Consequences of the Littelmann Path Theory for the Structure of the Kashiwara $B(\infty)$ Crystal

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Abstract These lectures give a review of results on the Kashiwara $B(\infty)$ crystal defined for a Kac–Moody Lie algebra, which can be obtained using the Littelmann path model. In this we do not need to assume, as does Kashiwara, that the Cartan matrix is symmetrizable.

First of all $B(\infty)$ is defined and shown to be upper normal. The latter is a deep result with no known easy proof. Again with respect to each simple root index i, it is shown that $B(\infty)$ has a canonical decomposition in the form $B^i \otimes B_i$, where B_i is the i^{th} elementary crystal. This allows one to establish in this greater generality the existence of an involution on $B(\infty)$, which extends the Kashiwara involution from the symmetrizable case. It is a new result.

Following earlier works of the author, the meaning of the Kashiwara involution to tensor product decomposition is described and the existence of combinatorial Demazure flags for certain tensor products, exhibited.

A paper of Nakashima and Zelevinski is reviewed. This proves an additivity property of $B(\infty)$ under a positivity hypothesis which they established in a few cases. It is noted that this positivity hypothesis immediately implies the upper normality of $B(\infty)$ and so is liable to be very difficult to establish in all generality.

Keywords Crystals · Tensor product · Demazure formula

Mathematics Subject Classification (2010) 17B37

Preamble Crystals arose from viewing the quantum parameter q in quantized enveloping algebras as the temperature in the belief that the $q \to 0$ limit used should result in a significant simplification.

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A. Joseph et al. (eds.), *Highlights in Lie Algebraic Methods*, Progress in Mathematics 295,
DOI 10.1007/978-0-8176-8274-3_2,

To define a $q \to 0$ limit mathematically one needs a lattice. This can be provided by a basis which in the present instance could be Lusztig's canonical basis. Here Kashiwara found a path to the resulting lattice without needing an explicit basis.

In some sense the $q \to 0$ limit strips a module of its linear structure, so we are reduced to combinatorics. Conversely linear structure is the "Pons Asinorum" of combinatorics—a bridge over which the donkey cannot pass, or as more poetically expressed by Robbie Burns—"a running stream they dare na cross". Actually the poet was referring to wizards and witches, his comment being a tip to anyone hotly pursued by a scantily clad young female in the depths of the night [3].

Now although wizards dare not cross the bridge to linear algebra, they can look across the running stream and try to imitate what is happening on the other side. This was the role of Littelmann [28–30], in constructing a purely combinatorial theory of crystals recovering many results previously obtained using the linear structure of representation theory.

The main theme of these lectures is a purely combinatorial analysis of Kashiwara's $B(\infty)$ crystal. The latter is supposed to represent the algebra of functions on the open Bruhat cell, thus one may already anticipate that it will admit several realizations corresponding to different Bott-Samelson desingularizations which must be shown to be equivalent. Again from the standpoint of global sections on invertible sheaves, $B(\infty)$ should provide realizations of crystals corresponding to integrable highest weight modules.

A fundamental fact is that $B(\infty)$ is a "highest weight" crystal. This was first proved by Kashiwara; but his argument was neither simple nor purely combinatorial. It used the quantized enveloping algebra and so required the Cartan matrix to be symmetrizable. The Littelmann path model does not need this condition. We already gave an exposition of this work [16], so will not repeat the details here, only draw the consequences.

The first consequence is the independence of $B(\infty)$ on presentation. A second is the recovery of the Littelmann closed family of normal highest weight crystals whereby such a family is shown to be unique. A third new consequence is a purely combinatorial construction of the Kashiwara involution which is also valid in the not necessarily symmetrizable case.

$B(\infty)$ has a particularly simple character suggesting it to possess the structure of an additive semigroup. We describe a result of Nakashima and Zelevinsky [33] which attempts to give this property and notably that $B(\infty)$ is highest weight. Unfortunately they need to impose a positivity hypothesis whose range of validity is unclear. Again this additive structure is hardly ever free. It would be interesting to have an algorithm to compute generators.

A further significant fact is that the crystal operators satisfy the Coxeter relations when restricted to $B(\infty)$. This leads naturally to "Demazure crystals". They are shown to satisfy a "string property" from which their characters can be computed via the Demazure algorithm. Certain of their tensor products admit a combinatorial version of a Demazure flag. The corresponding module theoretic fact has been established for semisimple Lie algebras by Mathieu [21] and in the simply-laced case [15]. A complete proof would require a better understanding of the possible matrix elements for simple root vectors acting on the canonical/global basis and this in turn requires a better understanding of $B(\infty)$, which parameterizes the latter.

1 Introductory Survey

1.1 A crystal is a very special graph built from a Cartan matrix A of a Kac–Moody Lie algebra $\mathfrak{g}_A$. Here one fixes a countable set I and takes A to have integer entries $a_{i,j} : i, j \in I$. For A to have reasonable properties one imposes that $I = I_{\text{re}} \sqcup I_{\text{im}}$ with the conditions

(1) $a_{i,i} = 2 : i \in I_{\text{re}}, a_{i,i} \in -\mathbb{N} : i \in I_{\text{im}}$,
(2) $a_{i,j} \in -\mathbb{N}$, if $i \neq j$,
(3) $a_{i,j} \neq 0 \Leftrightarrow a_{j,i} \neq 0$.

When $I_{\text{im}} \neq \emptyset$, we call this the Borcherds case.

The Cartan matrix A is said to be symmetrizable if there exist positive integers $d_i : i \in I$ such that $\{d_i a_{i,j}\}$ is a symmetric matrix. In this case the relations in $\mathfrak{g}_A$ are known. They are customarily referred to as the Serre relations. Also in this case one may find a quantization $U_q(\mathfrak{g}_A)$ of the enveloping algebra $U(\mathfrak{g}_A)$ of $\mathfrak{g}_A$. Below we shall often omit the A subscript.

1.2 Crystal theory arose partly from Lusztig's theory of canonical bases [31]. The latter have some striking properties. For example let $\delta M(0)$ denote the $\mathscr{O}$ dual of the Verma module of highest weight 0. (If $\dim \mathfrak{g}_A < \infty$, that is if $\mathfrak{g}_A$ is semisimple, one may identify $\delta M(0)$ with the algebra of regular functions on the open Bruhat cell.) Now fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and let P^+ denote the set of dominant weights. Let $V(\lambda)$ denote the simple $U(\mathfrak{g})$ module of highest weight λ and $\mathbb{C}_{-\lambda}$ the one dimensional $\mathfrak{b}$ module of weight $-\lambda$.

A relatively easy fact [7, 2.2] is that there exists a unique $U(\mathfrak{b})$ module embedding $\varphi_\lambda : V(\lambda)|_{U(\mathfrak{b})} \otimes \mathbb{C}_{-\lambda} \hookrightarrow \delta M(0)|_{U(\mathfrak{b})}$. A much deeper fact is that $\delta M(0)$ admits a basis (the dual canonical basis) such that $\operatorname{Im} \varphi_\lambda$ is spanned by a subbasis. Therefore in particular, the $\operatorname{Im} \varphi_\lambda : \lambda \in P^+$ form a distributive lattice of subspaces [11, 6.2.20]. From this one may prove both the Kostant and Richardson separation theorems and extend both to the quantum case [2], [11, 7.3.8].

A further important fact is that a Demazure module [23], which is the space of global functions on an ample invertible sheaf over a Schubert variety, admits a basis formed from a subset of the canonical basis. Moreover this subset is described by a "Demazure crystal" which has some interesting properties (3.4.9, 3.5.2).

1.3 Crystals may be regarded as a certain $q \to 0$ limit of simple highest weight modules in which the structure of latter "crystallize" to a simpler form as the "temperature" q goes to zero. In this the Kashiwara theory [22] is quite elementary though extremely complicated. Moreover by a process reversing the $q \to 0$ limit used, called globalization, Kashiwara constructed a global basis [22] for a simple highest weight module, which Grojnowski and Lusztig [4] showed coincided with the canonical basis of Lusztig.

1.4 The Kashiwara construction was further extended to the Borcherds case by Kashiwara in collaboration with Jeong and Kang [19]. This requires A to be symmetrizable (so that $U_q(\mathfrak{g}_A)$ can be constructed).

1.5 From the above Kashiwara formulated an abstract notion of a crystal [18, 23] which may be viewed as the "skeleton" of a simple highest weight module, where notably linear structure is eliminated (so allowing many other possible crystals). In addition, Kashiwara gave a tensor structure on the set of crystals. It is associative, but not commutative. The rules used to describe tensor structure appear rather ad hoc, though they do come from the tensor structure on direct sums of simple highest weight modules through the above $q \to 0$ limit.

1.6 Littelmann [28] discovered a purely abstract construction of certain crystals in which each element is a piecewise linear path. The Kashiwara operators resulting from the simple root vectors which are generators of $U_q(\mathfrak{g}_A)$, become operations on paths. Notably concatenation of paths gives tensor product structure and remarkably the Kashiwara rules are obtained, moreover in a natural fashion.

1.7 The Littelmann theory does not require A to be symmetrizable. A key point is how to choose families of paths giving crystals corresponding to the simple integrable highest weight $U(\mathfrak{g}_A)$ modules $V(\lambda) : \lambda \in P^+$. Here Littelmann was motivated by Lakshmibai-Seshadri theory of standard monomial bases for such modules which these authors had constructed in type A and several other cases. Littelmann calls the resulting paths *LS* paths. A major success of the Littelmann path theory [29] was to obtain via Lusztig's quantum Frobenius map (which requires A to be symmetrizable) a construction of standard monomial bases (which are not unique) for simple highest weight module for $\mathfrak{g}_A$ given A symmetrizable (and not of Borcherds type).

1.8 Recently Lamprou and myself [17] extended the Littelmann path model to the Borcherds case (again without the assumption that A is symmetrizable). That this works is rather surprising because in this extension there is now less connection with the ideas which led to *LS* paths.

1.9 One now has two parallel theories of crystals modeled on the family $V(\lambda) : \lambda \in P^+$, namely that of Kashiwara and that of Littelmann. It turns out that there are two ways of proving [14, Chaps. 8, 10] an isomorphism between these two sets of crystals; but only one works in the Borcherds case. This proof involves a family $\mathbb{F}_A = \{B(\lambda) : \lambda \in P^+\}$ of so-called normal highest weight crystals closed under tensor product. Once diagonal entries of A have been fixed, the existence of such a family requires A to satisfy (2) and (3) of 1.1.

In Sect. 2.5 we prove uniqueness of the family $\mathbb{F}_A$. This involves an "abstract" version of the Kashiwara $B(\infty)$ crystal defined without the condition that A be symmetrizable.

1.10 At present there are two methods to construct $\mathbb{F}_A$. The first (valid only for A symmetrizable) is due to Kashiwara taking a $q \to 0$ limit. The second due to Littelmann is via his path model. Both are elementary but rather complicated. Here we point out (2.5.9) that to construct $\mathbb{F}_A$ we only need to know that the abstract $B(\infty)$ crystal is upper normal. However we cannot say just from this that the family is closed. (No doubt this will be possible through a little extra work. For the moment it only follows by proving that it coincides with Littelmann's family and then applying the analysis of Littelmann, [28], [11, Sect. 6.4].)

Analyzing work of Nakashima and Zelevinsky [33], we found (Remark 3 of 3.6.4) that this upper normality results in a much easier fashion from their results. Unfortunately at present they must impose a positivity hypothesis which has not been verified in general. Assuming positivity, their work also shows that $B(\infty)$ admits (several) additive semigroup structures and it would be interesting to give a meaning to these structures, as well as to give an algorithm for finding generators (unfortunately free generators hardly ever exist).

1.11 We give (2.5.13–2.5.25) a purely combinatorial construction of the Kashiwara involution on $B(\infty)$. This extends this involution to the not necessarily symmetrizable case.

1.12 Following Littelmann we show that the crystal operators acting on $\mathbb{F}_A$ or on $B(\infty)$ give rise to a singular Hecke algebra. (We suggest that this also holds for the Nakashima–Zelevinsky operators—3.6.6). This Hecke algebra leads naturally to "Demazure crystals" which model the Demazure modules discussed in 1.2. Demazure had introduced what we call a string property and he had mistakenly suggested that this holds for Demazure modules [5, 6]. From this he found a family of operators on the group algebra of the weight lattice which also generate a singular Hecke algebra. Here following Kashiwara we show (Theorem 3.4.6) that the "Demazure crystals" admit the string property and that thereby their characters are given through the Demazure operators. For the Demazure modules themselves, the Demazure character formula was first proved for most characteristics and for $\mathfrak{g}_A$ semisimple by Andersen [1] using the Steinberg module. It was later proved by Kumar [24] and Mathieu [32] in the arbitrary Kac–Moody setting but in characteristic zero. Using globalization, Kashiwara obtained the Demazure character formula in all characteristics [23]; but assuming A symmetrizable. However, the problem of determining the more precise information encoded in the characters of their $\mathfrak{n}$ homology is still open [10, 25].

1.13 Certain tensor products of Demazure modules admit a filtration whose quotients are again Demazure modules (at least for $\mathfrak{g}_A$ semisimple—Mathieu–Polo [21], or if A is simply-laced [15]). Here we note (3.5) a combinatorial version of Demazure flags, though we do not give the details of the main theorem (3.5.3) as this has been amply explained in other notes of ours [16] available on the web.

1.14 We give an interpretation of the Kashiwara involution in terms of highest weight elements in the tensor product of crystals from $\mathbb{F}_A$, reviewing 3.2.1–3.2.4 earlier work of ours [13, Sect. 4]. At first this seemed a pure crystal phenomenon. However we note in 3.2.5 that this has a direct module theoretic analogue. In principle this should relate the left and right Brylinski–Kostant filtrations [8]; but this is highly optimistic and we do not pursue the matter.

1.15 Last but not least we describe a theorem of Nakashima and Zelevinsky [33] showing that $B(\infty)$ is defined by linear inequalities and hence that it has an additive semigroup structure. At present this is not completely general and requires a positivity hypothesis. Moreover it seems entirely ad hoc having no module theoretic interpretation. Yet as noted in Remark 3 of 3.6.4 positivity implies that $B(\infty)$ is upper normal. Thus when it holds we obtain by 2.5.9 a further way to construct the family $\mathbb{F}_A$ of normal highest weight crystals, though one cannot expect that this will be enough to establish their closure property.

1.16 Further Reading Several texts on crystals have appeared. We shall mainly refer to our own notes [14, 15]. However one may also consult the book of Hong and Kang [8] devoted almost entirely to crystal theory.

2 Basic Definitions, Tensor Structure and the Uniqueness Theorem

2.1 The Weyl Group

2.1.1 Recall 1.1. Let A be a Cartan matrix with countable index set I. Unless specifically noted we assume that all the diagonal entries $a_{i,i} : i \in I$ of A are equal to 2, that is $I = I_{\text{re}}$.

2.1.2 Although not at first sight obvious, the Weyl group W plays a fundamental role in the theory of crystals, particularly in the construction of $\mathbb{F}_A$.

To define W it is useful though not essential to realize the entries $a_{i,j} : i, j \in I$ as follows. Let $\mathfrak{h}$ be a $\mathbb{Q}$ vector space admitting linearly independent elements $\alpha_i^\vee :$ $i \in I$, called simple coroots, and define $\alpha_j \in \mathfrak{h}^* : j \in I$, called simple roots, to satisfy $\alpha_i^\vee(\alpha_j) = a_{i,j}, \forall i, j \in I$. Augment $\mathfrak{h}$ if necessary to ensure that the $\alpha_j : j \in I$ are also linearly independent. (If $n := |I| < \infty$, it is sufficient that $\dim \mathfrak{h} \geq 2n - \text{rk}\, A$.)

2.1.3 For all $i \in I$, define $s_i \in \text{Aut}(\mathfrak{h}^*)$ by the formula

$$s_i(\lambda) = \lambda - \alpha_i^\vee(\lambda)\alpha_i,$$

and let $W = \langle s_i : i \in I \rangle$ be the subgroup of $\text{Aut}(\mathfrak{h}^*)$ they generate. It is known that W is a Coxeter group with $S = \{s_i\}_{i \in I}$ as its set of generators. The relations in W are of two types.

First the Coxeter relations defined for each pair (i, j) of distinct elements of I, as follows. Set $m_{i,j} = a_{i,j}a_{j,i}$. The Coxeter relations take the form

$$s_is_j = s_js_i, \quad \text{if } m_{i,j} = 0; \qquad s_is_js_i = s_js_is_j, \quad \text{if } m_{i,j} = 1,$$
$$(s_is_j)^{m_{i,j}} = (s_js_i)^{m_{i,j}}, \quad \text{if } m_{i,j} = 2, 3,$$

with no relation if $m_{i,j} \geq 4$.

Second the relations $s_i^2 = 1 : i \in I$.

Given $w \in W$ we can write $w = s_{i_1}s_{i_2} \cdots s_{i_l} : i_j \in I$. It is called a reduced expression if l takes its smallest possible value, called the reduced length $l(w)$ of w.

2.1.4 There are a number of combinatorial results concerning Weyl groups which we shall review briefly. For proofs one may consult [11, Sect. A.1].

