u^i} \int_{t_0}^{\infty} (P_i \varpi^i - c^i[u_i(s)]) \exp[-r(s - t_0)] ds, \quad \text{for } i \in N, \quad (2.25)$$

subject to (2.23).

Example 2.4 A popular specification of the response function is

$$f^i[u_i(s)] = \beta_i [u_i(s)]^{\alpha_i},$$

with $\beta_i > 0$ and $\alpha_i \in (0, 1]$.

The cost of advertising

$$c^i[u_i(s)] = c_i [u_i(s)]^2,$$

where c_i is a positive constant.

Case (1979), Chintagunta and Jain (1995), Chintagunta and Vilcassim (1994), Erickson (1985, 1993, 1992, 1997), Fruchter (1999a, 1999b, 2001), Fruchter and Kalish (1998), Fruchter et al. (2001), Mesak and Calloway (1995), Mesak and Darat (1993), Olsder (2001), and Sorger (1989) developed and analyzed market share models along this line.

A model closely related to the market shares model is the sales response model. A sales response game model specifies the rate of change of a firm's sales rate $\varpi^i(s)$ as a function of the marketing instruments of all the firms in the market. Let $u_i(s) \in R^m$ be the marketing instruments of firm i ; a general specification of the sales dynamics is

$$\begin{aligned}\dot{\varpi}^i(s) &= f^i[s, \varpi^1(s), \varpi^2(s), \dots, \varpi^n(s), u_1(s), u_2(s), \dots, u_n(s)], \\ \varpi^i(t_0) &= \varpi_0^i, \quad \text{for } i \in N.\end{aligned}\tag{2.26}$$

Example 2.5 Mukundan and Elsner (1975) presented a model with sales dynamics

$$\dot{\varpi}^i(s) = \gamma_i u_i(s) \left[1 - \frac{\varpi^i(s)}{\varpi^1(s) + \varpi^1(s)} \right] - \delta_i \varpi^i(s), \quad \text{for } i \in \{1, 2\}.$$

Erickson (1995) presented a model with sales dynamics

$$\dot{\varpi}^i(s) = \gamma_i \sqrt{u_i(s)} \left[\bar{m}(s) - \sum_{j=1}^n \varpi^j(s) \right] - \delta_i \varpi^i(s), \quad \text{for } i \in N,$$

where $\bar{m}(s)$ is the time-varying market potential.

Deal (1979), Feichtinger and Dockner (1984), Jørgensen (1982), Little (1979), Sethi (1973), and Wang and Wu (2001) presented various sales response models.

(ii) New Product Diffusion Models

New product diffusion models are paradigms in which new products or services are introduced and their reputations built up in the market. The cumulative sales affect the current instantaneous sales as the market becomes more mature and the knowledge of the products becomes more available. Using $x^i(s)$ to denote the cumulative sales of product i at time s , the time derivative of $x^i(s)$ then represents the sales rate at time s . A general diffusion process governing the sales dynamics can be expressed as

$$\begin{aligned}\dot{x}^i(s) &= f^i[s, u_1(s), u_2(s), \dots, u_n(s), x^1(s), x^2(s), \dots, x^n(s)], \\ x^i(t_0) &= x_0^i,\end{aligned}\tag{2.27}$$

for $i \in N$, where $u_j(s)$ are the advertising strategies of firm j at time s .

The function f^i is assumed to satisfy the conditions $f_{u_i}^i > 0$, $f_{u_i u_j}^i < 0$. In a market with all products being substitutes of each other, $f_{u_j}^i < 0$ for $i \neq j$.

The instantaneous profit to firm i is

$$\pi^i[\dot{x}^i(s), u_i(s), x^i(s)].$$

In particular, $\pi^i[\dot{x}^i(s), u_i(s), x^i(s)]$ can take on a formulation like $R^i[\dot{x}^i(s), x^i(s)] - c^i[u_i(s)]$, where $R^i[\dot{x}^i(s), x^i(s)]$ is the instantaneous net revenue from sales $\dot{x}^i(s)$ and $c^i[u_i(s)]$ is the cost of advertising. The cumulative sales may affect the cost of production if experience counts.

A general dynamic game model can be formulated as

$$\max_{u^i} \int_{t_0}^T \{\pi^i[\dot{x}^i(s), u_i(s), x^i(s)]\} \exp\left[-\int_{t_0}^s r(y) dy\right] ds, \quad \text{for } i \in N, \quad (2.28)$$

subject to the dynamics in (2.27).

Example 2.6 Consider an oligopolistic extension of the Horsky and Simon (1983) model in which firm i seeks to maximize

$$\int_{t_0}^T \{\pi_i[\dot{x}^i(s)] - u_i(s)\} \exp[-r(s - t_0)] ds, \quad \text{for } i \in N,$$

where π_i is the nonnegative unit margin of firm i 's product.

The sales dynamics is

$$\dot{x}^i(s) = \left[\alpha + \beta \ln(u_i(s)) + \gamma \sum_{j=1}^n x^j(s) \right] \left[\bar{m} - \sum_{j=1}^n x^j(s) \right], \quad \text{for } i \in N.$$

Industry-wide positive effects are realized as the new products' cumulative sales increase.

For other new product diffusion models one can see Dockner and Jørgensen (1988, 1992).

(iii) Goodwill Models

Another class of advertising games is one that deals with the accumulation of a stock of goodwill or brand image. Let $G^i(s)$ denote the stock of goodwill of firm i at time s . A general form of the dynamics of the goodwill of firm i is

$$\dot{G}^i(s) = h^i[s, u_i(s), G^i(s), x^i(s)], \quad \text{for } i \in N, \quad (2.29)$$

where $u_i(s)$ is the effort on the creation of goodwill and $x^i(s)$ is the market share or sales rate of firm i .

The market share (sales rate) of firm i may be affected by all the firms' goodwill stocks and market shares. The dynamics of the market share or sales rate of firm i

yields the relationships

$$\dot{x}^i(s) = f^i[s, u_1(s), u_2(s), \dots, u_n(s), x^1(s), x^2(s), \dots, x^n(s), G^1(s), G^2(s), \dots, G^n(s)], \quad (2.30)$$

for $i \in N$.

Firm i seeks to maximize

$$\int_{t_0}^T \{\pi^i[x^i(s), u_i(s), G^1(s), G^2(s), \dots, G^n(s)]\} \exp\left[-\int_{t_0}^s r(y) dy\right] ds, \quad (2.31)$$

for $i \in N$,

subject to (2.29) and (2.30).

The term $\pi^i[x^i(s), u_i(s), G^1(s), G^2(s), \dots, G^n(s)]$ represents the instantaneous net revenue of firm i .

An infinite-horizon game problem can be formulated with $T = \infty$, a constant discount rate, autonomous versions of the goodwill dynamics in (2.29), and of the market share dynamics in (2.30).

Example 2.7 Fornell et al. (1985) exploited the concept of consumption as a form of production and assumed that production learning took place. This resulted in consumption experience. In an oligopolistic market, brand-specific consumption experience stocks are denoted by $G^1, G^2, \dots, G^n$. The dynamics of these experience stocks is

$$\dot{G}^i(s) = x^i(s) - \delta G^i(s), \quad G^i(t_0) = G_0^i, \quad \text{for } i \in N,$$

where $x^i(s)$ is the market share of firm i .

Firm i controls the ratio of its advertising expenditure to unit sales $a^i(s)$ and the ration of its promotion expense to unit sales $b^i(s)$. The market shares of firm i evolve according to

$$\begin{aligned} \dot{x}^i(s) = & \sum_{j=1}^n x^i(s)x^j(s)[f[a^i(s), x(s), G^i(s)] \\ & - f[a^j(s), x(s), G^j(s)] + g[b^i(s)] - g[b^j(s)]], \\ x^i(t_0) = & x_0^i, \quad \text{for } i \in N, \end{aligned}$$

where $x(s) = \{x^1(s), x^2(s), \dots, x^n(s)\}$.

Example 2.8 Consider a duopoly in which the goodwill dynamics is

$$\dot{G}^i(s) = \sqrt{u_i(s)} - \delta G^i(s), \quad G^i(t_0) = G_0^i, \quad \text{for } i \in \{1, 2\}.$$

The sales rate of firm i at time s is

$$\varpi^i[G^1(s), G^1(s)] = \alpha_i G^i(s) - \beta_i G^j(s) - \gamma_i [G^i(s)]^2 + \theta_i [G^j(s)]^2 \\ + \varsigma_i G^i(s) G^j(s),$$

for $i, j \in \{1, 2\}$ and $i \neq j$, where $\alpha_i, \beta_i, \gamma_i, \theta_i$, and ς_i are positive constants.

The objectives of the duopolists are

$$\int_{t_0}^{\infty} (\pi_i \varpi^i[G^1(s), G^2(s)] - u_i(s)) \exp[-r(s - t_0)] ds, \quad \text{for } i, j \in \{1, 2\},$$

where π_i is the constant unit margin of firm i .

Chintagunta (1993), Feichtinger et al. (1994), Fershtman (1984), Sethi and Thompson (2000), and Tapiero (1979) considered games involving goodwill.

2.1.3 Market Equilibrium

The outcome in the economic system (often known as market outcome when the system is driven by markets) is characterized by an equilibrium in which each participant is maximizing its objective given the other participants' optimal choices of controls/strategies.

A set of strategies $\{v_1^*(s), v_2^*(s), \dots, v_n^*(s)\}$ is said to constitute a *noncooperative Nash equilibrium solution* for the n -person differential game equations (2.1) and (2.2), if the following inequalities are satisfied for all $v_i(s) \in U^i, i \in N$:

$$\int_{t_0}^T g^i[s, x^*(s), v_1^*(s), v_2^*(s), \dots, v_n^*(s)] \exp\left[-\int_{t_0}^s r(y) dy\right] ds \\ + \exp\left[-\int_{t_0}^T r(y) dy\right] q^i(x^*(T)) \\ \geq \int_{t_0}^T g^i[s, \hat{x}^i(s), v_1^*(s), v_2^*(s), \dots, v_{i-1}^*(s), v_i(s), v_{i+1}^*(s), \dots, v_n^*(s)] \\ \times \exp\left[-\int_{t_0}^s r(y) dy\right] ds + \exp\left[-\int_{t_0}^T r(y) dy\right] q^i(\hat{x}^i(T)), \quad (2.32)$$

where, on the time interval $[t_0, T]$,

$$\dot{x}^*(s) = f[s, x^*(s), v_1^*(s), v_2^*(s), \dots, v_n^*(s)], \quad x^*(t_0) = x_0, \quad \text{and} \\ \dot{\hat{x}}^i(s) = f[s, \hat{x}^i(s), v_1^*(s), v_2^*(s), \dots, v_{i-1}^*(s), v_i(s), v_{i+1}^*(s), \dots, v_n^*(s)], \\ \hat{x}(t_0) = x_0.$$

Similarly, a set of strategies $\{v_1^*(s), v_2^*(s), \dots, v_n^*(s)\}$ constitutes a noncooperative Nash equilibrium solution for the infinite-horizon n -person differential game in (2.3) and (2.4), if there exists a set of inequalities similar to (2.32) with $T = \infty$, discount factor $\exp[-r(s - t_0)]$, objective functions g^i , and state growth f as in (2.3) and (2.4), and the omission of the terminal condition q^i .

