Chapter 2
Clifford Algebra in Euclidean 3-Space

2.1 Reflections, Rotations, and Quaternions in E^3

2.1.1 Using Square Matrices to Represent Vectors

One frequently represents a vector $\mathbf{x}$ in the 3-dimensional Euclidean space E^3 by $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ or (x, y, z). However, neither of these notations easily generalize to higher dimensions. Alternate notations which do easily generalizes to higher dimensions are $\mathbf{x} = x^1\mathbf{i}_1 + x^2\mathbf{i}_2 + x^3\mathbf{i}_3$ and

$$\mathbf{x} = \left(x^1, x^2, x^3\right) = x^1\,(1, 0, 0) + x^2\,(0, 1, 0) + x^3\,(0, 0, 1)\,. \tag{2.1}$$

These alternate notations have their own problem. In most areas of mathematics, we expect a superscript to designate an exponent. You might think that we could reserve superscripts for exponents and use subscripts to designate different coordinates or other labels. This approach is sometimes used for so-called flat spaces. However, if we accept Einstein's Theory of General Relativity, we live in a space that is curved. To reserve superscripts for exponents in the study of curved spaces is simply too restrictive and inconvenient.

So how can you distinguish a superscript representing an exponent from a superscript representing some kind of label? If you see a superscript outside of some bracket (usually round), you can be confident that it represents an exponent. For example,

$$(a)^2 = aa.$$

On the other hand, if the meaning is clear from the context, the brackets may be omitted. For example, in the next chapter, I will write c to represent the speed of light and c^2 to represent the square of the speed of light.

I now turn to another issue. Usually, one represents a vector as a linear combinations of unit row vectors as in (2.1), or a linear combination of unit column

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vectors. However, as we shall soon see, it is sometimes useful to represent a vector as a linear combination of square matrices. For example, we could write

$$\mathbf{x} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3, \tag{2.2}$$

where

$$\mathbf{e}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \text{and}$$

$$\mathbf{e}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1. \end{bmatrix} \tag{2.3}$$

At first sight, this may seem to be a pointless variation. However, representing a vector in terms of these square matrices enables us to multiply vectors in a way that would not otherwise be possible. We should first note that these matrices have some special algebraic properties. In particular,

$$(\mathbf{e}_1)^2 = (\mathbf{e}_2)^2 = (\mathbf{e}_3)^2 = \mathbf{I}. \tag{2.4}$$

where $\mathbf{I}$ is the identity matrix. Furthermore,

$$\mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_3\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = 0. \tag{2.5}$$

A set of matrices that satisfy (2.4) and (2.5) is said to form the basis for the *Clifford algebra* associated with Euclidean 3-space. There are matrices other than those presented in (2.3) that satisfy (2.4) and (2.5). (See Prob. 2.) In the formalism of Clifford algebra, one never deals with the components of any specific matrix representation. We have introduced the matrices of (2.3) only to demonstrate that there exist entities that satisfy (2.4) and (2.5).

Now let us consider the product of two vectors. Suppose $\mathbf{y} = y^1\mathbf{e}_1 + y^2\mathbf{e}_2 + y^3\mathbf{e}_3$, then

$$\begin{aligned} \mathbf{xy} = {} & (x^1y^1 + x^2y^2 + x^3y^3)\mathbf{I} + x^2y^3\mathbf{e}_2\mathbf{e}_3 + x^3y^2\mathbf{e}_3\mathbf{e}_2 \\ & + x^3y^1\mathbf{e}_3\mathbf{e}_1 + x^1y^3\mathbf{e}_1\mathbf{e}_3 + x^1y^2\mathbf{e}_1\mathbf{e}_2 + x^2y^1\mathbf{e}_2\mathbf{e}_1. \end{aligned}$$

Using the relations of (2.5), we have

$$\begin{aligned} \mathbf{xy} = {} & (x^1y^1 + x^2y^2 + x^3y^3)\mathbf{I} + (x^2y^3 - x^3y^2)\mathbf{e}_2\mathbf{e}_3 \\ & + (x^3y^1 - x^1y^3)\mathbf{e}_3\mathbf{e}_1 + (x^1y^2 - x^2y^1)\mathbf{e}_1\mathbf{e}_2. \end{aligned} \tag{2.6}$$

(Note $\mathbf{xy} \neq \mathbf{yx}$.)

From (2.6), we can construct formulas for the familiar *scalar product* $\langle \mathbf{x}, \mathbf{y} \rangle$ and the less familiar *wedge product* $\mathbf{x} \wedge \mathbf{y}$. In particular,

$$\langle \mathbf{x}, \mathbf{y} \rangle \, \mathbf{I} = \frac{1}{2} (\mathbf{xy} + \mathbf{yx}) = \left(x^1 y^1 + x^2 y^2 + x^3 y^3 \right) \mathbf{I}, \text{ and} \tag{2.7}$$

$$\begin{aligned} \mathbf{x} \wedge \mathbf{y} = \frac{1}{2} (\mathbf{xy} - \mathbf{yx}) &= (x^2 y^3 - x^3 y^2) \mathbf{e}_2 \mathbf{e}_3 \\ &+ (x^3 y^1 - x^1 y^3) \mathbf{e}_3 \mathbf{e}_1 + (x^1 y^2 - x^2 y^1) \mathbf{e}_1 \mathbf{e}_2. \end{aligned} \tag{2.8}$$

With a slight abuse of notation, we frequently omit the $\mathbf{I}$ that appears in (2.7).

We note that the coefficients of $\mathbf{e}_2\mathbf{e}_3$, $\mathbf{e}_3\mathbf{e}_1$, and $\mathbf{e}_1\mathbf{e}_2$ that appear in the wedge product $\mathbf{x} \wedge \mathbf{y}$ are the three components of the cross product $\mathbf{x} \times \mathbf{y}$.

2.1.2 1-Vectors, 2-Vectors, 3-Vectors, and Clifford Numbers

By considering all possible products of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, one obtains an 8-dimensional space spanned by $\{\mathbf{I}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_3\mathbf{e}_1, \mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\}$, where

$$\mathbf{e}_2\mathbf{e}_3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_3\mathbf{e}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{e}_1\mathbf{e}_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

One might think that one could obtain higher order products. However, any such higher order product will collapse to a scalar multiple of one of the eight matrices already listed. For example:

$$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2(\mathbf{e}_3\mathbf{e}_2) = -\mathbf{e}_1\mathbf{e}_2(\mathbf{e}_2\mathbf{e}_3) = -\mathbf{e}_1(\mathbf{e}_2\mathbf{e}_2)\mathbf{e}_3 = -\mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_3\mathbf{e}_1.$$

In this fashion, we have obtained an 8-dimensional vector space that is closed under multiplication. A vector space closed under multiplication is called an *algebra*. An algebra that arises from a vector space with a scalar product in the same manner as this example does from E^3 is called a *Clifford algebra.* (We will give a more formal definition of a Clifford algebra in Chap. 4.)

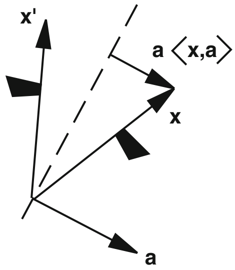


Fig. 2.1 The vector **x**' is the result of reflecting **x** with respect to the plane perpendicular to the unit vector **a**

I label the matrices $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ to be *Dirac vectors*. Any linear combination of Dirac vectors is a *1-vector*. A linear combination of $\mathbf{e}_2\mathbf{e}_3$, $\mathbf{e}_3\mathbf{e}_1$, and $\mathbf{e}_1\mathbf{e}_2$ is a *2-vector*. In the same vein, a scalar multiple of $\mathbf{I}$ is a *0-vector* and any scalar multiple of $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is a *3-vector*. A general linear combination of vectors of possibly differing type is a *Clifford number*.

