

# Chapter 2

## Sequences and Series

### 2.1 Sequences

A sequence is a function from the positive integers (possibly including 0) to the reals. A typical example is  $a_n = 1/n$  defined for all integers  $n \geq 1$ . The notation  $a_n$  is different from the standard notation for functions  $a(n)$  but means the same thing. We now define convergence for sequences.

#### Convergent sequence

The sequence  $a_n$  is said to converge to a real number  $a$  if for every  $\epsilon > 0$ , there is a natural number  $N$  (that usually depends on  $\epsilon$ ) such that if  $n \geq N$ , then

$$|a_n - a| < \epsilon.$$

$a_n$  converges to  $a$  is denoted by  $\lim_{n \rightarrow \infty} a_n = a$ . The number  $a$  is called the limit of the sequence  $a_n$ .

Note that for sequences, one looks only at the limit as  $n$  goes to infinity. A sequence that does not converge is said to diverge. We will now study several examples.

*Example 2.1* Consider a constant sequence  $a_n = c$  for all  $n \geq 1$ , where  $c$  is a constant. Then, of course,  $a_n$  converges to  $c$ : take any  $\epsilon > 0$ , take  $N = 1$ ; then for every  $n \geq N$ , we have  $|a_n - c| = 0 < \epsilon$ . Therefore, by definition,  $a_n$  converges to  $c$ .

*Example 2.2* Let  $a_n = 1/n$ . Intuitively, it is clear that  $a_n$  converges to 0. We now prove this. Let  $\epsilon > 0$ . By the Archimedean property there is a natural  $N > 1/\epsilon$ . For  $n \geq N$ , we have  $1/n \leq 1/N < \epsilon$  (why?). Thus,

$$1/n = |a_n - 0| < \epsilon \quad \text{for all } n \geq N,$$

and we have proved that  $a_n$  converges to 0.

*Example 2.3* Assume that the sequence  $a_n$  converges to  $\ell$ . Let  $b_n = a_{n+1}$ . Let us show that  $b_n$  converges to the same  $\ell$ .

Since  $a_n$  converges to  $\ell$ , for any  $\epsilon > 0$ , there is a natural  $N$  such that if  $n \geq N$ , then  $|a_n - \ell| < \epsilon$ . Note that if  $n \geq N$ , then  $n + 1 \geq N$  and  $|a_{n+1} - \ell| < \epsilon$  as well. That is, if  $n \geq N$ , we have  $|b_n - \ell| < \epsilon$ . This proves that  $b_n$  converges to  $\ell$  as well.

In order to prove that a sequence converges to 0, the following is sometimes useful.

### Convergence to 0

A sequence  $a_n$  converges to 0 if and only if the sequence  $|a_n|$  converges to 0.

Assume first that  $a_n$  converges to 0. For any  $\epsilon > 0$ , there is  $N$  such that if  $n \geq N$ , then

$$|a_n - 0| < \epsilon.$$

We have that

$$|a_n - 0| = |a_n| = ||a_n| - 0|.$$

Hence,

$$||a_n| - 0| < \epsilon$$

for all  $n \geq N$ . This proves that  $|a_n|$  converges to 0.

Assume now that  $|a_n|$  converges to 0. For any  $\epsilon > 0$ , there is  $N$  such that if  $n \geq N$ , then

$$||a_n| - 0| < \epsilon.$$

But  $||a_n| - 0| = |a_n - 0|$ . Thus,

$$|a_n - 0| < \epsilon$$

for all  $n \geq N$ . This proves that  $a_n$  converges to 0 and completes the proof.

*Example 2.4* Assume that  $a_n$  converges to  $\ell$ , and let  $c$  be a constant. Then  $ca_n$  converges to  $c\ell$ .

If  $c = 0$ , then  $ca_n$  is the constant sequence 0, and it converges to  $0 = 0 \cdot \ell$ . So the property holds when  $c = 0$ . Assume now that  $c \neq 0$ . Let  $\epsilon > 0$ ; since  $a_n$  converges to  $\ell$ , there is a natural  $N$  such that if  $n \geq N$ , we have

$$|a_n - \ell| < \frac{\epsilon}{|c|}.$$

Multiplying across this inequality by  $|c|$ , we get

$$|ca_n - c\ell| < \epsilon.$$

This proves that  $ca_n$  converges to  $c\ell$ , and we are done.

The next example shows that not all oscillating sequences diverge.

*Example 2.5* Let  $a_n = (-1)^n/n$ . Show that  $a_n$  converges to 0.

Observe that  $|a_n| = 1/n$  converges to 0. Thus,  $a_n$  converges to 0.

We now turn to the notion of bounded sequence

### **Bounded sequences**

A sequence  $a_n$  is said to be bounded if there exists a real number  $K$  such that

$$|a_n| < K \quad \text{for all } n.$$

There is an important relation between bounded sequences and convergent sequences.

### **A convergent sequence is bounded**

If a sequence converges, then it must be bounded.

We now prove that a convergent sequence is bounded. Assume that the sequence  $a_n$  converges to some limit  $a$ . Take  $\epsilon = 1$ ; since  $a_n$  converges to  $a$ , there is a natural  $N$  such that if  $n \geq N$ , then  $|a_n - a| < \epsilon = 1$ . Thus, by the triangle inequality we have

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a| \quad \text{for all } n \geq N.$$

The inequalities above show that the sequence is bounded for  $n \geq N$ . We now take care of  $n < N$ . We know that a finite set of real numbers always has a maximum and a minimum. Thus, let

$$M = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|\},$$

and let  $K = \max(M, 1 + |a|)$ . Then, we claim that

$$|a_n| < K \quad \text{for all } n \geq 1.$$

For if  $n < N$ , then  $|a_n| < M \leq K$ , while if  $n \geq N$ , then  $|a_n| < |a| + 1 \leq K$ . This completes the proof that a convergent sequence is bounded.

*Example 2.6* Let  $c$  such that  $|c| > 1$ . Consider the sequence  $b_n = c^n$  for  $n \geq 0$ . Let us show that  $b_n$  does not converge.

Let  $|c| = 1 + a$  where  $a = |c| - 1 > 0$ . By Bernoulli's inequality we have

$$|b_n| = |c|^n = (1 + a)^n \geq 1 + na.$$

Note that the sequence  $1 + na$  is not bounded (why?), and therefore  $b_n$  is not bounded either. Therefore, it cannot converge.

A bounded sequence does not necessarily converge. In order to show that a sequence does not converge, the following notion of subsequence is quite useful.

### Subsequences

Let  $1 \leq j_1 < j_2 < \cdots < j_n < \cdots$  be a strictly increasing sequence of natural numbers. Let  $a_n$  be a sequence of real numbers. Then  $a_{j_n}$  defines a new sequence of real numbers (it is in fact the composition of sequences  $a_n$  and  $j_n$ ) and is called a subsequence of  $a_n$ . The name comes from the fact that all terms  $a_{j_n}$  are in the original sequence  $a_n$ .

Useful examples of strictly increasing sequences of natural numbers are  $j_n = n$ ,  $j_n = 2n$ ,  $j_n = 2n + 1$ , and  $j_n = 2^n$ .

### Subsequences and convergence

A sequence  $a_n$  converges to  $a$  if and only if all the subsequences of  $a_n$  converge to  $a$ .

The proof is easy. First, the direct implication. Assume that  $a_n$  converges to  $a$ , and let  $a_{j_n}$  be a subsequence of  $a_n$ . For any  $\epsilon > 0$ , there is  $N$  such that if  $n \geq N$ , then  $|a_n - a| < \epsilon$ . It is easy to show (see the exercise) that  $j_n \geq n$  for all  $n \geq 1$ . Thus, if  $n \geq N$ , then  $j_n \geq n \geq N$  and  $|a_{j_n} - a| < \epsilon$ . Therefore,  $a_{j_n}$  converges to  $a$ . This proves that any subsequence of  $a_n$  converges to  $a$ .

We now prove the converse. We assume that all subsequences of  $a_n$  converge to  $a$ . But  $a_n$  is a subsequence of itself (take  $j_n = n$ ), and thus  $a_n$  converges to  $a$  and we are done.

We use the preceding result in the next example.

*Example 2.7* Let  $a_n = (-1)^n$ . Intuitively, it is clear that  $a_n$  does not converge (it oscillates between  $-1$  and  $1$ ). We prove this. Note that the subsequence  $a_{2n}$  is the constant sequence  $1$  and so converges to  $1$ . On the other hand,  $a_{2n+1}$  is the constant sequence  $-1$  and so converges to  $-1$ . That is, we have found two subsequences that converge to two distinct limits. This proves that  $(-1)^n$  does not converge.

The following principle is sometimes quite useful.

### Squeezing principle

Assume that the three sequences  $a_n$ ,  $b_n$ , and  $c_n$  are such that

$$a_n \leq b_n \leq c_n$$

and that the sequences  $a_n$  and  $c_n$  converge to the *same* limit  $\ell$ . Then  $b_n$  converges to  $\ell$  as well.