Since the game is being played over time, the conditions on the commitment of the agents' strategies at the beginning of the game duration has to be specified. If economic agents choose to commit their strategies from the outset, they are using *open-loop* strategies. If economic agents can revise their strategies contingent upon the state variables, they are using feedback strategies.

2.2 Market Outcomes Under Open-Loop Nash Equilibria

If the agents have to commit their strategies from the outset, the agents' information structure can be seen as an *open-loop* pattern in which $\eta^i(s) = \{x_0\}$, $s \in [t_0, T]$. Their strategies become functions of the initial state x_0 and time s and can be expressed as $\{u_i(s) = v_i(s, x_0)$, for $i \in N\}$.

2.2.1 Characterization of Open-Loop Equilibria

An open-loop Nash equilibrium for the game in (2.1) and (2.2) is characterized as follows.

Theorem 2.1 *If a set of strategies $\{u_i^*(s) = \zeta_i^*(s, x_0)$, for $i \in N\}$ provides an open-loop Nash equilibrium solution to the game in (2.1) and (2.2), and $\{x^*(s)$, $t_0 \leq s \leq T\}$ is the corresponding optimal state trajectory, then there exist m costate functions $\Lambda^i(s) : [t_0, T] \rightarrow R^m$, for $i \in N$, such that the following relations are satisfied:*

$$\begin{aligned}
 & \zeta_i^*(s, x_0) \\
 & \equiv u_i^*(s) \\
 & = \arg \max_{u_i \in U^i} \left\{ g^i \left[s, x^*(s), u_1^*(s), u_2^*(s), \dots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \dots, u_n^*(s) \right] \right. \\
 & \quad \times \exp \left[- \int_{t_0}^s r(y) dy \right] \\
 & \quad \left. + \Lambda^i(s) f \left[s, x^*(s), u_1^*(s), u_2^*(s), \dots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \dots, u_n^*(s) \right] \right\}, \\
 & \dot{x}^*(s) = f \left[s, x^*(s), u_1^*(s), u_2^*(s), \dots, u_n^*(s) \right], \quad x^*(t_0) = x_0,
 \end{aligned} \tag{2.33}$$

$$\begin{aligned}\dot{\Lambda}^i(s) &= -\frac{\partial}{\partial x^*} \left\{ g^i[s, x^*(s), u_1^*(s), u_2^*(s), \dots, u_n^*(s)] \exp \left[-\int_{t_0}^s r(y) dy \right] \right. \\ &\quad \left. + \Lambda^i(s) f[s, x^*(s), u_1^*(s), u_2^*(s), \dots, u_n^*(s)] \right\}, \\ \Lambda^i(T) &= \frac{\partial}{\partial x^*} q^i(x^*(T)) \exp \left[-\int_{t_0}^T r(y) dy \right];\end{aligned}$$

for $i \in N$.

Proof Consider the problem of choosing a control path $u_i^*(s) = u_i^*(s) = \zeta_i^*(s, x_0)$ that maximizes

$$\begin{aligned}&\int_{t_0}^T g^i[s, x(s), u_1^*(s), u_2^*(s), \dots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \dots, u_n^*(s)] \\ &\quad \times \exp \left[-\int_{t_0}^s r(y) dy \right] ds + \exp \left[-\int_{t_0}^T r(y) dy \right] q^i(x(T)),\end{aligned}$$

subject to the state dynamics

$$\begin{aligned}\dot{x}(s) &= f[s, x(s), u_1^*(s), u_2^*(s), \dots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \dots, u_n^*(s)], \\ x(t_0) &= x_0,\end{aligned}$$

for $i \in N$.

This is a standard optimal control problem for agent i , treating $u_j^*(s)$ for $j \in N$ and $j \neq i$ as time paths given at the beginning of the game.

Invoking Theorem A.3 in the Technical Appendixes, the conditions for a maximum for agent i 's problem is characterized by the i th set of equalities in Theorem 2.1. Since the set of equalities for all n agents holds, a Nash equilibrium as in (2.32) will arise. $\square$

There may be multiple Nash equilibria. We assume that the agents will choose an equilibrium at time t_0 and stick with the corresponding strategies for the entire game interval.

The derivation of open-loop equilibria in nonzero-sum deterministic differential games first appeared in Berkovitz (1964) and Ho et al. (1965), with open-loop and feedback Nash equilibria in nonzero-sum deterministic differential games being presented in Case (1967, 1969) and Starr and Ho (1969a, 1969b).

In the case when the game horizon approaches infinity, we can characterize an open-loop equilibrium solution to the infinite-horizon game in (2.3) and (2.4) as follows.

Theorem 2.2 *If a set of strategies $\{u_i^*(s) = \zeta_i^*(s, x_i), \text{ for } i \in N\}$ provides an open-loop Nash equilibrium solution to the infinite-horizon game in (2.3) and (2.4) and $\{x^*(s), t \leq s \leq T\}$ is the corresponding optimal state trajectory, then there exist m*

costate functions $\Lambda^i(s) : [t, T] \rightarrow R^m$, for $i \in N$, such that the following relations are satisfied:

$$\begin{aligned} \zeta_i^*(s, x) &\equiv u_i^*(s) \\ &= \arg \max_{u_i \in U^i} \left\{ g^i[x^*(s), u_1^*(s), u_2^*(s), \dots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \dots, u_n^*(s)] \right. \\ &\quad \left. + \lambda^i(s) f[x^*(s), u_1^*(s), u_2^*(s), \dots, u_{i-1}^*(s), u_i(s), \right. \\ &\quad \left. u_{i+1}^*(s), \dots, u_n^*(s)] \right\}, \end{aligned} \quad (2.34)$$

$$\dot{x}^*(s) = f[x^*(s), u_1^*(s), u_2^*(s), \dots, u_n^*(s)], \quad x^*(t) = x_t,$$

$$\begin{aligned} \dot{\lambda}^i(s) &= r\lambda(s) - \frac{\partial}{\partial x^*} \left\{ g^i[x^*(s), u_1^*(s), u_2^*(s), \dots, u_n^*(s)] \right. \\ &\quad \left. + \lambda^i(s) f[x^*(s), u_1^*(s), u_2^*(s), \dots, u_n^*(s)] \right\}, \quad \text{for } i \in N. \end{aligned}$$

Proof Consider the problem of choosing a control path $v_i^*(s) = u_i^*(s)$ that maximizes

$$\begin{aligned} &\int_t^\infty g^i[x(s), u_1^*(s), u_2^*(s), \dots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \dots, u_n^*(s)] \\ &\quad \times \exp[-r(s-t)] ds \end{aligned}$$

subject to the state dynamics

$$\dot{x}(s) = f[x(s), u_1^*(s), u_2^*(s), \dots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \dots, u_n^*(s)], \quad x(t) = x,$$

for $i \in N$.

This is an infinite-horizon optimal control problem for agent i , treating $u_j^*(s)$, for $j \in N$ and $j \neq i$ as time paths given at the beginning of the game.

Invoking Theorem A.4 in the Technical Appendixes, the conditions for a maximum for agent i 's problem is characterized by the i th set of equalities in Theorem 2.2. Since the set of equalities for all n agents hold, a Nash equilibrium as in (2.32) will arise. $\square$

A detailed account of the applications of open-loop equilibria in marketing, economics, and management science can be found in Zaccour (2003) and Dockner et al. (2000).

2.2.2 Open-Loop Solution in Competitive Advertising

Consider the competitive dynamic advertising game in Sorger (1989). There are two firms in a market and the profits of firm 1 and that of firm 2 are, respectively,

$$\int_0^T \left[q_1 x(s) - \frac{c_1}{2} u_1(s)^2 \right] \exp(-rs) ds + \exp(-rT) S_1 x(T),$$

and

$$\int_0^T \left[q_2(1 - x(s)) - \frac{c_2}{2} u_2(s)^2 \right] \exp(-rs) ds + \exp(-rT) S_2 [1 - x(T)], \quad (2.35)$$

where r, q_i, c_i, S_i , for $i \in \{1, 2\}$, are positive constants, $x(s)$ is the market share of firm 1 at time s , $[1 - x(s)]$ is that of firm 2's, and $u_i(s)$ is the advertising rate for firm $i \in \{1, 2\}$.

It is assumed that market potential is constant over time. The only marketing instrument used by the firms is advertising. Advertising has diminishing returns since there are increasing marginal costs of advertising as reflected through the quadratic cost function. The dynamics of firm 1's market share is governed by

$$\dot{x}(s) = u_1(s)[1 - x(s)]^{1/2} - u_2(s)x(s)^{1/2}, \quad x(0) = x_0. \quad (2.36)$$

Consider that the firms would like to seek an open-loop solution. Using open-loop strategies requires the firms to determine their action's path at the outset. This is realistic only if there are restrictive commitments concerning advertising. Invoking Theorem 2.1, an open-loop solution to the game in (2.35) and (2.36) has to satisfy the following conditions:

$$\begin{aligned} u_1^*(s) = \arg \max_{u_1} & \left\{ \left[q_1 x^*(s) - \frac{c_1}{2} u_1(s)^2 \right] \exp(-rs) \right. \\ & \left. + \Lambda^1(s) (u_1(s)[1 - x^*(s)]^{1/2} - u_2(s)x^*(s)^{1/2}) \right\}, \\ u_2^*(s) = \arg \max_{u_2} & \left\{ \left[q_2(1 - x^*(s)) - \frac{c_2}{2} u_2(s)^2 \right] \exp(-rs) \right. \\ & \left. + \Lambda^2(s) (u_1(s)[1 - x^*(s)]^{1/2} - u_2(s)x^*(s)^{1/2}) \right\}, \end{aligned} \quad (2.37)$$

$$\dot{x}^*(s) = u_1^*(s)[1 - x^*(s)]^{1/2} - u_2^*(s)x^*(s)^{1/2}, \quad x^*(0) = x_0,$$

$$\dot{\Lambda}^1(s) = \left\{ -q_1 \exp(-rs) + \Lambda^1(s) \left(\frac{1}{2} u_1^*(s)[1 - x^*(s)]^{-1/2} + \frac{1}{2} u_2^*(s)x^*(s)^{-1/2} \right) \right\},$$

$$\dot{\Lambda}^2(s) = \left\{ q_2 \exp(-rs) + \Lambda^2(s) \left(\frac{1}{2} u_1^*(s)[1 - x^*(s)]^{-1/2} + \frac{1}{2} u_2^*(s)x^*(s)^{-1/2} \right) \right\},$$

$$\Lambda^1(T) = \exp(-rT) S_1,$$

$$\Lambda^2(T) = -\exp(-rT) S_2.$$

Using (2.37), we obtain

$$u_1^*(s) = \frac{\Lambda^1(s)}{c_1} [1 - x^*(s)]^{1/2} \exp(rs) \quad \text{and} \quad u_2^*(s) = \frac{\Lambda^2(s)}{c_2} [x^*(s)]^{1/2} \exp(rs).$$

Substituting $u_1^*(s)$ and $u_2^*(s)$ into (2.37) yields

$$\begin{aligned} \dot{\Lambda}^1(s) &= \left\{ -q_1 \exp(-rs) + \left(\frac{[\Lambda^1(s)]^2}{2c_1} + \frac{\Lambda^1(s)\Lambda^2(s)}{2c_2} \right) \right\}, \\ \dot{\Lambda}^2(s) &= \left\{ q_2 \exp(-rs) + \left(\frac{[\Lambda^2(s)]^2}{2c_2} + \frac{\Lambda^1(s)\Lambda^2(s)}{2c_1} \right) \right\}, \end{aligned} \quad (2.38)$$

with boundary conditions $\Lambda^1(T) = \exp(-rT)S_1$ and $\Lambda^2(T) = -\exp(-rT)S_2$.

