2

Nonlinear Model Equations and Variational Principles

True Laws of Nature cannot be linear.

Albert Einstein

...the progress of physics will to a large extent depend on the progress of nonlinear mathematics, of methods to solve nonlinear equations ... and therefore we can learn by comparing different nonlinear problems.

Werner Heisenberg

Our present analytical methods seem unsuitable for the solution of the important problems arising in connection with nonlinear partial differential equations and, in fact, with virtually all types of nonlinear problems in pure mathematics. The truth of this statement is particularly striking in the field of fluid dynamics...

John Von Neumann

2.1 Introduction

This chapter deals with the basic ideas and many major nonlinear model equations which arise in a wide variety of physical problems. Included are one-dimensional wave, Klein–Gordon (KG), sine–Gordon (SG), Burgers, Fisher, Korteweg–de Vries (KdV), Boussinesq, modified KdV, nonlinear Schrödinger (NLS), Benjamin–Ono (BO), Benjamin–Bona–Mahony (BBM), Ginzburg–Landau (GL), Burgers–Huxley (BH), KP, concentric KdV, Whitham, Davey–Stewartson, Toda lattice, Camassa–Holm (CH), and Degasperis–Procesi (DP) equations. This is followed by variational principles and the Euler–Lagrange equations. Also included are Plateau's problem, Hamilton's principle, Lagrange's equations, Hamilton's equations, the variational principle for nonlinear Klein–Gordon equations, and the variational principle for nonlinear water waves. Special attention is given to the Euler equation of motion,

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the continuity equation, the associated energy equation and energy flux, linear water wave problems and their solutions, nonlinear finite amplitude waves (the Stokes waves), gravity waves, gravity-capillary waves, and linear and nonlinear dispersion relations. Finally, the modern theory of nonlinear water waves is formulated.

2.2 Basic Concepts and Definitions

The most general first-order nonlinear partial differential equation in two independent variables x and y has the form

$$F(x, y, u, u_x, u_y) = 0. \tag{2.2.1}$$

The most general second-order nonlinear partial differential equation in two independent variables x and y has the form

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \tag{2.2.2}$$

Similarly, the most general first-order and second-order nonlinear equations in more independent variables can be introduced.

More formally, it is possible to write these equations in the operator form

$$L_{\mathbf{x}} u(\mathbf{x}) = f(\mathbf{x}), \tag{2.2.3}$$

where $L_{\mathbf{x}}$ is a partial differential operator and $f(\mathbf{x})$ is a given function of two or more independent variables $\mathbf{x} = (x, y, \ldots)$. It has already been indicated in Section 1.2 that if $L_{\mathbf{x}}$ is *not* a linear operator, (2.2.3) is called a *nonlinear partial differential equation*. Equation (2.2.3) is called an *inhomogeneous nonlinear equation* if $f(\mathbf{x}) \neq 0$. On the other hand, (2.2.3) is called a *homogeneous nonlinear equation* if $f(\mathbf{x}) = 0$.

In general, the linear superposition principle can be applied to linear partial differential equations if certain convergence requirements are satisfied. This principle is usually used to find a new solution as a linear combination of a given set of solutions. For nonlinear partial differential equations, however, the linear superposition principle *cannot* be applied to generate a new solution. So, because most solution methods for linear equations cannot be applied to nonlinear equations, there is no general method of finding analytical solutions of nonlinear partial differential equations, and numerical techniques are usually required for their solution. A transformation of variables can sometimes be found that transforms a nonlinear equation into a linear equation, or some other ad hoc method can be used to find a solution of a particular nonlinear equation. In fact, new methods are usually required for finding solutions of nonlinear equations.

Methods of solution for nonlinear equations represent only one aspect of the theory of nonlinear partial differential equations. Like linear equations, questions of existence, uniqueness, and stability of solutions of nonlinear partial differential equations are of fundamental importance. These and other aspects of nonlinear equations have led the subject into one of the most diverse and active areas of modern mathematics.

2.3 Some Nonlinear Model Equations

Nonlinear partial differential equations arise frequently in formulating fundamental laws of nature and in the mathematical analysis of a wide variety of physical problems. Listed below are some important model equations of most common interest.

Example 2.3.1. The simplest first-order *nonlinear wave* (*or kinematic wave*) *equation* is

$$u_t + c(u)u_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2.3.1}$$

where $c(u)$ is a given function of u. This equation describes the propagation of a nonlinear wave (or disturbance). A large number of nonlinear problems governed by equation (2.3.1) include waves in traffic flow on highways (Lighthill and Whitham 1955; Richards 1956), shock waves, flood waves, waves in glaciers (Nye 1960, 1963), chemical exchange processes in chromatography, sediment transport in rivers (Kynch 1952), and waves in plasmas.

Example 2.3.2. The *nonlinear Klein–Gordon equation* is

$$u_{tt} - c^2\nabla^2 u + V'(u) = 0, \tag{2.3.2}$$

where c is a constant, and $V'(u)$ is a nonlinear function of u usually chosen as the derivative of the potential energy $V(u)$. It arises in many physical problems including nonlinear dispersion (Scott 1969; Whitham 1974) and nonlinear meson theory (Schiff 1951).

Example 2.3.3. The *sine-Gordon equation*

$$u_{tt} - c^2 u_{xx} + \kappa \sin u = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2.3.3}$$

where c and κ are constants, has arisen classically in the study of differential geometry, and in the propagation of a 'slip' dislocation in crystals (Frenkel and Kontorova 1939). More recently, it arises in a wide variety of physical problems including the propagation of magnetic flux in Josephson-type super conducting tunnel junctions, the phase jump of the wave function of superconducting electrons along long Josephson junctions (Josephson 1965; Scott 1969), a chain of rigid pendula connected by springs (Scott 1969), propagation of short optical pulses in resonant laser media (Arecchi et al. 1969; Lamb 1971), stability of fluid motions (Scott et al. 1973; Gibbon 1985), in ferromagnetism and ferroelectric materials, in the dynamics of certain molecular chains such as DNA (Barone et al. 1971), in elementary particle physics (Skyrme 1958, 1961; Enz 1963), and in weakly unstable baroclinic wave packets in a two-layer fluid (Gibbon et al. 1979).

Example 2.3.4. The Burgers equation is

$$u_t + uu_x = \nu u_{xx}, \quad x \in \mathbb{R},\ t > 0, \tag{2.3.4}$$

where ν is the kinematic viscosity. This is the simplest nonlinear model equation for diffusive waves in fluid dynamics. It was first introduced by Burgers (1948) to describe one-dimensional turbulence, and it also arises in many physical problems including sound waves in a viscous medium (Lighthill 1956), waves in fluid-filled viscous elastic tubes, and magnetohydrodynamic waves in a medium with finite electrical conductivity.

Example 2.3.5. The *Fisher equation*

$$u_t - \nu u_{xx} = k\left(u - \frac{u^2}{\kappa}\right), \quad x \in \mathbb{R},\ t > 0, \tag{2.3.5}$$

where ν, k, and κ are constants, is used as a nonlinear model equation to study wave propagation in a large number of biological and chemical systems. Fisher (1936) first introduced this equation to investigate wave propagation of a gene in a population. It is also used to study logistic growth–diffusion phenomena. In recent years, the Fisher equation has been used as a model equation for a large variety of problems which include gene-culture waves of advance (Aoki 1987), chemical wave propagation (Arnold et al. 1987), neutron population in a nuclear reactor (Canosa 1969, 1973), and spread of early farming in Europe (Ammerman and Cavalli-Sforva 1971). It also arises in the theory of combustion, nonlinear diffusion, and chemical kinetics (Kolmogorov et al. 1937; Aris 1975; and Fife 1979).

Example 2.3.6. The *Boussinesq equation*

$$u_{tt} - u_{xx} + \left(3u^2\right)_{xx} - u_{xxxx} = 0 \tag{2.3.6}$$

describes one-dimensional weakly nonlinear dispersive water waves propagating in both positive and negative x-directions (Peregrine 1967; Toda and Wadati 1973; Zakharov 1968a, 1968b; Ablowitz and Haberman 1975; and Prasad and Ravindran 1977). It also arises in one-dimensional lattice waves (Zabusky 1967) and ion-acoustic solitons (Kako and Yajima 1980). In recent years, considerable attention has been given to new forms of Boussinesq equations (Madsen et al. 1991; Madsen and Sorensen 1992, 1993) dealing with water wave propagation and to modified Boussinesq equations (Nwogu 1993; Chen and Liu 1995a, 1995b) in terms of a velocity potential on an arbitrary elevation and free surface displacement of water.

Example 2.3.7. The *Korteweg–de Vries* (*KdV*) *equation*

$$u_t + \alpha u u_x + \beta u_{xxx} = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2.3.7}$$

where α and β are constants, is a simple and useful model for describing the long time evolution of dispersive wave phenomena in which the steepening effect of the nonlinear term is counterbalanced by the dispersion. It was originally introduced by Korteweg and de Vries (1895) to describe the propagation of unidirectional shallow water waves.

It admits the exact solution called the *soliton*. This equation arises in many physical problems including water waves (Johnson 1980, 1997; Debnath 1994), internal gravity waves in a stratified fluid (Benney 1966; Redekopp and Weidman 1968), ion-acoustic waves in a plasma (Washimi and Taniuti 1966), pressure waves in a liquid-gas bubble (Van Wijngaarden 1968), and rotating flow in a tube (Leibovich 1970). There are other physical systems to which the KdV equation applies as a long wave approximation, including acoustic-gravity waves in a compressible heavy liquid, axisymmetric waves in a nonuniformly rotating fluid, acoustic waves in anharmonic crystals, nonlinear waves in cold plasmas, axisymmetric magnetohydrodynamic waves, and longitudinal dispersive waves in elastic rods.

Example 2.3.8. The *modified KdV* (*mKdV*) *equation*

$$u_t - 6u^2 u_x + u_{xxx} = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2.3.8}$$

describes nonlinear acoustic waves in an anharmonic lattice (Zabusky 1967) and Alfvén waves in a collisionless plasma (Kakutani and Ono 1969). It also arises in many other physical situations.

Example 2.3.9. The *nonlinear Schrödinger* (*NLS*) *equation*

$$iu_t + u_{xx} + \gamma |u|^2 u = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2.3.9}$$

where γ is a constant, describes the evolution of water waves ((Benney and Roskes 1969; Hasimoto and Ono 1972; Davey 1972; Davey and Stewartson 1974; Peregrine 1983); Zakharov 1968a, 1968b; Chu and Mei 1970; Yuen and Lake 1975; Infeld et al. 1987; Johnson 1997). It also arises in some other physical systems which include nonlinear optics (Kelley 1965; Talanov 1965; Bespalov and Talanov 1966; Karpman and Krushkal 1969; Asano et al. 1969; Hasegawa and Tappert 1973), hydromagnetic and plasma waves (Ichikawa et al. 1972; Schimizu and Ichikawa 1972; Taniuti and Washimi 1968; Fulton 1972; Hasegawa 1990; Ichikawa 1979; Weiland and Wiljelmsson 1977; Weiland et al. 1978), the propagation of a heat pulse in a solid (Tappert and Varma 1970), nonlinear waves in a fluid-filled viscoelastic tube (Ravindran and Prasad 1979), nonlinear instability problems (Stewartson and Stuart 1971; Nayfeh and Saric 1971), and the propagation of solitary waves in piezoelectric semiconductors (Pawlik and Rowlands 1975).

Example 2.3.10. The *Benjamin–Ono* (*BO*) *equation* is

$$u_t + uu_x + \mathcal{H}\{u_{xx}\} = 0, \tag{2.3.10}$$

where $\mathcal{H}\{f(\xi,t)\} = \tilde{f}(x,t)$ is the Hilbert transform of $f(\xi,t)$ defined by

$$\mathcal{H}\{f(\xi,t)\} = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(\xi,t)\, d\xi}{\xi - x}, \tag{2.3.11}$$

where P stands for the Cauchy principal value. This equation arises in the study of weakly nonlinear long internal gravity waves (Benjamin 1967; Davis and Acrivos 1967; and Ono 1975) and belongs to the class of weakly nonlinear models.

Example 2.3.11. The *Benjamin–Bona–Mahony* (*BBM*) *equation*

$$u_t + u_x + uu_x - u_{xxt} = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2.3.12}$$

represents another nonlinear model for long water waves. The KdV equation can be written as

$$u_t + u_x + uu_x + u_{xxx} = 0, \quad x \in \mathbb{R},\ t > 0. \tag{2.3.13}$$

The basic mathematical difference between the BBM and KdV equations can readily be determined by comparing the approximate dispersion relations for the respective linearized equations. We seek a plane wave solution of both linearized equations of the form

$$u(x,t) \sim \exp\big[i(\omega t - kx)\big]. \tag{2.3.14}$$

The dispersion relation of the linearized KdV equation is then given by

$$\omega = k - k^3. \tag{2.3.15}$$

The phase and group velocities are given by

$$C_p = \frac{\omega}{k} = 1 - k^2 \quad \text{and} \quad C_g = \frac{d\omega}{dk} = 1 - 3k^2, \tag{2.3.16ab}$$

which become negative for $k^2 > 1$. This means that all waves of large wavenumbers (small wavelengths) propagate in the *negative* x-direction in contradiction to the original assumption that waves travel only in the positive x-direction. This is an undesirable physical feature of the KdV equation. To eliminate this unrealistic feature of the KdV equation, Benjamin et al. (1972) proposed equation (2.3.12). The dispersion relation of the linearized version of (2.3.12) is

$$\omega = \frac{k}{(1+k^2)}. \tag{2.3.17}$$

Thus the phase and group velocities of waves associated with this model are given by

$$C_p = \frac{\omega}{k} = \big(1+k^2\big)^{-1}, \qquad C_g = \big(1-k^2\big)\big(1+k^2\big)^{-2}. \tag{2.3.18ab}$$

Both C_p and C_g tend to zero, as $k \to \infty$, showing that short waves do not propagate. In other words, the BBM model has the approximate features of responding only significantly to short wave components introduced in the initial wave form. Thus, the BBM equation seems to be a preferable model. However, the fact that the BBM model is a better model than the KdV model has not been fully confirmed, yet.

Example 2.3.12. The *Ginzburg–Landau* (*GL*) *equation* is

$$A_t + aA_{xx} = bA + cA|A|^2, \tag{2.3.19}$$

where a and b are complex constants determined by the dispersion relation of linear waves, and c is determined by the weakly nonlinear interaction (Stewartson and Stuart 1971). This equation describes slightly unstable nonlinear waves and has arisen originally in the theories of superconductivity and phase transitions.

The complex Ginzburg–Landau equation simplifies significantly if all of the coefficients are real. The real Ginzburg–Landau equation has been extensively investigated in problems dealing with phase separation in condensed matter physics (Ben-Jacob et al. 1985; Van Saarloos 1989; Balmforth 1995).

Example 2.3.13. The *Burgers–Huxley* (*BH*) *equation*

$$u_t + \alpha u u_x - \nu u_{xx} = \beta(1-u)(u-\gamma)u, \quad x \in \mathbb{R},\ t > 0, \tag{2.3.20}$$

where α, $\beta \geq 0$, γ $(0 < \gamma < 1)$, and ν are parameters, describes the interaction between convection, diffusion, and reaction. When $\alpha = 0$, equation (2.3.20) reduces to the Hodgkin and Huxley (1952) equation which describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals (Scott 1977; Satsuma 1987a, 1987b; Wang 1985, 1986; Wang et al. 1990). Because of the complexity of the Huxley equation, the FitzHugh–Nagumo equations (FitzHugh 1961; Sleeman 1982; Nagumo et al. 1962) proposed simple, analytically tractable, and particularly useful model equations which contain the key features of the Huxley model. On the other hand, when $\beta = 0$, equation (2.3.20) reduces to the Burgers equation (2.3.4) describing diffusive waves in nonlinear dissipating systems. Satsuma (1987a, 1987b) obtained solitary wave solutions of (2.3.20) by using Hirota's method in soliton theory.

Example 2.3.14. The *Kadomtsev–Petviashvili* (*KP*) *equation*

$$(u_t - 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0, \tag{2.3.21}$$

is a two-dimensional generalization of the KdV equation. Kadomtsev and Petviashvili (1970) first introduced this equation to describe slowly varying nonlinear waves in a dispersive medium (Johnson 1980, 1997). Equation (2.3.21) with $\sigma^2 = +1$ arises in the study of weakly nonlinear dispersive waves in plasmas and also in the modulation of weakly nonlinear long water waves (Ablowitz and Segur 1979) which travel nearly in one dimension (that is, nearly in a vertical plane). Satsuma (1987a, 1987b) showed that the KP equation has N line-soliton solutions which describe the oblique interaction of solitons. The equation with $\sigma^2 = -1$ arises in acoustics and admits unstable soliton solutions, whereas for $\sigma^2 = +1$ the solitons are stable. Freeman (1980) presented an interesting review of soliton interactions in two dimensions. Recently, Chen and Liu (1995a, 1995b) have derived the unified KP (uKP) equation for surface and interfacial waves propagating in a rotating channel with varying topography and sidewalls. This new equation includes most of the existing KP-type equations in the literature as special cases.

