
EPILOGUES

Applied Mathematics Old and New

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A LONG WITH NATURAL LANGUAGES, mathematics is part of the symbolic infrastructure of civilization. Build a bridge, conduct an election, study the galaxies and in some way you will engage mathematics. Make an investment; take out a loan or an insurance policy, and mathematics turns up. Consider DNA profiling and you enter the field of mathematical genetics.

Mathematics is a subject that is one of the finest, most profound intellectual creations, a subject full of splendid architectures of thought most of which, sadly, require specialist training. This has resulted in a popular view of the subject that hardly goes beyond the multiplication table and despite the ubiquity of hand held computers, has given rise to the frequent admonition “Now you do the math!”

Despite the indifference, the world is being mathematized, computerized, chipified at an increasing rate and the public is hardly aware that this is going on. Mathematics is a method and a language employed in increasing amounts to probe, to predict, to create order and to format our social, economic and political lives. It is a method and an attitude that has diffused into medicine, cognitive science, war, entertainment, art, aesthetics, law, sports; it is a mode of thought that has created schools of philosophy, and has given support to views of cosmology, mysticism, and theology.

Allied to mathematics is the computer, a physical device whose inner logic is based on mathematical symbolisms. The products of the computer and their infusion (or intrusion) into our daily lives represent the greatest social triumph of the mathematical spirit since the ancient Chinese computed with colored rods or since the Babylonians toted up the prices of goats and onions on clay tablets. The ACM's (Association for Computer Machinery) classification system for research in computer science gives us a vivid understanding of its tremendous scope. Under the main heading “Applied computing,” it lists as its secondary headings, “Enterprise information systems,” “Physical sciences and engineering,” “Life and medical sciences,” “Law, social and behavioral sciences,” “Arts and humanities,” “Computers in other domains [Publishing, government, military, etc.],” “Operations research,” “Education,” “Docu-

ment management and text processing” and “Electronic commerce.” Moreover there are tertiary headings.

Broadly speaking, the applications of mathematics at the time of the first edition of the *Mathematical Experience* fall into the category of mathematical physics or engineering. New developments in these disciplines have been pursued vigorously and recent natural events such as earthquakes, tsunamis, tornados, floods, will bring forth new theoretical material as well as experience with computer models. Practically any technological innovation, e.g., cell phones, has some mathematical element in it. We are indeed living in an increasingly techno-mathematized world. A recent hospitalization for a minor complaint drove this home to me. I was subjected to a battery of tests carried out on a variety of devices each of which produced either numbers or a waveform. The medical attendant marked down all the numbers and perhaps a fast Fourier transform was applied to the waveform to obtain more numbers. As a patient, I was transfigured—some might say dehumanized—into a multicomponented vector.

Since the appearance of *The Mathematical Experience*, a number of new applications of mathematics of a variety of natures and of different mathematical depths or complexity have become prominent. In what follows, I will list a few and comment on one of them. Search engines, product striping, bioinformatics (such as DNA sequencing, indentifying and interpreting), jurimath (i.e., probability applied to legal evidence), pattern recognition and computer vision, interactive military training via computer images, epidemiological data mining for causes of diseases. Computational finance (some of which, of course, may have contributed to the crash three years ago), designer drugs, interpretation of all manner of radiological scans, programs dealing with security and privacy, game and auction theory. The latter feeds into search engines in that, e.g., whenever Google answers a query, it carries out a virtual auction among sponsors who have bid to place ads on the results pages. The pursuit of computational finance involves, at the very least, linear algebra, multivariable calculus, differential equations, and probability and statistics.

Every technological innovation has its down side. This is the message of the Myth of Prometheus who stole fire from the gods and gave it to the humans. Mathematics and its applications have a downside. It tends to replace experience by logic. In the name of logic, mathematics can create seeming impossibilities and nonsense. It frequently transforms what are subjective opinions into so-called objective conclusions that bear the cachet of absolute truth.