Set $\pi^\vee := \{\alpha_i^\vee\}_{i\in I}$, $\pi := \{\alpha_i\}_{i\in I}$, $\Delta_{\text{re}} := W\pi \subset \mathfrak{h}^*$, $\Delta_{\text{re}}^\pm := \Delta_{\text{re}} \cap \pm\mathbb{N}\pi$. A remarkable fact [20, Lemma 3.7] is that $\Delta_{\text{re}} = \Delta_{\text{re}}^+ \cup \Delta_{\text{re}}^-$, that is every element of Δ_{re} written as a sum of simple roots has either only non-negative or only non-positive coefficients. It leads (cf. [11, A.1.1 (iv)]) to the following formula for $l(w)$. Set $S(w) := \{\alpha \in \Delta_{\text{re}}^+ | w\alpha \in \Delta_{\text{re}}^-\}$. Then

$$l(w) = \text{card}\, S(w).$$

2.1.5 Given $\gamma \in \Delta_{\text{re}}^+$ one may write $\gamma = w\alpha_i$, for some $w \in W$, $\alpha_i \in \pi$. After Kac [20, Sect. 5.1] ws_iw^{-1} is independent of the choice of the pair w, α_i and one sets $s_\gamma = ws_{\alpha_i}w^{-1}$. Given $\gamma \in \Delta_{\text{re}}^+$, $w \in W$ one has $l(s_\gamma w) > l(w)$ if and only if $w^{-1}\gamma \in \Delta_{\text{re}}^+$. If $l(w_2) = 1 + l(w_1)$ and $w_2 = s_\gamma w_1$, we write $w_1 \xrightarrow{\gamma} w_2$. Note that γ is uniquely determined by the pair w_1, w_2 (if it exists) and can be omitted. Write $w < w'$, if there exists a sequence (called a Bruhat sequence) $w = w_1 \to w_2 \to \cdots \to w_m = w'$. It is an order relation on W called the Bruhat order.

Bruhat sequences play a major role in the Littelmann path model and as a consequence in describing the Kashiwara crystal $B(\infty)$—see 2.4.

2.2 Crystals

2.2.1 Set $P = \{\lambda \in \mathfrak{h}^* | \alpha^\vee(\lambda) \in \mathbb{Z},\ \forall\alpha^\vee \in \pi^\vee\}$ (resp. $P^+ = \{\lambda \in P | \alpha^\vee(\lambda) \geq 0,\ \forall\alpha^\vee \in \pi^\vee\}$) called the set of integral (resp. integral and dominant) weights. Observe that $Q := \mathbb{Z}\pi \subset P$.

2.2.2 A crystal B is a set with maps

(i) $\text{wt} : B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$, $\forall i \in I$,
(ii) $e_i, f_i : B \cup \{0\} \to B \cup \{0\}$, with $e_i0 = f_i0 = 0$, $\forall i \in I$,

satisfying

(C1) For all $b \in B$, $i \in I$ one has $\varphi_i(b) = \varepsilon_i(b) + \alpha_i^\vee(\text{wt}\, b)$,
(C2) For all $b, e_ib \in B$, $i \in I$ one has $\text{wt}(e_ib) = \text{wt}\, b + \alpha_i$, $\varepsilon_i(e_ib) = \varepsilon_i(b) - 1$,

(C3) For all $b, b' \in B$, $i \in I$, one has $b' = e_i b \Leftrightarrow b = f_i b'$,
(C4) For all $b \in B$, $i \in I$ with $\varphi_i(b) = -\infty$, one has $e_i b = f_i b = 0$.

These rules need modifying in the Borcherds case [19].

2.2.3 One may view a crystal as a graph with vertices $b \in B$ and directed edges labelled by the elements of I such that if one suppresses all edges except those corresponding to fixed $i \in I$, then the graph decomposes into a disjoint union of linear graphs.

A crystal morphism is a map $\psi : B \cup \{0\} \to B' \cup \{0\}$ satisfying the following properties, for all $i \in I$.

(1) $\psi(0) = 0$,
(2) For all $b \in B$ with $\psi(b) \neq 0$, $i \in I$ one has $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, $\varphi_i(\psi(b)) = \varphi_i(b)$, $\mathrm{wt}\,\psi(b) = \mathrm{wt}\,b$,
(3) For all $b \in B$, $i \in I$ with $\psi(b) \neq 0$ and $\psi(e_i b) \neq 0$, one has $\psi(e_i b) = e_i \psi(b)$,
(4) For all $b \in B$, $i \in I$ with $\psi(b) \neq 0$ and $\psi(f_i b) \neq 0$, one has $\psi(f_i b) = f_i \psi(b)$.

Notice that say (3) permits $e_i \psi(b) \neq 0$ even if $e_i b = 0$. One calls B a subcrystal of B' if ψ is an embedding. In this case as a graph B is obtained from B' by deleting the vertices in $B' \setminus B$ and the edges joining them to vertices in B'. An embedding is called strict if it commutes with the $e_i, f_i : i \in I$, that is as a graph B is a component of B'.

2.2.4 Crystals are fairly arbitrary and in particular the Cartan matrix A is not invoked except in relating φ_i, ε_i which is just a matter of definitions. However, following Kashiwara we define a crystal to be upper (resp. lower) normal if $\varepsilon_i(b) = \max\{n | e_i^n b \neq 0\}$ (resp. $\varphi_i(b) = \max\{n | f_i^n b \neq 0\}$), for all $i \in I$. If both hold, a crystal is called normal. Already normality implies that $\alpha_i^\vee(\alpha_i) = 2, \forall i \in I$.

Again let B be a crystal and set $B_\varpi = \{b \in B | \mathrm{wt}\,b = \varpi\}$. If $\mathrm{card}\,B_\varpi < \infty$, $\forall \varpi \in P$ we may define the formal character $\mathrm{ch}\,B$ of B by

$$\mathrm{ch}\,B = \sum_{\varpi \in P} (\mathrm{card}\,B_\varpi) e^\varpi$$

as an element of $\mathbb{Z}P$.

If B is a normal crystal then $\mathrm{ch}\,B$ is W invariant. Indeed suppressing all indices except $\{i\}$ reduces B to a disjoint union of linear graphs, or i-strings, each of which has a character stable under s_i. Kashiwara further defined an action of W on elements of $\mathbb{F}_A$ (for an exposition see [16, 16.9] which follows Littelmann). However this will not be needed here. As described in [16, 16.12] Littelmann used it to compute a combinatorial character formula (see 3.1.2) for elements of $\mathbb{F}_A$.

2.2.5 Let $\mathscr{E}$ (resp. $\mathscr{F}$) denote the monoid generated by the e_i (resp. f_i): $i \in I$. A crystal $B(\lambda)$ is said to be of highest weight $\lambda \in P$ if there exists $b_\lambda \in B(\lambda)_\lambda$ such that

(1) $e_i b_\lambda = 0,\ \forall i \in I$,
(2) $\mathscr{F} b_\lambda = B(\lambda)$.

Although this seems completely analogous to the notion of a highest weight module, it is in fact a much weaker condition. For example, any crystal B admits a highest weight subcrystal $B(\lambda)$ if $B_\lambda \neq \emptyset$. Indeed, choose $b_\lambda \in B_\lambda$ and suppress all vertices not in $\mathscr{F} b_\lambda$ and all edges joining $B \setminus \mathscr{F} B_\lambda$ to $\mathscr{F} B_\lambda$.

However, already requiring a highest weight crystal $B(\lambda)$ to be normal forces $\lambda \in P^+$. More surprisingly we have the following rather easy result [16, 11.15] involving the particular form that the Cartan matrix may take.

Lemma *Suppose there exists a highest weight normal crystal for every* $\lambda \in P^+$. *Then* $\alpha_i^\vee(\alpha_j) \leq 0$, *for* i, j *distinct and* $\alpha_i^\vee(\alpha_j) \neq 0 \Leftrightarrow \alpha_j^\vee(\alpha_i) \neq 0$.

2.2.6 We sketch how to construct for each $\lambda \in P^+$, a normal highest weight crystal $B(\lambda)$ which at least for A symmetrizable has the same character as the integrable simple module $V(\lambda)$ with highest weight $\lambda \in P^+$. However without supplementary conditions there may be normal highest weight crystals which do not have this property. For example take $\pi = \{\alpha_1, \alpha_2\}$ of type A_2. Then there exists a normal highest weight crystal $B(\alpha_1 + \alpha_2)$ with card $B(\alpha_1 + \alpha_2)_0 = 1$. However the simple module $V(\alpha_1 + \alpha_2)$ of this highest weight satisfies $\dim V(\alpha_1 + \alpha_2)_0 = 2$.

2.3 Tensor Product Structure

2.3.1 Let B_2, B_1 be crystals. Kashiwara gave the Cartesian product $B_2 \times B_1$ a crystal structure. The resulting crystal is denoted as $B_2 \otimes B_1$. The tensor product is associative but not commutative.

Given $b = b_2 \otimes b_1 \in B_2 \otimes B_1$, define $\mathrm{wt}\, b = \mathrm{wt}\, b_2 + \mathrm{wt}\, b_1$,

$$\varepsilon_i(b) = \max\{\varepsilon_i(b_2),\ \varepsilon_i(b_1) - \alpha_i^\vee(\mathrm{wt}\, b_2)\} = \max\{\varphi_i(b_2), \varepsilon_i(b_1)\} - \alpha_i^\vee(\mathrm{wt}\, b_2),$$

with $\varphi_i(b)$ given by (C1), that is

$$\varphi_i(b) = \max\{\varphi_i(b_2), \varepsilon_i(b_1)\} + \alpha_i^\vee(\mathrm{wt}\, b_1).$$

Finally the action of the $e_i, f_i : i \in I$, is defined by

$$e_i(b_2 \otimes b_1) = \begin{cases} e_i b_2 \otimes b_1, & \text{if } \varphi_i(b_2) \geq \varepsilon_i(b_1), \\ b_2 \otimes e_i b_1, & \text{if } \varphi_i(b_2) < \varepsilon_i(b_1). \end{cases}$$

$$f_i(b_2 \otimes b_1) = \begin{cases} f_i b_2 \otimes b_1, & \text{if } \varphi_i(b_2) > \varepsilon_i(b_1), \\ b_2 \otimes f_i b_1, & \text{if } \varphi_i(b_2) \leq \varepsilon_i(b_1). \end{cases}$$

Remark The difference between the rule for e_i and for f_i can be remembered by noting that the former prefers and eventually goes into the left hand factor whilst the opposite is true for the latter. This leads to an important principle which will be exploited several times—see 2.5.11, 2.5.18, 3.5.4.

2.3.2 Now let us pass to the tensor product of finitely many crystals $B_i : i = 1, 2, \dots, n$.

Given $b = b_n \otimes b_{n-1} \otimes \cdots \otimes b_1 \in B_n \otimes B_{n-1} \otimes \cdots \otimes B_1$, $i \in I$, define the Kashiwara functions $b \mapsto r_i^k(b)$ through

$$r_i^k(b) = \varepsilon_i(b_k) - \sum_{j>k} \alpha_i^\vee(\mathrm{wt}\, b_j). \qquad (*)$$

Define $\mathrm{wt}\, b = \sum_{k=1}^n \mathrm{wt}\, b_k$, $\varepsilon_i(b) = \max_k\{r_i^k(b)\}$. Define $\varphi_i(b)$ through (C1).

To define $e_i, f_i : i \in I$ on the Cartesian product, let $s_i(b)$ (resp. $l_i(b)$) to be the smallest (resp. largest) value of k such that $r_i^k(b) = \varepsilon_i(b)$. Then

(1) $e_i(b_n \otimes b_{n-1} \otimes \cdots \otimes b_1) = b_n \otimes \cdots \otimes e_i b_l \otimes \cdots \otimes b_1$, where $l = l_i(b)$,
(2) $f_i(b_n \otimes b_{n-1} \otimes \cdots \otimes b_1) = b_n \otimes \cdots \otimes f_i b_s \otimes \cdots \otimes b_1$, where $s = s_i(b)$.

This can be expressed as saying that e_i (resp. f_i) goes in at the $l_i(b)^{\mathrm{th}}$ (resp. $s_i(b)^{\mathrm{th}}$) place.

One must check that this makes the tensor product a crystal. Set $s = s_j(b)$ and suppose $f_j b_s \neq 0$. Then $f_j b \neq 0$, and

$$r_i^k(f_j b) = \begin{cases} r_i^k(b), & \text{if } k > s, \\ r_i^k(b) + 1, & \text{if } k = s, \\ r_i^k(b) + \alpha_i^\vee(\alpha_j), & \text{if } k < s. \end{cases} \qquad (*)$$

Take $i = j$ in the above. Here the definition of s forces $r_i^k(b) < r_i^s(b)$ for $k < s$. Thus the condition $e_i f_i b = b$ forces $r_i^s(b) + 1 \geq r_i^k(b) + \alpha_i^\vee(\alpha_i)$, for $k < s$. Normality has already imposed that $\alpha_i^\vee(\alpha_i) = 2$, thus we require that $r_i^k(b) \leq r_i^s(b) - 1$, for $k < s$. This follows from the previous strict inequality as long as r_i^k takes integer values. In view of $(*)$ this forces the Cartan matrix to have integer entries. Given this one may further check that the tensor product does satisfy the crystal rules and is associative.

One may conclude from 2.2.5 and the above that enough normal crystals together with the above rule for the tensor product force A to be a Cartan matrix of a Kac–Moody algebra; but not necessarily symmetizable. Thus in the Borcherds case the tensor product rule needs to be modified [18], for $i \in I_{\mathrm{im}}$, so that a tensor product of crystals is again a crystal.

2.3.3 The tensor product structure on crystals has several quite astonishing consequences. First it allows one to build interesting crystals, and in particular Kashiwara's $B(\infty)$ crystal, from very simple ones. Secondly it allows one to formulate and prove a uniqueness theorem for crystals [11, 6.4.21]. Thirdly it allows one to formulate and prove Littelmann's remarkable path independence theorem [28]. Fourthly it led via Lusztig's quantum Frobenius map to the construction of standard monomial bases [29]. Fifthly it gives an elementary proof of the refined PRV conjecture [26, 3.5] which in turn extends the Chevalley restriction theorem [12], and finally it leads to a combinatorial version of a result asserting that certain tensor products of Demazure modules admit a Demazure flag [13, 15, 21]. Some of these constructions and perhaps all go over to the Borcherds case, though here significant extra efforts are needed [17].

2.3.4 We shall certainly not discuss all these results here. In general we shall avoid the details of Littelmann theory which although very beautiful is nevertheless too much of a casse-tête chinois to inflict on an amiable audience. What we do mention is that Kashiwara's tensor product which arose from a suitable $q \to 0$ limit, does seem rather mysterious if not obscure. However, in Littelmann's theory it is an immediate and straightforward consequence of applying the Littelmann crystal operators to concatenated paths.

2.3.5 In the next section we shall want to give a countably infinite Cartesian product $\cdots \times B_2 \times B_1$, a crystal structure. In order for the summation in 2.3.2 to make sense we need to restrict to subsets of the above Cartesian product having elements $b = \{\ldots, b_2, b_1\}$ satisfying $\operatorname{wt} b_j = 0$ for all but finitely many j. In order for $\varepsilon_i(b)$ to be well-defined for each $i \in I$ we must assume that $\{\varepsilon_i(b_j)\}_{j\in\mathbb{N}}$ is bounded above. We shall also assume that for all $i \in I$ one has $e_i b_j = 0$ except for finitely many j. Then if $l_i(b)$ as defined in 2.3.2 (1) is infinite, it does not matter at what point to the far left e_i enters since we still get $e_i b = 0$. In this situation we shall say that e_i enters at an infinite place.

2.4 Elementary Crystals and $B(\infty)$

2.4.1 Fix $i \in I$. The i^{th} elementary crystal denoted B_i is the set $\{b_i(-n) : n \in \mathbb{N}\}$ satisfying $\operatorname{wt} b_i(-n) = -n\alpha_i$, $\varphi_i(b_i(-n)) = -n$, $e_i b_i(0) = 0$, $e_i b_i(-n) = b_i(-(n-1))$, $f_i b_i(-n) = b_i(-(n+1))$ when $n \geq 0$, $\varphi_j(b_i(-n)) = -\infty$, $\varepsilon_j(b_i(-n)) = -\infty$, $e_j(b_i(-n)) = 0$, $f_j(b_i(-n)) = 0$, for $j \neq i$.

Notice that $\varepsilon_i(b_i(-n)) = 2n - n = \max\{k | e_i^k b_i(-n) \neq 0\}$. Thus B_i is upper normal just with respect to i. It is not at all lower normal. For brevity we often simply write $b_i(-n)$ as $-n$.

Remark Our definition modifies that of Kashiwara [23] who takes B_i to be infinite in both directions.

2.4.2 Fix a sequence $J = \{i_1, i_2, \ldots\}$ of elements of I such that every element of I appears infinitely many times. We define a crystal B_J to be the subset of the countably infinite Cartesian product $\cdots \times B_{i_2} \times B_{i_1}$, in which all but finitely many entries are zero, and in particular have weight zero. Thus if b belongs to this Cartesian product the Kashiwara function $r_i^k(b)$ reduces to a finite sum and so is well defined. Then the rules in 2.3.2 give B_J a crystal structure. The unique element in which every entry is zero is denoted by b_∞. Obviously $e_i b_\infty = 0$, $\forall i \in I$.

Let $B_J(\infty)$ be the subcrystal of B_J generated by b_∞. It is a strict subcrystal in the sense of 2.2.3.

Obviously $B_J(\infty)_\varpi \neq 0$ implies $\varpi \in -\mathbb{N}\pi$. It follows easily from the crystal rules that $B_J(\infty) = \mathscr{F}(B_J(\infty)^{\mathscr{E}})$, where $B_J(\infty)^{\mathscr{E}} := \{b \in B_J(\infty) | e_i b = 0, \forall i \in I\}$. Obviously $\{b_\infty\} \subset B_J(\infty)^{\mathscr{E}}$, but equality is far from obvious.

