CHAPTER 2

THE PRIMES VIEWED AT LARGE

Introduction

Not very much is known about the distribution of the primes. On one hand, their distribution in short intervals seems extremely irregular. This is the reason why it appears impossible to find a simple formula describing the distribution of the primes in any detail. On the other hand, the distribution of the primes, viewed at large, can be very well approximated by simple formulas.

As a mathematical theory, the distribution of prime numbers is quite inhomogeneous. Although Euclid has proved that there are infinitely many primes, and Legendre and Gauss, as early as about 1800, conjectured some of the basic theorems, the theory is still a mixture of unsolved problems, more or less reasonable conjectures and a few proved theorems. The proved theorems mostly cover only simple cases, as compared with existing conjectures, and their proofs are often extremely complicated. Many of the proofs are not elementary, relying upon theorems in the theory of functions. This is the reason why in this and the next chapter we sometimes have to refrain from proving even some of the fundamental theorems.—We shall also discuss some of the existing conjectures, together with theoretical or numerical evidence which appears to support or to contradict the conjecture in question.

No Polynomial Can Produce Only Primes

In the search for formulas yielding all primes (and no other numbers) some remarkable polynomials have been found, whose values contain a surprisingly large proportion of primes. One of these is $P(x) = x^2 - x + 17$ which is prime for $x = 0, 1, 2, 3, \ldots, 16$ but obviously is composite for $x = 17$, since $P(17) = 17^2 - 17 + 17$ must be divisible by 17. Still more remarkable is the polynomial $x^2 - x + 41$, found by Euler, yielding primes for $x = 0, 1, 2, \ldots, 40$ but being composite for $x = 41$, since $41 | 41^2 - 41 + 41$.—We here take the opportunity to indicate a connection between these remarkable polynomials and those quadratic fields $\mathbf{Q}(\sqrt{D})$ in which the theorem of unique factorization into prime factors is valid. The two polynomials mentioned above as examples of polynomials rich in primes have precisely the discriminants $D = -67$ and $D = -163$,

H. Riesel, *Prime Numbers and Computer Methods for Factorization: Second Edition*,
Modern Birkhäuser Classics, DOI 10.1007/978-0-8176-8298-9_2,

mentioned in Theorem A4.4 on p. 295.—Another polynomial, being even richer in primes than $x^2 - x + 41$ is Edgar Karst's $2x^2 - 199$, yielding 150 primes (and the number 1) for $x = 0, 1, 2, \ldots, 198$. Other such polynomials are $x^2 + x + 27941$, discovered by Beeger in 1938, and some more recently found $x^2 + x + A$ for the following four values of A: 72491, 247757, 85403497, and 132874279528931, the last value of A representing the most prime-dense quadratic polynomial so far constructed.—See [1′].

The proof that no (non-constant) polynomial can yield only primes is quite simple. The assertion follows from the fact that any polynomial is unbounded when the variables tend to infinity. (It may happen that the values of a polynomial are bounded when the variables tend to infinity in *certain directions*, but not in *all* directions.) Suppose, there were a polynomial in n variables, $P(x, y, z, \ldots)$, yielding only primes for integer values of the variables. First, write $P = Q_n(x, y, z, \ldots)+$ terms of lower degree $+\ a$, where Q_n is a *homogenous* polynomial of degree n, representing all terms of the highest degree n of P, and a is the constant term, which we to begin with shall assume is $\neq 0$ and $\neq \pm 1$. If $n > 0$, then Q_n tends to infinity when the variables do so in at least one direction $(\xi, \eta, \zeta, \ldots)$ in n-space because if Q_n is considered as a polynomial of one of its variables only, it has this property. Next, because Q_n is continuous, Q_n tends to infinity not only in the direction mentioned, but also in some narrow cone, with $(\xi, \eta, \zeta, \ldots)$ as axis, and the same is true for $P(x, y, z, \ldots)$, being dominated by its highest degree terms, Q_n, as all the variables tend to infinity in the direction considered. Finally, if all the variables are chosen as integer multiples of the constant term a, clearly $a \mid P(x, y, z, \ldots)$. Since any cone will, only if we proceed far enough from the origin, contain points belonging to any point lattice $(al, am, an, \ldots)$, where $l, m, n, \ldots$ all are integers, the above construction leads to *integer* values of the variables $(x, y, z, \ldots)$ for which $a \mid P(x, y, z, \ldots)$ with $P(x, y, z, \ldots)$ large, i.e. with the quotient $|P(x, y, z, \ldots)/a| > 1$, showing that the value of $P(x, y, z, \ldots)$ is *composite* for the particular set of variables arrived at in this way. (It is only at this very last point of the proof that we have to make use of the assumption that $a \neq 0$ or $\neq \pm 1$.)

If, on the contrary, we assume that $a = 0$ or ± 1, then we start by moving the origin to a point $(b, c, d, \ldots)$ with integer coordinates, for which the value of $P(b, c, d, \ldots) = s$ is large. That such a point exists is clear from the proof given above. This transformation $x' = x - b$, $y' = y - c$, $z' = z - d, \ldots$ gives a new polynomial $P'(x', y', z', \ldots)$ with its constant term $P'(0, 0, 0, \ldots) = P(b, c, d, \ldots) = s$, now a *large* integer and thus $\neq 0$ or ± 1. Since $x, y, z, \ldots$ and $x', y', z', \ldots$ take integer values at the same time, the sets of values of P and of P' for integer values of the variables are also the same. Applying our proof for the case when $a \neq 0$ or ± 1 on P' we arrive at the conclusion that neither in one of these cases can P take only prime values. This concludes the proof that no (non-constant) polynomial can give only primes for integer values of the variables.

Formulas Yielding All Primes

There *exist* certain formulas which yield all the primes and no other numbers. However, these are misleading, in that they either presuppose, in some latent way, the knowledge of each individual prime, or rely on the bogus definition of a prime as a non-composite number.—As an example of a formula of the first kind consider the following algorithm: Taking the number $x = 0.203050701101301701902302 9...$ as a starting point, extract a suitable total of adjacent digits and the primes will emerge! This is quite obviously cheating, since all the individual primes must be known in advance before the number x can be exploited.

