Rational Linking and Contact Geometry

Kenneth Baker and John Etnyre

This paper is dedicated to Oleg Viro on the occasion of his 60th birthday

Abstract In the note we study Legendrian and transverse knots in rationally null-homologous knot types. In particular, we generalize the standard definitions of self-linking number, Thurston–Bennequin invariant, and rotation number. We then prove a version of Bennequin's inequality for these knots and classify precisely when the Bennequin bound is sharp for fibered knot types. Finally, we study rational unknots and show that they are weakly Legendrian and transversely simple.

Keywords Legendrian knot • Transverse knot • Contact geometry • Self-linking • Open book

In this note we extend the self-linking number of transverse knots and the Thurston–Bennequin invariant and rotation number of Legendrian knots to the case of rationally null-homologous knots. This allows us to generalize many of the classical theorems concerning Legendrian and transverse knots (such as the Bennequin inequality) as well as to put other theorems in a more naturalcontext (such as the result in [10] concerning exactness in the Bennequin bound). Moreover, due

K. Baker
Department of Mathematics, University of Miami, PO Box 249085, Coral Gables, FL 33124-4250, USA
e-mail: k.baker@math.miami.edu

J. Etnyre (✉)
School of Mathematics, Georgia Institute of Technology, 686 Cherry St., Atlanta, GA 30332-0160, USA
e-mail: etnyre@math.gatech.edu

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to recent work on the Berge conjecture [3] and surgery problems in general, it has become clear that one should consider rationally null-homologous knots even when studying classical questions about Dehn surgery on knots in S^3. Indeed, the Thurston–Bennequin number of Legendrian rationally null-homologous knots in lens spaces has been examined in [2]. There is also a version of the rational Thurston–Bennequin invariants for links in rational homology spheres that was previously defined and studied in [13].

We note that there has been work on relative versions of the self-linking number (and other classical invariants) in the case of general (even non-null-homologous) knots; cf. [4]. While these relative invariants are interesting and useful, many of the results considered here do not have analogous statements. So rationally null-homologous knots seems to be one of the largest classes of knots to which one can generalize classical results in a straightforward manner.

There is a well-known way to generalize the linking number between two null-homologous knots to rationally null-homologous knots; see, for example, [11]. We recall this definition of a rational linking number in Sect. 1 and then proceed to define the rational self-liking number $\mathrm{sl}_{\mathbb{Q}}(K)$ of a transverse knot K and the rational Thurston–Bennequin invariant $\mathrm{tb}_{\mathbb{Q}}(L)$ and rational rotation number $\mathrm{rot}_{\mathbb{Q}}(L)$ of a Legendrian knot L in a rationally null-homologous knot type. We also show the expected relation between these invariants of the transverse pushoff of a Legendrian knot and those of stabilizations of Legendrian and transverse knots. This leads to one of our main observations, a generalization of Bennequin's inequality.

Theorem 2.1. *Let (M,ξ) be a tight contact manifold and suppose K is a transverse knot in it of order $r>0$ in homology. Further suppose that Σ is a rational Seifert surface of K. Then*

$$\mathrm{sl}_{\mathbb{Q}}(K) \leq -\frac{1}{r}\chi(\Sigma).$$

Moreover, if K is Legendrian, then

$$\mathrm{tb}_{\mathbb{Q}}(K) + |\mathrm{rot}_{\mathbb{Q}}(K)| \leq -\frac{1}{r}\chi(\Sigma).$$

In [10], bindings of open book decompositions that satisfy equality in the Bennequin inequality were classified. We generalize that result to the following.

Theorem 4.2. *Let K be a rationally null-homologous fibered transverse knot in a contact 3-manifold (M,ξ) such that ξ is tight when restricted to the complement of K. Denote by Σ a fiber in the fibration of $M-K$ and let r be the order of K. Then $r\,\mathrm{sl}_{\mathbb{Q}}^{\xi}(K,\Sigma) = -\chi(\Sigma)$ if and only if either ξ agrees with the contact structure supported by the rational open book determined by K or it is obtained from it by adding Giroux torsion along tori that are incompressible in the complement of K.*

A rational unknot in a manifold M is a knot K with a disk as a rational Seifert surface. One may easily check that if M is irreducible, then for M to admit a rational unknot (that is not actually an unknot), it must be diffeomorphic to a lens space.

Theorem 5.1. *Rational unknots in tight contact structures on lens spaces are weakly transversely simple and Legendrian simple.*

In Sect. 5 we also given an example of the classification of Legendrian rational unknots (and hence transverse rational unknots) in $L(p,1)$ when p is odd. The classification of Legendrian and transverse rational unknots in a general lens space can easily be worked out in terms of the classification of tight contact structures on the given lens space. The example we give illustrates this.

In Sect. 6, we briefly discuss the generalization of our results to the case of links.

1 Rational Linking and Transverse and Legendrian Knots

Let K be an oriented knot of $\mathbb{Z}$-homological order $r > 0$ in a 3-manifold M and denote a tubular neighborhood of it by $N(K)$. By $X(K)$ denote the knot exterior $\overline{M \setminus N(K)}$. We fix a framing on $N(K)$. We know that half the $\mathbb{Z}$-homology of $\partial X(K)$ dies when included in the $\mathbb{Z}$-homology of $X(K)$. Since K has order r, it is easy to see that there is an embedded (r,s)-curve on $\partial X(K)$ that bounds an oriented connected surface Σ° in $X(K)$. We can radially cone $\partial \Sigma^\circ \subset \partial X(K) = \partial N(K)$ in $N(K)$ to get a surface Σ in M whose interior is embedded in M and whose boundary wraps r times around K. Such a surface Σ will be called a *rational Seifert surface* for K, and we say that K *r-bounds* Σ. We also sometime say that Σ is of *order r* along K. We also call $\Sigma \cap \partial N(K)$ the *Seifert cable of* K. Notice that Σ may have more than one boundary component. Specifically, Σ will have $\gcd(r,s)$ boundary components. We call the number of boundary components of Σ the *multiplicity* of K. Notice that Σ defines a $\mathbb{Z}$-homology chain Σ and $\partial \Sigma = rK$ in the homology 1-chains. In particular, as $\mathbb{Q}$-homology chains, $\partial(\frac{1}{r}\Sigma) = K$.