Lemma *Admit that $B_J(\infty)$ is upper normal. Then $B_J(\infty)^{\mathscr{E}} = \{b_\infty\}$. In particular, $B_J(\infty)$ is a highest weight crystal.*

Proof Under the hypothesis, $b \in B_J(\infty)^{\mathscr{E}}$ implies that $\varepsilon_i(b) = 0$, $\forall i \in I$. From the definition of ε_i and the Kashiwara function, this forces $b = b_\infty$. □

2.4.3 A further deep property of $B_J(\infty)$ is that it is independent of J as a crystal, though not as a subset of $-\mathbb{N}^{\mathbb{N}}$. We write it simply as $B(\infty)$. Finally a further remarkable result is that

$$\operatorname{ch} B(\infty) = \prod_{\alpha \in \Delta} \left(1 - e^{-\alpha}\right)^{-m_\alpha},$$

where m_α denote the dimension of the root subspace corresponding to α in $\mathfrak{g}_A$. This result is due to Kashiwara [22] in the symmetrizable case. In general it is obtained by combining a combinatorial character formula of Littelmann (see [16, 16.12] for an example) with the Weyl denominator identity which was proved in the required generality by Kumar [27] and by Mathieu [32].

In these lectures we shall also discuss some further remarkable properties of $B(\infty)$.

2.4.4 The above definition of $B(\infty)$ is very straightforward but not very useful for establishing many of its properties. The goal of the next section is to give a second more complicated definition which will establish these properties. We mention that our approach is a little different to that of Kashiwara in [22]. The latter construction required the Kashiwara involution which we deduce as a consequence—see 2.5.14–2.5.25.

2.5 Closed Families of Normal Highest Weight Crystals

2.5.1 Recall 2.2.5. Let $B(\lambda)$, $B(\mu)$ be highest weight crystals. From the crystal rules it is clear that $e_i(b_\lambda \otimes b_\mu) = 0$, $\forall i \in I$. On the other hand it is not at all obvious that $\mathscr{F}(b_\lambda \otimes b_\mu)$ is a subcrystal of $B(\lambda) \otimes B(\mu)$, in other words that it is $\mathscr{E}$ stable and hence a highest weight crystal.

2.5.2 Let $\mathbb{F}_A = \{B(\lambda)|\lambda \in P^+\}$ be a family of normal highest weight crystals. We say that $\mathbb{F}_A$ is closed if for all $\lambda, \mu \in P^+$, $\mathscr{F}(b_\lambda \otimes b_\mu)$ is $\mathscr{E}$ stable and isomorphic to $B(\lambda+\mu)$. A main result is that such a family is unique up to isomorphism. Moreover from it we construct $B(\infty)$ and show that it has the properties described in Sect. 2.4.

2.5.3 A closed family in the above sense was first constructed by Kashiwara [22] in the symmetrizable case by taking a $q \to 0$ limit of integrable modules over the quantized enveloping algebra. It is elementary; but involves a rather long and very intricate induction argument. A much simpler purely combinatorial argument can be obtained from the Littelmann path model [28]. It does not require symmetrizability.

2.5.4 Let us now admit the existence of a closed family $\mathbb{F}_A$ of normal highest weight crystal $\{B(\lambda)|\lambda \in P^+\}$ and prove its uniqueness, that is to say that for all $\lambda \in P^+$, the crystal $B(\lambda)$ is uniquely determined up to isomorphism. The strategy is the following. First we give a new definition of $B(\infty)$ as a direct limit of elements of $\mathbb{F}_A$. Secondly we prove an embedding theorem which allows us to express $B(\infty)$ as $B_J(\infty)$, for any sequence J defined as in 2.4.2. Finally we recover the family $\mathbb{F}_A$ from any $B_J(\infty)$. Since $B_J(\infty)$ is canonically determined by J alone, this will prove uniqueness.

2.5.5 For each $\lambda \in P^+$, let $S(-\lambda) = \{s_{-\lambda}\}$ denote the one element crystal defined by $\operatorname{wt} s_{-\lambda} = -\lambda$ and $\varepsilon_i(s_{-\lambda}) = 0$, $\forall i \in I$.

Let $\mathbb{F}_A = \{B(\lambda)|\lambda \in P^+\}$ be a closed family of normal highest weight crystals. Let $\lambda, \mu \in P^+$ be dominant weights. Let $\psi_{\lambda,\lambda+\mu} : B(\lambda) \otimes S(-\lambda) \to B(\lambda) \otimes B(\mu) \otimes S(-(\lambda+\mu))$ be defined by $\psi_{\lambda,\lambda+\mu}(b \otimes s_{-\lambda}) = b \otimes b_\mu \otimes s_{-(\lambda+\mu)}$.

Lemma *The map $\psi_{\lambda,\lambda+\mu}$ is a crystal embedding commuting with $\mathscr{E}$. Moreover* $\operatorname{Im}\psi_{\lambda,\lambda+\mu} \subset B(\lambda+\mu) \otimes S(-(\lambda+\mu))$.

Proof Let $\lambda, \mu \in P^+$ be dominant weights. Clearly $\psi_{\lambda,\lambda+\mu}(b \otimes s_{-\lambda}) \neq 0$ for $b \otimes s_{-\lambda} \in B(\lambda) \otimes S(-\lambda)$. One has $\varepsilon_i(b_\mu) = \varepsilon_i(s_{-\lambda}) = 0$, whilst $\varphi_i(b) \geq 0$ for a normal crystal. For all $i \in I$, one has by the tensor product rule given in 2.3.2 that

$$\varepsilon_i(b \otimes s_{-\lambda}) = \max\left\{\varepsilon_i(b), -\alpha_i^\vee(\operatorname{wt} b)\right\}$$

and

$$\varepsilon_i(b \otimes b_\mu \otimes s_{-(\lambda+\mu)}) = \max\left\{\varepsilon_i(b), -\alpha_i^\vee(\operatorname{wt} b), -\alpha_i^\vee(\operatorname{wt} b) - \alpha_i^\vee(\mu)\right\}.$$

Now for all $b \in B(\lambda)$ one has $-\alpha_i^\vee(\operatorname{wt} b) - \alpha_i^\vee(\mu) \leq -\alpha_i^\vee(\operatorname{wt} b)$, since μ is dominant, and one has that $-\alpha_i^\vee(\operatorname{wt} b) = \varepsilon_i(b) - \varphi_i(b) \leq \varepsilon_i(b)$, by (C1) and normality. Thus

$$\varepsilon_i\left(\psi_{\lambda,\lambda+\mu}(b \otimes s_{-\lambda})\right) = \varepsilon_i(b \otimes b_\mu \otimes s_{-(\lambda+\mu)}) = \varepsilon_i(b) = \varepsilon_i(b \otimes s_{-\lambda}). \qquad (*)$$

Hence $\psi_{\lambda,\lambda+\mu}$ commutes with ε_i for all $i \in I$. Obviously

$$\operatorname{wt}\psi_{\lambda,\lambda+\mu}(b \otimes s_{-\lambda}) = \operatorname{wt} b - \lambda = \operatorname{wt}(b \otimes s_{-\lambda})$$

for all $i \in I$. Thus by (C1) and the above, $\psi_{\lambda,\lambda+\mu}$ commutes with φ_i for all $i \in I$.

The tensor product formulae given in 2.3.2, implies that for all $b \otimes s_{-\lambda} \in B(\lambda) \otimes S(-\lambda)$ and $i \in I$ one has

$$\begin{aligned} e_i \psi_{\lambda,\lambda+\mu}(b \otimes s_{-\lambda}) &= e_i(b \otimes b_\mu \otimes s_{-(\lambda+\mu)}) = e_i b \otimes b_\mu \otimes s_{-(\lambda+\mu)} \\ &= \psi_{\lambda,\lambda+\mu}\big(e_i(b \otimes s_{-(\lambda+\mu)})\big). \end{aligned}$$

Thus $\psi_{\lambda,\lambda+\mu}$ commutes with $\mathscr{E}$. Similarly, commutation with f_i only fails when $\varphi_i(b) = 0$. For a normal crystal, this is equivalent to $f_i b = 0$.

Therefore $\psi_{\lambda,\lambda+\mu}$ is a crystal embedding of $B(\lambda) \otimes S(-\lambda)$ into $B(\lambda) \otimes B(\mu) \otimes S(-(\lambda+\mu))$ and it commutes with $\mathscr{E}$.

Now for the second claim of the lemma observe that one has

$$\begin{aligned} \psi_{\lambda,\lambda+\mu}\big(B(\lambda) \otimes S(-\lambda)\big) &= \psi_{\lambda,\lambda+\mu}\big(\mathscr{F}(b_\lambda \otimes s_{-\lambda})\big) \subset \mathscr{F}\psi_{\lambda,\lambda+\mu}(b_\lambda \otimes s_{-\lambda}) \\ &= \mathscr{F}(b_\lambda \otimes b_\mu \otimes s_{-(\lambda+\mu)}) = \mathscr{F}(b_\lambda \otimes b_\mu) \otimes S\big(-(\lambda+\mu)\big) \\ &= B(\lambda+\mu) \otimes S\big(-(\lambda+\mu)\big), \end{aligned}$$

where the penultimate step follows from the normality of $B(\mu)$ as above, and the last step results from $\mathbb{F}_A$ being a closed family. □

Remark The reader should be aware that $s_{-\lambda} \otimes s_{-\mu}$ does not quite identify with $s_{-\lambda-\mu}$ since $\varepsilon_i(s_{-\lambda-\mu}) = 0$ while $\varepsilon_i(s_{-\lambda} \otimes s_{-\mu}) = \alpha_i^\vee(\lambda)$, which is in general nonzero.

2.5.6 View P^+ as a directed set through the order relation $\lambda \succcurlyeq \mu$ given $\lambda - \mu \in P^+$. Through the embeddings defined in 2.5.5 we may form the set theoretic direct limit

$$B(\infty) = \varinjlim\big(B(\lambda) \otimes S(-\lambda)\big).$$

We give $B(\infty)$ the structure of crystal as follows. Take $b \in B(\infty)$. Then $b \in B(\lambda) \otimes S(-\lambda)$ for some $\lambda \in P^+$, which is assumed to be sufficiently large so that the actions of the crystal operations (particularly $\mathscr{F}$) do not depend on λ. From Lemma 2.5.5, we obtain the following

Proposition *The above construction endows $B(\infty)$ with the structure of an upper normal highest weight crystal of highest weight* 0.

2.5.7 Recall the elementary crystals $B_i : i \in I$ defined in 2.4.1. The following result is due to Kashiwara in the symmetrizable case [22]. However the present proof is rather different and valid in general. It relies on the existence of the family $\mathbb{F}_A$ of 2.5.5.

Theorem *Fix $i \in I$.*

(i) *There exists a unique crystal embedding*

$$\psi_i : B(\infty) \hookrightarrow B(\infty) \otimes B_i,$$

sending b_∞ to $b_\infty \otimes b_i(0)$.

(ii) *If $b \otimes b_i(-n) \in \operatorname{Im} \psi_i$, then $\varphi_i(b) \geq 0$.*
(iii) *The crystal embedding ψ_i is strict.*

Proof (i) Uniqueness is obvious since b_∞ is a generator of $B(\infty)$.

Take $\lambda, \mu \in P^+$. By the closure property of $\mathbb{F}_A$, the elements $b_\lambda \otimes b_\mu$ and $b_{\lambda+\mu}$ generate the same highest weight crystal, namely $B(\lambda + \mu)$.

Let $b \in B(\infty)$. Then $b = fb_\infty$ for some $f \in \mathscr{F}$. Choose $\gamma \in P^+$ such that $\alpha_j^\vee(\gamma) : j \in I$ are positive and are sufficiently large relative to f. Write $\gamma = \lambda + \mu$ for $\lambda, \mu \in P^+$ such that $\alpha_i^\vee(\lambda) = 0$ and $\alpha_j^\vee(\mu) = 0$, $\forall j \in I \setminus \{i\}$. Then $fb_\infty = fb_{\lambda+\mu} = f(b_\lambda \otimes b_\mu) = f'b_\lambda \otimes f_i^m b_\mu$ for some $f' \in \mathscr{F}$ and $m \in \mathbb{N}$. Define $\psi_i(b) := f'b_\infty \otimes b_i(-m)$.

First we show that ψ_i is well-defined and commutes with $\mathscr{F}$. In particular, we must show that if applying f to $b_\infty \otimes b_i(0)$ gives $f'b_\infty \otimes b_i(-m)$, for some $f' \in \mathscr{F}$, then applying f to $b_\lambda \otimes b_\mu$ gives $f'b_\lambda \otimes f_i^m b_\mu$, and the latter is non-zero. Assume the claim holds for some $f \in \mathscr{F}$ and verify it for $f_j f : j \in I$.

Suppose $j \in I \setminus \{i\}$. Then we must show that f_j enters in the left hand factor in both cases, that is for $B(\infty) \otimes B_i$ and for $B(\lambda) \otimes B(\mu)$. This is trivial in the first case. For the second case we must show that $\varphi_j(f'b_\lambda) > \varepsilon_j(f_i^m b_\mu)$. Now $e_j f_i^m b_\mu = 0$, since otherwise it is a non-zero element of $B(\mu)$ of weight $\mu - m\alpha_i + \alpha_j$. Since $\alpha_i \neq \alpha_j$, this does not lie in $\mu - \mathbb{N}\pi$ and hence not in the set of weights of $B(\mu)$. By the upper normality of $B(\mu)$ one obtains $\varepsilon_j(f_i^m b_\mu) = 0$. Now by (C1) one has

$$\varphi_j(f'b_\lambda) = \varepsilon_j(f'b_\lambda) + \alpha_j^\vee(f'b_\lambda) \geq \alpha_j^\vee(f'b_\lambda) = \alpha_j^\vee(\operatorname{wt} f' + \lambda),$$

since $\varepsilon_j(f'b_\lambda) \geq 0$, by the upper normality of $B(\lambda)$. Thus it suffices that $\alpha_j^\vee(\lambda) > -\alpha_j^\vee(\operatorname{wt} f')$. Moreover the resulting element of $B(\lambda) \otimes B(\mu)$ is non-zero.

Finally suppose that $j = i$. Since $\alpha_i^\vee(\lambda) = 0$, it follows from (C1) and 2.5.5 $(*)$ that $\varphi_i(f'b_\lambda) = \varphi_i(f'b_\infty)$, which is independent of λ. Again $\varepsilon_i(b_i(-m)) = m = \varepsilon_i(f_i^m b_\mu)$, by the upper normality of $B(\mu)$. Then through the tensor product rules it follows that f_i enters in the same factor for both cases. Moreover the resulting element of $B(\lambda) \otimes B(\mu)$ is non-zero given that $\alpha_i^\vee(\mu) \geq m + 1$. This proves (i).

(ii) now follows from the normality of $B(\lambda)$, which forces $\varphi_i(b) \geq 0$, for $b \in B(\lambda)$ and that viewed as an element of $B(\infty)$ the value of $\varphi_i(b)$ is decreased by $\alpha_i^\vee(\lambda) = 0$.

For (iii) one must show that all the crystal operators commute with the crystal maps. This was already shown in the proof of (i) for the elements of $\mathscr{F}$ which are also shown to act injectively. Again since ε, φ, wt commute with ψ_i on the generators, they must also commute on all elements.

It remains to consider the $e_j : j \in I$. Clearly $e_j\psi_i(b) = \psi_i(e_j b)$ if $b = f_j b'$, equivalently if $e_j b \neq 0$. Thus we need only show that $e_j b = 0$ implies $e_j\psi_i(b) = 0$. Since $B(\infty)$ is upper normal, it is enough to show that $\varepsilon_j(\psi_i(b)) = 0$ implies $e_j\psi_i(b) = 0$. Let we write $\psi(b) = b' \otimes b_i(-n)$. Then since $\varepsilon_j(b') \geq 0$, by the upper normality of $B(\infty)$, the condition $\varepsilon_j(\psi_i(b)) = 0$, forces e_j to enter the left hand factor with $\varepsilon_j(b') = 0$. This forces $e_j b' = 0$, again by upper normality and so $e_j\psi_i(b) = 0$, as required. □

2.5.8 From Theorem 2.5.7, it follows that for all $n \in \mathbb{N}^+$ and all $i_1, i_2, \dots, i_n \in I$, there exists a unique strict embedding $B(\infty) \hookrightarrow B(\infty) \otimes B_{i_n} \otimes \cdots \otimes B_{i_1}$, sending b_∞ to $b_\infty \otimes (b_{i_n}(0) \otimes \cdots \otimes b_{i_1}(0))$. If $f \in \mathscr{F}$ applied to such an expression goes into the right hand factor, then for all $j \in I$ and $m > n$ for which $i_m = j$, the tensor product rule implies that $f_j f$ goes into the right hand factor of $b_\infty \otimes (b_{i_m}(0) \otimes \cdots \otimes b_{i_1}(0))$. Now take J as in 2.4.2. Then we obtain a unique strict embedding of $B(\infty)$ into $B(\infty) \otimes B_J$ sending b_∞ to $b_\infty \otimes (\cdots \otimes b_{i_m}(0) \otimes \cdots \otimes b_{i_1}(0))$. From this it is clear that $B_J(\infty)$ is isomorphic to $B(\infty)$ as defined in 2.5.6, hence is upper normal and independent of J.

2.5.9 We now recover $\mathbb{F}_A$ from $B(\infty)$ as defined in 2.5.6.

Fix $\lambda \in P^+$ and $S(\lambda) = \{s_\lambda\}$ denote the one-element crystal defined by $\operatorname{wt} s_\lambda = \lambda$ and $\varphi_i(s_\lambda) = 0, \ \forall i \in I$. Note the subtle difference with the definition of $S(-\lambda)$ given in 2.5.5.