The game equilibrium state dynamics becomes

$$\dot{x}^*(s) = \frac{\Lambda^1(s) \exp(rs)}{c_1} [1 - x^*(s)] - \frac{\Lambda^2(s) \exp(rs)}{c_2} x^*(s), \quad x^*(0) = x_0. \quad (2.39)$$

Solving the system of differential equations in (2.38) gives the solution time paths of $\Lambda^1(s)$ and $\Lambda^2(s)$. Using these time paths in (2.39), a solution time path for $x^*(s)$ can be derived. Substituting these solution paths into $u_1^*(s)$ and $u_2^*(s)$ yields the open-loop game equilibrium strategies.

2.3 Market Outcomes Under Feedback Equilibria

In many economic analyses we could not assume that agents would commit to fixed control paths at the outset of the game, as in the case of the open-loop solution. In particular, there are hardly any means that can prevent the agents from revising their strategies during duration of the game. Instead, agents would consider adopting feedback strategies, which are decision rules that are dependent upon the current state $x(t)$ and current time t for $t_0 \leq t \leq s$.

2.3.1 Characterization of Feedback Equilibria

For the n -person differential game of (2.1) and (2.2), an n -tuple of feedback strategies $\{u_i^*(s) = \phi_i^*(s, x) \in U^i, \text{ for } i \in N\}$ constitutes a Nash equilibrium solution if the following relations for each $i \in N$ are satisfied:

$$\begin{aligned} & \int_t^T g^i[s, x^*(s), \phi_1^*(s, x^*(s)), \phi_2^*(s, x^*(s)), \dots, \phi_n^*(s, x^*(s))] \\ & \times \exp\left[-\int_{t_0}^s r(y) dy\right] ds + q^i(x^*(T)) \exp\left[-\int_{t_0}^T r(y) dy\right] \end{aligned}$$

$$\begin{aligned}
&\geq \int_t^T g^i[s, x^i(s), \phi_1^*(s, x^i(s)), \phi_2^*(s, x^i(s)), \dots, \phi_{i-1}^*(s, x^i(s)), \phi_i(s, x^i(s)), \\
&\quad \phi_{i+1}^*(s, x^i(s)), \dots, \phi_n^*(s, x^i(s))] \\
&\quad \times \exp\left[-\int_{t_0}^s r(y) dy\right] ds + q^i(x^i(T)) \exp\left[-\int_{t_0}^T r(y) dy\right], \\
&\forall \phi_i^*(s, x) \in U^i, x \in R^m,
\end{aligned} \tag{2.40}$$

where, on the interval $[t_0, T]$,

$$\dot{x}^*(s) = f[s, x^*(s), \phi_1^*(s, x^*(s)), \phi_2^*(s, x^*(s)), \dots, \phi_n^*(s, x^*(s))], \quad x^*(t) = x;$$

and

$$\begin{aligned}
\dot{x}^i(s) &= f[s, x^i(s), \phi_1^*(s, x^i(s)), \phi_2^*(s, x^i(s)), \dots, \phi_{i-1}^*(s, x^i(s)), \phi_i(s, x^i(s)), \\
&\quad \phi_{i+1}^*(s, x^i(s)), \dots, \phi_n^*(s, x^i(s))], \quad x^i(t) = x, \text{ for } i \in N.
\end{aligned}$$

One salient feature of the concept introduced above is that if an n -tuple $\{\phi_i^*; i \in N\}$ provides a feedback Nash equilibrium solution (FNES) to an N -person differential game with duration $[t_0, T]$, its restriction to the time interval $[t, T]$ provides an FNES to the same differential game defined on the shorter time interval $[t, T]$, with the initial state taken as $x(t)$, and this being so for all $t_0 \leq t \leq T$. An immediate consequence of this observation is that feedback Nash equilibrium strategies will depend only on the time variable and the current value of the state, but not on memory (including the initial state x_0). Therefore the agents' strategies can be expressed as $\{u_i(s) = \phi_i(s, x), \text{ for } i \in N\}$. The following theorem provides a set of conditions characterizing a feedback Nash equilibrium solution for the game in (2.1) and (2.2) and is characterized as follows.

Theorem 2.3 *An n -tuple of strategies $\{u_i^*(s) = \phi_i^*(t, x) \in U^i, \text{ for } i \in N\}$ provides a feedback Nash equilibrium solution to the game in (2.1) and (2.2) if there exist continuously differentiable functions $V^{(t_0)i}(t, x) : [t_0, T] \times R^m \rightarrow R, i \in N$, satisfying the following set of partial differential equations:*

$$\begin{aligned}
-V_t^{(t_0)i}(t, x) &= \max_{u_i} \left\{ g^i[t, x, \phi_1^*(t, x), \phi_2^*(t, x), \dots, \phi_{i-1}^*(t, x), u_i(t, x), \right. \\
&\quad \left. \phi_{i+1}^*(t, x), \dots, \phi_n^*(t, x)] \exp\left[-\int_{t_0}^t r(y) dy\right] \right. \\
&\quad \left. + V_x^{(t_0)i}(t, x) f[t, x, \phi_1^*(t, x), \phi_2^*(t, x), \dots, \phi_{i-1}^*(t, x), u_i(t, x), \right. \\
&\quad \left. \phi_{i+1}^*(t, x), \dots, \phi_n^*(t, x)] \right\}
\end{aligned}$$

$$\begin{aligned}
&= g^i[t, x, \phi_1^*(t, x), \phi_2^*(t, x), \dots, \phi_n^*(t, x)] \exp\left[-\int_{t_0}^t r(y) dy\right] \\
&\quad + V_x^{(t_0)i}(t, x) f[t, x, \phi_1^*(t, x), \phi_2^*(t, x), \dots, \phi_n^*(t, x)], \\
V^{(t_0)i}(T, x) &= q^i(x) \exp\left[-\int_{t_0}^T r(y) dy\right], \quad i \in N.
\end{aligned}$$

Proof Invoking Theorem A.1 in the Technical Appendixes, $V^{(t_0)i}(t, x)$ is the maximized payoff associated with the optimal control problem of agent i for given strategies $\{u_j^*(s) = \phi_j^*(t, x) \in U^j, \text{ for } j \in N \text{ and } j \neq i\}$ of the other $n - 1$ agents. The conditions in Theorem 2.3 imply the expressions in (2.40), and hence yield a Nash equilibrium. $\square$

Again, there may be multiple Nash equilibria; the agents are assumed to choose an equilibrium at time t_0 and stick with the corresponding strategies for the entire game interval. Moreover, $V^{(t_0)i}(t, x)$ is the game equilibrium payoff of agent i at time $t \in [t_0, T]$ with the state being x , that is,

$$\begin{aligned}
V^{(t_0)i}(t, x) &= \int_t^T g^i[s, x^*(s), \phi_1^*(s, x^*(s)), \phi_2^*(s, x^*(s)), \dots, \phi_n^*(s, x^*(s))] \\
&\quad \times \exp\left[-\int_{t_0}^s r(y) dy\right] ds + q^i(x^*(T)) \exp\left[-\int_{t_0}^T r(y) dy\right].
\end{aligned}$$

We also call it the value function of agent i in the game.

A remark that will be utilized in the subsequent analysis is given below.

Remark 2.1 Let $V^{(\tau)i}(t, x)$ denote the value function of agent i in a game with the payoffs in (2.1) and dynamics in (2.2), which starts at time τ for $\tau \in [t_0, T]$. Note that the equilibrium feedback strategies are Markovian in the sense that they depend on the current time and current state. One can readily verify that

$$\begin{aligned}
&\exp\left[\int_{t_0}^\tau r(y) dy\right] V^{(t_0)i}(t, x) \\
&= \exp\left[\int_{t_0}^\tau r(y) dy\right] \\
&\quad \times \int_t^T g^i[s, x^*(s), \phi_1^*(s, x^*(s)), \phi_2^*(s, x^*(s)), \dots, \phi_n^*(s, x^*(s))] \\
&\quad \times \exp\left[-\int_{t_0}^s r(y) dy\right] ds \\
&= \int_t^T g^i[s, x^*(s), \phi_1^*(s, x^*(s)), \phi_2^*(s, x^*(s)), \dots, \phi_n^*(s, x^*(s))]
\end{aligned}$$

$$\begin{aligned} & \times \exp\left[-\int_{\tau}^s r(y) dy\right] ds \\ & = V^{(\tau)i}(t, x), \end{aligned}$$

for $\tau \in [t_0, T)$.

We now turn to the infinite-horizon autonomous game in (2.3) and (2.4). First, consider the infinite-horizon subgame that starts at time $\tau \in [t_0, \infty)$ with initial state $x(\tau) = x$

$$\max_{u_i} \int_{\tau}^{\infty} g^i[x(s), u_1(s), u_2(s), \dots, u_n(s)] \exp[-r(s - \tau)] ds, \quad \text{for } i \in N, \quad (2.41)$$

subject to the dynamics

$$\dot{x}(s) = f[x(s), u_1(s), u_2(s), \dots, u_n(s)], \quad x(\tau) = x. \quad (2.42)$$

The infinite-horizon autonomous game in (2.41) and (2.42) is independent of the choice of τ and dependent only upon the state at the starting time, that is, x .