Natural languages are symbolic systems that have raised humans from the level of Caliban brutes; the same is true of mathematics. It is a language that has transformed our lives for good but it can go hog wild

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when it becomes an adjunct of new and unprecedented dimensions of human cruelty. The ethical issues raised by science and technology are in the daily papers. The ethical issues involved in mathematical thinking should also be recognized and pondered so as—in the words of the mathematician/philosopher Bertrand Russell—“to tip the balance on the side of hope against vast forces.”

Acknowledgment

I wish to thank Ernest S. Davis for suggesting a number of striking examples.

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REUBEN HERSH

SINCE WE WROTE *The Mathematical Experience* (ME) thirty years ago, professional academic philosophy of mathematics has begun to recognize actual mathematical practice as a legitimate philosophical topic (see, e.g., [Mancosu 2008]). A major philosophical dilemma was presented in ME—“fictionalism” vs. “realism” with regard to the nature of mathematical entities. But this had actually already found a resolution by the famous anthropologist Leslie White (in [Hersh 2006]). Alongside two traditional philosophical realms—subjective (private or interior) and physical (material or exterior), there exists a third major realm: the “cultural” (or “public” or “intersubjective”). This social or historical-cultural realm was long ago excluded from philosophy by Plato, because it is transient and ephemeral, whereas he considered that true knowledge must be eternal and unchanging. But of course knowledge of social reality is possible; indeed, it is crucially important in everyone’s daily life. It has long since become recognized as a valid realm of scientific study (anthropology, history, sociology, economics, etc.). It can’t be ignored, and it can’t be reduced to either the mental or the physical. It is in this public or intersubjective realm that mathematical entities are present, and open to coherent, empirically verifiable analysis. White’s insight that this is the realm of existence of

mathematical entities was adopted by one of us, and developed and extended in [Hersh 1997] and also in [Hersh 2006].

Recently, remarkable studies of the nature of mathematical practice have been made by the mathematicians William Byers and Alexandre Borovik, and by the linguists George Lakoff and Rafael Núñez.

Reuben Hersh, *What Is Mathematics, Really?*, Oxford University Press, New York, 1997.

Reuben Hersh, ed., *18 Unconventional Essays on the Nature of Mathematics*, Springer, New York, 2006.

Paolo Mancosu, *The Philosophy of Mathematical Practice*, Oxford University Press, New York, 2008.

William Byers, *How Mathematicians Think*, Princeton University Press, Princeton, 2007.

Alexandre Borovik, *Mathematics Under the Microscope*, American Mathematical Society, Providence, 2010.

George Lakoff and Rafael Núñez, *Where Mathematics Comes From*, Basic Books, New York, 2000.

New Mathematics

From the tremendous outflow of new mathematics in the last few decades, we can only briefly mention a few famous instances that are directly related to topics in this book. These are Andrew Wiles' proof of Fermat's last theorem; Grisha Perelman's proof of the Poincaré conjecture and the "Thurston program" in four-dimensional topology; wavelets as a generalization of Fourier analysis; fractals and the Mandelbrot set as a new kind of non-Euclidean geometry; and random matrices, in connection with Riemann's hypothesis on the zeroes of the zeta function.

Fermat's Last Theorem

Fermat's last theorem was proved by the English mathematician Andrew Wiles, who had become fascinated by the problem at the age of ten, after reading about it in E. T. Bell's *Men of Mathematics*. He worked on it secretly and privately for seven years, and then, after announcing success, had to go back to work to fill a major gap. This took another year to accomplish, with help from Richard Taylor.

Simon Singh, *Fermat's Last Theorem*, 1997, Fourth Estate, London

C. J. Mozzochi, *The Fermat Diary*, 2000, The American Mathematical Society, Providence

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The Poincaré Conjecture

The Poincaré conjecture was a famous open problem in topology: “any three-manifold (which can be thought of as embedded in four-space) which satisfies a certain simple and natural condition, is homeomorphic to the 3-sphere.” This conjecture had been extended by the U.S. mathematician William Thurston to a general conjecture that all 3-manifolds are topologically equivalent to combinations of eight fundamental types, each of which can be represented within three-dimensional non-Euclidean (hyperbolic) geometry. The proof of this program of Thurston’s was sensationally completed by a young Russian, Grigori (Grisha) Perelman.