Lemma *$\mathscr{F}(b_\infty \otimes s_\lambda)$ is a strict subcrystal of $B(\infty) \otimes S(\lambda)$ and is isomorphic to the crystal $B(\lambda)$ of the family $\mathbb{F}_A$.*

Proof Since

$$\varepsilon_i(s_\lambda) = \varphi_i(s_\lambda) - \alpha_i^\vee(\lambda) = -\alpha_i^\vee(\lambda),$$

the tensor product rules give

$$f_i(b \otimes s_\lambda) = f_i b \otimes s_\lambda \quad \Leftrightarrow \quad \varphi_i(b) + \alpha_i^\vee(\lambda) > 0.$$

On the other hand by the lower normality of $B(\lambda) \in \mathbb{F}_A$, and in view of the shift by λ in the value of $\operatorname{wt} b$, the image of $b \in B(\lambda)$ in $B(\infty)$ satisfies $f_i b = 0$, if and only if $\varphi_i(b) + \alpha_i^\vee(\lambda) \leq 0$.

Again since $\varphi_i(s_\lambda) = 0$, the upper normality of $B(\infty)$ ensures that the $e_i : i \in I$ always enter the first factor and that the $\varepsilon_i : i \in I$ are preserved. Obviously wt is preserved and hence so are the φ_i.

This proves the first assertion.

The last assertion follows easily from the presentation of $B(\infty)$ given in 2.5.6 and the above observations, the essential point being that the crystal operators $e_i, f_i : i \in I$ enter the first factor in both cases with the slight exception noted for the $f_i : i \in I$. $\square$

Remark Again as in Remark 2.5.10 one cannot identify $s_\lambda \otimes s_\mu$ with $s_{\lambda+\mu}$. However for $\lambda \in P^+$ we *can* identify $s_{-\lambda} \otimes s_\lambda$ with s_0. This again proves that the embedding $b \mapsto b \otimes s_{-\lambda}$ of $B(\lambda)$ into $B(\infty)$ composed with $- \otimes s_\lambda$ on its image is a crystal automorphism of $B(\lambda)$.

2.5.10

Corollary *The closed family $\mathbb{F}_A$ of normal highest weight crystals is unique up to crystal isomorphism.*

Proof Lemma 2.5.9 recovers $B(\lambda)$ from $B(\infty)$ and hence by 2.5.8 from $B_J(\infty)$. Yet $B_J(\infty)$ is defined (2.4.2) in a manner independent of $\mathbb{F}_A$. Thus any given member of the family $B(\lambda)$ is uniquely determined by $\lambda \in P^+$. □

Remark If we start from $B_J(\infty)$ constructed as in 2.4.2 and admit that it is upper normal, then from Lemma 2.4.2 and pursuing the reasoning in Lemma 2.5.9 it follows easily that $B(\lambda) := \mathscr{F}(b_\infty \otimes s_\lambda)$ is a normal crystal of highest weight λ. On the other hand it is not so obvious that the resulting family $\{B(\lambda) : \lambda \in P^+\}$ is closed.

2.5.11 Using upper normality we can obtain a more precise version of 2.5.7.

If B' is a subset of a crystal which is $\mathscr{E}$ stable, then it makes sense to ask if B' is upper normal. Let $\mathscr{E}^i$ (resp. $\mathscr{F}^i$) denote the monoid generated by the e_j (resp. f_j) : $j \in I - \{i\}$. Define ψ_i by the conclusion of 2.5.7. For each $i \in I$, let B^i denote the subset of $B(\infty)$ defined by

$$B^i = \left\{b \in B(\infty) | b \otimes b_i(-n) \in \operatorname{Im}\psi_i, \text{ for some } n \in \mathbb{N}\right\}.$$

Define $B' = B^i \otimes B_i \subset B(\infty) \otimes B_i$. Clearly $\operatorname{Im}\psi_i \subset B'$.

Lemma *Fix $i \in I$.*

(i) *B^i is $\mathscr{E}$ stable and upper normal,*
(ii) *B^i is $\mathscr{F}^i$ stable,*
(iii) *$B(\infty) = B^i \times B_i$ as a set.*

Proof Obviously $f_j B^i \subset B^i$ and $f_j B' \subset B'$, if $j \in I \setminus \{i\}$. This proves (ii).

Again $e_j B^i \subset B^i$ and $e_j B' \subset B'$, if $j \in I \setminus \{i\}$. Let us show this also hold for i. Take $b \in B^i$ and $n \in \mathbb{N}$ such that $b \otimes b_i(-n) \in \operatorname{Im}\psi_i$. We claim that after sufficiently many applications e_i enters into the left hand factor. Otherwise $b \otimes b_i(0) \in \operatorname{Im}\psi_i$. Yet $\varepsilon_i(b_i(0)) = 0 \le \varphi_i(b)$, by (ii) of Theorem 2.5.7. Then $e_i(b \otimes b_i(0)) = e_i b \otimes b_i(0) \in \operatorname{Im}\psi_i$ and so $e_i b \in B^i$, as required.

We conclude that B^i and hence B' is $\mathscr{E}$ stable. Thus it makes sense to consider if B' is upper normal. Observe that $b \otimes b_i(-m) \in \operatorname{Im}\psi_i$, means that $b \in B(\infty)$. Thus the upper normality of $B(\infty)$ implies that B' is upper normal with respect to all indices except possibly i.

Let us show that B' is i-upper normal. Observe that the value of r_i^k on $\{b \otimes b_i(-n) : n \in \mathbb{N}\}$ is independent of n for $k > 1$. Suppose $b \otimes b_i(-n) \in \operatorname{Im}\psi_i$ and set $t = r_i^1(b \otimes b_i(-n)) - \max_{k>1} r_i^k(b \otimes b_i(-n)) = n - \alpha_i^\vee(\operatorname{wt} b) - \varepsilon_i(b) = n - \varphi_i(b) \le n$, since $\varphi_i(b) \ge 0$, by (ii) of 2.5.7. If $t > 0$, then $e_i^t(b \otimes b_i(-n)) = b \otimes b_i(-(n-t)) \in \operatorname{Im}\psi_i$ and $r_i^1(b \otimes b_i(-(n-t)) = \max_{k>1} r_i^k(b \otimes b_i(-(n-t))$. Thus it suffices to establish upper normality of B' when $r_i^1(b \otimes b_i(-n)) \le \max_{k>1} r_i^k(b \otimes b_i(-n))$. In this case e_i goes into the left hand factor. Moreover the value of r_i^1 decreases by $\alpha_i^\vee(\alpha_i) = 2$, whilst the value of $\max_{k>1} r_i^k$ decreases by 1. Thus powers of e_i continue to go into the left hand factor. (This is the principle referred to in Remark 2.3.1.) We conclude that the upper normality of $B(\infty)$ implies the upper normality of B'. This proves (i).

By the upper normality of $B' \subset B_J$ it follows as in the proof of Lemma 2.4.2 that $B'^{\mathscr{E}} = \{b_\infty\}$. Yet the set of weights of B' lie in $-\mathbb{N}\pi$, so for all $b \in B'$ one has $b_\infty \in \mathscr{E}b$. Combined with our previous assertion this implies that $B' = \mathscr{F}B'^{\mathscr{E}} = \mathscr{F}b_\infty = B(\infty)$. Hence (iii). □

2.5.12 By the above lemma we can give B^i a crystal structure by taking the induced structure from $B(\infty)$, except with respect to f_i. Now given $b \in B^i$ one has $b \otimes b_i(0) \in B(\infty)$, by (iii) above. Then $f_i(b \otimes b_i(0)) = f_i b \otimes b_i(0)$ and so defines $f_i b$ except if $\varphi_i(b) = 0$. In the latter case we redefine $f_i b$ to be equal to zero.

Lemma *Fix $i \in I$.*

(i) *B^i is f_i stable and i-lower normal,*
(ii) *B^i is a subcrystal of $B(\infty)$,*
(iii) *$B(\infty) = B^i \otimes B_i$ as a crystal.*

Proof Consider $b' = b \otimes b_i(0)$, $b \in B^i$. By the tensor product rules one has $f_i b' = f_i b \otimes b_i(0)$, unless $\varphi_i(b) = \varepsilon_i(b_i(0)) = 0$. In the latter case, one has that $f_i b = 0$ by our definition. Hence B^i is f_i stable. By (ii) of Theorem 2.5.7 we obtain $\varphi_i(b) = \max\{\varphi_i(b'), 0\} = \varphi_i(b')$, whilst by construction $f_i b = 0 : b \in B^i$, if and only if $\varphi_i(b) = 0$. Thus B^i is i-lower normal. Hence (i).

The fact that B^i is a subcrystal of $B(\infty)$ now follows from (i) and Lemma 2.5.11. Moreover this also shows that $B(\infty) = B^i \otimes B_i$ as a crystal. □

2.5.13

Lemma *Fix $i \in I$. Then B^i is the unique i-normal subcrystal of $B(\infty)$ such that the crystal embedding $B^i \hookrightarrow B(\infty)$ commutes with $\mathscr{E}, \mathscr{F}^i$.*

Proof Since B^i is $\mathscr{E}$ stable, it contains b_∞. Since B^i is a subcrystal of $B(\infty) = \mathscr{F}b_\infty$, it is generated by b_∞. Since it is i-lower normal, we have for $b \in B^i$ that $\varphi_i(b) = 0$ implies $f_i b = 0$. Clearly B^i is the unique subset with this property which is f_i-stable and commutes with $\mathscr{E}, \mathscr{F}^i$. □

2.5.14 In view of Lemma 2.5.12 we may define a new pair of operators $e_i^\star, f_i^\star$ on $B(\infty) = B^i \otimes B_i$ as just e_i, f_i acting on the right hand factor. Then

$$B(\infty)^{e_i^\star} = B^i = \left\{b \otimes b_i(0) \in \operatorname{Im}\psi_i\right\}.$$

Since $\mathrm{wt}(b \otimes b_i(-m)) = \mathrm{wt}\, b - m\alpha_i$, it follows that the first part of (C2) holds.

Further set $\mathrm{wt}^\star = \mathrm{wt}$ and $\varepsilon_j^\star(b) = \max\{m \in \mathbb{N}^+ | e_j^{\star m} b \neq 0\}$. Observe that $\varepsilon_i^\star(b) = n$ for $b = b' \otimes b_i(-n)$. However we do not know how to calculate $\varepsilon_j^\star(b) : j \in I \setminus \{i\}$ in this presentation of $B(\infty)$. The importance of this question is discussed in 3.2.6.

Define $\varphi_i^\star(b)$ through (C1).

By construction $B(\infty)$ with the maps $e_i^\star, f_i^\star, \varepsilon_i^\star, \varphi_i^\star, \mathrm{wt}^\star := \mathrm{wt}$ is an upper normal crystal. However since we have no presentation of $B(\infty)^\star$ in which we can simultaneously calculate the values of the $\varepsilon_j^\star : j \in I$ we cannot immediately conclude as in 2.4.2 that $B(\infty)^{\mathscr{E}^\star} = \{b_\infty\}$.

When we want to emphasize the $\star$ crystal structure of $B(\infty)$ we shall write it as $\mathrm{s}B(\infty)^\star$.

Let $\mathscr{E}^\star$ (resp. $\mathscr{F}^\star$) denote the monoid generated by the $e_j^\star$ (resp. $f_j^\star) : j \in I$.

2.5.15 We may observe (as did Kashiwara in the symmetrizable case) that the above crystal structures are (almost!) independent. This is expressed by the

Lemma *Take $i, j \in I$ distinct. Then the pairs $e_i, f_j^\star$; $f_i, f_j^\star$; $e_i^\star, f_j$; $e_i^\star, f_j$ commute.*

Proof Consider $e_i, f_j^\star$. Identify $B(\infty)$ with its image under the embedding ψ_j. Write $b \in B(\infty)$ in the form $f_j^{\star m} b'$ with $e_j^\star b' = 0$. Thus $e_i f_j^\star b = e_i f_j^{\star(m+1)} b' = e_i(b' \otimes b_j(-(m+1)) = e_i b' \otimes b_j(-(m+1))$. We conclude that $e_i b' \in B(\infty)^{e_j^\star}$ and this last expression equals $f_j^\star e_i b$. Since $b \in B(\infty)$ is arbitrary it follows that $e_i, f_j^\star$ commute on $B(\infty)$. The remaining cases are similar. $\square$

2.5.16 By contrast to 2.5.15, the $e_i, e_i^\star$ do not commute. On the other hand we showed in the proof of 2.5.11 that B^i which now identifies with $B(\infty)^{e_i^\star}$ is $\mathscr{E}$ stable and upper normal. Hence $B(\infty)^{\mathscr{E}^\star} = \bigcap_{i \in I} B(\infty)^{e_i^\star}$ is $\mathscr{E}$ stable.

Lemma $B(\infty)^{\mathscr{E}^\star} = \{b_\infty\}$, $B(\infty) = \mathscr{F}^\star b_\infty$.

Proof Obviously $B(\infty)^{\mathscr{E}^\star} \supset \{b_\infty\}$. Suppose b belongs to the complement. Since $B(\infty)^{\mathscr{E}^\star}$ is $\mathscr{E}$ stable and $B(\infty)^{\mathscr{E}} = \{b_\infty\}$ we can find $e \in \mathscr{E}$ such that $eb = b_\infty$. Choose $i \in I$ such that $e = e_i e'$, for some $e' \in \mathscr{E}$. Then $b' := e'b \in B(\infty)^{\mathscr{E}^\star}$. Yet $e_i b' = b_\infty$, so $b = f_i b_\infty = f_i(b_\infty \otimes b_i(0)) = b_\infty \otimes b_i(-1)$, by the tensor product rules. Yet obviously $b_\infty \otimes b_i(-1) \notin B(\infty)^{\mathscr{E}^\star}$. Hence a contradiction. This proves the first part. The second part is a consequence of the first part. $\square$

Hence $B(\infty)$ with the maps $e_i^\star, f_i^\star, \varepsilon_i^\star, \varphi_i^\star, \mathrm{wt}^\star := \mathrm{wt}$ is a highest weight crystal of highest weight zero.

2.5.17 Let $\mathscr{E}^{\star i}$ (resp. $\mathscr{F}^{\star i}$) denote the monoid generated by the $e_j^\star$ (resp. $f_j^\star) : j \in I - \{i\}$. Then $B(\infty)^{e_i}$ is $\mathscr{E}^{\star i}$ and $\mathscr{F}^{\star i}$ stable by 2.5.15. It remains to consider the action of $e_i^\star$ and $f_i^\star$.

Lemma *Take $b \in B(\infty)^{e_i}$. Then*

(i) $e_i^\star b \in B(\infty)^{e_i}$,
(ii) $\varphi_i^\star(b) \geq 0$,
(iii) $f_i^\star b \in B(\infty)^{e_i} \Leftrightarrow \varphi_i^\star(b) > 0$.

Proof We use the presentation $B(\infty) = B^i \otimes B_i$. Take $b = b' \otimes b_i(-m) \in B(\infty)^{e_i}$ and recall that $\varphi_i(b') \geq 0$, by (ii) of Theorem 2.5.7. Then $e_i b = 0$, if and only if $e_i b = e_i b' \otimes b_i(-m)$ and $e_i b' = 0$. In particular $\varphi_i(b') \geq \varepsilon_i(b_i(-m)) = m$.

Now $e_i^\star b = b' \otimes b_i(-(m-1))$ and $\varphi_i(b') \geq m > m-1 = \varepsilon_i(b_i(-(m-1)))$. Hence $e_i e_i^\star b = e_i b' \otimes b_i(-(m-1)) = 0$. This proves (i).

For (ii) recall that $\varepsilon_i^\star(b' \otimes b_i(-m)) = m$. Since $e_i b' = 0$, as in the first part we have $\varepsilon_i(b') = 0$, by the upper normality of $B(\infty)$ and so $\alpha_i^\vee(\mathrm{wt}\, b') = \varphi_i(b') - \varepsilon_i(b') = \varphi_i(b') \geq m$. Now $\alpha_i^\vee(\mathrm{wt}\, b) = \alpha_i^\vee(\mathrm{wt}\, b' - m\alpha_i) = \varphi_i(b') - 2m$. Hence $\varphi_i^\star(b) = \varepsilon_i^\star(b) + \alpha_i^\vee(\mathrm{wt}\, b) = m + \varphi_i(b') - 2m = \varphi_i(b') - m \geq 0$. Hence (ii).

Suppose $f_i^\star b \notin B(\infty)^{e_i}$. By definition $f_i^\star b = b' \otimes b_i(-(m+1))$ and so $e_i f_i^\star b \neq 0$, implies that $e_i(f_i^\star b) = b' \otimes e_i b_i(-(m+1)) = b' \otimes b_i(-m)$. In particular $\varphi_i(b') < \varepsilon_i(b_i(-(m+1))) = m+1$. Yet $\varphi_i(b') \geq m$, by the first part and so $\varphi_i(b') = m$, implying $\varphi_i^\star(b) = 0$. Conversely, if $f_i^\star b \in B(\infty)^{e_i}$ then $0 = e_i f_i^\star b = e_i b' \otimes b_i(-m)$ and $\varphi_i(b') \geq \varepsilon_i(b_i(-(m+1))) = m+1$, implying $\varphi_i^\star(b) > 0$. This gives (iii). □

2.5.18 We can give $B(\infty)^{e_i}$ a $\star$ crystal structure by setting $f_i^\star b$ to be equal to zero when $\varphi_i^\star(b) = 0$. Denote this crystal by $B^{i\star}$. Then $B^{i\star}$ is a subcrystal of $B(\infty)^\star$ with the rule $\varphi_i^\star(b) = 0 \Rightarrow f_i^\star b = 0$. In particular the embedding $B^{i\star} \hookrightarrow B(\infty)^\star$ commutes with $\mathscr{E}^\star$ and $\mathscr{F}^{\star i}$.