We now define the *rational linking number* of another oriented knot K' with K (and Seifert surface Σ) to be

$$\mathrm{lk}_{\mathbb{Q}}(K,K') = \frac{1}{r}\Sigma \cdot K',$$

where $\cdot$ denotes the algebraic intersection of Σ and K'. It is not hard to check that $\mathrm{lk}_{\mathbb{Q}}$ is well defined given the choice of $[\Sigma] \in H_2(X(K), \partial X(K))$. Choosing another rational Seifert surface for K representing a different relative second homology class in $X(K)$ may change this rational linking number by a multiple of $\frac{1}{r}$. To emphasize this, one may prefer to write $\mathrm{lk}_{\mathbb{Q}}((K,[\Sigma]),K')$. Notice that if there exist rational Seifert surfaces Σ_1 and Σ_2 for which $\mathrm{lk}_{\mathbb{Q}}((K,[\Sigma_1]),K') \neq \mathrm{lk}_{\mathbb{Q}}((K,[\Sigma_2]),K')$, then K' is not rationally null-homologous.

Moreover, if K' is also rationally null-homologous, then it r'-bounds a rational Seifert surface Σ'. In $M \times [0,1]$ with Σ and Σ' thought of as subsets of $M \times \{1\}$, we can perturb them relative to the boundary to make them transverse. Then one may also check that

$$\mathrm{lk}_{\mathbb{Q}}(K,K') = \frac{1}{rr'}\Sigma \cdot \Sigma'.$$

From this one readily sees that the rational linking number of rationally null-homologous links is symmetric.

1.1 Transverse Knots

Let (M,ξ) be a contact 3-manifold (with orientable contact structure ξ) and K a (positively) transverse knot. Given a rational Seifert surface Σ for K with $\partial\Sigma = rK$, we can trivialize ξ along Σ. More precisely, we can trivialize the pullback $i^*\xi$ to Σ, where $i : \Sigma \to M$ is the inclusion map. Notice that the inclusion map restricted to $\partial\Sigma$ is an r-fold covering map of $\partial\Sigma$ to K. We can use the exponential map to identify a neighborhood of the zero section of $i^*\xi|_{\partial\Sigma}$ with an r-fold cover of a tubular neighborhood of K. Let v be a nonzero section of $i^*\xi$. By choosing v generically and suitably small, the image of $v|_{\partial\Sigma}$ gives an embedded knot K' in a neighborhood of K that is disjoint from K. We define the *rational self-linking number* to be

$$\mathrm{sl}_{\mathbb{Q}}(K) = \mathrm{lk}_{\mathbb{Q}}(K,K').$$

It is standard to check that $\mathrm{sl}_{\mathbb{Q}}$ is independent of the trivialization of $i^*\xi$ and the section v. Moreover, the rational self-linking number depends only on the relative homology class of Σ. When this dependence is important to note, we denote the rational self-linking number by

$$\mathrm{sl}_{\mathbb{Q}}(K,[\Sigma]).$$

Just as in the case of the self-linking number one can compute it by considering the characteristic foliation on Σ. To this end we can always isotop Σ so that its characteristic foliation Σ_ξ is generic (in particular has only elliptic and hyperbolic singularities) and we denote by $e_\pm$ the number of $\pm$-elliptic singular points, and similarly, $h_\pm$ denotes the number of $\pm$-hyperbolic points.

Lemma 1.1. *Suppose K is a transverse knot in a contact manifold (M,ξ) that r-bounds the rational Seifert surface Σ. Then*

$$\mathrm{sl}_{\mathbb{Q}}(K,[\Sigma]) = \frac{1}{r}\left((e_- - h_-) - (e_+ - h_+)\right). \tag{1}$$

Proof. We begin by constructing a nice neighborhood of Σ in (M,ξ). To this end, notice that for suitably small ϵ, K has a neighborhood N that is contactomorphic to the image C_ϵ of $\{(r,\theta,z) : r \leq -\epsilon\}$ in $(\mathbb{R}^3, \ker(\mathrm{d}z + r^2\,\mathrm{d}\theta))$ modulo the action $z \mapsto z+1$. Let C' be the r-fold cover of C_ϵ. Taking ϵ sufficiently small, we can assume that $\Sigma \cap \partial N$ is a transverse curve T. Thinking of T as sitting in C_ϵ, we can take its lift T' to C'. Let N' be a small neighborhood of $\overline{\Sigma - (N \cap \Sigma)}$. We can glue N' to C_ϵ along a neighborhood of T to get a model neighborhood U for Σ in M. Moreover, we can glue N' to C' along a neighborhood of T' to get a contact manifold U' that will map onto U so that C' r-fold covers C_ϵ and N' in U' maps diffeomorphically to N' in U. Inside U' we have $K' = \partial\Sigma$, which r-fold covers K in U. The transverse knot K' is a null-homologous knot in U'. According to a well-known formula that easily follows by interpreting $\mathrm{sl}(K')$ as a relative Euler class, see [5], we have that

$$\mathrm{sl}(K') = (e_- - h_-) - (e_+ - h_+),$$

where $e_\pm$ and $h_\pm$ are as in the statement of the theorem. Now one easily sees that $\mathrm{sl}_{\mathbb{Q}}(K) = \frac{1}{r}\,\mathrm{sl}(K')$, from which the lemma follows. $\square$

1.2 Legendrian Knots

Let (M,ξ) be a contact 3-manifold (with orientable contact structure ξ) and K a Legendrian knot. Choose a framing on K. Given a rational Seifert surface Σ for K, the Seifert cable of K is $K_{(r,s)}$.

