Preface

A two-week Summer School on "Structures in Lie Theory, Crystals, Derived Functors, Harish-Chandra Modules, Invariants and Quivers" was held at Jacobs University Bremen during 9–22 August 2009 on both the geometric and algebraic aspects of Lie Theory. The participants were mainly from European countries with strong contingents from Germany, Russia, and Israel. Several high-level graduate courses were given, containing recent or original results on topics of particular current interest. The detailed notes of five of these lecture courses are reproduced in this volume. They not only provide a welcome reminder of the content of these courses to the participants, but enable all those who did not attend the meeting to profit from insights of leading specialists into the latest developments in some exciting fields of research. Even those participants who followed the courses attentively may profit from the details presented in this book.

Besides the presenters of the graduate courses, some younger researchers were invited to speak at the meeting and to submit a manuscript for publication. Thus the present volume also contains five original articles.

All the texts reproduced here were subject to a strict refereeing procedure befitting any mathematical journal.

The Courses

Spherical Varieties by Michel Brion, Grenoble

This course discusses and interrelates several classes of complex algebraic varieties equipped with an action of a connected algebraic group G. For example, the homogeneous varieties (in which G acts transitively), the spherical varieties (for which G is reductive and for which a Borel subgroup of G has an open orbit), the symmetric spaces G/G^θ (where again G is reductive, and θ is an involution of G), and the "wonderful varieties" in the sense of De Concini and Procesi, are examined in this course. Classification theorems are presented, especially in the case of complete varieties.

A significant aspect of this course is the systematic use of the notion of a log homogeneous variety X. This is a smooth G-variety equipped with a divisor D with normal crossings such that D is G-stable and the associated "logarithmic tangent bundle" is generated by the Lie algebra $\mathfrak{g}$ (acting on X via vector fields that preserve D). This condition implies that $X \setminus D$ is a single G orbit.

A local structure theorem is given for a complete log homogeneous variety X along a closed G-orbit, showing in particular that X is spherical if G is linear. Conversely, it is stated that a spherical homogeneous space G/H always admits a log homogeneous completion. Moreover, if H is its own normalizer in G, then the closure of the G-orbit of $\mathfrak{h}$ in the variety of Lie subalgebras of $\mathfrak{g}$ is a log homogeneous completion with a unique closed G-orbit.

An alternative construction of this "wonderful completion" is presented in the case of a semisimple group G with trivial center, viewed as a homogeneous space under $G \times G$. Let V be a simple G-module with a regular highest weight. Then $G \times G$ acts on the projectivization $\mathbb{P}(\mathrm{End}\, V)$, and the orbit through the class of the identity is isomorphic to G. By a result of De Concini and Procesi, the closure of that orbit is the desired wonderful completion of G and is independent of the choice of V.

The above result is generalized to an arbitrary symmetric space G/G^θ as follows. Assume that V is spherical (i.e., it admits a G^θ-invariant vector) and "regular" for that property. Then, as shown by De Concini and Procesi, the orbit closure of the corresponding point of $\mathbb{P}(V)$ is again a wonderful completion of G/G^θ, independent of V.

Finally, the notion of a wonderful variety is discussed. It is stated that there are only finitely wonderful varieties for a given semisimple group G, and there is an ongoing program to classify them.

Consequences of the Littelmann Path Model for the Structure of the Kashiwara $B(\infty)$ Crystal by Anthony Joseph, Weizmann Institute

This course starts with the observation that the $B(\infty)$ crystal of Kashiwara associated to a Kac–Moody algebra $\mathfrak{g}$ is given by a purely combinatorial construction. The properties of $B(\infty)$ were determined by taking a $q \to 0$ limit of highest weight modules over the quantized enveloping $U_q(\mathfrak{g})$.

The Littelmann path construction associates a crystal to a highest weight of $\mathfrak{g}$. Then $B(\infty)$ can be obtained by a limiting process on the highest weight. This gives a purely combinatorial way to analyze the structure of $B(\infty)$, which has the advantage that there is no need to assume that $\mathfrak{g}$ is symmetrizable. In particular the key properties that $B(\infty)$ is upper normal and canonically determined by a tensor product construction are established. Furthermore it is shown that $B(\infty)$ admits an involution which coincides with the Kashiwara involution in the symmetrizable case. Conversely, it is shown that Littelmann's crystals may be recovered from $B(\infty)$, but so far it is not known how to show that the resulting "highest weight crystals" satisfy tensor product decomposition without appealing to the Littelmann path model (or the Lusztig–Kashiwara theory of bases).