It will be helpful to use an abbreviated notation for products of Dirac vectors. In particular, let

$$\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_{23}, \quad \mathbf{e}_3\mathbf{e}_1 = \mathbf{e}_{31}, \quad \mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_{12}, \quad \text{and} \quad \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_{123}.$$

2.1.3 Reflection and Rotation Operators

The algebraic properties of Clifford numbers provide us with a convenient way of representing reflections and rotations. Suppose **a** is a vector of unit length perpendicular to a plane passing through the origin and **x** is an arbitrary vector in E^3. (See Fig. 2.1.) In addition, suppose $\acute{\mathbf{x}}$ is the vector obtained from **x** by reflection of **x** with respect to the plane corresponding to **a**. Then

$$\acute{\mathbf{x}} = \mathbf{x} - 2\,\langle \mathbf{a}, \mathbf{x} \rangle\, \mathbf{a}. \tag{2.9}$$

From (2.7), it is clear that

$$2\,\langle \mathbf{a}, \mathbf{x} \rangle\, \mathbf{a} = (\mathbf{ax} + \mathbf{xa})\,\mathbf{a} = \mathbf{axa} + \mathbf{x}(\mathbf{a})^2 = \mathbf{axa} + \mathbf{x}.$$

So (2.9) becomes

$$\acute{\mathbf{x}} = -\mathbf{axa} \tag{2.10}$$

A rotation is the result of two successive reflections (See Fig. 2.2). From Fig. 2.2, it is clear that $\acute{\acute{\mathbf{x}}}$ is the vector that results from rotating vector **x** through the angle 2ψ about an axis with the direction of the axial vector $\mathbf{a} \times \mathbf{b}$. We can rewrite this relation in the form:

$$\acute{\acute{\mathbf{x}}} = -\mathbf{b}\acute{\mathbf{x}}\mathbf{b} = \mathbf{baxab}, \text{ or}$$

$$\acute{\acute{\mathbf{x}}} = \mathbf{R}^{-1}\mathbf{x}\mathbf{R} \quad \text{where} \quad \mathbf{R} = \mathbf{ab}. \tag{2.11}$$

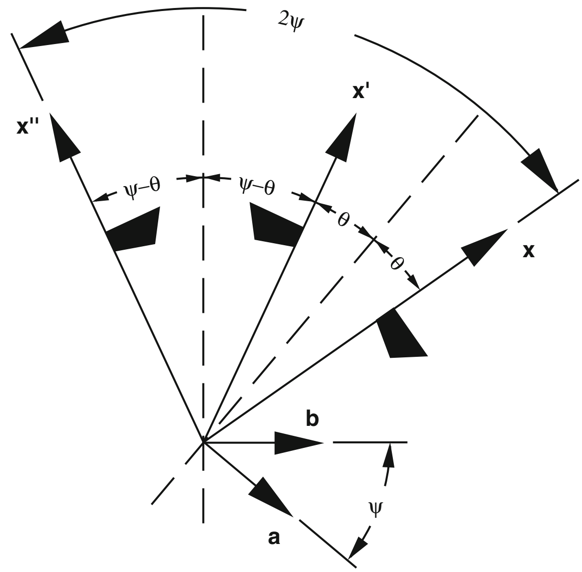


Fig. 2.2 When **x** is subjected to two successive reflections first with respect to a plane perpendicular to **a** and then with respect to a plane perpendicular to **b**, the result is a rotation of **x** about an axis in the direction of $\mathbf{a} \times \mathbf{b}$. The angle of rotation is twice the angle between **a** and **b**

It is useful to explicitly compute the product **ab** and interpret the separate components. If

$$\mathbf{a} = a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3,$$

and

$$\mathbf{b} = b^1\mathbf{e}_1 + b^2\mathbf{e}_2 + b^3\mathbf{e}_3,$$

then from (2.7) and (2.8):

$$\begin{aligned}\mathbf{R} = \mathbf{ab} &= \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) + \frac{1}{2}(\mathbf{ab} - \mathbf{ba}) \\ &= \mathbf{I}\,\langle \mathbf{a}, \mathbf{b} \rangle + \mathbf{a} \wedge \mathbf{b}.\end{aligned}$$

Since both **a** and **b** are vectors of unit length, $\langle \mathbf{a}, \mathbf{b} \rangle = \cos\psi$. Furthermore, the magnitude of $\mathbf{a} \times \mathbf{b}$ is $\sin\psi$. Although $\mathbf{a} \wedge \mathbf{b}$ unlike $\mathbf{a} \times \mathbf{b}$ is a 2-vector, $\mathbf{a} \wedge \mathbf{b}$ has the same three components as $\mathbf{a} \times \mathbf{b}$. For this reason, we can write

$$\mathbf{a} \wedge \mathbf{b} = \left(n^1\mathbf{e}_{23} + n^2\mathbf{e}_{31} + n^3\mathbf{e}_{12}\right)\sin\psi,$$

where n^1, n^2, and n^3 are the direction cosines of the axial vector $\mathbf{a} \times \mathbf{b}$. With this thought in mind, we have

$$\mathbf{R} = \mathbf{I}\cos\psi + (n^1\mathbf{e}_{23} + n^2\mathbf{e}_{31} + n^3\mathbf{e}_{12})\sin\psi.$$

Note! These ideas can be generalized to higher dimensions. For higher dimensions the entity $\mathbf{a} \wedge \mathbf{b}$ remains well defined, while $\mathbf{a} \times \mathbf{b}$ becomes meaningless. In higher dimensions, you no longer have an axis of rotation; so you must think of the rotation as occurring in the 2-dimensional plane spanned by **a** and **b**.

We should note that ψ represents $\frac{1}{2}$ the angle of rotation. If θ is the actual angle of rotation, we then have

$$\mathbf{R} = \mathbf{I}\cos\frac{\theta}{2} + (n^1\mathbf{e}_{23} + n^2\mathbf{e}_{31} + n^3\mathbf{e}_{12})\sin\frac{\theta}{2}. \tag{2.12}$$

To obtain $\mathbf{R}^{-1}$ from $\mathbf{R}$, one can replace θ by $-\theta$ or reverse the order of the Dirac vectors. In either case,

$$\mathbf{R}^{-1} = \mathbf{I}\cos\frac{\theta}{2} - (n^1\mathbf{e}_{23} + n^2\mathbf{e}_{31} + n^3\mathbf{e}_{12})\sin\frac{\theta}{2}. \tag{2.13}$$

Returning to (2.11), we see that there appears to be two representations for the same rotation. In the context of (2.11), $\mathbf{R}$ is equivalent to $-\mathbf{R}$. From (2.12), we see that changing the sign of $\mathbf{R}$ is equivalent to replacing θ by $\theta + 2\pi$. Indeed, the operator $\mathbf{R}$ does not have the expected periodicity of 2π, but it does have a periodicity of 4π. One's first reaction is to think that Clifford algebra has introduced an undesirable complication. In the context of (2.11), this may be the case. However, there are circumstance for which this "complication" corresponds to physical reality. We will discuss this point in the next section.

Meanwhile, we note that for k reflections:

$$\acute{\mathbf{x}} = (-1)^k\mathbf{a}_k\mathbf{a}_{k-1}\ldots\mathbf{a}_1\mathbf{x}\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_k = (-1)^k\mathbf{T}^{-1}\mathbf{x}\mathbf{T}. \tag{2.14}$$

2.1.4 Quaternions

Using quaternions, you can represent a rotation operator in a form essentially identical to that which appears in (2.12). What are quaternions? They were invented (discovered?) by William Rowan Hamilton (1805–1865) in 1843. Before that time, it had been observed that the multiplication of complex numbers could be interpreted as the multiplication of points in a 2-dimensional plane. This was first done by Casper Wessel (1745–1818) in 1797 and then again independently by Jean Robert Argand (1768–1822) in 1806 (Kramer 1981, pp. 72–73). In particular, instead of writing:

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc), \text{ one can write,}$$
$$(a, b)(c, d) = (ac - bc, ad + bc).$$

The question that Hamilton asked himself was, "Could there be a 3-dimensional version of this multiplication that would be useful for the study of physics?" Since his idea was to generalize the notion of complex numbers, he was investigating triples of the form: $a + \mathbf{i}b + \mathbf{j}c$. You can invent all kinds of multiplication rules, but he was looking for a rule that would be meaningful and useful for the study of physics. Starting in 1828, he spent 15 years on this project without success. Finally on October 16, 1843 (a Monday), he had an eureka experience. He was walking along side of the Royal Canal in Dublin with his wife to preside at a Council meeting of the Royal Irish Academy. Then it dawned on him that he should introduce a fourth dimension. In this joyful moment, he carved the formulas for multiplying numbers of the form: $a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$ on a stone of the Broome Bridge (or Brougham Bridge as he called it). ((O'Connor and Robertson: Hamilton) and (Boyer 1968, p. 625)).