The restriction $\xi|_K$ induces a framing on the normal bundle of K. Define the *(rational) Thurston–Bennequin number* of the Legendrian knot K to be

$$\mathrm{tb}_{\mathbb{Q}}(K) = \mathrm{lk}_{\mathbb{Q}}(K, K'),$$

where K' is a copy of K obtained by pushing off using the framing coming from ξ.

We now assume that K is oriented. Recall that the inclusion $i\colon \Sigma \hookrightarrow M$ is an embedding on the interior of Σ and an r-to-1 cover $\partial\Sigma \to K$. As above, we can trivialize ξ along Σ. That is, we can trivialize the pullback $i^*\xi$ to Σ. The oriented tangent vectors T_K give a section of $\xi|_K$. Thus i^*T_K gives a section of $\mathbb{R}^2 \times \partial\Sigma$. Define the *rational rotation number* of the Legendrian knot K to be the winding number of i^*T_K in $\mathbb{R}^2$ divided by r:

$$\mathrm{rot}_{\mathbb{Q}}(K) = \frac{1}{r}\mathrm{winding}(i^*T_K, \mathbb{R}^2).$$

Recall [8] that given a Legendrian knot K, we can always form the *(positive) transverse pushoff of* K, denoted by $T(K)$, as follows: the knot K has a neighborhood contactomorphic to the image of the x-axis in $(\mathbb{R}^3, \ker(\mathrm{d}z - y\,\mathrm{d}x))$ modulo the action $x \mapsto x+1$ such that the orientation on the knot points toward increasing x-values. The curve $\{(x,\epsilon,0)\}$ for $\epsilon > 0$ small enough will give the transverse pushoff of K.

Lemma 1.2. *If K is a rationally null-homologous Legendrian knot in a contact manifold (M,ξ), then*

$$\mathrm{sl}_{\mathbb{Q}}(T(K)) = \mathrm{tb}_{\mathbb{Q}}(K) - \mathrm{rot}_{\mathbb{Q}}(K).$$

Proof. Notice that in pulling K back to a cover U' similar to the one constructed in the proof of Lemma 1.1, we get a null-homologous Legendrian knot K'. Here we have the well-known formula (see [8])

$$\mathrm{sl}(T(K')) = \mathrm{tb}(K') - \mathrm{rot}(K').$$

One easily computes that $r\,\mathrm{sl}(T(K')) = \mathrm{sl}_{\mathbb{Q}}(T(K)), r\mathrm{tb}(K') = \mathrm{tb}_{\mathbb{Q}}(K')$ and $r\mathrm{rot}(K') = \mathrm{rot}(K)$. The lemma follows. □

We can also construct a Legendrian knot from a transverse knot. Given a transverse knot K, it has a neighborhood as constructed in the proof of Lemma 1.1. It is clear that the boundary of a sufficiently small closed neighborhood of K of the appropriate size will have a linear characteristic foliation by longitudes of K. One of the leaves in this characteristic foliation will be called a *Legendrian pushoff of K*. We note that this pushoff is not unique, but that different Legendrian pushoffs are related by negative stabilizations; see [9].

1.3 Stabilization

Recall that stabilization of a transverse and Legendrian knot is a local procedure near a point on the knot, so it can be performed on any transverse or Legendrian knot whether null-homologous or not.

There are two types of stabilization of a Legendrian knot K: positive and negative stabilization, denoted by $S_+(K)$ and $S_-(K)$, respectively. Recall that if one identifies a neighborhood of a point on a Legendrian knot with a neighborhood of the origin in $(\mathbb{R}^3, \ker(dz - y\,dx))$ so that the Legendrian knot is mapped to a segment of the x-axis and the orientation induced on the x-axis from K is toward increasing x-values, then $S_+(K)$, respectively $S_-(K)$, is obtained by replacing the segment of the x-axis by a "downward zigzag," respectively "upward zigzag"; see [8, Fig. 19]. One may similarly define stabilization of a transverse knot K, and we denote it by $S(K)$. Stabilizations have the same effect on the rationally null-homologous knots as they have on null-homologous ones.

Lemma 1.3. *Let K be a rationally null-homologous Legendrian knot in a contact manifold. Then*

$$\mathrm{tb}_{\mathbb{Q}}(S_\pm(K)) = \mathrm{tb}_{\mathbb{Q}}(K) - 1 \quad \textit{and} \quad \mathrm{rot}_{\mathbb{Q}}(S_\pm(K)) = \mathrm{rot}_{\mathbb{Q}}(K) \pm 1.$$

Let K be a rationally null-homologous transverse knot in a contact manifold. Then

$$\mathrm{sl}_{\mathbb{Q}}(S(K)) = \mathrm{sl}_{\mathbb{Q}}(K) - 2.$$

Proof. One may check that if K' is a pushoff of K by some framing $\mathcal{F}$ and K'' is the pushoff of K by a framing $\mathcal{F}''$ such that the difference between $\mathcal{F}$ and $\mathcal{F}'$ is -1, then

$$\mathrm{lk}_{\mathbb{Q}}(K, K'') = \mathrm{lk}_{\mathbb{Q}}(K, K') - 1.$$

Indeed, by noting that $r\,\mathrm{lk}_{\mathbb{Q}}(K, K')$ can easily be computed by intersecting the Seifert cable of K on the boundary of a neighborhood of K, $T^2 = \partial N(K)$, with the curve $K' \subset T^2$, the result easily follows. From this one obtains the change in $\mathrm{tb}_{\mathbb{Q}}$.

Given a rational Seifert surface Σ that is r-bounded by K, a small Darboux neighborhood N of a point $p \in K$ intersects Σ in r disjoint disks. Since the stabilization can be performed in N, it is easy to see that Σ is altered by adding r small disks, each containing a positive elliptic point and negative hyperbolic point (see [8]). The result for $\mathrm{sl}_{\mathbb{Q}}$ follows.

Finally, the result for $\mathrm{rot}_{\mathbb{Q}}$ follows by a similar argument or from the previous two results, Lemma 1.2, and the following lemma (whose proof does not explicitly use the rotation number results from this lemma). □

The proof of the following lemma is given in [9].