The result above not only proves the convergence of  $b_n$ , but it also gives the limit. We now prove it. Recall that

$$|x| < a \iff -a < x < a.$$

We start by writing that  $a_n$  and  $c_n$  converge to  $\ell$ . For any  $\epsilon > 0$ , there are naturals  $N_1$  and  $N_2$  such that if  $n \geq N_1$ , then

$$|a_n - \ell| < \epsilon \iff \ell - \epsilon < a_n < \ell + \epsilon,$$

and if  $n \geq N_2$ , then

$$|c_n - \ell| < \epsilon \iff \ell - \epsilon < c_n < \ell + \epsilon.$$

Define  $N = \max(N_1, N_2)$ , so that the two double inequalities above hold for  $n \geq N$ . Thus, for  $n \geq N$ , using that  $b_n$  is squeezed between  $a_n$  and  $c_n$ , we have

$$\ell - \epsilon < a_n \leq b_n \leq c_n < \ell + \epsilon.$$

But this implies that

$$\ell - \epsilon < b_n < \ell + \epsilon \iff |b_n - \ell| < \epsilon \quad \text{for all } n \geq N.$$

That is,  $b_n$  converges to  $\ell$ , and the squeezing principle is proved.

Next we give an application of the squeezing principle.

*Example 2.8* Assume that  $a_n$  converges to 0 and that  $b_n$  is bounded. Then  $a_n b_n$  converges to 0.

Since  $b_n$  is bounded, there is  $B$  such that  $|b_n| < B$  for all  $n$ . Then,

$$0 \leq |a_n b_n| \leq B|a_n|.$$

That is, the sequence  $|a_n b_n|$  is squeezed between the constant sequence 0 and the sequence  $B|a_n|$ . But they both converge to 0 (why?). By the squeezing principle,  $|a_n b_n|$  converges to 0, and so does  $a_n b_n$ .

*Example 2.9* Consider the sequence  $a_n = n^r$  where  $r$  is a rational number. Discuss the convergence of  $a_n$  in function of  $r$ .

If  $r = 0$ , then  $a_n$  is the constant sequence 1, and it converges to 1.

Assume that  $r > 0$ . By contradiction, assume that the sequence  $a_n = n^r$  is bounded. Then there is  $B > 0$  such that for all  $n$ ,  $n^r < B$ . Since  $1/r > 0$ , the function  $x \rightarrow x^{1/r}$  is increasing, and we have  $n < B^{1/r}$  for all naturals  $n$ . This contradicts the Archimedean property and shows that  $a_n$  cannot be bounded. Hence, if  $r > 0$ ,  $a_n$  does not converge.

Let  $r > 0$ . We now show that  $n^{-r}$  converges to 0. Take any  $\epsilon > 0$  and let  $N > 1/\epsilon^{1/r}$ ; then for  $n \geq N$ , we have

$$|n^{-r} - 0| = 1/n^r \leq 1/N^r < \epsilon.$$

This proves that  $n^{-r}$  converges to 0.

The following is a useful property of least upper bounds.

### Least upper bound and sequences

Let  $A$  be a subset in the reals with a least upper bound  $m$ . There exists a sequence  $a_n$  in  $A$  that converges to  $m$ .

The important part of this property is that the sequence is *in*  $A$ . The same property holds for the greatest lower bound, and the proof is left to the reader.

We now prove the property. Since  $m$  is the least upper bound of  $A$ ,  $m - 1$  is not an upper bound of  $A$ . Thus, there is at least one  $a$  in  $A$  such that  $a > m - 1$ . We pick one such  $a$ , and we denote it by  $a_1$ . Similarly,  $m - 1/2$  cannot be an upper bound of  $A$ , so there is at least one  $a$  in  $A$  such that

$$a > m - 1/2.$$

We pick one such  $a$ , and we call it  $a_2$ . More generally, for every natural  $n$ ,  $m - 1/n$  is not an upper bound of  $A$ , and we may pick  $a_n$  in  $A$  such that

$$a_n > m - 1/n.$$

Hence, there exists a sequence  $a_n$  in  $A$  such that  $a_n > m - 1/n$  for all  $n \geq 1$ . On the other hand, since the sequence  $a_n$  is in  $A$ , we must have  $a_n \leq m$  for all  $n \geq 1$ . Thus,

$$m - 1/n < a_n \leq m.$$

It is easy to prove that  $m - 1/n$  converges to  $m$  (it is also a consequence of the operations on limits to be proved in the next section). Hence, by the squeezing principle, the sequence  $a_n$  converges to  $m$ .

### Exercises

- (a) Let  $a_n$  be a sequence of reals converging to  $\ell$ . Let  $b_n = a_{n-1}$ . Show that  $b_n$  converges to  $\ell$  as well.  
(b) State a generalization of the result in (a) and prove your claim.
- Let  $1 \leq j_1 < j_2 < \cdots < j_n < \cdots$  be a sequence of natural numbers. Prove (by induction) that  $j_n \geq n$  for all  $n \geq 1$ .
- (a) Show that if  $a_n$  converges to  $a$ , then  $|a_n|$  converges to  $|a|$  (use that  $||x| - |y|| \leq |x - y|$ ).  
(b) Is it true that if  $|a_n|$  converges, then  $a_n$  converges? Prove it or give a counterexample.
- (a) Assume that the real  $a$  is such that  $|a| < \epsilon$  for any  $\epsilon > 0$ . Prove that  $a = 0$ .  
(b) Prove that a limit is unique (assume that a sequence  $a_n$  has two limits  $a$  and  $b$ , and show that  $|a - b| < \epsilon$  for any  $\epsilon > 0$ .)
- Assume that  $a_n$  converges to  $a$ . Prove that for any  $\epsilon > 0$ ,  $a_n$  is in  $(a - \epsilon, a + \epsilon)$  for all  $n$  except possibly finitely many.
- Assume that  $a_n$  converges to 1.  
(a) Show that there is  $N$  such that if  $n \geq N$ , then  $a_n < 2$ .

(b) Show that there is  $N_1$  such that if  $n \geq N_1$ , then  $a_n > 1/2$ .

7. Assume that the three sequences  $a_n$ ,  $b_n$ , and  $c_n$  are such that

$$a_n \leq b_n \leq c_n \quad \text{for } n \geq 1000$$

and that the sequences  $a_n$  and  $c_n$  converge to the *same* limit  $\ell$ . Explain how to modify the proof of the squeezing principle so that the conclusion still holds.

8. Let  $A$  be a subset in the reals with a greatest lower bound  $k$ . Show that there exists a sequence  $a_n$  in  $A$  that converges to  $k$ .

9. Assume that  $a_n$  is positive and converges to a limit  $\ell$ .

(a) Assume that  $\ell = 0$ . Show that  $\sqrt{a_n}$  converges to 0.

(b) Assume that  $\ell > 0$ . Show that  $\sqrt{a_n}$  converges to  $\sqrt{\ell}$ .

$$\left( \text{Write that } |\sqrt{a_n} - \sqrt{\ell}| = \frac{|a_n - \ell|}{\sqrt{a_n} + \sqrt{\ell}}. \right)$$

10. Let  $a_n$  be a sequence of reals, and  $\ell$  be a real. Let  $b_n = |a_n - \ell|$ . Show that  $a_n$  converges to  $\ell$  if and only if  $b_n$  converges to 0.

11. Assume that  $a_n$  takes values in the set  $\{0, 1, 2\}$  for every  $n \geq 1$ . That is,  $a_n$  can take only the values 0, 1, or 2. Suppose, moreover, that  $a_n$  converges to 1. Show that there is a natural  $N$  such that if  $n \geq N$ , then  $a_n = 1$ .

12. Consider a sequence  $a_n$  that takes values in the naturals.

(a) Give an example of such a sequence.

(b) Show that  $a_n$  converges if and only if it is stationary, that is, if and only if there is a natural  $N$  such that if  $n \geq N$ , then  $a_n = a_N$ .

13. Consider the sequence  $a_n = n^{1/n}$ . Let  $b_n = a_n - 1$ .

(a) Show that  $b_n > 0$  for  $n \geq 2$ .

(b) Show that

$$n = (b_n + 1)^n \geq \frac{n(n-1)}{2} b_n^2.$$

(Use the binomial expansion

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

where all  $\binom{n}{k}$  are natural numbers and  $\binom{n}{2} = n(n-1)/2$ .)

(c) Use (a) and (b) to get

$$0 < b_n \leq \sqrt{\frac{2}{n-1}}.$$

(d) Show that  $n^{1/n}$  converges to 1.

14. (a) Assume that  $a_n$  does not converge to  $a$ . Show that there exists an  $\epsilon > 0$  and a subsequence  $a_{j_n}$  such that

$$|a_{j_n} - a| > \epsilon \quad \text{for all } n.$$

(b) Let  $b_n$  be a sequence of reals. Assume that there exists a  $b$  such that any subsequence of  $b_n$  has a further subsequence that converges to  $b$ . Show that  $b_n$  converges to  $b$ . (Do a proof by contradiction and use (a).)

## 2.2 Monotone Sequences, Bolzano–Weierstrass Theorem, and Operations on Limits

We start with the notion of monotone sequence.

### Monotone sequences

A sequence  $a_n$  is said to be increasing if for every natural  $n$ , we have  $a_{n+1} \geq a_n$ . A sequence  $a_n$  is said to be decreasing if for every natural  $n$ , we have  $a_{n+1} \leq a_n$ . A sequence which is increasing or decreasing is said to be monotone.

The sequence  $a_n = 1/n$  is decreasing. The sequence  $b_n = n^2$  is increasing, and the sequence  $c_n = (-1)^n$  is not monotone. The following notions are crucial for monotone sequences.

### Bounded below and above

A sequence  $a_n$  is said to be bounded below if there is a real number  $m$  such that  $a_n > m$  for all  $n$ . A sequence  $a_n$  is said to be bounded above if there is a real number  $M$  such that  $a_n < M$  for all  $n$ .

We are now ready to state an important convergence criterion for monotone sequences.

### Convergent monotone sequences

An increasing sequence converges if and only if it is bounded above. A decreasing sequence is convergent if and only if it is bounded below.