Time has obliterated the original carving but in 1958, the Royal Irish Academy erected a plaque commemorating the event:

Here as he walked by
on the 16th of October1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$
and cut it in a stone on this bridge.

From the formula that Hamilton carved in stone, it can be shown that

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \text{ and } \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

(See Prob. 3.)

Due to this achievement, William Hamilton is known as the founder of modern "abstract algebra."

In the theory of quaternions, a rotation operator corresponding to that which appears in (2.12) is written in the form:

$$\mathbf{R} = \mathbf{I}\cos\frac{\theta}{2} - (n^1\mathbf{i} + n^2\mathbf{j} + n^3\mathbf{k})\sin\frac{\theta}{2}. \tag{2.15}$$

Comparison with (2.12) suggests that we can identify identify **i**, **j,** and **k**, respectively, with $-\mathbf{e}_{23}$, $-\mathbf{e}_{31}$, and $-\mathbf{e}_{12}$. As mentioned above, the binary relations for quaternion multiplication are:

$$(\mathbf{i})^2 = (\mathbf{j})^2 = (\mathbf{k})^2 = -1, \tag{2.16}$$

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \tag{2.17}$$

$$\mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \text{ and} \tag{2.18}$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}. \tag{2.19}$$

You should check that the same equations hold for the corresponding 2-vectors associated with E^3. Namely:

$$(-\mathbf{e}_{23})^2 = (-\mathbf{e}_{31})^2 = (-\mathbf{e}_{12})^2 = -\mathbf{I}, \tag{2.20}$$

$$(-\mathbf{e}_{31})(-\mathbf{e}_{12}) = -(-\mathbf{e}_{12})(-\mathbf{e}_{31}) = (-\mathbf{e}_{23}), \tag{2.21}$$

$$(-\mathbf{e}_{12})(-\mathbf{e}_{23}) = -(-\mathbf{e}_{23})(-\mathbf{e}_{12}) = (-\mathbf{e}_{31}), \tag{2.22}$$

$$\text{and} \quad (-\mathbf{e}_{23})(-\mathbf{e}_{31}) = -(-\mathbf{e}_{31})(-\mathbf{e}_{23}) = (-\mathbf{e}_{12}). \tag{2.23}$$

In Hamilton's formulation, a vector $\mathbf{x}$ is represented as $x^1\mathbf{i}+x^2\mathbf{j} + x^3\mathbf{k}$ and the rotated vector $\acute{\mathbf{x}}$ is computed by the quaternion version of (2.12).

Neither the usual vector formulation nor the Hamilton approach makes a good distinction between an ordinary vector and an *axial* or *pseudo-vector*.

As we have seen, in the formalism of Clifford algebra, an ordinary vector appears as a l-vector and a plane of rotation appears as a 2-vector. In three dimensions, a 1-vector and a 2-vector both have three components. In the usual vector formalism, they both appear as 1-vectors. In the quaternion formulation, they both appear as 2-vectors.

The distinction between the two entities arises if we consider a reflection. If, for example, we consider a reflection with respect to the y-z plane, we have

$$\acute{\mathbf{x}} = -\mathbf{e}_1\mathbf{x}\mathbf{e}_1.$$

If

$$\mathbf{x} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3, \text{ then}$$

$$\mathbf{x}' = -x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3.$$

On the other hand, under the same reflection the 2-vector

$$\mathbf{X} = x^1\mathbf{e}_{23} + x^2\mathbf{e}_{31} + x^3\mathbf{e}_{12} = x^1\mathbf{e}_2\mathbf{e}_3 + x^2\mathbf{e}_3\mathbf{e}_1 + x^3\mathbf{e}_1\mathbf{e}_2$$

becomes

$$\acute{\mathbf{X}} = x^1(-\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1)(-\mathbf{e}_1\mathbf{e}_3\mathbf{e}_1) + x^2(-\mathbf{e}_1\mathbf{e}_3\mathbf{e}_1)(-\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1) + x^3(-\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1)(-\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1) \text{ or}$$

$$\acute{\mathbf{X}} = x^1\mathbf{e}_{23} - x^2\mathbf{e}_{31} - x^3\mathbf{e}_{12}.$$

This same distinction is carried out in the usual vector formulation but in a somewhat awkward fashion. Let us consider the cross product $\mathbf{x} \times \mathbf{y}$. Suppose

$$\mathbf{x} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3, \text{ and}$$

$$\mathbf{y} = y^1\mathbf{e}_1 + y^2\mathbf{e}_2 + y^3\mathbf{e}_3, \text{ then}$$

$$\mathbf{x} \times \mathbf{y} = (x^2y^3 - x^3y^2)\mathbf{e}_1 + (x^3y^1 - x^1y^3)\mathbf{e}_2 + (x^1y^2 - x^2y^1)\mathbf{e}_3.$$

How should the cross product transform under a reflection with respect to the y-z plane? If we treat $\mathbf{x} \times \mathbf{y}$ as an ordinary vector, then

$$(\mathbf{x} \times \mathbf{y})' = -(x^2y^3 - x^3y^2)\mathbf{e}_1 + (x^3y^1 - x^1y^3)\mathbf{e}_2 + (x^1y^2 - x^2y^1)\mathbf{e}_3.$$

On the other hand, if we carry out the same reflection on $\mathbf{x}$ and $\mathbf{y}$ before computing the cross product, we have

$$\acute{\mathbf{x}} = -\mathbf{x}^1\mathbf{e}_1 + \mathbf{x}^2\mathbf{e}_2 + \mathbf{x}^3\mathbf{e}_3,$$

$$\acute{\mathbf{y}} = -\mathbf{y}^1\mathbf{e}_1 + \mathbf{y}^2\mathbf{e}_2 + \mathbf{y}^3\mathbf{e}_3, \text{ and}$$

$$\acute{\mathbf{x}} \times \acute{\mathbf{y}} = (x^2y^3 - x^3y^2)\mathbf{e}_1 - (x^3y^1 - x^1y^3)\mathbf{e}_2 - (x^1y^2 - x^2y^1)\mathbf{e}_3.$$

When this second interpretation of the impact of a reflection on $\mathbf{x} \times \mathbf{y}$ is applied, $\mathbf{x} \times \mathbf{y}$ is said to be an *axial* or *pseudo-vector*. In the context of Clifford algebra a pseudo-vector is a 2-vector and this awkwardness disappears. Similarly, the entity $\langle \mathbf{x} \times \mathbf{y}, \mathbf{z} \rangle$, which is referred to as a *pseudo-scalar* in the usual vector formulation, becomes a 3-vector in Clifford algebra.

In three dimensions, it is still useful to use the usual *cross product*, when one seeks a vector that is perpendicular to a plane spanned by two vectors such as $\mathbf{x}$ and $\mathbf{y}$. Thus, we will still use the usual definition:

$$\mathbf{x} \times \mathbf{y} = (x^2y^3 - x^3y^2)\mathbf{e}_1 + (x^3y^1 - x^1y^3)\mathbf{e}_2 + (x^1y^2 - x^2y^1)\mathbf{e}_3.$$

However, we will also need the notion of a *wedge product* that we defined in (2.8). Namely:

$$\mathbf{x} \wedge \mathbf{y} = \frac{1}{2}(\mathbf{xy} - \mathbf{yx}) = (x^2y^3 - x^3y^2)\mathbf{e}_{23} + (x^3y^1 - x^1y^3)\mathbf{e}_{31} + (x^1y^2 - x^2y^1)\mathbf{e}_{12}.$$

In closing this section, we wish to bring to your attention the notion of *orthogonal transformations*. An orthogonal transformation is simply a product of reflections. This terminology is chosen when one wishes to focus on the fact that the standard scalar product in E^n is preserved. In this chapter, we have restricted ourselves to E^3. In this context, it is appropriate that you verify the fact that products of reflections do indeed preserve the scalar product (at least in E^3). (See Probs. 6 and 7.)