In the infinite-horizon optimization problem in Sect. A.1 in the Technical Appendixes, the feedback control is shown to be a function the state variable x only. With the validity of the game equilibrium $\{u_i^*(s) = \phi_i^*(x) \in U^i, \text{ for } i \in N\}$ to be verified later, we first define the following.

Definition 2.1 For the n -person differential game in (2.41) and (2.42), an n -tuple of feedback strategies $\{u_i^*(s) = \phi_i^*(x) \in U^i, \text{ for } i \in N\}$ constitutes a *feedback Nash equilibrium solution* if the following relations for each $i \in N$ are satisfied:

$$\begin{aligned} & \int_t^{\infty} g^i[x^*(s), \phi_1^*(x^*(s)), \phi_2^*(x^*(s)), \dots, \phi_n^*(x^*(s))] \exp[-r(s - \tau)] ds \\ & \geq \int_t^{\infty} g^i[x^i(s), \phi_1^*(x^i(s)), \phi_2^*(x^i(s)), \dots, \phi_{i-1}^*(x^i(s)), \phi_i(x^i(s)), \\ & \quad \phi_{i+1}^*(x^i(s)), \dots, \phi_n^*(x^i(s))] \exp[-r(s - \tau)] ds, \\ & \forall \phi_i(\cdot) \in \Gamma^i, x \in R^m, \end{aligned} \quad (2.43)$$

where on the interval $[\tau, \infty)$,

$$\begin{aligned} \dot{x}^*(s) &= f[x^*(s), \phi_1^*(x^*(s)), \phi_2^*(x^*(s)), \dots, \phi_n^*(x^*(s))], \quad x^*(s) = x; \\ \dot{x}^i(s) &= f[x^i(s), \phi_1^*(x^i(s)), \phi_2^*(x^i(s)), \dots, \phi_{i-1}^*(x^i(s)), \phi_i(x^i(s)), \\ & \quad \phi_{i+1}^*(x^i(s)), \dots, \phi_n^*(x^i(s))], \quad x^i(t) = x. \end{aligned}$$

We can express the value function of agent i as

$$V^{(\tau)i}(t, x) = \exp[-r(t - \tau)] \int_t^\infty g^i[x^*(s), \phi_1^*(x^*(s)), \phi_2^*(x^*(s)), \dots, \phi_n^*(x^*(s))] \\ \times \exp[-r(s - t)] ds,$$

for $x(t) = x^*(t) = x$.

Since $\int_t^\infty g^i[x^*(s), \phi_1^*(x^*(s)), \phi_2^*(x^*(s)), \dots, \phi_n^*(x^*(s))] \exp[-r(s - t)] ds$ is independent of the choice of t and dependent only upon the state at the starting time x , we can write

$$\hat{V}^i(x) = \int_t^\infty g^i[x^*(s), \phi_1^*(x^*(s)), \phi_2^*(x^*(s)), \dots, \phi_n^*(x^*(s))] \exp[-r(s - t)] ds.$$

It follows that

$$\begin{aligned} V^{(\tau)i}(t, x) &= \exp[-r(t - \tau)] \hat{V}^i(x), \\ V_t^{(\tau)i}(t, x) &= -r \exp[-r(t - \tau)] \hat{V}^i(x), \quad \text{and} \\ V_x^{(\tau)i}(t, x) &= \exp[-r(t - \tau)] \hat{V}_x^i(x), \quad \text{for } i \in N. \end{aligned} \tag{2.44}$$

A feedback Nash equilibrium solution for the infinite-horizon autonomous game in (2.41) and (2.42) can be characterized as follows.

Theorem 2.4 *An n -tuple of strategies $\{u_i^* = \phi_i^*(\cdot) \in U^i, \text{ for } i \in N\}$, provides a feedback Nash equilibrium solution to the infinite-horizon game in (2.3) and (2.4) if there exist continuously differentiable functions $\hat{V}^i(x) : R^m \rightarrow R, i \in N$, satisfying the following set of partial differential equations:*

$$\begin{aligned} r \hat{V}^i(x) &= \max_{u_i} \{ g^i[x, \phi_1^*(x), \phi_2^*(x), \dots, \phi_{i-1}^*(x), u_i, \phi_{i+1}^*(x), \dots, \phi_n^*(x)] \\ &\quad + \hat{V}_x^i(x) f[x, \phi_1^*(x), \phi_2^*(x), \dots, \phi_{i-1}^*(x), u_i, \phi_{i+1}^*(x), \dots, \phi_n^*(x)] \} \\ &= \{ g^i[x, \phi_1^*(x), \phi_2^*(x), \dots, \phi_n^*(x)] \\ &\quad + \hat{V}_x^i(x) f[x, \phi_1^*(x), \phi_2^*(x), \dots, \phi_n^*(x)] \}, \end{aligned}$$

for $i \in N$.

Proof By Theorem A.2 in the Technical Appendixes, $\hat{V}^i(x)$ is the value function associated with the optimal control problem of agent $i, i \in N$. Together with the expressions in Definition 2.1, the conditions in Theorem 2.4 imply a Nash equilibrium. $\square$

Since time t is not explicitly involved in the partial differential equations in Theorem 2.4, the validity that the feedback Nash equilibrium $\{u_i^* = \phi_i^*(x), \text{ for } i \in N\}$, are functions independent of time is obtained.

Substituting the game equilibrium strategies in Theorem 2.4 into (2.4) yields the game equilibrium dynamics of the state path as

$$\dot{x}(s) = f[x(s), \phi_1^*(x(s)), \phi_2^*(x(s)), \dots, \phi_n^*(x(s))], \quad x(t_0) = x_0.$$

Solving the above dynamics yields the optimal state trajectory $\{x^*(t)\}_{t \geq t_0}$ as

$$x^*(t) = x_0 + \int_{t_0}^t f[x^*(s), \phi_1^*(x^*(s)), \phi_2^*(x^*(s)), \dots, \phi_n^*(x^*(s))] ds, \\ \text{for } t \geq t_0. \quad (2.45)$$

We denote term $x^*(t)$ by x_t^* . The feedback Nash equilibrium strategies for the infinite-horizon game in (2.3) and (2.4) can be obtained as

$$[\phi_1^*(x_t^*), \phi_2^*(x_t^*), \dots, \phi_n^*(x_t^*)], \quad \text{for } t \geq t_0.$$

2.3.2 Feedback Equilibria in Resource Extraction

Consider an economy endowed with a renewable resource and with $n \geq 2$ resource extractors (firms). The lease for resource extraction begins at time t_0 and ends at time T . Let $u_i(s)$ denote the rate of resource extraction of firm i at time s , $i \in N = \{1, 2, \dots, n\}$, where each extractor controls its rate of extraction. Let U^i be the set of admissible extraction rates and $x(s)$ the size of the resource stock at time s . In particular, we have $U^i \in R^+$ for $x > 0$ and $= \{0\}$ for $x = 0$. The extraction cost for firm $i \in N$ depends on the quantity of the resource extracted $u_i(s)$, the resource stock size $x(s)$, and a parameter c .

In particular, the extraction cost can be specified as $C^i = cu_i(s)/x(s)^{1/2}$. The market price of the resource depends on the total amount extracted and supplied to the market. The price-output relationship at time s is given by the following downward-sloping inverse demand curve $P(s) = Q(s)^{-1/2}$, where $Q(s) = \sum_{j=1}^n u_j(s)$ is the total amount of the resource extracted and marketed at time s . A terminal bonus $w x(T)^{1/2}$ is offered to each extractor and r is a discount rate that is common to all extractors. Extractor i seeks to maximize the present value of the profits

$$\int_{t_0}^T \left[\left(\sum_{j=1}^n u_j(s) \right)^{-1/2} u_i(s) - \frac{c}{x(s)^{1/2}} u_i(s) \right] \exp[-r(s - t_0)] ds \\ + \exp[-r(T - t_0)] w x(T)^{1/2}, \quad \text{for } i \in N, \quad (2.46)$$

subject to the resource dynamics

$$\dot{x}(s) = ax(s)^{1/2} - bx(s) - \sum_{j=1}^n u_j(s), \quad x(t_0) = x_0 \in X. \quad (2.47)$$

The model is a deterministic version of the Jørgensen and Yeung (1996) fishery game model. Invoking Theorem 2.3, a set of feedback strategies $\{u_i^*(t) = \phi_i^*(t, x); i \in N\}$ constitutes a feedback Nash equilibrium solution for the game in (2.46) and (2.47), if there exist functions $V^{(t_0)i}(t, x) : [t_0, T] \times R \rightarrow R$ for $i \in N$, which satisfy the following set of partial differential equations:

$$\begin{aligned} -V_t^{(t_0)i}(t, x) = \max_{u_i \in U^i} & \left\{ \left[u_i \left(\sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(t, x) + u_i \right)^{-1/2} - \frac{c}{x^{1/2}} u_i(t) \right] \exp[-r(t - t_0)] \right. \\ & \left. + V_x^{(t_0)i} \left[ax^{1/2} - bx - \sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(t, x) - u_i \right] \right\}, \quad \text{and} \end{aligned} \quad (2.48)$$

$$V^{(t_0)i}(T, x) = \exp[-r(T - t_0)]wx^{1/2}.$$

Applying the maximization operator on the right-hand side of the first equation in (2.49) for agent i yields the condition for a maximum as

$$\begin{aligned} & \left[\left(\sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(t, x) + \frac{1}{2}\phi_i^*(t, x) \right) \left(\sum_{j=1}^n \phi_j^*(t, x) \right)^{-3/2} - \frac{c}{x^{1/2}} \right] \exp[-r(t - t_0)] \\ & - V_x^{(t_0)i} = 0, \end{aligned} \quad (2.49)$$

for $i \in N$.

Summing over $i = 1, 2, \dots, n$ in (2.49) yields

$$\left(\sum_{j=1}^n \phi_j^*(t, x) \right)^{1/2} = \left(n - \frac{1}{2} \right) \left(\sum_{j=1}^n \left[\frac{c}{x^{1/2}} + \exp[r(t - t_0)]V_x^{(t_0)j} \right] \right)^{-1}. \quad (2.50)$$

Substituting (2.50) into (2.49) produces

$$\begin{aligned} & \left(\sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(t, x) + \frac{1}{2}\phi_i^*(t, x) \right) \left(n - \frac{1}{2} \right)^{-3} \left(\sum_{j=1}^n \left[\frac{c}{x^{1/2}} + \exp[r(t - t_0)]V_x^{(t_0)j} \right] \right)^3 \\ & - \frac{c}{x^{1/2}} - \exp[r(t - t_0)]V_x^{(t_0)i} = 0, \quad \text{for } i \in N. \end{aligned} \quad (2.51)$$

Rearranging the terms in (2.51) yields

$$\begin{aligned} & \left(\sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(t, x) + \frac{1}{2}\phi_i^*(t, x) \right) \\ & = \left(n - \frac{1}{2} \right)^3 \frac{[c + \exp[r(t - t_0)]V_x^{(t_0)i}x^{1/2}]x}{(\sum_{j=1}^n [c + \exp[r(t - t_0)]V_x^{(t_0)j}x^{1/2}])^3}, \end{aligned} \quad (2.52)$$

for $i \in N$.