Example 2.3.15. The *concentric KdV equation*

$$2u_R + \frac{1}{R}u + 3uu_\xi + \frac{1}{3}u_{\xi\xi\xi} = 0 \tag{2.3.22}$$

describes concentric waves on the free surface of water that have decreasing amplitude with increasing radius. This is also called the *cylindrical KdV equation* which was first derived in another context by Maxon and Viecelli (1974). The inverse scattering transform for equation (2.3.22) involves a linearly increasing potential which yields eigenfunctions based on the Airy function (see Calogero and Degasperis 1978). A discussion of this equation and its solution can also be found in Johnson (1997) and Freeman (1980).

Example 2.3.16. The *nearly concentric KdV equation* (or the *Johnson equation*)

$$\left(2u_R + \frac{1}{R}u + 3uu_\xi + \frac{1}{3}u_{\xi\xi\xi}\right)_\xi + \frac{1}{R^2}u_{\theta\theta} = 0 \tag{2.3.23}$$

describes the nearly concentric surface waves incorporating weak dependence on the angular coordinate θ. In the absence of θ-dependence, equation (2.3.23) reduces to (2.3.22). This equation was first derived by Johnson (1980) in his study of problems of nonlinear water waves.

Example 2.3.17. The *Davey–Stewartson* (*DS*) *equations*

$$-2ikc_pA_\tau + aA_{\zeta\zeta} - c_pc_gA_{yy} + bA|A|^2 + ck^2Af_\zeta = 0, \tag{2.3.24}$$

$$\left(1 - c_g^2\right)f_{\zeta\zeta} + f_{yy} = d\left(|A|^2\right)_\zeta, \tag{2.3.25}$$

where a, b, c, d are functions of δk (see Davey and Stewartson 1974; Johnson 1997), describe weakly nonlinear dispersive waves propagating in the x-direction with a slowly varying structure in both the x- and y-directions. In the absence of y-dependence with $f_\zeta \equiv 0$, the DS equations recover the NLS equation for water waves (see Hasimoto and Ono 1972) in the form

$$-2ikc_pA_\tau + aA_{\zeta\zeta} + bA|A|^2 = 0. \tag{2.3.26}$$

This is similar to (2.3.9).

Example 2.3.18. The *Whitham* (1974) *nonlinear nonlocal integrodifferential equation*

$$\eta_t + d\eta\eta_x + \int_{-\infty}^{\infty} K(x-\xi)\eta_\xi(\xi,t)\,d\xi = 0 \tag{2.3.27}$$

can describe symmetric waves that propagate without change of shape and peak at a critical height, as well as asymmetric waves that invariably break. The kernel $K(x)$ is given by the inverse Fourier transform of the phase velocity $c(k) = \frac{\omega}{k}$ in the form

$$K(x) = \mathcal{F}^{-1}\{c(k)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}c(k)\,dk. \tag{2.3.28}$$

It is a well known fact that the nonlinear shallow water equations which neglect dispersion altogether lead to breaking of the typical hyperbolic kind, with development of a vertical slope and a multivalued wave profile. It is clear that the third derivative dispersion term in the KdV equation (2.3.7) prevents wave breaking. Whitham formulated his equation (2.3.27) to describe the observed phenomena of solitary and periodic cnoidal waves as well as peaking and breaking of water waves. The Whitham equation is a kind of generalization of the KdV equation that takes $c(k) = c_0 - \gamma k^2$ and $K(x) = c_0\delta(x) + \delta''(x)$, $c_0^2 = gh_0$. The detailed analysis of Whitham's analysis is given in Section 7.8 in Chapter 7.

Example 2.3.19. The *Camassa and Holm* (*CH*) *Equation* for the free surface elevation $u(x,t)$ over a flat rigid bottom is

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xx}, \quad x \in \mathbb{R},\ t > 0. \tag{2.3.29}$$

It describes the propagation of nonlinear dispersive shallow water equation to capture the essential features of wave breaking. It is integrable in the sense that there exists a Lax pair, and has infinitely many conservation laws. The CH equation admits stable solitary wave solutions with a peak at their crests; these waves are called *peakons*. A more elaborate discussion of this equation (Camassa and Holm 1993) and its various extensions are presented in Section 9.13.

Example 2.3.20. The *Degasperis and Procesi* (*DP*) *equation* is

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad x \in \mathbb{R},\ t > 0. \tag{2.3.30}$$

It also describes the propagation of nonlinear dispersive shallow water waves. Its solutions are singular, leading to wave breaking. The DP equation admits a shock-peakon solution which is significantly different from the peakon solutions of the CH equation. Both the CH and DP equations have soliton solutions which develop singularities in finite time (or solutions blow-up in finite time). Both the CH and DP equations can be combined into a $(1+1)$-dimensional b-family equation for fluid velocity $u(x,t)$ in the form

$$m_t + um_x + bmu_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2.3.31}$$

where $m = (u - u_{xx})$ and $u = g * m$ is the convolution product given by

$$u(x) = \int_{\mathbb{R}} g(x-\xi)m(\xi)\,d\xi, \qquad g(x) = \frac{1}{2}\exp\bigl(-|x|\bigr), \tag{2.3.32}$$

which determines the traveling wave shape and length scale for equation (2.3.31) and the constant b is a balance (or bifurcation) parameter. Degasperis and Procesi (1999) showed that (2.3.31) cannot be completely integrable unless $b = 2$ or $b = 3$. When $b = 2$, equation (2.3.31) reduces to the CH equation (2.3.29) and when $b = 3$, (2.3.31) becomes the DP equation (2.3.30). A more detailed discussion on these equations can be found in Section 9.13.

Example 2.3.21 (The Toda Lattice Equation *in a mass–spring system).* A mass–spring lattice is an infinite chain of identical masses m interconnected by nonlinear springs. We assume that the springs have potential $V(r)$, where r is the increase in distance between adjacent masses from the rest value at which the spring energy is minimum and its force $(F = -\frac{dV}{dr})$ is zero. If y_n is the longitudinal displacement of the nth mass from its equilibrium position, it follows from the Newton second law of motion that

$$m\frac{d^2y_n}{dt^2} = V'(y_{n+1} - y_n) - V'(y_n - y_{n-1}). \tag{2.3.33}$$

With $r_n = (y_{n+1} - y_n)$, this gives an infinite set of differential equations

$$m\ddot{r}_n = \left[\frac{dV(r_{n+1})}{dr_{n+1}} - \frac{dV(r_n)}{dr_n}\right] - \left[\frac{dV(r_n)}{dr_n} - \frac{dV(r_{n-1})}{dr_{n-1}}\right], \tag{2.3.34}$$

where $n \in \mathbb{N}$.

In his celebrated paper, Toda (1967a, 1967b) investigated a mass–spring lattice system with an anharmonic potential in the form

$$V(r) = \frac{a}{b}\left(e^{-br} + br - 1\right), \quad a, b > 0. \tag{2.3.35}$$

With unit masses $(m = 1)$, equation (2.3.34) reduces to the form

$$\ddot{r}_n = a\left(2e^{-br_n} - e^{-br_{n+1}} - e^{-br_{n-1}}\right). \tag{2.3.36}$$

This is known as the *Toda lattice equation*.

In the limit as $b \to 0$ with finite ab, equation (2.3.36) reduces to the linear differential-difference equation

$$\ddot{r}_n = ab(r_{n+1} - 2r_n + r_{n-1}). \tag{2.3.37}$$

This has solutions with a long wavelength velocity of $\sqrt{ab}$ lattice points per unit time.

When b is *not* small, the Toda lattice equation (2.3.36) admits exact solitary wave solutions of the form (see Section 11.13)

$$r_n = -\frac{1}{b}\log\left[1 + \sinh^2\kappa\,\mathrm{sech}^2\left\{\kappa\left(n \pm t\frac{\sqrt{ab}}{\kappa}\sinh\kappa\right)\right\}\right], \tag{2.3.38}$$

where the velocity of the lattice wave is expressed in terms of the amplitude parameter κ in the form

$$v = \frac{\sinh\kappa}{\kappa}\sqrt{ab}, \tag{2.3.39}$$

and the minus sign in (2.3.38) implies that the Toda lattice soliton (TLS) is a compression wave.

As the amplitude of the TLS is reduced to zero (by letting $\sinh\kappa$ approach zero), it reduces to a solution of the linear equation (2.3.37) traveling with velocity $v = \sqrt{ab}$.

We close this section by mentioning the *Yang–Mills field equations* which seem to be a useful model unifying electromagnetic and weak forces. They have solutions, called *instantons*, localized in space and time, which are interpreted as quantum-mechanical transitions between different states of a particle. Recently, it has been shown that the *self-dual Yang–Mills equations* are multidimensional integrable systems, and these equations admit reductions to well-known soliton equations in $(1+1)$ dimensions, that is, the sine-Gordon, NLS, KdV, and Toda lattice equations (Ward 1984, 1985, 1986).

2.4 Variational Principles and the Euler–Lagrange Equations

Many physical systems are often characterized by their extremum (minimum, maximum, or saddle point) property of some associated physical quantity that appears as an integral in a given domain, known as a *functional*. Such a characterization is a variational principle leading to the Euler–Lagrange equation which optimizes the related functional. For example, light rays travel along a path from one point to another in a minimum time. The shortest distance between two points on a plane curve is a straight line. A physical system is in equilibrium if its potential energy is minimum. So the main problem is to optimize a physical quantity (time, distance, or energy) in most real-world problems. These problems belong to the subject of the calculus of variations.

The classical Euler–Lagrange variational problem is to determine the extremum value of the functional

$$I(u) = \int_a^b F(x, u, u')\,dx, \quad u' = \frac{du}{dx}, \tag{2.4.1}$$

with the boundary conditions

$$u(a) = \alpha \quad \text{and} \quad u(b) = \beta, \tag{2.4.2ab}$$

where α and β are given numbers and $u(x)$ belongs to the class $C^2([a, b])$ of functions which have continuous derivatives up to the second order in $a \le x \le b$ and the integrand F has continuous second derivatives with respect to all of its arguments.

We assume that $I(u)$ has an extremum at some $u \in C^2([a, b])$. Then we consider the set of all variations $u + \varepsilon v$ for finite u, and arbitrary v belonging to $C^2([a, b])$ such that $v(a) = 0 = v(b)$. We next consider the variation δI of the functional $I(u)$

$$\begin{aligned}\delta I &= I(u + \varepsilon v) - I(u)\\ &= \int_a^b \big[F(x, u + \varepsilon v, u' + \varepsilon v') - F(x, u, u')\big]\,dx\end{aligned}$$

which, by the Taylor series expansion,

$$\begin{aligned}&= \int_a^b \bigg[F(x, u, u') + \varepsilon\left(v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'}\right)\\ &\quad + \frac{\varepsilon^2}{2!}\left(v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'}\right)^2 + \cdots - F(x, u, u')\bigg]\,dx\\ &= \int_a^b \varepsilon\left(v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'}\right)dx + O\big(\varepsilon^2\big).\end{aligned} \tag{2.4.3}$$

Thus, a necessary condition for the functional $I(u)$ to have an extremum (or for $I(u)$ to be stationary) for an arbitrary ε is

$$0 = \delta I = \int_a^b \left(v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'}\right)dx, \tag{2.4.4}$$

which, integrating the second term by parts, is

$$= \int_a^b v\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right] dx + \left[v\frac{\partial F}{\partial u'}\right]_a^b. \tag{2.4.5}$$

Since v is arbitrary with $v(a) = 0 = v(b)$, the last term of (2.4.5) vanishes and consequently, the integrand of the integral in (2.4.5) must vanish, that is,

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0. \tag{2.4.6}$$

This is called the *Euler–Lagrange equation* of the variational problem involving one independent variable. Using the result

$$d\left(\frac{\partial F}{\partial u'}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'}\right) dx + \frac{\partial}{\partial u}\left(\frac{\partial F}{\partial u'}\right) du + \frac{\partial}{\partial u'}\left(\frac{\partial F}{\partial u'}\right) du', \tag{2.4.7}$$

the Euler–Lagrange equations (2.4.6) can be written in the form

$$F_u - F_{xu'} - u' F_{uu'} - u'' F_{u'u'} = 0. \tag{2.4.8}$$

This is a second-order nonlinear ordinary differential equation for u provided $F_{u'u'} \neq 0$ and, hence, there are two arbitrary constants involved in the solution. However, when F does not depend explicitly on one of its variables x, u, or u', the Euler–Lagrange equation assumes a simplified form. Evidently, there are three possible cases:

1. If $F = F(x, u)$, then (2.4.6) reduces to $F_u(x, u) = 0$, which is an algebraic equation.
2. If $F = F(x, u')$, then (2.4.6) becomes

$$\frac{\partial F}{\partial u'} = \text{const.} \tag{2.4.9}$$

3. If $F = F(u, u')$, then (2.4.6) takes the form

$$F - u' F_{u'} = \text{const.} \tag{2.4.10}$$

This follows from the fact that

$$\begin{aligned}\frac{d}{dx}(F - u'F_{u'}) &= \frac{dF}{dx} - u'\frac{d}{dx}F_{u'} - u''F_{u'}\\ &= F_x + u'F_u + u''F_{u'} - u'\frac{d}{dx}F_{u'} - u''F_{u'}\\ &= u'\left(F_u - \frac{d}{dx}F_{u'}\right) = 0 \quad \text{by (2.4.6)}.\end{aligned}$$

The Euler–Lagrange variational problem involving two independent variables is to determine a function $u(x, y)$ in a domain $D \subset \mathbb{R}^2$ satisfying the boundary conditions prescribed on the boundary ∂D of D and extremizing the functional

$$I\big[u(x,y)\big] = \iint_D F(x,y,u,u_x,u_y)\,dx\,dy, \tag{2.4.11}$$

where the function F is defined over the domain D and assumed to have continuous second-order partial derivatives.

Similarly, for functionals depending on a function of two independent variables, the first variation δI of I is defined by

$$\delta I = I(u+\varepsilon v) - I(u). \tag{2.4.12}$$

In view of Taylor's expansion theorem, this reduces to

$$\delta I = \iint_D \big[\varepsilon(vF_u + v_x F_p + v_y F_q) + O\big(\varepsilon^2\big)\big]\,dx\,dy, \tag{2.4.13}$$

where $v = v(x,y)$ is assumed to vanish on ∂D and $p = u_x$ and $q = u_y$.

A necessary condition for the functional I to have an extremum is that the first variation of I vanishes, that is,

$$\begin{aligned}
0 = \delta I &= \iint_D (vF_u + v_x F_p + v_y F_q)\,dx\,dy \\
&= \iint_D v\left(F_u - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q\right)dx\,dy \\
&\quad + \iint_D \left[v\left(\frac{\partial}{\partial x}F_p + \frac{\partial}{\partial y}F_q\right) + (v_x F_p + v_y F_q)\right]dx\,dy \\
&= \iint_D v\left(F_u - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q\right)dx\,dy \\
&\quad + \iint_D \left[\frac{\partial}{\partial x}(vF_p) + \frac{\partial}{\partial y}(vF_q)\right]dx\,dy.
\end{aligned} \tag{2.4.14}$$

We assume that the boundary curve ∂D has a piecewise, continuously moving tangent so that Green's theorem can be applied to the second double integral in (2.4.14). Consequently, (2.4.14) reduces to

$$\begin{aligned}
0 = \delta I &= \iint_D v\left(F_u - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q\right)dx\,dy \\
&\quad + \int_{\partial D} v(F_p\,dy - F_q\,dx).
\end{aligned} \tag{2.4.15}$$

Since $v = 0$ on ∂D, the second integral in (2.4.15) vanishes. Moreover, since v is an arbitrary function, it follows that the integrand of the first integral in (2.4.15) must vanish. Thus, the function $u(x,y)$ extremizing the functional defined by (2.4.11) satisfies the partial differential equation

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q = 0. \tag{2.4.16}$$

This is called the *Euler–Lagrange equation* for the variational problem involving two independent variables.

The above variational formulation can readily be generalized for functionals depending on functions of three or more independent variables. Many physical problems require determining a function of several independent variables which will lead to an extremum of such functionals.