The product of an even number of reflections (a rotation) is called a *proper orthogonal transformation*, while the product of and odd number of reflections is called an *improper orthogonal transformation*.

Problem 1. From the form of (2.11), it is clear that if the rotation operators $\mathbf{R}$ and $\acute{\mathbf{R}}$ represent two successive rotations, then the combined rotation is represented by the product $\mathbf{R}\acute{\mathbf{R}}$. Use this fact and (2.12) to show that a 90^0 rotation about the y-axis followed by a 90^0 rotation about the x-axis is equivalent to a 120^0 rotation about the axis, which has the direction of the vector $(1, 1, 1)$.

Problem 2. There are many representations that can be used for $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. One convenient representation is that using *Pauli matrices* $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, and $\boldsymbol{\sigma}_3$. That is, we can let

$$\mathbf{e}_1 = \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{e}_2 = \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Show that in this representation, (2.4) and (2.5) are satisfied.

Problem 3. If you assume associativity for the multiplication of quaternions, then using the equations that appears on Hamilton's plaque, we have

$$\mathbf{ijk} = -1 \Rightarrow (\mathbf{i})^2\mathbf{jk} = -\mathbf{i} \Rightarrow -\mathbf{jk} = -\mathbf{i} \Rightarrow \mathbf{jk} = \mathbf{i}.$$

(a) In a similar fashion, show

$$\mathbf{ki} = \mathbf{j} \ \text{ and } \ \mathbf{ij} = \mathbf{k}.$$

(b) Also show that

$$\mathbf{kj} = -\mathbf{i}, \ \mathbf{ik} = -\mathbf{j}, \ \text{and} \ \mathbf{ji} = -\mathbf{k}.$$

Problem 4. In the representation introduced in Prob. 2, the quaternions $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are represented by complex 2×2 matrices. In particular,

$$\mathbf{i} = -\mathbf{e}_{23} = -i\boldsymbol{\sigma}_1 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \ \mathbf{j} = -\mathbf{e}_{31} = -i\boldsymbol{\sigma}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\text{and} \ \mathbf{k} = -\mathbf{e}_{12} = -i\boldsymbol{\sigma}_3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}.$$

In this representation, the rotation operator

$$\begin{aligned} \mathbf{R} &= \mathbf{I}\cos\frac{\theta}{2} + (\mathbf{e}_{23}n^1 + \mathbf{e}_{31}n^2 + \mathbf{e}_{12}n^3)\sin\frac{\theta}{2} \\ &= \begin{bmatrix} \cos\frac{\theta}{2} + in^3\sin\frac{\theta}{2} & (n^2 + in^1)\sin\frac{\theta}{2} \\ -(n^2 - in^1)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} - in^3\sin\frac{\theta}{2} \end{bmatrix}. \end{aligned}$$

Show that in this representation, the matrix representing $\mathbf{R}$ is unitary and has determinant equal to 1. (From this result, it is clear that the algebraic properties of the double-valued rotation operators for three dimensions can be ascertained by studying the algebraic properties of 2×2 unitary matrices whose determinant is 1. For this reason, the group of double-valued rotation operators is labeled $\mathbf{SU}(2)$. The letter $\mathbf{U}$ indicates "unitary". The letter $\mathbf{S}$ indicates "special", which in the context of group representation theory means the determinant is 1.)

Problem 5. Suppose

$$\mathbf{R} = \mathbf{I}\cos\frac{\theta}{2} + \hat{\mathbf{n}}\sin\frac{\theta}{2}, \quad \text{where}$$
$$\hat{\mathbf{n}} = n^1\mathbf{e}_{23} + n^2\mathbf{e}_{31} + n^3\mathbf{e}_{12}.$$

(a) Using the fact that

$$(n^1)^2 + (n^2)^2 + (n^3)^2 = 1, \text{ show that}$$
$$(\hat{\mathbf{n}})^2 = -1.$$

(b) Show that $\exp\left[\hat{\mathbf{n}}(\frac{\theta}{2})\right] = \mathbf{R}$. Hint: represent $\exp\left[\hat{\mathbf{n}}(\frac{\theta}{2})\right]$ by a Taylor's series and then separate the odd and even odd and even powers $\hat{\mathbf{n}}$.

Problem 6. Suppose $\acute{\mathbf{x}} = -\mathbf{axa}$ and $\acute{\mathbf{y}} = -\mathbf{aya}$, where $\mathbf{a}$ is a unit vector. Show $\langle\acute{\mathbf{x}}, \acute{\mathbf{y}}\rangle = \langle\mathbf{x}, \mathbf{y}\rangle$. (Remember from (2.7), $\langle\mathbf{x}, \mathbf{y}\rangle\,\mathbf{I} = \frac{1}{2}(\mathbf{xy} + \mathbf{yx})$.)

Problem 7. Suppose $\acute{\mathbf{x}} = (-1)^k\mathbf{a}_k\mathbf{a}_{k-1}\ldots\mathbf{a}_1\mathbf{x}\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_k$ and $\mathbf{y}' = (-1)^k\mathbf{a}_k\mathbf{a}_{k-1}\ldots\mathbf{a}_1\mathbf{y}\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_k$. Show $\langle\acute{\mathbf{x}}, \acute{\mathbf{y}}\rangle = \langle\mathbf{x}, \mathbf{y}\rangle$.

2.2 The 4π Periodicity of the Rotation Operator

From the consequences of the last section, we see that if the vector $\mathbf{x}(\theta)$ represents the result of rotating vector $\mathbf{x}(0)$ through an angle θ, then we can represent the rotation in the form:

$$\mathbf{x}(\theta) = \mathbf{R}^{-1}(\theta)\mathbf{x}(0)\mathbf{R}(\theta), \quad \text{where}$$
$$\mathbf{R}(\theta) = \mathbf{I}\cos\frac{\theta}{2} + \hat{\mathbf{n}}\sin\frac{\theta}{2},$$
$$\hat{\mathbf{n}} = n^1\mathbf{e}_{23}+n^2\mathbf{e}_{31}+n^3\mathbf{e}_{12}, \quad \text{and}$$

n^1, n^2, along with n^3 are the direction cosines for the axis of rotation.

Although $\mathbf{x}(\theta)$ has a period of 2π, $\mathbf{R}(\theta)$ has a period of 4π! With the development of quantum mechanics in the 1920s, it became recognized that a 4π periodicity sometimes occurs in nature. To explain the observed structure of the hydrogen energy spectrum, it was necessary to attribute to the electron a spin of $\frac{1}{2}$ and a periodicity of 4π. Later, it became recognized that some objects larger than electrons also have a 4π periodicity (Bolker 1973). A demonstration of this fact has been put forward by Edgar Riefin (1979).

For an object to display a 4π periodicity, it is necessary that it be in some sense attached to its surroundings.