We prove the statement about increasing sequences, the other one is similar and is left as an exercise. Assume that the sequence  $a_n$  converges. Then we know that it must be bounded and therefore bounded above. This proves one implication.

For the other implication, assume that the increasing sequence  $a_n$  is bounded above by a number  $M$ . Let  $A$  be the set

$$A = \{a_n, n \in \mathbb{N}\}.$$

The set  $A$  has an upper bound  $M$  and is not empty. The fundamental property of the reals applies: there is a least upper bound  $\ell$  for  $A$ . We are now going to show that the sequence  $a_n$  converges to  $\ell$ . Take any  $\epsilon > 0$ ; since  $\ell$  is the *least* upper bound,  $\ell - \epsilon$  cannot be an upper bound. That is, there is an element in  $A$  strictly larger than  $\ell - \epsilon$ . Therefore, there is  $N$  such that  $a_N > \ell - \epsilon$ . Note that up to this point we have not used that  $a_n$  is an increasing sequence. We now do. Take  $n \geq N$ . Then

$$a_n \geq a_N > \ell - \epsilon.$$

Hence,

$$\ell - a_n < \epsilon.$$

Since  $\ell$  is larger than all  $a_n$ , we have

$$|a_n - \ell| = \ell - a_n < \epsilon \quad \text{for all } n \geq N.$$

That is,  $a_n$  converges to  $\ell$ .

*Example 2.10* Let  $a > 1$ . We show that  $a_n = a^{1/n}$  converges and find its limit.

Since  $\frac{1}{n+1} < \frac{1}{n}$  and  $a > 1$ , we have that  $a_{n+1} < a_n$ . That is,  $a_n$  is decreasing. Moreover, we have  $a^{1/n} > 1^{1/n} = 1$  (the function  $x \rightarrow x^{1/n}$  is increasing). Therefore,  $a_n$  is decreasing and bounded below by 1, and thus it converges.

We are now going to find the limit  $\ell$  of  $a_n$ . Consider the subsequence  $a_{2n}$ . It must converge to  $\ell$  as well. On the other hand  $a_{2n} = a^{\frac{1}{2n}} = (a_n)^{1/2}$  converges to  $\ell^{1/2}$  (see Exercise 9 in Sect. 2.1). Therefore,  $\ell = \ell^{1/2}$ . Either  $\ell = 0$  or  $\ell = 1$ , but  $\ell$  cannot be 0 (why not?), therefore it is 1.

*Example 2.11* Let  $c$  be in  $(0, 1)$  and define  $a_n = c^n$ . We show that  $c_n$  converges to 0.

Since  $c < 1$ , we have  $c^n c < c^n$ . That is,  $a_{n+1} < a_n$ . The sequence  $a_n$  is strictly decreasing. It is also a positive sequence, and therefore it is bounded below by 0. Hence, the sequence  $a_n$  converges to some  $\ell$ . We know that  $a_{n+1}$  converges to the same limit  $\ell$ . However,  $a_{n+1} = ca_n$ , and since  $a_n$  converges to  $\ell$ , we know that  $ca_n$  converges to  $c\ell$ . Hence,  $a_{n+1}$  converges to  $\ell$  and to  $c\ell$ . We need to have  $\ell = c\ell$ . Therefore, either  $c = 1$  (but we know that  $c < 1$ ) or  $\ell = 0$ . Hence,  $\ell = 0$ .

As we have seen, not all bounded sequences converge. However, the following weaker statement holds and is very important.

### **Bolzano–Weierstrass Theorem**

A bounded sequence has always a convergent subsequence.

Bolzano–Weierstrass is one of the fundamental theorems in analysis. We will apply it in Chap. 5, for instance, to prove the extreme value theorem.

In order to prove this theorem, we first show that every sequence (bounded or not) has a monotone subsequence.

**Lemma** *Every sequence has a monotone subsequence.*

We prove this lemma. Consider a sequence  $a_n$ . Let

$$A = \{m \in \mathbf{N} : \text{for all } n > m, a_n \leq a_m\}.$$

There are three possibilities:  $A$  may be empty, finite, or infinite. Assume first that  $A$  is finite. It has a maximum  $N$  (this is true for any finite set). Set  $n_1 = N + 1$ . Since

$n_1$  is not in  $A$ , there exists at least one  $n_2 > n_1$  such that  $a_{n_2} > a_{n_1}$ . Assume that we have found  $n_1 < n_2 < \cdots < n_k$  such that

$$a_{n_1} < a_{n_2} < \cdots < a_{n_k}.$$

Using that  $n_k$  is not in  $A$ , there is  $n_{k+1} > n_k$  such that

$$a_{n_{k+1}} > a_{n_k}.$$

That is, we may construct inductively a strictly increasing sequence when  $A$  is finite. Note that in case  $A$  is empty, the same construction works for  $N = 1$ .

Assume now that  $A$  is infinite. Since  $A$  is a nonempty set of naturals, it has a minimum  $n_1$ . Let  $n_2 > n_1$  be another element in  $A$ . Since  $n_1$  is in  $A$ , we have

$$a_{n_2} \leq a_{n_1}.$$

Assume that we have found  $n_1 < n_2 < \cdots < n_k$  in  $A$  such that

$$a_{n_1} \geq a_{n_2} \geq \cdots \geq a_{n_k}.$$

Since  $A$  is infinite, there is  $n_{k+1}$  in  $A$  such that  $n_{k+1} > n_k$ . Using that  $n_k$  is in  $A$ , we have

$$a_{n_{k+1}} \leq a_{n_k}.$$

That is, we may construct inductively a decreasing sequence when  $A$  is infinite. This completes the proof of the lemma.

It is now very easy to prove the Bolzano–Weierstrass theorem. Let  $a_n$  be a bounded sequence. According to the lemma,  $a_n$  has a monotone subsequence  $a_{n_k}$ . Since  $a_n$  is bounded, so is  $a_{n_k}$ . Hence,  $a_{n_k}$  is a monotone and bounded sequence. Therefore, it converges. This completes the proof of the theorem.

*Example 2.12* Show that if  $a_n$  is a sequence in  $[0, 1]$ , then it has a subsequence that converges.

Since for all  $n \geq 1$ ,  $a_n$  is in  $[0, 1]$ , we have  $|a_n| \leq 1$ . Hence  $a_n$  is bounded, and by Bolzano–Weierstrass, it has a convergent subsequence.

We now turn to the operations on limits.

### Operations on limits

Assume that the sequences  $a_n$  and  $b_n$  converge, respectively, to  $a$  and  $b$ . Then

- (i)  $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$ .
- (ii)  $\lim_{n \rightarrow \infty} a_n b_n = ab$ .
- (iii) Assume that  $b_n$  is never 0 and that  $b$  is not 0. Then

$$\lim_{n \rightarrow \infty} a_n / b_n = a / b.$$

- (iv) Assume that for all  $n$ ,  $a_n \leq b_n$ . Then  $a \leq b$ .

We prove the statements above. Since  $a_n$  and  $b_n$  converge, for any  $\epsilon > 0$ , there are natural numbers  $N_1$  and  $N_2$  such that if  $n \geq N_1$ , we have

$$|a_n - a| < \epsilon/2,$$

and if  $n \geq N_2$ , we have

$$|b_n - b| < \epsilon/2.$$

Take  $N = \max(N_1, N_2)$  so that both statements above hold for  $n \geq N$  and use the triangle inequality to get

$$|(a_n + b_n) - (a + b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This proves (i).

We now deal with (ii). The case  $b = 0$  can be dealt with by using the result in Example 2.8, Sect. 2.1. Since  $a_n$  converges, it is bounded. We proved already that the product of a bounded sequence and a sequence converging to 0 converges to 0.

We now assume that  $b \neq 0$ . We have

$$|a_n b_n - ab| = |a_n(b_n - b) + a_n b - ab| \leq |a_n||b_n - b| + |b||a_n - a|. \quad (2.1)$$

Since  $a_n$  converges, it is bounded, so there is  $A > 0$  such that  $|a_n| < A$  for every  $n$ . Take  $\epsilon > 0$ ; there is  $N_1$  such that if  $n \geq N_1$ , then

$$|a_n - a| < \frac{\epsilon}{2b}.$$

There is also  $N_2$  such that if  $n \geq N_2$ , then

$$|b_n - b| < \frac{\epsilon}{2A}.$$

We use the two preceding inequalities in (2.1) to get that if  $n \geq \max(N_1, N_2)$ ,

$$|a_n b_n - ab| \leq |a_n||b_n - b| + |b||a_n - a| \leq A \frac{\epsilon}{2A} + |b| \frac{\epsilon}{2b} = \epsilon.$$

This completes the proof of (ii).

To prove (iii), it is enough to prove that  $1/b_n$  converges to  $1/b$ . We can then use (ii) to show that  $a_n/b_n = a_n(1/b_n)$  converges to  $a(1/b) = a/b$ . We start with

$$|1/b_n - 1/b| = \left| \frac{b_n - b}{b_n b} \right|.$$

The only difficulty is to show that  $b_n$  is bounded away from 0. Since  $b_n$  converges to  $b$ , there is a natural number  $N_1$  such that if  $n \geq N_1$ , then

$$|b_n - b| \leq |b|/2,$$

where we are using the definition of convergence with  $\epsilon = |b|/2 > 0$  (since  $b \neq 0$ ). By the triangle inequality we get, for  $n \geq N_1$ ,

$$|b| = |b - b_n + b_n| \leq |b - b_n| + |b_n| < |b|/2 + |b_n|.$$

That is,

$$|b_n| > |b|/2 \quad \text{for all } n \geq N_1.$$

In particular, we have

$$\frac{1}{|b_n b|} < \frac{2}{b^2} \quad \text{for all } n \geq N_1.$$

On the other hand, since  $b_n$  converges to  $b$ , for any  $\epsilon > 0$ , there is  $N_2$  such that if  $n \geq N_2$ , then

$$|b_n - b| < \frac{\epsilon b^2}{2}.$$

Therefore, for  $n \geq \max(N_1, N_2)$ , we have

$$|1/b_n - 1/b| = \left| \frac{b_n - b}{b_n b} \right| = \frac{|b_n - b|}{|b_n b|} < \frac{\epsilon b^2}{2} \frac{2}{b^2} = \epsilon.$$

That is,  $1/b_n$  converges to  $1/b$ , and the proof of (iii) is complete.