Since $B(\infty)^\star = \mathscr{F}^\star b_\infty$ by Lemma 2.5.16 we may define a map of $B(\infty)^\star$ into $B^{i\star} \otimes B_i$ by sending $f^\star b_\infty \mapsto f^\star(b_\infty \otimes b_i(0))$.

Lemma $B(\infty)^\star \xrightarrow{\sim} B^{i\star} \otimes B_i$.

Proof Take $b = b' \otimes b_i(-n) \in B^{i\star} \otimes B_i$. Obviously each element of $\mathscr{E}^{\star i}$ enters the left hand factor. By (ii) of Lemma 2.5.17 this is eventually true of $e_i^\star$, as shown in the proof (given in (ii) of Lemma 2.5.11) of the analogous result for e_i. Now suppose $b \in (B^{i\star} \otimes B_i)^{\mathscr{E}^\star}$. By the above each $e_j : j \in I$ enters the left hand factor and so $b' \in (B(\infty)^\star)^{\mathscr{E}^\star} = \{b_\infty\}$, by 2.5.16. Thus $b' = b_\infty$, which further implies that $n = 0$. As in the proof of (iii) of Lemma 2.5.11, this proves surjectivity. □

2.5.19 Thus we have a crystal embedding $B(\infty)^\star \xrightarrow{\sim} B^{i\star} \otimes B_i \hookrightarrow B(\infty)^\star \otimes B_i$. Since this holds for all $i \in I$, we conclude that $B(\infty)^\star$ also satisfies 2.5.8. This presentation implies that $B(\infty)^\star$ with the $\star$ action is isomorphic to $B(\infty)$ with the previous action. Thus we obtain

Theorem *$B(\infty)^\star$ is isomorphic to $B(\infty)$.*

2.5.20 Let us show how the above result allows us to obtain the Kashiwara involution on $B(\infty)$ in the general, not necessarily symmetrizable, case.

Given $b \in B(\infty)$, we can write $b = f_{i_1}^{m_1} f_{i_2}^{m_2} \cdots f_{i_k}^{m_k} b_\infty$. Define $b^\star$ by replacing $f_{i_j}^{m_j}$ by $f_{i_j}^{\star m_j}$. Since the $f_i^\star : i \in I$ act injectively $b^\star \neq 0$. By the isomorphism theorem above, the result is independent of choice of representatives. This defines a map $b \mapsto b^\star$ of $B(\infty)$ into $B(\infty)^\star$. Notice that we have $(fb)^\star = f^\star b^\star$, for all $f \in \mathscr{F}$,

$b \in B(\infty)$. A map inverse to $\star$ may be similarly defined by expressing b as an element of $\mathscr{F}^\star b_\infty$ and replacing $f_i^\star$ by f_i. Hence $\star$ is bijective and intertwines the two crystal structures on $B(\infty)$. Comparison with [11, 6.1.12, 6.1.13] shows that $\star$ coincides with the Kashiwara involution if A is symmetrizable.

2.5.21 We would like to show that the map $\star$ as defined above is an involution in the not necessarily symmetrizable case. First we will study the relationship between these two different crystal structures on $B(\infty)$.

Let us recall the decomposition $B(\infty)^\star \xrightarrow{\sim} B^{i\star} \otimes B_i$ given by the star action. We may identify $B(\infty)^\star$ with $B(\infty)$ as sets, and we also have the decomposition $B(\infty) \xrightarrow{\sim} B^i \otimes B_i$.

Take $b \in B(\infty)^\star$ and write $b = b' \otimes b_i(-m)$ as an element of $B^{i\star} \otimes B_i$. Now $\varepsilon_i^\star(b) = \max\{m, \varphi_i^\star(b')\} - \alpha_i^\vee(\operatorname{wt} b')$. Since $\alpha_i^\vee(\operatorname{wt} b') = 2m + \alpha_i^\vee(\operatorname{wt} b)$, we obtain

$$\varepsilon_i^\star(b) \geq -m - \alpha_i^\vee(\operatorname{wt} b), \text{ with equality} \Leftrightarrow \varphi_i^\star(b') \leq m \Leftrightarrow f_i^\star b = f_i^{\star\star} b. \qquad (*)$$

Indeed the second equivalence follows from the fact that by the definition of $f_i^{\star\star}$ it always enters the right hand factor, whilst $f_i^\star$ enters the right hand factor (and hence $f_i^\star b = f_i^{\star\star} b$) if and only if $\varphi_i^\star(b') \leq \varepsilon_i^\star(b_i(-m)) = m$.

Now also write $b = b'' \otimes b_i(-n)$ as an element of $B^i \otimes B_i$. Then as in $(*)$ we obtain

$$\varepsilon_i(b) \geq -n - \alpha_i^\vee(\operatorname{wt} b), \text{ with equality} \Leftrightarrow \varphi_i(b'') \leq n \Leftrightarrow f_i b = f_i^\star b. \qquad (**)$$

By the definition of $\star$ crystal structure $\varepsilon_i^\star(b) = n$.

Lemma *In the presentation $b = b' \otimes b_i(-m) \in B^{i\star} \otimes B_i$, one has $\varepsilon_i(b) = m$.*

Proof Given $\lambda \in \mathbb{N}\pi$, let $|\lambda|$ denote the sum of its coefficients. The proof is by induction on $|-\operatorname{wt}(b)|$. It is trivial when $|-\operatorname{wt}(b)| = 0$.

Recall (2.5.16) that $B(\infty)^\star = \mathscr{F}^\star b_\infty$. If $j \in I \backslash \{i\}$, then $f_j^\star$ enters the first factor and moreover by 2.5.15 and upper normality one has $\varepsilon_i(f_j^\star b) = \varepsilon_i(b)$. Thus it remains to show that the conclusion of the lemma holds for $f_i^\star b$ given that it holds for b.

Case (1). One has $f_i^\star(b' \otimes b_i(-m)) = b' \otimes b_i(-(m+1))$.

By the tensor product rule this holds if and only if $\varphi_i^\star(b') \leq m$. On the other hand $\varepsilon_i^\star(b) = n$, whilst $\varepsilon_i(b) = m$ by the induction hypothesis. Substitution into $(*)$ above gives $m + n = -\alpha_i^\vee(\operatorname{wt} b)$. Further substitution into $(**)$ above gives $f_i b = f_i^\star b$. Consequently $\varepsilon(f_i^\star b) = \varepsilon_i(f_i b) = \varepsilon_i(b) + 1$, as required.

Case (2). One has $f_i^\star(b' \otimes b_i(-m)) = f_i^\star b' \otimes b_i(-(m))$.

By the tensor product rule this holds if and only if $\varphi_i^\star(b') > m$. Substituting in $(*)$ and $(**)$ above and using the induction hypothesis as before gives $\varphi_i(b'') > n$. Thus $f_i^\star b = b'' \otimes b_i(-(n+1))$, and then

$$\varepsilon_i(f_i^\star b) = \max\bigl\{\varphi_i(b''), n+1\bigr\} - \alpha_i^\vee(\operatorname{wt} b'') = \varphi_i(b'') - \alpha_i^\vee(\operatorname{wt} b'') = \varepsilon_i(b),$$

as required. $\square$

2.5.22

Corollary *Let $b \in B(\infty)$ and identify $B(\infty)$ with $B(\infty)^\star$ as sets. Then the following are equivalent*:

(i) $f_i b = f_i^\star b$,
(ii) $f_i^\star b = f_i^{\star\star} b$,
(iii) *Equality holds in* $\varepsilon_i^\star(b) \geq -\varepsilon_i(b) - \alpha_i^\vee(\mathrm{wt}\, b) = -\varphi_i(b)$.

Proof Since $\varepsilon_i^\star(b) = n$, whilst $\varepsilon_i(b) = m$, by the previous lemma, the assertion follows by substitution into $(*)$ and $(**)$. □

2.5.23 Take $b \in B(\infty)$ and let us write $b = b'' \otimes b_i(-n) \in B^i \otimes B_i$ and $b = b' \otimes b_i(-m) \in B^{i\star} \otimes B_i$, as before.

Lemma

(a) *The following are equivalent*
  (i) $\varphi_i(b'') \neq n$,
  (ii) $e_i^\star f_i b = f_i e_i^\star b$,
  (iii) $f_i^\star e_i b = e_i f_i^\star b$.
(b) *The following are equivalent*
  (i) $\varphi_i^\star(b') \neq m$,
  (ii) $e_i^\star f_i^{\star\star} b = f_i^{\star\star} e_i^\star b$,
  (iii) $f_i^\star e_i^{\star\star} b = e_i^{\star\star} f_i^\star b$.

Proof For (i) ⇔ (ii) of (a), write $b = b'' \otimes b_i(-n) \in B^i \otimes B_i$. By definition $e_i^\star$ always enters the right hand factor and so (ii) holds exactly when f_i enters the same factor for both terms. Now f_i enters the left hand factor of b (resp. of $e_i^\star b$) if and only if $\varphi_i(b'') > n$ (resp. $\varphi_i(b'') > n - 1$). Thus commutation fails exactly when $\varphi_i(b'') = n$. The proof of (i) ⇔ (iii) in (a) is similar. Finally (b) obtains from (a) by translating all arguments by $\star$. □

2.5.24 Unfortunately the above result is not free of the presentation of b. The following is a weaker result independent of presentation.

Corollary *If $b \in B(\infty)$ such that $f_i b \neq f_i^\star b$ (or equivalently $f_i^\star b \neq f_i^{\star\star} b$), then for all $k \in \mathbb{N} - \{0\}$ we have*

(i) $e_i^\star f_i b = f_i e_i^\star b$,
(ii) $f_i^\star e_i b = e_i f_i^\star b$,
(iii) $e_i^\star f_i^{\star\star} b = f_i^{\star\star} e_i^\star b$,
(iv) $f_i^\star e_i^{\star\star} b = e_i^{\star\star} f_i^\star b$.

Proof For (i) and (ii) observe that $\varphi_i(b'') \leq n \Leftrightarrow \varphi_i(b) + \varepsilon_i^\star(b) = 0$ and apply 2.5.22 to the lemma. For (iii) and (iv) observe that $\varphi_i^\star(b') \leq m \Leftrightarrow \varphi_i^\star(b) + \varepsilon_i(b) = \varphi_i(b) + \varepsilon_i^\star(b) = 0$, and apply 2.5.22 to the lemma. □

2.5.25

Theorem $\star$ *is an involution. In particular,* $f_i b = f_i^{\star\star} b$ *for all* $b \in B(\infty)$.

Proof We prove this by contradiction. Given $\lambda \in \mathbb{N}\pi$, let $|\lambda|$ denote the sum of its coefficients. Suppose that there exists $b \in B(\infty)$ such that $f_i b \neq f_i^{\star\star} b$ and choose this b to be minimal with respect to $|-\mathrm{wt}(b)|$.

If $f_i b = f_i^\star b$ then $f_i b = f_i^\star b = f_i^{\star\star} b$ by 2.5.22, so under our hypothesis $f_i b \neq f_i^\star b$. Then by 2.5.24 we have $e_i^\star f_i b = f_i e_i^\star b$ and $e_i^\star f_i^{\star\star} b = f_i^{\star\star} e_i^\star b$. Also by 2.5.15 we have that $e_j^\star f_i b = f_i e_j^\star b$ and $e_j^\star f_i^{\star\star} b = f_i^{\star\star} e_j^\star b$ for all $j \in I - \{i\}$.

Now if $j \in I$ is such that $e_j^\star b \neq 0$, then $|-\mathrm{wt}(e_j^\star b)| = |-\mathrm{wt}(b) - \alpha_j| = |-\mathrm{wt}(b)| - 1$. So by our assumption on the minimality of $|-\mathrm{wt}(b)|$ we have $f_i e_j^\star b = f_i^{\star\star} e_j^\star b$. But then by the previous paragraph we have that $e_j^\star f_i b = f_i e_j^\star b = f_i^{\star\star} e_j^\star b = e_j^\star f_i^{\star\star} b$. By applying $f_j^\star$ to both sides of this equation we obtain $f_i b = f_i^{\star\star} b$, which contradicts our assumption.

Hence we are reduced to the case that $b \in B(\infty)^{\mathscr{E}^\star} = \{b_\infty\}$, by 2.5.16. Yet $f_i b_\infty = f_i^\star b_\infty = f_i^{\star\star} b_\infty$ trivially. This contradiction proves that $f_i b = f_i^{\star\star} b$, for all $b \in B(\infty)$. $\square$

2.5.26 The previous analysis shows that the properties of $B(\infty)$ established above, namely that $B_J(\infty)$ is independent of the choice of J, that $B(\infty) \otimes S_\lambda$ admits a strict normal subcrystal $B(\lambda)$ of highest weight λ, and that $B(\infty)$ admits a non-trivial involution $\star$ such that the induced crystal structure almost commutes (2.5.15, 2.5.24) with the original crystal structure, all follow from the upper normality of $B_J(\infty)$ and the decompositions $B(\infty) = B^i \otimes B_i : i \in I$. However even upper normality is apparently rather non-trivial as the following example indicates.

Example Take $T = \{\alpha_1, \alpha_2\}$ of type A_2, that is $A = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. Take J with $i_j = 1$, if j is odd and $i_j = 2$, if j is even. Then $\cdots b_{i_4}(-m_4) \otimes b_{i_3}(-m_3) \otimes b_{i_2}(-m_2) \otimes b_{i_1}(-m_1) \in B_J(\infty)$ if and only if $m_3 \leq m_2$ and $m_j = 0$, for $j > 3$. One might think therefore that $B_J(\infty)$ is the subcrystal of $B_1 \otimes B_2 \otimes B_1$ generated by $b_\infty := b_1(0) \otimes b_2(0) \otimes b_1(0)$. However this is false! Consider $b = b_1(-1) \otimes b_2(-1) \otimes b_1(0)$, which belongs to $\mathscr{F} b_\infty$. Obviously $e_2 b = b_1(-1) \otimes b_2(0) \otimes b_1(0)$. This does not belong to $\mathscr{F} b_\infty$. Moreover $e_2^2 b = 0$ and also $e_1 e_2 b = 0$. Thus $b \in (\mathscr{E}\mathscr{F} b_\infty)^{\mathscr{E}}$. In particular the crystal generated by b_∞ in $B_1 \otimes B_2 \otimes B_1$ is not a highest weight crystal. By contrast if we set $b' = b_2(0) \otimes b_1(-1) \otimes b_2(-1) \otimes b_1(0)$, then e_2 goes into the first place of b' and so $e_2 b' = 0$. The point is that the crystal embedding theorem actually shows that $B(\infty)$ is a subcrystal of $b_\infty \otimes B_1 \otimes B_2 \otimes B_1$, which is slightly different from $B_1 \otimes B_2 \otimes B_1$. Instead of carrying b_∞ in the left hand factor it is enough to put $b_2(0)$ on the left. In general if $\dim \mathfrak{g}_A < \infty$, we may just take $|\Delta^+|$ terms in J determined as we shall see by any reduced decomposition of the longest element w_0 of the Weyl group W) as long as we carry b_∞ in the extreme left hand side. Alternatively one can replace b_∞ by $\bigotimes_{i \in I} b_i(0)$, with the product taken in any order. (This was not possible in the original Kashiwara theory since he took the elementary crystals to have infinite extent in both directions.)

3 Further Properties of $B(\infty)$

3.1 Character Formulae

3.1.1 Surprisingly it is extremely difficult to calculate $\operatorname{ch} B(\infty)$ especially when one considers that $B(\infty)$ is supposed to represent the dual of a Verma module of highest weight zero, and that the character formula of the latter is easily shown to be a product over the negative roots of the Lie algebra. One consequence of this product formula is nevertheless easy to prove directly for $\operatorname{ch} B(\infty)$ and we start with this. Recall the translated action of W on P defined by extending the formula

$$s_i.\lambda := s_i\lambda - \alpha_i, \quad \forall i \in I,$$

to all $w \in W$.

Lemma *For all $w \in W$, one has*

$$w.\operatorname{ch} B(\infty) = (-1)^{l(w)} \operatorname{ch} B(\infty).$$

Proof Recall $B^i : i \in I$ defined in 2.5.7. It is clear from the construction in 2.5.7 that

$$B^i = \varinjlim_{\lambda \in P^+ | \alpha_i^\vee(\lambda)=0} \big(T_{-\lambda}\big(B(\lambda)\big)\big)$$

where $T_{-\lambda}$ signifies translating weights by $-\lambda$. Since $B(\lambda)$ is a normal crystal, it is W invariant (see 2.2.4). Yet $\alpha_i^\vee(\lambda) = 0$, so $\operatorname{ch} T_{-\lambda} B(\lambda)$ is s_i invariant and hence so is $\operatorname{ch}\ B^i$.