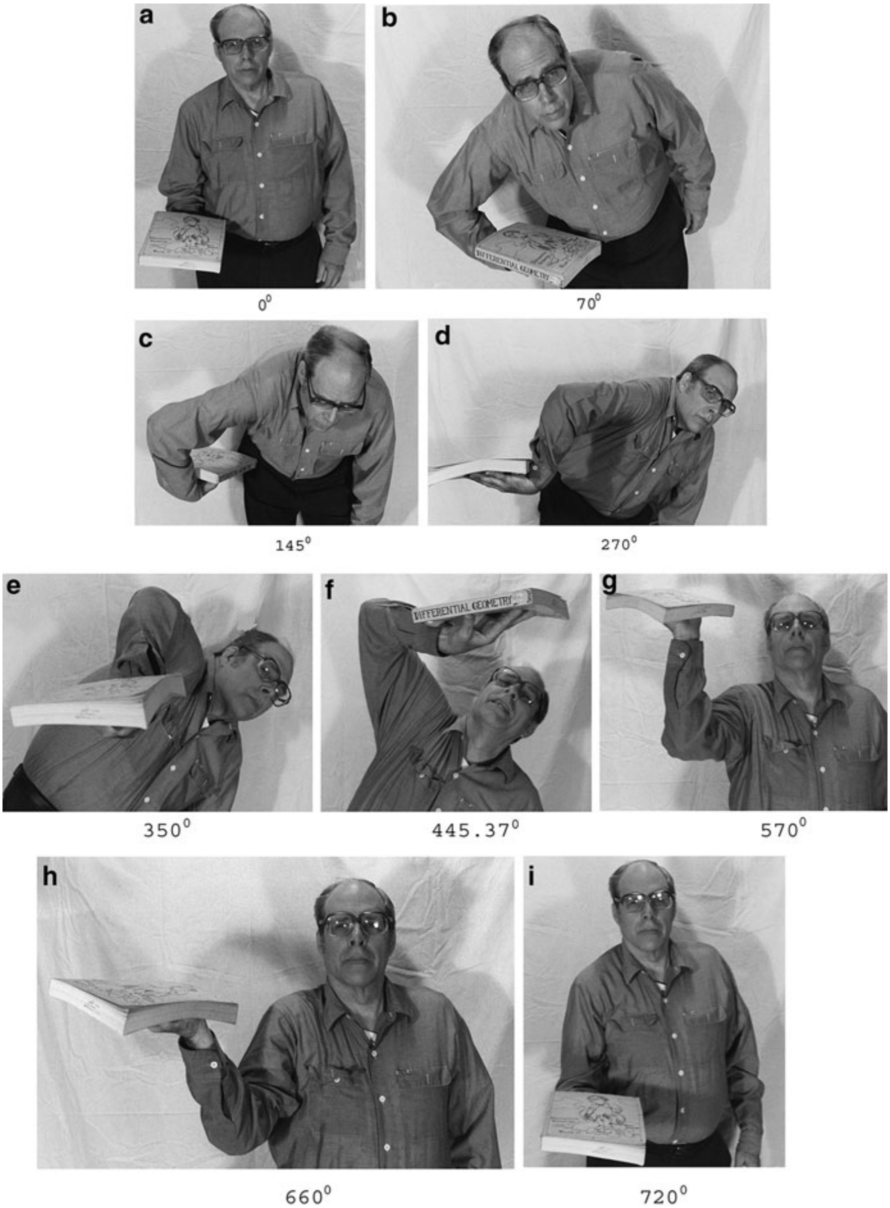


Fig. 2.3 A book with a 4π periodicity

To illustrate this, you may wish to carry out a demonstration. First, hold a glass of water in the palm of your hand. The hand holding the glass may be left or right but it is important that your hand be under the glass with palm up. Then maintain a firm grip on the glass and rotate it 360^0 without moving your feet or spilling any water.

When you have completed this maneuver, you will find yourself in an awkward position with the glass slightly above your head and your elbow pointed upward. Clearly, the relationship of the glass to you is quite different from what it was in its initial position. However, if you continue the rotation, you may be surprised to find that your arm will unwind itself and the glass will return to its initial position with its initial relationship to you. Thus, the glass attached to your arm does not have a 2π periodicity but it does have a 4π periodicity.

This demonstration is shown in Fig. 2.3 where a book is used in place of a glass of water.

2.3 *The Point Groups for the Regular Polyhedrons

One aspect of geometry, which attracts a lot of attention in physics, is symmetry groups. The symmetry of a body can be characterized by the set of transformations that maintain distances between points and bring the body into its original space of occupation. Quite reasonably, these are called *symmetry transformations*. For infinite bodies (for example an infinite crystal lattice), the set of symmetry transformations may contain translations.

But for finite bodies, symmetry transformations are restricted to rotations and products of rotations and reflections. For this reason, Clifford algebra is a good tool to attack the mathematics of symmetry for finite bodies.

Before getting very deep into this topic, it is useful to prove a theorem by Élie Cartan (1938, pp. 13–17; 1966, pp. 10–12). His theorem states that in an n-dimensional space (real or complex), a transformation consisting of any finite number of reflections can also be obtained by a number of reflections that does not exceed n.

In this text, we only need the real 3-dimensional version and that is the only version we will prove.

Theorem 8. *Suppose* $\acute{\mathbf{x}} = (-1)^k \mathbf{a}_k\, \mathbf{a}_{k-1} \ldots \mathbf{a}_1 \mathbf{x} \mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_k$. *That is we have a transformation consisting of k reflections. Then this same transformation (in* E^3*) can be achieved by three or fewer reflections.*

Proof. Case 1. The number of reflections k is even. If we multiply an even number of 1-vectors, we get a linear combination of the 0-vector $\mathbf{I}$ and the three 2-vectors $\mathbf{e}_{23}$, $\mathbf{e}_{31}$, and $\mathbf{e}_{21}$. That is

$$\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_k = \mathbf{I}\alpha + \mathbf{e}_{23}\beta^1 + \mathbf{e}_{31}\beta^2 + \mathbf{e}_{12}\beta^3.$$

(This already looks like a rotation operator!) The operator $\mathbf{a}_k \mathbf{a}_{k-1} \ldots \mathbf{a}_1$ is essentially the same as $\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_k$ except for the fact that the underlying Dirac vectors are in reverse order. Thus,

$$\mathbf{a}_k \mathbf{a}_{k-1} \ldots \mathbf{a}_1 = \mathbf{I}\alpha + \mathbf{e}_{32}\beta^1 + \mathbf{e}_{13}\beta^2 + \mathbf{e}_{21}\beta^3 = \mathbf{I}\alpha - \mathbf{e}_{23}\beta^1 - \mathbf{e}_{31}\beta^2 - \mathbf{e}_{12}\beta^3.$$

Since $(\mathbf{a}_1)^2 = (\mathbf{a}_2)^2 = \ldots = (\mathbf{a}_k)^2 = \mathbf{I}$, $(\mathbf{a}_k\mathbf{a}_{k-1}\ldots\mathbf{a}_1)(\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_k) = \mathbf{I}$, and

$$\begin{aligned}\mathbf{I} &= \left(\mathbf{I}\alpha - \mathbf{e}_{23}\beta^1 - \mathbf{e}_{31}\beta^2 - \mathbf{e}_{12}\beta^3\right)\left(\mathbf{I}\alpha + \mathbf{e}_{23}\beta^1 + \mathbf{e}_{31}\beta^2 + \mathbf{e}_{12}\beta^3\right)\\ &= \mathbf{I}\left((\alpha)^2 + (\beta^1)^2 + (\beta^2)^2 + (\beta^3)^2\right).\end{aligned}$$

Since $(\alpha)^2 + (\beta^1)^2 + (\beta^2)^2 + (\beta^3)^2 = 1$, there exists an angle ψ such that

$$\cos\psi = \alpha \quad \text{and} \quad \sin\psi = \sqrt{(\beta^1)^2 + (\beta^2)^2 + (\beta^3)^2}.$$

Furthermore, if at least one of the β^k's is not zero, we can define the direction cosines for the axis of rotation by

$$n^k = \beta^k / \sqrt{(\beta^1)^2 + (\beta^2)^2 + (\beta^3)^2} = \beta^k / \sin\psi \quad \text{for } k = 1, 2, \text{and } 3.$$

(Note! this definition guarantees that $(n^1)^2 + (n^2)^2 + (n^3)^2 = 1$.) We now have shown:

$$\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_k = \mathbf{I}\cos\psi + (n^1\mathbf{e}_{23} + n^2\mathbf{e}_{31} + n^3\mathbf{e}_{12})\sin\psi.$$

If the $\sin\psi = 0$, $\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_k = \pm\mathbf{I}$. Otherwise, we have a nontrivial rotation operator. From Fig. 2.2, it is clear that this rotation operator can be replaced by a product of two reflections.

Case 2. The number of reflections k is odd.

In this case, we can multiply out the first k-1 reflections to get a rotation operator and we then have:

$$\begin{aligned}\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_k &= \left[\mathbf{I}\cos\psi + (n^1\mathbf{e}_{23} + n^2\mathbf{e}_{31} + n^3\mathbf{e}_{12})\sin\psi\right]\mathbf{a}_k\\ &= \left[\mathbf{I}\cos\psi + (n^1\mathbf{e}_{23} + n^2\mathbf{e}_{31} + n^3\mathbf{e}_{12})\sin\psi\right](k^1\mathbf{e}_1 + k^2\mathbf{e}_2 + k^3\mathbf{e}_3).\end{aligned}$$

If $\sin\psi = 0$ or $k^1n^1 + k^2n^2 + k^3n^3 = 0$, our product $\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_k$ reduces to a 1-vector. Otherwise after factoring the rotation into two reflections, we have the product of three reflections. □

Now we are in a position to have a reasonably intelligent discussion of symmetry groups. Generally, the set of multiple reflections that bring a particular finite body into its original position in space is called a *point group* for two reasons. One is due to the fact that at least one point remains fixed under all symmetry transformations associated with a particular body. The second is due to the fact that the set of the symmetry transformations identified with a particular body forms a mathematical structure known as a *group*.