A formula of the second kind mentioned is the polynomial given below, *whose positive values consist of all the primes, when the variables range over all non-negative integers.* The polynomial also yields negative values (as a matter of fact it does so for most values of the variables), but these are not necessarily (negative) primes.

$$\begin{aligned}
&(k+2)\left\{1-[wz+h+j-q]^2-[(gk+2g+k+1)(h+j)+h-z]^2-\right.\\
&\quad-[16(k+1)^3(k+2)(n+1)^2+1-f^2]^2-[2n+p+q+z-e]^2-\\
&\quad-[e^3(e+2)(a+1)^2+1-o^2]^2-[(a^2-1)y^2+1-x^2]^2-\\
&\quad-[16r^2y^4(a^2-1)+1-u^2]^2-[n+l+v-y]^2-\\
&\quad-[(a^2-1)l^2+1-m^2]^2-[ai+k+1-l-i]^2-\\
&\quad-\left[\left\{(a+u^2(u^2-a))^2-1\right\}(n+4dy)^2+1-(x+cu)^2\right]^2-\\
&\quad-[p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m]^2-\\
&\quad-[q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x]^2-\\
&\quad\left.-[z+pl(a-p)+t(2ap-p^2-1)-pm]^2\right\}.
\end{aligned}$$

For the deduction of these types of formula, see [1].—The reader might wonder how this expression, being the product of two factors, $(k+2)$ and the complicated factor within the large curly brackets, can produce any primes at all. Well, this is merely an apparent paradox, since the only *positive* value assumed by the second factor happens to be the value 1. Looking closer at the second factor the reader will notice that it has the form

$$1-\sum_{i=1}^{14}(\text{expression}_i)^2$$

A factor of this form can obviously take only the values 1, 0, and *negative* values, and thus, once again we have been deceived, the whole thing being the following statement in disguise:

$k+2$ is prime if and only if the following diophantine system of 14 equations in 26 variables has a positive integral solution:

$$\begin{cases} wz+h+j-q=0 \\ (gk+2g+k+1)(h+j)+h-z=0 \\ 16(k+1)^3(k+2)(n+1)^2+1-f^2=0 \\ 2n+p+q+z-e=0 \\ e^3(e+2)(a+1)^2+1-o^2=0 \\ (a^2-1)y^2+1-x^2=0 \\ 16r^2y^4(a^2-1)+1-u^2=0 \\ n+l+v-y=0 \\ (a^2-1)l^2+1-m^2=0 \\ ai+k+1-l-i=0 \\ \left\{(a+u^2(u^2-a))^2-1\right\}(n+4dy)^2+1-(x+cu)^2=0 \\ p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m=0 \\ q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x=0 \\ z+pl(a-p)+t(2ap-p^2-1)-pm=0. \end{cases}$$

It is the author's hope that the reader has not been greatly disappointed by this revelation of the true nature of this prime-producing polynomial. Remember that *no polynomial can produce only primes*, so that there must be some trick involved in arriving at a polynomial producing all positive primes (and a large number of negative composite integers).

The Distribution of Primes Viewed at Large. Euclid's Theorem

We now give Euclid's extremely elegant proof of the infinitude of primes. (This proof, as a matter of fact, is frequently given as an example of indirect proof.) Suppose there exist only a finite number of primes, $p_1, p_2, \ldots, p_n$. Now, consider the integer $N = p_1 p_2 \cdots p_n + 1$. None of the existing primes divides N, since the division N/p_i will always give the remainder 1. Thus either N is a (new) prime number, or N contains a (new) prime factor, which is different from all the ones given. Therefore we conclude that there must be an infinitude of primes.

Example. The following construction starts with the prime 2 and yields at least one new prime in each step:

$$N_2 = 2+1 = 3 \text{ (prime)}$$

$$N_3 = 2\cdot 3+1 = 7 \text{ (prime)}$$

$$N_4 = 2\cdot 3\cdot 7+1 = 43 \text{ (prime)}$$

$N_5 = 2 \cdot 3 \cdot 7 \cdot 43 + 1 = 1807 = 13 \cdot 139$ (yielding two primes)

$N_6 = 2 \cdot 3 \cdot 7 \cdot 43 \cdot 139 + 1 = 251035 = 5 \cdot 50207$ (yielding two primes)

$N_7 = 2 \cdot 3 \cdot 7 \cdot 43 \cdot 139 \cdot 50207 + 1 = 126\,03664039 =$
$= 23 \cdot 1607 \cdot 340999$ (yielding three primes)

$N_8 = 2 \cdot 3 \cdot 7 \cdot 43 \cdot 139 \cdot 50207 \cdot 340999 + 1 = 4298368\,33293963 =$
$= 23 \cdot 79 \cdot 23653\,47734339$ (yielding three primes)

$N_9 = 2 \cdot 3 \cdot 7 \cdot 43 \cdot 139 \cdot 50207 \cdot 340999 \cdot 23652\,47734339 + 1 =$
$= 10165\,87861619\,05754590\,68761119 =$
$= 17 \cdot 1277\,70091783 \cdot 4\,68022256\,41471129$ (yielding three primes)

The Formulas of Gauss and Legendre for $\pi(x)$

The Prime Number Theorem

Let us start as Legendre and Gauss did and try to estimate the number of primes $\leq x$ by counting the number s of primes in suitable intervals. We choose the intervals $[0.95 \cdot 10^n,\ 1.05 \cdot 10^n]$ for $n = 3(1)7$ and the intervals $[10^n,\ 10^n + 150000]$ for $n = 8(1)15$. In each of these intervals of length d we calculate the proportion $(s/d) \times 10,000$ of prime numbers. The result is shown in the table on the next page, where the values given in the last two columns have been rounded. By studying the figures, we observe that the density of primes in an interval, centered around x, slowly decreases as x grows. Which law does this function obey? Comparing the values found for $x = 10^n$ and $x = 10^{2n}$, we find that the density of primes is approximately halved when x is squared. Mathematically, this is described by the function $1/\ln x$, since $1/\ln(x^2) = 0.5/\ln x$. Let us compare the density of primes with $1/\ln x$ (natural logarithms!), the values of which we have given in the last column of the table above. We see that both columns agree well except for the smaller values of x. This disagreement is obviously caused by local irregularities in the distribution of primes, which more heavily influence the number of primes in short than in long intervals. This striking agreement between the density of primes in an interval centred around x and the function $1/\ln x$ was discovered independently by Legendre and by Gauss, who formulated the following approximations to $\pi(x)$:

$$\pi(x) \approx \frac{x}{\ln x - B}, \quad B = 1.08366 \quad \text{(Legendre)} \tag{2.1}$$

$$\pi(x) \approx \int_2^x \frac{dx}{\ln x} \qquad \text{(Gauss)} \tag{2.2}$$

The number s of primes in different intervals $[x-d/2, x+d/2]$				
x	d	s	$\frac{10000s}{d}$	$\frac{10000}{\ln x}$
10^3	10^2	15	1500	1448
10^4	10^3	107	1070	1086
10^5	10^4	867	867	869
10^6	10^5	7227	723	724
10^7	10^6	62031	620	620
$10^8+75000$	150000	8154	544	543
$10^9+75000$	150000	7242	483	483
$10^{10}+75000$	150000	6511	434	434
$10^{11}+75000$	150000	5974	398	395
$10^{12}+75000$	150000	5433	362	362
$10^{13}+75000$	150000	5065	338	334
$10^{14}+75000$	150000	4643	310	310
$10^{15}+75000$	150000	4251	283	290

Nowadays, the latter approximation is usually replaced by the so-called logarithmic integral, defined by

$$\operatorname{li} x = \int_0^x \frac{dx}{\ln x},$$

where this improper integral has to be interpreted as

$$\operatorname{li} x = \lim_{\epsilon \to +0} \left(\int_0^{1-\epsilon} \frac{dx}{\ln x} + \int_{1+\epsilon}^{x} \frac{dx}{\ln x} \right). \tag{2.3}$$

The approximation by Gauss (2.2) and the logarithmic integral differ only by a constant, $\operatorname{li} 2 = 1.045$.—The approximations by Gauss and Legendre are in fact related, since

$$\begin{aligned} \int \frac{dx}{\ln x} &= \frac{x}{\ln x} + \int \frac{dx}{(\ln x)^2} = \\ &= \frac{x}{\ln x} + \frac{x}{(\ln x)^2} + 2\int \frac{dx}{(\ln x)^3} = \\ &= \frac{x}{\ln x - 1} - \frac{x}{(\ln x)^2(\ln x - 1)} + 2\int \frac{dx}{(\ln x)^3}. \end{aligned} \tag{2.3A}$$

The last two terms of (2.3A) are of smaller order of magnitude than the leading term $x/(\ln x - 1)$ as $x \to \infty$, and hence we have

$$\lim_{x\to\infty} \frac{\operatorname{li} x}{x/\ln x} = \lim_{x\to\infty} \frac{\operatorname{li} x}{x/(\ln x - 1)} = 1.$$

We may thus expect that the approximations given by Legendre and by Gauss should be about equally good, unless there is some particular reason in favour of one or the other.