Example 2.4.1. Find $u(x,y)$ which extremizes the functional

$$I\big[u(x,y)\big] = \iint_D \big(u_x^2 + u_y^2\big)\,dx\,dy, \quad D \subset \mathbb{R}^2. \tag{2.4.17}$$

The Euler–Lagrange equation with $F = u_x^2 + u_y^2 = p^2 + q^2$ is

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) = 0,$$

or

$$u_{xx} + u_{yy} = 0. \tag{2.4.18}$$

This is a two-dimensional Laplace equation. Similarly, the functional

$$I\big[u(x,y,z)\big] = \iiint_D \big(u_x^2 + u_y^2 + u_z^2\big)\,dx\,dy\,dz, \quad D \subset \mathbb{R}^3, \tag{2.4.19}$$

will lead to the three-dimensional Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0. \tag{2.4.20}$$

In this way, we can derive the n-dimensional Laplace equation

$$\nabla^2 u = u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n} = 0. \tag{2.4.21}$$

Example 2.4.2 (Plateau's Problem). Find the surface S in the (x,y,z)-space of minimum area passing through a given plane curve C.

The direction cosine of the angle between the z-axis and the normal to the surface $z = u(x,y)$ is $(1 + u_x^2 + u_y^2)^{-\frac{1}{2}}$. The projection of the element dS of the area of the surface onto the (x,y)-plane is given by $(1 + u_x^2 + u_y^2)^{-\frac{1}{2}}\,dS = dx\,dy$. The area A of the surface S is given by

$$A = \iint_D \big(1 + u_x^2 + u_y^2\big)^{\frac{1}{2}}\,dx\,dy, \tag{2.4.22}$$

where D is the area of the (x,y)-plane bounded by the curve C.

The Euler–Lagrange equation with $F = (1 + p^2 + q^2)^{\frac{1}{2}}$ is given by

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+p^2+q^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1+p^2+q^2}}\right) = 0. \tag{2.4.23}$$

This is the *equation of minimal surface*, which reduces to the nonlinear elliptic partial differential equation

$$\left(1+u_y^2\right)u_{xx} - 2u_x u_y u_{xy} + \left(1+u_x^2\right)u_{yy} = 0. \tag{2.4.24}$$

Therefore, the desired function $u(x,y)$ should be determined as the solution of the *nonlinear Dirichlet problem* for (2.4.24). This is difficult to solve. However, if the equation (2.4.23) is linearized around the zero solution, the square root term is replaced by one, and then the Laplace equation is obtained.

Example 2.4.3 (Lagrange's Equation in Mechanics). According to the Hamilton principle in mechanics, the first variation of the time integral of the Lagrangian $L = L(q_i, \dot{q}_i, t)$ of any dynamical system must be stationary, that is,

$$0 = \delta I = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt, \tag{2.4.25}$$

where $L = T - V$ is the difference between the kinetic energy, T, and the potential energy, V. In coordinate space, there are infinitely many possible paths joining any two positions. From all these paths, which start at a point A at time t_1 and end at another point B at time t_2, nature selects the path $q_i = q_i(t)$ for which $\delta I = 0$. Consequently, in this case, the Euler–Lagrange equation (2.4.6) reduces to

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \quad i = 1, 2, \ldots, n. \tag{2.4.26}$$

In classical mechanics, these equations are universally known as the *Lagrange equations of motion*.

The *Hamilton function* (or simply *Hamiltonian*) H is defined in terms of the generalized coordinates q_i, generalized momentum $p_i = \frac{\partial L}{\partial \dot{q}_i}$, and L by

$$H = \sum_{i=1}^{n} (p_i \dot{q}_i - L) = \sum_{i=1}^{n} \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L(q_i, \dot{q}_i)\right). \tag{2.4.27}$$

It readily follows that

$$\frac{dH}{dt} = \frac{d}{dt}\left[\sum_{i=1}^{n} (p_i \dot{q}_i - L)\right] = \sum_{i=1}^{n} \dot{q}_i \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}\right) = 0. \tag{2.4.28}$$

Thus, H is a constant, and hence, the Hamiltonian is the constant of motion.

Example 2.4.4 (Hamilton's Equations in Mechanics). To derive Hamilton equations of motion, we use the concepts of generalized momentum p_i and generalized force F_i defined by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \text{and} \tag{2.4.29a}$$

$$F_i = \frac{\partial L}{\partial q_i}. \tag{2.4.29b}$$

Consequently, the Lagrange equations of motion (2.4.26) reduce to

$$\frac{\partial L}{\partial q_i} = \frac{dp_i}{dt} = \dot{p}_i. \tag{2.4.30}$$

In general, the Lagrangian $L = L(q_i, \dot{q}_i, t)$ is a function of q_i, $\dot{q}_i$, and t where $\dot{q}_i$ enters through the kinetic energy as a quadratic term. It then follows from the definition (2.4.27) of the Hamiltonian that $H = H(p_i, q_i, t)$, and hence, its differential is

$$dH = \sum \frac{\partial H}{\partial p_i}\, dp_i + \sum \frac{\partial H}{\partial q_i}\, dq_i + \frac{\partial H}{\partial t}\, dt. \tag{2.4.31}$$

Differentiating (2.4.27) with respect to t gives

$$\frac{dH}{dt} = \sum p_i \frac{d}{dt}\dot{q}_i + \sum \dot{q}_i \frac{d}{dt} p_i - \sum \frac{\partial L}{\partial q_i}\frac{d}{dt} q_i - \sum \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\dot{q}_i - \frac{\partial L}{\partial t},$$

or equivalently,

$$dH = \sum p_i\, d\dot{q}_i + \sum \dot{q}_i\, dp_i - \sum \frac{\partial L}{\partial q_i}\, dq_i - \sum \frac{\partial L}{\partial \dot{q}_i}\, d\dot{q}_i - \frac{\partial L}{\partial t}\, dt,$$

which, due to equation (2.4.29a), is

$$= \sum \dot{q}_i\, dp_i - \sum \frac{\partial L}{\partial q_i}\, dq_i - \frac{\partial L}{\partial t}\, dt. \tag{2.4.32}$$

We next equate the coefficients of the two identical expressions (2.4.31) and (2.4.32) to obtain

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad -\frac{\partial L}{\partial q_i} = \frac{\partial H}{\partial q_i}, \qquad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}. \tag{2.4.33}$$

Using the Lagrange equations (2.4.30), the first two equations in (2.4.33) give

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{2.4.34ab}$$

These are universally known as the *Hamilton canonical equations* of motion.

Example 2.4.5 (Law of Conservation of Energy). The kinetic energy of a mechanical system described by a set of generalized coordinates q_i is defined by

$$T = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{2} a_{ij}\dot{q}_i\dot{q}_j, \tag{2.4.35}$$

where a_{ij} are known functions of q_i, and $\dot{q}_i$ is the generalized velocity.

In general, the potential energy $V = V(q_i, \dot{q}_i, t)$ is a function of q_i, $\dot{q}_i$, and t. We assume here that V is independent of $\dot{q}_i$. For such a mechanical system, the Lagrangian is defined by $L = T - V$.

Using the above definitions, the Hamilton principle states that, between any two points t_1 and t_2, the actual motion takes place along the path $q_i = q_i(t)$ such that the functional

$$I\big(q_i(t)\big) = \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} (T - V)\,dt, \tag{2.4.36}$$

is stationary (that is, the functional is an extremum). Or equivalently, the Hamilton principle can be stated as

$$\delta I = \delta \int_{t_1}^{t_2} (T - V)\,dt = 0. \tag{2.4.37}$$

The integral I defined by (2.4.36) is often called the *action integral* of the system.

Since the potential energy V does not depend on $\dot{q}_i$, it follows that

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{n} a_{ij}\dot{q}_j,$$

and the Hamiltonian H defined by (2.4.27) becomes

$$H = \sum_{i=1}^{n} p_i\dot{q}_i - L = \sum_{i=1}^{n} \dot{q}_i\left(\sum_{j=1}^{n} a_{ij}\dot{q}_j\right) - L = 2T - L = T + V. \tag{2.4.38}$$

This proves that the Hamiltonian H is equal to the total energy. By (2.4.28), H is a constant, thus, the total energy of the system is constant. This is the celebrated *law of conservation of energy*.

Example 2.4.6 (Motion of a Particle Under the Action of a Central Force). Consider the motion of a particle of mass m under the action of a central force $-mF(r)$ where r is the distance of the particle from the center of force. The kinetic energy T is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\big(\dot{x}^2 + \dot{y}^2\big),$$

which, in terms of polar coordinates,

$$= \frac{1}{2}m\left[\left\{\frac{d}{dt}(r\cos\theta)\right\}^2 + \left\{\frac{d}{dt}(r\sin\theta)\right\}^2\right] = \frac{1}{2}m\big(\dot{r}^2 + r^2\dot{\theta}^2\big). \tag{2.4.39}$$

Since $\mathbf{F} = \nabla V$, the potential is given by

$$V(r) = \int^{r} F(r)\,dr. \tag{2.4.40}$$

Then the Lagrangian L is given by

$$L = T - V = \frac{1}{2}m\left[\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - 2\int^r F(r)\,dr\right]. \tag{2.4.41}$$

Thus the Hamilton principle requires that the functional

$$I(r,\theta) = \int_{t_1}^{t_2} L\,dr = \int_{t_1}^{t_2} (T - V)\,dt \tag{2.4.42}$$

be stationary, that is, $\delta I = 0$. Consequently, the Euler–Lagrange equations are given by

$$L_r - \frac{d}{dt}L_{\dot{r}} = 0 \quad \text{and} \quad L_\theta - \frac{d}{dt}L_{\dot{\theta}} = 0, \tag{2.4.43}$$

or equivalently,

$$\ddot{r} - r\dot{\theta}^2 = -F(r) \quad \text{and} \quad \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0. \tag{2.4.44}$$

These equations describe the planar motion of the particle.

It follows immediately from the second equation of (2.4.44) that

$$r^2\dot{\theta} = \text{const.} = h. \tag{2.4.45}$$

In this case, $r\dot{\theta}$ represents the transverse velocity of the particle and $mr^2\dot{\theta} = mh$ is the constant angular momentum of the particle about the center of force.

Introducing $r = \frac{1}{u}$, we find

$$\begin{aligned}
\dot{r} &= \frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt} = -h\frac{du}{d\theta},\\
\ddot{r} &= \frac{d^2r}{dt^2} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}.
\end{aligned}$$

Substituting these into the first equation of (2.4.44) gives

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{h^2u^2}F\left(\frac{1}{u}\right). \tag{2.4.46}$$

This is the differential equation of the central orbit, and it can be solved by standard methods.

In particular, if the law of force is the attractive inverse square $F(r) = \mu/r^2$ so that the potential $V(r) = -\mu/r$, the differential equation (2.4.46) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}. \tag{2.4.47}$$

If the particle is projected initially from the distance a with velocity v at an angle α that the direction of motion makes with the outward radius vector, then the constant h in (2.4.45) is $h = av\sin\alpha$.

The angle ϕ between the tangent and radius vector of the orbit at any point is given by

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} = u\frac{d}{d\theta}\left(\frac{1}{u}\right) = -\frac{1}{u}\frac{du}{d\theta}.$$

The initial conditions at $t = 0$ are

$$u = \frac{1}{a}, \qquad \frac{du}{d\theta} = -\frac{1}{a}\cot\alpha, \quad \text{when } \theta = 0. \tag{2.4.48}$$

The general solution of (2.4.47) is

$$u = \frac{\mu}{h^2}\big[1 + e\cos(\theta+\varepsilon)\big], \tag{2.4.49}$$

where e and ε are constants to be determined by the initial data.

Finally, the solution can be written as

$$\frac{\ell}{r} = 1 + e\cos(\theta+\varepsilon), \tag{2.4.50}$$

where

$$\ell = \frac{h^2}{\mu} = \frac{1}{\mu}(av\sin\alpha)^2. \tag{2.4.51}$$

This represents a conic section of semilatus rectum ℓ and eccentricity e with its axis inclined at the point of projection.

The initial conditions (2.4.48) give

$$\frac{\ell}{a} = 1 + e\cos\varepsilon, \qquad -\frac{\ell}{a}\cot\alpha = -e\sin\varepsilon,$$

so that

$$\begin{aligned}
\tan\varepsilon &= \left(\frac{\ell}{\ell-a}\right)\cot\alpha,\\
e^2 &= \left(\frac{\ell}{a}-1\right)^2 + \frac{\ell^2}{a^2}\cot^2\alpha = 1 - \frac{2\ell}{a} + \frac{\ell^2}{a^2}\operatorname{cosec}^2\alpha\\
&= 1 - \frac{1}{\mu}\big(2av^2\sin^2\alpha\big) + \frac{1}{\mu^2}\big(a^2v^4\sin^2\alpha\big)\\
&= 1 + \left(\frac{av\sin\alpha}{\mu}\right)^2\left(v^2 - \frac{2\mu}{a}\right).
\end{aligned} \tag{2.4.52}$$

Thus, the central orbit is an ellipse, parabola, or hyperbola accordingly as $e < 1$, $= 1$, or > 1, that is, $v^2 < (2\mu/a)$, $= (2\mu/a)$, or $> (2\mu/a)$.

Example 2.4.7 (The Wave Equation of a Vibrating String). We assume that, initially, the string of length ℓ and line density ρ is stretched along the x-axis from $x = 0$ to $x = \ell$. The string will be given a small lateral displacement, which is denoted by $u(x,t)$ at each point along the x-axis at time t. The kinetic energy T of the string is given by

$$T = \frac{1}{2} \int_0^\ell \rho u_t^2 \, dx, \tag{2.4.53}$$

and the potential energy is given by

$$T = \frac{T^*}{2} \int_0^\ell u_x^2 \, dx, \tag{2.4.54}$$

where T^* is the constant tension of the string.

According to the Hamilton principle

$$\begin{aligned} 0 &= \delta \int_{t_1}^{t_2} L \, dt = \delta \int_{t_1}^{t_2} (T - V) \, dt \\ &= \delta \int_{t_1}^{t_2} \frac{1}{2} \int_0^\ell \left(\rho u_t^2 - T^* u_x^2\right) dx \, dt. \end{aligned} \tag{2.4.55}$$

In this case, $L = \frac{1}{2}(\rho u_t^2 - T^* u_x^2)$ which does not depend explicitly on x, t, or u, and hence, the Euler–Lagrange equation is given by

$$\frac{\partial}{\partial t}(\rho u_t) - \frac{\partial}{\partial x}(T^* u_x) = 0,$$

or

$$u_{tt} - c^2 u_{xx} = 0, \tag{2.4.56}$$

where $c^2 = (T^*/\rho)$. This is the wave equation of the vibrating string.

Example 2.4.8 (Two-Dimensional Wave Equation of Motion for Vibrating Membrane). We consider the motion of a vibrating membrane occupying the domain D under the action of a prescribed lateral force $f(x, y, t)$ and subject to the homogeneous boundary conditions $u = 0$ on the boundary ∂D.

The kinetic energy T and the potential energy V are given by

$$T = \frac{1}{2}\rho \iint_D u_t^2 \, dx \, dy, \qquad V = \frac{1}{2}\mu \iint_D \left(u_x^2 + u_y^2\right) dx \, dy, \tag{2.4.57}$$

where ρ is the surface density, μ is the elastic modulus of the membrane, and $u = u(x, y, t)$ is the displacement function. The Lagrangian functional is of the form

$$L = \iint_D \mathcal{L} \, dx \, dy, \tag{2.4.58}$$

where the Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\rho u_t^2 - \frac{1}{2}\mu\left(u_x^2 + u_y^2\right) - u f(x, y, t). \tag{2.4.59}$$

According to the Hamilton principle, the first variation of the Lagrangian L must be stationary, that is,

$$0 = \delta \int_{t_1}^{t_2} L\,dt = \delta \int_{t_1}^{t_2} \iint_D \left[\frac{1}{2}\rho u_t^2 - \frac{1}{2}\mu\big(u_x^2 + u_y^2\big) - uf\right] dx\,dy. \tag{2.4.60}$$

The Euler–Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\mathcal{L}_{u_x} - \frac{\partial}{\partial y}\mathcal{L}_{u_y} - \frac{\partial}{\partial t}\mathcal{L}_{u_t} = 0, \tag{2.4.61}$$

or equivalently,

$$-f + \mu(u_{xx} + u_{yy}) - \rho u_{tt} = 0. \tag{2.4.62}$$

This leads to the nonhomogeneous wave equation

$$\mu \nabla^2 u - \rho u_{tt} = f(x, y, t). \tag{2.4.63}$$

This is the two-dimensional nonhomogeneous wave equation that can be solved with the initial conditions

$$u(x, y, t = 0) = \phi(x, y) \quad \text{and} \quad u_t(x, y, t = 0) = \psi(x, y) \quad \text{at } t = 0. \tag{2.4.64}$$

Example 2.4.9 (Three-Dimensional Nonhomogeneous Wave Equation). In three-dimensional wave propagation in elastic media, the traveling waves exhibit various modes of vibration including longitudinal and transverse waves. To derive the appropriate equations of motion in continuous media, we need to extend the Hamilton principle by considering the displacement vector $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$. We use symmetric motion given by $\mathbf{u} = \mathbf{u}(u_1, u_2, u_3)$ where $u_i = u_i(x_1, x_2, x_3, t)$, $i = 1, 2, 3$, and denote the particle velocity by $u_t = (u_{1,t}, u_{2,t}, u_{3,t})$. Using this notation and tensor summation convention, the kinetic energy T and the potential energy V are given by

$$T = \frac{1}{2}\rho u_{i,t} u_{i,t} \quad \text{and} \quad V = \frac{1}{2}\mu u_{i,j} u_{i,j}, \tag{2.4.65}$$

where $u_{i,t} = \frac{\partial u_i}{\partial t}$ and $u_{i,j} = \frac{\partial u_i}{\partial x_j}$.