Definition 9. A group is a set of elements with a binary operation $\circ$ having the following properties:

(1) Closure: $g_1 \in G, g_2 \in G \Rightarrow g_1 \circ g_2 \in G$.
(2) Associativity: $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.

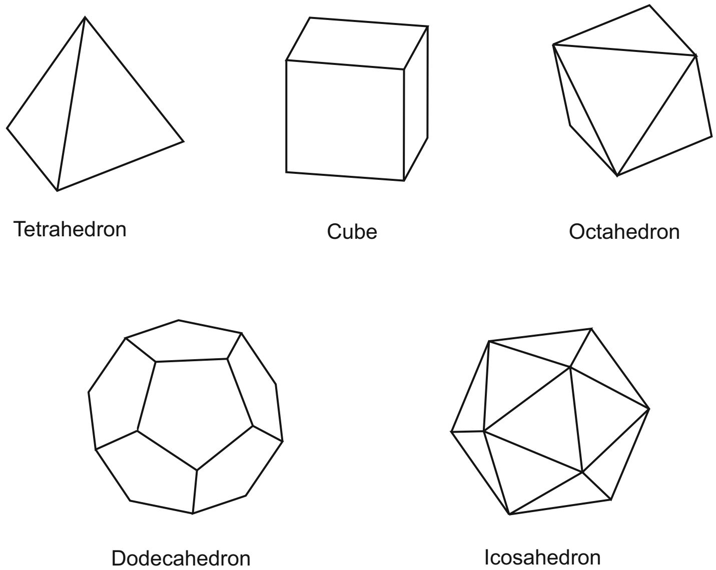


Fig. 2.4 The five regular polyhedrons

(3) Identity element: $\exists$ an element $e \in G$ such that $\forall\ g \in G$, $e \circ g = g \circ e = g$.
(4) Inverse: $\forall\ g \in G$, $\exists\ g^{-1} \in G$ such that $g \circ g^{-1} = g^{-1} \circ g = e$.

Examples of groups include the integers under addition, the positive rational numbers under multiplication, and nonsingular n×n matrices under matrix multiplication.

We will only give a short description of a few point groups – in particular the five point groups associated with the five regular polyhedrons. (See Fig. 2.4.) For each of the polyhedrons, we have a finite symmetry group. One way to verify we have a group is to run through the check list in the definition above.

The elements of a symmetry group for a finite solid are finite products of reflections. It is clear that the multiplication of two finite products results in a finite product, which preserves the original position of the relevant solid. Thus, the set of symmetry transformations satisfy the property of closure.

The identity element corresponds to the transformation that does nothing or rotates the solid some integral multiple of 360^0.

To obtain the inverse of a product of reflections, one simply constructs the product of the same reflections in the reverse order.

To show that the symmetry groups for the regular polyhedrons have only a finite number of members, let us consider the example of the cube. (See Fig. 2.5.) Applying Cartan's theorem, we know that an even number of reflections (a proper orthogonal transformation) can be reduced to either the identity element or a rotation. The possible symmetry rotations are not difficult to count.

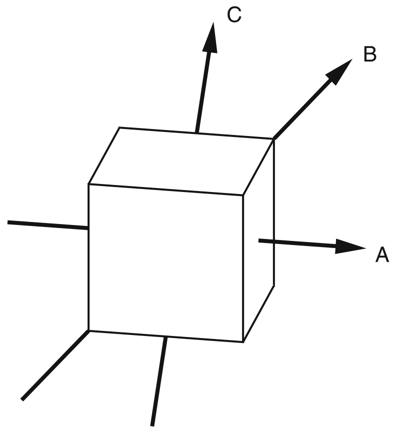


Fig. 2.5 Some symmetry axes of rotation for the cube

Perhaps, the most obvious symmetry rotations are those that correspond to the fourfold axes that pass through the centers of opposite faces. Not counting the 360^0 identity rotation, we have symmetry rotations of 90^0, 180^0, and 270^0. Since there are three such axes, this gives us $3 \times 3 = 9$ elements.

We also have some twofold axes that pass through the midpoints of opposite edges. Since there are 12 edges, there are six such axes and corresponding to each of these axes is a symmetry rotation of 180^0. This accounts for six more elements in the group. Then there are four threefold axes that pass through opposite vertices. This adds another eight members to the group.

Finally, there is the identity transformation. Thus, the total number of proper orthogonal members for the point group associated with the cube is $9 + 6 + 8 + 1 = 24$. (Because any product of reflections has two representations in the Clifford formalism ($\pm$), there are 48 Clifford numbers in the Clifford version of the proper orthogonal group for the cube.)

To obtain the number of improper orthogonal transformations by simply counting them is difficult because some members of this set are not simple reflections but products of three reflections. To complete our counting problem, we wish to apply the following theorem:

Theorem 10. *For a finite point group, the number of improper orthogonal transformations (products of an odd number of reflections) is equal to the number of proper transformations (products of an even number of reflections). Note! For those familiar with group theory, what is proven below is that the set of improper orthogonal transformations is a coset of the subgroup of proper orthogonal transformations.*

Proof. To establish the truth of this theorem, we choose a unit vector **a** corresponding to a simple reflection in the group and then show that any improper orthogonal transformation can be represented uniquely (aside from the sign ambiguity) in the form **Ra** where **R** is a rotation or $\pm$ the identity element **I**.

Consider a product of an odd number of reflections $\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_k$. If k is odd, we can multiply out the first k-1 reflections to get a rotation operator $\mathbf{R}$. So we have

$$\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_k = \mathbf{R}\mathbf{a}_{k.}$$

If $\mathbf{a}_k = \mathbf{a}$, we are incredibly lucky. Otherwise,

$$\mathbf{R}\mathbf{a}_k = \mathbf{R}\mathbf{a}_k(\mathbf{a})^2 = \mathbf{R}(\mathbf{a}_k\mathbf{a})\mathbf{a} = \mathbf{R}\acute{\mathbf{R}}\mathbf{a} = \acute{\acute{\mathrm{R}}}\mathbf{a}, \text{ where}$$

$$\acute{\acute{\mathrm{R}}} = \mathbf{R}\acute{\mathbf{R}}. \text{ Thus, we have}$$

$$\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_k = \acute{\acute{\mathrm{R}}}\mathbf{a}.$$

To show that this representation is unique aside from the sign ambiguity, suppose

$$\mathbf{R}\mathbf{a} = \pm\acute{\mathbf{R}}\mathbf{a}. \text{ Multiply both sides by } \mathbf{a} \text{ to get}$$

$$\mathbf{R}(\mathbf{a})^2 = \pm\acute{\mathbf{R}}(\mathbf{a})^2 \text{ or } \mathbf{R} = \pm\acute{\mathbf{R}}. \qquad \square$$

Applying this theorem to the cube, we see that the point group for the cube has 48 members (96 for the double valued Clifford version).

Using the terminology of group theory, we say the *order* of the point group for the cube is 48.

To get the orders for the point groups of the other polyhedrons, the chief problem is counting the edges and vertices. For example, the dodecahedron is constructed by assembling 12 regular pentagons. Before assembly, the 12 pentagons have a total of $12 \times 5 = 60$ edges. When assembled, one edge from one pentagon and one edge from a second pentagon align to become a single edge of the dodecahedron. Thus, the dodecahedron has $60/2 = 30$ edges, which correspond to $30/2 = 15$ twofold axes. Similarly, the 60 vertices of the 12 pentagons become $60/3 = 20$ vertices for the dodecahedron. In turn, this corresponds to ten threefold axes.

For four of the five regular polyhedrons, the axes of symmetry pass through pairs of faces, pairs of edges, or pairs of vertices. The one exception is the tetrahedron. For the tetrahedron, the twofold axes do indeed correspond to pairs of edges. However for the threefold axes, the situation is different. For the tetrahedron, each threefold axis passes through one vertex and one face.