The first mathematician to *prove* something in the direction of these formulas was Chebyshev, who around 1850 proved that

$$A \operatorname{li} x < \pi(x) < B \operatorname{li} x \tag{2.4}$$

for some suitably chosen values of the constants A and B. This establishes the fact that $\pi(x)$ has the same order of magnitude as $\operatorname{li} x$ (and thus also as $x/\ln x$) as $x \to \infty$. (In honour of this mathematician, $\pi(x) \approx \operatorname{li} x$ is often called Chebyshev's approximation.)

In 1896, Hadamard and de la Vallée-Poussin independently of each other proved *The Prime Number Theorem:*

$$\pi(x) \sim \operatorname{li} x, \quad \text{as} \quad x \to \infty. \tag{2.5}$$

(This formula reads: $\pi(x)$ is asymptotically equal to $\operatorname{li} x$.) The Prime Number Theorem can be reformulated as

$$\lim_{x\to\infty} \frac{\pi(x)}{\operatorname{li} x} = 1. \tag{2.6}$$

The Prime Number Theorem provides information about the error introduced by Gauss' approximation. It tells us that the relative error of the approximation, $(\operatorname{li} x - \pi(x))/\pi(x)$, tends to 0 as x tends to infinity. The *absolute error*, $\operatorname{li} x - \pi(x)$, however, may be large, something which will be discussed later.

Unfortunately, the scope of this book does not allow for a proof of the Prime Number Theorem. We must refer the reader to [2], which gives an elementary (but very tedious) proof, or to [3], which provides a proof based on the theory of functions.

Exercise 2.1. Computing $\operatorname{li} x$. Write a `FUNCTION li(x)` for $\operatorname{li} x$, utilizing the continued fraction expansion

$$\operatorname{li}(e^z) = \frac{e^z|}{|z} - \frac{1|}{|1} - \frac{1|}{|z} - \frac{2|}{|1} - \frac{2|}{|z} - \frac{3|}{|1} - \frac{3|}{|z} - \cdots$$

Compute the continued fraction backwards, starting with the term $10/z$. Test values can be found in Table 3 (compute $\operatorname{li} x$ as $\pi(x) + (\operatorname{li} x - \pi(x))$).

The Chebyshev Function $\theta(x)$

A variation of the prime number theorem which is sometimes useful depends on the *Chebyshev function* $\theta(x) = \sum_{p\le x} \ln p$. Using the prime number theorem, this function can be estimated in the following way:

$$\theta(x) = \int_{2-0}^{x} \ln x \, d\pi(x) = [\pi(x)\ln x]_{2-0}^{x} - \int_{2}^{x} \frac{\pi(x)}{x}\, dx =$$
$$= \pi(x)\ln x + O\left(\frac{x}{\ln x}\right),$$

and thus

$$\theta(x) = \sum_{p\le x} \ln p \sim \pi(x)\ln x \sim x, \tag{2.6A}$$

which is equivalent to the prime number theorem. In order to see how good this approximation for $\theta(x)$ is, we give, for some selected values of x, the value of $\theta(x)$:

x	10^2	10^3	10^4	10^5
$\theta(x)$	83.73	956.25	9895.99	99685.4

See also p. 57 for some sharp estimates of $\theta(x)$.

The Riemann Zeta-function

The approximations to $\pi(x)$ by Gauss and by Legendre were found by empiric methods. Riemann was the first who with great success systematically deduced relations between the primes and already known mathematical functions. Riemann's starting point was a relation discovered already by Euler,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{1-p^{-s}}, \tag{2.7}$$

where the infinite product is taken over all primes. The function $\zeta(s)$ is called the Riemann zeta-function and (2.7) is a highly informative formula from which many properties of the primes can be deduced. It is very important because it relates each individual prime p to the simple sum $\sum n^{-s}$. Thus the properties of the primes are via (2.7) transformed into properties of the sum and this *without the necessity of specifying each individual prime!*—(2.7) can be deduced in the following manner: Write each factor of the infinite product as a (convergent) geometric series

$$\frac{1}{1-p^{-s}} = 1 + p^{-s} + p^{-2s} + p^{-3s} + \cdots$$

Multiply all these series together to obtain the result

$$\sum (2^{\alpha_1} 3^{\alpha_2} \dots p_n^{\alpha_n})^{-s}$$

where the summation must cover all combinations of non-negative integer exponents α_i and all primes p_i. Now the fundamental theorem of arithmetic tells us that the products so obtained are precisely all the positive integers, raised to the power $-s$, because each integer has a unique representation in the form $2^{\alpha_1} 3^{\alpha_2} \dots p_n^{\alpha_n}$. But this is exactly what appears in $\sum n^{-s}$.

Riemann's basic idea was to put the so-called theory of analytic functions (differentiable functions of one complex variable) to work. This is effected by extending the variable s, which in (2.7) is restricted to $s > 1$, to a complex variable $s = \sigma + it$. In order for $\sum_{n=1}^{\infty} n^{-s}$ to converge, σ must be > 1. However, with so-called analytic continuation, Riemann was able to extend the function to all real and complex values of s except $s = 1$, which is a singularity, and for which $|\zeta(s)| = \infty$. The extension of the definition of $\zeta(s)$ to all s with $\sigma > 0$ is achieved by considering

$$\begin{aligned}
(1 - 2 \cdot 2^{-s})\zeta(s) &= \zeta(s) - 2 \cdot 2^{-s}\zeta(s) = \\
&= 1 + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + \cdots - \\
&\quad - 2 \cdot 2^{-s} - 2 \cdot 4^{-s} - 2 \cdot 6^{-s} - \cdots = \\
&= 1 - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.
\end{aligned}$$

This series converges for all s with $\sigma > 0$. Thus $\zeta(s)$ may be written as

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad \text{when } \sigma > 0, \text{ if } s \neq 1. \tag{2.8}$$

For $\sigma \leq 0$ the so-called functional equation

$$\zeta(1 - s) = 2^{1-s}\pi^{-s} \cos \frac{\pi s}{2}\, \Gamma(s)\zeta(s) \tag{2.9}$$

can be used to obtain the values of $\zeta(s)$. Here $\Gamma(s)$ is the gamma function, defined for $\sigma > 0$ as

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x}\, dx. \tag{2.10}$$

Exercise 2.2. Computing $\zeta(s)$. $\zeta(s)$ may be computed by aid of the Euler–Maclaurin sum formula:

$$\begin{aligned}
\zeta(s) &= \sum_{n=1}^{N-1} n^{-s} + \frac{1}{s-1} N^{1-s} + \frac{1}{2} N^{-s} + \frac{s}{12} N^{-s-1} - \\
&\quad - \frac{s(s+1)(s+2)}{720} N^{-s-3} + \frac{s(s+1)\cdots(s+4)}{30240} N^{-s-5} - \cdots
\end{aligned}$$

This is a so-called semiconvergent asymptotic series. The truncation error made in breaking off the summation is always less than the immediately following term. Write a `FUNCTION zeta(s)` performing the summation of the series. In single precision arithmetic (about 8 decimal digits), a suitable value of N is 10. Break off the series immediately before the last term written out above. Check your values against the known exact values of $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$, $\zeta(8) = \pi^8/9450$ and $\zeta(10) = \pi^{10}/93555$.

