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VARIETIES OF MATHEMATICAL EXPERIENCE

The Current Individual and Collective Consciousness

“The whole cultural world, in all its forms, exists through tradition.”

“Tradition is the forgetting of the origins.”

Edmund Husserl, *“The Origin of Geometry”*

THERE IS A LIMITED amount of knowledge, practice, and aspiration which is currently manifested in the thoughts and activities of contemporary mathematicians. The mathematics that is frequently used or is in the process of emerging is part of the current consciousness. This is the material which—to use a metaphor from computer science—is in the high speed memory or storage cells. What is done, created, practiced, at any given moment of time can be viewed in two distinct ways: as part of the larger cultural and intellectual consciousness and milieu, frozen in time, or as part of a changing flow of consciousness.

What was in Archimedes’ head was different from what was in Newton’s head and this, in turn, differed from what was in Gauss’s head. It is not just a matter of “more,” that Gauss knew more mathematics than Newton who, in turn, knew more than Archimedes. It is also a matter of “different.” The current state of knowledge is woven into a network of different motivations and aspirations, different interpretations and potentialities.

Archimedes, Newton, and Gauss all knew that in a triangle the sum of the angles adds to 180° . Archimedes knew this as a phenomenon of nature as well as a conclusion deduced on the basis of the axioms of Euclid. Newton knew the statement as a deduction and as application, but he might also have pondered the question of whether the statement is so true, so bound up with what is right in the universe, that God Almighty could not set it aside. Gauss knew that the statement was sometimes valid and sometimes invalid depending on how one started the game of deduction, and he worried about what other strange contradictions to Euclid could be derived on a similar basis.

Take a more elementary example. Counting and arithmetic can be and have been done in a variety of ways: by stones, by abacuses, by counting beads, by finger reckoning, with pencil and paper, with mechanical adding machines, with hand-held digital computers. Each of these modes leads one to a slightly different perception of, and a different relationship to, the integers. If there is an outcry today against children doing their sums by computer, the criers are correct in asserting that things won't be the same as they were when one struggled with pencil and paper arithmetic and all its nasty carryings and borrowings. They are wrong in thinking that pencil and paper arithmetic is ideal, and that what replaces it is not viable.

To understand the mathematics of an earlier period requires that we penetrate the contemporary individual and collective consciousness. This is a particularly difficult task because the formal and informal mathematical writings that come down to us do not describe the network of consciousness in any detail. It is unlikely that the meaning of mathematics could be reconstructed on the basis of the printed record alone. The sketches that follow are intended to give some insight into the inner feelings that can lie behind mathematical engagement.

The Ideal Mathematician

WE WILL CONSTRUCT a portrait of the “ideal mathematician.” By this we do not mean the perfect mathematician, the mathematician without defect or limitation. Rather, we mean to describe the most mathematician-like mathematician, as one might describe the ideal thoroughbred greyhound, or the ideal thirteenth-century monk. We will try to construct an impossibly pure specimen, in order to exhibit the paradoxical and problematical aspects of the mathematician’s role. In particular, we want to display clearly the discrepancy between the actual work and activity of the mathematician and his own perception of his work and activity.

The ideal mathematician’s work is intelligible only to a small group of specialists, numbering a few dozen or at most a few hundred. This group has existed only for a few decades, and there is every possibility that it may become extinct in another few decades. However, the mathematician regards his work as part of the very structure of the world, containing truths which are valid forever, from the beginning of time, even in the most remote corner of the universe.

He rests his faith on rigorous proof; he believes that the difference between a correct proof and an incorrect one is an unmistakable and decisive difference. He can think of no condemnation more damning than to say of a student, “He doesn’t even know what a proof is.” Yet he is able to give no coherent explanation of what is meant by rigor, or what is required to make a proof rigorous. In his own work, the line between complete and incomplete proof is always somewhat fuzzy, and often controversial.

To talk about the ideal mathematician at all, we must have a name for his “field,” his subject. Let’s call it, for instance, “non-Riemannian hypersquares.”

He is labeled by his field, by how much he publishes, and especially by whose work he uses, and by whose taste he follows in his choice of problems.

He studies objects whose existence is unsuspected by all except a handful of his fellows. Indeed, if one who is not an initiate asks him what he studies, he is incapable of showing or telling what it is. It is necessary to go through an arduous apprenticeship of several years to understand the theory to which he is devoted. Only then would one's mind be prepared to receive his explanation of what he is studying. Short of that, one could be given a "definition," which would be so recondite as to defeat all attempts at comprehension.

The objects which our mathematician studies were unknown before the twentieth century; most likely, they were unknown even thirty years ago. Today they are the chief interest in life for a few dozen (at most, a few hundred) of his comrades. He and his comrades do not doubt, however, that non-Riemannian hypersquares have a real existence as definite and objective as that of the Rock of Gibraltar or Halley's comet. In fact, the proof of the existence of non-Riemannian hypersquares is one of their main achievements, whereas the existence of the Rock of Gibraltar is very probable, but not rigorously proved.

It has never occurred to him to question what the word "exist" means here. One could try to discover its meaning by watching him at work and observing what the word "exist" signifies operationally.

In any case, for him the non-Riemannian hypersquare exists, and he pursues it with passionate devotion. He spends all his days in contemplating it. His life is successful to the extent that he can discover new facts about it.

He finds it difficult to establish meaningful conversation with that large portion of humanity that has never heard of a non-Riemannian hypersquare. This creates grave difficulties for him; there are two colleagues in his department who know something about non-Riemannian hypersquares, but one of them is on sabbatical, and the other is much more interested in non-Eulerian semirings. He goes to conferences, and on summer visits to colleagues, to meet

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people who talk his language, who can appreciate his work and whose recognition, approval, and admiration are the only meaningful rewards he can ever hope for.

At the conferences, the principal topic is usually “the decision problem” (or perhaps “the construction problem” or “the classification problem”) for non-Riemannian hypersquares. This problem was first stated by Professor Nameless, the founder of the theory of non-Riemannian hypersquares. It is important because Professor Nameless stated it and gave a partial solution which, unfortunately, no one but Professor Nameless was ever able to understand. Since Professor Nameless’ day, all the best non-Riemannian hypersquarers have worked on the problem, obtaining many partial results. Thus the problem has acquired great prestige.

Our hero often dreams he has solved it. He has twice convinced himself during waking hours that he had solved it but, both times, a gap in his reasoning was discovered by other non-Riemannian devotees, and the problem remains open. In the meantime, he continues to discover new and interesting facts about the non-Riemannian hypersquares. To his fellow experts, he communicates these results in a casual shorthand. “If you apply a tangential mollifier to the left quasi-martingale, you can get an estimate better than quadratic, so the convergence in the Bergstein theorem turns out to be of the same order as the degree of approximation in the Steinberg theorem.”

This breezy style is not to be found in his published writings. There he piles up formalism on top of formalism. Three pages of definitions are followed by seven lemmas and, finally, a theorem whose hypotheses take half a page to state, while its proof reduces essentially to “Apply Lemmas 1–7 to definitions A–H.”