Introducing an external force term $f(x_i, t)$ so that the Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\rho u_{i,t} u_{i,t} - \frac{1}{2}\mu u_{i,j} u_{i,j} - u_i f(x_i, t), \tag{2.4.66}$$

the Lagrangian functional is of the form

$$L = \iiint_D \mathcal{L}\,dx_j. \tag{2.4.67}$$

The generalized Hamilton principle for a three-dimensional continuum for various modes of wave propagation described by $\mathbf{u}(\mathbf{x}_j, t)$ takes the form

$$0 = \delta I(\mathbf{u}) = \delta \int_{t_1}^{t_2} dt \iiint_D \mathcal{L}\,dx_j = \int_{t_1}^{t_2} dt \iiint_D \delta\mathcal{L}\,dx_j. \tag{2.4.68}$$

This means that the function $\mathbf{u} = \mathbf{u}(x_j, t)$ makes the functional $I(\mathbf{u})$ an extremum.

Since $\mathcal{L}$ is a function of u_i and $u_{i,t}$, and the operator δ acts on the function u_i and $u_{i,t}$, we expand $\mathcal{L}$ to obtain

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial u_i}\delta u_i + \frac{\partial \mathcal{L}}{\partial u_{i,t}}\delta u_{i,t} + \frac{\partial \mathcal{L}}{\partial u_{i,j}}\delta u_{i,j}. \tag{2.4.69}$$

We next substitute (2.4.69) into (2.4.68) and then integrate by parts with respect to t to obtain

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial u_{i,t}}\delta u_{i,t}\,dt = -\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial u_{i,t}}\right)\delta u_{i,t}\,dt. \tag{2.4.70}$$

Interchanging $\frac{\partial}{\partial x_j}$ and the δ variations in the integrals involving the spatial derivatives of u_i, it turns out that

$$\iiint_D \frac{\partial \mathcal{L}}{\partial u_{i,j}}\delta u_{i,j}\,dx_j = \iiint_D \frac{\partial \mathcal{L}}{\partial u_{i,j}}\left(\frac{\partial \delta u_i}{\partial x_j}\right)dx_j,$$

which, by integrating by parts, is

$$= \frac{\partial \mathcal{L}}{\partial u_{i,t}}\delta u_i - \iiint_D \frac{d}{dx_j}\left(\frac{\partial \mathcal{L}}{\partial u_{i,j}}\right)\delta u_i\,dx_j. \tag{2.4.71}$$

Since u_i vanishes at t_1 and t_2, the integrated term also vanishes. Using (2.4.69)–(2.4.71) in (2.4.68) gives

$$0 = \delta I(\mathbf{u}) = \delta\int_{t_1}^{t_2} dt \iiint_D \delta u_i\left[\frac{\partial \mathcal{L}}{\partial u_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial u_{i,t}}\right) - \frac{d}{dx_j}\left(\frac{\partial \mathcal{L}}{\partial u_{i,j}}\right)\right]dx_j. \tag{2.4.72}$$

This is true only if the coefficients of each of the linearly independent displacements δu_i vanish. Consequently, (2.4.72) leads to the Euler–Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial u_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial u_{i,t}}\right) - \frac{d}{dx_j}\left(\frac{\partial \mathcal{L}}{\partial u_{i,j}}\right) = 0, \tag{2.4.73}$$

where the summation over j is used.

In particular, if the Lagrangian $\mathcal{L}$ is of the form (2.4.66), (2.4.73) gives the non-homogeneous wave equations

$$\mu\nabla^2 u_i - \rho u_{i,tt} = f(x_i, t). \tag{2.4.74}$$

In the case of equilibrium, the Euler–Lagrange equations (2.4.74) reduce to the Poisson equation

$$\mu\nabla^2 u_i = f(x_i, t). \tag{2.4.75}$$

We close this section by adding an important comment. Many equations in applied mathematics and mathematical physics can be derived from the Euler–Lagrange variational principle, the Hamilton principle, or from some appropriate variational principle.

2.5 The Variational Principle for Nonlinear Klein–Gordon Equations

The nonlinear Klein–Gordon equation is

$$u_{tt} - c^2 u_{xx} + V'(u) = 0, \tag{2.5.1}$$

where $V'(u)$ is some nonlinear function of u chosen as the derivative of the potential energy $V(u)$.

The variational principle for equation (2.5.1) is given by

$$\delta \iint L(u, u_t, u_x)\, dt\, dx = 0, \tag{2.5.2}$$

where L is the associated Lagrangian density

$$L(u, u_t, u_x) = \frac{1}{2}\left(u_t^2 - c^2 u_x^2\right) - V(u). \tag{2.5.3}$$

The Euler–Lagrange equation associated with (2.5.2) is

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_t}\right) = 0, \tag{2.5.4}$$

which can be simplified to obtain the Klein–Gordon equation (2.5.1).

We consider the variational principle

$$\delta \iint L\, dx\, dt = 0, \tag{2.5.5}$$

with the Lagrangian L given by

$$L \equiv \frac{1}{2}\left(u_t^2 - c^2 u_x^2 - d^2 u^2\right) - \gamma u^4, \tag{2.5.6}$$

where γ is a constant. The Euler–Lagrange equation associated with (2.5.5) gives the special case of the Klein–Gordon equation

$$u_{tt} - c^2 u_{xx} + d^2 u + 4\gamma u^3 = 0. \tag{2.5.7}$$

2.6 The Variational Principle for Nonlinear Water Waves

In his pioneering work, Whitham (1965a, 1965b) first developed a general approach to linear and nonlinear dispersive waves using a Lagrangian. It is well known that most of the general ideas about dispersive waves have originated from the classical problems of water waves. So it is important to have a variational principle for water waves. Luke (1967) first explicitly formulated a variational principle for two-dimensional water waves and showed that the basic equations and boundary and free surface conditions can be derived from the Hamilton principle.

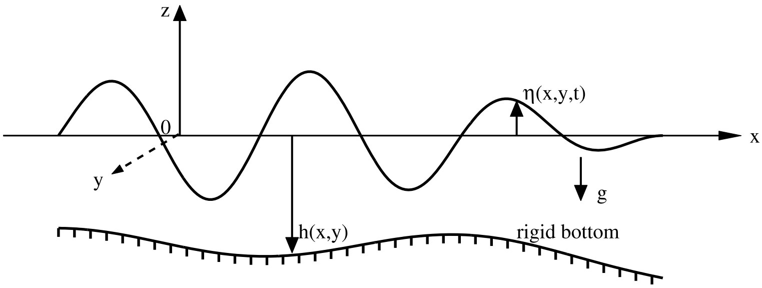

Fig. 2.1 A general surface gravity wave problem.

We now formulate the *variational principle* for three-dimensional water waves in the form

$$\delta I = \delta \iint_D L \, d\mathbf{x} \, dt = 0, \tag{2.6.1}$$

where the *Lagrangian* L is assumed to be equal to the pressure, so that

$$L = -\rho \int_{-h(x,y)}^{\eta(x,t)} \left[\phi_t + \frac{1}{2}(\nabla\phi)^2 + gz \right] dz, \tag{2.6.2}$$

where D is an arbitrary region in the $(\mathbf{x}, t)$ space, ρ is the density of water, g is the gravitational acceleration, and $\phi(\mathbf{x}, z, t)$ is the velocity potential in an unbounded fluid lying between the rigid bottom at $z = -h(x, y)$ and the free surface $z = \eta(x, y, t)$ as shown in Figure 2.1. The functions $\phi(\mathbf{x}, z, t)$ and $\eta(\mathbf{x}, t)$ are allowed to vary subject to the restrictions $\delta\phi = 0$ and $\delta\eta = 0$ at x_1, x_2, y_1, y_2, t_1, and t_2.

Using the standard procedure in the calculus of variations, (2.6.1) becomes

$$\begin{aligned} 0 &= -\delta \iint_D \frac{L}{\rho} \, d\mathbf{x} \, dt \\ &= \iint_D \left\{ \left[\phi_t + \frac{1}{2}(\nabla\phi)^2 + gz \right]_{z=\eta} \delta\eta \right. \\ &\quad \left. + \int_{-h}^{\eta} [\phi_x \delta\phi_x + \phi_y \delta\phi_y + \phi_z \delta\phi_z + \delta\phi_t] \, dz \right\} d\mathbf{x} \, dt, \end{aligned} \tag{2.6.3}$$

which, integrating the z-integral by parts, is

$$\begin{aligned} &= \iint_D \left\{ \left[\phi_t + \frac{1}{2}(\nabla\phi)^2 + gz \right]_{z=\eta} \delta\eta \right. \\ &\quad + \left[\frac{\partial}{\partial t} \int_{-h}^{\eta} \delta\phi \, dz + \frac{\partial}{\partial x} \int_{-h}^{\eta} \phi_x \delta\phi \, dz + \frac{\partial}{\partial y} \int_{-h}^{\eta} \phi_y \delta\phi \, dz \right] \\ &\quad - \int_{-h}^{\eta} (\phi_{xx} + \phi_{yy} + \phi_{zz}) \delta\phi \, dz - \left[(\eta_t + \eta_x \phi_x + \eta_y \phi_y - \phi_z) \delta\phi \right]_{z=\eta} \\ &\quad \left. + \left[(\phi_x h_x + \phi_y h_y + \phi_z) \delta\phi \right]_{z=-h} \right\} d\mathbf{x} \, dt. \end{aligned} \tag{2.6.4}$$

The second term within the square brackets integrates out to the boundaries ∂D of D and vanishes if $\delta\phi$ is chosen to be zero on ∂D. If we take $\delta\eta = 0$, $[\delta\phi]_{z=\eta} = [\delta\phi]_{z=-h} = 0$, since $\delta\phi$ is otherwise arbitrary; it turns out that

$$\nabla^2\phi = 0, \quad -\infty < x, y < \infty, \ -h < z < \eta. \tag{2.6.5}$$

Since $\delta\eta$, $[\delta\phi]_{z=\eta}$, $[\delta\phi]_{z=-h}$ may be given arbitrary independent values, it follows that

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + g\eta = 0 \quad \text{on } z = \eta, \tag{2.6.6}$$

$$\eta_t + \eta_x\phi_x + \eta_y\phi_y - \phi_z = 0 \quad \text{on } z = \eta, \tag{2.6.7}$$

$$\phi_x h_x + \phi_y h_y + \phi_z = 0 \quad \text{on } z = -h. \tag{2.6.8}$$

Thus, (2.6.5)–(2.6.8) represent the well-known nonlinear system of equations for classical water waves. Finally, this analysis is in perfect agreement with that of Luke (1967) and Whitham (1965a, 1965b, 1974) for two-dimensional waves on water of arbitrary but uniform depth h.

It may be relevant to mention Zakharov's (1968a, 1968b) Hamiltonian formulation. The Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\int_{-\infty}^{\infty}\left(g\eta^2 + \int_{-h}^{\eta}(\nabla\phi)^2\,dz\right)dx. \tag{2.6.9}$$

On the other hand, Benjamin and Olver (1982) have described the Hamiltonian structure, symmetrics, and conservation laws for water waves. Olver (1984a, 1984b) has discussed Hamiltonian and non-Hamiltonian models for water waves, and Hamiltonian perturbation theory and nonlinear water waves.

2.7 The Euler Equation of Motion and Water Wave Problems

The Euler equation of motion and the equation of continuity have provided the fundamental basis of the study of modern theories of water waves, which are the most common observable phenomena in nature. Water wave motions are of great importance as they range from waves generated by wind or solar heating at the surface of oceans to flood waves in rivers, from waves caused by a moving ship in a channel to tsunami (tidal waves) generated by earthquakes, and from solitary waves on the surface of a channel generated by a disturbance to waves generated by underwater explosions, to mention only a few.

Making reference to Debnath's book (1994), *Nonlinear Water Waves*, the Euler equation of motion in an inviscid and incompressible fluid of constant density ρ under the action of body force $\mathbf{F} = (0, 0, -g)$ where g is the constant acceleration of gravity and the equation of continuity are given by

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{F}, \tag{2.7.1}$$

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.7.2}$$

where $\mathbf{x} = (x, y, z)$ is the rectangular Cartesian coordinates and $\mathbf{u} = (u, v, w)$ is the velocity vector, p is the pressure field, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}. \tag{2.7.3}$$

These equations constitute a closed system of four *nonlinear* partial differential equations for four unknowns u, v, w, and p. So these equations with appropriate initial and boundary conditions are sufficient to determine the velocity field $\mathbf{u}$ and pressure p uniquely.

In the study of water waves, the body force is always the acceleration due to gravity, that is, $\mathbf{F} = (0, 0, -g)$. It is convenient to write the three components of the Euler equation in the form

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2.7.4}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{2.7.5}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{2.7.6}$$

and the continuity equation (2.7.2).

In cylindrical polar coordinates $\mathbf{x} = (r, \theta, z)$ with the velocity vector $\mathbf{u} = (u, v, w)$, the Euler equations and the continuity equation are given by

$$\frac{Du}{Dt} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{2.7.7}$$

$$\frac{Dv}{Dt} + \frac{uv}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta}, \tag{2.7.8}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{2.7.9}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0, \tag{2.7.10}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}. \tag{2.7.11}$$

One of the fundamental properties of a fluid flow is called the *vorticity*, which is defined by the *curl* of the velocity field so that $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u} = \nabla \times \mathbf{u}$. The vorticity vector $\boldsymbol{\omega}$ measures the local spin or rotation of individual fluid particles. Evidently, fluid flows in which $\boldsymbol{\omega} = \mathbf{0}$ are called *irrotational*. In the real world, fluid flows are hardly irrotational anywhere; however, for many flows the vorticity is very small almost everywhere and the fluid motion may be treated as irrotational. In problems of water waves, the motion of fluid is considered unsteady and irrotational which implies that the vorticity $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u} = \nabla \times \mathbf{u} = \mathbf{0}$. So there exists a single-valued velocity potential ϕ so that $\mathbf{u} = \nabla\phi$. The continuity equation (2.7.2) then reduces to the Laplace equation

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{2.7.12}$$

Using the vector identity, $(\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{2}\nabla u^2 - \mathbf{u}\times\boldsymbol{\omega}$ combined with $\boldsymbol{\omega} = \mathbf{0}$ and $\mathbf{u} = \nabla\phi$, the Euler equation (2.7.1) may be written in the form

$$\nabla\left[\phi_t + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} + gz\right] = 0. \tag{2.7.13}$$

This can be integrated with respect to the space variables to obtain the equation

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} + gz = c(t), \tag{2.7.14}$$

where $c(t)$ is an arbitrary function of time only $(\nabla c = 0)$ determined by the pressure imposed at the boundaries of the fluid flow. Since only the pressure gradient affects the flow, a function of t alone added to the pressure field p has virtually no effect on the motion. So, without loss of generality, we can set $c(t) \equiv 0$ in (2.7.14). Consequently, equation (2.7.14) becomes

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} + gz = 0. \tag{2.7.15}$$

This equation is known as *Bernoulli's equation* (or the *pressure equation*), which completely determines the pressure in terms of the velocity potential ϕ. Thus, the Laplace equation (2.7.12) and (2.7.15) are used to determine ϕ and p, and hence the velocity components u, v, w, and the pressure p.