The connection between the Riemann zeta-function and the primes is evident from the infinite product in (2.7). To write $\pi(x)$ with the aid of $\zeta(s)$ will require theorems and techniques from the theory of functions that we have to omit. A detailed deduction can be found in [4]. We can only hint at some of the highlights of the theory. The first is that Riemann found the function

$$f(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \frac{1}{4}\pi(x^{\frac{1}{4}}) + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}\pi(x^{\frac{1}{n}}) \tag{2.11}$$

to be of more fundamental importance than $\pi(x)$ itself for the study of the desired relation between $\pi(x)$ and $\zeta(s)$. This sum is only formally infinite, since $\pi(x^{1/n}) = 0$, as soon as $x^{1/n}$ decreases below 2, which will happen as soon as $n > \ln x/\ln 2$. $f(x)$ has jump discontinuities with jumps $1/r$ when x passes a prime power p^r. (When x passes a prime p, this is regarded as the prime power p^1.) Let us now (as is usual in working with trigonometric series) modify the definition (2.11) to read

$$f(p^r) = \lim_{\epsilon \to +0} \frac{f(p^r - \epsilon) + f(p^r + \epsilon)}{2}. \tag{2.12}$$

This means that all jumps have been split into two equal halves. The graph of the function $f(x)$, modified in this way, is shown on p. 48. Each time x is a prime power p^r, $f(x)$ increases by the amount $1/r$, but $f(x)$ is constant between the jumps.

The simplest relation between the Riemann zeta-function and $f(x)$ is

$$\frac{\ln \zeta(s)}{s} = \int_1^{\infty} f(x) x^{-s-1}\, dx, \tag{2.13}$$

which can be deduced as follows:

$$\ln \zeta(s) = \ln \prod_p (1 - p^{-s})^{-1} = -\sum_p \ln(1 - p^{-s}) =$$

$$= \sum_p p^{-s} + \frac{1}{2}\sum_p p^{-2s} + \frac{1}{3}\sum_p p^{-3s} + \cdots \tag{2.14}$$

Using Stieltjes' integrals (see Appendix 11 for this important tool!) and performing integration by parts, we obtain

$$s\int_1^\infty \pi(x)x^{-s-1}\,dx = \left[-x^{-s}\pi(x)\right]_1^\infty + \int_1^\infty x^{-s}\,d\pi(x) = \sum_p p^{-s}, \tag{2.15}$$

since for $s > 1$ the integrated term vanishes both at $x = \infty$ and $x = 1$. In an analogous manner, we find that

$$s\int_1^\infty \pi(x^{\frac{1}{n}})x^{-s-1}\,dx = \int_1^\infty x^{-s}\,d\pi(x^{\frac{1}{n}}) = \sum_p p^{-ns}. \tag{2.16}$$

Next, inserting the definition of $f(x)$ from (2.11) in the integral of (2.13), and using (2.15) and (2.16) as well as taking (2.14) into account, the integral is reduced to $\ln\zeta(s)/s$.—Formula (2.13) can, in fact, be transformed in several different ways. One form, in which the so-called Mellin transform has been used, is the following:

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{x^s}{s}\ln\zeta(s)\,ds = \begin{cases}\displaystyle\sum_{j=1}^\infty \frac{1}{j}\pi(x^{\frac{1}{j}}), & \text{if } x \neq p^m\\ \displaystyle\sum_{j=1}^\infty \frac{1}{j}\pi(x^{\frac{1}{j}}) - \frac{1}{2m}, & \text{if } x = p^m.\end{cases} \tag{2.17}$$

In this equation $f(x)$ has been expressed with the aid of known functions. However, the integral is very difficult to determine with high accuracy; thus (2.17) is of no immediate value in the computation of $f(x)$ and $\pi(x)$. An efficient formula for numerical computations has been devised by Lagarias and Odlyzko, and used to calculate $\pi(x)$ for large values of x; see p. 33!

The Zeros of the Zeta-function

There is also a reasonably simple connection between $f(x)$ and the *zeros* of the Riemann zeta-function:

$$f(x) = \operatorname{li} x - \sum_\rho \operatorname{li}(x^\rho) + \int_x^\infty \frac{dt}{(t^2-1)t\ln t} - \ln 2. \tag{2.18}$$

This formula was published by Riemann in 1859 and proved by von Mangoldt in 1895. Here ρ denotes all the complex zeros of the Riemann zeta-function, and $\operatorname{li}(x^\rho) = \operatorname{li}(e^{\rho\ln x})$ is the logarithmic integral of a complex variable, defined by

$$\operatorname{li}(e^{u+iv}) = \int_{-\infty+iv}^{u+iv}\frac{e^z}{z}\,dz, \quad v \neq 0. \tag{2.19}$$