Lemma 1.4. *Two transverse knots in a contact manifold are transversely isotopic if and only if they have Legendrian pushoffs that are Legendrian isotopic after each has been negatively stabilized some number of times. The same statement is true with "transversely isotopic" and "Legendrian isotopic" both replaced by "contactomorphic."* □

We similarly have the following result.

Lemma 1.5. *Two Legendrian knots representing the same topological knot type are Legendrian isotopic after each has been positively and negatively stabilized some number of times.* □

While this is an interesting result in its own right, it clarifies the range of possible values for $\mathrm{tb}_{\mathbb{Q}}$. More precisely, the following result is an immediate corollary.

Corollary 1.6. *If two Legendrian knots represent the same topological knot type, then the difference in their rational Thurston–Bennequin invariants is an integer.*

2 The Bennequin Bound

Recall that in a tight contact structure, the self-linking number of a null-homologous knot K satisfies the well-known Bennequin bound

$$\mathrm{sl}(K) \leq -\chi(\Sigma)$$

for any Seifert surface Σ for K; see [6]. We have the analogous result for rationally null-homologous knots.

Theorem 2.1. *Suppose K is a transverse knot in a tight contact manifold (M,ξ) that r-bounds the rational Seifert surface Σ. Then*

$$\mathrm{sl}_{\mathbb{Q}}(K,[\Sigma]) \leq -\frac{1}{r}\chi(\Sigma). \tag{1}$$

If K is a Legendrian knot, then

$$\mathrm{tb}_{\mathbb{Q}}(K) + |\mathrm{rot}_{\mathbb{Q}}(K)| \leq -\frac{1}{r}\chi(\Sigma).$$

Proof. The proof is essentially the same as the one given in [6]; see also [7]. The first thing we observe is that if v is a vector field that directs Σ_ξ, that is, v is zero only at the singularities of Σ_ξ and points in the direction of the orientation of the nonsingular leaves of Σ_ξ, then v is a generic section of the tangent bundle of Σ and points out of Σ along $\partial\Sigma$. Thus the Poincaré–Hopf theorem implies

$$\chi(\Sigma) = (e_+ - h_+) + (e_- - h_-).$$

Adding this equality to r times equation (1) gives

$$r\,\mathrm{sl}_{\mathbb{Q}}(K,[\Sigma]) + \chi(\Sigma) = 2(e_- - h_-).$$

So if we can isotop Σ relative to the boundary so that $e_- = 0$, then we clearly have the desired inequality. Recall that if an elliptic point and a hyperbolic point of the same sign are connected by a leaf in the characteristic foliation, then they may be canceled (without introducing any further singular points).

Thus we are left to show that for every negative elliptic point we can find a negative hyperbolic point that cancels it. To this end, given a negative elliptic point p, consider the *basin* of p, that is, the closure of the set of points in Σ that limit under the flow of v in backward time to p. Denote this set by B_p. Since the flow of v goes out the boundary of Σ, it is clear that B_p is contained in the interior of Σ. Thus we may analyze B_p exactly as in [6, 7] to find the desired negative hyperbolic point.

We briefly recall the main points of this argument. First, if there are repelling periodic orbits in the characteristic foliation, then add canceling pairs of positive elliptic and hyperbolic singularities to eliminate them. This prevents any periodic orbits in B_p, and thus one can show that B_p is the immersed image of a polygon that is an embedding on its interior. If B_p is the image of an embedding, then the boundary consists of positive elliptic singularities and hyperbolic singularities of either sign and flow lines between these singularities. If one of the hyperbolic singularities is negative, then we are done, since it is connected to B_p by a flow line. If none of the hyperbolic points are negative, then we can cancel them all with the positive elliptic singularities in ∂B_p, so that ∂B_p becomes a periodic orbit in the characteristic foliation and, more to the point, the boundary of an overtwisted disk. In the case that B_p is an immersed polygon, one may argue similarly; see [6, 7].

The inequality for Legendrian K clearly follows from considering the positive transverse pushoff of K and $-K$ and Lemma 1.2 together with the inequality in the transverse case. □

3 Rational Open Book Decompositions and Cabling

A *rational open book decomposition* of a manifold M is a pair (L,π) consisting of

- an oriented link L in M and
- a fibration $\pi\colon (M\setminus L)\to S^1$

such that no component of $\pi^{-1}(\theta)$ meets a component of L meridionally for any $\theta\in S^1$. We note that $\pi^{-1}(\theta)$ is a rational Seifert surface for the link L. If $\pi^{-1}(\theta)$ is actually a Seifert surface for L, then we say that (L,π) is an *open book decomposition* of M (or sometimes we will say an *integral* or *honest* open book decomposition of M). We call L the *binding* of the open book decomposition and $\pi^{-1}(\theta)$ a *page*.

The rational open book decomposition (L,π) of M *supports* a contact structure ξ if there is a contact form α for ξ such that

- $\alpha(v)>0$ for all positively pointing tangent vectors $v\in TL$ and
- $d\alpha$ is a volume form when restricted to (the interior of) each page of the open book.

Generalizing the work of Thurston and Winkelnkemper [14], the authors of this paper in work with Van Horn-Morris proved the following result.

Theorem 3.1 (Baker et al. 2008 [1]). *Let (L,π) be any rational open book decomposition of M. Then there exists a unique contact structure $\xi_{(L,\pi)}$ that is supported by (L,π).*

It is frequently useful to deal with only honest open book decompositions. One may easily pass from a rational open book decomposition to an honest one using cables, as we now demonstrate.

Given any knot K, let $N(K)$ be a tubular neighborhood of K, choose an orientation on K and an oriented meridian μ linking K positively once, and choose some oriented framing (i.e., longitude) λ on K such that $\{\lambda,\mu\}$ give longitude-meridian coordinates on $\partial N(K)$. The (p,q)*-cable* of K is the embedded curve (or collection of curves if p and q are not relatively prime) on $\partial N(K)$ in the homology class $p\lambda+q\mu$. Denote this curve (these curves) by $K_{p,q}$. We say that a cabling of K is *positive* if the cabling coefficients have slope greater than the Seifert slope of K. (The *slope* of the homology class $p\lambda+q\mu$ is q/p.)