When you determine the orders of the point groups (See Prob. 12.), you will see that the order of the point group for the cube is identical to the order of the point group for the octahedron. This raises the possibility that the two groups are *isomorphic*. Two groups are said to be isomorphic if one can set up a one-to-one correspondence between the groups is such a way that if $\mathbf{x}$ in one group corresponds to $\acute{\mathbf{x}}$ in the second group and $\mathbf{y}$ corresponds to $\acute{\mathbf{y}}$ then $\mathbf{x} \circ \mathbf{y}$ corresponds to $\acute{\mathbf{x}} \circ \acute{\mathbf{y}}$. For the cube and the octahedron, this is plausible because the numbers of fourfold, threefold, and twofold axes match up in the two groups. Nonetheless, it would

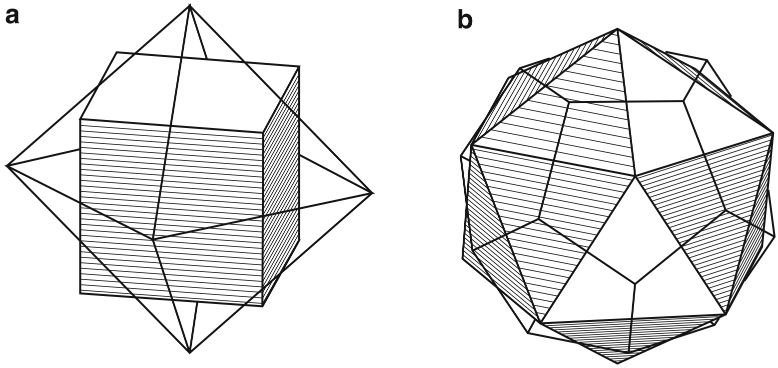


Fig. 2.6 (**a**) A cube aligned with a skeleton frame of an octahedron. (**b**) An icosahedron aligned with a skeleton frame of a dodecahedron

be very difficult to determine an isomorphic correspondence without resorting to geometry. However using geometry, it becomes a trivial exercise to establish the isomorphism. One merely matches the vertices of one with the face centers of the other. In Fig. 2.6a, we have aligned a cube with the skeleton frame of an octahedron is such a way that the symmetry axes of rotation for the two polyhedrons coincide. Thus we see that a proper symmetry transformation for one of the polyhedrons is a proper symmetry transformation for the other. The two point groups also contain the same improper symmetry transformations. (See Prob. 15.) Thus, the two point groups are isomorphic.

In Fig. 2.6b, we have aligned an icosahedron with the skeleton frame of a dodecahedron with similar consequences.

One can also demonstrate geometrically that the point group for the tetrahedron is a subgroup of the point groups for the other polyhedrons so that any symmetry transformation of the tetrahedron is also a symmetry transformation of the other polyhedrons.

One can imbed a tetrahedron inside a cube so that the threefold axes for the two polyhedrons coincide. (See Fig. 2.7a.) The twofold axes of the tetrahedron do not coincide with the twofold axes of the cube. However, the twofold axes of the tetrahedron do coincide with the fourfold axes of the cube. Thus, it becomes clear that any proper orthogonal transformation in the point group for the tetrahedron belongs to the point group for the cube. It can also be said that any improper transformation belonging to the point group for the tetrahedron is also an improper transformation belonging to the point group for the cube. (See Prob. 16.) Thus, it is clear that the point group for the tetrahedron is a subgroup of the point group for the cube.

It is more difficult to visualize but the point group for the tetrahedron is also a subgroup of the dodecahedron (or icosahedron). (See Fig. 2.7b.)

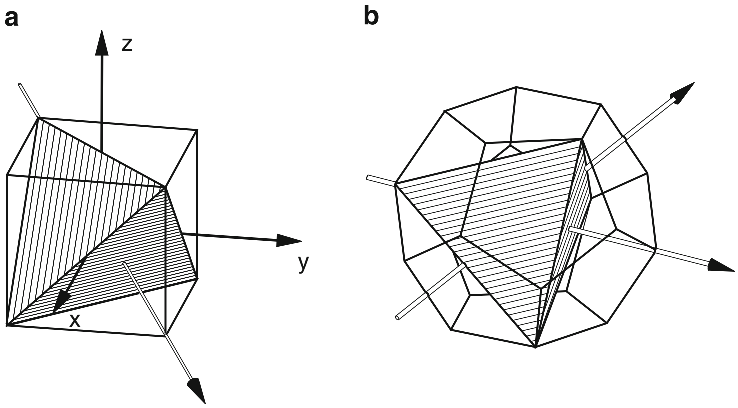
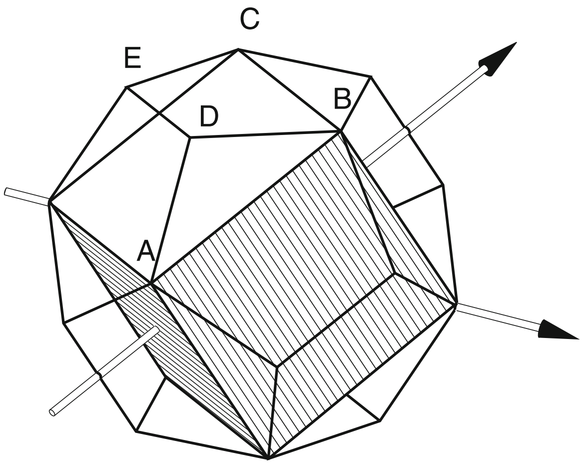


Fig. 2.7 (**a**) A tetrahedron aligned with the skeleton frame of a cube. (**b**) A tetrahedron aligned with the skeleton frame of a dodecahedron

Fig. 2.8 A cube aligned with the skeleton frame of a dodecahedron.



It is also enlightening to examine Fig. 2.8. You may not be convinced that connecting some of the vertices of the dodecahedron as shown in Fig. 2.8 results in the edges of a cube. However, it should be clear that the direction of line segment AB is perpendicular to the direction of line segment DE. Furthermore, line segment DE is parallel to line segment BC. Thus, the edges of our suspect cube do indeed meet at right angles at each vertex. By studying the alignment of the various symmetry axes of rotation in Fig. 2.8, we reach the conclusion that the intersection of the point group for the cube (or octahedron) and the point group for the dodecahedron (or icosahedron) is the point group for the tetrahedron.

Problem 11. Prove that there are no more than five regular polyhedrons. Hint: What is the maximum number of equilateral triangles that can share a single vertex?

Problem 12. Determine the orders of the point groups for the tetrahedron, octahedron, dodecahedron, and icosahedron. Are your results consistent with Figs. 2.6a and 2.6b?

Problem 13. How does the result of Prob. 1 relate to the point group for the cube? What is the consequence of two successive 90^0 rotations about two non-aligned fourfold axes?

Problem 14. In view of Fig. 2.7a, the three twofold axes of the tetrahedron can be aligned with the **x**, **y**, and **z** axes. Suppose we designate a 180^0 rotation about the x-axis by $\mathbf{R}_x = \pm\mathbf{e}_{23}$. Suppose we also define $\mathbf{R}_y$ and $\mathbf{R}_z$ in a similar manner. Complete the following table:

$\circ$	$\mathbf{I}$	$\mathbf{R}_x$	$\mathbf{R}_y$	$\mathbf{R}_z$
$\mathbf{I}$				
$\mathbf{R}_x$	$\mathbf{R}_x$			
$\mathbf{R}_y$				
$\mathbf{R}_z$				

You will find that the 180^0 rotations commute, although the Clifford representations do not.

Problem 15. Consider Fig. 2.6a.

(a) Draw the figure with the cube and octahedron aligned with the **x**, **y**, and **z** axes.
(b) Describe a plane of reflection that is common to both the cube and octahedron.
(c) It has already been pointed out that if the cube and the octahedron are aligned as in Fig. 2.6a, the proper orthogonal symmetry transformations for the two point groups are identical. Use your result in part b) to show that the improper symmetry transformations for the two point groups are identical.
(d) Explain why the improper symmetry transformations for the icosahedron are the same as the improper symmetry transformations for the dodecahedron.