Condition (2.52) represents a system of equations that is linear in $\{\phi_1^*(t, x), \phi_2^*(t, x), \dots, \phi_n^*(t, x)\}$. Solving (2.52) yields

$$\phi_i^*(t, x) = \frac{x(2n-1)^2}{2[\sum_{j=1}^n [c + \exp[r(t-t_0)]V_x^{(t_0)j}x^{1/2}]]^3} \left\{ \sum_{j \neq i}^n \left[c + \frac{V_x^{(t_0)j}x^{1/2}}{\exp[-r(t-t_0)]} \right] \right. \\ \left. - \left(n - \frac{3}{2} \right) \left[c + \frac{V_x^{(t_0)i}x^{1/2}}{\exp[-r(t-t_0)]} \right] \right\}, \quad \text{for } i \in N. \quad (2.53)$$

Substituting $\phi_i^*(t, x)$ in (2.53) into (2.49); upon solving it yields the following.

Proposition 2.1 *The system in (2.49) admits a solution*

$$V^{(t_0)i}(t, x) = \exp[-r(t-t_0)][A(t)x^{1/2} + B(t)], \quad \text{for } i \in N, \quad (2.54)$$

where $A(t)$ and $B(t)$ satisfy

$$\dot{A}(t) = \left[r + \frac{b}{2} \right] A(t) - \frac{(2n-1)}{2n^2} \left(c + \frac{A(t)}{2} \right)^{-1} \\ + \frac{c(2n-1)^2}{4n^3} \left(c + \frac{A(t)}{2} \right)^{-2} + \frac{(2n-1)^2 A(t)}{8n^2 (c + \frac{A(t)}{2})^2}, \quad (2.55) \\ \dot{B}(t) = rB(t) - \frac{a}{2} A(t), \\ A(T) = w \quad \text{and} \quad B(T) = 0.$$

Proof Substituting $V^i(t, x)$ and the relevant derivatives $V_t^i(t, x)$ and $V_x^i(t, x)$ into (2.53) and (2.49) yields the results in Proposition 2.1. $\square$

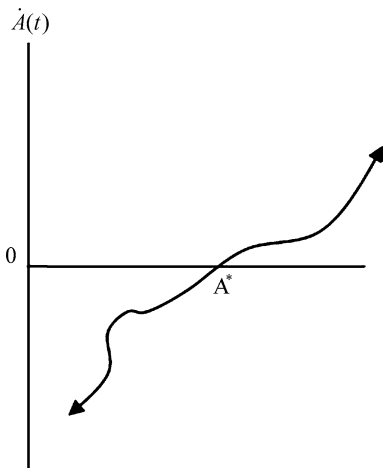
The first equation in (2.55) can be further reduced to

$$\dot{A}(t) = \left\{ \left(r + \frac{1}{8}\sigma^2 + \frac{b}{2} \right) \frac{[A(t)]^3}{4} + \left(r + \frac{1}{8}\sigma^2 + \frac{b}{2} \right) c [A(t)]^2 \right. \\ \left. + \left[\left(r + \frac{1}{8}\sigma^2 + \frac{b}{2} \right) c^2 + \frac{(4n^2 - 8n + 3)}{8n^2} \right] A(t) - \frac{(2n-1)c}{4n^3} \right\} \\ \left/ \left(c + \frac{A(t)}{2} \right)^2 \right. \quad (2.56)$$

The denominator of the right-hand side of (2.56) is always positive. Denote the numerator of the right-hand side of (2.56) by

$$F[A(t)] = \frac{(2n-1)c}{4n^3}. \quad (2.57)$$

Fig. 2.1 Phase diagram for $\dot{A}(t)$ and $A(t)$



In particular, $F[A(t)]$ is a polynomial function in $A(t)$ of degree 3. Moreover, $F[A(t)] = 0$ for $A(t) = 0$, and for any $A(t) \in (0, \infty)$,

$$\begin{aligned} \frac{dF[A(t)]}{dA(t)} &= \left(r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right) \frac{3[A(t)]^2}{4} + 2\left(r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right)c[A(t)] \\ &\quad + \left[\left(r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right)c^2 + \frac{(4n^2 - 8n + 3)}{8n^2}\right] > 0. \end{aligned} \quad (2.58)$$

Therefore, there exists a unique level of $A(t)$, denoted by A^* , at which

$$F[A^*] - \frac{(2n-1)c}{4n^3} = 0. \quad (2.59)$$

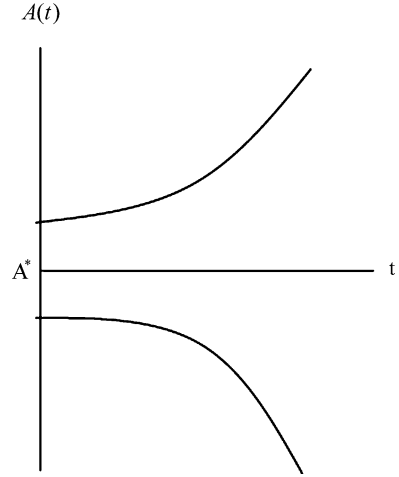
If $A(t)$ equals A^* , $\dot{A}(t) = 0$. For values of $A(t)$ less than A^* , $\dot{A}(t)$ is negative. For values of $A(t)$ greater than A^* , $\dot{A}(t)$ is positive. A phase diagram depicting the relationship between $\dot{A}(t)$ and $A(t)$ is provided in Fig. 2.1, while the time paths of $A(t)$ in relation to A^* are illustrated in Fig. 2.2.

For a given value of w that is less than A^* , the time path $\{A(t)\}_{t=t_0}^T$ will start at a value $A(t_0)$, which is greater than w and less than A^* . The value of $A(t)$ will decrease over time and reach w at time T . On the other hand, for a given value of w that is greater than A^* , the time path $\{A(t)\}_{t=t_0}^T$ will start at a value $A(t_0)$, which is less than w and greater than A^* . The value of $A(t)$ will increase over time and reach w at time T . Therefore $A(t)$ is a monotonic function and $A(t) > 0$, for $t \in [t_\tau, T]$.

Using $A(t)$, the solution to $B(t)$ can be readily obtained as

$$B(t) = \exp(rt) \left(K - \int_{t_0}^t \frac{a}{2} A(s) \exp(-rs) ds \right), \quad (2.60)$$

where $K = \int_{t_0}^T \frac{a}{2} A(s) \exp(-rs) ds$.

Fig. 2.2 Time paths of $A(t)$ 

Substituting the relevant derivatives of the value functions in Proposition 2.1 into the game equilibrium strategies of (2.53) gives the feedback Nash equilibrium of the resource extraction game of (2.46) and (2.47).

2.3.3 Feedback Solution in Competitive Advertising

Consider the competitive advertising game in Sect. 2.2.2. Instead of an open-loop solution we seek a feedback Nash equilibrium. Invoking Theorem 2.3, a set of feedback strategies $\{u_i^*(t) = \phi_i^*(t, x); i \in N\}$ constitutes a feedback Nash equilibrium solution for the game in (2.35) and (2.36), if there exist functions $V^{(t_0)i}(t, x) : [t_0, T] \times R \rightarrow R$ for $i \in \{1, 2\}$, which satisfy the following set of partial differential equations:

$$\begin{aligned} -V_t^{(t_0)1}(t, x) &= \max_{u_1} \left\{ \left(q_1 x - \frac{c_1}{2} u_1^2 \right) \exp(-rt) \right. \\ &\quad \left. + V_x^1(t, x) [u_1 (1-x)^{1/2} - \phi_2^*(t, x) x^{1/2}] \right\}, \\ V^{(t_0)1}(T, x) &= \exp(-rT) S_1 x \end{aligned}$$

and

$$\begin{aligned} -V_t^{(t_0)2}(t, x) &= \max_{u_1} \left\{ \left[q_2 (1-x) - \frac{c_2}{2} u_2^2 \right] \exp(-rt) \right. \\ &\quad \left. + V_x^2(t, x) [\phi_1^*(t, x) (1-x)^{1/2} - u_2 x^{1/2}] \right\}, \quad (2.61) \\ V^{(t_0)2}(T, x) &= \exp(-rT) S_2 (1-x). \end{aligned}$$

Performing the indicated maximization in (2.62) yields the condition for a maximum as

$$\begin{aligned}\phi_1^*(t, x) &= \frac{\exp rt}{c_1} V_x^{(t_0)1}(t, x)(1-x)^{1/2} \quad \text{and} \\ \phi_2^*(t, x) &= \frac{\exp rt}{c_2} V_x^{(t_0)2}(t, x)x^{1/2}.\end{aligned}\tag{2.62}$$

Substituting $\phi_1^*(t, x)$ and $\phi_2^*(t, x)$ into (2.62) and solving it yields

$$\begin{aligned}-V_t^{(t_0)1}(t, x) &= \left(q_1 x - \frac{(\exp rt)^2}{2c_1} [V_x^{(t_0)1}(t, x)]^2 (1-x) \right) \exp(-rt) \\ &\quad + V_x^{(t_0)1}(t, x) \left[\frac{\exp rt}{c_1} V_x^{(t_0)1}(t, x)(1-x) - \frac{\exp rt}{c_2} V_x^{(t_0)2}(t, x)x \right], \\ V^{(t_0)1}(T, x) &= \exp(-rT) S_1 x; \\ -V_t^{(t_0)2}(t, x) &= \left(q_2 (1-x) - \frac{(\exp rt)^2}{2c_2} [V_x^{(t_0)2}(t, x)]^2 x \right) \exp(-rt) \\ &\quad + V_x^{(t_0)2}(t, x) \left[\frac{\exp rt}{c_1} V_x^{(t_0)1}(t, x)(1-x) - \frac{\exp rt}{c_2} V_x^{(t_0)2}(t, x)x \right], \\ V^{(t_0)2}(T, x) &= \exp(-rT) S_2 (1-x).\end{aligned}\tag{2.63}$$

Proposition 2.2 *The system in (2.63) admits a solution*

$$\begin{aligned}V^{(t_0)1}(t, x) &= \exp[-r(t)] [A_1(t)x + B_1(t)], \\ V^{(t_0)2}(t, x) &= \exp[-r(t)] [A_2(t)x + B_2(t)],\end{aligned}$$

where $A(t)$ and $B(t)$ satisfy

$$\begin{aligned}\dot{A}_1(t) &= r A_1(t) - q_1 + \frac{[A_1(t)]^2}{2c_1} + \frac{A_1(t)A_2(t)}{c_2}, \quad A_1(T) = S_1, \\ \dot{B}_1(t) &= r B_1(t) - \frac{[A_1(t)]^2}{2c_1}, \quad B_1(T) = 0; \\ \dot{A}_2(t) &= r A_2(t) - q_2 + \frac{[A_2(t)]^2}{2c_2} + \frac{A_1(t)A_2(t)}{c_1}, \quad A_2(T) = S_2, \\ \dot{B}_2(t) &= r B_2(t) - \frac{[A_2(t)]^2}{2c_2}, \quad B_2(T) = 0.\end{aligned}$$

Proof Substituting $V^i(t, x)$ and the relevant derivatives $V_t^i(t, x)$ and $V_x^i(t, x)$ into (2.63) yields the results in Proposition 2.2. $\square$

With the value functions in Proposition 3.2, one can characterize the game equilibrium strategies in (2.62) over the game interval $[t_0, T]$, the equilibrium state path, and the profits of the firms over time.