Masha Gessen, *Perfect Rigor*, Houghton–Mifflin–Harcourt, Boston, 2009.

George G. Szpiro, *Poincaré’s Prize* Penguin Group, New York, 2008.

Wavelets

Wavelets are a powerful new tool in applied mathematics, discovered and developed by engineers. Mathematicians noticed what the engineers were up to, and followed up with sophisticated theories. Ingrid Daubechies of Belgium was a leader in this work.

Wavelets are defined as having three of the simple properties that are possessed by the sines and cosines of classical Fourier analysis. One of these essential properties is closure under addition (forming a complete linear space. We can add or subtract any sine or cosine function to any other.) Secondly, when a sine or cosine function is shifted to the right or left, the shifted function is still a member of the space of sines and cosines. Thirdly, the set of sine and cosine functions is preserved under change of scale: $\sin(nx)$ or $\cos(mx)$, where m or n is any real number, is again a member of the space of sines and cosines. A wavelet space is simply a space of functions built up from some carefully chosen basic function, by including all expansions and contractions of scale, along with all shifts and linear combinations. One can start with a very simple basic function—just a single “sawtooth,” made of two connected line segments, one rising from height 0 to height 1, and the second descending back down to 0. Then from this define a space of “wavelets”—all linear combinations of functions obtained by shifting, blowing up or shrinking this single saw tooth. Such a wavelet space turns out to be a powerful, convenient way to approximate functions arising in engineering and applied mathematics.

Ingrid Daubechies, *Ten Lectures on Wavelets*, Society for Industrial and Applied Mathematics, Philadelphia, 1992.

Michael Frazier, *An Introduction to Wavelets through Linear Algebra*, Springer, New York, 1999.

Fractals

Fractals have become very widely known as a computer-produced art form. Their graphs yield fascinating pictures that are sometimes useful in motion pictures—for instance, for creating a synthetic landscape on an unknown planet. Two old examples of fractals have often been taught to undergraduates. One example is Brownian motion (or “a sample path of the Wiener measure”). It can be described intuitively as the path of a particle that follows a continuous trajectory, but at every instant randomly and discontinuously changes direction, having a speed that is (almost always) infinite. This intuitive description is not easy to make mathematically clear, but this model has important physical and technical applications, and is a central concept in modern probability theory. A second example is the “Cantor middle-thirds set,” which is obtained from the unit interval by first deleting the “middle third” (all numbers greater than $1/3$ but less than $2/3$), then from each of the remaining two pieces again deleting the middle third, and so on to infinity. What’s left after all these middle thirds have been removed, the so-called “Cantor dust,” turns out to be an interesting object; for instance, while it is uncountably infinite, it has measure zero. We mention together these two “anti-intuitive” mathematical creatures because they both have the property of “self-similarity.” If you take a tiny piece of the Cantor middle-third set, or of a Brownian trajectory, and then scale it up, you get back the whole original set. Under change of scale, the original set is identical to an arbitrarily small subset!

The Franco-American mathematician Benoit Mandelbrot used computer simulations to discover many interesting mathematical structures having self-similarity, and he used them to describe real-world phenomena which have a “rough” or “uneven” boundary between two sets. A famous example is “the seashore of England,” which is very, very wiggly, no matter how closely you look at it. The human circulatory system, with its ever-tinier capillary veins, and the lungs of humans or other mammals, can be studied using notions from fractal geometry. Simple iteration procedures in the complex plane generate the fascinating fractal set called “the Mandelbrot set,” which is universal, in the sense that it contains within itself all the fractals generated by all such iterations.

Benoit B. Mandelbrot, *Fractals Form, Chance, and Dimension*, W. H. Freeman, San Francisco, 1977.

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Heinz-Otto Peitgen, Hartmut Jurgens, and Dietmar Saupe, *Chaos and Fractals*, Springer, New York, 2004.