In cylindrical polar coordinates $\mathbf{x} = (r,\theta,z)$ with the velocity field $u = (\phi_r, \frac{1}{r}\phi_\theta, \phi_z)$, the Laplace equation becomes

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}(r\phi_r) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{2.7.16}$$

We assume that the fluid occupies the region $-h \le z \le 0$ with the plane $z = -h$ as the rigid *bottom boundary* and the plane $z = 0$ as the *upper* (*free surface*) *boundary* in the undisturbed state. We suppose that the upper boundary is the surface exposed to a constant atmospheric pressure p_a. Since the free surface is exposed to the constant atmospheric pressure p_a, we have $p = p_a$ on this surface. After the motion is set up, we denote this surface by S with the equation $z = \eta(x,y,t)$ where η is an unknown function of x, y, and t that tends to zero as $t \to 0$. The function $\eta(x,y,t)$ is referred to as the *free surface elevation*.

The rate of change of η, following a fluid particle, is equal to the vertical component of $\nabla\phi$ at the surface, that is,

$$\eta_t + \mathbf{u} \cdot \nabla \eta = \phi_z \quad \text{on } z = \eta.$$

Or equivalently, this free surface condition reads as

$$\eta_t + \phi_x \eta_x + \phi_y \eta_y - \phi_z = 0 \quad \text{on } z = \eta. \tag{2.7.17}$$

This is called the *kinematic free surface condition*.

Since $p = p_a$ on S, after absorbing $\frac{p_a}{\rho}$ and $c(t)$ into ϕ_t, equation (2.7.14) can be rewritten as

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla \phi)^2 + gz = 0 \quad \text{on } S \text{ for } t \geq 0. \tag{2.7.18}$$

Since S is a free boundary surface, it contains the same fluid particles for all times, that is, S is a material surface. Hence, it follows from (2.7.18) that

$$\frac{D}{Dt}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla \phi)^2 + gz\right] = 0 \quad \text{on } S \text{ for } t \geq 0.$$

Or equivalently, on S for $t \geq 0$,

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \nabla \phi \cdot \nabla\right)\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla \phi)^2 + gz\right] \\
&= \phi_{tt} + 2\nabla \phi \cdot \nabla(\phi_t) + \frac{1}{2}\nabla \phi \cdot \nabla(\nabla \phi)^2 + g\phi_z = 0.
\end{aligned} \tag{2.7.19}$$

Since the bottom boundary $z = -h$ is a rigid solid surface at rest, the condition to be satisfied at this boundary is

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = -h, \ t \geq 0. \tag{2.7.20}$$

Thus, Laplace's equation (2.7.12) together with the free surface boundary conditions (2.7.17), (2.7.19) and the bottom boundary condition (2.7.20) determine the velocity potential ϕ and the free surface elevation η. Because of the presence of the nonlinear terms in the free surface boundary conditions (2.7.17) and (2.7.19), the determination of ϕ and η in the general case is a difficult task. We restrict our discussion to two particular cases because of the great importance of water wave motions.

Example 2.7.1 (Small Amplitude Water Waves). We consider plane waves propagating in the x-direction whose amplitude varies in the z-direction with the gravitational force as the only body force. We first consider the case where the motion is linear so that nonlinear terms in velocity components may be neglected. In this case, no distinction is made between the initial and the current states of the free surface boundary, and the boundary conditions (2.7.18) and (2.7.19) are given in the linearized forms

$$\phi_t + g\eta = 0 \quad \text{on } z = 0, \ t > 0, \tag{2.7.21}$$

$$\phi_{tt} + g\phi_z = 0 \quad \text{on } z = 0, \ t > 0. \tag{2.7.22}$$

These conditions yield

$$\eta_t = \phi_z \quad \text{on } z = 0,\ t > 0. \tag{2.7.23}$$

For a plane wave propagating in the x-direction with frequency ω and wavenumber k, we seek a solution for $\phi(x, z, t)$ in the form

$$\phi = \Phi(z) \exp\big[i(\omega t - kx)\big], \tag{2.7.24}$$

where $\Phi(z)$ is a function to be determined.

Substituting (2.7.24) in the Laplace equation (2.7.12) with no y dependence gives an equation for Φ as

$$\Phi_{zz} = k^2\Phi. \tag{2.7.25}$$

The general solution of this equation is

$$\Phi(z) = Ae^{kz} + Be^{-kz}, \tag{2.7.26}$$

where A and B are arbitrary constants. Using the boundary condition (2.7.20), we find $A\exp(-kh) = B\exp(kh)$ so that the solution (2.7.26) takes the form

$$\Phi = C\cosh k(z + h), \tag{2.7.27}$$

where $C = 2A\exp(-kh) = 2B\exp(kh)$ is an arbitrary constant so that the solution (2.7.24) becomes

$$\phi = C\cosh k(z + h)\exp\big[i(\omega t - kx)\big]. \tag{2.7.28}$$

Using (2.7.28) in (2.7.21) yields

$$\eta = a\exp\big[i(\omega t - kx)\big], \tag{2.7.29}$$

where $a = (C\omega/ig)\cosh kh = \max|\eta|$ is the *amplitude*. Thus, the solution (2.7.28) assumes the final form

$$\phi = \left(\frac{iag}{\omega}\right)\frac{\cosh k(z + h)}{\cosh kh}\exp\big[i(\omega t - kx)\big]. \tag{2.7.30}$$

Substituting (2.7.30) into (2.7.22) gives the following *dispersion relation* between the frequency and wavenumber:

$$\omega^2 = gk\tanh kh. \tag{2.7.31}$$

Thus, the phase velocity, $c_p = (\frac{\omega}{k})$, can be obtained from (2.7.31) as

$$c_p^2 = \frac{\omega^2}{k^2} = \frac{g}{k}\tanh(kh). \tag{2.7.32}$$

This result shows that the phase velocity c_p depends on the wavenumber k, depth h, and the gravity g. Hence, water waves are dispersive in nature. This means that, as the time passes, the waves would disperse (spread out) into different groups such that each group would consist of waves having approximately the same wavelength. The quantity $\frac{d\omega}{dk}$ represents the velocity of such a group in the direction of propagation and is called the *group velocity*, c_g. It follows from (2.7.31) that

$$c_g = \frac{d\omega}{dk} = \left(\frac{g}{2\omega}\right)\left(\tanh kh + kh \sec h^2 kh\right), \tag{2.7.33}$$

which, by using (2.7.32), is

$$= \frac{1}{2}c_p\left[1 + \frac{2kh}{\sinh 2kh}\right]. \tag{2.7.34}$$

Evidently, the group velocity is different from the phase velocity.

In the case where wavelength $2\pi/k$ is large compared with the depth h, such waves are called *long waves* (or *shallow water waves*), $kh \ll 1$ so that $\tanh kh \approx kh$, and hence, $\sinh 2kh \approx 2kh$. In such a situation, results (2.7.32) and (2.7.34) give

$$c_g = c_p \approx \sqrt{gh} = c. \tag{2.7.35}$$

Thus, shallow water waves are nondispersive and their speed varies as the square root of the depth.

In the other limiting case, where the wavelength is very small compared with the depth, such waves are called *short waves* (or *deep water waves*), $kh \gg 1$. In the limit $kh \to \infty$, $[\cosh k(z+h)/\cosh kh] \to \exp(kz)$, and the corresponding solutions for ϕ and η become

$$\begin{aligned}\phi &= \mathrm{Re}\left(\frac{iag}{\omega}\right)\exp\left[kz + i(\omega t - kx)\right] \\ &= \left(\frac{ag}{\omega}\right)\exp(kz)\sin(kx - \omega t), \qquad (2.7.36)\\ \eta &= \mathrm{Re}\, a \exp\left[i(\omega t - kx)\right] = a\cos(\omega t - kx). \qquad (2.7.37)\end{aligned}$$

In the limit $kh \to \infty$, $\tanh kh \to 1$ so that the dispersion relation becomes

$$\omega^2 = gk. \tag{2.7.38}$$

Consequently,

$$\begin{aligned}c_p &= \left(\frac{g}{k}\right)^{\frac{1}{2}} = \left(\frac{g\lambda}{k\pi}\right)^{\frac{1}{2}}, \qquad (2.7.39)\\ c_g &= \frac{1}{2}c_p. \qquad (2.7.40)\end{aligned}$$

Evidently, deep water waves are dispersive and the phase velocity is proportional to the square root of their wavelength. Also the group velocity is equal to one-half of the phase velocity.

Example 2.7.2 (The Stokes' Waves or Nonlinear Finite Amplitude Waves). We consider the Stokes' waves where the motion is nonlinear and the amplitude of the waves is *not* small. We recall Bernoulli's equation (2.7.18) and (2.7.19) and write them for ready reference in the form

$$\eta = -\frac{1}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2\right]_{z=\eta}, \tag{2.7.41}$$

$$[\phi_{tt} + g\phi_z]_{z=\eta} + 2[\nabla\phi\cdot\nabla\phi_t]_{z=\eta} + \frac{1}{2}\big[\nabla\phi\cdot\nabla(\nabla\phi)^2\big]_{z=\eta} = 0. \tag{2.7.42}$$

A systematic procedure can be employed to rewrite these boundary conditions by using Taylor's series expansions of the potential ϕ and its derivatives in the form

$$\phi(x,y,z=\eta,t) = [\phi]_{z=0} + \eta[\phi_z]_{z=0} + \frac{1}{2}\eta^2[\phi_{zz}]_{z=0} + \cdots, \tag{2.7.43}$$

$$\phi_z(x,y,z=\eta,t) = [\phi_z]_{z=0} + \eta[\phi_{zz}]_{z=0} + \frac{1}{2}\eta^2[\phi_{zzz}]_{z=0} + \cdots. \tag{2.7.44}$$

Substituting these and similar Taylor's expansions into (2.7.41) gives

$$\begin{aligned}\eta &= -\frac{1}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2\right]_{z=0} + \eta\left[-\frac{1}{g}\left\{\phi_t + \frac{1}{2}(\nabla\phi)^2\right\}_z\right]_{z=0} + \cdots \\ &= -\frac{1}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2\right]_{z=0} \\ &\quad + \frac{1}{g^2}\left[\left\{\phi_t + \frac{1}{2}(\nabla\phi)^2\right\}\left\{\phi_t + \frac{1}{2}(\nabla\phi)^2\right\}_z\right]_{z=0} + \cdots \\ &= -\frac{1}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{g}\phi_t\phi_{zt}\right]_{z=0} + O\big(\phi^3\big).\end{aligned} \tag{2.7.45}$$

Similarly, condition (2.7.42) gives

$$\begin{aligned}&[\phi_{tt} + g\phi_z]_{z=0} + \eta\big[(\phi_{tt} + g\phi_z)_z\big] + \frac{1}{2}\eta^2\big[(\phi_{tt} + g\phi_z)_{zz}\big]_{z=0} \\ &\quad + \cdots + 2[\nabla\phi\cdot\nabla\phi_t]_{z=0} + 2\eta\big[\{\nabla\phi\cdot\nabla\phi_t\}_z\big]_{z=0} + \eta^2\big[\{\nabla\phi\cdot\nabla\phi_t\}_{zz}\big]_{z=0} \\ &\quad + \cdots + \frac{1}{2}\big[\{\nabla\phi\cdot\nabla(\nabla\phi)^2\}\big]_{z=0} + \frac{1}{2}\eta\big[\{\nabla\phi\cdot\nabla(\nabla\phi)^2\}_z\big]_{z=0} \\ &\quad + \frac{1}{4}\eta^2\big[\{\nabla\phi\cdot\nabla(\nabla\phi)^2\}_{zz}\big]_{z=0} + \cdots = 0.\end{aligned} \tag{2.7.46}$$

We substitute (2.7.45) for η into (2.7.46) to obtain

$$\begin{aligned}&[\phi_{tt} + g\phi_z]_{z=0} - \frac{1}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{g}\phi_t\phi_{zt}\right]_{z=0}\big[(\phi_{tt} + g\phi_z)_z\big]_{z=0} \\ &\quad + \frac{1}{2g^2}\left[\left\{\phi_t + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{g}\phi_t\phi_{zt}\right\}^2\right]_{z=0}\big[(\phi_{tt} + g\phi_z)_{zz}\big]_{z=0}\end{aligned}$$

$$+2\big[(\nabla\phi)\cdot\nabla\phi_t\big]_{z=0}-\frac{2}{g}\left[\phi_t+\frac{1}{2}(\nabla\phi)^2-\frac{1}{g}\phi_t\phi_{zt}\right]_{z=0}\big[(\nabla\phi\cdot\nabla\phi_t)_z\big]_{z=0}$$
$$+\frac{1}{2}\big[\nabla\phi\cdot\nabla(\nabla\phi)^2\big]_{z=0}-\frac{2}{g}\left[\phi_t+\frac{1}{2}(\nabla\phi)^2-\frac{1}{g}\phi_t\phi_{zt}\right]_{z=0}$$
$$\times\big[\big\{\nabla\phi\cdot\nabla(\nabla\phi)^2\big\}_z\big]_{z=0}=0. \tag{2.7.47}$$

The first-, second-, and third-order boundary conditions on $z=0$ are respectively given by

$$(\phi_{tt}+g\phi_z)=0+O\big(\phi^2\big), \tag{2.7.48}$$
$$(\phi_{tt}+g\phi_z)+2[\nabla\phi\cdot\nabla\phi_t]-\frac{1}{g}\phi_t(\phi_{tt}+g\phi_z)_z=0+O\big(\phi^3\big), \tag{2.7.49}$$
$$(\phi_{tt}+g\phi_z)+2[\nabla\phi\cdot\nabla\phi_t]+\frac{1}{2}\big[\nabla\phi\cdot\nabla(\nabla\phi)^2\big]$$
$$-\frac{1}{g}\phi_t\big[\phi_{tt}+g\phi_z+2(\nabla\phi\cdot\nabla\phi_t)\big]_z-\frac{1}{g}\left[\frac{1}{2}\big(\nabla\phi^2\big)-\frac{1}{g}\phi_t\phi_{zt}\right][\phi_{tt}+g\phi_z]_z$$
$$+\frac{1}{2g^2}[\phi_t]^2\big[(\phi_{tt}+g\phi_z)_{zz}\big]=0+O\big(\phi^4\big), \tag{2.7.50}$$

where $O\,(\cdot)$ indicates the order of magnitude of the neglected terms. These results can be determined by the third-order expansion of plane progressive surface waves.

As indicated before, the first-order plane wave potential ϕ in deep water is given by (2.7.36). Direct substitution of the first-order velocity potential (2.7.36) in the second-order boundary condition (2.7.49) reveals that the second-order terms in (2.7.49) vanish. Thus, the first-order potential is a solution of the second-order boundary-value problem, and we can state that

$$\phi=\left(\frac{ga}{\omega}\right)e^{kz}\sin(kx-\omega t)+O\big(a^3\big). \tag{2.7.51}$$

Substitution of this result into (2.7.45) leads to the second-order result for η in the form

$$\eta=a\cos(kx-\omega t)-\frac{1}{2}ka^2+ka^2\cos^2(kx-\omega t)+\cdots$$
$$=a\cos(kx-\omega t)+\frac{1}{2}ka^2\cos\big\{2(kx-\omega t)\big\}+\cdots. \tag{2.7.52}$$

The second term in (2.7.52), which represents the second-order correction to the surface profile, is positive at the *crests* $kx-\omega t=0, 2\pi, 4\pi, \dots$, and negative at the *troughs* $kx-\omega t=\pi, 3\pi, 5\pi, \dots$. But the crests are steeper, and the troughs flatter as a result of the nonlinear effect. The notable feature of solution (2.7.52) is that the wave profile is no longer sinusoidal. The actual shape of the wave profile is a curve known as a *trochoid* (see Figure 2.2), whose crests are steeper and troughs are flatter than those of the sinusoidal wave.

Substituting the wave potential (2.7.51) in the third-order boundary condition (2.7.50) reveals that all nonlinear terms vanish identically except one term, $(\frac{1}{2})\nabla\phi\,\cdot$

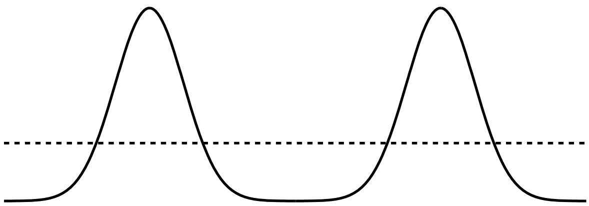

Fig. 2.2 The surface wave profile.