His writing follows an unbreakable convention: to conceal any sign that the author or the intended reader is a human being. It gives the impression that, from the stated definitions, the desired results follow infallibly by a purely mechanical procedure. In fact, no computing machine has ever been built that could accept his definitions as inputs. To read his proofs, one must be privy to a whole subcul-

ture of motivations, standard arguments and examples, habits of thought and agreed-upon modes of reasoning. The intended readers (all twelve of them) can decode the formal presentation, detect the new idea hidden in lemma 4, ignore the routine and uninteresting calculations of lemmas 1, 2, 3, 5, 6, 7, and see what the author is doing and why he does it. But for the noninitiate, this is a cipher that will never yield its secret. If (heaven forbid) the fraternity of non-Riemannian hypersquarers should ever die out, our hero's writings would become less translatable than those of the Maya.

The difficulties of communication emerged vividly when the ideal mathematician received a visit from a public information officer of the University.

P.I.O. I appreciate your taking time to talk to me. Mathematics was always my worst subject.

I.M.: That's O.K. You've got your job to do.

P.I.O. I was given the assignment of writing a press release about the renewal of your grant. The usual thing would be a one-sentence item, "Professor X received a grant of Y dollars to continue his research on the decision problem for non-Riemannian hypersquares." But I thought it would be a good challenge for me to try and give people a better idea about what your work really involves. First of all, what is a hypersquare?

I.M. I hate to say this, but the truth is, if I told you what it is, you would think I was trying to put you down and make you feel stupid. The definition is really somewhat technical, and it just wouldn't mean anything at all to most people.

P.I.O. Would it be something engineers or physicists would know about?

I.M. No. Well, maybe a few theoretical physicists. Very few.

P.I.O. Even if you can't give me the real definition, can't you give me some idea of the general nature and purpose of your work?

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- I.M.:* All right, I'll try. Consider a smooth function f on a measure space Ω taking its value in a sheaf of germs equipped with a convergence structure of saturated type. In the simplest case . . .
- P.I.O.:* Perhaps I'm asking the wrong questions. Can you tell me something about the applications of your research?
- I.M.:* Applications?
- P.I.O.:* Yes, applications.
- I.M.:* I've been told that some attempts have been made to use non-Riemannian hypersquares as models for elementary particles in nuclear physics. I don't know if any progress was made.
- P.I.O.:* Have there been any major breakthroughs recently in your area? Any exciting new results that people are talking about?
- I.M.:* Sure, there's the Steinberg-Bergstein paper. That's the biggest advance in at least five years.
- P.I.O.:* What did they do?
- I.M.:* I can't tell you.
- P.I.O.:* I see. Do you feel there is adequate support in research in your field?
- I.M.:* Adequate? It's hardly lip service. Some of the best young people in the field are being denied research support. I have no doubt that with extra support we could be making much more rapid progress on the decision problem.
- P.I.O.:* Do you see any way that the work in your area could lead to anything that would be understandable to the ordinary citizen of this country?
- I.M.:* No.
- P.I.O.:* How about engineers or scientists?
- I.M.:* I doubt it very much.
- P.I.O.:* Among pure mathematicians, would the majority be interested in or acquainted with your work?
- I.M.:* No, it would be a small minority.
- P.I.O.:* Is there anything at all that you would like to say about your work?
- I.M.:* Just the usual one sentence will be fine.

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P.I.O.: Don't you want the public to sympathize with your work and support it?

I.M.: Sure, but not if it means debasing myself.

P.I.O.: Debasing yourself?

I.M.: Getting involved in public relations gimmicks, that sort of thing.

P.I.O.: I see. Well, thanks again for your time.

I.M.: That's O.K. You've got a job to do.

Well, a public relations officer. What can one expect? Let's see how our ideal mathematician made out with a student who came to him with a strange question.

Student: Sir, what is a mathematical proof?

I.M.: You don't know *that*? What year are you in?

Student: Third-year graduate.

I.M.: Incredible! A proof is what you've been watching me do at the board three times a week for three years! That's what a proof is.

Student: Sorry, sir, I should have explained. I'm in philosophy, not math. I've never taken your course.

I.M.: Oh! Well, in that case—you have taken *some* math, haven't you? You know the proof of the fundamental theorem of calculus—or the fundamental theorem of algebra?

Student: I've seen arguments in geometry and algebra and calculus that were called proofs. What I'm asking you for isn't *examples* of proof, it's a definition of proof. Otherwise, how can I tell what examples are correct?

I.M.: Well, this whole thing was cleared up by the logician Tarski, I guess, and some others, maybe Russell or Peano. Anyhow, what you do is, you write down the axioms of your theory in a formal language with a given list of symbols or alphabet. Then you write down the hypothesis of your theorem in the same symbolism. Then you show that you can transform the hypothesis step by step, using the rules of logic, till you get the conclusion. That's a proof.

Student: Really? That's amazing! I've taken elementary



Alfred Tarski
1902–1983

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- and advanced calculus, basic algebra, and topology, and I've never seen that done.
- I.M.:* Oh, of course no one ever really *does* it. It would take forever! You just show that you could do it, that's sufficient.
- Student:* But even that doesn't sound like what was done in my courses and textbooks. So mathematicians don't really do proofs, after all.
- I.M.:* Of course we do! If a theorem isn't proved, it's nothing.
- Student:* Then what is a proof? If it's this thing with a formal language and transforming formulas, nobody ever proves anything. Do you have to know all about formal languages and formal logic before you can do a mathematical proof?
- I.M.:* Of course not! The less you know, the better. That stuff is all abstract nonsense anyway.
- Student:* Then really what *is* a proof?
- I.M.:* Well, it's an argument that convinces someone who knows the subject.
- Student:* Someone who knows the subject? Then the definition of proof is subjective; it depends on particular persons. Before I can decide if something is a proof, I have to decide who the experts are. What does that have to do with proving things?
- I.M.:* No, no. There's nothing subjective about it! Everybody knows what a proof is. Just read some books, take courses from a competent mathematician, and you'll catch on.
- Student:* Are you sure?
- I.M.:* Well—it is possible that you won't, if you don't have any aptitude for it. That can happen, too.
- Student:* Then *you* decide what a proof is, and if I don't learn to decide in the same way, you decide I don't have any aptitude.
- I.M.:* If not me, then who?

Then the ideal mathematician met a positivist philosopher.

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- P.P.:* This Platonism of yours is rather incredible. The silliest undergraduate knows enough not to multiply entities, and here you've got not just a handful, you've got them in uncountable infinities! And nobody knows about them but you and your pals! Who do you think you're kidding?
- I.M.:* I'm not interested in philosophy, I'm a mathematician.
- P.P.:* You're as bad as that character in Molière who didn't know he was talking prose! You've been committing philosophical nonsense with your "rigorous proofs of existence." Don't you know that what exists has to be observed, or at least observable?
- I.M.:* Look, I don't have time to get into philosophical controversies. Frankly, I doubt that you people know what you're talking about; otherwise you could state it in a precise form so that I could understand it and check your argument. As far as my being a Platonist, that's just a handy figure of speech. I never thought hypersquares existed. When I say they do, all I mean is that the axioms for a hypersquare possess a model. In other words, no formal contradiction can be deduced from them, and so, in the normal mathematical fashion, we are free to postulate their existence. The whole thing doesn't really mean anything, it's just a game, like chess, that we play with axioms and rules of inference.
- P.P.:* Well, I didn't mean to be too hard on you. I'm sure it helps you in your research to imagine you're talking about something real.
- I.M.:* I'm not a philosopher, philosophy bores me. You argue, argue and never get anywhere. My job is to prove theorems, not to worry about what they mean.