For (iv), we do a proof by contradiction. Assume that  $a > b$  and let  $\epsilon = \frac{a-b}{2} > 0$ . Since  $a_n$  converges to  $a$ , there exists  $N_1$  such that if  $n \geq N_1$ , then

$$|a_n - a| < \epsilon.$$

In particular,  $a_n > a - \epsilon = \frac{a+b}{2}$  if  $n \geq N_1$ . On the other hand, since  $b_n$  converges to  $b$ , there exists  $N_2$  such that if  $n \geq N_2$ , then

$$|b_n - b| < \epsilon.$$

In particular,  $b_n < b + \epsilon = \frac{a+b}{2}$  if  $n \geq N_2$ . Thus, if  $n > \max(N_1, N_2)$ , then

$$b_n < \frac{a+b}{2} < a_n,$$

contradicting the assumption  $a_n \leq b_n$  for all  $n$ . This completes the proof of (iv).

*Example 2.13* Assume that  $a_n$  converges to  $a$  and that  $a_n + b_n$  converges to  $c$ . We show that  $b_n$  converges.

We start by writing

$$b_n = (b_n + a_n) + (-a_n).$$

We have assumed that  $b_n + a_n$  converges to  $c$ . By (ii),  $(-a_n)$  converges to  $-a$  (why?). Hence, by (i),  $b_n = (b_n + a_n) + (-a_n)$  converges to  $c - a$ . This completes the proof.

## Exercises

1. Show that if  $a_n$  and  $b_n$  converge to  $a$  and  $b$ , respectively, then  $ca_n + db_n$  converges for any constants  $c$  and  $d$ .
2. Assume that  $a_n$  is increasing. Show that if  $n > m$ , then  $a_n \geq a_m$ .
3. Prove that a decreasing sequence converges if and only if it is bounded below.
4. Assume that  $a_n$  is an increasing sequence,  $b_n$  is a decreasing sequence, and  $a_n \leq b_n$  for all  $n \geq 1$ .
  - (a) Show that  $a_n$  and  $b_n$  converge.

- (b) If in addition to the previous hypotheses, we have that  $b_n - a_n$  converges to 0, prove that  $a_n$  and  $b_n$  converge to the same limit.
5. Assume that  $a_n < b_n$  for all naturals  $n$ . Assume that  $a_n$  and  $b_n$  converge to  $a$  and  $b$ , respectively. Is it true that  $a < b$ ? Prove it or give a counterexample.
6. Assume that  $a_n$  converges to  $\ell$  and is bounded below by  $m$ . Show that  $\ell \geq m$ .
7. Show that for any  $a > 0$ ,  $a^{1/n}$  converges to 1 (you may use the fact that the result has been proved for  $a > 1$  in Example 2.10).
8. Suppose that  $a_n$  is increasing. Show that  $a_n$  converges if and only if it has a subsequence which converges.
9. Assume that  $a_n$  converges to  $a \in (0, 1)$ .
- (a) Show that there is  $N$  such for  $n \geq N$ , we have  $a_n$  in  $(0, 1)$ .
- (b) Is the statement in (a) true if we replace  $(0, 1)$  by  $[0, 1)$ ?
10. We define a sequence  $a_n$  recursively by setting  $a_1 = 2$  and

$$a_{n+1} = \frac{2a_n - 1}{a_n}.$$

- (a) Compute the first five terms of this sequence.
- (b) Show that for all  $n \geq 1$ , if  $a_n > 1$ , then  $a_{n+1} > 1$ .
- (c) Use (b) to show that the sequence  $a_n$  is well defined and that  $a_n$  is bounded below by 1.
- (d) Show that  $a_n$  is decreasing.
- (e) Show that  $a_n$  converges to some limit  $\ell \geq 1$ .
- (f) Show that

$$\frac{2a_n - 1}{a_n} \quad \text{converges to} \quad \frac{2\ell - 1}{\ell}.$$

- (g) Find  $\ell$ .
11. Assume that  $a_n - a$  converges to 0. Show that  $a_n$  converges to  $a$ .
12. Assume that  $|a_n - a| \leq 1/n$  for all  $n \geq 1$ . Show that  $a_n$  converges to  $a$ .
13. Assume that  $a_n - b_n$  converges and that  $b_n$  converges. Show that  $a_n$  converges.
14. Consider a sequence  $a_n$  that converges to  $a > 0$ . Show that there is a natural  $N$  such that if  $n \geq N$ , then  $a_n > 0$ .
15. Assume that  $|c| < 1$ . Show that the sequence  $c^n$  converges to 0.
16. A set  $C$  in  $\mathbf{R}$  is said to be closed if for every convergent sequence in  $C$ , the limit is also in  $C$ .
- (a) Give an example of a closed set.
- (b) An open set  $O$  in  $\mathbf{R}$  is a set whose complement (all the elements not in  $O$ ) is closed. Show that if  $O$  is open, then for every  $a$  in  $O$ , it is possible to find  $\epsilon > 0$  such that  $(a - \epsilon, a + \epsilon) \subset O$ .

## 2.3 Series

A series, as we are going to see, is the sum of a sequence.

**Definition**

Let  $a_n$  be a sequence of real numbers. Define the sequence  $S_n$  by

$$S_n = \sum_{k=1}^n a_k.$$

$S_n$  is the sequence of partial sums. The series  $\sum_{k=1}^{\infty} a_k$  is said to converge if the sequence  $S_n$  converges. If that is the case, then the limit of  $S_n$  is denoted by  $\sum_{k=1}^{\infty} a_k$ .

The series above starts at  $k = 1$ , but it may actually start at any positive or 0 integer. A series that does not converge is said to diverge.

*Example 2.14* Let  $r$  be a real number in  $(-1, 1)$ , and  $a_n = r^n$ . Recall that for any real numbers  $a, b$  and natural number  $n$ , we have

$$a^{n+1} - b^{n+1} = (a - b)(a^n + a^{n-1}b + \cdots + ab^{n-1} + b^n).$$

Thus,

$$r^{n+1} - 1 = (r - 1)(r^n + r^{n-1} + \cdots + r + 1).$$

Since  $r \neq 1$ , we have

$$\sum_{k=0}^n r^k = \frac{r^{n+1} - 1}{r - 1}.$$

Defining  $S_n = \sum_{k=0}^n a_k$ , we get that

$$S_n = \frac{r^{n+1} - 1}{r - 1}.$$

By Example 2.3 in Sect. 2.1 we know that  $r^{n+1}$  converges to 0 as  $n$  goes to infinity for  $|r| < 1$ . Therefore,

$$\lim_{n \rightarrow \infty} S_n = \frac{1}{1 - r} \quad \text{for } |r| < 1.$$

In other words, the series  $\sum_{k=0}^{\infty} a_k$  converges, and its sum is  $\frac{1}{1-r}$ .

The series considered in Example 2.14 is called a geometric series. It can be computed exactly, and as the reader will see, this is rather rare. We now turn to convergence issues. We start by a divergence test.

**Divergence test**

If the series  $\sum_{k=1}^{\infty} a_k$  converges, then the sequence  $a_n$  converges to 0.

This test as indicated by its name can be used to show that a series diverges; it can *never* be used to show convergence.

If  $a_n$  does not converge to 0, the divergence test implies that the series  $\sum_{k=1}^{\infty} a_k$  diverges. On the other hand, if the sequence  $a_n$  does converge to 0, then the test cannot be used: the series  $\sum_{k=1}^{\infty} a_k$  may converge or may diverge. We will see examples of both situations.

We now prove the divergence test. Assume that the series  $\sum_{k=1}^{\infty} a_k$  converges. By definition this means that  $S_n = \sum_{k=1}^n a_k$  converges. Therefore,  $S_{n-1}$  converges to the same limit (why?), and so

$$a_n = S_n - S_{n-1} \quad \text{converges to 0.}$$

This completes the proof of the divergence test.

*Example 2.15* Consider the geometric series  $\sum_{k=0}^{\infty} r^k$  with  $|r| \geq 1$ . Then the sequence  $a_n = |r^n| = |r|^n$  is bounded below by 1 and therefore cannot converge to 0. Thus, the series  $\sum_{k=0}^{\infty} r^k$  diverges by the divergence test if  $|r| \geq 1$ .

*Example 2.16* The series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.