Random Matrices and the Riemann Hypothesis

In 1972 a lucky accident at lunch at the Institute for Advanced Study revealed an amazing connection between the zeroes of Riemann's zeta function and the energy levels of atomic nuclei of heavy elements. Hugh Montgomery, a number theorist from the University of Michigan, had been introduced to the famous physicist Freeman Dyson, and was telling Dyson about his work, studying the statistical distribution of the gaps between zeros of the zeta function, on the critical line $\text{Re } z = 1/2$. Dyson saw with astonishment that they matched a distribution he was familiar with from quantum mechanics, that of the eigenvalues of random matrices that are used to model the interaction of elementary particles inside the nuclei of heavy atoms. Subsequently, numerical calculations by Andrew Odlyzko and others confirmed the match, out to *billions* of zeta zeroes and billions of random matrix eigenvalues. The more accurate the numerical calculations, the closer the two sets of numbers matched—one from analytic number theory, the other from quantum physics. The reasons for this astonishing relationship remain a mystery. A great amount of numerical and theoretical work was stimulated by this discovery, in search of a proof of Riemann's hypothesis, that all the nontrivial zeroes of the zeta function lie on the critical line. But decades have passed, and the hoped-for proof is still missing.

Francesco Mezzadri and Nina C. Snaith, *Recent Perspectives in Random Matrix Theory and Number Theory*, Cambridge University Press, Cambridge, UK, 2005.

Dan Rockmore, *Stalking the Riemann Hypothesis*, Pantheon Books, New York, 2005.

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ELENA ANNE CORIE MARCHISOTTO

IN THE MID-1990S, I was invited to join Reuben Hersh and Philip J. Davis in writing a sequel to *The Mathematical Experience* intended for use in “mathematics appreciation” general education classes, as well as in “capstone” courses for students majoring in mathematics, science, and philosophy of science, and for prospective teachers of these subjects. The sequel, the Study Edition, was reviewed by Ken Millett in the notices of the American Mathematical Society (“*The Mathematical Experience: A Book Review*”: <http://www.ams.org/notices/199710/comm-millett.pdf>).

At California State University Northridge (CSUN), the study edition has been used continuously, since it was published in 1995, in different versions of the above-mentioned courses. In recent years it has served as the text in a freshman general education course for nonmajors delivered in “hybrid” format (half online and half in the classroom) and in an upper division general education course for majors and nonmajors that is delivered totally online. The chapters of the Study Edition are partitioned into the following themes on websites designed to support these courses:

1. The Mathematical Landscape: What and Where is Mathematics?
2. The Course of Mathematical Evolution: The Role of the Individual and the Culture.
3. The Growth of Mathematics: Invention vs. Discovery.
4. The Aesthetic Appeal of Mathematics: Mathematicians as Pattern Finders.
5. Cognitive Styles and the Learning and Practice of Mathematics.
6. Mathematical Realities.

Lower Division General Education Course

The design of the online course is similar to the lower division course, but increases the coverage of materials from the Study Edition and includes more complex assignments. It confronts the challenge to motivate students who are anxious about mathematics and the need for teachers themselves to become more comfortable about teaching the sub-

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ject. but for the graduate student instructors teaching it as well. In particular, the goals for the course included (but were not limited to) the following:

- that students will benefit from the greater flexibility in scheduling “learning” (online instruction is not “real time”), and from working on the projects at their own pace, but within a certain time frame. The hybrid design exposes students to the vast supply of online resources that involve the application of mathematics to their fields. The activities structured through the course website are designed to encourage students to take individual responsibility for learning (via individual assignment, self-tests, etc.), as well as experience the power of collaborative work (online chat rooms, group responses to discussion questions, etc.). The web activities are intended to provide the context for the classroom experience so that there is a shared responsibility for learning among the individual student, the students groups, and the instructor.
- that hybrid course will strengthen the experience and teaching preparation of Mathematics graduate student instructors by familiarizing them with the expository and popular literature that invites contemplation of the interaction between mathematics and different aspects of their chosen discipline (e.g., the history and philosophy of mathematics). They will be exposed to applications of mathematics that they will not generally encounter in traditional classes. They will develop valuable skills in explaining mathematics to audiences of nonmathematicians.