The ideal mathematician feels prepared, if the occasion should arise, to meet an extragalactic intelligence. His first effort to communicate would be to write down (or other-

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wise transmit) the first few hundred digits in the binary expansion of π . He regards it as obvious that any intelligence capable of intergalactic communication would be mathematical and that it makes sense to talk about mathematical intelligence apart from the thoughts and actions of human beings. Moreover, he regards it as obvious that binary representation and the real number π are both part of the intrinsic order of the universe.

He will admit that neither of them is a natural object, but he will insist that they are discovered, not invented. Their discovery, in something like the form in which we know them, is inevitable if one rises far enough above the primordial slime to communicate with other galaxies (or even with other solar systems).

The following dialogue once took place between the ideal mathematician and a skeptical classicist.

S.C.: You believe in your numbers and curves just as Christian missionaries believed in their crucifixes. If a missionary had gone to the moon in 1500, he would have been waving his crucifix to show the moon-men that he was a Christian, and expecting them to have their own symbol to wave back.* You're even more arrogant about your expansion of π .

I.M.: Arrogant? It's been checked and rechecked, to 100,000 places!

S.C.: I've seen how little you have to say even to an American mathematician who doesn't know your game with hypersquares. You don't get to first base trying to communicate with a theoretical physicist; you can't read his papers any more than he can

* Cf. the description of Coronado's expedition to Cibola, in 1540: "... there were about eighty horsemen in the vanguard besides twenty-five or thirty foot and a large number of Indian allies. In the party went all the priests, since none of them wished to remain behind with the army. It was their part to deal with the friendly Indians whom they might encounter, and they especially were bearers of the Cross, a symbol which . . . had already come to exert an influence over the natives on the way" (H. E. Bolton, *Coronado*, University of New Mexico Press, 1949).

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read yours. The research papers in your own field written before 1910 are as dead to you as Tutankhamen's will. What reason in the world is there to think that you could communicate with an extragalactic intelligence?

I.M.: If not me, then who else?

S.C.: Anybody else! Wouldn't life and death, love and hate, joy and despair be messages more likely to be universal than a dry pedantic formula that nobody but you and a few hundred of your type will know from a hen-scratch in a farmyard?

I.M.: The reason that my formulas are appropriate for intergalactic communication is the same reason they are not very suitable for terrestrial communication. Their content is not earthbound. It is free of the specifically human.

S.C.: I don't suppose the missionary would have said quite that about his crucifix, but probably something rather close, and certainly no less absurd and pretentious.

The foregoing sketches are not meant to be malicious; indeed, they would apply to the present authors. But it is a too obvious and therefore easily forgotten fact that mathematical work, which, no doubt as a result of long familiarity, the mathematician takes for granted, is a mysterious, almost inexplicable phenomenon from the point of view of outsider. In this case, the outsider could be a layman, a fellow academic, or even a scientist who uses mathematics in his own work.

The mathematician usually assumes that his own view of himself is the only one that need be considered. Would we allow the same claim to any other esoteric fraternity? Or would a dispassionate description of its activities by an observant, informed outsider be more reliable than that of a participant who may be incapable of noticing, not to say questioning, the beliefs of his coterie?

Mathematicians know that they are studying an objective reality. To an outsider, they seem to be engaged in an esoteric communion with themselves and a small clique of

friends. How could we as mathematicians prove to a skeptical outsider that our theorems have meaning in the world outside our own fraternity?

If such a person accepts our discipline, and goes through two or three years of graduate study in mathematics, he absorbs our way of thinking, and is no longer the critical outsider he once was. In the same way, a critic of Scientology who underwent several years of “study” under “recognized authorities” in Scientology might well emerge a believer instead of a critic.

If the student is unable to absorb our way of thinking, we flunk him out, of course. If he gets through our obstacle course and then decides that our arguments are unclear or incorrect, we dismiss him as a crank, crackpot, or misfit.

Of course, none of this proves that we are not correct in our self-perception that we have a reliable method for discovering objective truths. But we must pause to realize that, outside our coterie, much of what we do is incomprehensible. There is no way we could convince a self-confident skeptic that the things we are talking about make sense, let alone “exist.”

A Physicist Looks at Mathematics

HOW DO PHYSICISTS view mathematics? Instead of answering this question by summarizing the writings of many physicists, we interviewed one physicist whose scientific feelings were judged to be representative. Since the summary which follows cannot represent his full and precise views, his name has been changed.

Professor William F. Taylor is an international authority in Engineering Science. He is actively engaged in teaching and research, and maintains extensive scientific connec-

tions. In August, 1977, the writer interviewed Professor Taylor in Wilmington, Vermont where he and his wife were on vacation enjoying tennis and the Marlboro Concerts. In the interview, an attempt was made not to confront the interviewee with opposing views and not to engage in argumentation.

Professor Taylor says that his professional field lies at the intersection of physics, chemistry, and materials science. He does not care to describe this combination by a single word. Although he uses mathematics extensively, he says he is definitely not an applied mathematician. He thinks, though, that many of his views would be held by applied mathematicians.

Taylor makes frequent computations. When asked whether he thought of himself as a creator or a consumer of mathematics, he answered that he was a consumer. He added that most of the mathematics he uses is of a nineteenth century variety. With respect to contemporary mathematical research he says that he feels drawn to it intellectually. It appears to unify a wide variety of complex structures. He is not, however, sufficiently motivated to learn any of it because he feels it has little applicability to his work. He thinks that much of the recently developed mathematics has gone beyond what is useful.

He seemed to be aware of the broad outline of the newly developed "nonstandard" analysis. He said,

That subject looks very interesting to me, and I wish I could take out the time to master it. There are numerous places in my field where one is confronted with things that are going on simultaneously at totally different size scales. They are very difficult to deal with by conventional methods. Perhaps nonstandard analysis with its infinitesimals might provide a handle for this sort of thing.

Taylor asserts that only seldom in his professional work does he think along philosophic lines. He has done a small amount of reading in the philosophy of science and the philosophy of physics, principally in the area of quantum physics. He finds questions as to how and to what extent processes are affected by the mode of observation particu-

larly interesting. He says that such questions have affected his professional work and outlook somewhat although he has not written anything of a formal nature about it.

Although his personal familiarity with the philosophy of science may be said to be slight, he believes it to be an important line of inquiry, and he welcomed the present interview and framed his answers thoughtfully and with gusto.

Taylor is unaware of the main classical issues of mathematical philosophy. In response to the question of whether there were or had been any crises in mathematics, he answered that he had heard of Russell's Paradox, but it seemed to be quite remote from anything he was interested in. "It was nothing I should worry about," he said.

Taylor's approach to science, to mathematics, and to a variety of related philosophic issues can be summed up by saying that he is a strong and eloquent spokesman for the model theory or approach. This holds that physical theories are provisional models of reality. He uses the word "model" frequently and brings around his arguments to this approach. Mathematics itself is a model, he says. Questions as to the truth or the indubitability of mathematics are not important to him because all scientific work of every kind is of a provisional nature. The question should be not how true it is but how good it is. In the interview, he elaborated at length on what he meant by "good" and this was done from the vantage point of models.

As part of his elaboration, he answered along the following lines. There are many situations in physics that are very messy. They may contain too many mutually interacting phenomena of equal degrees of importance. In such a situation there is no hope whatever of setting up something which can be asserted to be the "real thing." The best one can hope for is a model which is a partial truth. It is a tentative thing and one hopes the best for it. All physical theories are models. A model should be able at the very least to describe certain phenomena fairly accurately. Even at this level one runs into trouble in constructing models. The models that one constructs are of course dependent upon one's state of knowledge. Ideally, a model should have predictive value. Therefore it is no good to construct

a model which is too complex to support reason. Whether it is or it is not too complex may depend upon the current state of the mathematical or computational art. But one has to be in a position to derive mathematical and hence physical consequences from the model, and if this is found to be impossible—and it may be so for a variety of reasons—then the model has little significance.