$\nabla(\nabla\phi)^2$. Thus the boundary condition for the third-order plane wave solution is given by

$$\phi_{tt} + g\phi_z + \frac{1}{2}\nabla\phi \cdot \nabla(\nabla\phi)^2 = 0 + O\big(\phi^4\big). \tag{2.7.53}$$

If the first-order solution (2.7.51) is substituted into the third-order boundary condition on $z = 0$, the dispersion relation with second-order effect is obtained in the form

$$\omega^2 = gk\big(1 + a^2k^2\big) + O\big(k^3a^3\big). \tag{2.7.54}$$

Note that this relation involves the amplitude in addition to frequency and wavenumber. This is called the *nonlinear dispersion relation* and it can be expressed in terms of the phase velocity as

$$c_p = \frac{\omega}{k} = \left(\frac{g}{k}\right)^{\frac{1}{2}} \big(1 + k^2a^2\big)^{\frac{1}{2}} \approx \left(\frac{g}{k}\right)^{\frac{1}{2}} \left(1 + \frac{1}{2}a^2k^2\right). \tag{2.7.55}$$

Thus the phase velocity depends on the wave amplitude, and waves of large amplitude travel faster than smaller ones. The dependence of c_p on amplitude is known as the *amplitude dispersion* in contrast to the *frequency dispersion* as given by (2.7.38).

It may be noted that Stokes' results (2.7.52), (2.7.54), and (2.7.55) can easily be approximated further to obtain solutions for long waves (or shallow water) and for short waves (or deep water).

We conclude this example by discussing the phenomenon of breaking of water waves, which is one of the most common observable phenomena on an ocean beach. A wave coming from deep ocean changes shape as it moves across a shallow beach. Its amplitude and wavelength also are modified. The wavetrain is very smooth some distance offshore, but as it moves inshore, the front of the wave steepens noticeably until, finally, it breaks. After breaking, waves continue to move inshore as a series of bores or hydraulic jumps, whose energy is gradually dissipated by means of the water turbulence. Of the phenomena common to waves on beaches, breaking is the physically most significant and mathematically least known. In fact, it is one of the most intriguing longstanding problems of water waves theory.

For waves of small amplitude in deep water, the maximum particle velocity is $v = a\omega = ack$. But the basic assumption of small amplitude theory implies that $\frac{v}{c} = ak \ll 1$. Therefore, wave breaking can never be predicted by the small amplitude wave theory. That possibility arises only in the theory of finite amplitude waves. It is

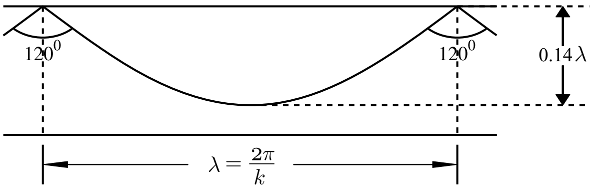

Fig. 2.3 The steepest wave profile.

to be noted that the Stokes' expansions are limited to relatively small amplitude and cannot predict the wavetrain of maximum height at which the crests are found to be very sharp. For a wave profile of constant shape moving at a uniform velocity, it can be shown that the maximum total crest angle as the wave begins to break is 120°; see Figure 2.3.

The upshot of the Stokes' analysis reveals that the inclusion of higher-order terms in the representation of the surface wave profile distorts its shape away from the linear sinusoidal curve. The effects of nonlinearity are likely to make crests narrower (sharper) and the troughs flatter as depicted in Figure 2.7 of Debnath (1994). The resulting wave profile more accurately portrays the water waves that are observed in nature. Finally, the sharp crest angle of 120° was first found by Stokes.

On the other hand, in 1865, Rankine conjectured that there exists a wave of extreme height. In a moving reference frame, the Euler equations are Galilean invariant, and the Bernoulli equation (2.7.14) on the free surface of water with $\rho = 1$ becomes

$$\frac{1}{2}|\nabla\phi|^2 + gz = E.$$

Thus, this equation represents the conservation of local energy, where the first term is the kinetic energy of the fluid and the second term is the potential energy due to gravity. For the wave of maximum height, $E = gz_{\max}$, where $z_{\max}$ is the maximum height of the fluid. Thus, the velocity is zero at the maximum height so that there will be a stagnation point in the fluid flow. Rankine conjectured that a cusp is developed at the peak of the free surface with a vertical slope so that the angle subtended at the peak is 120° as also conjectured by Stokes (1847). Toland (1978) and Amick et al. (1982) have proved rigorously the existence of a wave of greatest height and the Stokes' conjecture for the wave of extreme form. However, Toland (1978) also proved that if the singularity at the peak is *not* a cusp, that is, if there is no vertical slope at the peak of the free surface, then the Stokes' remarkable conjecture of the crest angle of 120° is true. Subsequently, Amick et al. (1982) confirmed that the singularity at the peak is *not* a cusp. Therefore, the full Euler equations exhibit singularities, and there is a limiting amplitude to the periodic waves.

We next formulate the modern mathematical theory of nonlinear water waves. It is convenient to take the free surface elevation above the undisturbed mean depth h as $z = \eta(x, y, t)$ so that the free surface of water is at $z = H = h + \eta$ and the horizontal rigid bottom is at $z = 0$ where the z-axis is vertical positive upwards.

It is also convenient to introduce nondimensional flow variables based on a typical horizontal length scale ℓ (which may be wavelength λ), typical vertical length scale h, typical horizontal velocity scale, $c = \sqrt{gh}$ (shallow water wave speed), typical time scale $(\frac{c}{\ell})$, typical vertical velocity scale $(\frac{hc}{\ell})$, and the typical pressure scale ρc^2. Using asterisks to denote nondimensional flow variables, we write

$$(x,y) = \ell\big(x^*, y^*\big), \qquad z = hz^*, \qquad t = \left(\frac{\ell}{c}\right)t^*, \tag{2.7.56}$$

$$(u,v) = c\big(u^*, v^*\big), \qquad w = \left(\frac{hc}{\ell}\right)w^*, \quad \text{and} \quad p = \rho c^2 p^*. \tag{2.7.57}$$

We next introduce two fundamental parameters δ and ε defined by

$$\delta = \frac{h^2}{\ell^2} \quad \text{and} \quad \varepsilon = \frac{a}{h}, \tag{2.7.58}$$

where δ is called the *long wavelength* (or *shallowness*) *parameter*, ε is called the *amplitude parameter*, and a is the typical wave amplitude. These two parameters play a crucial role in the modern theory of water waves.

In terms of the amplitude parameter, the free surface at $z = H = h + \eta$ and the bottom boundary surface at $z = 0$ of the fluid can be written as the nondimensional form

$$z = 1 + \varepsilon\eta \quad \text{and} \quad z = 0, \tag{2.7.59}$$

where $(\frac{\eta}{a})$ is replaced by the nondimensional value η.

The variable pressure field P representing the deviation from the hydrostatic pressure $g\rho(h - z)$ is given by

$$p = p_a + g\rho(z - h) + g\rho h P, \tag{2.7.60}$$

where p_a is the constant atmospheric pressure and the scale $g\rho h$ of pressure is based on the pressure at the depth $z = h$.

In terms of the nondimensional variables, the Euler equations (2.7.4)–(2.7.6) and the continuity equation (2.7.2) can be written in the form, dropping the asterisks and replacing P by p,

$$\frac{Du}{Dt} = -\frac{\partial p}{\partial x}, \qquad \frac{Dv}{Dt} = -\frac{\partial p}{\partial y}, \qquad \delta\frac{Dw}{Dt} = -\frac{\partial p}{\partial z}, \tag{2.7.61}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2.7.62}$$

The kinematic free surface and the dynamic free surface conditions (see Debnath 1994, pp. 6–7) are expressed in the nondimensional form, dropping the asterisks,

$$w = \varepsilon(\eta_t + u\eta_x + v\eta_y) \quad \text{on } z = 1 + \varepsilon\eta, \tag{2.7.63}$$

$$p = \varepsilon\eta \quad \text{on } z = 1 + \varepsilon\eta. \tag{2.7.64}$$

The bottom boundary condition is

$$w = 0 \quad \text{on } z = 0. \tag{2.7.65}$$

It follows from the free surface conditions that both w and p are proportional to the amplitude parameter ε. In the limit as $\varepsilon \to 0$, both w and p tend to zero, indicating that there is no disturbance at the free surface.

Consistent with the governing equations and the boundary conditions, we introduce a set of scaled flow variables

$$(u, v, w, p) \to \varepsilon(u, v, w, p) \tag{2.7.66}$$

so that the governing equations (2.7.61) and (2.7.62) and the boundary conditions (2.7.63)–(2.7.65) become

$$\frac{Du}{Dt} = -\frac{\partial p}{\partial x}, \qquad \frac{Dv}{Dt} = -\frac{\partial p}{\partial y}, \qquad \delta\frac{Dw}{Dt} = -\frac{\partial p}{\partial z}, \tag{2.7.67}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.7.68}$$

$$w = \eta_t + \varepsilon(u\eta_x + v\eta_y), \quad p = \eta \quad \text{on } z = 1 + \varepsilon\eta, \tag{2.7.69}$$

$$w = 0 \quad \text{on } z = 0, \tag{2.7.70}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \varepsilon\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right), \tag{2.7.71}$$

and parameters δ and ε are given by (2.7.58).

In general, there are two most commonly adopted and useful approximations: (i) $\varepsilon \to 0$, that is, small amplitude water waves governed by the linearized equations, and (ii) $\delta \to 0$, that is, shallow water wave equations (or long water waves). These approximate models and their solutions constitute the classical theory of water waves (see Debnath 1994).

In the first case, the linearized equations of water waves are obtained from (2.7.67)–(2.7.71) in the limit as $\varepsilon \to 0$ in the form

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}, \qquad \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y}, \qquad \delta\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z}, \tag{2.7.72}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.7.73}$$

$$w = \eta_t \quad \text{and} \quad p = \eta \quad \text{on } z = 1, \tag{2.7.74}$$

$$w = 0 \quad \text{on } z = 0. \tag{2.7.75}$$

In the second case, the shallow water equations (long water waves) are described in the sense that $\sqrt{\delta} = (\frac{h}{\ell})$ is small so that $\delta \to 0$ with fixed amplitude parameter ε. Consequently, the governing equations and the boundary conditions are obtained from (2.7.67)–(2.7.71) in the limit as $\delta \to 0$ in the form

$$\frac{Du}{Dt} = -\frac{\partial p}{\partial x}, \qquad \frac{Dv}{Dt} = -\frac{\partial p}{\partial y}, \qquad \frac{\partial p}{\partial z} = 0, \tag{2.7.76}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{2.7.77}$$

$$w=\eta_t+\varepsilon(u\eta_x+v\eta_y) \quad \text{and} \quad p=\eta \quad \text{on } z=1+\varepsilon\eta, \tag{2.7.78}$$

$$w=0 \quad \text{on } z=0, \tag{2.7.79}$$

where $\frac{D}{Dt}$ is given by (2.7.71).

Finally, equations that describe small amplitude waves $\varepsilon \to 0$ and long waves $(\delta \to 0)$ are obviously consistent with (2.7.72)–(2.7.75) for the first case, and also with equations (2.7.76)–(2.7.79) with

$$\frac{\partial p}{\partial z}=0; \quad p=\eta \quad \text{on } z=1 \tag{2.7.80}$$

or (2.7.76)–(2.7.79) with $\varepsilon \to 0$.

The solutions of these various approximate governing equations describe the classical water waves (see Debnath 1994).

Example 2.7.3 (Solution of a Linearized Water Wave Problem). We consider the propagation of a plane harmonic water wave in the x-direction in a fluid of constant depth. With no y-dependence, the governing equations and the boundary conditions are obtained from (2.7.72)–(2.7.75) in the form

$$u_t=-p_x, \qquad \delta w_t=-p_z,\ u_x+w_z=0, \tag{2.7.81}$$

$$w=\eta_t \quad \text{and} \quad p=\eta \quad \text{on } z=1, \tag{2.7.82}$$

$$w=0 \quad \text{on } z=0. \tag{2.7.83}$$

We assume a plane wave solution in the form

$$(u,w,p)=\big[u^*(z),w^*(z),p^*(z)\big]\exp\big[i(\omega t-kx)\big]. \tag{2.7.84}$$

The free surface elevation is given by

$$\eta(x,t)=a\exp\big[i(\omega t-kx)\big]+c.c., \tag{2.7.85}$$

where a is a constant wave amplitude and $c.c.$ denotes the complex conjugate. Obviously, (2.7.85) represents a plane harmonic wave whose initial form at $t=0$ is given by $\eta(x,0)$.

Substituting the solution (2.7.84) into (2.7.81) gives, dropping asterisks,

$$u=\frac{k}{\omega}p, \qquad p'=-i\omega\delta w, \qquad w'=iku, \tag{2.7.86}$$

where the prime denotes the derivative with respect to z.

It readily follows from (2.7.86) that

$$w''=iku'=\frac{ik^2}{\omega}p'=\big(\delta k^2\big)w. \tag{2.7.87}$$

Thus, the general solution of (2.7.87) is

$$w = A\exp\big(\sqrt{\delta}kz\big) + B\exp\big(-\sqrt{\delta}kz\big), \tag{2.7.88}$$

where A and B are arbitrary constants to be determined from (2.7.82), (2.7.83) which give

$$w(1) = i\omega a, \qquad p(1) = a, \qquad w(0) = 0. \tag{2.7.89}$$

Consequently, the solution (2.7.88) assumes the final form

$$w(z) = \mathrm{Re}(i\omega a)\frac{\sinh(\sqrt{\delta}kz)}{\sinh(\sqrt{\delta}k)}. \tag{2.7.90}$$

It follows from the boundary conditions that

$$a = p(1) = \frac{\omega}{k}u(1) = \frac{\omega}{ik^2}w'(1) = \left(\frac{a\omega^2}{k}\right)\sqrt{\delta}\coth\big(k\sqrt{\delta}\big). \tag{2.7.91}$$

This leads to the dispersion relation

$$\omega^2 = \left(\frac{k}{\sqrt{\delta}}\right)\tanh\big(k\sqrt{\delta}\big), \tag{2.7.92}$$

where $\sqrt{\delta} = h\lambda^{-1}$ which is equal to dimensional (physical) quantity $(\frac{h}{\ell})$ and $k\sqrt{\delta}$ is equal to dimensional quantity kh.

As before, the dispersion relation determines the frequency $\omega = \omega(k)$ and the phase velocity

$$c_p^2 = \left(\frac{\omega}{k}\right)^2 = \big(k\sqrt{\delta}\big)^{-1}\tanh\big(k\sqrt{\delta}\big). \tag{2.7.93}$$

The group velocity c_g is given by

$$c_g = \frac{d\omega}{dk} = \frac{1}{2\omega\sqrt{\delta}}\big[\tanh\big(k\sqrt{\delta}\big) + k\sqrt{\delta}\sec h^2\big(k\sqrt{\delta}\big)\big]$$

which, by (2.7.93), is

$$= \frac{1}{2}c_p\left[1 + \frac{2k\sqrt{\delta}}{\sinh(2k\sqrt{\delta})}\right]. \tag{2.7.94}$$

In the case of shallow water waves, $k\sqrt{\delta} \to 0$ so that $\tanh(k\sqrt{\delta}) \approx k\sqrt{\delta}$. Hence, results (2.7.93), (2.7.94) lead to $c_p = c_g = 1$. Both the phase and group velocities are independent of the wavelength. So, the shallow water waves are nondispersive. In terms of the dimensional variables, the phase velocity is

$$c_p = \pm c = \pm\sqrt{gh}. \tag{2.7.95}$$

This confirms the choice of the velocity scale c adopted before.

For deep water waves, $k\sqrt{\delta} \to \infty$ so that $\tanh(k\sqrt{\delta}) \to 1$. Consequently,

$$\omega^2 = \frac{k}{\sqrt{\delta}}, \qquad c_p^2 = \big(k\sqrt{\delta}\big)^{-1}, \quad \text{and} \quad c_g = \frac{1}{2}c_p. \tag{2.7.96}$$

Example 2.7.4 (Small Amplitude Gravity-Capillary Surface Waves on Water of depth h). The governing equation for the two-dimensional linearized gravity-capillary surface waves on water of constant depth h with the free surface at $z = 0$ are given by

$$\nabla^2\phi = \phi_{xx} + \phi_{zz} = 0, \quad -h \le z < 0,\ t > 0, \tag{2.7.97}$$

where $\phi(x, z, t)$ is the velocity potential.

Representing the free surface elevation function by $\eta = \eta(x, t)$, the linearized kinematic and dynamic free surface conditions are

$$\eta_t - \phi_z = 0 \quad \text{on } z = 0,\ t > 0, \tag{2.7.98}$$

$$\phi_t + g\eta - \frac{T}{\rho}\eta_{xx} = 0 \quad \text{on } z = 0,\ t > 0, \tag{2.7.99}$$

where g is the acceleration of gravity and T is the surface tension, and ρ is the constant density of water.