If K is also a transverse knot with respect to a contact structure on M, then using the contactomorphism in the proof of Lemma 1.1 between the neighborhood

$N = N(K)$ and C_ϵ for sufficiently small ϵ, we may assume that the cable $K_{p,q}$ on ∂N is also transverse. As such, we call $K_{p,q}$ the *transverse* (p,q)*-cable*.

If $L = K_1 \cup \cdots \cup K_n$ is a link, then we can fix framings on each component of L and choose n pairs of integers (p_i, q_i). Then after setting $(\mathbf{p}, \mathbf{q}) = ((p_1, q_1), \ldots, (p_n, q_n))$, we denote by $L_{(\mathbf{p},\mathbf{q})}$ the result of (p_i, q_i)-cabling K_i for each i. It is easy to check (see, for example, [1]) that if L is the binding of a rational open book decomposition of M, then so is $L_{(\mathbf{p},\mathbf{q})}$, unless a component K_i of L is nontrivially cabled by curves of the fibration's restriction to $\partial N(K_i)$.

The following lemma says how the Euler characteristic of the fiber changes under cabling as well as the multiplicity and order of a knot.

Lemma 3.2. *Let L be a (rationally null-homologous) fibered link in M. Suppose K is a component of L for which the fibers in the fibration approach as (r,s)-curves (in some framing on K). Let L' be the link formed from L by replacing K by the (p,q)-cable of K, where $p \neq \pm 1, 0$ and $(p,q) \neq (kr, ks)$ for any $k \in \mathbb{Q}$. Then L' is fibered. Moreover, the Euler characteristic of the new fiber is*

$$\chi(\Sigma_{\text{new}}) = \frac{1}{\gcd(p,r)} \left(|p| \chi(\Sigma_{\text{old}}) + |ps - qr| (1 - |p|) \right),$$

where Σ_{new} is the fiber of L' and Σ_{old} is the fiber of L. The multiplicity of each component of the cable of K is

$$\gcd\left(\frac{r}{\gcd(p,r)}, \frac{p(rq - sp)}{\gcd(p,r)\gcd(p,q)} \right)$$

and the order of Σ_{new} along each component of the cable of K is

$$\frac{r}{\gcd(p,r)}.$$

□

The proof of this lemma may be found in [1], but it follows easily by observing that one may construct Σ_{new} by taking $|\frac{p}{\gcd(p,r)}|$ copies of Σ_{old} and $|\frac{rq-sp}{\gcd(p,r)}|$ copies of meridional disks to K and connecting them via $|\frac{p(rq-sp)}{\gcd(p,r)}|$ half-twisted bands.

Now suppose we are given a rational open book decomposition (L, π) of M. Suppose K is a rational binding component of an open book (L, π) whose page approaches K in an (r,s)-curve with respect to some framing on K. (Note that $r \neq 1$ and that r is not necessarily coprime to s.) For any $l \neq s$, replacing K in L by the (r,l)-cable of K gives a new link $L_{K_{(r,l)}}$ that by Lemma 3.2 is the binding of a (possibly rational) open book for M and has $\gcd(r,l)$ new components each having order and multiplicity 1. This is called the (r,l)*-resolution of L along K*. In the resolution, the new fiber is created using just one copy of the old fiber following the construction of the previous paragraph. Thus after resolving L along the other rational binding components, we have a new fibered link L' that is the binding of an integral open book (L', π'). This is called an *integral resolution of L*.

If we always choose the cabling coefficients (r,l) to have slope greater than the original coefficients (r,s), then we say that we have constructed a *positive (integral) resolution of L*.

Theorem 3.3 (Baker et al. 2008 [1]). *Let (L,π) be a rational open book for M supporting the contact structure ξ. If L' is a positive resolution of L, then L' is the binding of an integral open book decomposition for M that also supports ξ.*

4 Fibered Knots and the Bennequin Bound

Recall that in [10], null-homologous (nicely) fibered links satisfying the Bennequin bound were classified. In particular, the following theorem was proven.

Theorem 4.1 (Etnyre and Van Horn-Morris, 2008 [10]). *Let L be a fibered transverse link in a contact 3-manifold (M,ξ) and assume that ξ is tight when restricted to $M \setminus L$. Moreover, assume that L is the binding of an (integral) open book decomposition of M with page Σ. Then $\mathrm{sl}_\xi(L,\Sigma) = -\chi(\Sigma)$ if and only if either*

1. *ξ is supported by (L,Σ) or*
2. *ξ is obtained from $\xi_{(L,\Sigma)}$ by adding Giroux torsion along tori that are incompressible in the complement of L.*

In this section we generalize this theorem to allow for any rationally null-homologous knots. The link case will be dealt with in Sect. 6.

Theorem 4.2. *Let K be a rationally null-homologous fibered transverse knot in a contact 3-manifold (M,ξ) such that ξ is tight when restricted to the complement of K. Denote by Σ a fiber in the fibration of $M-K$ and let r be the order of K. Then $r\,\mathrm{sl}_{\mathbb{Q}}^{\xi}(K,\Sigma) = -\chi(\Sigma)$ if and only if either*

1. *ξ agrees with the contact structure supported by the rational open book determined by K or*
2. *ξ is obtained from the contact structure by adding Giroux torsion along tori that are incompressible in the complement of K.*

Proof. Let K' be a positive integral resolution of K. Then from Theorem 3.3 we know that K' and K support the same contact structure. In addition, the following lemma (with Lemma 3.2) implies that if $r\,\mathrm{sl}_{\mathbb{Q}}^{\xi}(K,\Sigma) = -\chi(\Sigma)$, then $\mathrm{sl}_\xi(K',\Sigma') = -\chi(\Sigma')$, where Σ' is a fiber in the fibration of $M-K'$. Thus the proof is finished by Theorem 4.1.