Character formulas are discussed, noting that there is not yet a purely combinatorial proof of the Weyl denominator formula if $\dim \mathfrak{g}$ is infinite.

Combinatorial Demazure flags are described. Here it is noted that the corresponding result for the tensor product of a Demazure module (global sections of sheaves on Schubert varieties) with a one-dimensional Demazure module is known only for semisimple $\mathfrak{g}$ (Mathieu) or if the root system is simply laced. The latter is notably by virtue of a positivity result in the multiplication of canonical basis elements due to Lusztig.

Under a positivity hypothesis, Nakashima and Zelevinsky showed that $B(\infty)$ admits an additive structure (which as yet has no module-theoretic interpretation). Their proof is reproduced, noting that this positivity hypothesis implies upper normality and so is liable to be rather difficult to establish.

Aside from the Littelmann construction, detailed proofs are given for most of these combinatorial results.

Structure and Representation Theory of Kac–Moody Superalgebras, by Vera Serganova, Berkeley

The essence of supersymmetry is the introduction of a sign rule in mathematical operations. Though definitions carry over to this situation in a seemingly innocent fashion, this idea leads to a new theory with many challenging open problems. From the mathematician's point of view, the "sign rule" leads to new notions such as supermanifold, Lie supergroup, and Lie superalgebra. The interrelated theories based on the notions of superanalysis, supergeometry, representation theory of Lie supergroups and Lie superalgebras started their rapid development in the works of Berezin, Bernstein, Kac, Kostant, Leites, and others.

In these lectures, Serganova concentrates on the theory of contragredient Lie superalgebras and their representations. She recalls Kac's classification of finite-dimensional simple Lie superalgebras, gives a brief introduction into Lie supergroups, and then proceeds to the definition of Kac–Moody Lie superalgebras. The existence of odd reflections ("odd" elements of the Weyl group) plays an important role when describing the structure of Kac–Moody Lie superalgebras. Serganova then presents the classification of finite-growth contragredient Lie superalgebras, due to Van de Leur in the symmetrizable case and to Hoyt and Serganova in the non-symmetrizable case.

The next topic discussed is integrable highest weight modules. In contrast with the case of contragredient Lie superalgebras, only a few finite-growth contragredient Lie superalgebras admit nontrivial highest weight modules; see Theorem 5 due to Kac and Wakimoto. Serganova proves the Weyl character formula for typical integrable highest weight modules. This formula goes back to Kac in the finite-dimensional case. (A related result, not discussed in the lectures, is Gorelik's recent proof of the "super" denominator identity–editorial note.)

In the final Sect. 4, Serganova concentrates on the case of a finite-dimensional contragredient superalgebra. Here she discusses the case of atypical finite-dimensional simple modules and, in particular, recalls the Kac–Wakimoto conjecture on the superdimension of such a module.

A point worthy of special attention is the discussion of geometric methods: the notion of an "odd associated variety" due to Duflo and Serganova, and the older Bott–Borel–Weil approach to flag supervarieties going back to Penkov.

The lectures are concluded by a list of current open problems.

Categories of Harish-Chandra Modules by Wolfgang Soergel, Freiburg

Recall that the Kazhdan–Lusztig polynomials evaluated at $q = 1$ give the Jordan–Hölder multiplicity $[M : L]$ of a simple quotient L in a Verma module M. However the polynomials themselves have two possible interpretations, either as describing the more precise data encoded as the dimensions of the extension groups $\mathrm{Ext}^k(M, L)$ or as the Jordan–Hölder multiplicities $[M_k, L]$ with M_k the kth graded component of M. Here the resulting matrices are not the same but related to one another by matrix inversion and appropriate sign changes. (This was first suggested in O. Gabber and A. Joseph, *Ann. Sci. École Norm. Sup.* (4) 14 (1981), no. 3, 261–302, where the gradation was to be given by the Jantzen filtration. Notably the second interpretation was proven for the socle filtration in R. S. Irving, *Ann. Sci. École Norm. Sup.* (4) 21 (1988), no. 1, 47–65. It had the important consequence that one can thereby describe the Duflo involutions through the Kazhdan–Lusztig polynomials—editorial note.)