Problem 16. (a) Prove that any improper orthogonal symmetry transformation for the tetrahedron is also an improper orthogonal symmetry transformation for the cube. (If you get stuck, review the approach used in the proof of Theorem 10.)
(b) Explain why any improper orthogonal symmetry for the tetrahedron is also an improper orthogonal symmetry transformation for the dodecahedron (or icosahedron).

Problem 17. If a tetrahedron is aligned with the **x**, **y**, and **z** axes as shown in Fig. 2.7a, then the rotations about the threefold axis shown are

$$\pm\left[\mathbf{I}\cos 60^0 + \sin 60^0\left(\frac{1}{\sqrt{3}}\mathbf{e}_{23} + \frac{1}{\sqrt{3}}\mathbf{e}_{31} - \frac{1}{\sqrt{3}}\mathbf{e}_{12}\right)\right] = \pm\frac{1}{2}\left[\mathbf{I} + \mathbf{e}_{23} + \mathbf{e}_{31} - \mathbf{e}_{12})\right]$$

and

$$\pm\left[\mathbf{I}\cos 120^0 + \sin 120^0\left(\frac{1}{\sqrt{3}}\mathbf{e}_{23} + \frac{1}{\sqrt{3}}\mathbf{e}_{31} - \frac{1}{\sqrt{3}}\mathbf{e}_{12}\right)\right] = \mp\frac{1}{2}\left[\mathbf{I} - \mathbf{e}_{23} - \mathbf{e}_{31} + \mathbf{e}_{12})\right]$$
$$= \pm\frac{1}{2}\left[\mathbf{I} - \mathbf{e}_{23} - \mathbf{e}_{31} + \mathbf{e}_{12})\right].$$

(a) List all of the rotations for both the twofold and threefold axes. (Don't compute them all – after computing a few, you should see patterns.)
(b) Write down the Clifford representation of a reflection and use this to construct a list of the improper orthogonal symmetry for the tetrahedron.
(c) In the list constructed in part b), which are simple reflections and which cannot be achieved by fewer than three reflections?

Problem 18. Euler's Formula

In 1750, Leonard Euler made the conjecture that for any convex polyhedron, $F - E + V = 2$, where F equals the number of faces, E equals the number of edges, and V equals the number of vertices (James 2002, p. 5). Determine whether this formula is valid for the five regular polyhedrons. Suppose you slice off a corner of a cube. Does the resulting solid satisfy Euler's formula?

2.4 *Élie Cartan 1869–1951

The way mathematicians deal with differential geometry was significantly altered by the work of Élie Cartan. In 1993, the American Mathematical Society published a 301-page translation from Russian of a summary of his work. This short biography is extracted from that source.

The authors of that summary are two Russian mathematicians: M.A. Akivis and B.A. Rosenfeld (1993). Élie Cartan's contributions to mathematics are so deep and broad that these two accomplished geometers felt compelled to include a virtual apology in their preface: "Of course the authors are only able to describe in detail Cartan's results connected with those branches of geometry in which the authors are experts." (Akivis and Rosenfeld 1993, p. xi).

Élie Joseph Cartan was born on April 9, 1869 in Dolomieu, a small village in southeastern France of less than 2,000 people. At the time of his birth, no one would have predicted that Élie Cartan would become a world renowned mathematician. His father was a blacksmith. His older sister, Jeanne-Marie, became a dressmaker, and his younger brother, Leon, would eventually join the family business as another blacksmith.

Élie seemed destined for a similar career in rural France until a fateful visit to Élie's elementary school by the up and coming politician, Antonin Dubost (1844–1921). This event would change Élie's direction in life.

When Élie's teachers described their very remarkable student to Dubost, Dubost encouraged the young Cartan to compete for a scholarship at a more competitive lycée. Antonin Dubost eventually became the Minister of Justice under one administration and later became President of the French Senate for what was essentially the last 14 years of his life. Throughout his life, Antonin Dubost maintained a fatherly interest in Cartan's career.

To help Élie obtain the desired scholarship, one of his teachers, M. Dupuis, supervised his preparation for the required exam. Cartan scored well on the exam, received the scholarship, and left home at the age of 10.

At the age of 17, Cartan decided to become a mathematician and enrolled at l'École Normale Supérieure in Paris. During the next three years, Cartan not only attended lectures at l'École Normale Supérieure but also at the Sorbonne. In this way, he became exposed to many outstanding mathematicians including Henri Poincaré. After graduation, he was drafted into the French army for one year. He then returned to Paris and received his doctorate at the Sorbonne two years later in 1894 while attracting the attention of prominent mathematicians including Sophus Lie at Leipzig University in Germany.

Early in his career, Cartan developed aspects of Lie groups and Lie algebras that could be applied to differential geometry. Later, his work on differential forms led him to develop methods that are now commonly used to deal with differential equations. In 1910, Cartan began to perfect the method of moving frames to deal with problems in differential geometry (Cartan 1910a, 1910b). (You will encounter this method in later chapters of this book.)

In 1915, when Cartan was 46, he was again drafted into the French army soon after World War I broke out. However, he was not sent to the front. Instead, he was assigned to a hospital set up in the building of l'École Normale Supérieure. This situation allowed him to continue his mathematical research during the war years.

During these same war years, Einstein living in Berlin, discovered that a slight variation of Riemannian geometry was necessary to express his general theory of relativity. After the war, Einstein and others sought out mathematical structures that could be used to construct a unified field theory. With this motivation, Cartan turned his attention to extracting properties of more general geometric spaces that might be useful. (His correspondence with Einstein was edited by Robert Debever and published by Princeton University Press in 1979 under the title *Élie Cartan and Albert Einstein: Letters on Absolute Parallelism, 1929–1932*.)

To summarize, Cartan was prolific. Akivis and Rosenfeld attribute over 200 publications to Cartan, and this includes several books that have been republished in recent years.

Cartan was also successful as a family person. In 1903, he married Marie-Louise Bianconi (1880–1950) and soon became the father of three sons: Henri (1904–2008), Jean (1906–1932), and Louis (1909–1943). Later Élie and Marie-Louise had a daughter Hélene (1917–1952). His first son, Henri, became a world renowned mathematician in his own right. (Henri Cartan died on August 13, 2008 at the age of 104!) His second son, Jean, seemed headed for a promising career as a music composer but he died of tuberculosis at the age of 25. The third son, Louis, was a talented physicist, but during World War II, he was arrested by Vichy government police for his activities in the French resistance. He was then turned over to the Germans who held him in captivity for 15 months before executing him by decapitation. The daughter Hélene taught mathematics at several lycées and authored several math papers before she died at the age of 34.

During most of his adult life, Élie Cartan made his home in Paris or within commuting distance of Paris. He had spent much of his boyhood away from his hometown but he always maintained his ties there. He encouraged his younger sister Anna to pursue a career in math education. She taught at several secondary schools for girls and authored two textbooks, which were reprinted many times.

In 1909, Cartan built a vacation home in Dolomieu and sometimes he could be seen at the family blacksmith shop helping his father and brother to blow the blacksmith bellows.

Cartan's sister Anna and daughter Hélene were not the only women to receive Cartan's encouragement to study mathematics. After he retired from his professorial position at the Sorbonne in 1940, he devoted the last years of his life in his 70s to teaching mathematics at the École Normale Supérieure for girls.

After a long illness, he died in Paris on May 6, 1951.

2.5 *Suggested Reading

Milton Hamermesh 1962. *Group Theory.* Reading, Massachusetts, U.S.A: Addison-Wesley Publishing Company, Inc. Also reprint edition 1990. New York. Dover Publications, Inc.

The second chapter is devoted to the point groups.

Leo Dorst, Chris Doran, and Joan Lasenby (Editors) 2002. *Applications of Geometric Algebra in Computer Science and Engineering*. Boston: Birkhäuser. Chapter I entitled "Point Groups and Space Groups in Geometric Algebra"by David Hestenes is devoted to the application of Geometric Algebra (Clifford Algebra) to the classification of symmetry groups.

D.M.Y. Sommerville 1958. *An Introduction to the Geometry of N Dimensions.* New York: Dover Publications, Inc.

This book includes a discussion of regular polyhedrons in higher dimensions.

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