Professor Taylor was asked to comment on the contemporary view that the scientific method can be summed up by the sequence: induction, deduction, verification, iterated as often as necessary. He replied that he went along with it in its broad outlines. But he wanted to elaborate.

Induction is related to my awareness of the observations of others and of existing theories. Deduction is related to the construction of a model and of physical conclusions drawn from it by means of mathematical derivations. Verification is related to predictions of phenomena not yet observed and to the hope that the experimentalist will look for new phenomena.

The experimentalist and the theoretician need one another. The experimentalist needs a model to help him lay out his experiments. Otherwise he doesn't know where to look. He would be working in the dark. The theoretician needs the experimentalist to tell him what is going on in the real world. Otherwise his theorizing would be empty. There must be adequate communication between the two and, in fact, I think there is.

When asked why the profession splits into two types—experimentalists and theoreticians—he said that apart from a general tendency to specialize, it was probably a matter of temperament. “But the gap is always bridged—usually by the theoretician.”

Professor Taylor was asked how he felt about the often quoted remark of a certain theoretician that he would rather his theories be beautiful than be right.

This cuts close to the bone. It really does. But as I see it, mere aesthetics doesn't pay dividends. In my experience, I should be inclined to replace the word “beautiful” by the word “analyzable.” I should like my models to be beautiful,

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effective, and predictive. But the real goal is the understanding of a situation. Therefore the models must be analyzable because understanding can come only through analyzability. If one has all of these things, then this is a great and rare achievement, but I should say that my immediate goal is analyzability.

What were his views on mathematical proof? Professor Taylor said that his papers rarely contain formal proofs of a sort that would satisfy a mathematician. To him, proofs were relatively uninteresting and they were largely unnecessary in his personal work. Yet, he felt that his work contained elements that could be described as mathematical reasoning or deduction. Truth in mathematics, he said, is reasoning that leads to correct physical relationships. Empirical demonstrations are possible. True reasoning should be capable of being put into the format of a mathematical proof. It is nice to have this done ultimately. Proof is for cosmetic purposes and also to reduce somewhat the edge of insecurity on which one always lives. However, for him to engage in mathematical proof would seriously take him away from his main interests and methodology.

In view of Professor Taylor's familiarity with computational procedures, he was asked to comment on the current opinion that the object of numerical science or numerical physics is to replace experimentation. He thought a while and then replied,

I think one has to distinguish here between the requirements of technology and those of pure science. To the former, I would reply a limited "yes"; to the latter "no." Consider a problem in technology. One has a pressure vessel which is subject to many many cycles of heating and cooling. How many cycles can it stand? Now, if one really knew the process that leads to failure (which is not yet the case) one could say that in a specific instance it might be much more effective to make a computer experiment than an actual experiment. Here one is dealing with something like a "production" situation.

On the other hand, in pure science, the elimination of experimentation is a contradiction in terms. The way one finds out what is going on in the universe is through ex-

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perimentation. This is where new experiences, new facts come from.

There is no point to run experiments on bodies falling in a vacuum. Newtonian mechanics is known to be an adequate model. But if one goes, say, to cosmology, where it isn't known whether existing models are adequate or are not adequate, then numerical computation is insufficient.

Asked whether it would be possible to imagine a kind of theoretical physics without mathematics, Professor Taylor answered that it would not be possible.

Asked the same question for technology, he answered again that it would not be possible.

He added that the mathematics of technology was perhaps more elementary and more completely studied than that of modern physics, but it was mathematics, nonetheless. The role of mathematics in physics or in technology is that of a powerful reasoning tool in complex situations.

He was then asked why mathematics was so effective in physics and technology. The interviewer underlined that the word "effective" was one used by Professor Eugene Wigner in a famous article, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences." "This has to do," he answered,

with our current convention or system of beliefs as to what constitutes understanding. In these fields we mean by 'understanding' precisely those things which are explainable or predictable by mathematics. You may think this is going around in circles, and so it may be. The question of course is fundamentally unanswerable, and this is the way I care to frame my answer. Understanding means understanding through mathematics.

"Do you rule out other types of understanding?"

There is what might be called humanistic or cross-cultural understanding. I have been reading Jacques Barzun and Theodore Roszak recently. What is the great concern with numbers and decimal points, they seem to be asking. One sees it in the old poem of Walt Whitman called "The Astronomer." Whitman had heard a lecture in astronomy in Cooper Union Hall. After the lecture he went outside,

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looked up at the heavens, and felt a certain release at being freed from theories and symbols. He felt the exhilaration of being confronted by naked experience, if you will.

Now this may be a valid point of view, but it leads to a different end result. Quantitative science—that is, science with mathematics—has proved effective in altering and controlling nature. The majority of society backs it up for this reason. At the present moment, they want nature altered and controlled—to the extent, of course that we can do it and the results are felicitous. The humanist point of view is a minority point of view. But it is influential—one sees this among young people. It seems to have a defensive nature to it, a chip on its shoulder, but because it is a minority point of view, it poses only a minor threat to quantitative science.

“With regard to the conflict of the ‘Two Worlds,’ which of the two, the scientist and the humanist, knows more about the other man’s business?”

The scientist very definitely knows more about the humanities than the other way around. The scientist—well, many that I know anyway—are forever reading novels, essays, criticism, etc., go to concerts, theatres, to art shows. The humanists very seldom read anything about science other than what they find in the newspaper. Part of the reason for this lies in the fact that the locus of the humanities is to be found in sound, vision, and common language. The language of science with its substantial sublanguage of mathematics poses a formidable barrier to the humanist.

The goals of society may change, of course. If they do, then the goals of quantitative science may be weakened. Science and mathematics might be pursued only by a small but interested minority. It might not be possible to make a living at it. We saw a very slight indication of this in the late sixties and early seventies.

“Can there be knowledge without words, without symbols?”

Knowledge, as I understand it in the technical sense, implies that it can be expressed in symbols. Moving towards humanistic questions, one might say that a skillful writer evokes a mood by his use of words. Or when a Mozart score

is played, it evokes a kind of conscious state. The symbolic words and the music are a model for the state.

“Does a cat have knowledge?”

“A cat knows certain things. But this is knowledge of a different kind. We are not dealing here with theoretical knowledge.”

“When a flower brings forth a blossom with six-fold symmetry, is it doing mathematics?”

“It is not.”

“Would you care to comment on the old Greek saying that God is a Mathematician?”

“This conveys nothing to me. It is not a useful concept.”

“What is scientific or mathematical intuition?”

“Intuition is an expression of experience. Stored experience. There is an inequality in people with respect to it. Some people gain intuition more rapidly than others.”

“To what extent can one be deceived by intuition?”

“This occurs not infrequently. It is a large part of my own work. I say to myself, this model seems to be sufficient, but it just doesn’t sit right. Or, I ask myself, is my model a better one than their model? And I probably have to answer on the basis of intuition.”

The final question put to Taylor was whether he is a mathematical Platonist in the sense that he believes that mathematical concepts exist in the world apart from the people that do mathematics. He replied that he was, but in a limited sense. Certainly not in a “theological” sense. He believed that certain concepts turn out to be so far superior to others that it is only a matter of time before these concepts prevail and are universally adopted. This is something like a Darwinian process, a survival of the fittest ideas, models, constructs. The evolution of mathematics and theoretical physics is something like the evolution of biosystems.