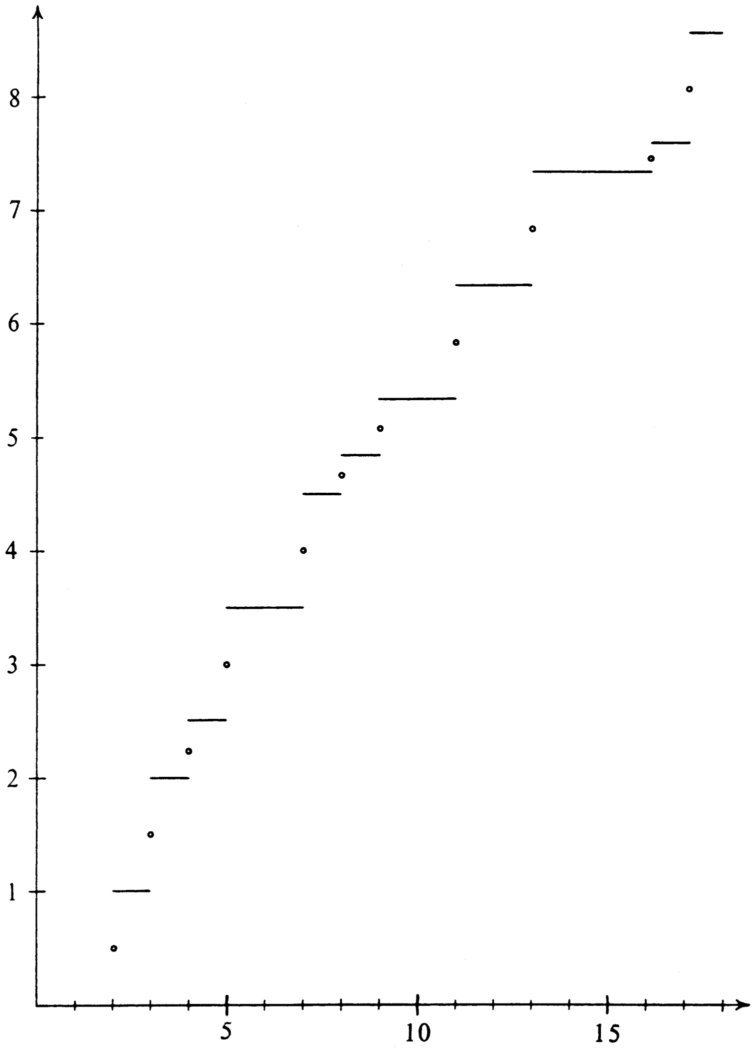

Figure 2.1. The step-function $f(x) = \sum_{n=1}^{\infty} \frac{\pi(x^{1/n})}{n}$

A complication is that $\sum \mathrm{li}(x^\rho)$ is only a conditionally convergent infinite series; thus the value of its sum is dependent on the order of summation of its terms. The summation has to be carried out in increasing order of magnitude of the complex zeros. It was in connection with these investigations that Riemann formulated his famous conjecture, for which no proof has been found so far. Riemann conjectured that the complex zeros of the zeta-function all have their real part $\sigma = 1/2$ and are all simple zeros. By means of extremely laborious but delicate computations, it has been proved by Brent, van de Lune, te Riele and Winter that the first 1,500,000,001 zeros on each side of the σ-axis all lie *exactly* on the line $\sigma = 1/2$ and are simple zeros, see [5] and [6]. This covers the segment $|t| < 545, 439, 823.215$ of the line $\sigma = 1/2$, since the number of zeros below t is $\approx u \ln u - u - 1/8$, with $u = t/2\pi$. A detailed error analysis of the computations executed in the computer program used proves the correctness of the computer's results.—Also, much larger intervals of substantial length have been searched for the zeros, see [6′], where a large interval about the 10^{20}th zero is studied.—*If* the Riemann hypothesis is true, then the order of magnitude of the terms $\mathrm{li}(x^\rho)$ in (2.18) will be $O(\sqrt{x})$, and the function $f(x)$ will be approximated by its leading term, $\mathrm{li}\, x$, with an error of the order of magnitude $O(\sqrt{x}\ln x)$, as can be proved by a detailed analysis. Thus, *assuming the truth of the Riemann hypothesis,* we have the conjectured error term

$$f(x) \stackrel{c}{=} \mathrm{li}\, x + O(\sqrt{x}\ln x). \tag{2.20}$$

Conversion From $f(x)$ Back to $\pi(x)$

If $\mathrm{li}\, x$ is a good approximation to $f(x)$, what can be said about $\pi(x)$? In the definition (2.11), $f(x)$ is a rather complicated function of $\pi(x)$. Fortunately, however, there exists an inversion formula by which $\pi(x)$ can be expressed in terms of $f(x)$:

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} f(x^{1/n}). \tag{2.21}$$

The function $\mu(n)$, which appears here, is called the Möbius function and is defined by the rules

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n \text{ contains some multiple prime factor} \\ (-1)^k, & \text{if } n \text{ is the product of } k \text{ distinct primes.} \end{cases} \tag{2.22}$$

The most important property of the Möbius function is

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1. \end{cases} \tag{2.23}$$

(Note that also the improper divisors $d = 1$ and $d = n$ of n have to be included in this formula!) To prove (2.23), suppose that $n = \prod_{i=1}^{s} p_i^{\alpha_i}$, with all p_i being different primes. Then $d|n$, and $\mu(d) = (-1)^k$ if d is a product of precisely k different members of the set of s primes p_i. This case will occur for $\binom{s}{k}$ different divisors d of n. All divisors d of n containing one or several of the primes p_i *twice or more* have $\mu(d) = 0$, according to the definition of $\mu(d)$. Thus

$$\sum_{d|n} \mu(d) = \sum_{k=0}^{s} (-1)^k \binom{s}{k} = (1-1)^s = 0, \quad \text{if} \quad s \geq 1.$$

The relation (2.23) has as one of its consequences that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}, \tag{2.24}$$

since

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{d=1}^{\infty} \frac{\mu(d)}{d^s} = \sum \frac{\sum_{d|md} \mu(d)}{(md)^s} =$$

$$= \sum_{n=1}^{\infty} \frac{\sum_{d|n} \mu(d)}{n^s} = 1 \cdot 1^{-s} = 1.$$

That (2.21) is equivalent to (2.11) is now proved in the following way:

$$\sum_{1}^{\infty} \frac{\mu(n)}{n} f(x^{\frac{1}{n}}) = \sum_{n=1}^{\infty} \left(\frac{\mu(n)}{n} \sum_{m=1}^{\infty} \frac{\pi(x^{1/mn})}{m} \right) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\mu\left(\frac{mn}{m}\right)}{mn} \pi(x^{1/mn})$$

$$= \sum_{u=1}^{\infty} \sum_{m|u} \frac{\mu\left(\frac{u}{m}\right)}{u} \pi(x^{1/u}) = \sum_{u=1}^{\infty} \left(\frac{\pi(x^{1/u})}{u} \sum_{d|u} \mu(d) \right) = \pi(x),$$

according to (2.23).

The Riemann Prime Number Formula

If $f(x)$ in (2.21) is approximated by $\operatorname{li} x$, we obtain Riemann's famous prime number formula:

$$R(x) = \sum_{1}^{\infty} \frac{\mu(n)}{n} \operatorname{li}(x^{\frac{1}{n}}) =$$

$$= \operatorname{li} x - \frac{1}{2} \operatorname{li}(x^{\frac{1}{2}}) - \frac{1}{3} \operatorname{li}(x^{\frac{1}{3}}) - \frac{1}{5} \operatorname{li}(x^{\frac{1}{5}}) + \frac{1}{6} \operatorname{li}(x^{\frac{1}{6}}) - \cdots \tag{2.25}$$

The leading term in (2.25) is the approximation of Gauss, $\operatorname{li} x$, and the maximum error of (2.25) is of the same order of magnitude as it is for the approximation $\operatorname{li} x$. For the numerical computation of $\operatorname{li} x$ and $R(x)$, it is convenient to use their power series expansions in the variable $\ln x$. The deduction for $\operatorname{li} x$ runs

$$\int \frac{dx}{\ln x} = \text{(putting } e^t = x\text{)} \int \frac{e^t}{t} dt = \int \sum_{n=0}^{\infty} \frac{t^{n-1}\, dt}{n!} =$$

$$= \ln t + \sum_{1}^{\infty} \frac{t^n}{n!\, n} + C_1 = \ln \ln x + \sum_{1}^{\infty} \frac{(\ln x)^n}{n!\, n} + C_1.$$