The above observation can be interpreted as an inversion formula for the Kazhdan–Lusztig matrix of polynomials. In the well known paper of A. Beilinson, V. Ginzburg, and W. Soergel, *J. AMS* 9 (1996), no. 2, pp. 473–526, this inversion formula has been given a categorical interpretation as Koszul self-duality of the category $\mathscr{O}$. In fact, Soergel sketches a proof of this result in the latter part of his lectures.

The main content of the course is a statement of a conjecture extending the above ideas to the category of Harish-Chandra modules. The starting point is the inversion formula in J. Adams, D. Barbasch, and D. A. Vogan, Progress in Mathematics, 104. Birkhaüser Boston, Inc., Boston, MA, 1992, involving two sets of matrices coming from Jordan–Hölder (JH) and intersection cohomology (IC) matrices.

Soergel's conjecture is that the category of Harish-Chandra modules is equivalent to a certain category of finite-dimensional modules determined entirely in terms of intersection cohomology. This gives the desired relationship between the JH and IC matrices and emphasizes the similarity between the category $\mathscr{O}$ and the category of

Harish-Chandra modules. It also shows how far-reaching the idea of Koszul duality is. Soergel's conjecture is proved so far for two cases, namely for $SL(2,\mathbb{R})$ and for complex groups.

Soergel discusses his conjecture in light of tilting modules. For example, in the $\mathcal{O}$ category tilting modules interpolate between projectives and simples, and can be used to establish the Bernstein–Gelfand–Gelfand reciprocity.

Generalized Harish-Chandra Modules by Gregg Zuckerman, Yale

A module M over a Lie algebra $\mathfrak{g}$ is defined to be "generalized Harish-Chandra" if there exists a Lie subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ such that M is locally finite as an $\mathfrak{l}$-module and has finite multiplicities as an $\mathfrak{l}$-module in an appropriate sense (see Definition 1.8 in the lectures). If $\mathfrak{g}$ is finite-dimensional and semisimple, $\mathfrak{l}$ is the fixed point of an involution θ of $\mathfrak{g}$, and M is finitely generated over $\mathfrak{g}$, then this reduces to the usual definition of a Harish-Chandra module.

Harish-Chandra modules first arose in the description of unitary representations of real Lie groups and have been intensively studied since the monumental work of Harish-Chandra. Notably Langlands gave a classification of simple Harish-Chandra modules. Key questions are to determine the precise multiplicities as an $\mathfrak{l}$-module and to determine when (in terms of the Langlands parameters) a simple Harish-Chandra module is unitarizable (still an open question!).

Recently there has been considerable interest in extending this well-developed theory to generalized Harish-Chandra modules, for which the course provides an introduction. A first case of (genuinely) generalized Harish-Chandra modules occurs when $\mathfrak{g}$ is semisimple and $\mathfrak{l}$ is a Cartan subalgebra. The classification of the simple modules is basically due to Fernando and Mathieu. An important fact is that the set $\mathfrak{g}[M]$ of elements of a finite-dimensional Lie algebra $\mathfrak{g}$ acting locally finitely on a $\mathfrak{g}$-module M forms a Lie subalgebra. (This assertion generally fails for infinite-dimensional Lie algebras—editorial note.)

Considerable emphasis in these lectures is put on the use of the Zuckerman functor, which roughly is the right derived functor of the functor of passing to the submodule of $\mathfrak{l}$-finite vectors. An important procedure is cohomological induction, where the Zuckerman functor is applied to a parabolically induced or produced module. Some general properties of the Zuckerman functor are described, for example, commutation with the tensor product by a finite-dimensional $\mathfrak{g}$-module and with the action of the center of $U(\mathfrak{g})$. These properties are crucial to the "translation principle." A finite multiplicity theorem is proved, which allows Zuckerman to apply his functors to the construction of generalized Harish-Chandra modules. Some information on multiplicities is provided by standard homological arguments, for example, Frobenius reciprocity and the Euler principle. In the last section Zuckerman describes his joint work with Penkov classifying simple generalized Harish-Chandra modules with "generic" minimal $\mathfrak{l}$-type when $\mathfrak{g}$ is semisimple and $\mathfrak{l}$ is reductive in $\mathfrak{g}$. (In the case of simple Harish-Chandra modules, the notion of minimal $\mathfrak{l}$-type and its role in the classification theory was extensively studied in D. A. Vogan, *Proc. Nat. Acad. Sci. USA* 74 (1977), no. 7, 2649–2650—editorial note.)