Let  $S_n = \sum_{k=1}^n \frac{1}{k}$ . We are going to prove by induction that

$$S_{2^n} \geq \frac{n+2}{2}.$$

Let  $P$  denote the set of natural numbers  $n$  for which the inequality holds. Note that  $S_2 = 1 + 1/2 = 3/2$  and that  $\frac{1+2}{2}$  is also  $3/2$ . So the inequality holds for  $n = 1$ . Assume that it holds for  $n$ . Then

$$S_{2^{n+1}} = S_{2^n} + \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \geq S_{2^n} + \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{2^{n+1}},$$

where the inequality comes from the fact that we replace all  $1/k$  for  $k$  between  $2^n$  and  $2^{n+1}$  by the smallest of them,  $1/2^{n+1}$ . Note that

$$\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{2^{n+1}} = (2^{n+1} - 2^n) \frac{1}{2^{n+1}} = \frac{1}{2}.$$

Therefore, by the induction hypothesis,

$$S_{2^{n+1}} \geq S_{2^n} + \frac{1}{2} \geq \frac{n+2}{2} + \frac{1}{2} = \frac{n+3}{2}.$$

So  $n+1$  belongs to  $P$ , and the inequality is proved.

The sequence  $S_{2^n}$  is larger than an unbounded sequence and therefore must be unbounded itself. Thus,  $S_{2^n}$  diverges.  $S_n$  has a subsequence that diverges and so cannot converge. Hence, the series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges. This series gives an example for which  $a_n = 1/n$  converges to 0 but the series  $\sum_{n=1}^{\infty} a_n$  diverges. This provides a counterexample showing that the converse of the divergence test does not hold.

**Operations on series**

Assume that the series  $\sum_{k=1}^{\infty} a_k$  and  $\sum_{k=1}^{\infty} b_k$  converge. Let  $c$  be a constant. Then

(i) The series  $\sum_{k=1}^{\infty} (a_k + b_k)$  converges, and

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$

(ii) The series  $\sum_{k=1}^{\infty} (ca_k)$  converges, and

$$\sum_{k=1}^{\infty} (ca_k) = c \sum_{k=1}^{\infty} a_k.$$

These properties are easily proved by applying the operations on limits of Sect. 2.2. Let

$$A_n = \sum_{k=1}^n a_k \quad \text{and} \quad B_n = \sum_{k=1}^n b_k.$$

Since the series  $\sum_{k=1}^{\infty} a_k$  and  $\sum_{k=1}^{\infty} b_k$  converge, we have that the sequences  $A_n$  and  $B_n$  converge. Hence, the sequence  $A_n + B_n$  converges. Note that

$$A_n + B_n = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k = \sum_{k=1}^n (a_k + b_k).$$

Hence, the convergence of the sequence  $A_n + B_n$  is equivalent to the convergence of the series  $\sum_{k=1}^{\infty} (a_k + b_k)$ . Moreover,

$$\lim_{n \rightarrow \infty} (A_n + B_n) = \lim_{n \rightarrow \infty} A_n + \lim_{n \rightarrow \infty} B_n.$$

That is,

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$

This proves (i). The proof of (ii) is similar and left to the reader.

We now turn to convergence tests for positive-term series. Positive-term series are easier to analyze because of the following obvious but important fact.

**Positive-term series**

Let  $a_n$  be a sequence, and  $S_n = \sum_{k=1}^n a_k$  be the corresponding partial sums. The sequence  $S_n$  is increasing if and only if  $a_n \geq 0$  for all  $n \geq 1$ .

Observe that

$$S_n - S_{n-1} = a_n,$$

and the result is obvious. As the reader will see, all the tests we will state in the rest of this section depend crucially on the properties of monotone sequences.

### Comparison test

Consider two sequences of real numbers  $a_n \geq 0$  and  $b_n \geq 0$  such that for some natural  $N$ ,

$$a_n \leq b_n \quad \text{for all } n \geq N.$$

- (1) If the series  $\sum_{n=0}^{\infty} b_n$  converges, so does the series  $\sum_{n=0}^{\infty} a_n$ .
- (2) If the series  $\sum_{n=0}^{\infty} a_n$  diverges, so does the series  $\sum_{n=0}^{\infty} b_n$ .

Let  $S_n = \sum_{k=0}^n a_k$  and  $T_n = \sum_{k=0}^n b_k$ . We first prove (1). We assume that  $T_n$  converges. Take  $n > N$ . Then

$$S_n = \sum_{k=0}^N a_k + \sum_{k=N+1}^n a_k \leq S_N + \sum_{k=N+1}^n b_k.$$

Note that  $\sum_{k=N+1}^n b_k = T_n - T_N$ . Therefore,

$$S_n \leq S_N - T_N + T_n.$$

Since the series  $\sum_{n=0}^{\infty} b_n$  converges, the sequence  $T_n$  is bounded and therefore bounded above by some  $K$ . Observe that since  $N$  is fixed,  $S_N - T_N$  is a constant (that is, it does not depend on the variable  $n$ ), and  $S_n$  is bounded above by  $S_N - T_N + K$ . On the other hand,  $S_{n+1} - S_n = a_{n+1} \geq 0$ , and thus  $S_n$  is an increasing sequence. An increasing sequence which is bounded above must converge, and (1) is proved.

Now we turn to (2). Assume that  $S_n$  diverges. We use the inequality proved above:

$$T_n \geq S_n - S_N + T_N.$$

As already observed,  $S_n$  is increasing; since it diverges, it cannot be bounded above. Thus,  $T_n$  is larger than  $S_n - S_N + T_N$ , which is not bounded above (why?), and so  $T_n$  cannot be bounded either. Therefore,  $T_n$  diverges, and (2) is proved.

*Example 2.17* Let  $a_n \geq 0$  and assume that the series  $\sum_{n=1}^{\infty} a_n$  diverges. We show that  $\sum_{n=1}^{\infty} \sqrt{a_n}$  diverges as well.

We consider two cases. If  $a_n$  does not converge to 0, then  $\sqrt{a_n}$  does not converge to 0 either (why?), and by the divergence test the series  $\sum_{n=1}^{\infty} \sqrt{a_n}$  diverges.

If  $a_n$  converges to 0, there is  $N$  such that if  $n \geq N$ , then

$$|a_n - 0| < \epsilon = 1.$$

In particular,  $0 \leq a_n < 1$  for  $n \geq N$ . Recall that  $\sqrt{x} \geq x$  for  $x$  in  $[0, 1]$ . Hence,

$$\sqrt{a_n} \geq a_n \quad \text{for } n \geq N.$$

Since the series  $\sum_{n=1}^{\infty} a_n$  diverges, so does the series  $\sum_{n=1}^{\infty} \sqrt{a_n}$  by the comparison test.

The following limit comparison test is sometimes easier to apply than the comparison test.

### Limit comparison test

Consider two sequences of positive real numbers  $a_n \geq 0$  and  $b_n > 0$  such that  $\frac{a_n}{b_n}$  converges to a strictly positive number. Then, the series  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=0}^{\infty} b_n$  both converge or both diverge.

We now prove the limit comparison test. Assume that  $a_n/b_n$  converges to some  $\ell > 0$ . By the definition of convergence with  $\epsilon = \ell/2$ , there is  $N$  such that if  $n \geq N$ ,

$$\left| \frac{a_n}{b_n} - \ell \right| < \ell/2$$

or, equivalently,

$$\ell/2 < \frac{a_n}{b_n} < 3\ell/2 \quad \text{for } n \geq N.$$

Assume that the series  $\sum_{n=0}^{\infty} a_n$  converges. Since

$$b_n < \frac{2}{\ell} a_n \quad \text{for } n \geq N,$$

the series  $\sum_{n=0}^{\infty} b_n$  converges by the comparison test.

Assume now that the series  $\sum_{n=0}^{\infty} a_n$  diverges. Since

$$b_n > \frac{2}{3\ell} a_n \quad \text{for } n \geq N,$$

the series  $\sum_{n=0}^{\infty} b_n$  diverges by the comparison test.

To complete the proof, we should now show that if the series  $\sum_{n=0}^{\infty} b_n$  converges, so does the series  $\sum_{n=0}^{\infty} a_n$ , and that if the series  $\sum_{n=0}^{\infty} b_n$  diverges, so does the series  $\sum_{n=0}^{\infty} a_n$ . Instead, we may observe that  $b_n/a_n$  converges to  $1/\ell$  (also a strictly positive real) and therefore  $a_n$  and  $b_n$  play symmetric roles here. This completes the proof of the limit comparison test.

*Example 2.18* Does the series  $\sum_{n=2}^{\infty} \frac{n}{n^2-1}$  converge?

Let  $a_n = \frac{1}{n}$  and  $b_n = \frac{n}{n^2-1}$ . It is easy to check that  $a_n/b_n$  converges to 1. Since the series  $\sum_{n=2}^{\infty} a_n$  diverges, so does the series  $\sum_{n=2}^{\infty} b_n$  by the limit comparison test.

*Example 2.19* Let  $a \leq 1$  be a real number. The series

$$\sum_{k=1}^{\infty} \frac{1}{k^a}$$

diverges.

Let  $a_n = \frac{1}{n}$  and  $b_n = \frac{1}{n^a}$ . Both sequences are positive, and since  $a \leq 1$ ,

$$a_n \leq b_n \quad \text{for all } n \geq 1.$$

We have already shown that the series  $\sum_{n=1}^{\infty} a_n$  diverges, so by the comparison test (with  $N = 1$ ) the series  $\sum_{n=1}^{\infty} b_n$  diverges as well.

We still need to analyze the case  $a > 1$  for the series  $\sum_{k=1}^{\infty} \frac{1}{k^a}$ . In this case,

$$\frac{1}{k} > \frac{1}{k^a}.$$

Since the series  $\sum_{k=1}^{\infty} \frac{1}{k}$  diverges, this comparison is useless. We need another test.

### Cauchy test

Consider a decreasing and positive sequence  $a_n$ . The series

$$\sum_{k=1}^{\infty} a_k$$

converges if and only if the series

$$\sum_{k=1}^{\infty} 2^k a_{2^k}$$

converges.