The other implication is obvious from the generalization of the Thurston–Winkelnkemper construction in [1] and (1), since the characteristic foliation on the page of a rational open book contains only positive singularities and while adding Giroux torsion adds negative singularities, they cancel in the computation of the rational self-linking number and the Euler characteristic of Σ. □

Lemma 4.3. *Let K be a rationally null-homologous transverse knot of order r in a contact 3-manifold. Fix some framing on K and suppose a rational Seifert surface Σ approaches K as a cone on an (r,s)-knot. Let K' be a (p,q)-cable of K that is positive and transverse in the sense described before Lemma 3.2 and let Σ' be the Seifert surface for K' constructed from Σ as in the previous section. Then*

$$\mathrm{sl}(K',[\Sigma']) = \frac{1}{\gcd(r,p)}\left(|p|r\,\mathrm{sl}_{\mathbb{Q}}(K,[\Sigma]) + |rq-sp|(-1+|p|)\right).$$

Proof. For each singular point in the characteristic foliation of Σ there are $\frac{|p|}{\gcd(p,r)}$ corresponding singular points on Σ' (coming from the $\frac{|p|}{\gcd(p,r)}$ copies of Σ used in the construction of Σ'). For each of the $\frac{|rq-sp|}{\gcd(p,r)}$ meridional disks used to construct Σ' we get one positive elliptic point in the characteristic foliation of Σ'. Finally, since cabling was positive, the $|\frac{p(rq-sp)}{\gcd(p,r)}|$ half-twisted bands added to create Σ' each have a single positive hyperbolic singularity in their characteristic foliation. (It is easy to check that the characteristic foliation is as described, since the construction takes place mainly in a solid torus neighborhood of K where we can write an explicit model for this construction.) The lemma follows now from Lemma 1.1. $\square$

5 Rational Unknots

A knot K in a manifold M is called a *rational unknot* if a rational Seifert surface D for K is a disk. Notice that the union of a neighborhood of K and a neighborhood of D is a punctured lens space. Thus the only manifold to have rational unknots (that are not actual unknots) are manifolds with a lens space summand. In particular, the only irreducible manifolds with rational unknots (that are not actual unknots) are lens spaces. So we restrict our attention to lens spaces in this section.

A knot K in a lens space is a rational unknot if and only if the complement of a tubular neighborhood of K is diffeomorphic to a solid torus. This, of course, implies that the rational unknots in $L(p,q)$ are precisely the cores of the Heegaard tori.

Theorem 5.1. *Rational unknots in tight contact structures on lens spaces are weakly transversely simple and Legendrian simple.*

A knot type is *weakly transversely simple* if it is up to contactomorphism (topologically) isotopic to the identity by its knot type and (rational) self-lining number. We have the analogous definition for *weakly Legendrian simple*. We will prove this theorem in the standard way. That is, we identify the maximal value for the rational Thurston–Bennequin invariant, show that there is a unique Legendrian knot with that rational Thurston–Bennequin invariant, and finally, show that any transverse unknot with nonmaximal rational Thurston–Bennequin invariant can be destabilized. The transverse result follows from the Legendrian result, as Lemma 1.4 shows.

5.1 Topological Rational Unknots

We explicitly describe $L(p,q)$ as follows: fix $p > q > 0$ and set

$$L(p,q) = V_0 \cup_\phi V_1,$$

where $V_i = S^1 \times D^2$ and we are thinking of S^1 and D^2 as the unit complex circle and disk, respectively. In addition the gluing map $\phi : \partial V_1 \to \partial V_0$ is given in standard longitude-meridian coordinates on the torus by the matrix

$$\begin{pmatrix} -p' & p \\ q' & -q \end{pmatrix},$$

where p' and q' satisfy $pq' - p'q = -1$ and $p > p' > 0$, $q \geq q' > 0$. We can find such p', q' by taking a continued fraction expansion of $-\frac{p}{q}$,

$$-\frac{p}{q} = a_0 - \cfrac{1}{a_1 - \cdots \frac{1}{a_{k-1} - \frac{1}{a_k}}},$$

with each $a_i \geq 2$ and then defining

$$-\frac{p'}{q'} = a_0 - \cfrac{1}{a_1 - \cdots \frac{1}{a_{k-1} - \frac{1}{a_k + 1}}}.$$

Since we have seen that a rational unknot must be isotopic to the core of a Heegaard torus, we clearly have four possible (oriented) rational unknots: $K_0, -K_0, K_1, -K_1$, where $K_i = S^1 \times \{pt\} \subset V_i$. We notice that K_0 represents a generator in the homology of $L(p,q)$, and $-K_0$ is the negative of that generator. So except in $L(2,1)$, the knots K_0 and $-K_0$ are not isotopic or homeomorphic via a homeomorphism isotopic to the identity. Similarly for K_1 and $-K_1$. Moreover, in homology, $q[K_0] = [K_1]$. So if $q \neq 1$ or $p-1$, then K_1 is not homeomorphic via a homeomorphism isotopic to the identity to K_0 or $-K_0$. We have established most of the following lemma.

Lemma 5.2. *The set of rational unknots up to homeomorphism isotopic to the identity in $L(p,q)$ is given by*

$$\{\textit{rational unknots in } L(p,q)\} = \begin{cases} \{K_1\}, & p = 2, \\ \{K_1, -K_1\}, & p \neq 2, q = 1 \textit{ or } p-1, \\ \{K_0, -K_0, K_1, -K_1\}, & q \neq 1 \textit{ or } p-1. \end{cases}$$

Proof. Recall that $L(p,q)$ is an S^1-bundle over S^2 if and only if $q = 1$ or $p-1$. In this case, K_0 and K_1 are both fibers in this fibration and hence are isotopic. We are left to see that K_0 and $-K_0$ are isotopic in $L(2,1) = \mathbb{R}P^3$. To this end, notice that K_0 can be thought of as an $\mathbb{R}P^1$. In addition, we have the natural inclusions $\mathbb{R}P^1 \subset \mathbb{R}P^2 \subset \mathbb{R}P^3$. It is easy to find an isotopy of $\mathbb{R}P^1 = K_0$ in $\mathbb{R}P^2$ that reverses the orientation. This isotopy easily extends to $\mathbb{R}P^3$. □

5.2 Legendrian Rational Unknots

For results concerning convex surfaces and standard neighborhoods of Legendrian knots we refer the reader to [9].