At CSUN, the freshman course had always been one that gave professors considerable flexibility in the choice of topics to be covered. In creating the hybrid, the design team was able retain that aspect of the course by adopting the Study Edition of *The Mathematical Experience*. Each theme associated with the book is addressed on the class website with a variety of activities (reading, problem and essay assignments, self-quizzes, and group discussion forums) that are timed to both precede and concur with the exploration of specific mathematics topics in the classroom. Some of the topics coordinated with the above described themes have been the following:

Theme 1: The real number system; the Pythagorean theorem; Pythagorean triples, Fermat’s last theorem; Euclidean geometry and taxicab geometry; the Monty Hall problem.

Theme 2: Sequences, recursion, Fibonacci numbers.

Theme 3: Fibonacci sequences; the golden rectangle; Frieze patterns, tessellations, patterns of primes.

Theme 4: Pi (its “faces” (geometric ratio, irrational number) and its “appearance” in number theory, geometry, astronomy and engineering); the prisoner’s dilemma; dice games and coin tosses; statistical inference and clinical trials; similarity and self-similarity.

Theme 5: The language of randomness and selected topics in probability (e.g., applying probability theory to solving traffic problems); equivalent and nonequivalent definitions (e.g., different definitions of “dimension”); determining one’s learning style (North Carolina State University self-test) and applications of Polya’s heuristics.

Theme 6: Fractals, chaos theory, hypercubes (using the geometer sketchpad), the role of technology in mathematics.

The choice of these specific topics reflects the interests of the current instructors teaching the course. But the themes can easily suggest other topics. For example, theme 6 would surely embrace a discussion of mathematics in the workplace.

One goal of the hybrid, as designed, is to help students understand applications of mathematics to their major fields. To that end, the students in the course are partitioned into groups according to major. Their final project involves researching the expository literature to find connections between their majors and mathematics. They are asked to write a paper on their research and give a short report in the classroom with their group. The website provides detailed lists of suggested connections and an extensive bibliography that provides access to journal articles from the expository literature for each.

Upper Division General Education Course

Students who enroll in the upper division general education course, delivered totally online at CSUN, are both mathematics majors and non-mathematics majors. A large number of the nonmajors are planning to teach mathematics in K–12.

It is perhaps not surprising that students who have not chosen science or mathematics-related majors, often exhibit some anxiety about mathematics. In some cases such anxiety often precludes the student from choosing major fields for which mathematics is a requirement. They frequently viewed themselves as “mathematically disabled,” and fail to acquire even the most minimal mathematical skills necessary in today’s technological world. Perhaps more serious is that many maintain an aversion to mathematics (and science) for life, and perpetuate such views to their children. Most prospective teachers encounter such students who are anxious about mathematics, so any course that addresses this issue will be beneficial to them. What was surprising to me in teach-

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ing the upper division general education course at CSUN was to discover that the future K–12 teachers themselves often have poor attitudes towards mathematics, and are often anxious about the prospect of teaching the subject.

The design of the online course is similar to the lower division course, but increases the coverage of materials from the Study Edition and includes more complex assignments. It confronts the challenge to motivate students who are anxious about mathematics and the need for teachers themselves to become more comfortable about teaching the subject. It not only seeks to demonstrate how mathematics relates to different fields and hobbies in order to give students a greater appreciation of how mathematics relates to daily life, but it also incorporates activities which expose prospective teachers to strategies that will motivate their future students to learn it. Students learn how to research the professional and popular literature for resources and collaborate with their colleagues on joint research.

The Wider Audience

The use of the Study Edition of *The Mathematical Experience* is not only applicable for the populations addressed above, but can be modeled for university and science courses, as well as some senior high school classes. The Study Edition as well as the materials developed for these courses can also be made accessible to the general public, to provide access for lifelong learning.

The intention is not only to open a dialogue regarding what might be some new perspectives on the teaching of mathematics and science. The use of the Study Edition and the component website also teaches subject content, encourages the use of technology, and enables the understanding of procedures for conducting library research.

In my experience, the Study Edition of *The Mathematical Experience* provides concrete ways to further the intentions of the original book in ways that open a window on mathematics and those who practice it, that encourage the learning of interesting mathematics, and that seek to foster improved attitudes toward mathematics.



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