I. R. Shafarevitch and the New Neoplatonism

ONE OF THE WORLD'S leading researchers in algebraic geometry is also a leading advocate of Russian nationalism, orthodox Christianity, and frank anti-Semitism. I. R. Shafarevitch discussed his views on the relation between mathematics and religion in a lecture he gave on the occasion of his receiving a prize from the Academy of Science at Göttingen, Germany. We quote from his lecture.

Shafarevitch discussed his views on the relation between mathematics and religion in a lecture he gave on the occasion of his receiving a prize from the Academy of Science at Göttingen, West Germany. We quote from his lecture.

“A superficial glance at mathematics may give an impression that it is a result of separate individual efforts of many scientists scattered about in continents and in ages. However, the inner logic of its development reminds one much more of the work of a single intellect, developing its thought systematically and consistently using the variety of human individualities only as a means. It resembles an orchestra performing a symphony composed by someone. A theme passes from one instrument to another, and when one of the participants is bound to drop his part, it is taken up by another and performed with irreproachable precision.

“This is by no means a figure of speech. The history of mathematics has known many cases when a discovery made by one scientist remains unknown until it is later reproduced by another with striking precision. In the letter written on the eve of his fatal duel, Galois made several assertions of paramount importance concerning integrals of algebraic functions. More than twenty years later Riemann, who undoubtedly knew nothing about the letter of Galois, found anew and proved exactly the same assertions. An-

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other example: after Lobachevski and Bolyai laid the foundation of non-Euclidean geometry independently of one another, it became known that two other men, Gauss and Schweikart, also working independently, had both come to the same results ten years before. One is overwhelmed by a curious feeling when one sees the same designs as if drawn by a single hand in the work done by four scientists quite independently of one another.

“One is struck by the idea that such a wonderfully puzzling and mysterious activity of mankind, an activity that has continued for thousands of years, cannot be a mere chance—it must have some goal. Having recognized this we inevitably are faced by the question: *What is this goal?*”

“Any activity devoid of a goal, by this very fact loses its sense. If we compare mathematics to a living organism, mathematics does not resemble conscious and purposeful activity. It is more like instinctive actions which are repeated stereotypically, directed by an external or internal stimulus.

“Without a definite goal, mathematics cannot develop any idea of its own form. The only thing left to it, as an ideal, is uncontrolled growth, or more precisely, expansion in all directions. Using another simile, one can say that the development of mathematics is different from the growth of a living organism which preserves its form and defines its own border as it grows. This development is much more akin to the growth of crystals or the diffusion of gas which will expand freely until it meets some outside obstacle.

“More than two thousand years of history have convinced us that mathematics cannot formulate for itself this final goal that can direct its progress. Hence it must take it from outside. It goes without saying that I am far from attempting to point out a solution of this problem, which is not only the inner problem of mathematics but the problem of mankind at large. I want only to indicate the main directions of the search for this solution.

Apparently there are two possible directions. In the first place one may try to extract the goal of mathematics from its practical applications. But it is hard to believe that a superior (spiritual) activity will find its justification in the in-



I. R. Shafarevitch

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ferior (material) activity. In the “Gospel according to Thomas” discovered in 1945,* Jesus says ironically:

If the flesh came for the sake of the spirit, it
is a miracle. But if the spirit for the sake
of the flesh—it is a miracle of miracles.

All the history of mathematics is a convincing proof that such a “miracle of miracles” is impossible. If we look upon the decisive moment in the development of mathematics, the moment when it took its first step and when the ground on which it is based came into being—I have in mind *logical proof*—we shall see that this was done with material that actually excluded the very possibility of practical applications. The first theorems of Thales of Miletus proved statements evident to every sensible man—for instance that a diameter divides the circle into two equal parts. Genius was needed not to be convinced of the justice of these statements, but to understand that they need proofs. Obviously the practical value of such discoveries is nought.

In ending, I want to express a hope that . . . mathematics may serve now as a model for the solution of the main problem of our epoch: to reveal a supreme religious goal and to fathom the meaning of the spiritual activity of mankind.”

Thus, Shafarevitch—a surprising statement to come from the lips of any contemporary mathematician in or out of Russia. But it is hardly a new statement. The Greek philosophers thought of mathematics as a bridge between theology and the perceptible, physical world, and this view was stressed and developed by the Neoplatonists. The quadrivium: arithmetic, music, geometry, astronomy, already known to Protagoras (d. 411 B.C.), was thought to

* (Footnote added by P.J.D.) The Gospel of Thomas is probably the most significant of the books discovered in the 1940s at Nag Hammadi in Egypt. It is a compilation of the “sayings of Jesus,” placed in a Gnostic context. Gnosticism asserts that there is a secret knowledge (gnosis) through which salvation can be achieved and that this knowledge is superior to ordinary faith. (See R. M. Grant, “Gnosticism, Marcion, Origen” in “The Crucible of Christianity,” A. Toynbee, ed., London: Thames and Hudson, 1969.)

lead the mind upward through mathematics to the heavenly sphere where the eternal movements were the perceptible form of the world soul.

Further Readings. See Bibliography

P. Merlan; I. R. Shafarevitch

Unorthodoxies

MOST MATHEMATICIANS have had the following experience and those whose activities are somewhat more public have had it often: an unsolicited letter arrives from an unknown individual and contained in the letter is a piece of mathematics of a very sensational nature. The writer claims that he has solved one of the great unsolved mathematical problems or that he has refuted one of the standard mathematical assertions. In times gone by, circle squaring was a favorite activity; in fact, this activity is so old that Aristophanes parodies the circle squarers of the world. In more recent times, proofs of Fermat's "Last Theorem" have been very popular. The writer of such a letter is usually an amateur, with very little training in mathematics. Very often he has a poor understanding of the nature of the problem he is dealing with, and an imperfect notion of just what a mathematical proof is and how it operates. The writer is usually male, frequently a retired person with leisure to pursue his mathematics, often he has achieved considerable professional status in the larger community and he exhibits his status symbols within the mathematical work itself.

Very often the correspondent not only "succeeds" in solving one of the great mathematical unsolvables, but has also found a way to construct an antigravity shield, to interpret the mysteries of the Great Pyramid and of Stone-

henge, and is well on his way to producing the Philosophers' Stone. This is no exaggeration.

If the recipient of such a letter answers it, he will generally find himself entangled with a person with whom he cannot communicate scientifically and who exhibits many symptoms of paranoia. One gets to recognize such correspondents on sight, and to leave their letters unanswered, thus unfortunately increasing the paranoia.

I have on my desk as I write a paper of just this sort which was passed on to me by the editor of one of the leading mathematical journals in the United States. For self-protection I shall change the personal details, retaining the flavor as best I can. The paper is nicely and expensively printed on glossy stock and comes from the Philippines. It is written in Spanish and purports to be a demonstration of Fermat's Last Theorem. There is a photograph of the author, a fine-looking gentleman in his eighties, who had been a general in the Philippine army. Along with the mathematics there is a lengthy autobiography of the author. It would appear that the author's ancestors were French aristocrats, that after the French Revolution the cadet branch was sent to the East, whence the family made its way to the Philippines, etc. There are also included in this paper on Fermat's Last Theorem, nice engravings of the last three reigning Louis of France and a long plea for the restoration of the Bourbon dynasty. After page one, the mathematics rapidly wanders into incomprehensibility. I spent ten minutes with this paper; your average editor would spend less. Why? The Fermat "Last Theorem" is at the time of this writing a great unsolved problem. Perhaps the man from the Philippines has solved it. Why did I not examine his work carefully?