Let

$$S_n = \sum_{k=1}^n a_k \quad \text{and} \quad T_n = \sum_{k=1}^n 2^k a_{2^k}$$

be the two partial sums.

Assume first that the series  $\sum_{k=1}^{\infty} a_k$  converges and therefore that the sequence  $S_n$  converges. Consider

$$S_{2^k} - S_{2^{k-1}} = \sum_{j=2^{k-1}+1}^{2^k} a_j.$$

Since the sequence  $a_n$  is decreasing for every natural  $j$  in  $[2^{k-1} + 1, 2^k]$ , we have  $a_j \geq a_{2^k}$ . Thus,

$$S_{2^k} - S_{2^{k-1}} \geq \sum_{j=2^{k-1}+1}^{2^k} a_{2^k} = 2^{k-1} a_{2^k},$$

where we use that in  $[2^{k-1} + 1, 2^k]$ , there are  $2^{k-1}$  naturals (why?). We sum the preceding inequality to get

$$\sum_{k=1}^n (S_{2^k} - S_{2^{k-1}}) \geq \sum_{k=1}^n 2^{k-1} a_{2^k} = \frac{1}{2} \sum_{k=1}^n 2^k a_{2^k} = T_n/2.$$

Observe that

$$\sum_{k=1}^n (S_{2^k} - S_{2^{k-1}}) = (S_2 - S_1) + (S_4 - S_2) + \cdots + S_{2^n} - S_{2^{n-1}} = S_{2^n} - S_1.$$

Hence,

$$S_{2^n} - S_1 \geq T_n/2.$$

That is,

$$T_n \leq 2S_{2^n} - 2S_1.$$

Since  $S_n$  converges, it must be bounded by some  $K$ . Thus,

$$T_n \leq 2K - 2S_1.$$

So  $T_n$  is bounded above. Since it is also increasing (why?), it must converge. We have proved the direct implication.

Assume now that  $T_n$  converges. Consider again

$$S_{2^k} - S_{2^{k-1}} = \sum_{j=2^{k-1}+1}^{2^k} a_j.$$

Since the sequence  $a_n$  is decreasing for every natural  $j$  in  $[2^{k-1} + 1, 2^k]$ , we have  $a_j \leq a_{2^{k-1}}$ . Thus,

$$S_{2^k} - S_{2^{k-1}} \leq \sum_{j=2^{k-1}+1}^{2^k} a_{2^{k-1}} = 2^{k-1} a_{2^{k-1}}.$$

It follows that

$$\sum_{k=1}^n (S_{2^k} - S_{2^{k-1}}) \leq \sum_{k=1}^n 2^{k-1} a_{2^{k-1}} = T_{n-1} + a_1.$$

Therefore,

$$S_{2^n} - S_1 \leq T_{n-1} + a_1,$$

and since  $S_1 = a_1$ , we have

$$S_{2^n} \leq T_{n-1} + 2a_1.$$

But  $n < 2^n$  for every natural  $n$ . Using that  $S_n$  is an increasing sequence, we have

$$S_n \leq S_{2^n}$$

for every  $n$ . Hence,

$$S_n \leq S_{2^n} \leq T_{n-1} + 2a_1.$$

Since  $T_n$  is increasing and convergent, it is bounded above, and so is  $S_n$ . We know that  $S_n$  is also increasing, so it converges. This proves the converse. The proof of the Cauchy test is complete.

*Example 2.20* Let  $p > 0$  be a real number and consider the series  $\sum_{k=1}^{\infty} \frac{1}{k^p}$ . We have seen already that this series diverges for  $p \leq 1$ . The Cauchy test will allow us to determine convergence in all cases. First, we need to check the hypotheses of the test. The sequence

$$a_k = \frac{1}{k^p}$$

is positive and decreasing for all  $p > 0$ . Note that

$$a_{2^k} = \frac{1}{(2^k)^p} = (2^{-p})^k$$

and that the series of general term

$$2^k a_{2^k} = (2^{-p+1})^k$$

is a geometric series with ratio  $r = 2^{-p+1}$ . In particular,  $r < 1$  if and only if  $p > 1$ . By the Cauchy test, the series  $\sum_{k=1}^{\infty} \frac{1}{k^p}$  converges if and only if  $p > 1$ .

We now have a complete result for series of the type  $\sum_{k=1}^{\infty} \frac{1}{k^p}$  for real  $p$ . Since this is an important class of series, we summarize our results below.

**$p$  Test**  
The series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges when  $p > 1$  and diverges when  $p \leq 1$ .

### Exercises

1. Let  $a_n = \frac{1}{n(n+1)}$  for  $n \geq 1$ .

- (a) Compute  $S_n = \sum_{k=1}^n a_k$  in function of  $n$  (observe that  $a_n = 1/n - 1/(n+1)$ ).
- (b) Show that the series  $\sum_{k=1}^{\infty} a_k$  converges by computing its sum.
2. Does  $\sum_{k=1}^{\infty} (-1)^k$  converge?
3. Compute  $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{4^{n+1}}$ .
4. Let  $a_n \geq 0$  and  $b_n = \sqrt{a_n}$ . Show that  $a_n$  converges to 0 if and only if  $b_n$  converges to 0.
5. (a) Let  $a_n \geq 0$  be such that  $\sum_{k=1}^{\infty} a_k$  converges. Show that  $\sum_{k=1}^{\infty} a_k^2$  converges as well.
- (b) Is the converse of (a) true?
- (c) In (a) you have shown that if  $\sum_{k=1}^{\infty} a_k$  converges, then  $\sum_{k=1}^{\infty} f(a_k)$  converges for  $f(x) = x^2$ . Find other functions  $f$  for which this is true.
6. Let  $a_n > 0$ . Show that  $\sum_{k=1}^{\infty} a_k$  converges if and only if

$$\sum_{k=1}^{\infty} \frac{a_k}{1 + a_k}$$

converges.

7. Show that  $\sum_{k=1}^{\infty} a_k$  converges if and only if  $\sum_{k=100}^{\infty} a_k$  converges.
8. Assume that the sequences  $a_n \geq 0$  and  $b_n > 0$  are such that  $a_n/b_n$  converges to 0.
  - (a) Show that if the series  $\sum_{k=1}^{\infty} b_k$  converges, so does  $\sum_{k=1}^{\infty} a_k$ .
  - (b) Show that if the series  $\sum_{k=1}^{\infty} a_k$  diverges, so does  $\sum_{k=1}^{\infty} b_k$ .
  - (c) Show by exhibiting counterexamples that the converses of (a) and (b) do not hold.
9. The sequence  $a_n$  is said to go to  $+\infty$  if for any real  $A > 0$ , there is a natural number  $N$  such that if  $n \geq N$ , then  $a_n > A$ .
  - (a) Give an example of a sequence that goes to  $+\infty$ . Prove your claim.
  - (b) Show that a sequence that goes to  $+\infty$  cannot be bounded.
  - (c) Is it true that any unbounded positive sequence goes to infinity?
10. Assume that the sequences  $a_n \geq 0$  and  $b_n > 0$  are such that  $a_n/b_n$  goes to infinity (see the definition in Exercise 9).
  - (a) Show that if the series  $\sum_{k=1}^{\infty} a_k$  converges, so does  $\sum_{k=1}^{\infty} b_k$ .
  - (b) Show that if the series  $\sum_{k=1}^{\infty} b_k$  diverges, so does  $\sum_{k=1}^{\infty} a_k$ .
  - (c) Show by exhibiting counterexamples that the converses of (a) and (b) do not hold.
11. Assume that the sequences  $a_n \geq 0$  and  $b_n \geq 0$  are such that  $\sum_{k=1}^{\infty} a_k$  converges and  $b_n$  is bounded. Prove that  $\sum_{k=1}^{\infty} a_k b_k$  converges.
12. Assume that  $a_n \geq 0$  and  $b_n \geq 0$  for all  $n \geq 1$ . Suppose that  $\sum_{k=1}^{\infty} a_k$  and  $\sum_{k=1}^{\infty} b_k$  converge. Prove that  $\sum_{k=1}^{\infty} a_k b_k$  converges.
13. Prove that in  $[2^{k-1} + 1, 2^k]$  there are  $2^{k-1}$  naturals.
14. (a) Show that if  $a_n$  converges, then  $a_n - a_{n-1}$  converges to 0.
- (b) Is the converse of (a) true? Prove it or give a counterexample.

15. Assume that the series  $\sum_{k=1}^{\infty} a_k$  converges. Let  $c$  be a constant. Show that the series  $\sum_{k=1}^{\infty} (ca_k)$  converges and that

$$\sum_{k=1}^{\infty} (ca_k) = c \sum_{k=1}^{\infty} a_k.$$

16. Let  $d_n$  be a sequence in  $\{0, 1, 2, \dots, 9\}$ . Show that the series

$$\sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

converges.

## 2.4 Absolute Convergence

We start with a definition.

### Absolute convergence

The series  $\sum_{n=0}^{\infty} a_n$  is said to converge absolutely if the series  $\sum_{n=0}^{\infty} |a_n|$  converges.

*Example 2.21* Let  $a_n = \frac{(-1)^n}{n^2}$ . Then  $|a_n| = \frac{1}{n^2}$ . By the  $p$  rule, the series  $\sum_{n=1}^{\infty} |a_n|$  converges. That is, the series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$  converges absolutely. Does this imply that the series  $\sum_{n=1}^{\infty} a_n$  also converges? As the following result shows, the answer to that question is yes.

### Absolute convergence implies convergence

If a series converges absolutely, then it converges.