If the limits of integration are chosen to be 0 and x, it can be shown that the constant of integration assumes the value γ = Euler's constant = 0.5772..., which gives

$$\operatorname{li} x = \gamma + \ln\ln x + \sum_{n=1}^{\infty} \frac{(\ln x)^n}{n!\,n}. \tag{2.26}$$

The function $R(x)$ can be transformed into the so-called Gram series:

$$R(x) = \sum_{1}^{\infty} \frac{\mu(n)}{n} \operatorname{li}(x^{1/n}) = \text{(putting } x = e^t\text{)} \sum_{1}^{\infty} \frac{\mu(n)}{n} \operatorname{li}(e^{t/n}) =$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left(\gamma + \ln \frac{t}{n} + \sum_{m=1}^{\infty} \frac{(t/n)^m}{m!\,m} \right) =$$

$$= (\gamma + \ln t) \sum_{1}^{\infty} \frac{\mu(n)}{n} - \sum_{1}^{\infty} \frac{\mu(n)\ln n}{n} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\mu(n) t^m}{n^{m+1} m!\,m} =$$

$$= 1 + \sum_{m=1}^{\infty} \left(\frac{t^m}{m!\,m} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{m+1}} \right) = 1 + \sum_{m=1}^{\infty} \frac{t^m}{m!\,m\,\zeta(m+1)}. \tag{2.27}$$

The deduction above is dependent on the following two limits:

$$\sum_{1}^{\infty} \frac{\mu(n)}{n} = \lim_{s\to 1} \sum_{1}^{\infty} \frac{\mu(n)}{n^s} = \lim_{s\to 1} \frac{1}{\zeta(s)} = 0, \tag{2.28}$$

and

$$\sum_{1}^{\infty} \frac{\mu(n)\ln n}{n} = \lim_{s\to 1} \sum_{1}^{\infty} \frac{\mu(n)\ln n}{n^s} = \lim_{s\to 1} \sum_{1}^{\infty} \mu(n) \frac{d}{ds}\left(-\frac{1}{n^s}\right) =$$

$$= \lim_{s\to 1} -\frac{d}{ds} \sum_{1}^{\infty} \frac{\mu(n)}{n^s} = \lim_{s\to 1} -\frac{d}{ds}\frac{1}{\zeta(s)} = \lim_{s\to 1} \frac{\zeta'(s)}{\zeta^2(s)} = -1. \tag{2.29}$$

The value of the last limit follows from the fact that $\zeta(s)$ in the vicinity of $s = 1$ has the leading term $1/(s-1)$, and that thus its derivative $\zeta'(s)$ has the leading term $-1/(s-1)^2$.

In order to provide the reader with some idea of the accuracy in the approximations $\operatorname{li} x$ and $R(x)$, we give some values in a small table:

Accuracy of the approximations $\operatorname{li} x$ and $R(x)$			
x	$\pi(x)$	$\operatorname{li} x - \pi(x)$	$R(x) - \pi(x)$
10^2	25	5	1
10^3	168	10	0
10^4	1,229	17	−2
10^5	9,592	38	−5
10^6	78,498	130	29
10^7	664,579	339	88
10^8	5,761,455	754	97
10^9	50,847,534	1,701	−79
10^{10}	455,052,511	3,104	−1,828
10^{11}	4,118,054,813	11,588	−2,318
10^{12}	37,607,912,018	38,263	−1,476
10^{13}	346,065,536,839	108,971	−5,773
10^{14}	3,204,941,750,802	314,890	−19,200
10^{15}	29,844,570,422,669	1,052,619	73,218
10^{16}	279,238,341,033,925	3,214,632	327,052
10^{17}	2,623,557,157,654,233	7,956,589	−598,255
10^{18}	24,739,954,287,740,860	21,949,555	−3,501,366

A denser table can be found at the end of the book. We conclude that for large values of x, $\operatorname{li} x$ and in particular $R(x)$ are close to $\pi(x)$. The more detailed table shows that $R(x) - \pi(x)$ has a great number of sign changes, which often characterizes a good approximation.

The Sign of $\operatorname{li} x - \pi(x)$

From the table above we might get the impression that $\operatorname{li} x$ is always $> \pi(x)$. This is, as a matter of fact, an old famous conjecture in the theory of primes. Judging only from the values given in the table, we might even try to estimate the order of magnitude of $\operatorname{li} x - \pi(x)$ and find it to be about $\sqrt{x}/\ln x$. However, *for large values of x, this is completely wrong!* On the contrary, Littlewood has proved that the function $\operatorname{li} x - \pi(x)$ changes sign infinitely often. Since those days people have tried to expose a specific example of $\operatorname{li} x < \pi(x)$. Moreover, Littlewood's bounds for $\operatorname{li} x - \pi(x)$ show that this difference for some large values of x can become much larger than $\operatorname{li} x - R(x)$, showing that $R(x)$ for such values of x is close to $\operatorname{li} x$ rather than close to $\pi(x)$. Thus the good approximation of $R(x)$ to $\pi(x)$ is also to a large extent deceptive. If our table could be continued far enough, there would be arguments x for which $\operatorname{li} x - \pi(x) = 0$, while $R(x) - \pi(x)$, for such values of x, would be fairly large!—The best result so far obtained in the attempts to expose an x, leading to a negative value of $\operatorname{li} x - \pi(x)$, is by Herman te Riele, who in 1986 showed that between $6.62 \cdot 10^{370}$ and $6.69 \cdot 10^{370}$ there are at least 10^{180} consecutive integers, for which $\operatorname{li} x < \pi(x)$. See [7′].

The study of the difference $\operatorname{li} x - \pi(x)$ shows that in some cases reasoning based on numerical evidence can lead to wrong conclusions, even if the evidence seems overwhelming!

The Influence of the Complex Zeros of $\zeta(s)$ on $\pi(x)$

As can be seen from the above table, not even $R(x)$ can reproduce the primes completely. The reason for this is that the contribution to $f(x)$ from the terms $-\operatorname{li}(x^{\rho})$ in (2.18) has not been taken into account in writing down the expression $R(x)$. (The other two terms have very little influence on $\pi(x)$, particularly for large values of x.) Since $\operatorname{li} x$ describes well the distribution of primes viewed at large, we may say that the complex zeros of the Riemann zeta-function induce the local variations in the distribution of the primes. It turns out that each pair $\rho_k = \frac{1}{2} \pm i\alpha_k$ of complex zeros of $\zeta(s)$ gives rise to a correction $C_k(x)$ to $f(x)$, of magnitude

$$C_k(x) = -2\Re(\operatorname{li} x^{\rho_k}) \approx \frac{-2\sqrt{x}\cos(\alpha_k \ln x - \arg \rho_k)}{|\rho_k| \ln x}. \tag{2.30}$$

Each of the corrections $C_k(x)$ to $f(x)$ is an oscillating function whose amplitude, which is $2\sqrt{x}/(|\rho_k| \ln x)$, increases very slowly with the value of x. The curves $y = C_k(x)$, for $k = 1, 2, \ldots, 5$, corresponding to the first five pairs of complex zeros of $\zeta(s)$, have been sketched in the diagrams on the next page.