Finally, it should be noted that generalized Harish-Chandra modules have been studied also by geometric methods (D-modules). In particular, geometric methods have been helpful in the computation by Penkov, Serganova, and Zuckerman of the Lie subalgebra $\mathfrak{g}[M]$ for certain M, as well as in the recent construction of bounded multiplicity $(\mathfrak{g}, \mathfrak{l})$-modules by Penkov and Serganova.

The Papers

B-Orbits of 2-Nilpotent Matrices and Generalizations, by Magdalena Boos and Markus Reineke, Wuppertal

Let B be the standard Borel subgroup of $GL(n, \mathbb{C})$. Melnikov showed that the set of B-orbits in the set of upper triangular matrices of square zero is finite, classified them in terms of involutions in the symmetric group S_n, and showed that the inclusion relations of their closures can be defined by link patterns. The authors obtain similar results for (the larger family of) B-orbits in the set of all matrices of square zero. Their methods are quite different to those of Melnikov. Indeed, the authors use a relation of indecomposable representations of quivers to these orbits. The former are classified using the Auslander–Reiten quiver. They use results of Zwara to translate the condition of inclusion under orbit closure into module theoretic terms. This gives the minimal orbit closure inclusion via *oriented* link patterns. Finally they discuss generic B-orbits in the set of all nilpotent matrices as well as describing semi-invariants for the action. Boos and Reineke suggest that these may generate all the semi-invariants.

Weyl Denominator Identity for Finite-Dimensional Lie Superalgebras, by Maria Gorelik, Weizmann Institute

The Weyl denominator identity for the algebras in the title was formulated by Kac and Wakimoto, who also proved it in some cases. The basic idea of Gorelik's proof (which is purely combinatorial) is to show that both sides of the identity are skew invariant for the Weyl group W and that each side forms just one orbit sum. However, unlike the Lie algebra case, there are several technical difficulties even to formulate the identity.

In general, one side of the identity is just a product over the positive roots as in the Lie algebra case but with sign changes corresponding to the odd roots. The second side is given by two sets of data $W^{\#}$ and S. The former is a subgroup of the Weyl group, and the latter is a certain maximal isotropic set of simple roots.

Moreover, there can be several choices of S even up to equivalence and therefore several denominator formulas.

In an extension of the above, the Weyl denominator identity for untwisted affine Lie superalgebras with nonzero dual Coxeter number has recently been established by Gorelik (Weyl denominator identity for affine Lie superalgebras with nonzero dual Coxeter number, *J. Algebra*, to appear), and the case of zero dual Coxeter number has been established by Gorelik and Reif (Denominator identity for affine Lie superalgebras with zero dual Coxeter number, arXiv 10125879).

Hopf Algebras and Frobenius Algebras in Finite Tensor Categories by Christoph Schweigert, Hamburg, and Jurgen Fuchs, Karlstad

This contribution reviews the importance of a canonical Hopf algebra object which exists in braided tensor categories that obey certain finiteness conditions. This Hopf algebra object has various relations to topological invariants. In particular, for any object U, the morphism space $\mathrm{Hom}(U, H)$ carries a projective representation of $SL(2, \mathbb{Z})$.

Mutation Classes of 3×3 *Generalized Cartan Matrices by Ahmet Seven, Ankara*

If B is a skew-symmetrizable matrix with integer entries, its diagram $\Gamma(B)$ is defined to be the directed graph with a directed edge from i to j whenever $B_{i,j} > 0$. The author derives a criterion, in terms of the value of the Markov constant, for the diagram of 3×3 skew-symmetrizable matrices to remain acyclic under mutation in the sense of cluster algebras.

Contractions and Polynomial Lie Algebras by Benjamin Wilson, Munich

Let $A = \mathbb{C}[t]/t^{\ell}$, and let $\mathfrak{g}$ denote a Lie algebra. Their tensor product $A \otimes \mathfrak{g}$ is again a Lie algebra and, under suitable hypotheses, possesses a highest-weight theory. A reducibility criterion for its Verma modules is derived using a contraction of a direct sum of copies of $\mathfrak{g}$.

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