There are many types of anomalous or idiosyncratic writing in mathematics. How does the community strain out what it wants? How does one recognize brilliance, genius, crankiness, madness? Anyone can make an honest error. Shortly after World War II, Professor Hans Rademacher of the University of Pennsylvania, one of the leading number theoreticians in the world, thought he had proved the famous Riemann Hypothesis. (See page 405 for

a statement of this conjecture.) The media got wind of this news and an account was published in *Time* magazine. It is not often that a mathematical discovery makes the popular press. But shortly thereafter, an error was found in Rademacher's work. The problem is still open as these words are being written.

This is an example of incorrect mathematics produced within the bounds of mathematical orthodoxy—and detected there as well. This happens to the best of us every day of the week. When the error is pointed out, one recognizes it as an error and acknowledges it. This kind of situation is dealt with routinely.

At the opposite pole, there is the type whose psychopathology has just been described above. This type of writing is usually dismissed at sight. The probability that it contains something of interest is extremely small and it is a risk that the mathematical community is willing to take. But it is not always easy to draw the line between the crank and the genius.

An obscure and poor young man from a little-known place in India writes a letter around 1913 to G. H. Hardy, the leading English mathematician of the day. The letter betrays signs of inadequate training, it is intuitive and disorganized, but Hardy recognizes in it brilliant pearls of mathematics. The Indian's name was Srinivasa Ramanujan. If Hardy had not arranged for a fellowship for Ramanujan, some very interesting mathematics might have been lost forever.

Then there was the case of Hermann Grassman (1809–1877). In 1844 Grassman published a book called *Die lineale Ausdehnungslehre*. This work is today recognized as a work of genius. It was an anticipation of what would be subsequently worked out as vector and tensor analysis and associative algebras (quaternions). But because Grassman's exposition was obscure, mystical, and unusually abstract for its period, this work repelled the mathematical community and was ignored for many years.

Less known than either Grassman or Ramanujan is the story of Józef Maria Wronski (1776–1853), whose personality and work combined elements from pretentious na-



Srinivasa Ramanujan
1887–1920

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iveté to genius near madness. Today Wronski is chiefly remembered for a certain determinant $W[u_1, u_2, \dots, u_n] =$

$$\begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u'_1 & u'_2 & \cdots & u'_n \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix} \quad \begin{array}{l} \text{formed from } n \text{ functions} \\ u_1, \dots, u_n. \end{array}$$

This determinant is related to theories of linear independence and is of importance in the theory of linear differential equations. Every student of differential equations has heard of the Wronskian.

Wronski was a Pole who fought with Kosciuszko for Polish independence, yet, dedicated his book “Introduction à la Philosophie des Mathématiques et Technie de l’Algorithmie” to His Majesty, Alexander I, Autocrat of all the Russias. A political realist, one would think.

On the 15th of August 1803, Wronski experienced a revelation which enabled him to conceive of “the absolute.” His subsequent mathematical and philosophical work was motivated by a drive to expound the absolute and its laws of unification. In addition to his mathematics and philosophy, Wronski pursued theosophy, political and cultural messianism (he wrote five books on this topic), promoted the ideas of arithmosophism, mathematical vitalism, and something which he called “séchelianisme” (from the Hebrew; sechel: reason). This latter purported to change Christianity from a revealed religion to a proved religion. Wronski distinguished three forces which control history: providence, fatality (destiny), and reason. He constructed almost all of his system around the negation of the principle of inertia. Inasmuch as the material has no inertia it does not compete with the spiritual. The scientific ideal would be a kind of panmathematism which unites the knowledge of the formation of mathematical systems with the laws of living beings.

Wronski’s philosophy is, apparently, not uninteresting and ties in with the later writings of Bergson.

What do we find, mathematically speaking, when we open up the first volume of his *Oeuvres Mathématiques*?

It appears, at a quick glance, to be mixture of the theory of infinite series, difference equations, differential equations, and complex variables. It is long, rambling, polemical, tedious, obscure, egocentric, and full of philosophical interpolations giving unifying schemata. The “Grand Law of the Generation of Quantities,” which contains the Key to the Universe, appears as equation (7). Wronski sold it to a wealthy banker. The banker did not pay up and Wronski aired his complaints publicly. Here is the Grand Law:

$$Fx = A_0\Omega_0 + A_1\Omega_1 + A_2\Omega_2 + A_3\Omega_3 + A_4\Omega_4 + \text{etc. à l'infini.}''$$

What does it mean? It appears to be a general scheme for the expansion of functions as linear combinations of other functions; a kind of generalized Taylor expansion which contains all expansions of the past and all future expansions.

It is not possible for me to grasp the essential spirit of Wronski's work; and it would take a profound student of eighteenth century mathematics to tell what, if anything, is new or useful in the four volumes. I am only too willing to accept the judgment of history that Wronski deserves to be remembered only for the Wronskian. The doors of the mathematical past are often rusted. If an inner chamber is difficult of access, it does not necessarily mean that treasure is to be found therein.

There is work, then, which is wrong, is acknowledged to be wrong and which, at some later date may be set to rights. There is work which is dismissed without examination. There is work which is so obscure that it is difficult to interpret and is perforce ignored. Some of it may emerge later. There is work which may be of great importance—such as Cantor's set theory—which is heterodox, and as a result, is ignored or boycotted. There is also work, perhaps the bulk of the mathematical output, which is admittedly correct, but which in the long run is ignored, for lack of interest, or because the main streams of mathematics did not choose to pass that way. In the final analysis, there can be no formalization of what is right and how we know it



Wronski's key to the universe; placed in a cartouche, sanctified by the zodiac, guarded by a sphinx, and printed on all his works.

right, what is accepted, and what the mechanism for acceptance is. As Hermann Weyl has written, "Mathematizing may well be a creative activity of man . . . whose historical decisions defy complete objective rationalization."

Further Readings. See Bibliography

J. M. Wronski

The Individual and the Culture

THE RELATIONSHIP BETWEEN the individual and society has never been of greater concern than it is today. The opposing tendencies of amalgamation versus fragmentation, of nationalism versus regionalism, of the freedom of the individual as opposed to the security within a larger group are acting out a drama on history's stage which may settle a direction for civilization for the next several centuries. Running perpendicularly to these struggles is the conflict between the "Two Cultures": the humanistic and the technological.

Mathematics, being a human activity, possesses all four components. It profits greatly from individual genius, but thrives only with the tacit approval of the wider community. As a great art form, it is humanistic; it is scientific-technological in its applications.

To understand just where and how mathematics fits into the human condition, it is important that we pay heed to all four of these components.

There are two extreme positions on the history of discovery. The first position holds that individual genius is the wellspring of discovery. The second position is that social and economic forces bring forth discovery. Most people do not hold with the one or with the other in a pure form, but try to find a mixture which is compatible with their own experiences.

The doctrine of the individual is the more familiar of the two, the easier of the two, and we are rather more comfortable with it. As teachers, we try our best to concentrate on the individual student; we do not attempt to teach people in their multitude. Methods of teaching en masse, through media of some sort, all postulate an individual at the receiving end. On the contrary, the word “indoctrination,” which implies a kind of group phenomenon, worries us.