By Lemma 2.5.11 one can write $B(\infty)$ as a Cartesian product

$$B(\infty) = B^i \times B_i,$$

where moreover $\operatorname{wt}(b_1, b_2) = \operatorname{wt} b_1 + \operatorname{wt} b_2$, $\forall b_1 \in B^i$, $b_2 \in B_i$. Hence $\operatorname{ch} B(\infty) = \operatorname{ch} B^i \operatorname{ch} B_i$. On the other hand it is clear that

$$\operatorname{ch} B_i = \big(1 - e^{-\alpha_i}\big)^{-1},$$

which satisfies

$$s_i.\operatorname{ch} B_i = -\operatorname{ch} B_i.$$

Combining the above observations the conclusion of the lemma results. □

3.1.2 Through a clever use of the path model, Littelmann obtained a "combinatorial character formula" for $B(\lambda) : \lambda \in P^+$ defining the family $\mathscr{F}$. We simply quote his result, an exposition of the proof of which we gave in [16, 16.11, 16.12]

Theorem (Littelmann)

$$\operatorname{ch} B(\lambda) = \frac{\sum_{w\in W}(-1)^{l(w)}e^{w.\lambda}}{\sum_{w\in W}(-1)^{l(w)}e^{w.0}}.$$

3.1.3 From 3.1.2 and the presentation of $B(\infty)$ as a direct limit (2.5.6) one obtains the

Corollary

$$\operatorname{ch} B(\infty) = \frac{1}{\sum_{w\in w}(-1)^{l(w)}e^{w.0}}.$$

3.1.4 Through the representation theory of finite dimensional simple highest weight modules for semisimple Lie algebras, Weyl first obtained an expression for the denominator occurring in 3.1.2 as a product over the negative roots. This was generalized by Kac [20, Chap. 7] using ideas of Bernstein–Gelfand–Gelfand to $\mathfrak{g}_A$, when A is symmetrizable, a key point being the use of the Casimir invariant. It is much more difficult to prove the Weyl denominator formula in general, though this was achieved independently by Kumar [27] and Mathieu [32]. Combined with 3.1.2 we obtain the formula for $\operatorname{ch} B(\infty)$ described in 2.4.3.

3.1.5 Littelmann's combinatorial character formula carries over to the Borcherds case [17, Sect. 9]. However, except when A is symmetrizable, it is not known if the analogue of the Weyl denominator formula holds.

3.2 Highest Weight Elements

3.2.1 Let $\mathbb{F}_A = \{B(\lambda) : \lambda \in P^+\}$ denote the closed family described in 2.5. Fix $\lambda, \mu \in P^+$ and set $B^\lambda(\mu) = \{b \in B(\mu) | e_i^{\alpha_i^\vee(\lambda)+1} b = 0,\ \forall i \in I\}$.

Lemma *One has*

$$\big(B(\lambda)\otimes B(\mu)\big)^{\mathscr{E}} = \big\{b_\lambda \otimes b | b \in B^\lambda(\mu)\big\}.$$

Proof Suppose $e_i(b_1 \otimes b_2) = 0$. If e_i goes in the right hand factor, then $e_i b_2 = 0$ so $\varepsilon_i(b_2) = 0$, by upper normality of $B(\mu)$. Yet $\varphi_i(b_1) \geq 0$, by the lower normality of $B(\lambda)$, which by the hypothesis contradicts the tensor product rule. Thus we can assume that e_i always enters the left hand factor. This forces $b_1 = b_\lambda$ and $\varepsilon_i(b_2) \leq \varphi_i(b_\lambda) = \alpha_i^\vee(\lambda),\ \forall i \in I$, that is $b_2 \in B^\lambda(\mu)$. Upper normality of $B(\mu)$ concludes the proof. □

Remark Since the proof only requires of $B(\mu)$ to be upper normal, it holds with $B(\mu)$ replaced by $B(\infty)$.

3.2.2 We remark that in Littelmann's path model, the paths lying in $B^\lambda(\mu)$ are exactly those lying entirely in the closure of the dominant chamber.

Take $b \in B^\lambda(\mu)$ and $\nu := \lambda + \mathrm{wt}\, b \in P^+$. It is not obvious that the crystal generated by b is a highest weight crystal or that the resulting crystal is isomorphic to $B(\nu)$. Already the case $b = b_\mu$ is quite difficult. The general case established by Littelmann [28] is significantly more complicated. It leads to the following decomposition theorem for the tensor product of elements of $\mathbb{F}_A$ valid in the not necessarily symmetrizable case. Kashiwara had obtained this result in the symmetrizable case by taking a $q \to 0$ limit.

Theorem *Fix $\lambda, \mu \in P^+$. One has the crystal decomposition*

$$B(\lambda) \otimes B(\mu) = \coprod_{b \in B^\lambda(\mu)} B(\lambda + \mathrm{wt}\, b).$$

Remark One might compare this to the corresponding Jordan-Holder series of tensor products in the family $\{V(\lambda) | \lambda \in P^+\}$ of simple highest weight $U(\mathfrak{g}_A)$ modules, known to be a direct sum in the symmetrizable case. Choose $\lambda, \mu, \nu \in P^+$ and set

$$V^\mu(\nu)_{\lambda-\mu} := \left\{ a \in V(\nu)_{\lambda-\mu} | x_{\alpha_i}^{\alpha_i^\vee(\mu)+1} a = 0,\ \forall i \in I \right\}.$$

Observe that $v_\lambda \otimes v_{-\mu}$ is a cyclic vector for $V(\lambda) \otimes V(\mu)^*$ under the diagonal action. Thus the map $\varphi \mapsto \varphi(v_\lambda \otimes v_{-\mu})$ of $\mathrm{Hom}_{\mathfrak{g}}(V(\lambda) \otimes V(\mu)^*, V(\nu))$ into $V(\nu)$ is injective. It may be shown to have image $V^\mu(\nu)_{\mu-\nu}$ irrespective of whether A is symmetrizable. Various authors have claims on this result. Zelobenko and Kostant for $\mathfrak{g}_A$ semisimple and Mathieu for the general case. A proof can be found in [11, 6.3.10] though as it is presented it is not obvious that it holds in the generality claimed here.

3.2.3 The result in 3.2.1 gives [13, Sect. 5] an interpretation of the Kashiwara involution which at first seemed rather surprising. Recall (2.5.9) that we have a strict embedding $B(\lambda) \hookrightarrow B(\infty) \otimes S(\lambda),\ \forall \lambda \in P^+$. Recall 3.2.1.

Proposition *Take $b \in B(\infty)$. Then for all $\lambda \in P^+$, one has $b \in B^\lambda(\infty)$ if and only if $b^\star \otimes s_\lambda \in B(\lambda)$.*

Proof Take $b \in B(\infty)$. By 2.5.7 there exist $n \in \mathbb{N}, i_1, \dots, i_n \in J, m_1, \dots, m_n \in \mathbb{N}$ such that

$$b = b_\infty \otimes b_{i_n}(-m_n) \otimes b_{i_{n-1}}(-m_{n-1}) \otimes \cdots \otimes b_{i_1}(-m_1) \in B(\infty) \otimes B_{i_n} \otimes \cdots \otimes B_{i_1}.$$

Through the Kashiwara function one checks that

$$\varepsilon_i(b) = \max_{s | i_s = i} \left\{ 0, m_s + \sum_{t > s} \alpha_i^\vee(\alpha_{i_t}) m_t \right\}.$$

Thus by 3.2.1 we conclude that

$$b \in B^\lambda(\infty) \quad \Leftrightarrow \quad \alpha_{i_s}^\vee(\lambda) \geq m_s + \sum_{t>s} \alpha_{i_s}^\vee(\alpha_{i_t}) m_t, \quad \forall s = 1, \ldots, n.$$

Through the crystal structure on $B(\infty)$ defined by the starred operators we may write every element of $B(\infty)$ uniquely in the form

$$b = f_{i_1}^{\star m_1} f_{i_2}^{\star m_2} \cdots f_{i_n}^{\star m_n} b_\infty,$$

with

$$e_{i_j}^\star f_{i_{j+1}}^{\star m_{j+1}} \cdots f_{i_n}^{\star m_n} b_\infty = 0. \tag{$*$}$$

Applying 2.5.14 we obtain

$$b = b_\infty \otimes b_{i_n}(-m_n) \otimes \cdots \otimes b_{i_1}(-m_1),$$

as above. (Of course the m_j cannot be arbitrary as $(*)$ must be satisfied.)

Set $F_s = f_{i_s}^{m_s} \cdots f_{i_n}^{m_n} : s = 1, 2, \ldots, n$, with $F_{n+1} = 1$. By definition of $\star$ we have $b^\star = F_1 b_\infty$. Now by 2.5.9 one has $b^\star \otimes s_\lambda \in B(\lambda)$, if and only if $F_1(b_\infty \otimes s_\lambda) = F_1 b_\infty \otimes s_\lambda$, that is the f_j all enter the first factor. Since $\varepsilon_i(s_\lambda) = -\alpha_i^\vee(\lambda),\ \forall i \in I$, the above holds if and only if

$$\varphi_{i_s}(F_{s+1} b_\infty) \geq -\alpha_{i_s}^\vee(\lambda) + m_s, \quad \forall s = 1, 2, \ldots, n.$$

On the other hand $(*)$ translates to give $e_{i_s} F_{s+1} b_\infty = 0$. Since $B(\infty)$ is upper normal, this is equivalent to $\varepsilon_{i_s}(F_{s+1} b_\infty) = 0$.

Thus $\varphi_{i_s}(F_{s+1} b_\infty) = \alpha_{i_s}^\vee(F_{s+1} b_\infty) = -\sum_{t>s} \alpha_{i_s}^\vee(\alpha_{i_t}) m_t$, for all $s = 1, 2, \ldots, n$. These are exactly the conditions that $b \in B^\lambda(\infty)$. This proves the proposition. □

3.2.4 Since $S(\lambda)$ is a one element crystal we may identify $B(\lambda)$ with a subset of $B(\infty)$ (with the understanding that weights are translated). By this slight abuse of notation we obtain the

Corollary $B^\lambda(\mu) = B(\mu) \cap B(\lambda)^\star$.

3.2.5 The above may be compared to the following result for $U(\mathfrak{g}_A)$ modules. Take a triangular decomposition $\mathfrak{g}_A = \mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, where $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) is spanned by the positive (resp. negative) root vectors. Set $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$ which is a Borel subalgebra.

The dual $\delta M(0)$ of a Verma module of highest weight 0 is defined to be $\mathfrak{h}$ locally finite elements of $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{b})^*$, given a left module structure through the Chevalley antiautomorphism κ. The latter may be identified with $S(\mathfrak{n}^-)$ as a $U(\mathfrak{g})$-algebra, that is as an algebra in which $\mathfrak{g}$ acts by derivations.

Now $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{b}$ identifies with $U(\mathfrak{n}^-)$ and so the left action of $U(\mathfrak{g})$ on $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{b}$ restricts to a left action of $\mathfrak{n}^-$ which is just left multiplication by $\mathfrak{n}^-$

and defines a left $\mathfrak{n} = \kappa(\mathfrak{n}^-)$ structure on $\delta M(0)$, which is the restriction of the left $\mathfrak{g}$ action. Right multiplication of $U(\mathfrak{n}^-)$ by $\mathfrak{n}^-$ gives a right action of $\mathfrak{n}$ on $\delta M(0)$ commuting with the left action of $\mathfrak{n}^-$ (but not with the left action of $\mathfrak{g}$).

Given $\mu \in P^+$, let $\mathbb{C}_{-\mu}$ denote the one-dimensional $\mathfrak{b}$ module of weight $-\mu$. Analogously to 2.5.9, one may show that $\delta M(0)|_{U(\mathfrak{b})}$ admits a unique $U(\mathfrak{b})$ submodule isomorphic to $V(\mu)|_{U(\mathfrak{b})} \otimes \mathbb{C}_{-\mu}$ and moreover the latter is given through the right action of $U(\mathfrak{n})$ via the formula [8, Theorem 2.6],

$$\left\{a \in \delta M(0) | a x_{\alpha_i}^{\alpha_i^\vee(\mu)+1} = 0,\ \forall i \in I\right\}.$$

Now the Kashiwara involution $\star$ was obtained (see for example [11, Sect. 6.1]) via the $q \to 0$ limit and the seemingly trivial antiautomorphism defined as the identity on the generators $x_{\alpha_i} : i \in I$ in the quantization $U_q(\mathfrak{n}^-)$ of $U(\mathfrak{n}^-)$. This interchanges left and right action in $U(\mathfrak{n}^-)$ and so interchanges $V(\mu)|_{U(\mathfrak{b})} \otimes \mathbb{C}_{-\mu}$ with $V(\mu)^\star \otimes \mathbb{C}_{-\mu} := \{a \in \delta M(0) | x_{\alpha_i}^{\alpha_i^\vee(\mu)+1} a = 0,\ \forall i \in I\}$. On the other hand by the remark in 3.2.2, $\mathrm{Hom}_{\mathfrak{g}}(V(\lambda) \otimes V(\mu)^*, V(\nu))$ identifies with $((V(\nu) \otimes \mathbb{C}_{-\nu}) \cap (V(\mu)^\star \otimes \mathbb{C}_{-\mu}))_{\lambda-\mu-\nu}$. Making the cyclic permutation $\lambda \to \nu \to \mu$, this is equivalent to

$$\dim \mathrm{Hom}_{\mathfrak{g}}\big(V(\nu), V(\lambda) \otimes V(\mu)\big) = \dim\big((V(\lambda)^\star \otimes \mathbb{C}_{-\lambda}) \cap \big(V(\mu) \otimes \mathbb{C}_{-\mu}\big)\big)_{\nu-\lambda-\mu}.$$

Whereas by 3.2.4, the number of copies of $B(\nu)$ in $B(\lambda) \otimes B(\mu)$ is exactly

$$\mathrm{card}\big\{b \in B(\infty)_{\nu-\lambda-\mu} | b^\star \otimes s_\lambda \in B(\lambda),\ b \otimes s_\mu \in B(\mu)\big\}.$$

3.2.6 Observe that B_J is an additive semigroup with respect to component-wise addition. One can ask if $B_J(\infty)$ is a subsemigroup. This question will be discussed further in 3.6.1–3.6.5 using the method of Nakashima and Zelevinsky [33]. However they do not obtain a complete answer since a positivity hypothesis still has to be verified. For the moment we mention another possible approach.

Following the conclusion resulting from 3.2.3 $(*)$, it is enough to show that $e_i^\star b = e_i^\star b' = 0$ implies $e_i^\star(b+b') = 0$. Here the sum refers to the additive structure on B_J. Since $B(\infty)$ is upper normal (by construction) it is enough to show that $\varepsilon_i^\star(b) = \varepsilon_i^\star(b') = 0$ implies $\varepsilon_i^\star(b+b') = 0$. Now suppose that there exists on B_J an analogue of the Kashiwara function $b \mapsto r_i^{\star k}(b)$, which is additive *with respect to the additive structure on* B_J such that $\varepsilon_i^\star(b) = \max_k(r_i^{\star k}(b))$. Assume further (as in the case of the Kashiwara function) that $r_i^{\star k}(b) = 0$, for all $k \in \mathbb{N}^+$ sufficiently large. Then $\varepsilon_i^\star(b) = 0$ (resp. $\varepsilon_i^\star(b') = 0$) implies $r_i^{\star k}(b) \le 0$ (resp. $r_i^{\star k}(b') \le 0$), for all $k \in \mathbb{N}^+$. Then $r_i^{\star k}(b+b') \le 0$, for all $k \in \mathbb{N}^+$, forcing through the second property that $\varepsilon_i^\star(b+b') = 0$, as required.

A further anticipated property of $B(\infty)$ which we failed to prove can be expressed through the following question.

Suppose

$$b = b_\infty \otimes b_{i_n}(-m_n) \otimes \cdots \otimes b_{i_1}(-m_1) \in B(\infty),$$

with $m_n > 0$. Then does

$$b = b_\infty \otimes b_{i_n}\big(-(m_n - 1)\big) \otimes \cdots \otimes b_{i_1}(-m_1) \in B(\infty)$$

hold?

3.3 Braid Relations

3.3.1 For each $i \in I$, set $\mathscr{E}_i = \bigcup_{n\in\mathbb{N}} e_i^n$, $\mathscr{F}_i = \bigcup_{n\in\mathbb{N}} f_i^n$, where by convention $e_i^0 = f_i^0 = 1$.

Given $w \in W$, set $r = l(w)$ and fix a reduced decomposition $\underline{w} = s_{i_r}s_{i_{r-1}}\cdots s_{i_1}$ of w. Set $\mathscr{E}_{\underline{w}} = \mathscr{E}_{i_r}\mathscr{E}_{i_{r-r}}\cdots\mathscr{E}_{i_1}$, $\mathscr{F}_{\underline{w}} = \mathscr{F}_{i_r}\mathscr{F}_{i_{r-1}}\cdots\mathscr{F}_{i_1}$.

Now let B be a crystal and take $b \in B$. One can ask if $\mathscr{E}_{\underline{w}}b$ and $\mathscr{F}_{\underline{w}}b$ depend only on w. One can hardly expect this to be true in an arbitrary crystal. However we show that this does hold for crystals in the family $\mathbb{F}_A$ defined in 2.5 and by taking the limit for $B(\infty)$.