2.3.4 Duopolistic Competition in Infinite Horizon

Consider a dynamic duopoly in which there are two publicly listed firms selling a homogeneous good. Since the value of a publicly listed firm is the present value of its discounted expected future earnings. The terminal time of the game T may be very far in the future and nobody knows when the firms will be out of business. Therefore, setting $T = \infty$ may very well be the best approximation for the true game horizon. Even if the firm's management restricts itself to considering profit maximization over the next year, it should value its asset positions at the end of the year by the earning potential of these assets in the years to come. There is a lag in price adjustment so the evolution of market price over time is assumed to be a function of the current market price and the price specified by the current demand condition. In particular, we follow Tsutsui and Mino (1990) and assume that

$$\dot{P}(s) = k[a - u_1(s) - u_2(s) - P(s)], \quad P(t_0) = P_0, \quad (2.64)$$

where $P(s)$ is the market price at time s , $u_i(s)$ is the output supplied firm $i \in \{1, 2\}$, the current demand condition is specified by the instantaneous inverse demand function $P(s) = [a - u_1(s) - u_2(s)]$ and $k > 0$ represents the price adjustment velocity.

The payoff of firm i is given as the present value of the stream of discounted profits

$$\int_{t_0}^{\infty} \{P(s)u_i(s) - cu_i(s) - (1/2)[u_i(s)]^2\} \exp[-r(s - t_0)] ds, \\ \text{for } i \in \{1, 2\}, \quad (2.65)$$

where $cu_i(s) + (1/2)[u_i(s)]^2$ is the cost of producing output $u_i(s)$ and r is the interest rate.

Once again, we consider the infinite-horizon game that starts at time $t \in [t_0, \infty)$ with initial state $P(t) = P$

$$\max_{u_i} \int_t^{\infty} \{P(s)u_i(s) - cu_i(s) - (1/2)[u_i(s)]^2\} \exp[-r(s - t)] ds, \\ \text{for } i \in \{1, 2\}, \quad (2.66)$$

subject to

$$\dot{P}(s) = k[a - u_1(s) - u_2(s) - P(s)], \quad P(t) = P. \quad (2.67)$$

The infinite-horizon game in (2.66) and (2.67) has autonomous structures and a constant rate. Therefore, we can apply Theorem 3.2 to characterize a feedback Nash equilibrium solution as

$$\begin{aligned} r\hat{V}^i(P) = \max_{u_i} \{ & [Pu_i - cu_i - (1/2)(u_i)^2] \\ & + \hat{V}_P^i[k(a - u_i - \phi_j^*(P) - P)] \}, \quad \text{for } i \in \{1, 2\}. \end{aligned} \quad (2.68)$$

Performing the indicated maximization in (2.68), we obtain

$$\phi_i^*(P) = P - c - k\hat{V}_P^i(P), \quad \text{for } i \in \{1, 2\}. \quad (2.69)$$

Substituting the results from (2.69) into (2.68), and upon solving (2.68), yields

$$\hat{V}^i(P) = \frac{1}{2}AP^2 - BP + C, \quad (2.70)$$

where

$$\begin{aligned} A &= \frac{r + 6k - \sqrt{(r + 6k)^2 - 12k^2}}{6k^2}, \\ B &= \frac{-akA + c - 2kcA}{r - 3k^2A + 3k}, \quad \text{and} \\ C &= \frac{c^2 + 3k^2B^2 - 2kB(2c + a)}{2r}. \end{aligned}$$

Again, one can readily verify that $\hat{V}^i(P)$ in (2.70) indeed solves (2.68) by substituting $\hat{V}^i(P)$ and its derivative into (2.68) and (2.69).

The game equilibrium strategy can then be expressed as

$$\phi_i^*(P) = P - c - k(AP - B), \quad \text{for } i \in \{1, 2\}.$$

Substituting the game equilibrium strategies above into (2.64) yields the game equilibrium state dynamics of the game in (2.64) and (2.65) as

$$\dot{P}(s) = k[a - 2(c + kB) - (3 - kA)P(s)], \quad P(t_0) = P_0.$$

Solving the above dynamics yields the optimal state trajectory as

$$P^*(t) = \left[P_0 - \frac{k[a - 2(c + kB)]}{k(3 - kA)} \right] \exp[-k(3 - kA)t] + \frac{k[a - 2(c + kB)]}{k(3 - kA)}.$$

We denote term $P^*(t)$ by P_t^* . The feedback Nash equilibrium strategies for the infinite-horizon game in (2.64) and (2.65) can be obtained as

$$\phi_i^*(P_t^*) = P_t^* - c - k(AP_t^* - B), \quad \text{for } i \in \{1, 2\}.$$

2.4 Dynamic Stochastic Interactive Economic System

One way to incorporate stochastic elements in dynamic interactive economic systems is to introduce stochastic dynamics. Uncertainties in the evolution of economic state variables are prevalent. For instance, the natural growth rate of renewable resources, the development of technology, capital accumulation, the build-up of goodwill, and special skills are often subject to stochastic impacts.

2.4.1 Game Formulation and Solution Characterization

A stochastic formulation of state dynamics is by adopting a vector-valued stochastic differential equation

$$\begin{aligned} dx(s) &= f[s, x(s), u_1(s), u_2(s), \dots, u_n(s)] ds + \sigma[s, x(s)] dz(s), \\ x(t_0) &= x_0, \end{aligned} \quad (2.71)$$

where $\sigma[s, x(s)]$ is a $m \times \Theta$ matrix, $z(s)$ is a Θ -dimensional Wiener process, and the initial state x_0 is given. Let $\Omega[s, x(s)] = \sigma[s, x(s)]$, $\sigma[s, x(s)]$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}[s, x(s)]$. Moreover, $E[dz_{\varpi}] = 0$, $E[dz_{\varpi} dt] = 0$ and $E[(dz_{\varpi})^2] = dt$ for $\varpi \in [1, 2, \dots, \Theta]$; $E[dz_{\varpi} dz_{\omega}] = 0$ for $\varpi \in [1, 2, \dots, \Theta]$, $\varpi \in [1, 2, \dots, \Theta]$, and $\varpi \neq \omega$.

Given the stochastic nature of the state dynamics, the economic agent i 's objective becomes

$$\begin{aligned} E_{t_0} \left\{ \int_{t_0}^T g^i[s, x(s), u_1(s), u_2(s), \dots, u_n(s)] \exp \left[- \int_{t_0}^s r(y) dy \right] ds \right. \\ \left. + \exp \left[- \int_{t_0}^T r(y) dy \right] q^i(x(T)) \right\}, \quad \text{for } i \in N, \end{aligned} \quad (2.72)$$

with $E_{t_0}\{\cdot\}$ denoting the expectation operation taken at time t_0 .

The system in (2.71) and (2.72) is a stochastic differential game. Basar (1977a, 1977b, 1980) was the first to derive explicit results for stochastic linear quadratic differential games. Examples of solvable stochastic differential games in economics include Clemhout and Wan (1985b), Kaitala (1993), Jørgensen and Yeung (1996, 1999), and Yeung (1998, 1999, 2001).

A Nash equilibrium of the stochastic game in (2.71) and (2.72) can be characterized as follows.

Theorem 2.5 *An N -tuple of feedback strategies $\{\phi_i^*(t, x) \in U^i; i \in N\}$ provides a Nash equilibrium solution to the game in (2.71) and (2.72) if there exist suitably*

smooth functions $V^{(t_0)i}(t, x) : [t_0, T] \times R^m \rightarrow R, i \in N$, satisfying the partial differential equations

$$\begin{aligned}
 & -V_t^{(t_0)i}(t, x) - \frac{1}{2} \sum_{h, \zeta=1}^m \Omega^{h\zeta}(t, x) V_{x^h x^\zeta}^{(t_0)i}(t, x) \\
 & = \max_{u_i} \left\{ g^i[t, x, \phi_1^*(t, x), \phi_2^*(t, x), \dots, \phi_{i-1}^*(t, x), u_i(t), \phi_{i+1}^*(t, x), \dots, \phi_n^*(t, x)] \right. \\
 & \quad \times \exp\left[-\int_{t_0}^s r(y) dy\right] + V_x^{(t_0)i}(t, x) \\
 & \quad \times f[t, x, \phi_1^*(t, x), \phi_2^*(t, x), \dots, \phi_{i-1}^*(t, x), u_i(t), \phi_{i+1}^*(t, x), \dots, \phi_n^*(t, x)] \Big\}, \\
 & V^{(t_0)i}(T, x) = q^i(x) \exp\left[-\int_{t_0}^T r(y) dy\right], \quad i \in N.
 \end{aligned}$$

Proof This result follows readily from the definition of the Nash equilibrium and from the stochastic control result in Theorem A.5 of the Technical Appendixes. $\square$

In particular, $V^{(t_0)i}(t, x)$ represents the expected game equilibrium payoff of agent i at time $t \in [t_0, T]$ with the state being x , that is,

$$\begin{aligned}
 E_{t_0} \left\{ V^{(t_0)i}(t, x) = \int_t^T g^i[s, x^*(s), \phi_1^*(s, x^*(s)), \phi_2^*(s, x^*(s)), \dots, \phi_n^*(s, x^*(s))] \right. \\
 \left. \times \exp\left[-\int_{t_0}^s r(y) dy\right] ds + q^i(x^*(T)) \exp\left[-\int_{t_0}^T r(y) dy\right] \right\}.
 \end{aligned}$$

A remark that will be utilized in the subsequent analysis is given below.