We now prove the result above. Assume that  $\sum_{n=1}^{\infty} |a_n|$  converges. Note that for all reals  $x$ , we have

$$-|x| \leq x \leq |x|,$$

and so

$$0 \leq x + |x| \leq 2|x|.$$

Hence, for all naturals  $n \geq 1$ ,

$$0 \leq a_n + |a_n| \leq 2|a_n|.$$

Since  $\sum_{n=1}^{\infty} |a_n|$  converges, so does  $\sum_{n=1}^{\infty} 2|a_n|$ . By the comparison test (which is valid only for positive-term series), the series  $\sum_{n=1}^{\infty} (a_n + |a_n|)$  converges as well.

Let  $S_n = \sum_{k=1}^n a_k$  and  $A_n = \sum_{k=1}^n |a_k|$ . We have just shown that  $S_n + A_n$  converges, and we know that  $A_n$  converges by hypothesis. Thus,

$$S_n = (S_n + A_n) - A_n$$

also converges (why?). This proves that the series  $\sum_{n=1}^{\infty} a_n$  converges, and we are done.

*Example 2.22* Consider the series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ . This series does not converge absolutely (why not?). However, as we will show below, it converges. Therefore, in general, convergence does not imply absolute convergence. Of course, if the  $a_n$  are all positive, then absolute convergence is the same as convergence.

We now turn to a useful test for absolute convergence.

### Ratio Test

Assume that  $a_n \neq 0$  for all  $n$  and that the sequence  $\left| \frac{a_{n+1}}{a_n} \right|$  converges to some limit  $\ell$ . If  $\ell < 1$ , then the series  $\sum_{n=1}^{\infty} a_n$  converges absolutely. If  $\ell > 1$ , then the series  $\sum_{n=1}^{\infty} a_n$  diverges. If  $\ell = 1$ , this test is inconclusive.

We prove the ratio test. Assume first that the limit  $\ell$  exists and that  $\ell < 1$ . Let  $\epsilon = (1 - \ell)/2 > 0$ . By the definition of convergence, there is a natural  $N$  such that if  $n \geq N$ , then

$$\left| \left| \frac{a_{n+1}}{a_n} \right| - \ell \right| < \epsilon.$$

Thus, for  $n \geq N$ ,

$$|a_{n+1}| < |a_n|(\epsilon + \ell) = |a_n|(1 + \ell)/2.$$

Let  $r = (1 + \ell)/2$  and note that  $r < 1$ . We have that

$$|a_{n+1}| < |a_n|r \quad \text{for all } n \geq N. \quad (2.2)$$

We will now show by induction that for all integers  $k \geq 1$ , we have

$$|a_{N+k}| < |a_N|r^k. \quad (2.3)$$

Let  $P$  be the set of naturals  $k$  for which (2.3) holds. Using (2.2) with  $n = N$ , we get

$$|a_{N+1}| < |a_N|r,$$

and 1 belongs to  $P$ . Assume now that natural  $k$  belongs to  $P$ . Using (2.2) with  $n = N + k$ , we have

$$|a_{N+k+1}| < |a_{N+k}|r.$$

By the induction hypothesis,

$$|a_{N+k}| < |a_N|r^k,$$

and so

$$|a_{N+k+1}| < |a_{N+k}|r < |a_N|r^k r = |a_N|r^{k+1}.$$

That is,  $k+1$  belongs to  $P$ , and (2.3) is proved by induction.

Note that (2.3) can also be written as

$$|a_n| < |a_N|r^{n-N} \quad \text{for } n > N.$$

Observe also that since  $r < 1$ ,

$$\sum_{n>N} |a_N|r^{n-N} = |a_N| \sum_{n \geq 1} r^n = |a_N|r \frac{1}{1-r}.$$

By the comparison test, the series  $\sum_{n=1}^{\infty} |a_n|$  converges, and therefore the series  $\sum_{n=1}^{\infty} a_n$  converges absolutely. The ratio test is proved for  $\ell < 1$ .

We now turn to  $\ell > 1$ . Let  $\epsilon = (\ell - 1)/2 > 0$ . By the definition of convergence, there is a natural  $N$  such that if  $n \geq N$ , then

$$\left| \frac{a_{n+1}}{a_n} - \ell \right| < \epsilon.$$

Thus, for  $n \geq N$ ,

$$|a_{n+1}| > |a_n|(-\epsilon + \ell) = |a_n|(1 + \ell)/2 > a_n.$$

It is then easy to show, by induction, that

$$|a_n| > |a_N| \quad \text{for all } n > N.$$

This shows that the sequence  $a_n$  does not converge to 0 (why?). By the divergence test, the series  $\sum_{n=1}^{\infty} a_n$  diverges.

The final claim to prove is that for  $\ell = 1$ , the test is not conclusive. In order to prove this claim, we need to find two examples with  $\ell = 1$ , one for which the series is convergent and one for which it is not. First, take  $a_n = \frac{1}{n^2}$ . Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n^2}{(n+1)^2} = \left( \frac{n}{n+1} \right)^2$$

converges to  $1^2 = 1$ . That is,  $\ell = 1$ . By the  $p$  test we know that this series converges.

To get a divergent series, take  $b_n = \frac{1}{n}$ . Then

$$\left| \frac{b_{n+1}}{b_n} \right| = \frac{n}{n+1}$$

converges to 1. We have again  $\ell = 1$ , but this time the series diverges.

The ratio test is proved.

The ratio test is especially useful when the general term of the series  $a_n$  involves  $n$  as an exponent or as a factorial. We now give two such examples.

*Example 2.23* Does  $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n^2}$  converge?

Let  $a_n = (-1)^n \frac{3^n}{n^2}$ . Then

$$\frac{|a_{n+1}|}{|a_n|} = 3 \frac{n^2}{(n+1)^2},$$

which converges to 3 as  $n$  goes to infinity. By the ratio test, the series diverges.

*Example 2.24* Does the series  $\sum_{n=0}^{\infty} \frac{1}{n!}$  converge?

Let  $b_n = \frac{1}{n!}$ . Then

$$\frac{|b_{n+1}|}{|b_n|} = \frac{n!}{(n+1)!} = \frac{1}{n+1},$$

which converges to 0 as  $n$  goes to infinity. By the ratio test,  $\sum_{n=0}^{\infty} \frac{1}{n!}$  converges.

*Example 2.25* The ratio test is really a comparison with a geometric series. This remark can be useful to compute numerical approximations. We now use this idea to compute a numerical approximation of the series  $\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}$ .

First, we check that this is a convergent series. Let

$$c_n = \frac{1}{n^2 2^n}.$$

Then

$$\frac{c_{n+1}}{c_n} = \frac{n^2}{(n+1)^2} \frac{2^n}{2^{n+1}} = \frac{1}{2} \frac{n^2}{(n+1)^2}$$

converges to  $1/2$ . By the ratio test, this series converges. We may use the first 10 terms (for instance) to get a numerical approximation of this series.

$$S_{10} = \sum_{n=1}^{10} \frac{1}{n^2 2^n} = \frac{946538429}{1625702400} \sim 0.58.$$

The error is

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} - S_{10} \right| = \sum_{n=11}^{\infty} \frac{1}{n^2 2^n}.$$

However,

$$\frac{1}{n^2 2^n} \leq \frac{1}{11^2 2^n} \quad \text{for all } n \geq 1.$$

Hence,

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} - S_{10} \right| \leq \frac{1}{11^2} \sum_{n=11}^{\infty} \frac{1}{2^n} = \frac{1}{11^2 2^{10}}.$$

Therefore, the approximation above has an error of at most  $\frac{1}{2^{10}}$ .

A test closely related to the ratio test is the following:

### Root Test

Assume that the sequence  $|a_n|^{1/n}$  converges to some limit  $\ell$ . If  $\ell < 1$ , then the series  $\sum_{n=1}^{\infty} a_n$  converges absolutely. If  $\ell > 1$ , then the series  $\sum_{n=1}^{\infty} a_n$  diverges. If  $\ell = 1$ , this test is inconclusive.

Assume first that  $|a_n|^{1/n}$  converges to some limit  $\ell < 1$ . Let  $r = \frac{1+\ell}{2} < 1$ . Take  $\epsilon = \frac{1-\ell}{2} > 0$ . There is  $N$  such that if  $n \geq N$ , then

$$||a_n|^{1/n} - \ell| < \epsilon = \frac{1-\ell}{2}.$$

Hence,

$$|a_n|^{1/n} < \frac{1+\ell}{2} = r \quad \text{for all } n \geq N$$

and

$$|a_n| < r^n \quad \text{for all } n \geq N.$$

Since the geometric series  $\sum_{n=1}^{\infty} r^n$  converges ( $r < 1$ ), so does the series  $\sum_{n=1}^{\infty} a_n$ . This proves the first half of the test.

For the second half, assume that  $|a_n|^{1/n}$  converges to some limit  $\ell > 1$ . Take  $\epsilon = \frac{\ell-1}{2} > 0$ . There is  $N$  such that if  $n \geq N$ , then

$$||a_n|^{1/n} - \ell| < \epsilon = \frac{\ell-1}{2}.$$

Hence,

$$|a_n|^{1/n} > \frac{\ell+1}{2} \quad \text{for all } n \geq N$$

and

$$|a_n| > \left(\frac{\ell+1}{2}\right)^n \quad \text{for all } n \geq N.$$

Since  $\frac{\ell+1}{2} > 1$ ,  $|a_n|$  does not converge to 0, and by the divergence test the series  $\sum_{n=1}^{\infty} a_n$  diverges. This completes the proof of the root test.