We study mathematical didactics and strategies of discovery as in Pólya’s books (See Chapter 6) and try to transfer some of the insights of a great mathematician to our students. We read biographies of great geniuses and study their works carefully.

One of the most striking statements of the doctrine of the individual in mathematics was put forward in an article by Alfred Adler. The author is a professional mathematician and his article is as eloquent as it is dramatic. The article is also a very personal statement; its views are romanticized, manic-depressive, and apocalyptic.

Adler begins by putting the case for an extreme form of elitism:

Each generation has its few great mathematicians, and mathematics would not even notice the absence of the others. They are useful as teachers, and their research harms no one, but it is of no importance at all. A mathematician is great or he is nothing.

This is accompanied by the statement of “The Happy Few.”

But there is never any doubt about who is and who is not a creative mathematician, so all that is required is to keep track of the activities of these few men.

“The Few”—or at least five of them—are then identified (as of 1972).

It is noted that the creation of mathematics appears to be a young man’s business:

The mathematical life of a mathematician is short. Work rarely improves after the age of twenty-five or thirty. If little has been accomplished by then, little will ever be ac-

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completed. If greatness has been attained, good work may continue to appear, but the level of accomplishment will fall with each decade.

Adler records the intense joy of the artist:

A new mathematical result, entirely new, never before conjectured or understood by anyone, nursed from the first tentative hypothesis through labyrinths of false attempted proofs, wrong approaches, unpromising directions, and months or years of difficult and delicate work—there is nothing, or almost nothing, in the world that can bring a joy and a sense of power and tranquillity to equal those of its creator. And a great new mathematical edifice is a triumph that whispers of immortality.

He winds up with a mathematical *Götterdämmerung*:

There is a constant awareness of time, of the certainty that mathematical creativity ends early in life, so that important work must begin early and proceed quickly if it is to be completed. There is the focus on problems of great difficulty, because the discipline is unforgiving in its contempt for the solution of easy problems and in its indifference to the solution of almost any problems but the most profound and difficult ones.

What is more, mathematics generates a momentum, so that any significant result points automatically to another new result, or perhaps to two or three new results. And so it goes—goes, until the momentum all at once dissipates. Then the mathematical career is, essentially, over; the frustrations remain, but the satisfactions have vanished.

And so we leave our ageing hero as he knocks tentatively on the gates of a Valhalla which itself may be illusory.

Lest any reader be deterred from a mathematical career by this dismal picture, we must report that there are many instances of mathematicians continuing to do first-class research past the age of fifty; for example, Paul Lévy, one of the creators of modern probability theory was close to forty when he wrote his first paper in this area; he continued doing profound, original work into his sixties.

When we speak of the culture as being the main source

of discovery, we are on grounds that are far more tenuous, far less well understood. This is the doctrine of "The Many." This is Hegel's *Zeitgeist*, the spirit of the age: the ideas, the attitudes, the conceptions, the needs, the modes of self-expression that are common to a time and to a place. These are the things that are "in the air." Read Tolstoy's retrospective final chapter of *War and Peace* and see how he comes to the conclusion that the trends initiated in Europe by the French Revolution would have worked themselves through with or without Napoleon. There is a tendency on the part of theoretical Marxists to favor the doctrine of the culture. So, for example, one might read how the British scientist and Marxist J. D. Bernal works it out in the area of the natural sciences.

We know in our bones that culture makes a difference. We know that there are cultures in which symphonic music has flourished and those in which it has not. But the explication by culture does not come easily. The record of a single man is easier to read than the traces of a whole civilization. Why did the small country of Hungary in the years since 1900 produce such a large number of first-rate mathematicians? Why have governments since 1940 supported mathematical research while prior to that date they did not? Why did the Early Christians find Christ and Euclid incompatible, while a thousand years later, Newton was able to embrace them both?

For contemporary history, where the facts are available or fresh in mind and where the principal actors might yet be alive, it would be possible to write easily and convincingly of the cultural reasons for this or that. So, for example, it might be possible—and very worthwhile—to spell out the extramathematical, extratechnological reasons which have led in one short generation to the development of the electronic computing machine. (See the book of H. Goldstine.) It would be rather harder to explain the rise of function algebras along the same lines. When it comes to the deep past, one puts it together by inference or by statistics as best as one can. A whole new subject, cliometrics—mathematical treatment of historical records—has just

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been born; but what comes out is as often as not romanticized fabrications, oversimplifications and misinterpretations.

The doctrine of the culture is buttressed, strangely, by the platonic view of mathematics. If, after all, $e^{\pi i} = -1$ is a fact of the universe, an immutable truth, existing for all time, then surely Euler's discovery of this fact was mere accident. He was merely the medium through which the fact was vented. Sooner or later, so the argument goes, it would have—of necessity it had to have—been discovered by any one of a hundred other mathematicians.

Neither of the extreme views presented is adequate. Why did mathematics go to sleep for at least 800 years from about 300 to 1100? Presumably the genes of mathematical genius were present in the Mediterranean populace of the year 600 as they were in the days of Archimedes. Or take Tolstoy's philosophy of history. Despite his relegation of Napoleon to historical nonnecessity, everything that is of interest in *War and Peace* derives from the perception of individuals in their uniqueness. Despite the penchant of Marxists for cultural explanations, the relevance of V. I. Lenin to the Russian Revolution is not for them a subject of silent contemplation.

In the final analysis, the dichotomy between the doctrine of the individual and the doctrine of the culture is a false dichotomy, something like the argument of mind over matter or of the spirit and the flesh. Attempts have been made to reconcile the extreme views in a variety of ways. There is the reconciliation by means of time scale. This opinion holds that in the short run (say less than 500 years) the individual is important. In the long run (say more than 500 years), the individual is no longer important, but the culture is.

An intermediate view of great appeal was put forward by the American psychologist and philosopher William James. In his essay "Great Men and Their Environment," James wrote,

The community stagnates without the impulse of the individual; the impulse dies away without the sympathy of the community.

Now this is a very simple and undramatic formulation stating what must be apparent to most observers, that both elements are necessary. I was brought up in a textile town and have my own private formulation of the Jamesian synthesis. Woven cloth consists of two perpendicular sets of interlaced threads: the warp and the woof. Neither holds without the other. Similarly, the warp of society requires the woof of the individual.

Having now summarized James' view of the matter in this brief quotation, we can now pose a major question.

Is it possible to write a history of mathematics along the lines suggested by this quotation?

It would be nice to think so, but it has not been done and it is not at all certain it can be done.

Further Readings. See Bibliography

A. Adler; J. D. Bernal; S. Bochner [1966]; P. J. Davis [1976]; B. Hessen.
For a rebuttal see G. N. Clark; W. James [1917], [1961]; M. Kline [1972]; T. S. Kuhn; R. L. Wilder [1978].

The relation between society and the physical sciences has been rather more intensively explored than with mathematics. Here are some books in that direction:

A. H. Dupree; G. Basalla; L. M. Marsak; J. Ziman.

Assignments and Problem Sets

Varieties of Mathematical Experience *The Ideal Mathematician. The Individual and the Culture. The Current Individual and Collective Consciousness. A Physicist Looks at Mathematics. I. R. Shafarevitch and the New Neoplatonism. Unorthodoxies*

Topics to Explore

1. Proof: why and how
2. Evidence, intuition, and proof
3. The Goldbach conjecture
4. Undecidability
5. The theorem of Pythagoras
6. Dissection proofs
7. Goals of mathematics
8. Mathematics and religion

Essay Assignments

1. In “The Ideal Mathematician,” what particular “difficulties of communication emerge vividly” from the exchange between the ideal mathematician and the public relations officer? Can you find any evidence of contradiction between what the ideal mathematician believes and what he can explain to the student? Describe the tone of this essay.