3.3.2 Suppose we replace e_i (resp. f_i) by the corresponding root vector x_{α_i} (resp. $x_{-\alpha_i}$). Since in the enveloping algebra $U(\mathfrak{g})$ we have linear structure at our disposal, the natural analogues of $\mathscr{E}_i$, $\mathscr{F}_i$ are $\mathbb{C}[x_{\alpha_i}]$, $\mathbb{C}[x_{-\alpha_i}]$ respectively. Then we may define analogues $U_{\underline{w}}$ (resp. $U^-_{\underline{w}}$) of $\mathscr{E}_{\underline{w}}$ (resp. $\mathscr{F}_{\underline{w}}$) as products in $U(\mathfrak{n})$ (resp. $U(\mathfrak{n}^-)$) of $\mathbb{C}[x_{\alpha_i}]$, $\mathbb{C}[x_{-\alpha_i}]$ respecting the order defined by $\underline{w}$. In this there are various ways to show that these subspaces depend only on w. For example, for each $\lambda \in P^+$, the subspace $U^-_{\underline{w}}v_\lambda \subset V(\lambda)$ is just the $U(\mathfrak{n})$ module generated by $v_{w\lambda}$ [11, Sect. 4.4]. This approach does not quite work in the crystal framework. Indeed by 2.2.4, there exists a unique element $b_{w\lambda} \in B(\lambda)$ of weight $w\lambda$, trivially $b_{w\lambda} \in \mathscr{F}_{\underline{w}}b_\lambda$ and via the use of $B(\infty)$ one may show that $\mathscr{F}_{\underline{w}}b_\lambda$ is $\mathscr{E}$ stable. Yet it is false that the resulting inclusion $\mathscr{E}b_{w\lambda} \subset \mathscr{F}_{\underline{w}}b_\lambda$ is an equality. (A simple example occurs in the adjoint representation of $\mathfrak{sl}(3)$.)

Nevertheless one can show that $\mathscr{F}_{\underline{w}}b_\lambda$ depends only on w. In the Kashiwara theory, this results from the fact that $U^-_w v_\lambda$ admits a basis formed from the subset of the global basis defined by $\mathscr{F}_{\underline{w}}b_\lambda$. Naturally this is extremely complicated and also requires A to be symmetrizable.

Littelmann gives a purely combinatorial proof that $\mathscr{F}_{\underline{w}}b_\lambda$ depends only on w. Indeed in his model, the subset $\mathscr{F}_{\underline{w}}b_\lambda$ is given by piecewise linear paths in $\mathbb{Q}P$ described by all Bruhat sequences beginning at the identity and ending in w, satisfying an integrability condition depending on λ. Here we admit this result (for an exposition—see [16, Sect. 16]) and show it implies that $\mathscr{F}_{\underline{w}}b : b \in B(\lambda)$ also only depends on w by an argument which is also due to Littelmann, but which does not involve paths but only the tensor product rule.

3.3.3 Given a monomial $f \in \mathscr{F}$, an element $f' \in \mathscr{F}$ in which some of the factors in f are deleted is called a submonomial.

The proof of the assertion in 3.3.1 results from Theorem 3.2.2 and the following rather easy

Lemma *Fix $\lambda, \mu \in P^+$, $b \in B(\lambda)$, $b' \in B(\mu)$.*

(i) *For each $f' \in \mathscr{F}_{\underline{w}}$, there exists submonomials $f'', f \in \mathscr{F}_{\underline{w}}$ of f' such that*

$$f'(b \otimes b') = f''b \otimes fb'.$$

(ii) *For each $f \in \mathscr{F}_{\underline{w}}$, there exists $f', f'' \in \mathscr{F}_{\underline{w}}$ with f'', f submonomials of f' such that*

$$f'(b \otimes b') = f''b \otimes fb'.$$

Proof (i) is a completely obvious consequence of the tensor product rule. For (ii), set $m = \max\{0, \varphi_i(b) - \varepsilon_i(b')\}$. Then by the tensor product rule $f_i^{m+t}(b \otimes b') = f_i^m(b) \otimes f_i^t b'$, $\forall t \in \mathbb{N}$. Thus by taking an appropriate power of f_i, any power of f_i may be brought into the second factor. Then (ii) follows by induction on $l(w)$. □

3.3.4 Take $\mu \in P^+$ and $b \in B(\mu)$. By 3.2.1, there exists $\lambda \in P^+$ such that $b_\lambda \otimes b \in (B(\lambda) \otimes B(\mu))^{\mathscr{E}}$. Set $\nu = \lambda + \mathrm{wt}\, b$. By 3.2.2, this element generates the highest weight crystal $B(\nu)$ and so by 3.3.2 one has

$$\mathscr{F}_{\underline{w}_1}(b_\lambda \otimes b) = \mathscr{F}_{\underline{w}_2}(b_\lambda \otimes b), \qquad (*)$$

for any two reduced decompositions $\underline{w}_1, \underline{w}_2$ of w.

Proposition *The subset $\mathscr{F}_{\underline{w}} b \subset B(\mu)$ depends only on w.*

Proof Given $f \in \mathscr{F}_{\underline{w}_1}$ choose f', f'' as in the conclusion 3.3.3 (ii). By $(*)$, there exists $\tilde{f}' \in \mathscr{F}_{\underline{w}_2}$ such that $f'(b_\lambda \otimes b) = \tilde{f}'(b_\lambda \otimes b)$ and let $\tilde{f}'', \tilde{f}$ be as in the conclusion of 3.3.3 (i). Then $f''b_\lambda \otimes fb = f'(b_\lambda \otimes b) = \tilde{f}'(b_\lambda \otimes b) = \tilde{f}''b_\lambda \otimes \tilde{f}b$, forcing $fb = \tilde{f}b$, and hence $\mathscr{F}_{\underline{w}_1} b \subset \mathscr{F}_{\underline{w}_2} b$. Interchanging $\underline{w}_1, \underline{w}_2$ gives the reverse inclusion. □

3.4 The String Property

3.4.1 The $U(\mathfrak{b})$ modules $F(w\lambda) := U(\mathfrak{n})v_{w\lambda}$ discussed in 3.3.2 were studied by Demazure partly because of their geometric significance being the algebra of global sections of an invertible sheaf $\mathcal{L}_\lambda$ defined by $\lambda \in P^+$ on the Schubert variety defined by w. It is known for example that for λ regular their inclusion relations are given by Bruhat order on W. Demazure calculated [5, 6] their characters based on a lemma of Verma implying a "string property" with respect to the $\mathfrak{sl}(2)$ subalgebras defined by the simple roots. As noted later by Kac the lemma is false and so is the string property. However it was shown by Kashiwara (and later by Littelmann in his path model) that the string property does hold for the corresponding "crystals" $\mathscr{F}_{\underline{w}} b_\lambda$. This we shall prove by the Kashiwara method profiting from the fact that the main technical tool is the Kashiwara involution, which we have shown to be defined in the general, not necessarily symmetrizable, case.

3.4.2 Before going further we remark that one may try to compute the characters of the homology spaces $H_*(\mathfrak{n}, F(w\lambda))$ which are $\mathfrak{h}$ modules. This is more detailed information than the $\operatorname{ch} F(w\lambda)$. In [9, 10] one may find information on these homology spaces which gave the first correct proof of the Demazure character formula for large λ and in characteristic zero. Later a correct proof for all λ was given by Anderson [1] using the Steinberg module. Andersen's proof is also valid in good characteristic. The Kashiwara theory [23] gives a proof which is characteristic free, entirely elementary but far from simple. One might hope it would also describe the homology spaces $H_*(\mathfrak{n}, F(w\lambda))$ but so far no success has been reported. Kumar [25] has translated the calculation of $H_*(\mathfrak{n}, F(w\lambda))$ to a problem in sheaf cohomology.

3.4.3 Adopt the notation of 3.3.1 and set $B_w(\infty) = \mathscr{F}_{\underline{w}} b_\infty$, which as noted in 3.3.2 is independent of the reduced decomposition $\underline{w} = s_{i_r} s_{i_{r-1}} \cdots s_{i_1}$ of w. Notice we can also assume that J is chosen to be $\dots i_r, i_{r-1}, \dots, i_1$ and consequently $B_w(\infty) \subset b_\infty \otimes B_{i_r} \otimes B_{i_{r-1}} \otimes \cdots \otimes B_{i_1}$. From this one easily obtains the

Lemma *For all $w \in W$, $B_w(\infty)$ is $\mathscr{E}$ stable.*

3.4.4 Suppose $w \in W$ and $i \in I$ satisfy $s_i w < w$. Without loss of generality we can assume that $i = i_r$ in the notation of 3.4.

Lemma *If $b \in B(\infty)$, satisfies $e_i b = 0$ and $f_i^k b \in B_w(\infty)$ for some $k \in \mathbb{N}$, then $b \in B_{s_i w}(\infty)$.*

Proof By the previous lemma $b = e_i^k f_i^k b \in B_w(\infty)$. Thus we can write $b = b_\infty \otimes b_{i_r}(-m_r) \otimes \cdots \otimes b_{i_1}(-m_1)$. Since $i = i_r$, one has $0 = \varepsilon_i(b) \geq \varepsilon_i(b_i(-m_i)) = m_i$. Thus $b \in B_{s_i w}(\infty)$. □

3.4.5 We make the following preliminary to establishing the string property. The argument is due to Kashiwara [23].

Proposition *Fix $w \in W$ and $i \in I$. Suppose $b \in B(\infty)$ satisfies $e_i^\star b = 0$, $f_i^\star b \in B_w(\infty)$. Then $f_i^{\star k} b \in B_w(\infty)$, for all $k \in \mathbb{N}$.*

Proof The proof is by induction on $l(w)$. If $l(w) = 0$, the hypothesis $f_i^\star b \in B_w(\infty) = \{b_\infty\}$, cannot be satisfied, so there is nothing to prove.

Suppose $l(w) > 0$. Then there exists $j \in I$ such that $l(s_j w) < l(w)$.

Suppose $j \neq i$. Then the assertion is an immediate consequence of 2.5.15. Indeed we can write $b = f_j^t b'$, with $e_j b' = 0$. Then $e_j f_i^\star b' = 0$, whilst $f_j^t f_i^\star b' = f_i^\star b \in B_w(\infty)$. Thus $f_i^\star b' \in B_{s_j w}(\infty)$, by 3.4.4. Again $0 = e_i^\star b = f_j^t e_i^\star b'$, so $e_i^\star b' = 0$, by the injectivity of f_j. Then $f_i^{\star k} b' \in B_{s_j w}(\infty)$, for all $k \in \mathbb{N}$, by the induction hypothesis. Then $f_i^{\star k} b = f_j^t f_i^{\star k} b' \in B_w(\infty)$, for all $k \in \mathbb{N}$, as required.

Suppose $j = i$. Since $e_i^\star b = 0$, we can write $\psi_i(b) = b \otimes b_i(0)$. From now one we omit the injection ψ_i for ease of notation. Then $f_i^\star b = b \otimes b_i(-1)$.

Suppose $\varphi_i(b) \leq \varepsilon_i(b_i(-1)) = 1$. Then powers of f_i enter in the right hand factor. Thus $f_i^{k-1} f_i^\star b = b \otimes b_i(-k) = f_i^{\star k} b$. By the hypothesis $f_i^\star b \in B_w(\infty)$ and since $s_i w < w$, $B_w(\infty)$ is $\mathscr{F}_i$ stable, hence $f_i^{\star k} b \in B_w(\infty)$, for all $k \in \mathbb{N}$.

Suppose $\varphi_i(b) > \varepsilon_i(b_i(-1)) = 1$. Write $b = f_i^t b'$ with $e_i b' = 0$. Then $\varphi_i(b') = \varphi_i(b) + t > t + 1$. Consequently $e_i(b' \otimes b_i(-1)) = 0$, whilst $f_i^t(b' \otimes b_i(-1)) = b \otimes b_i(-1) = f_i^\star b \in B_w(\infty)$. Consequently $b' \otimes b_i(-1) \in B_{s_i w}(\infty)$. Yet $e_i^\star b' = 0$ and $f_i^\star b' = b' \otimes b_i(-1) \in B_{s_i w}(\infty)$, so by the induction hypothesis $f_i^{\star k} b' \in B_{s_i w}(\infty)$, for all $k \in \mathbb{N}$. Then

$$f_i^{\star k} b = f_i^{\star k} f_i^t b' = f_i^{\star k} f_i^t \big(b' \otimes b_i(0)\big) = f_i^{\star k} \big(f_i^t b' \otimes b_i(0)\big) = f_i^t b' \otimes f_i^k b_i(0).$$

Apply powers of e_i to this expression. Through the last part of Theorem 2.5.7 as noted in the proof of 2.5.11, e_i^u will enter the right hand factor for some $u : 0 \leq u \leq k$ and from then on powers of e_i will enter the left hand factor. Thus we can write the expression as $f_i^{t+u}(b' \otimes f_i^{k-u} b_i(0))$. The latter equals $f_i^{t+u}(f_i^{\star(k-u)} b') \in f_i^{t+u} B_{s_i w}(\infty) \subset B_w(\infty)$. This gives the required result. □

3.4.6 Suppose $b \in B_w(\infty)$. Through the presentation given in 3.4.3, we can write $b = f_{i_r}^{m_r} \cdots f_{i_1}^{m_1} b_\infty$ with $e_{i_s} f_{i_{s-1}}^{m_{s-1}} \cdots f_{i_1}^{m_1} b_\infty = 0$, for all $s = 1, 2, \ldots, r$. Then $b^\star \in B_{w^{-1}}(\infty)$ by 2.5.14. This leads to the following

Theorem *Fix $w \in W$ and $i \in I$. If $b \in B(\infty)$ satisfies $f_i b \in B_w(\infty)$, then $f_i^k b \in B_w(\infty)$, for all $k \in \mathbb{N}$.*

Proof By 3.4.3 we can assume $e_i b = 0$. By definition of $\star$ one has $(e_i b)^\star = e_i^\star b^\star$ and $(f_i^k b)^\star = f_i^{\star k} b^\star$. Thus the hypotheses of 3.4.5 are satisfied with replacing b by $b^\star$ and w by w^{-1}. From its conclusion $f_i^{\star k} b^\star \in B_{w^{-1}}(\infty)$, for all $k \in \mathbb{N}$ and applying $\star$ we obtain $f_i^k b \in B_w(\infty)$, as required. □

3.4.7 Through the strict embedding $B(\lambda) \hookrightarrow B(\infty) \otimes S_\lambda$ obtained in 2.5.9 we obtain the

Corollary *Fix $w \in W$, $i \in I$, $\lambda \in P^+$. If $f_i b \in B_w(\lambda)$, then $f_i^k b \in B_w(\lambda)$, for all $k \in \mathbb{N}$.*

3.4.8 The above result is just the string property of $B_w(\lambda)$, which Demazure had attempted to prove for the $U(\mathfrak{b})$ module $F(w\lambda)$. Its interest lies in the fact that it leads inductively to the formula for $\operatorname{ch} B_w(\lambda)$ described below.

3.4.9 Fix $i \in I$. For each $\lambda \in P$, set

$$\Delta_i e^\lambda = \big(1 - e^{-\alpha_i}\big)^{-1} \big(e^\lambda - e^{s_i \lambda - \alpha_i}\big)$$

which one may check belongs to $\mathbb{Z}P$. Hence Δ_i is a $\mathbb{Z}$ linear endomorphism of $\mathbb{Z}P$, called the Demazure operator.

Call $S \subset B(\lambda)$ an i-string if $S = \mathscr{F}_i s$, for some $s \in B(\lambda)$ satisfying $e_i s = 0$. Through the normality of $B(\lambda)$, one checks that

$$\operatorname{ch} S = \Delta_i e^{wt(s)}.$$

On the other hand one may also check that $\Delta_i^2 = \Delta_i$. Thus

$$\operatorname{ch} S = \Delta_i \operatorname{ch} S.$$

These two marvellous formulae combined with the conclusion of 3.4.7 give the following

Lemma *Fix $w \in W$, $i \in I$ such that $s_i w > w$. Then for all $\lambda \in P^+$, one has*

$$\operatorname{ch} B_{s_i w}(\lambda) = \Delta_i \operatorname{ch} B_w(\lambda).$$

Proof It remains to observe that the condition $s_i w > w$, implies that $B_{s_i w}(\lambda)$ is $\mathscr{F}_i$ stable, thus a disjoint union of i-strings each of which either already lie in $B_w(\lambda)$ or whose intersection with $B_w(\lambda)$ consists of the unique element annihilated by e_i. □

3.4.10 This result already has an interesting corollary resulting from the independence of $\mathscr{F}_{\underline{w}} b_\lambda = B(w\lambda)$ on the reduced decomposition $\underline{w}$ of w.

Corollary *The $\Delta_i : i \in I$ satisfy the Coxeter relations.*

Remark Of course this may also be checked by explicit computation; but be warned that Demazure [6] has described the calculation in G_2 as being "épouvantable".

3.4.11 Suppose $|W| < \infty$. Then it admits a unique longest element w_0. It follows from 3.4.10 and the idempotence of the Δ_i, that $\Delta_i \Delta_{w_0} = \Delta_{w_0}$, for all $i \in I$.

Given $a \in \mathbb{Z}P$ one easily checks that $\Delta_i a = a$, if and only if $s_i a = a$. In particular we conclude that $\operatorname{ch} B(\lambda) = \Delta_{w_0} e^\lambda$ is W invariant. From this and the particular form of Δ_{w_0}, Demazure [6] noted that $\Delta_{w_0} e^\lambda$ is just the Weyl character formula. Unfortunately the corresponding result and argument fails in the case that W is infinite.

3.4.12 The algebra with generators $T_i : i \in I$ and relations $T_i^2 = T_i : i \in I$, together with the Coxeter relations is called the singular Hecke algebra. So far we have met two examples. The first is given by the $\mathscr{F}_i : i \in I$ acting on $B(\infty)$, the second by the Demazure operators $\Delta_i : i \in I$ acting on $\mathbb{Z}P$. One may remark that as a consequence any monomial $T_{i_1} T_{i_2} \cdots T_{i_n}$ can be written as T_y where $T_y = T_{j_1} T_{j_2} \cdots T_{j_m}$ with $s_{j_1} s_{j_2} \cdots s_{j_m}$ a reduced decomposition of y.

