Preface

This book was written with the intention that it be used as a text for an undergraduate course. The end result is not only suitable for an undergraduate course but also ideal for a masters level course directed toward future high school math teachers. It is also appropriate for anyone who wants to acquaint himself or herself with the usefulness of Clifford algebra. In that context, instructors teaching Ph.D. students may want to use it as a source book.

Most introductory books on differential geometry are restricted to three dimensions. I use the notation used by geometers in n-dimensions. Admittedly, most of my examples are in three dimensions. However in Chap. 3, I present some aspects of the four-dimensional theory of special relativity. One can present some fun aspects of special relativity without mention of such concepts as "force", "momentum", and "energy." For example, adding speeds possibly near the speed of light results in a sum that is always less than the speed of light.

The capstone topic for this book is Einstein's general theory of relativity. Here again, knowledge of Newtonian physics is not a prerequisite. Because of the geometric nature of Einstein's theory, some interesting aspects can be presented without knowledge of Newtonian physics. In particular, I discuss the possibility of twins aging at different rates, the precession of Mercury, and the bending of light rays passing near the Sun or some other massive body.

The only topic that does require some knowledge of Newtonian physics is Huygen's isochronous pendulum clock, and the relevant section should be considered optional.

My strategy for writing this book had three steps:

Step 1: Steal as many good pedagogical ideas from as many authors as possible.
Step 2: Improve on them if I could.
Step 3: Choose topics that are fun for me and fit together in a coherent manner.

Frequently I was able to improve on the presentation of others using Clifford algebra.

The selection of this book may introduce a hurdle for some instructors. It is likely they will have to learn something new – Clifford algebra. Paradoxically, the use of Clifford algebra will make differential geometry *more* accessible to students who have completed a course in linear algebra. That is because in this book, Clifford algebra replaces the more complicated and less powerful formalism of differential forms. Anyone who is familiar with the concept of non-commutative matrix multiplication will find it easy to master the Clifford algebra presented in this text. Using Clifford algebra, it becomes unnecessary to discuss mappings back and forth between the space of tangent vectors and the space of differential forms. With Clifford algebra, everything takes place in one space.

The fact that Clifford algebra (otherwise known as “geometric algebra”) is not deeply embedded in our current curriculum is an accident of history. William Kingdon Clifford wrote two papers on the topic shortly before his early death in 1879 at the age of 33. Although Clifford was recognized worldwide as one of England’s most distinguished mathematicians, he chose to have the first paper published in what must have been a very obscure journal at the time. Quite possibly it was a gesture of support for the efforts of James Joseph Sylvester to establish the first American graduate program in mathematics at Johns Hopkins University. As part of his endeavors, Sylvester founded the *American Journal of Mathematics* and Clifford’s first paper on what is now known as Clifford algebra appeared in the very first volume of that journal.

The second paper was published after his death in unfinished form as part of his collected papers. Both of these papers were ignored and soon forgotten. As late as 1923, math historian David Eugene Smith discussed Clifford’s achievements without mentioning “geometric algebra” (Smith, David Eugene 1923). In 1928, P.A.M. Dirac reinvented Clifford algebra to formulate his equation for the electron. This equation enabled him to predict the discovery of the positron in 1931.

In 1946 and 1958, Marcel Riesz published some results on Clifford algebra that stimulated David Hestenes to investigate the subject. In 1966, David Hestenes published a thin volume entitled *Space-time Algebra* (Hestenes 1966). And 18 years later, with his student Garret Sobezyk, he wrote a more extensive book entitled *Clifford Algebra to Geometric Calculus – A Unified Language for Mathematics and Physics* (Hestenes and Sobczyk 1984). Since then, extensive research has been carried out in Clifford algebra with a multitude of applications.

Had Clifford lived longer, “geometric algebra” would probably have become mainstream mathematics near the beginning of the twentieth century. In the decades following Clifford’s death, a battle broke out between those who wanted to use quaternions to do physics and geometry and those who wanted to use vectors. Quaternions were superior for dealing with rotations, but they are useless in dimensions higher than three or four without grafting on some extra structure. Eventually vectors won out.

Since the structure of both quaternions and vectors are contained in the formalism of Clifford algebra, the debate would have taken a different direction had Clifford lived longer. While alive, Clifford was an articulate spokesman and his writing for popular consumption still gets published from time to time. Had Clifford participated in the quaternion–vector debate, “geometric algebra” would have received more serious consideration.

The advantage that quaternions have for dealing with rotations in three dimensions can be generalized to higher dimensions using Clifford algebra. This is important for dealing with the most important feature of a surface in any dimension – namely its curvature.

Suppose you were able to walk from the North Pole along a curve of constant longitude to the equator, then walk east along the equator for 37° and finally return to the North Pole along another curve of constant longitude. In addition, suppose at the start of your trek, you picked up a spear, pointed it in the south direction and then avoided any rotation of the spear with respect to the surface of the earth during your long journey. If you were careful, the spear would remain pointed south during the entire trip. However, on your return to the North Pole, you would discover that your spear had undergone a 37° rotation from its initial position. This rotation is a measure of the curvature of the Earth's surface.

The components of the Riemann tensor, used to measure curvature, are somewhat abstract in the usual formalism. Using Clifford algebra, the components of the Riemann tensor can be interpreted as components of an infinitesimal rotation operator that indicates what happens when a vector is "parallel transported" around an infinitesimal loop in a curved space.

In many courses on differential geometry, the Gauss–Bonnet Formula is the capstone result. Exploiting the power of Clifford algebra, a proof appears slightly less than halfway through this book. If optional intervening historical digressions were eliminated, the proof of the Gauss–Bonnet Formula would appear on approximately p. 115.

This should leave time to cover other topics that interest the instructor or the instructor's students. I hope that instructors endeavor to cover enough of the theory of general relativity to discuss the precession of Mercury. The general theory of relativity is essentially geometric in nature.

Whatever topics are chosen, I hope people have fun.

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