The ratio test is usually easier to use than the root test. However, the root test is useful in series having terms  $n^n$  such as in the following example.

*Example 2.26* Consider the convergence and numerical approximation of the series  $\sum_{n=1}^{\infty} \frac{1}{n^n}$ .

Let  $a_n = \frac{1}{n^n}$ . Then

$$|a_n|^{1/n} = \frac{1}{n}$$

converges to 0. By the root test, this series converges absolutely. We now find a numerical approximation. We have

$$S_4 = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} = \frac{8923}{6912}.$$

The error is

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^n} - S_4 \right| = \sum_{n=5}^{\infty} \frac{1}{n^n}.$$

For  $n \geq 5$ , we have

$$n^n \geq 5^n,$$

and so

$$\frac{1}{n^n} \leq \frac{1}{5^n}.$$

Hence, by taking only the first four terms the error is less than

$$\sum_{n=5}^{\infty} \frac{1}{5^n} = \frac{1}{5^5} \frac{1}{1 - 1/5} = \frac{1}{2500}.$$

We now turn to alternating series.

### Alternating Series Theorem

Assume that  $a_n \geq 0$  for all  $n \geq 1$ ,  $a_n$  is a decreasing sequence, and that  $a_n$  converges to 0. Then the series

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges.

Let  $S_n = \sum_{k=1}^n (-1)^k a_k$  and  $E_n = S_{2n}$ . We have that

$$\begin{aligned} E_{n+1} - E_n &= S_{2n+2} - S_{2n} = (-1)^{2n+2} a_{2n+2} + (-1)^{2n+1} a_{2n+1} \\ &= a_{2n+2} - a_{2n+1} \leq 0, \end{aligned}$$

where the inequality comes from the assumption that  $a_n$  is a decreasing sequence. Since  $E_{n+1} - E_n \leq 0$ ,  $E_n$  is also decreasing. We now show that it is bounded below. Note that

$$E_1 = -a_1 + a_2 \geq -a_1.$$

Similarly,

$$E_2 = -a_1 + (a_2 - a_3) + a_4 \geq -a_1$$

since  $a_2 - a_3 \geq 0$  and  $a_4 \geq 0$ . More generally, taking  $n \geq 3$ , we have that

$$E_n = -a_1 + (a_2 - a_3) + (a_4 - a_5) + \cdots + (a_{2n-2} - a_{2n-1}) + a_{2n} \geq -a_1.$$

That is, the sequence  $E_n$  is decreasing and bounded below by  $-a_1$ ; therefore, it converges to some limit  $\ell$ . Let  $F_n = S_{2n-1}$  for  $n \geq 1$ . Then

$$E_n - F_n = a_{2n} \geq 0.$$

Since  $a_n$  converges to 0, so does  $a_{2n}$  (why?), and by the squeezing principle,  $E_n - F_n$  converges to 0 as well. Therefore,  $F_n$  converges to  $\ell$ . At this point we have that  $S_{2n}$  and  $S_{2n+1}$  converge to the same limit  $\ell$ . We now show that this implies that the whole sequence  $S_n$  converges to  $\ell$ . Take any  $\epsilon > 0$ . There are  $N_1$  and  $N_2$  such that if  $n \geq N_1$ , then

$$|S_{2n} - \ell| < \epsilon,$$

and if  $n \geq N_2$ , then

$$|S_{2n-1} - \ell| < \epsilon.$$

Let  $N = \max(N_1, N_2)$  and take  $n \geq 2N$ . Either  $n$  is even and  $n = 2k$  with  $k \geq N \geq N_1$ , so

$$|S_{2k} - \ell| < \epsilon,$$

or  $n$  is odd and  $n = 2k - 1$  with  $k \geq N \geq N_2$ , so

$$|S_{2k-1} - \ell| < \epsilon.$$

We have shown that for any  $\epsilon > 0$ , there is  $2N$  such that if  $n \geq 2N$ , then

$$|S_n - \ell| < \epsilon.$$

That is, the alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges.

Numerical approximations are easy to compute for alternating series.

### Numerical approximations

Assume that  $a_n \geq 0$  for all  $n \geq 1$ ,  $a_n$  is a decreasing sequence, and that  $a_n$  converges to 0. Then, for any natural  $k$ , we have

$$\sum_{n=1}^{2k-1} (-1)^n a_n \leq \sum_{n=1}^{\infty} (-1)^n a_n \leq \sum_{n=1}^{2k} (-1)^n a_n.$$

That is, if the alternating series theorem applies, then a partial sum gives an approximation whose error is less than the first omitted term.

In order to prove the double inequality above, we use the notation and the results above. In particular, we have shown that  $E_n = S_{2n}$  is decreasing and convergent. Therefore, the limit  $\ell$  of  $E_n$  (which is also the infinite series) is less than  $E_n$  for every  $n \geq 1$ . This proves the upper bound. For the lower bound, consider  $F_n = S_{2n-1}$ . We have

$$F_{n+1} - F_n = -a_{2n+1} + a_{2n} \geq 0$$

since  $a_n$  is decreasing. That is,  $F_n$  is increasing. We also know that  $F_n$  converges to the same  $\ell$ . Therefore,  $\ell$  is larger than  $F_n$  for all  $n \geq 1$ . This provides the lower bound above, and we are done.

*Example 2.27* Consider  $\sum_{n=1}^{\infty} (-1)^n a_n$  with  $a_n = 1/n$ . Note that  $a_n > 0$  for all  $n$ ,  $a_n$  is decreasing, and that it converges to 0. Therefore,  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  converges. Moreover, we have

$$\begin{aligned} S_1 &= -1, & S_2 &= -1 + 1/2 = -1/2, & S_3 &= -1/2 - 1/3 = -5/6, \\ S_4 &= -5/6 + 1/4 = -7/12. \end{aligned}$$

In particular,

$$-5/6 \leq \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \leq -7/12.$$

## Exercises

1. Show that for every real  $x$ , we have

$$0 \leq x + |x| \leq 2|x|.$$

2. Assume that the sequences  $a_n + b_n$  and  $a_n$  converge. Show that the sequence  $b_n$  converges as well.
3. Assume that  $a_n \neq 0$  for all  $n$ .
  - (a) Show that if  $|a_{n+1}| > |a_n|$  for every  $n \geq N$ , then  $|a_n| > |a_N|$  for all  $n > N$ .
  - (b) Show that if the conclusion of (a) holds, then the sequence  $a_n$  does not converge to 0.
4. Does the series  $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}$  converge?
5. (a) Does the series  $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$  converge?  
 (b) Use (a) to find the limit of the sequence  $\frac{n!}{(2n)!}$ .
6. Does the series  $\sum_{n=1}^{\infty} \frac{n!}{2^n}$  converge?
7. Let  $a_n = 1/3^n$  if  $n$  is not a prime and  $a_n = n^2/3^n$  if  $n$  is a prime. Does the series  $\sum_{n=1}^{\infty} a_n$  converge?
8. Under the assumptions of the alternating series theorem, show that

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

both converge.

9. Find a numerical approximation for the series  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ .
10. Assume that  $|a_n|^{1/n}$  converges to some  $\ell$ .
- (a) Show that if  $\ell = 1$ , then the root test is not conclusive (you may use that  $n^{1/n}$  converges to 1).
- (b) Show that if  $\ell > 1$ , then there is  $r > 1$  and a natural  $N$  such that if  $n \geq N$ , then

$$|a_n| \geq r^n.$$

11. (a) Assume that  $a_{3n}, a_{3n+1}, a_{3n+2}$  all converge to the same limit  $\ell$ . Show that the sequence  $a_n$  converges.
- (b) Generalize the result in (a).
12. Assume that  $a_n \geq 0$  for every  $n$  and that the series  $\sum_{n=1}^{\infty} a_n$  converges.
- (a) Show that  $\sum_{n=1}^{\infty} (-1)^n a_n$  converges as well.
- (b) Does (a) hold if the sequence  $a_n$  is not assumed to be positive?
13. Assume that  $|\frac{a_{n+1}}{a_n}|$  converges to some limit  $\ell > 1$ .
- (a) Show that there exist a real  $r > 1$  and a natural  $N$  such that for  $n > N$ , we have that

$$|a_n| > r^{n-N} |a_N|.$$

- (b) Show that  $|a_n|$  goes to infinity as  $n$  goes to infinity. That is, show that for every  $A$ , there is  $M$  such that if  $n \geq M$ , then  $|a_n| > A$ .
14. How many terms do we need to take to have a numerical approximation of  $\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}$  with an error less than  $10^{-10}$ ? Prove your claim.
15. (a) Find an upper bound for

$$\sum_{n=k+1}^{\infty} \frac{1}{n^n}$$

as a function of  $k$ .

- (b) Find a bound on the error when  $\sum_{n=1}^{\infty} \frac{1}{n^n}$  is approximated by  $\sum_{n=1}^{10} \frac{1}{n^n}$ .
16. In the exercises of Sect. 2.3 it was shown that  $a_n \geq 0$  for every  $n$  and that if the series  $\sum_{n=1}^{\infty} a_n$  converges, then  $\sum_{n=1}^{\infty} a_n^2$  converges as well. Does this hold if the sequence is not assumed to be positive?
17. (a) Assume that the series  $\sum_{k=1}^{\infty} |a_k - a_{k-1}|$  converges. Show that the sequence  $a_n$  converges.
- (b) Suppose that for every  $k \geq 1$ , we have

$$|a_k - a_{k-1}| \leq \frac{1}{2^k}.$$

Show that the sequence  $a_n$  converges.



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