3.5 Combinatorial Demazure Flags

3.5.1 Some 25 years ago, I suggested to my then Ph.D. student P. Polo that the category of Demazure modules may admit a tensor structure, namely that for all $\lambda, \mu \in P^+$, $y, w \in W$, the tensor product $F(y\lambda) \otimes F(w\mu)$ admits a Demazure flag, that is a $U(\mathfrak{b})$ filtration whose quotients are again Demazure modules. This turned out to be false; but it appears to hold if we take $y = Id$. Indeed for $\mathfrak{g}$ semisimple (equivalently if W is finite) Polo checked most cases. Then somewhat in a spirit of competition O. Mathieu proved this to be true for all $\mathfrak{g}$ semisimple and in all characteristics—see [21] for an exposition. The proof occupies an entire book! The method which depended on a universality property of Demazure modules broke down for arbitrary Kac–Moody Lie algebras. However I later [15] showed it still hold in all characteristics for any $\mathfrak{g}_A$ with A symmetric and simply-laced. This needed the Kashiwara $q \to 0$ limit of these modules over the quantized enveloping algebra (requiring A symmetrizable) together with a positivity property in the multiplication of canonical bases due to Lusztig requiring A simply-laced. An essential though elementary step in the proof was a corresponding decomposition for "Demazure crystals" which holds for all A. This result is described below.

3.5.2 For all $w \in W, \lambda \in P^+$, set $B_w^\lambda(\infty) = B^\lambda(\infty) \cap B_w(\infty)$. Given $b \in B_w^\lambda(\infty)$ set

$$\mathscr{F}_{w,b}^\lambda := \{f \in \mathscr{F} \mid f(b_\lambda \otimes b) \subset b_\lambda \otimes B_w(\infty)\}.$$

Lemma

$$b_\lambda \otimes B_w(\infty) = \coprod_{b \in B_w^\lambda(\infty)} \mathscr{F}_{w,b}^\lambda (b_\lambda \otimes b).$$

Proof The inclusion $\supset$ is by construction. Conversely given $b_\lambda \otimes b' \in B_w(\infty)$, then $\mathscr{E}(b_\lambda \otimes b') = b_\lambda \otimes \mathscr{E}b'$, which by 3.4.3 contains an element of the form $b_\lambda \otimes b : b \in B_w^\lambda(\infty)$ such that $b_\lambda \otimes b' \in \mathscr{F}(b_\lambda \otimes b)$. Thus $b_\lambda \otimes b' \in \mathscr{F}_{w,b}^\lambda(b_\lambda \otimes b)$ by definition of $\mathscr{F}_{w,b}^\lambda$. Finally the union is disjoint by Theorem 3.2.2. □

3.5.3 Of course 3.5.2 does not say too much as we have to be able to compute $\mathscr{F}_{w,b}^\lambda$. This is provided by the following.

Fix a reduced decomposition $\underline{w} = s_{i_1} s_{i_2} \cdots s_{i_n}$ of $w \in W$. Given $b \in B_w(\infty)$ we may write $b = b_\infty \otimes b_{i_1}(-m_1) \otimes \cdots \otimes b_{i_n}(-m_n)$ for some $n \in \mathbb{N}$, $m_j \in \mathbb{N}$, $j = 1, 2, \ldots, n$.

At this point it is worthwhile to mention that the Kashiwara function $r_i^k(b)$ is not so interesting for $i_k \neq i$, since it simply equals $-\infty$. Instead we consider the function $b \mapsto r_{i_j}^j(b)$ which always takes a finite value.

Given $b \in B_{\underline{w}}^\lambda(\infty)$, set

$$J_{\underline{w},b}^\lambda = \{j \in \{1, 2, \ldots, n\} \mid r_{i_j}^j(b) \leq \alpha_{i_j}^\vee(\lambda)\}.$$

In the monoid

$$\mathscr{F}_{\underline{w}} = \mathscr{F}_{i_1}\mathscr{F}_{i_2}\cdots\mathscr{F}_{i_n}$$

suppress the $\mathscr{F}_{i_j}$, for all $j \in J^{\lambda}_{\underline{w},b}$. By 3.4.12 the resulting set can be written uniquely in the form

$$\mathscr{F}_{y^{\lambda}_{\underline{w},b}}, \quad \text{for some } y^{\lambda}_{\underline{w},b} \in W.$$

We remark that it is a property of Bruhat order that $y^{\lambda}_{\underline{w},b} \leq w$.

Theorem *For all* $\lambda \in P^+$, $w \in W$, $b \in B^{\lambda}_{w}(\infty)$ *one has*

(i) $\mathscr{F}^{\lambda}_{w,b} \supset \mathscr{F}_{y^{\lambda}_{\underline{w},b}}$,

(ii) $\mathscr{F}^{\lambda}_{w,b}(b_\lambda \otimes b) = \mathscr{F}_{y^{\lambda}_{\underline{w},b}}(b_\lambda \otimes b)$,

(iii) $y^{\lambda}_{\underline{w},b}$ *is independent of the reduced decomposition* $\underline{w}$ *of* w.

Remark 1 Equality may fail in (i).

Remark 2 Set $y = y^{\lambda}_{\underline{w},b}$. Then the right hand side of (ii) equals $B_y(\nu)$, where $\nu = \lambda + \mathrm{wt}\, b$.

Remark 3 Combined with 3.5.2 we obtain a decomposition of $b_\lambda \otimes B_w(\infty)$ as a disjoint union of the "Demazure crystals" $B_y(\nu)$. In order to obtain the corresponding decomposition for $B_w(\mu)$, we simply restrict the elements of $B^{\lambda}_{w}(\infty)$ to lie in the first factor of the image of $B(\mu)$ in $B(\infty) \otimes S_\mu$.

3.5.4 A proof of Theorem 3.5.3 is given in [16, 19.3]. A similar result in the language of paths was proved by Littelmann [30]. A key point in the above proof is that if e_i (resp. f_i) enters at the j^{th} place in b, then further powers of e_i (resp. f_i) enter at the j^{th} place or to the left (resp. right) of the j^{th} place.

3.6 Additive Structure

3.6.1 Fix a sequence J as in 2.4.2. It is clear that B_J is a semigroup under component-wise addition. One can then ask if $B_J(\infty)$ is a subsemigroup. The (over-ambitious) goal here was to find a choice of J such that $B_J(\infty)$ is free, say on generators $\{b_k\}_{k\in K}$ running over some generally infinite set K. From this we would obtain $\mathrm{ch}\, B(\infty)$ in the required form, namely

$$\mathrm{ch}\, B(\infty) = \prod_{k\in K}\left(1 - e^{wt(b_k)}\right)^{-1}.$$

By comparison with the Weyl denominator formula, we would conclude that the $\mathrm{wt}(b_k) : k \in K$ are just the set of negative roots, giving the latter a purely combinatorial interpretation. Unfortunately it seems that freeness almost never holds, though it does hold in type A for a very particular choice of J, namely for $J = \{\alpha_1, \alpha_2, \alpha_1, \alpha_3, \alpha_2, \alpha_1, \dots\}$. Then the weights of the generators are in natural correspondence with the negative roots. However this case is very special. Notice in the above J is always an acceptable choice even if I is infinite (and countable).

3.6.2 We describe a result of Nakashima and Zelevinsky [33] which implies the required semigroup structure for $B_J(\infty)$, under a positivity hypothesis. It is disarmingly simple.

3.6.3 Fix a sequence J as above. View each $b \in B_J$ as a sequence $\underline{m} = (\dots, m_2, m_1) : m_i \in \mathbb{N}$ and hence as a free semigroup under component-wise addition. One can therefore speak of linear forms on B_J, that is functions $\varphi : B_J \to \mathbb{Z}$ such that $\varphi(\underline{m} + \underline{m}') = \varphi(\underline{m}) + \varphi(\underline{m}'),\ \forall \underline{m}, \underline{m}' \in B_J$. Recall 3.5.3 and observe that the modified Kashiwara function $r^k_{i_k}(\underline{m})$ which we recall is given by

$$r^k_{i_k}(\underline{m}) = m_k + \sum_{j>k} \alpha^\vee_{i_k}(\alpha_{i_j}) m_j$$

is a linear form on B_J. Write $r^k_{j_k}$ simply as r_k.

Again for all k there exists a co-ordinate form on B_J, x_k defined by $x_k(\underline{m}) = m_k$. Then every linear form φ on B_J can be written as an infinite sum

$$\varphi = \sum_{k \in \mathbb{N}^+} \varphi_k x_k.$$

In this notation

$$r_k = x_k + \sum_{j>k} \alpha^\vee_{i_k}(\alpha_{i_j}) x_j.$$

For all $k \in \mathbb{N}^+$, let k_+ be the smallest $j > k$ such that $i_j = i_k$ (which exists by the definition of J). If $i_j \neq i_k$, for all $j < k$ set $k_- = 0$. Otherwise let k_- be the largest integer $< k$ such that $i_j = i_k$.

Let t^+_k, t^-_k be the linear forms on B_J defined by

$$t^+_k := r_k - r_{k_+} = x_k + \sum_{k_+ > j > k} \alpha^\vee_{i_k}(\alpha_{i_j}) x_j + x_{k_+},$$

$$t^-_k := r_{k_-} - r_k = x_{k_-} + \sum_{k > j > k_-} \alpha^\vee_{i_k}(\alpha_{i_j}) x_j + x_k.$$

The key to the Nakashima–Zelevinski theory is the family of piecewise linear operators $S_k : k \in \mathbb{N}^+$ on linear forms given by

$$S_k(\varphi) = \begin{cases} \varphi - \varphi_k\, t^+_k & : \varphi_k \geq 0 \\ \varphi - \varphi_k\, t^-_k & : \varphi_k \leq 0. \end{cases}$$

Now one has

$$(t_k^+)_j = \begin{cases} 1 & : j \in \{k_+, k\} \\ \alpha_{i_k}^\vee(\alpha_{i_j}) & : k_+ > j > k \\ 0 & : \text{otherwise.} \end{cases}$$

$$(t_k^-)_j = \begin{cases} 1 & : j \in \{k, k_-\} \\ \alpha_{i_k}^\vee(\alpha_{i_j}) & : k > j > k_- \\ 0 & : \text{otherwise.} \end{cases}$$

In particular $(t_k^+)_k = (t_k^-)_k = 1$. Thus $(\varphi - \varphi_k\ t_k^+)_k = (\varphi - \varphi_k\ t_k^-)_k = 0$. Consequently $S_k^2(\varphi) = S_k(\varphi)$, for all linear forms φ on B_J.

3.6.4 Through the operators $S_k : k \in \mathbb{N}^+$ and the co-ordinate linear forms $x_l : l \in \mathbb{N}^+$ one may generate a family $\mathcal{L} = \{S_{k_1} S_{k_2} \cdots S_{k_t} x_{k_{t+1}} : t \in \mathbb{N}, k_i \in \mathbb{N}^+\}$ of linear forms on B_J.

The positivity hypothesis on $\mathcal{L}$ is that for all $\varphi \in \mathcal{L}$ one has $\varphi(k) \geq 0$, for all k for which $k_- = 0$. The following result is due to Nakashima and Zelevinsky [33, Theorem 3.5]

Theorem *Assume $\mathcal{L}$ satisfies the positivity hypothesis. Then*

$$B_J(\infty) = \left\{\underline{m} \in B_J | \varphi(\underline{m}) \geq 0, \textit{for all } \varphi \in \mathcal{L}\right\}.$$

Proof We first show that the right hand side $B'_J(\infty)$ is stable under $\mathscr{F}, \mathscr{E}$.

Take $i \in I$, $\underline{m} \in B'_J(\infty)$ and show that $f_i\underline{m} \in B'_J$. Choose $k \in \mathbb{N}^+$, so that f_i enters at the k^{th} place, thus sending m_k to m_{k+1}. In particular $\varphi(f_i\underline{m}) = \varphi(\underline{m}) + \varphi_k \geq \varphi_k$, since $\underline{m} \in B'_J(\infty)$. Thus to show $\varphi(f_i\underline{m}) \geq 0$, it is enough to consider the case when $\varphi_k < 0$. By the positivity hypothesis this means $k_- \geq 1$.

The condition that f_i enters in the k^{th} place implies by 2.3.2 (2) that $r_k(\underline{m}) > r_{k_-}(\underline{m})$. Thus $t_k^-(\underline{m}) = r_{k_-}(\underline{m}) - r_k(\underline{m}) \leq -1$. Consequently $\varphi(f_i\underline{m}) = \varphi(\underline{m}) + \varphi_k \geq \varphi(\underline{m}) - \varphi_k t_k^-(\underline{m}) = (S_k\varphi)(\underline{m}) \geq 0$, since $\underline{m} \in B'_J(\infty)$.

Next we show that $e_i\underline{m} \in B'_J \cup \{0\}$. Suppose that x_i enters in the k^{th} place. If $m_k = 0$, then $e_i\underline{m} = 0$, so suppose that $m_k \geq 1$. Then $\varphi(e_i\underline{m}) = \varphi(\underline{m}) - \varphi_k \geq -\varphi_k$, so it is enough to consider the case when $\varphi_k > 0$. By the condition that x_i enters in the k^{th} place one has $r_k(\underline{m}) > r_{k_+}(\underline{m})$, so then $t_k^+(\underline{m}) \geq 1$. Consequently

$$\varphi(e_i\underline{m}) = \varphi(\underline{m}) - \varphi_k \geq \varphi(\underline{m}) - \varphi_k t_k^+(\underline{m}) = (S_k\varphi)(\underline{m}) \geq 0,$$

since $\underline{m} \in B'_J(\infty)$.

Finally suppose that $\underline{m} \neq \underline{0}$. Let l be the maximal index such that $m_l > 0$. Set $i = i_l$ and suppose that $e_i\underline{m} = 0$. Suppose e_i enters at the k^{th} place. Since $r_l(\underline{m}) = r_{i_l}^l(\underline{m}) = m_l > 0$ and $r_{l'}(\underline{m}) = 0$ for $l' > l$, one has $k \leq l$. Then $e_i\underline{m} = 0$, implies $m_k = 0$. Consider the co-ordinate form $\varphi = x_k$. One has $\varphi_k = 1$, so $S_k x_k = x_k - t_k^+$. Then since $\underline{m} \in B'_J(\infty)$, we must have

$$0 \leq (S_k x_k)(\underline{m}) = x_k(\underline{m}) - t_k^+(\underline{m}) = -t_k^+(\underline{m}),$$

forcing $r_k(\underline{m}) - r_{k_+}(\underline{m}) = t_k^+(\underline{m}) \leq 0$. On the other hand $t_k^+(\underline{m}) \geq 1$, by the assumption the e_i enters at the k^{th} place. This contradiction implies that $e_i\underline{m} \neq 0$.

It follows that $(B'_J(\infty))^{\mathscr{E}} = \{0\} = b_\infty$. Consequently $B'_J(\infty) = \mathscr{F} b_\infty = B_J(\infty)$, as required. □

Remark 1 This use of x_k, $S_k x_k$ does not occur in the original argument of Nakashima and Zelevinsky ([33], end of proof of Theorem 3.5). Either they forgot to include it, or there was a gap in their reasoning.

Remark 2 Showing that $k \leq l$, avoids the awkwardness of having e_i enter at an infinite place (see 2.3.5).

Remark 3 Notice that we can turn the argument of the last part around to show that $B'_J(\infty)$ (and hence $B_J(\infty)$) is upper normal! Indeed suppose $e_i\underline{m} = 0$. If e_i enters at a finite place, say at the k^{th} place, then the above reasoning gives a contradiction. Thus e_i must enter at an infinite place. This means that $\varepsilon_i(\underline{m}) = 0$. Hence we have shown that $B'_J(\infty)$ is upper normal. Indeed already in B_J (which is not upper normal) one has $\infty > \varepsilon_i(b) \geq 0$, $\forall b \in B_J$. Thus if $b \in B'_J(\infty)$ satisfies $\varepsilon_i(b) > 0$ one has $e_i b \in B'_J$ and then $\varepsilon_i(e_i(b)) = \varepsilon_i(b) - 1$ by (C2). Induction then establishes that $\varepsilon_i(b) = \max_k\{e_i^k b \neq 0\}$.

3.6.5 The result noted in Remark 3 and the discussion in 2.5.26 means that the above theory can be used to construct $\mathbb{F}_A$ without recourse to the Littelmann theory. However we still need the positivity hypothesis on $\mathscr{L}$—see 3.6.4, Nakashima and Zelevinsky [33] note that it holds in all the examples they study. However it maybe be that for a given A, there may be no choices of J for which the positivity hypothesis can be verified.

3.6.6 Assume the positivity hypothesis holds for a given A and a given J. Then we obtain the immediate corollary of Theorem 3.6.4.

Corollary $B_J(\infty)$ *is a semigroup under component-wise addition.*

Remark 4 It would be interesting to give an algorithm for finding generators.

3.6.7 A further question that one may ask is whether $S_k : k \in \mathbb{N}^+$ of 3.6.3 generate a singular Hecke algebra corresponding to an affinization of W.

Acknowledgements I would like to thank Crystal Hoyt for editing the manuscript which served as a basis for a Master Class at the Jacobs University, Bremen during August 2009. During this process two significant errors in the main new result, namely the combinatorial construction of the Kashiwara involution and its extension to the non-symmetrizable case, were detected. These were corrected with her help.

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Highlights in Lie Algebraic Methods
Joseph, A.; Melnikov, A.; Penkov, I. (Eds.)
2012, XV, 227 p. 4 illus., Hardcover
ISBN: 978-0-8176-8273-6
A product of Birkhäuser Basel