Recall that in the classification of tight contact structures on $L(p,q)$ given in [12], the following lemma was proven as part of Proposition 4.17.

Lemma 5.3. *Let N be a standard neighborhood of a Legendrian knot isotopic to the rational unknot K_1 in a tight contact structure on $L(p,q)$. Then there is another neighborhood with convex boundary N' such that $N \subset N'$ and $\partial N'$ has two dividing curves parallel to the longitude of V_1. Moreover, any two such solid tori with convex boundary each having two dividing curves of infinite slope have contactomorphic complements.*

We note that N' from this lemma is the standard neighborhood of a Legendrian knot L topologically isotopic to K_1. Moreover, one easily checks that

$$\mathrm{tb}_{\mathbb{Q}}(L) = -\frac{p'}{p},$$

where $p' < p$ is defined as in the previous sections. The next possible larger value for $\mathrm{tb}_{\mathbb{Q}}$ is $-\frac{p'}{p} + 1 > -\frac{1}{p}$, which violates the Bennequin bound.

Theorem 5.4. *The maximum possible value for the rational Thurston–Bennequin invariant for a Legendrian knot isotopic to K_1 is $-\frac{p'}{p}$, and it is uniquely realized, up to contactomorphism isotopic to the identity. Moreover, any Legendrian knot isotopic to K_1 with nonmaximal rational Thurston–Bennequin invariant destabilizes.*

Proof. The uniqueness follows from the last sentence in Lemma 5.3. The first part of the same lemma also establishes the destabilization result, since it is well known (see [9]) that if the standard neighborhood of a Legendrian knot is contained in the standard neighborhood of another Legendrian knot, then the first is a stabilization of the second. □

To finish the classification of Legendrian knots in the knot type of K_1, we need to identify the rational rotation number of the Legendrian knot L in the knot type of K_1 with maximal rational Thurston–Bennequin invariant. To this end, notice that if we fix the neighborhood N' from Lemma 5.3 as the standard neighborhood of the maximal rational Thurston–Bennequin invariant Legendrian knot L, then we can choose a nonzero section s of $\xi|_{\partial N'}$. This allows us to define a relative Euler class for $\xi|_{N'}$ and $\xi|_C$, where $C = \overline{L(p,q) - N'}$. One easily sees that the Euler class of $\xi|_{N'}$ vanishes and the Euler class $e(\xi)$ is determined by its restriction to the solid torus C. In particular, $\xi|_C$ is determined by

$$e(\xi)(D) = e(\xi|_C, s)(D) \mod p,$$

where D is the meridional disk of C and the generator of two-chains in $L(p,q)$.

Thinking of D as the rational Seifert surface for L, we can arrange the foliation near the boundary to be by Legendrian curves parallel to the boundary (see [12, Fig. 1]). From this we see that we can take a Seifert cable L_c of L to be Legendrian and satisfy

$$\mathrm{rot}_{\mathbb{Q}}(L) = \frac{1}{p}\mathrm{rot}(L_c).$$

By taking the foliation on $\partial N' = \partial C$ to be such that $D \cap \partial C$ is a ruling curve, we see that

$$\mathrm{rot}_{\mathbb{Q}}(L) = \frac{1}{p}\mathrm{rot}(L_c) = \frac{1}{p} e(\xi|_C, s)(D).$$

By the classification of tight contact structures on solid tori [12], we see that the number $e(\xi|_C, s)(D)$ is always a subset of $\{p' - 1 - 2k : k = 0, 1, \ldots, p' - 1\}$ and determined by the Euler class of ξ. To give a more precise classification we need to know the range of possible values for the Euler class of tight ξ on $L(p,q)$. This is in principal known, but difficult to state in general. We consider several cases in the next subsection.

We clearly have the analog of Theorem 5.4 for $-K_1$. That is all Legendrian knots in the knot type $-K_1$ destabilize to the unique maximal representative L with $\mathrm{tb}_{\mathbb{Q}}(L) = -\frac{p'}{p}$ and rotation number the negative of the rotation number for the maximal Legendrian representative of K_1.

Proof (Proof of Theorem 5.1). Notice that if $q^2 \equiv \pm 1 \mod p$, we have a diffeomorphism $\psi : L(p,q) \to L(p,q)$ that exchanges the Heegaard tori, and if $q = 1$ or $p - 1$, then this diffeomorphism is isotopic to the identity. Thus when $p \neq 2$ and $q = 1$ or $p - 1$, we have competed the proof of Theorem 5.1. Note also that we always have the diffeomorphism $\psi' : L(p,q) \to L(p,q)$ that preserves each of the Heegaard tori but acts by complex conjugation on each factor of each Heegaard torus (recall that the Heegaard tori are $V_i = S^1 \times D^2$, where S^1 and D^2 are a unit circle and disk in the complex plane, respectively). If $p = 2$, then this diffeomorphism is also isotopic to the identity. Thus we have finished the proof of Theorem 5.1 in this case.

We are left to consider the case that $q \neq 1$ or $p-1$. In this case we can understand K_0 and $-K_0$ by reversing the roles of V_0 and V_1. That is, we consider using the gluing map

$$\phi^{-1} = \begin{pmatrix} q & p \\ q' & p' \end{pmatrix}$$

to glue ∂V_0 to ∂V_1. $\square$

5.3 Classification Results

To give some specific classification results we recall that for the lens space $L(p,1)$, p odd, there is a unique tight contact structure for any given Euler class not equal to the zero class in $H_2(L(p,q);\mathbb{Z})$. From this, the fact that $p' = p-1$ in this case, and the discussion in the previous subsection we obtain the following theorem.