The boundary condition at the horizontal rigid bottom at $z = -h$

$$\phi_z = 0 \quad \text{at } z = -h. \tag{2.7.100}$$

We seek the same solution (2.7.28) for $\phi(z, x, t)$ and (2.7.29) for $\eta(x, t)$ so that $\phi(x, z, t)$ assumes the same form (2.7.30). In view of (2.7.98), (2.7.99) reduces to the form

$$\phi_{tt} + g\phi_z - \frac{T}{\rho}\phi_{zxx} = 0 \quad \text{on } z = 0,\ t > 0. \tag{2.7.101}$$

Substituting (2.7.30) into (2.7.101) gives the *dispersion relation for the gravity-capillary waves* in the form

$$\omega^2 = gk\left(1 + \frac{Tk^2}{\rho g}\right)\tanh kh. \tag{2.7.102}$$

Or equivalently, this gives the phase velocity c_p of the surface gravity-capillary waves on water of finite depth h

$$c_p^2 = \frac{\omega^2}{k^2} = \left(\frac{g}{k} + \frac{T}{\rho}k\right)\tanh kh. \tag{2.7.103}$$

It can easily be recognized that result (2.7.102) or (2.7.103) is *exactly the same* as result (2.7.31) or (2.7.32) with g replaced by $(g + \rho^{-1}Tk^2)$. This means that all the properties of gravity-capillary waves can be described correctly when this replacement is made in the results of pure gravity waves.

Introducing the *Froude number* F and the *Bond number* τ by

$$F = \frac{c_p}{\sqrt{gh}} \quad \text{and} \quad \tau = \frac{T}{g\rho h^2}, \tag{2.7.104}$$

the dimensionless form of the dispersion relation (2.7.103) is

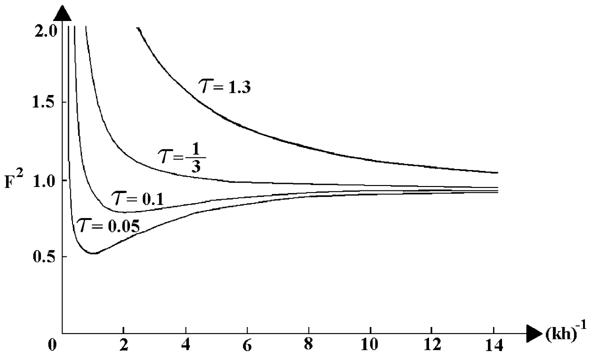

Fig. 2.4 The square of the Froude number, F^2, against $(kh)^{-1}$.

$$F^2 = \left(\frac{1}{kh} + \tau kh\right) \tanh kh. \tag{2.7.105}$$

The square of the Froude number is plotted against $(kh)^{-1} = (\frac{\lambda}{2\pi h})$ for four values of the Bond number $\tau = 1.3,\ \frac{1}{3},\ 0.1$, and 0.05 in Figure 2.4. This figure shows that F^2 decreases monotonically with $(kh)^{-1}$ when $\tau > \frac{1}{3}$, and it has a minimum for $\tau < \frac{1}{3}$. As $kh = 2\pi(\frac{h}{\lambda}) \to 0$ (or $\frac{\lambda}{h} \to \infty$), $F \to 1$.

In case of water of infinite depth ($kh \to \infty, \tanh kh \to 1$), (2.7.102) or (2.7.103) leads to

$$\omega^2 = gk\left(1 + \frac{Tk^2}{g\rho}\right) \quad \text{or} \quad c_p^2 = \left(\frac{g}{k} + \frac{Tk}{\rho}\right). \tag{2.7.106}$$

Thus, for pure surface gravity waves ($T = 0,\ g \neq 0$) in deep water, (2.7.106) reduces to (2.7.38). Similarly, for pure surface capillary waves ($g = 0,\ T \neq 0$), the dispersion relation is

$$\omega^2 = \frac{Tk^3}{\rho} \quad \text{or} \quad c_p^2 = \frac{Tk}{\rho}. \tag{2.7.107}$$

It is convenient to write (2.7.106) as

$$\omega^2 = gk(1 + T^*) \quad \text{or} \quad c_p^2 = \frac{g}{k}(1 + T^*), \tag{2.7.108}$$

where the parameter $T^* = (Tk^2/g\rho)$ represents the relative importance of surface tension and gravity.

It also follows from (2.7.106) that the phase velocity c_p has a minimum value at $k = k_m = \sqrt{g\rho/T}$ (or $T^* = 1$) with the corresponding minimum value for c_p is

$$(c_p)_m = \left(\frac{4Tg}{\rho}\right)^{\frac{1}{4}} \tag{2.7.109}$$

at the wavelength $\lambda = \lambda_m = 2\pi(T/g\rho)^{\frac{1}{2}}$.

The inequality $k \ll k_m$ is the condition for the waves to be effectively pure gravity waves with negligible surface tension. This is equivalent to large wavelength

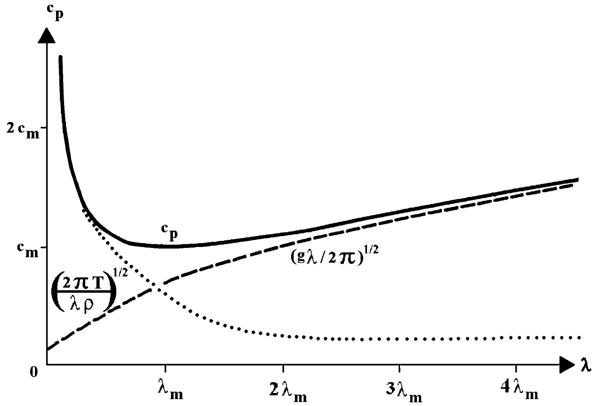

Fig. 2.5 The solid curve represents the phase velocity c_p for capillary-gravity waves against λ. (From Lighthill 1978.)

$\lambda > \lambda_m = (\frac{2\pi}{k_m}) = 2\pi(\frac{T}{\rho g})^{\frac{1}{2}}$. The phase velocity c_p in (2.7.106) for gravity-capillary waves in deep water is shown by the solid curve in Figure 2.5 against λ with minimum $(c_p)_m$ attained at $\lambda = \lambda_m$. The dotted curve corresponds to (2.7.107) for pure capillary waves (or *ripples*) in deep water dominated by surface tension for small $\lambda < \lambda_m$. The dashed curve corresponds to $c_p = (g/k)^{\frac{1}{2}}$ for pure gravity waves for large wavelengths $\lambda > \lambda_m$.

The group velocity for gravity-capillary waves can be calculated from the dispersion relation (2.7.102) and is given by

$$c_g = \frac{g}{2\omega}\left[\left(1+\frac{3Tk^2}{g\rho}\right)\tanh kh + \left(1+\frac{Tk^2}{g\rho}\right)kh\,\mathrm{sech}^2\,kh\right] \tag{2.7.110}$$

$$= \frac{g}{2\omega}\big[(1+3T^*)\tanh kh + (1+T^*)kh\,\mathrm{sech}^2\,kh\big]. \tag{2.7.111}$$

We next multiply the numerator by kc_p and the denominator by $\omega(= kc_p)$, then replace ω^2 by (2.7.106) to obtain

$$c_g = \frac{1}{2}c_p\left(\frac{1+3T^*}{1+T^*} + \frac{2kh}{\sinh 2kh}\right). \tag{2.7.112}$$

In the deep water limit, $kh \to \infty$, the second term in (2.7.112) tends to zero, and hence, the group velocity of gravity-capillary waves is

$$c_g = \frac{1}{2}c_p\left(\frac{1+3T^*}{1+T^*}\right). \tag{2.7.113}$$

This reduces to the result (2.7.96) for pure gravity waves $(T^* = 0)$ in deep water

$$c_g = \frac{1}{2}c_p = \frac{1}{2}\sqrt{\frac{g}{k}}, \tag{2.7.114}$$

and for pure capillary waves in deep water ($g \to 0$ and $T^* \to \infty$), the group velocity is

$$c_g = \frac{3}{2} c_p = \frac{3}{2}\left(\frac{Tk}{\rho}\right)^{\frac{1}{2}}. \tag{2.7.115}$$

It follows from the definition of group velocity (2.7.33) that the group and phase velocities are related by a simple formula

$$c_g = \frac{d\omega}{dk} = \frac{d}{dk}(kc_p) = c_p + k\frac{dc_p}{dk} = c_p - \lambda \frac{dc_p}{d\lambda}. \tag{2.7.116}$$

It is clear from (2.7.116) that $c_g \neq c_p$. However, if c_p does not depend on the wavelength, λ (or wavenumber, k), then $c_g = c_p$. If $\frac{dc_p}{dk} > 0$, then $c_g > c_p$, and if $\frac{dc_p}{dk} < 0$, then $c_g < c_p$. If c_p is minimum for some k, then $\frac{dc_p}{dk} = 0$, and hence, $c_g = c_p$. For shallow water waves, $\omega^2 = (gh)k^2$ or $c_p = \sqrt{gh}$, and then $c_g = c_p$.

In case of gravity-capillary waves in deep water, c_p has a minimum for $k = k_{\min} = \sqrt{g\rho/T}$, and hence, $c_g = (c_p)_m$. Figure 2.5 reveals that on the left of the minimum, $\frac{dc_p}{dk} > 0$, and hence, result (2.7.116) confirms that $c_g > c_p$, whereas on the right of the minimum, $\frac{dc_p}{dk} < 0$, and hence, $c_g < c_p$.

Finally, formula (2.7.116) can also be written in the form

$$c_g = c_p\left(1 - k\frac{dc_p}{d\omega}\right)^{-1} = c_p\left(1 - \frac{\omega}{c_p}\frac{dc_p}{d\omega}\right)^{-1}. \tag{2.7.117}$$

This is known as the *Rayleigh formula* for one-dimensional dispersive waves. The general theory of dispersive waves was developed by Whitham in 1960s that will be discussed in Chapter 7.

Example 2.7.5 (Total Energy of Pure Gravity Waves on Water of Constant Depth). We calculate the potential energy and the kinetic energy of pure gravity waves on water constant depth h. The potential energy over a single wavelength λ is given by

$$V = \frac{g\rho}{2}\int_0^\lambda \eta^2\, dx = \frac{1}{4} g\rho a^2, \tag{2.7.118}$$

where the free surface elevation η given by (2.7.37) is used to obtain the above value V.

The kinetic energy T is given by

$$T = \frac{1}{2}\rho \int_0^\lambda dx \int_{-h}^{n} \left(\phi_x^2 + \phi_z^2\right) dz$$

which can be transformed into

$$T = -\frac{1}{2}\rho \int \phi \frac{\partial \phi}{\partial n}\, dS = \frac{1}{2}\rho \int_0^\lambda \left(\phi\frac{\partial \phi}{\partial z}\right)_{z=0} dx$$

which is, by using (2.7.30),

$$= \frac{1}{2}\rho g a^2 \int_0^\lambda \sin^2(kx - \omega t)\, dx = \frac{1}{4} g\rho a^2 \lambda. \tag{2.7.119}$$

Hence, the total energy per unit wavelength is

$$E = T + V = \frac{1}{2} g\rho a^2 \lambda. \tag{2.7.120}$$

Thus, the total energy is half kinetic and half potential.

The horizontal and vertical velocity components of water particles are

$$u = \phi_x = \left(\frac{agk}{\omega}\right) \frac{\cosh k(z+h)}{\cosh kh} \exp\bigl[i(\omega t - kx)\bigr], \tag{2.7.121}$$

$$v = \phi_z = i\left(\frac{agk}{\omega}\right) \frac{\sinh k(z+h)}{\cosh kh} \exp\bigl[i(\omega t - kx)\bigr]. \tag{2.7.122}$$

So, it is possible to determine the actual path of a fluid particle in motion from (2.7.121)–(2.7.122). In terms of particle displacements X and Z of a particle whose mean motion is (x, z), we get $\dot{X} = u$ and $\dot{Z} = v$ in which terms of the second order are neglected. So integration gives

$$X = \left(\frac{agk}{\omega^2}\right) \frac{\cosh k(z+h)}{\cosh kh} \sin(\omega t - kx) + X_0, \tag{2.7.123}$$

$$Z = \left(\frac{agk}{\omega^2}\right) \frac{\sinh k(z+h)}{\cosh kh} \cos(\omega t - kx) + Z_0, \tag{2.7.124}$$

where X_0 and Z_0 are constants of integration, and they move the origin of X and Z. Eliminating $(\omega t - kx)$ from (2.7.123)–(2.7.124) gives the equation of a particle path as

$$\frac{(X - X_0)^2}{\cosh^2 k(z+h)} + \frac{(Z - Z_0)^2}{\sinh^2 k(z+h)} = \frac{a^2}{\sinh^2 kh}. \tag{2.7.125}$$

This represents an ellipse with the semi-major axis in the x-direction of magnitude $a \operatorname{cosech} kh \cosh k(z+h)$ and with semi-minor axis in the z-direction of magnitude $a \operatorname{cosech} kh \cosh k(z+h)$. Both semi-axes decrease with depth. When $X_0 = Z_0 = 0$ and $z = -h$, $X \neq 0$, $Z = 0$, and particles oscillate along the bottom. But for a real liquid, viscosity would prevent such oscillations.

In deep water $(kh \to \infty)$, both $\cosh k(z+h)/\cosh kh$ and $\cosh k(z+h)/\sinh kh$ tend to $\exp(kz)$; hence, (2.7.123)–(2.7.124) give

$$X - X_0 = ae^{kz} \sin(\omega t - kx), \tag{2.7.126}$$

$$Z - Z_0 = ae^{kz} \cos(\omega t - kx). \tag{2.7.127}$$

These results show that the paths of the fluid particles are circles of radius ae^{kz}. Clearly, the radius decreases exponentially with increasing depth.

2.8 The Energy Equation and Energy Flux

In dealing with surface gravity waves, it is important and useful to derive an equation that describes the flow of energy in the fluid. Thus, an energy equation is obtained by taking the scalar product of the velocity vector $\mathbf{u}$ with the respective terms of the momentum equation (2.7.13), with $\nabla\phi$ replaced by $\mathbf{u}$ so that

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}u^2 + \frac{p}{\rho} + gz\right) = 0. \tag{2.8.1}$$

We take the scalar product of $\mathbf{u}$ with (2.8.1) and use $\mathbf{u}\cdot\mathbf{u} = u^2$ to obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) + \mathbf{u}\cdot\nabla\left(\frac{1}{2}u^2 + \frac{p}{\rho} + gz\right) = 0. \tag{2.8.2}$$

Since $\operatorname{div}\mathbf{u} = 0$, we can add $(\frac{1}{2}u^2 + p/\rho + gz)\operatorname{div}\mathbf{u}$ to (2.8.2) and use $\partial z/\partial t = 0$ and the formula for $\operatorname{div}(a\mathbf{u})$ with any scalar a to derive, multiplying by ρ,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho g z\right) + \operatorname{div}\left[\mathbf{u}\left(\frac{1}{2}\rho u^2 + p + g\rho z\right)\right] = 0. \tag{2.8.3}$$

The terms $\frac{1}{2}\rho u^2$ and $\rho g z$ represent the kinetic and potential energies, respectively, and equation (2.8.3) describes a balance between the rate of change of energy and energy flux terms, including convection by the velocity and the rate of working of the pressure. In fact, the rate of change of energy per unit volume is described in terms of the divergence of the *energy flux* $\mathfrak{F}$

$$\mathfrak{F} = \mathbf{u}\left(\frac{1}{2}\rho u^2 + p + g\rho z\right). \tag{2.8.4}$$

Equation (2.8.3) gives, writing $E = \frac{1}{2}\rho u^2 + \rho g z$,

$$\frac{\partial E}{\partial t} + \operatorname{div}\mathfrak{F} = 0. \tag{2.8.5}$$

This is usually called the *law of conservation of energy*.

In order to see some physical meaning of (2.8.3), we integrate it over some volume V enclosed by a closed surface S. By using the Gauss divergence theorem, we can transform the volume integral over V to a surface integral over S. Consequently,

$$\begin{aligned}\frac{\partial}{\partial t}\int_V \rho\left(\frac{1}{2}u^2 + gz\right)dV &= -\int_V \operatorname{div}\left[\rho\mathbf{u}\left(\frac{1}{2}u^2 + \frac{p}{\rho} + gz\right)\right]dV \\ &= -\int_S \left(\frac{1}{2}\rho u^2 + p + \rho g z\right)\mathbf{u}\cdot\mathbf{n}\,dS. \end{aligned}\tag{2.8.6}$$

This represents the rate of change of the total energy in a volume V that is equal to the amount of energy flowing out of this volume across the surface S per unit time.