Remark 2.2 Let $V^{(\tau)i}(t, x)$ denote the value function of nation i in a game with stochastic dynamics found in (2.71) and expected payoffs in (2.72), which starts at time τ for $\tau \in [t_0, T]$. Note that the equilibrium feedback strategies are Markovian in the sense that they depend on the current time and the current state. One can readily verify that

$$\begin{aligned}
 & \exp\left[\int_{t_0}^\tau r(y) dy\right] V^{(t_0)i}(t, x) \\
 & = \exp\left[\int_{t_0}^\tau r(y) dy\right] \\
 & \quad \times E_{t_0} \left\{ \int_t^T g^i[s, x^*(s), \phi_1^*(s, x^*(s)), \phi_2^*(s, x^*(s)), \dots, \phi_n^*(s, x^*(s))] \right.
 \end{aligned}$$

$$\begin{aligned}
& \times \exp \left[- \int_{t_0}^s r(y) dy \right] ds \Big\} \\
& = E_t \left\{ \int_t^T g^i [s, x^*(s), \phi_1^*(s, x^*(s)), \phi_2^*(s, x^*(s)), \dots, \phi_n^*(s, x^*(s))] \right. \\
& \quad \times \exp \left[- \int_\tau^s r(y) dy \right] ds \Big\} = V^{(\tau)i}(t, x), \quad \text{for } \tau \in [t_0, T].
\end{aligned}$$

In the case when the terminal horizon T approaches infinity, an autonomous game structure with constant discounting will replace (2.71) and (2.72). In particular, the game becomes

$$\begin{aligned}
& \max_{u_i} E_{t_0} \left\{ \int_{t_0}^\infty g^i [x(s), u_1(s), u_2(s), \dots, u_n(s)] \exp[-r(s - t_0)] ds \right\}, \\
& \text{for } i \in N,
\end{aligned} \tag{2.73}$$

subject to the stochastic dynamics

$$dx(s) = f[x(s), u_1(s), u_2(s), \dots, u_n(s)] ds + \sigma[x(s)] dz(s), \quad x(t_0) = x_0. \tag{2.74}$$

Consider the alternative infinite-horizon game that starts at time $t \in [t_0, \infty)$ with initial state $x(t) = x$

$$\max_{u_i} E_t \left\{ \int_t^\infty g^i [x(s), u_1(s), u_2(s), \dots, u_n(s)] \exp[-r(s - t)] ds \right\}, \tag{2.75}$$

for $i \in N$, subject to the stochastic dynamics

$$dx(s) = f[x(s), u_1(s), u_2(s), \dots, u_n(s)] ds + \sigma[x(s)] dz(s), \quad x(t) = x_t. \tag{2.76}$$

Let $\Omega[x(s)] = \sigma[x(s)]\sigma[x(s)]^T$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}[x(s)]$.

The infinite-horizon autonomous game in (2.75) and (2.76) is independent of the choice of t and dependent only upon the state at the starting time, that is, x .

A Nash equilibrium solution for the infinite-horizon stochastic differential game in (2.75) and (2.76) can be characterized as follows.

Theorem 2.6 *An n -tuple of strategies $\{u_i^* = \phi_i^*(\cdot) \in U^i, \text{ for } i \in N\}$, provides a Nash equilibrium solution to the game in (2.75) and (2.76) if there exist continuously twice differentiable functions $\hat{V}^i(x) : R^m \rightarrow R, i \in N$, satisfying the following set of partial differential equations:*

$$\begin{aligned}
& r \hat{V}^i(x) - \frac{1}{2} \sum_{h, \zeta=1}^m \Omega^{h\zeta}(x) \hat{V}_{x^h x^\zeta}^i(x) \\
& = \max_{u_i} \{ g^i [x, \phi_1^*(x), \phi_2^*(x), \dots, \phi_{i-1}^*(x), u_i(x), \phi_{i+1}^*(x), \dots, \phi_n^*(x)] \}
\end{aligned}$$

$$\begin{aligned}
& + \hat{V}_x^i(x) f[x, \phi_1^*(x), \phi_2^*(x), \dots, \phi_{i-1}^*(x), u_i(x), \phi_{i+1}^*(x), \dots, \phi_n^*(x)] \} \\
& = \{ g^i[x, \phi_1^*(x), \phi_2^*(x), \dots, \phi_n^*(x)] + \hat{V}_x^i(x) f[x, \phi_1^*(x), \phi_2^*(x), \dots, \phi_n^*(x)] \},
\end{aligned}$$

for $i \in N$.

Proof This result follows readily from the definition of a Nash equilibrium and from the infinite-horizon stochastic control Theorem A.6 in the Technical Appendixes. $\square$

2.4.2 An Application of Stochastic Differential Games in Resource Extraction

Consider the resource extraction game in Sect. 2.3.2. To present a stochastic model we replace the deterministic state dynamics with a stochastic dynamics

$$\begin{aligned}
dx(s) &= \left[ax(s)^{1/2} - bx(s) - \sum_{j=1}^n u_j(s) \right] ds + \sigma x(s) dz(s), \\
x(t_0) &= x_0 \in X.
\end{aligned} \tag{2.77}$$

In the absence of human harvesting, the resource stock will grow according to the dynamics

$$dx(s) = [ax(s)^{1/2} - bx(s)] ds + \sigma x(s) dz(s).$$

The deterministic part of the natural growth function is $G(x) = ax^{1/2} - bx = x[ax^{-1/2} - b]$. This function represents pure compensation, viz., the proportional growth rate $G(x)/x$ is a decreasing function of x . The stochastic term reflects the randomness in the deathrate b . The resource stock has a nondegenerate stationary equilibrium level, which is characterized by the stationary density function $\varphi(x)$ (see Jørgensen and Yeung 1996)

$$\varphi(x) = \{ K / [\sigma^2 x^{2(1+b/\sigma^2)}] \} \exp[-(4a/\sigma^2)x^{-1/2}],$$

where K is a normalization factor such that $\int_0^\infty \varphi(x) dx = 1$.

Extractor i seeks to maximize the expected payoff

$$\begin{aligned}
E_{t_0} \left\{ \int_{t_0}^T \left[\left(\sum_{j=1}^n u_j(s) \right)^{-1/2} u_i(s) - \frac{c}{x(s)^{1/2}} u_i(s) \right] \exp[-r(t - t_0)] ds \right. \\
\left. + \exp[-r(T - t_0)] w x(T)^{1/2} \right\}, \quad \text{for } i \in N,
\end{aligned} \tag{2.78}$$

subject to the resource dynamics in (2.77).

Invoking Theorem A.5 in the Technical Appendixes, a set of feedback strategies $\{u_i^*(t) = \phi_i^*(t, x); i \in N\}$ constitutes a Nash equilibrium solution for the game in (2.77) and (2.78), if there exist functions $V^{(t_0)i}(t, x) : [t_0, T] \times R \rightarrow R$, for $i \in N$, which satisfy the following set of partial differential equations:

$$\begin{aligned}
 & -V_t^{(t_0)i}(t, x) - \frac{1}{2}\sigma^2 x^2 V_{xx}^{(t_0)i}(t, x) \\
 & = \max_{u_i \in U^i} \left\{ \left[u_i \left(\sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(t, x) + u_i \right)^{-1/2} - \frac{c}{x^{1/2}} u_i(t) \right] \exp[-r(t - t_0)] \right. \\
 & \quad \left. + V_x^{(t_0)i} \left[ax^{1/2} - bx - \sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(t, x) - u_i \right] \right\}, \quad \text{and}
 \end{aligned} \tag{2.79}$$

$$V^{(t_0)i}(T, x) = \exp[-r(T - t_0)]wx^{1/2}.$$

Applying the maximization operator on the right-hand side of the first equation in (2.79) for agent i yields the condition for a maximum as

$$\begin{aligned}
 & \left[\left(\sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(t, x) + \frac{1}{2}\phi_i^*(t, x) \right) \left(\sum_{j=1}^n \phi_j^*(t, x) \right)^{-3/2} - \frac{c}{x^{1/2}} \right] \exp[-r(t - t_0)] \\
 & - V_x^{(t_0)i} = 0,
 \end{aligned} \tag{2.80}$$

for $i \in N$.

Following the analysis in Sect. 2.3.2, we obtain

$$\begin{aligned}
 \phi_i^*(t, x) = & \frac{x(2n-1)^2}{2[\sum_{j=1}^n [c + \exp[r(t - t_0)]V_x^{(t_0)j}x^{1/2}]]^3} \left\{ \sum_{\substack{j=1 \\ j \neq i}}^n \left[c + \frac{V_x^{(t_0)j}x^{1/2}}{\exp[-r(t - t_0)]} \right] \right. \\
 & \left. - \left(n - \frac{3}{2} \right) \left[c + \frac{V_x^{(t_0)i}x^{1/2}}{\exp[-r(t - t_0)]} \right] \right\}, \quad \text{for } i \in N.
 \end{aligned} \tag{2.81}$$

Substituting $\phi_i^*(t, x)$ in (2.81) into (2.79), and upon solving it, yields the following.

Proposition 2.3 *The system in (2.79) admits a solution*

$$V^{(t_0)i}(t, x) = \exp[-r(t - t_0)][A(t)x^{1/2} + B(t)], \quad \text{for } i \in N, \tag{2.82}$$

where $A(t)$ and $B(t)$ satisfy

$$\begin{aligned}\dot{A}(t) &= \left[r + \frac{1}{8}\sigma^2 + \frac{b}{2} \right] A(t) - \frac{(2n-1)}{2n^2} \left(c + \frac{A(t)}{2} \right)^{-1} \\ &\quad + \frac{c(2n-1)^2}{4n^3} \left(c + \frac{A(t)}{2} \right)^{-2} + \frac{(2n-1)^2 A(t)}{8n^2 (c + \frac{A(t)}{2})^2}, \\ \dot{B}(t) &= rB(t) - \frac{a}{2} A(t), \\ A(T) &= w, \quad \text{and} \\ B(T) &= 0.\end{aligned}\tag{2.83}$$

Proof Substituting $V^{(t_0)i}(t, x)$ and the relevant derivatives $V_t^{(t_0)i}(t, x)$, $V_x^{(t_0)i}(t, x)$, and $V_{xx}^{(t_0)i}(t, x)$ into (2.81) and (2.79) yields the results in Proposition 2.3. $\square$

Substituting the relevant derivatives of the value functions in Proposition 2.3 into the game equilibrium strategies in (2.81) gives a Nash equilibrium of the stochastic resource extraction game in (2.77) and (2.78).