Finally, all these corrections to $f(x)$ produce, in turn, corrections to $\pi(x)$ which are constructed following the same pattern as that exhibited in formula (2.25) for $R(x)$, but with $C_k(x)$ substituted for $\operatorname{li} x$. Adding up the first ten corrections of the type shown in the graphs on the next page, we arrive at the approximation $R_{10}(x)$ of $\pi(x)$, depicted on p. 55. Comparing $R_{10}(x)$ with $\pi(x)$, in the same figure, we see that $R_{10}(x)$ reproduces $\pi(x)$ quite faithfully.—The influence of the first 29 pairs of zeros, covering the segment $|t| \le 100$ of the line $\sigma = 1/2$, has been studied in detail in [7] by the author of this book.

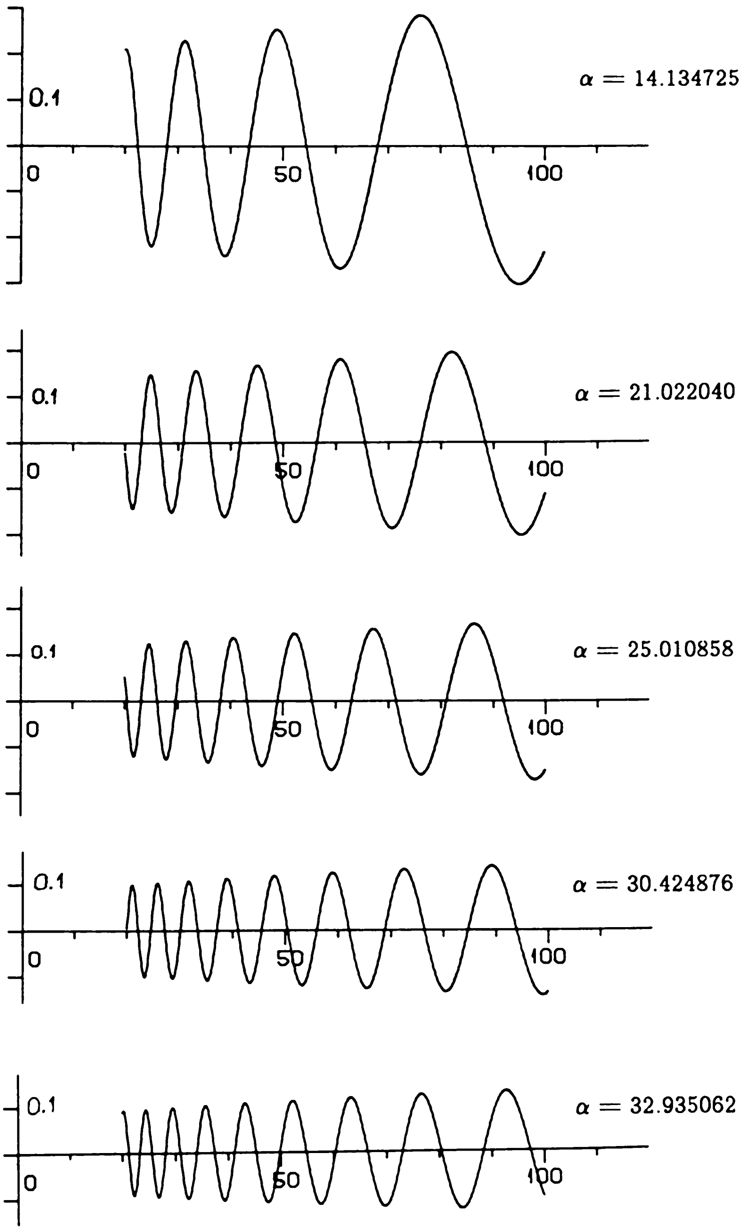

Figure 2.2. The functions $C_k(x) = -\frac{2\sqrt{x}}{|\rho_k| \ln x} \cdot \cos(\alpha_k \ln x - \arg \rho_k)$, $\rho_k = \frac{1}{2} + i\alpha_k$

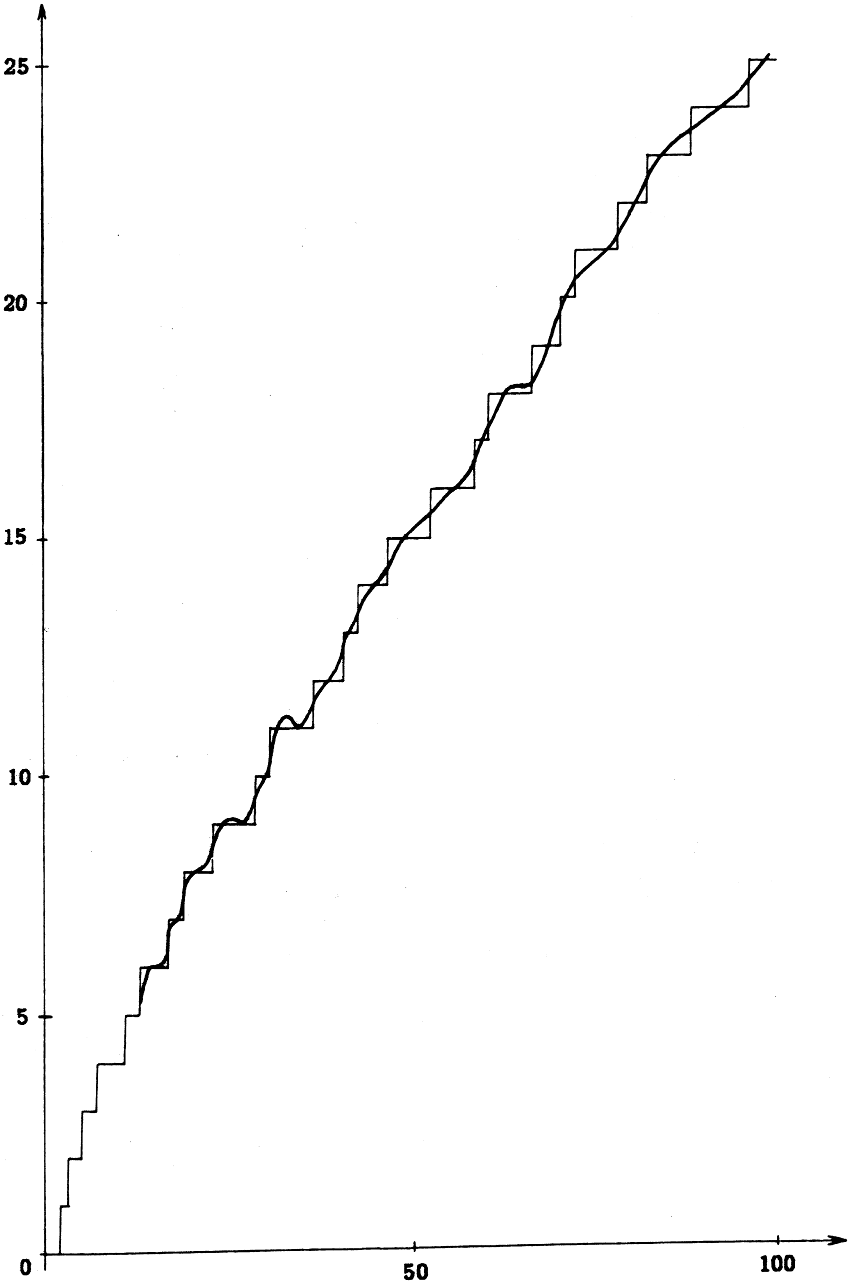

Figure 2.3. $\pi(x)$ vs. $R_{10}(x)$

The Remainder Term in the Prime Number Theorem

We have mentioned that assuming the Riemann hypothesis to be true, the Prime Number Theorem with remainder term states that