For this reason $\mathfrak{F}$ is called the *energy flux density vector*. Its magnitude represents the amount of energy passing across a unit surface area normal to the velocity field $\mathbf{u}$ per unit time. We may rewrite the right-hand side of (2.8.6) as

$$-\int_S \mathbf{u}\left(\frac{1}{2}\rho u^2\right)\cdot\mathbf{n}\,dS-\int_S p\mathbf{u}\cdot\mathbf{n}\,dS-\int_S \rho g z\mathbf{u}\cdot\mathbf{n}\,dS. \tag{2.8.7}$$

The first term is the kinetic energy transported across S per unit time by the fluid; the second term is the work done by the pressure forces on the fluid within the surface, and the third term is the work done by the gravitational force acting on the system.

2.9 Exercises

1. Use the Hamilton principle to derive
   (i) the Newton second law of motion, and
   (ii) the equation for a simple harmonic oscillator.
2. Derive the equation of motion of an elastic beam of length ℓ, line density ρ, cross-sectional moment of inertia I, and modulus of elasticity E which is fixed at each end and performs small transverse oscillations in the horizontal (x,t)-plane.
3. Apply the variational principle

   $$\delta\iint L\,dx\,dt=0,$$

   where the Lagrangian $L=\frac{1}{2}(u_{xx}^2+u_t u_x)+u_x^3$, to derive the equation

   $$u_{xt}+6u_x u_{xx}+u_{xxxx}=0.$$

   Show that this equation leads to the KdV equation when $\eta=u_x$.
4. Use the variational principle

   $$\delta\iint\left(1-u_t^2+u_x^2\right)^{\frac{1}{2}}\,dx\,dt=0$$

   to derive the Born and Infeld (1934) equation

   $$\left(1-u_t^2\right)u_{xx}+2u_x u_t u_{xt}-\left(1+u_x^2\right)u_{tt}=0.$$

5. Show that the variational principle (Whitham 1967a, 1967b)

   $$\delta\iint\left\{\frac{1}{2}\psi_x\psi_t+\frac{1}{2}c_0\psi_x^2+\frac{1}{6}c_0\psi_x^3+\frac{1}{12}c_0h_0^2\left(\chi^2+2\chi_x\psi_x\right)\right\}dx\,dt=0$$

   gives the coupled equations

   $$\psi_{xt}+c_0(1+\psi_x)\psi_{xx}+\frac{1}{6}c_0h_0^2\chi_{xx}=0,\qquad \psi_{xx}-\chi=0,$$

   where c_0 and h_0 are constants.

6. Show that the variational principle (Whitham 1967a, 1967b)

$$\delta \iiint \left\{ \phi_t + \alpha\beta_t + \frac{1}{2}\left(u^2 + v^2\right) \right\} dx\, dy\, dt = 0$$

leads to the equations

$$u_x + v_y = 0, \qquad \frac{D\alpha}{Dt} + fv = 0, \qquad \frac{D\beta}{Dt} - u = 0,$$

where $u = \phi_x + \alpha\beta_x - \alpha$, $v = \phi_y + \alpha\beta_y - f\beta$, and

$$-p = \phi_t + \alpha\beta_t + \frac{1}{2}\left(u^2 + v^2\right).$$

7. If $\mathcal{L} = \mathcal{L}(\omega, k)$ where $\omega = -\theta_t$ and $k = \theta_x$, show that the variational principle (Whitham 1965a, 1965b; Lighthill 1967)

$$\delta \iint \mathcal{L}(\omega, k)\, dt\, dx = 0$$

gives the Euler–Lagrange equation

$$\frac{\partial}{\partial t}(\mathcal{L}_\omega) = \frac{\partial}{\partial x}(\mathcal{L}_k).$$

Show also that this equation reduces to a second-order quasi-linear partial differential equation for $\theta(x, t)$

$$\mathcal{L}_{\omega\omega}\theta_{tt} - 2\mathcal{L}_{\omega k}\theta_{xt} + \mathcal{L}_{kk}\theta_{xx} = 0.$$

8. Derive the Boussinesq equation for water waves

$$u_{tt} - c^2 u_{xx} - \mu u_{xxtt} = \frac{1}{2}\left(u^2\right)_{xx}$$

from the variational principle

$$\delta \iint L\, dx\, dt = 0,$$

where $L \equiv \frac{1}{2}\phi_t^2 - \frac{1}{2}c^2\phi_x^2 + \frac{1}{2}\mu\phi_{xt}^2 - \frac{1}{6}\phi_x^3$ and ϕ is the potential for u where $u = \phi_x$.

9. Show that the Euler equation of the variational principle

$$\delta I\left[u(x, y)\right] = \delta \iint_D F(x, y, u, p, q, l, m, n)\, dx\, dy = 0$$

is

$$F_u - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q + \frac{\partial^2}{\partial x^2}F_l + \frac{\partial^2}{\partial x \partial y}F_m + \frac{\partial^2}{\partial y^2}F_n = 0,$$

where

$$p = u_x, \qquad q = u_y, \qquad l = u_{xx}, \qquad m = u_{xy}, \quad \text{and} \quad n = u_{yy}.$$

10. In each of the following cases, apply the variational principle or its simple extension with appropriate boundary conditions to derive the corresponding equations:
    (a) $F = u_{xx}^2 + u_{yy}^2 + u_{xy}^2$,
    (b) $F = \frac{1}{2}[u_t^2 - \alpha^2(u_x^2 + u_y^2) - \beta^2 u^2]$,
    (c) $F = \frac{1}{2}(u_t u_x + \alpha u_x^2 + \beta u_{xx}^2)$,
    (d) $F = \frac{1}{2}(u_t^2 + \alpha^2 u_{xx}^2)$,
    (e) $F = p(x)u'^2 + \frac{d}{dx}(q(x)u^2) - [r(x) + \lambda s(x)]u^2$,

    where p, q, r, and s are given functions of x, and α and β are constants.
11. Derive the Schrödinger equation from the variational principle

$$\delta \iiint_D \left[\frac{\hbar^2}{2m} \left(\psi_x^2 + \psi_y^2 + \psi_z^2 \right) + (V - E)\psi^2 \right] dx\, dy\, dz = 0,$$

    where $h = 2\pi\hbar$ is the Planck constant, m is the mass of a particle moving under the action of a force field described by the potential $V(x, y, z)$, and E is the total energy of the particle.
12. Derive the Poisson equation $\nabla^2 u = F(x, y)$ from the variational principle with the functional

$$I(u) = \iint_D \left[u_x^2 + u_y^2 + 2uF(x, y) \right] dx\, dy,$$

    where $u = u(x, y)$ is given on the boundary ∂D of D.
13. Prove that the Euler–Lagrange equation for the functional

$$I = \iiint_D f(x, y, z, u, p, q, r, l, m, n, a, b, c)\, dx\, dy\, dz$$

    is

$$F_u - \frac{\partial}{\partial x} F_p - \frac{\partial}{\partial y} F_q - \frac{\partial}{\partial z} F_r + \frac{\partial^2}{\partial x^2} F_l + \frac{\partial^2}{\partial y^2} F_m + \frac{\partial^2}{\partial z^2} F_n$$
$$+ \frac{\partial^2}{\partial x \partial y} F_a + \frac{\partial^2}{\partial y \partial z} F_b + \frac{\partial^2}{\partial z \partial x} F_c = 0,$$

    where $(p, q, r) = (u_x, u_y, u_z)$, $(l, m, n) = (u_{xx}, u_{yy}, u_{zz})$, and $(a, b, c) = (u_{xy}, u_{yz}, u_{zx})$.
14. Derive the equation of motion of a vibrating string of length l under the action of an external force $F(x, t)$ from the variational principle

$$\delta \int_{t_1}^{t_2} \int_0^l \left[\left(\frac{1}{2} \rho u_t^2 - T^* u_x^2 \right) + \rho u F(x, t) \right] dx\, dt = 0,$$

    where ρ is the line density and $T^*$is the constant tension of the string.

15. The kinetic and potential energies associated with the transverse vibration of a thin elastic plate of constant thickness h are

$$T = \frac{1}{2}\rho \iint_D \dot{u}^2\,dx\,dy,$$
$$V = \frac{1}{2}\mu_0 \iint_D \big[(\nabla u)^2 - 2(1-\sigma)\big(u_{xx}u_{yy} - u_{xy}^2\big)\big]\,dx\,dy,$$

where ρ is the surface density and $\mu_0 = 2h^3E/3(1-\sigma^2)$.
Use the variational principle

$$\delta \int_{t_1}^{t_2} \iint_D \big[(T-V) + fu\big]\,dx\,dy\,dt = 0$$

to derive the equation of motion of the plate

$$\rho\ddot{u} + \mu_0 \nabla^4 u = f(x,y,t),$$

where f is the transverse force per unit area acting on the plate.

16. The kinetic and potential energies associated with wave motion in elastic solids are

$$T = \frac{1}{2}\iiint_D \rho\big(u_t^2 + v_t^2 + w_t^2\big)\,dx\,dy\,dz,$$
$$V = \frac{1}{2}\iiint_D \big[\lambda(u_x + v_y + w_z)^2 + 2\mu\big(u_x^2 + v_y^2 + w_z^2\big)$$
$$+ \mu\big\{(v_x + u_y)^2 + (w_y + v_z)^2 + (u_z + w_x)^2\big\}\big]\,dx\,dy\,dz.$$

Use the variational principle

$$\delta \int_{t_1}^{t_2} \iiint_D (T-V)\,dx\,dy\,dz\,dt = 0$$

to derive the equation of wave motion in an elastic medium

$$(\lambda+\mu)\,\mathrm{grad\,div}\,\mathbf{u} + \mu\nabla^2\mathbf{u} = \rho\mathbf{u}_{tt},$$

where $\mathbf{u} = (u,v,w)$ is the displacement vector.

17. Apply the variational principle (2.5.2) with the Lagrangian $L = \frac{1}{2}(u_t u_x - u_{xx}^2 - 2u_x^3)$ to derive $u_{xt} - 6u_x u_{xx} + u_{xxxx} = 0$. Show that this equation leads to the KdV equation when $u_x = \eta$.

18. An inviscid and incompressible fluid flow under the conservative force field $\mathbf{F} = -\nabla\Omega$ with a potential $\Omega = gz$, where g is the constant acceleration due to gravity, is governed by the Euler equation (2.7.1).
    (a) Show that
$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{p}{\rho} + \Omega\right) - \mathbf{u}\times\boldsymbol{\omega} = 0,$$
    where $\nabla\times\mathbf{u} = \boldsymbol{\omega}$ is the vorticity vector.

(b) Taking the scalar product of the above equation with $\mathbf{u}$, derive the result

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)+(\mathbf{u}\cdot\nabla)\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}+\frac{p}{\rho}+\Omega\right)=0.$$

(c) Derive the energy equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+\rho\Omega\right)+\nabla\cdot\left[\mathbf{u}\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+p+\rho\Omega\right)\right]=0.$$

Explain the significance of each term of the energy equation.

19. The three-dimensional Plateau problem is governed by the functional

$$I\big[u(x,y,z)\big]=\iiint_D\left(1+p^2+q^2+r^2\right)^{\frac{1}{2}}\,dx\,dy\,dz$$

where $p=u_x$, $q=u_y$, and $r=u_z$.
Find the Euler–Lagrange equation of this functional.

20. (a) Show that the Euler–Lagrange equation for the functional

$$I(\mathbf{u})=\int_a^b F(x,\mathbf{u},\mathbf{u}')\,dx,$$

where $u=(u_1,u_2,\ldots,u_n)$, $u_i\in C^2[a,b]$, $u_i(a)=a_i$, and $u_i(b)=b_i$, $i=1,2,\ldots,n$, is a system of n ordinary differential equations

$$F_{u_i}-\frac{d}{dx}F_{u_i'}=0,\quad i=1,2,\ldots,n.$$

(b) If F in (a) does not depend explicitly on x, then show that

$$F-\sum_{i=1}^{n}u_i'F_{u_i'}=\text{const.}$$

21. Consider a simple pendulum of length ℓ with a bob of mass m suspended from a frictionless support. Apply the Hamilton principle to the functional

$$I[\theta(t)]=\int_{t_1}^{t_2}(T-V)\,dt,$$

where $T=\frac{1}{2}m\ell^2\dot{\theta}^2$ and $V=mg(\ell-\ell\cos\theta)$ to derive the equation of the simple pendulum

$$\ddot{\theta}+\omega^2\sin\theta=0,\quad \omega^2=\frac{g}{\ell}.$$

22. Derive the Euler–Lagrange equation for the functional

$$I(y(x))=\int_a^b F(x,y,y')\,dx,$$

where

(a) $F(x,y,y') = u(x,y)\sqrt{1+y'^2}$,
(b) $F(x,y,y') = \frac{1}{\sqrt{2g}}(\frac{1+y'^2}{y_1-y})^{\frac{1}{2}}$ with $y(a) = y_1$, $y(b) = y_2 < y_1$ (Brachistochrone problem).

23. The Fermat principle in optics states that light travels from one point $A(x_1, y_1)$ to another point $B(x_2, y_2)$ in an optically homogeneous medium along a path in a minimum (least) time. The time taken for the light beam to travel from A to B is

$$I\big(y(x)\big) = \int_{t_1}^{t_2} dt = \int_{x_1}^{x_2} \left(\frac{dt}{ds}\right)\left(\frac{ds}{dx}\right) dx = \frac{1}{c}\int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx,$$

where $c = \frac{ds}{dt}$ is the constant velocity of light.
Apply the variational principle

$$\delta I = \frac{1}{c}\delta \int \sqrt{1+y'^2}\, dx = 0$$

to derive the *Snell law of refraction of light*, $n\sin\phi =$ const., where $n = \frac{1}{c}$ is the refractive index of the medium and ϕ is the angle made by the tangent to the minimum path with the vertical y-axis.

24. (a) Derive the *principle of least action* for a conservative system

$$\delta \int_{t_1}^{t_2} 2T\, dt = 0,$$

where the time integral of $2T$ is called the *action* of the system.
(b) Explain the significance of this principle.

25. Show that the Euler–Lagrange equation of the variational principle

$$\delta I = \delta \int_a^b F\big(x, y, y', y'', \ldots, y^{(n)}\big)\, dx = 0$$

is an ordinary differential equation of order $2n$

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} - \cdots + (-1)^n \frac{d^n}{dx^n}F_{y^{(n)}} = 0.$$

26. The electrostatic potential $\phi(x,y,z)$ is defined in terms of the electrostatic field $\mathbf{E}$ so that $\mathbf{E} = -\nabla\phi$ in a domain D of volume V, where ϕ is specified on ∂D. Show that ϕ that minimizes the electric energy functional

$$I[\phi] = \frac{e_0}{2}\iiint_V E^2 dv = \frac{e_0}{2}\iiint_V (\nabla\phi)^2\, dv$$

satisfies the Laplace equation.

27. Use the Lagrangian $L = T - V$ and the Lagrange equation to derive the Newton laws of motion of a particle of mass m moving under a force, $F = -\nabla V$, in

(a) one dimension,
(b) a two dimensional plane in Cartesian coordinates,
(c) a two dimensional plane in polar coordinates.

28. Seek a traveling wave solution

$$r_n = A\cos\theta = A\cos(\omega t - kn)$$

of the linearized Toda lattice equation

$$m\ddot{r}_n = (ab)(r_{n+1} - 2r_n + r_{n-1}),$$

where $r_n = (y_{n+1} - y_n)$ and y_n is the displacement of the nth mass. Show that the dispersion relation is

$$\omega^2 = \left(\frac{4ab}{m}\right)\sin^2\left(\frac{k}{2}\right).$$

29. Consider the Ablowitz and Ladik (AL) equation for the lattice system (1976a, 1976b)

$$i\frac{d\phi_n}{dt} + (\phi_{n+1} + \phi_{n-1})\left(1 + \frac{\gamma}{2}|\phi_n|^2\right) = 0.$$

(a) Using $\phi_n = e^{2it}\psi_n$, show that the solution of the AL equation reduces to that of the NLS equation

$$i\psi_t + \psi_{xx} + \gamma|\psi|^2\psi = 0,$$

as the ratio of anharmoncity to dispersion (γ) tends to zero.
(b) Show that the solution of the above AL equation is

$$\phi_n(t) = Acn\big[\beta(n - vt); k\big]\exp\big[-i(\omega t + \alpha n + \phi_0)\big],$$

selecting the units of ϕ_n so that $\frac{\gamma}{2} = 1$, the parameters A, ω and v are given by

$$A = \frac{ksn(\beta;k)}{dn(\beta;k)}, \qquad \omega = -\frac{2cn(\beta;k)\cos\alpha}{dn^2(\beta;k)}, \qquad v = -\frac{2sn(\beta;k)\sin\alpha}{\beta\, dn(\beta;k)},$$

where $0 < \beta < \infty$, $-\pi \le \alpha \le \pi$, and $0 < k < 1$.

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