2. In what way is the mathematician described in the essay “The Ideal Mathematician” ideal? Read “The Man Who Loves Only Numbers” by Paul Hoffmann in *The Atlantic Monthly*, November 1987. Would you describe Erdős as ideal?

3. You meet the ideal mathematician at a party. Make up a possible conversation you might have with him.

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4. There is a common feeling that mathematicians are introverted and socially misfit. How does this agree or disagree with your experience?

5. How would you describe the view of mathematics presented in the essay “A Physicist Looks at Mathematics”? How does this view compare to that expressed by the ideal mathematician?

6. How does Shafarevitch see the relation between religion and mathematics in the essay “I. R. Shafarevitch and the New Neoplatonism”? What role do applications play in Shafarevitch’s understanding of the goal of mathematics?

7. How does mathematics relate to your religion, or to whatever religion you are most familiar with?

8. How do the unorthodoxies described in the essay of the same name impact the development of mathematics? Can you think of analogies in the evolution of other disciplines?

9. What do you think of the attitude to unorthodoxies conveyed in this book? Is it fair and reasonable?

10. From your reading of “The Individual and the Culture” what do you believe influences the course of mathematical discovery? Summarize any dichotomy between the individual and culture when you explain your conclusion.

11. Read Mark Kac’s essay “Marginalia: How I Became a Mathematician.” Compare Kac to the ideal mathematician or to your perception of a typical mathematician.

12. Read “Women Mathematicians” by Dubriel-Jacotin (in *Mathematics, People-Problems-Results*, vol. 1, edited by Douglas Campbell and John Higgins) or the biography of Sophie Kovalevsky (in *A Convergence of Lives* by Koblitz). Choose one mathematician and compare her to the ideal mathematician or to your perception of a typical mathematician.

13. Read the biography of mathematician Jean van Heijenoort, who was secretary to Leon Trotsky (*Politics, Logic and Love: The Life of Jean van Heijenoort* by A. B. Fefferman). Consider the relationship between political idealism and the idealism expressed by mathematics.

Assignments and Problem Sets

14. Talk about the Pythagorean theorem. Why do mathematicians continue to prove it? How does it demonstrate the role of conjecture in mathematics?

15. Describe what undecidability is to your younger brother. Try to help him understand the connection between truth and provability in mathematics.

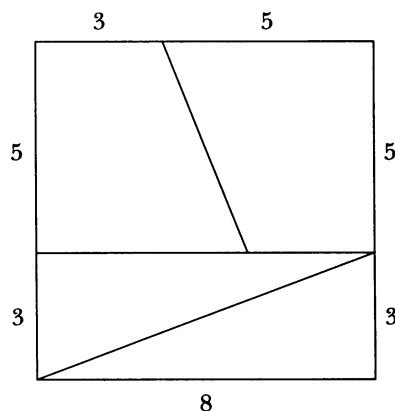
16. Write an article for your local newspaper describing the difference between a conjecture and a proof. Explain the role of evidence and intuition in each. Give examples.

17. You are a mathematician who thought you proved the Goldbach conjecture. Unfortunately, someone has found a flaw in your proof. You are now trying a different approach to the problem. *The Atlantic Monthly* wants to interview you. Describe the Goldbach conjecture to your readers and explain why you continue to try to prove this conjecture that has eluded mathematicians for more than 100 years.

Problems

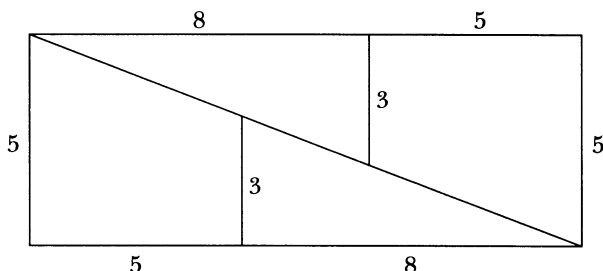
1. Create your own dissection proof of the Pythagorean theorem.

2. Construct an 8-inch square. Dissect it as follows:



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Rearrange the pieces of your square into a 5×13 -inch rectangle:



Calculate the areas of each of your figures. What conjecture can you make about the outcome of your experiment? Have you proved that $5 \times 13 = 8 \times 8$?

Computer Problem

Write a program that exhibits every even integer up to, say, 1000, as the sum two prime numbers (Goldbach's conjecture).

Suggested Readings

"Some Mathematicians I Have Known" by George Pólya in *Mathematics, People-Problems-Results*, Vol. 1, Douglas Campbell and John Higgins, eds. (Belmont, MA: Wadsworth International, 1984).

"Women Mathematicians" by Marie-Louise Dubriel-Jacotin in *Mathematics, People-Problems-Results*, Vol. 1, Douglas Campbell and John Higgins, eds. (Belmont, MA: Wadsworth International, 1984).

A Convergence of Lives by Anne H. Koblitz (Boston: Birkhäuser, 1983).

The Mathematical Tourist by I. Peterson (New York: W.H. Freeman and Company, 1988).

"Marginalia: How I Became a Mathematician" by Mark Kac in *The American Scientist*, Vol. 72.

"Mathematics—Our Invisible Culture" by Allen Hammond in *Mathematics, People-Problems-Results*, Douglas Campbell and John Higgins, eds. (Belmont, MA: Wadsworth International, 1984). (Also in Kac, *Enigma of Chance*, Berkeley: University of California Press, 1987.)

"Math Proof Refuted During Berkeley Scrutiny" by Gina Kolata in *Science*, Vol. 234, December 1986.

"Proving Is Convincing and Explaining" by Reuben Hersh in *Educational Studies in Mathematics*, Vol. 24, 1993.

Mathematical People—Profiles and Interviews by D. J. Albers and G. L. Alexanderson (Cambridge: Birkhäuser Boston, 1985).

Assignments and Problem Sets

More Mathematical People—Contemporary Conversations by D. J. Albers and G. L. Alexanderson (Cambridge: Brace, Jovanovich, 1990).

The Mathematics of Great Amateurs by J. L. Coolidge (New York: Oxford University Press, 1990).

“Proving Is Convincing and Explaining” by R. Hersh in *Educational Studies in Mathematics* 24, 1993.

Georg Cantor—His Mathematics and Philosophy of the Infinite by J. W. Dauben (Cambridge: Harvard University Press, 1979).

I Want to Be a Mathematician—An Automathography by Paul Halmos (New York: Springer-Verlag, 1985).

John von Neumann and Norbert Wiener—From Mathematics to the Technologies of Life and Death by Steve Heims (Cambridge: The MIT Press, 1981).

The Mathematical Career of Pierre de Fermat by Michael S. Mahoney (Princeton: Princeton University Press, 1973).

MATHEQUALS—Biographies of Women Mathematicians and Related Activities by Teri Perl (Menlo Park: Addison-Wesley, 1978).

“The Man Who Loves Only Numbers” by Paul Hoffmann in *The Atlantic Monthly*, November 1987.

Politics, Logic and Love: The Life of Jean van Heijenoort by A. B. Fefferman (Wellesley: AK Peters, 1993).

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