$$\pi(x) = \operatorname{li} x + h(x), \quad \text{with} \quad h(x) \stackrel{c}{=} O(\sqrt{x}\ln x), \tag{2.31}$$

where "$\stackrel{c}{=}$" indicates that the result is only conjectured, not proved. Until the Riemann hypothesis is proved, however, only weaker estimates of the remainder term $h(x)$ are known to be valid. If e.g. it could be proved that the upper limit of the real part σ of the zeros of the zeta-function is θ, for some θ between $\frac{1}{2}$ and 1, then it would follow that

$$\pi(x) = \operatorname{li} x + O(x^{\theta}). \tag{2.32}$$

Some actually *proved* estimates of the remainder term are given in the following expressions:

$$\pi(x) = \operatorname{li} x + O\left(xe^{-\sqrt{\ln x}/15}\right) \tag{2.33}$$

$$\pi(x) = \operatorname{li} x + O\left(xe^{-0.009(\ln x)^{3/5}/(\ln\ln x)^{1/5}}\right). \tag{2.34}$$

For proofs, see [8] or [9].—In these estimates of the error term, the numerical values of the constants latent in the O-notations are not given. These constants are theoretically computable, but the computations are so complicated that nobody has undertaken them. In spite of this lack of precision, the formulas are very useful in theoretical investigations, but mainly useless when it comes to computing the number of primes less than, say, 10^{100}. In such a situation it is not sufficient to know just only the order of magnitude of $\pi(x) - \operatorname{li} x$, when $x \to \infty$, but the numerical values of all the constants in the remainder term must also be known. This remark leads us over to

Effective Inequalities for $\pi(x)$, p_n, and $\theta(x)$

In 1962 some elegant inequalities for $\pi(x)$ have been discovered by J. Barkley Rosser and Lowell Schoenfeld [10]–[12]. We quote (without proving) the following:

$$\underset{x \ge 59}{\frac{x}{\ln x}\left(1+\frac{1}{2\ln x}\right) < \pi(x)} \quad \underset{x \ge 1}{< \frac{x}{\ln x}\left(1+\frac{3}{2\ln x}\right)} \tag{2.35}$$

$$\underset{x \ge 67}{\frac{x}{\ln x - \frac{1}{2}} < \pi(x)} \quad \underset{x \ge e^{\frac{3}{2}} = 4.48169}{< \frac{x}{\ln x - \frac{3}{2}}} \tag{2.36}$$

$$\begin{array}{ccccc} \operatorname{li} x - \operatorname{li} \sqrt{x} & \leq & \pi(x) & \leq & \operatorname{li} x. \\ & 11 \leq x \leq 10^8 & & 2 \leq x \leq 10^8 & \end{array} \tag{2.37}$$

The domain of validity is indicated below the inequality sign for each of the inequalities. We also give some estimates of the nth prime, p_n, the first one given found by Rosser and the second by Robin [13]:

$$\begin{array}{ccccc} n\left(\ln(n \ln n) - \frac{3}{2}\right) & < & p_n & < & n\left(\ln(n \ln n) - \frac{1}{2}\right) \\ & n \geq 2 & & n \geq 20 & \end{array} \tag{2.38}$$

$$\begin{array}{ccccc} n(\ln(n \ln n) - 1.0072629) & \leq & p_n & < & n(\ln(n \ln n) - 0.9385). \\ & n \geq 2 & & n \geq 7022 & \end{array} \tag{2.39}$$

In [11] the following inequalities are proved for the Chebyshev function $\theta(x) = \sum_{p \leq x} \ln p$:

$$\begin{array}{ccccc} 0.998684x & < & \theta(x) & < & 1.001102x, \\ & 1319007 \leq x & & 0 \leq x & \end{array} \tag{2.40}$$

and for smaller values of x

$$\begin{array}{llll} \theta(x) > 0.985x & \text{if} \quad x \geq 11927, & \theta(x) > 0.990x & \text{if} \quad x \geq 32057 \\ \theta(x) > 0.995x & \text{if} \quad x \geq 89387, & \theta(x) > 0.998x & \text{if} \quad x \geq 487381. \end{array} \tag{2.41}$$

These inequalities are useful when it comes to analysing strategies for the optimization of various algorithms in factorization and primality testing.

The Number of Primes in Arithmetic Progressions

Suppose that a and b are positive integers, and consider all integers forming an arithmetic progression $an + b$, $n = 0, 1, 2, 3, \ldots$ How many of these numbers $\leq x$ are primes? Denote this total by $\pi_{a,b}(x)$. In order for $an + b$ to contain any primes at all, it is apparent that the greatest common divisor (a, b) of a and b must be $= 1$ (except in the obvious case when b is a prime and a is chosen a multiple of b, in which instance $an + b$ will contain just one prime, b). If this condition is fulfilled, a certain proportion of all primes, $1/\varphi(a)$, where $\varphi(a)$ is Euler's function (see Appendix 2, p. 269), belong to the arithmetic series as $x \to \infty$. Utilizing the Prime Number Theorem this gives the following theorem of Dirichlet, the proof of which was not completed until de la Vallée-Poussin gave his proof of the Prime Number Theorem:

$$\lim_{x \to \infty} \frac{\pi_{a,b}(x)}{\operatorname{li} x} = \frac{1}{\varphi(a)}. \tag{2.42}$$

Analogous to the remainders of the Prime Number Theorem presented in (2.33)–(2.34), it has been proved that

$$\pi_{a,b}(x) = \frac{\operatorname{li} x}{\varphi(a)} + O\left(xe^{-\sqrt{\ln x}/15}\right) \tag{2.43}$$

etc. Proofs are furnished in [8]. Dirichlet's theorem states that the primes are approximately equi-distributed among those arithmetic series of the form $an+b$, for a fixed value of a, which contain several primes. Thus in the limit half of the primes are found in each of the two series $4n-1$ and $4n+1$, or in $6n-1$ and $6n+1$, and 25% of the primes are found in each of the four series $10n \pm 1$, $10n \pm 3$, etc. Dirichlet's theorem also tells us that every arithmetic series $an+b$ with $(a, b) = 1$ contains infinitely many primes. To give just one example: There are infinitely many primes ending in 33 333, such as 733 333, 1 133 333, 2 633 333, 2 833 333, 3 233 333, 3 433 333, 3 733 333, 4 933 333, 5 633 333, 6 233 333, ..., 1 000 133 333 ..., because the series $100\,000n + 33\,333$ contains, in the long run, $1/40\,000$ of all primes.

A readable account in which many of the topics of this chapter are discussed in greater detail is [14], which also contains a large bibliography.

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