Theorem 5.5. *For p odd and any integer $l \in \{p-2-2k : k = 0,1,\dots,p-2\}$, there is a unique tight contact structure ξ_l on $L(p,1)$ with $e(\xi_l)(D) = l$ (here D is again the 2-cell in the CW-decomposition of $L(p,1)$ given in the last subsection). In this contact structure, the knot types K_1 and $-K_1$ are weakly Legendrian simple and transversely simple. Moreover, the rational Thurston–Bennequin invariants realized by Legendrian knots in the knot type K_1 are*

$$\left\{-\frac{p-1}{p} - k : k \text{ a nonpositive integer}\right\}.$$

The range for Legendrian knots in the knot type $-K_1$ is the same. The range of rotation numbers realized for a Legendrian knot in the knot type K_1 with rational Thurston–Bennequin invariant $-\frac{p-1}{p} - k$ is

$$\left\{\frac{l}{p} + k - 2m : m = 0,\dots,k\right\},$$

and for $-K_1$, the range is

$$\left\{\frac{-l}{p} + k - 2m : m = 0,\dots,k\right\}.$$

The range of possible rational self-linking numbers for transverse knots in the knot type K_1 is

$$\left\{-\frac{p+l-1}{p} - k : k \text{ a nonpositive integer}\right\},$$

and in the knot type $-K_1$ is

$$\left\{-\frac{p-l-1}{p}-k : k \ \text{ a nonpositive integer}\right\}.$$

Results for other $L(p,q)$ can easily be written down after the range of Euler classes for the tight contact structures is determined.

6 Rationally Null-Homologous Links and Uniform Seifert Surfaces

Much of our previous discussion for rational knots also applies to links, but many of the statements are a bit more awkward (or even uncertain) if we do not restrict to certain kinds of rational Seifert surfaces.

Let $L = K_1 \cup \cdots \cup K_n$ be an oriented link of $\mathbb{Z}$-homological order $r > 0$ in a 3-manifold M, and let us denote a tubular neighborhood of L by $N(L) = N(K_1) \cup \cdots \cup N(K_n)$. By $X(L)$ we denote the link exterior $\overline{M \setminus N(L)}$. Fix a framing for each $N(K_i)$. Since L has order r, there is an embedded (r, s_i)-curve on $\partial N(K_i)$ for each i, and together, they bound an oriented surface Σ° in $X(L)$. Radially coning $\partial\Sigma^\circ \subset N(L)$ to L gives a surface Σ in M whose interior is embedded and for which $\partial\Sigma|_{K_i}$ wraps r times around K_i. By tubing if needed, we may take Σ to be connected. Such a surface Σ will be called a *uniform rational Seifert surface* for L, and we say that L *r-bounds Σ*.

Notice that as $\mathbb{Z}$-homology chains, $\partial\Sigma = rL = 0$. Since as 1-chains there may exist varying integers r_i such that $r_1K_1 + \cdots + r_nK_n = 0$, the link L may have other rational Seifert surfaces that are not uniform. However, only for a uniform rational Seifert surface Σ do we have that $\partial(\frac{1}{r}\Sigma) = L$ as $\mathbb{Q}$-homology chains.

With respect to uniform rational Seifert surfaces, the definition of rational linking number for rationally null-homologous links extends directly: If L is an oriented link that r-bounds Σ and L' is another oriented link, then

$$\mathrm{lk}_{\mathbb{Q}}(L, L') = \frac{1}{r}\Sigma \cdot L'$$

with respect to $[\Sigma]$. If L' is rationally null-homologous and r'-bounds Σ', then this linking number is symmetric and independent of choice of Σ and Σ'.

It now follows that the entire content of Sects. 1 and 2 extends in a straightforward manner to transverse/Legendrian links L that r-bound a uniform rational Seifert surface Σ in a contact manifold. The generalization of Theorem 4.2 is straightforward as well, but relies on the generalized statements of Lemmas 3.2 and 4.3. Rather than record the general statements of these lemmas (which becomes cumbersome for arbitrary cables), we present them only for integral resolutions of rationally null-homologous links with uniform rational Seifert surfaces.

Lemma 6.1. *Let L be a link in M that r-bounds a uniform rational Seifert surface Σ for $r > 0$. Choose a framing on each component K_i of L, $i = 1, \dots, n$, such that Σ approaches K_i as (r, s_i)-curves. Let L' be the link formed by replacing each K_i by its (r, q_i)-cable, where $q_i \neq s_i$. If L is a rationally fibered link with fiber Σ, then L' is a (null-homologous) fibered link bounding a fiber Σ' with*

$$\chi(\Sigma') = \chi(\Sigma) + (1-r)\sum_{i=1}^{n} |s_i - q_i|.$$

Furthermore, assume that M is endowed with a contact structure ξ and that L is a transverse link. If the integral resolution L' of L is positive and transverse, then

$$\mathrm{sl}(L', [\Sigma']) = r\,\mathrm{sl}_{\mathbb{Q}}(K, [\Sigma]) + (-1+r)\sum_{i=1}^{n} |s_i - q_i|.$$

Proof. The construction of Σ' is done by attaching $|s_i - q_i|$ copies of meridional disks of $N(K_i)$ with $r|s_i - q_i|$ half-twisted bands to Σ for each i. Now follow the proof of Lemma 4.3. □

Theorem 6.2. *Let L be a rationally null-homologous fibered transverse link in a contact 3-manifold (M, ξ) such that ξ is tight when restricted to the complement of L. Suppose L r-bounds the fibers of the fibration of $M - L$ and let Σ be a fiber. Then $r\,\mathrm{sl}_{\mathbb{Q}}^{\xi}(L, \Sigma) = -\chi(\Sigma)$ if and only if either ξ agrees with the contact structure supported by the rational open book determined by L and Σ or ξ is obtained from the contact structure by adding Giroux torsion along tori that are incompressible in the complement of L.*

Proof. Follow the proof of Theorem 4.2 using Lemma 6.1 instead of Lemmas 3.2 and 4.3. □

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