2.4.3 Infinite-Horizon Resource Extraction

Consider the infinite-horizon game in which extractor i seeks to maximize the expected payoff

$$\begin{aligned}E_{t_0} \left\{ \int_{t_0}^{\infty} \left[\left(\sum_{j=1}^n u_j(s) \right)^{-1/2} u_i(s) - \frac{c}{x(s)^{1/2}} u_i(s) \right] \exp[-r(s-t_0)] ds \right\}, \\ \text{for } i \in N,\end{aligned}\tag{2.84}$$

subject to the resource dynamics

$$\begin{aligned}dx(s) &= \left[ax(s)^{1/2} - bx(s) - \sum_{j=1}^n u_j(s) \right] ds + \sigma x(s) dz(s), \\ x(t_0) &= x_0 \in X.\end{aligned}\tag{2.85}$$

Consider the alternative problem that starts at time $t \in [t_0, \infty)$ with initial state $x(t) = x$

$$\begin{aligned}E_t \left\{ \int_t^{\infty} \left[\left(\sum_{j=1}^n u_j(s) \right)^{-1/2} u_i(s) - \frac{c}{x(s)^{1/2}} u_i(s) \right] \exp[-r(s-t)] ds \right\}, \\ \text{for } i \in N,\end{aligned}\tag{2.86}$$

subject to the resource dynamics

$$\begin{aligned} dx(s) &= \left[ax(s)^{1/2} - bx(s) - \sum_{j=1}^n u_j(s) \right] ds + \sigma x(s) dz(s), \\ x(t) &= x \in X. \end{aligned} \quad (2.87)$$

Invoking Theorem 2.6, we obtain a set of feedback strategies $\{\phi_i^*(x), i \in N\}$ constituting of a Nash equilibrium solution for the game in (2.86) and (2.87) if there exist functions $\hat{V}^i(x) : R \rightarrow R$ for $i \in N$ that satisfy the following set of partial differential equations:

$$\begin{aligned} & r\hat{V}^i(x) - \frac{1}{2}\sigma^2 x^2 W_{xx}^i(x) \\ &= \max_{u_i \in U^i} \left\{ \left[u_i \left(\sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(x) + u_i \right)^{-1/2} - \frac{c}{x^{1/2}} u_i \right] \right. \\ & \quad \left. + \hat{V}_x^i \left[ax^{1/2} - bx - \sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(x) - u_i \right] \right\}, \quad \text{for } i \in N. \end{aligned} \quad (2.88)$$

Applying the maximization operator in (2.88) for agent i yields the condition for a maximum as

$$\begin{aligned} & \left[\left(\sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(x) + \frac{1}{2}\phi_i^*(x) \right) \left(\sum_{j=1}^n \phi_j^*(x) \right)^{-3/2} - \frac{c}{x^{1/2}} \right] - \hat{V}_x^i = 0, \\ & \text{for } i \in N. \end{aligned} \quad (2.89)$$

Summing over $i = 1, 2, \dots, n$ in (2.89) yields

$$\left(\sum_{j=1}^n \phi_j^*(x) \right)^{1/2} = \left(n - \frac{1}{2} \right) \left(\sum_{j=1}^n \left[\frac{c}{x^{1/2}} + \hat{V}_x^j \right] \right)^{-1}. \quad (2.90)$$

Substituting (2.90) into (2.89) produces

$$\begin{aligned} & \left(\sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(x) + \frac{1}{2}\phi_i^*(x) \right) \left(n - \frac{1}{2} \right)^{-3} \left(\sum_{j=1}^n \left[\frac{c}{x^{1/2}} + \hat{V}_x^j \right] \right)^3 - \frac{c}{x^{1/2}} - \hat{V}_x^i = 0, \\ & \text{for } i \in N. \end{aligned} \quad (2.91)$$

Rearranging the terms in (2.91) yields

$$\left(\sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(x) + \frac{1}{2} \phi_i^*(x) \right) = \left(n - \frac{1}{2} \right)^3 \frac{[c + \hat{V}_x^i x^{1/2}]x}{(\sum_{j=1}^n [c + \hat{V}_x^j x^{1/2}])^3},$$

for $i \in N$.

(2.92)

The condition in (2.92) represents a system of equations that is linear in $\{\phi_1^*(x), \phi_2^*(x), \dots, \phi_n^*(x)\}$. Solving (2.92) yields the game equilibrium strategies

$$\phi_i^*(x) = \frac{x(2n-1)^2}{2[\sum_{j=1}^n [c + \hat{V}_x^j x^{1/2}]]^3} \left\{ \sum_{\substack{j=1 \\ j \neq i}}^n [c + \hat{V}_x^j x^{1/2}] - \left(n - \frac{3}{2} \right) [c + \hat{V}_x^i x^{1/2}] \right\},$$

for $i \in N$.

(2.93)

Substituting $\phi_i^*(t, x)$ in (2.93) into (2.88), and upon solving it, yields the following.

Proposition 2.4 *The system in (2.88) admits a solution*

$$\hat{V}^i(x) = [Ax^{1/2} + B], \quad \text{for } i \in N, \quad (2.94)$$

where A and B satisfy

$$0 = \left[r + \frac{1}{8}\sigma^2 + \frac{b}{2} \right] - \frac{(2n-1)}{2n^2} \left(c + \frac{A}{2} \right)^{-1} + \frac{c(2n-1)^2}{4n^3} \left(c + \frac{A}{2} \right)^{-2} \\ + \frac{(2n-1)^2 A}{8n^2 (c + \frac{A}{2})^2}, \quad (2.95)$$

$$B = \frac{a}{2r} A.$$

Proof Substituting $\hat{V}^i(x)$ and the relevant derivatives $\hat{V}_x^i(x)$ and $\hat{V}_{xx}^i(x)$ into (2.93) and (2.88) yields the results in Proposition 2.4. $\square$

A feedback Nash equilibrium can be readily obtained by substituting the relevant derivatives of the value functions in Proposition 2.4 into the game equilibrium strategies of (2.93).

2.5 Exercises

2.1 Consider the competitive dynamic advertising game in which there are two firms in a market. The profits of firm 1 and that of firm 2 are, respectively,

$$\int_0^5 [10x(s) - 2u_1(s)^2] \exp(-0.05s) ds + \exp[(-0.05)5] 12x(5),$$

and

$$\int_0^5 [8(1 - x(s)) - u_2(s)^2] \exp(-0.05s) ds + \exp[(-0.05)5]9[1 - x(5)],$$

where $x(s)$ is the market share of firm 1 at time s , $[1 - x(s)]$ is that of firm 2, and $u_i(s)$ is the advertising rate for firm $i \in \{1, 2\}$.

It is assumed that market potential is constant over time. The only marketing instrument used by the firms is advertising. Advertising has diminishing returns since there are increasing marginal costs of advertising as reflected through the quadratic cost function. The dynamics of firm 1's market share is governed by

$$\dot{x}(s) = u_1(s)[1 - x(s)]^{1/2} - u_2(s)x(s)^{1/2}, \quad x(0) = 0.6.$$

Derive an open-loop solution for the market equilibrium.

2.2 Consider an economy endowed with a renewable resource and with $n \geq 2$ resource extractors (firms). The lease for resource extraction begins at time 0 and ends at time 10. Let $u_i(s)$ denote the rate of resource extraction of firm i at time s , $i \in \{1, 2\}$, and $x(s)$ is the size of the resource stock. The extraction cost is $C^i = 2u_i(s)/x(s)^{1/2}$ for firm $i \in \{1, 2\}$. The demand for the resource is $P(s) = Q(s)^{-1/2}$, where $Q(s) = \sum_{j=1}^2 u_j(s)$. A terminal bonus $4x(T)^{1/2}$ is offered to each extractor and the discount rate is 0.1. Extractor i seeks to maximize the present value of profits

$$\int_0^4 \left[\left(\sum_{j=1}^2 u_j(s) \right)^{-1/2} u_i(s) - \frac{2}{x(s)^{1/2}} u_i(s) \right] \exp(-0.1s) ds \\ + \exp[-0.1(10)]4x(T)^{1/2},$$

for $i \in \{1, 2\}$, subject to the resource dynamics

$$\dot{x}(s) = 5x(s)^{1/2} - x(s) - \sum_{j=1}^2 u_j(s), \quad x(0) = 100.$$

Derive a feedback Nash equilibrium solution.

2.3 Consider a dynamic duopoly in which there are two publicly listed firms selling a homogeneous good. The payoff of firm i is given as the present value of the stream of discounted profits

$$\int_0^\infty \{P(s)u_i(s) - 2u_i(s) - (1/2)[u_i(s)]^2\} \exp[-0.05s] ds, \quad \text{for } i \in \{1, 2\},$$

where $2u_i(s) + (1/2)[u_i(s)]^2$ is the cost of the producing output $u_i(s)$ and the interest rate is 0.05.

There is a lag in price adjustment so the evolution of the market price over time is assumed to be a function of the current market price and the price specified by the current demand condition. In particular, the price dynamics follows

$$\dot{P}(s) = 0.5[50 - u_1(s) - u_2(s) - P(s)], \quad P(0) = 5.$$

Characterize a feedback equilibrium for the duopoly.

2.4 Consider an economy endowed with a renewable resource and with two resource extraction firms. The lease for resource extraction begins at time 0 and ends at time 3. Let $u_i(s)$ denote the rate of resource extraction of firm i at time $s \in [0, 3]$, $i \in \{1, 2\}$, where each extractor controls its rate of extraction. Let $x(s)$ denote the size of the resource stock at time s ; the resource growth dynamics is stochastic. The extraction cost depends on the quantity of the resource extracted $u_i(s)$ and the resource stock size $x(s)$. In particular, the extraction cost can be specified as $C^i = u_i(s)/x(s)^{1/2}$ for firm $i \in \{1, 2\}$. The market price of the resource depends on the total amount extracted and supplied to the market. The price-output relationship at time s is given by the following downward-sloping inverse demand curve $P(s) = 0.5Q(s)^{-1/2}$, where $Q(s) = \sum_{j=1}^2 u_j(s)$ is the total amount of the resource extracted and marketed at time s . A terminal bonus $2x(T)^{1/2}$ is offered to each extractor and the discount rate is 0.1. Extractor $i \in \{1, 2\}$ seeks to maximize the present value of the expected profits

$$E_0 \left\{ \int_0^3 \left[0.5 \left(\sum_{j=1}^2 u_j(s) \right)^{-1/2} u_i(s) - \frac{u_i(s)}{x(s)^{1/2}} \right] \exp(-rs) ds + \exp[-3(0.1)] 2x(T)^{1/2} \right\}$$

subject to the stochastic resource dynamics

$$dx(s) = \left[10x(s)^{1/2} - x(s) - \sum_{j=1}^2 u_j(s) \right] ds + 0.4x(s) dz(s), \quad x(0) = 120.$$

Derive a feedback equilibrium for the above stochastic dynamic economy.

Subgame Consistent Economic Optimization
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Yeung, D.W.K.; Petrosyan, L.A.

2012, XVI, 395 p. 2 illus., Hardcover

ISBN: 978-0-8176-8261-3

A product of Birkhäuser Basel